Commun Nonlinear Sci Numer Simulat 103 (2021) 105995

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Research paper

Mathematical and asymptotic analysis of thermoelastic shells in normal damped response contact

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ARTICLE INFO

Article history:
Available online 14 August 2021

MSC:
41A60
74A15
35Q74
74K25
74K15
74M15

Keywords:
Thermoelastodynamics
Asymptotic analysis
Shells
Membrane
Contact

ABSTRACT

The purpose of this paper is twofold. We first provide the mathematical analysis of a dynamic contact problem in thermoelasticity, when the contact is governed by a normal damped response function and the constitutive thermoelastic law is given by the Duhamel-Neumann relation. Under suitable hypotheses on data and using a Faedo-Galerkin strategy, we show the existence and uniqueness of solution for this problem. Then, we study the particular case when the deformable body is, in fact, a shell and use asymptotic analysis to study the convergence to a 2D limit problem when the thickness tends to zero.

1. Introduction

Over the last decades, asymptotic methods have played an important role in the derivation and justification of reduced models for problems in solid mechanics involving three-dimensional bodies that feature one or two dimensions much smaller than the others. Take for example the case of rods, where the length is much bigger than the diameter of the cross section, or the case of plates and shells, who have a very small thickness compared to the extension of the middle section.

The basis to these methods was introduced by Lions in [1] and they were first applied to obtain models for plate bending problems (see [2,3]). Also, a thorough compilation of models for plates can be found in [4].

But the application of asymptotic methods goes far beyond plate models, having been used also to justify simplified models for elastic shells, or beams. The literature is quite vast, and we cite here just some classic works that use the asymptotic expansion method for the derivation of several beam models [5–8] while presenting also convergence results. The asymptotic modelling of rods in linearized thermoelasticity was also studied in [7].

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https://doi.org/10.1016/j.cnsns.2021.105995

A complete theory for elastic shells models was presented in [9], including models for membranes (elliptic and generalized) as well as flexural shells. The asymptotic method applied to these problems leads to two-dimensional problems which provide an accurate modelisation of the mechanical phenomena of interest, just like their three-dimensional counterparts, while showing better properties from the practical point of view, namely they can be numerically solved more efficiently, (see, for example [10,11]). In spite of its background history and growing number of contributions, the theory of asymptotic analysis for shell structures is far from being a closed subject, particularly when involving contact conditions. That is the reason why, in a series of recent papers, we studied the asymptotic analysis of viscoelastic shells [12–15]. Two-dimensional equations have been derived in [16,17] for dynamic shell problems by using asymptotic analysis, although no strong convergence results are provided. Recently, the asymptotic limit of the dynamic problem for thermoelastic elliptic shells was obtained in [18]. Nevertheless, none of above cited works on shells consider contact conditions on a part of the boundary, a void which is starting to be dealt with in the recent years, as we shall address below.

The variety of applications to real life problems with contact conditions has drawn the attention of many researchers throughout the years, and many references deal with its modeling, mathematical analysis and numerical approximation as long as with the study of the variational inequalities associated to them (see for example [19–23] and the references therein). Normal compliance and normal damped response conditions are two particular boundary conditions that have been used in the modelling of contact problems. For example, normal compliance has been used in [24,25] and [26] to obtain error estimates and convergence results in various situations, like wear and adhesion phenomena for elastic beams, and normal damped response has been considered in [27–28] to obtain results on the existence and uniqueness of solutions to some mechanical contact problems.

Also, focusing on dynamic contact problems one can find that several types of material and different boundary contact conditions have been considered. As an example, one can cite the works of Cocou et. al. in [30–32] for viscoelastic bodies with Signorini conditions and nonlocal friction for general and cracked bodies respectively. Also in [33] a viscoelastic body is considered under normal compliance conditions, and in [34] the authors proved some existence and uniqueness results concerning dynamic contact problems with friction for a viscoelastic material. With respect to dynamic thermo-viscoelastic contact problems, in [35] the author considered Coulomb friction and in [36] the authors extended the results given in [33] to this kind of materials.

In the most recent years we have devoted some efforts to the application of the asymptotic method to the analysis of several contact problems for elastic shells. In particular, in [37] the case of an elastic shell in frictionless unilateral contact with a rigid foundation was considered, finding a classification of the two dimensional limit problems as membranes or flexural shells which is just a natural extension of that obtained by Ciarlet et al. in [9] for the case without contact. Then, in [38] we studied the case of an elliptic membrane shell with arbitrary surface forces and gap function, and provided rigorous convergence results. Even more recently, in [39] we followed the work in [40] to obtain error estimates for linearly elastic shells in unilateral frictionless contact with a rigid foundation, by using several corrector techniques and in [41] a convergence result for elastic elliptic membrane shells under normal compliance contact conditions is presented.

In this paper, we will apply the asymptotic method for the first time to a contact problem for elliptic membrane shells in thermoelasticity, thus obtaining a two-dimensional limit problem whose validity is confirmed by providing rigorous convergence results. The combination of contact conditions and temperature evolution in a mechanical problem of this kind leads to a system of coupled nonlinear variational equations for which, to the best of our knowledge, no previous results of existence and uniqueness of solution were available in the literature (see the seminal paper [36], where a variety of contact problems in thermodynamics for general domains are studied). That is the reason why we devote the first part of the paper to analyze the existence and uniqueness of solution for both the three dimensional contact problem and its corresponding two dimensional limit. Then, we pass to the study of the convergence of the solution of the three-dimensional model to the two-dimensional one, as the thickness (small parameter) tends to zero, in the particular case of an elliptic membrane.

From the mathematical point of view, the main new challenges in comparison with our previous works are related with the (weak) convergence of sequences of boundary terms (where the contact conditions are defined). We cannot rely on the compacity for the trace operator in our functional framework, as we shall see, due to the lack of regularity on the transversal direction. The contact is modeled with a normal damped response function (see for example [42]) and the constitutive law follows the Duhamel-Neumannn relation (see [7] and references therein). Other choices can be made in both cases, thus leading to a vast yet unexplored field of mathematical problems, each one of them with its own mathematical challenges and with useful applications in real life situations. The present paper will serve of basis for future research in this direction.

The outline of the paper is the following: in Section 2, we present a result of existence and uniqueness of solution for the proposed contact problem in cartesian coordinates in a general domain, after describing its mechanical and variational formulations. In Section 3, the deformable body is assumed to be a shell, and the variational formulation of the contact problem is reformulated in curvilinear coordinates, and scaled to be posed on a reference domain whose thickness does not depend on the small parameter. Next, Section 4 is devoted to give a brief description of the formal asymptotic analysis which leads to the obtention of a two dimensional limit problem. In Section 5 the existence and uniqueness of solution for that problem is proved. Also, Section 5 contains the main result of the paper, namely the rigorous convergence result for the elliptic membrane case. Finally, we prove the convergence of the solution to the re-scaled version of this problem which bears the true physical meaning. The paper ends with conclusions and a compilation of possible future work.

2. A three-dimensional dynamic contact problem for thermoelastic bodies. The normal damped response case

In this section we will present the mathematical model arising when we consider a three-dimensional dynamic problem for a thermoelastic body when a contact condition is modeled by a normal damped response law. We will also provide a variational formulation and study existence and uniqueness of solution.

2.1. Statement of the problem

Let $\hat{\Omega}^\varepsilon$ be a three-dimensional bounded domain which is the reference configuration of a deformable body made of an homogeneous and isotropic elastic material, with Lamé coefficients $\hat{\lambda}^\varepsilon \geq 0, \hat{\mu}^\varepsilon > 0$. Let $\hat{\Gamma}^\varepsilon = \partial\hat{\Omega}^\varepsilon$ denote the boundary of the body, which is divided into three disjoint parts $\hat{\Gamma}_+^\varepsilon$, $\hat{\Gamma}_C^\varepsilon$ and $\hat{\Gamma}_0^\varepsilon$, where the measure of the latter is strictly positive. A generic point of $\bar{\hat{\Omega}}^\varepsilon$ shall be denoted by $\hat{\boldsymbol{x}}^\varepsilon = (\hat{x}_i^\varepsilon)$ but notice that through the whole paper, the explicit dependence of various functions of space and time will be omitted when there is no ambiguity. Let $\hat{\alpha}_T^\varepsilon$, $\hat{\beta}^\varepsilon$, $\hat{k}^\varepsilon$ and $\hat{\rho}^\varepsilon$ denote thermal dilatation coefficient, thermal conductivity coefficient, specific heat coefficient and specific mass density, respectively. The constitutive equation relating the stress tensor components $\hat{\sigma}_{ij}^\varepsilon$ to the linearized strain tensor $\hat{e}_{ij}^\varepsilon(\hat{\boldsymbol{u}}^\varepsilon)$ components, and the temperature $\hat{\vartheta}^\varepsilon$ is given by the linearized Duhamel-Neumann law (see [7]):

$$\hat{\sigma}_{ij}^\varepsilon = \hat{\lambda}^\varepsilon \hat{e}_{kk}^\varepsilon(\hat{\boldsymbol{u}}^\varepsilon)\delta_{ij} + 2\hat{\mu}^\varepsilon \hat{e}_{ij}^\varepsilon(\hat{\boldsymbol{u}}^\varepsilon) - \hat{\alpha}_T^\varepsilon(3\hat{\lambda}^\varepsilon + 2\hat{\mu}^\varepsilon)\hat{\vartheta}^\varepsilon\delta_{ij}, \tag{1}$$

where $\hat{e}_{ij}^\varepsilon(\hat{\boldsymbol{v}}^\varepsilon) = \frac{1}{2}(\hat{\partial}_j\hat{v}_i^\varepsilon + \hat{\partial}_i\hat{v}_j^\varepsilon)$ denotes the linearized deformation operator being $\hat{\partial}_i$ the partial derivative with respect to the $i-th$ component. Also, δ_{ij} represents the Kronecker's symbol. We assume that the body is fixed on a part of the boundary, so we consider that the displacements field vanishes on $\hat{\Gamma}_0^\varepsilon$. Further, the body is assumed to be under the effect of a heat source $\hat{q}^\varepsilon$, volume forces of density $\hat{\boldsymbol{f}}^\varepsilon = (\hat{f}^{i,\varepsilon})$ and traction forces of density $\hat{\boldsymbol{h}}^\varepsilon = (\hat{h}^{i,\varepsilon})$ applied on $\hat{\Gamma}_+^\varepsilon$. On the remaining part of the boundary, $\hat{\Gamma}_C^\varepsilon$, the body may enter in contact with a deformable solid, and the distance between both bodies, measured along the direction of the normal $\hat{\boldsymbol{n}}^\varepsilon = (\hat{n}_i^\varepsilon)$, at the initial time is given by a known function $\hat{s}^\varepsilon$. For simplicity, in the following, we shall take $\hat{s}^\varepsilon = 0$.

We assume that the normal response on the contact surface only happens when the surface element is moving towards the foundation, and vanishes when it is moving away. Thus to model contact in the normal direction we are using the so-called normal damped response (see [23] and references therein). Therefore,

$$-\hat{\sigma}_n^\varepsilon = \hat{p}^\varepsilon(\dot{\hat{u}}_n^\varepsilon) \text{ on } \hat{\Gamma}_C^\varepsilon \times [0,T],$$

where $T > 0$ is the time period of observation, a dot above indicates time derivative and $\hat{p}^\varepsilon : \mathbb{R} \to \mathbb{R}_+$ is a non negative function which vanishes when its argument (the surface velocity) is nonpositive. Specifically, one may use

$$\hat{p}^\varepsilon(r) = \hat{\kappa}^\varepsilon r_+, \tag{2}$$

where $\hat{\kappa}^\varepsilon > 0$ stands for the normal damping coefficient, and we denote by $r_+ = \max\{r, 0\}$ for any $r \in \mathbb{R}$. The mathematical assumptions for $\hat{p}^\varepsilon(\cdot) : \mathbb{R} \to \mathbb{R}^+$ are the following:

$$\begin{cases} \hat{p}^\varepsilon(r) = 0 \text{ if } r \leq 0, \\ \text{There exists } L_p > 0 \text{ such that } |\hat{p}^\varepsilon(r_1) - \hat{p}^\varepsilon(r_2)| \leq L_p|r_1 - r_2| \forall\ r_1, r_2 \in \mathbb{R}, \\ (\hat{p}^\varepsilon(r_1) - \hat{p}^\varepsilon(r_2))(r_1 - r_2) \geq 0\ \forall\ r_1, r_2 \in \mathbb{R}. \end{cases} \tag{3}$$

In particular, hypotheses (3) are verified by (2). Regarding initial and boundary conditions, they will be all considered as homogeneous conditions in the aim to simplify the exposition of the asymptotic analysis which follows. Then, the equations of the three-dimensional dynamic thermoelastic frictionless contact problem between a regular three-dimensional solid and a deformable foundation with normal damped response are the following:

Problem 1. Find the displacements field $\hat{\boldsymbol{u}}^\varepsilon = (\hat{u}_i^\varepsilon)$ and the temperature field $\hat{\vartheta}^\varepsilon$ verifying

$$\begin{aligned}
&\hat{\rho}^\varepsilon \ddot{\hat{\boldsymbol{u}}}^\varepsilon - \mathrm{div}\hat{\boldsymbol{\sigma}}^\varepsilon = \hat{\boldsymbol{f}}^\varepsilon && \text{in } \hat{\Omega}^\varepsilon \times (0,T),\\
&\hat{\beta}^\varepsilon \dot{\hat{\vartheta}}^\varepsilon = \partial_j(\hat{k}^\varepsilon \hat{\partial}_j^\varepsilon \hat{\vartheta}^\varepsilon) - \hat{\alpha}_T^\varepsilon(3\hat{\lambda}^\varepsilon + 2\hat{\mu}^\varepsilon)\hat{e}_{kk}^\varepsilon(\dot{\hat{\boldsymbol{u}}}^\varepsilon) + \hat{q}^\varepsilon && \text{in } \hat{\Omega}^\varepsilon \times (0,T),\\
&\hat{\boldsymbol{u}}^\varepsilon = 0 && \text{on } \hat{\Gamma}_0^\varepsilon \times (0,T),\\
&\hat{\vartheta}^\varepsilon = 0 && \text{on } \hat{\Gamma}_0^\varepsilon \times (0,T),\\
&\hat{\sigma}^\varepsilon \hat{\boldsymbol{n}}^\varepsilon = \hat{\boldsymbol{h}}^\varepsilon && \text{on } \hat{\Gamma}_+^\varepsilon \times (0,T),\\
&-\hat{\sigma}_n^\varepsilon = \hat{p}^\varepsilon(\dot{\hat{u}}_n^\varepsilon),\ \hat{\boldsymbol{\sigma}}_t^\varepsilon = (\hat{\sigma}_{ti}^\varepsilon) = 0 && \text{on } \hat{\Gamma}_C^\varepsilon \times (0,T),\\
&\hat{k}^\varepsilon \hat{\partial}_j^\varepsilon \hat{\vartheta}^\varepsilon n_j = 0 && \text{on } (\hat{\Gamma}_+^\varepsilon \cup \hat{\Gamma}_C^\varepsilon) \times (0,T),\\
&\dot{\hat{\boldsymbol{u}}}^\varepsilon(\cdot,0) = \hat{\boldsymbol{u}}^\varepsilon(\cdot,0) = 0 && \text{in } \hat{\Omega}^\varepsilon,\\
&\hat{\vartheta}^\varepsilon(\cdot,0) = 0 && \text{in } \hat{\Omega}^\varepsilon,
\end{aligned}$$

where $\hat{\boldsymbol{\sigma}}^\varepsilon = (\hat{\sigma}_{ij}^\varepsilon)$ is described in (1).

2.2. Variational formulation of the problem

Let $V(\hat{\Omega}^\varepsilon)$ and $S(\hat{\Omega}^\varepsilon)$ be the spaces of admissible displacements and temperatures, defined by

$$V(\hat{\Omega}^\varepsilon) := \{\hat{\boldsymbol{v}}^\varepsilon = (\hat{v}_i^\varepsilon) \in [H^1(\hat{\Omega}^\varepsilon)]^3; \hat{\boldsymbol{v}}^\varepsilon = \boldsymbol{0} \text{ on } \hat{\Gamma}_0^\varepsilon\},$$
$$S(\hat{\Omega}^\varepsilon) := \{\hat{\varphi}^\varepsilon \in H^1(\hat{\Omega}^\varepsilon); \hat{\varphi}^\varepsilon = 0 \text{ on } \hat{\Gamma}_0^\varepsilon\},$$

respectively. Both of them are Hilbert spaces equipped with the inner products

$$(\hat{\boldsymbol{v}}^\varepsilon, \hat{\boldsymbol{w}}^\varepsilon)_{V(\hat{\Omega}^\varepsilon)} = \int_{\hat{\Omega}^\varepsilon} \hat{e}_{ij}^\varepsilon(\hat{\boldsymbol{v}}^\varepsilon)\hat{e}_{ij}^\varepsilon(\hat{\boldsymbol{w}}^\varepsilon)d\hat{x}^\varepsilon, \ \ \forall \hat{\boldsymbol{v}}^\varepsilon, \hat{\boldsymbol{w}}^\varepsilon \in V(\hat{\Omega}^\varepsilon)$$
$$(\hat{\varphi}^\varepsilon, \hat{\zeta}^\varepsilon)_{S(\hat{\Omega}^\varepsilon)} = \int_{\hat{\Omega}^\varepsilon} \hat{\partial}_j^\varepsilon \hat{\varphi}^\varepsilon \hat{\partial}_j^\varepsilon \hat{\zeta}^\varepsilon d\hat{x}^\varepsilon, \ \ \forall \hat{\varphi}^\varepsilon, \hat{\zeta}^\varepsilon \in S(\hat{\Omega}^\varepsilon),$$

respectively. Besides, as long as there is no room for confusion, we shall avoid specifying the domain in the subindices for the corresponding norms notation. Further, for the sake of simplicity in the formulations to come, we define the following operators. The bilinear, continuous and coercive forms $a^{V,\varepsilon} : V(\hat{\Omega}^\varepsilon) \times V(\hat{\Omega}^\varepsilon) \to \mathbb{R}$ and $a^{S,\varepsilon} : S(\hat{\Omega}^\varepsilon) \times S(\hat{\Omega}^\varepsilon) \to \mathbb{R}$ defined by

$$(\hat{\boldsymbol{u}}^\varepsilon, \hat{\boldsymbol{v}}^\varepsilon) \to a^{V,\varepsilon}(\hat{\boldsymbol{u}}^\varepsilon, \hat{\boldsymbol{v}}^\varepsilon) = \int_{\hat{\Omega}^\varepsilon} \hat{A}^{ijkl,\varepsilon}\hat{e}_{kl}^\varepsilon(\hat{\boldsymbol{u}}^\varepsilon)\hat{e}_{ij}^\varepsilon(\hat{\boldsymbol{v}}^\varepsilon)d\hat{x}^\varepsilon,$$
$$(\hat{\varphi}^\varepsilon, \hat{\psi}^\varepsilon) \to a^{S,\varepsilon}(\hat{\varphi}^\varepsilon, \hat{\psi}^\varepsilon) = \int_{\hat{\Omega}^\varepsilon} \hat{k}^\varepsilon \hat{\partial}_j^\varepsilon \hat{\varphi}^\varepsilon \hat{\partial}_j^\varepsilon \hat{\psi}^\varepsilon d\hat{x}^\varepsilon,$$

where $\hat{A}^{ijkl,\varepsilon} = \hat{\lambda}^\varepsilon \delta^{ij}\delta^{kl} + \hat{\mu}^\varepsilon(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})$ denotes the elasticity fourth-order tensor. The continuous form $c^\varepsilon : S(\hat{\Omega}^\varepsilon) \times V(\hat{\Omega}^\varepsilon) \to \mathbb{R}$ defined by

$$(\hat{\varphi}^\varepsilon, \hat{\boldsymbol{v}}^\varepsilon) \to c^\varepsilon(\hat{\varphi}^\varepsilon, \hat{\boldsymbol{v}}^\varepsilon) = \int_{\hat{\Omega}^\varepsilon} \hat{\alpha}_T^\varepsilon(3\hat{\lambda}^\varepsilon + 2\hat{\mu}^\varepsilon)\hat{\varphi}^\varepsilon \hat{e}_{kk}^\varepsilon(\hat{\boldsymbol{v}}^\varepsilon)d\hat{x}^\varepsilon.$$

The nonlinear map $\hat{P}^\varepsilon : [H^1(\hat{\Omega}^\varepsilon)]^3 \to [H^1(\hat{\Omega}^\varepsilon)]^{3'}$ such that

$$\left\langle \hat{P}^\varepsilon(\hat{\boldsymbol{u}}^\varepsilon), \hat{\boldsymbol{v}}^\varepsilon \right\rangle = \int_{\hat{\Gamma}_C^\varepsilon} \hat{p}^\varepsilon(\hat{u}_n^\varepsilon)\hat{v}_n^\varepsilon \, d\hat{\Gamma}^\varepsilon \ \ \forall \hat{\boldsymbol{u}}^\varepsilon, \hat{\boldsymbol{v}}^\varepsilon \in [H^1(\hat{\Omega}^\varepsilon)]^3.$$

Above and below we use the notation for a duality pair $\langle \cdot, \cdot \rangle$ in $V'(\hat{\Omega}^\varepsilon) \times V(\hat{\Omega}^\varepsilon)$ (also for $S'(\hat{\Omega}^\varepsilon) \times S(\hat{\Omega}^\varepsilon)$). The functional $\hat{f}^\varepsilon(\cdot)$ is defined a.e. in $(0, T)$ as

$$\left\langle \hat{f}^\varepsilon(t), \hat{\boldsymbol{v}}^\varepsilon \right\rangle = \int_{\hat{\Omega}^\varepsilon} \hat{f}^{i,\varepsilon}(t)\hat{v}_i^\varepsilon \, d\hat{x}^\varepsilon + \int_{\hat{\Gamma}_+^\varepsilon} \hat{h}^{i,\varepsilon}(t)\hat{v}_i^\varepsilon \, d\hat{\Gamma}^\varepsilon, \qquad \forall \hat{\boldsymbol{v}}^\varepsilon \in V(\hat{\Omega}^\varepsilon),$$

and similarly,

$$\left\langle \hat{Q}^\varepsilon(t), \hat{\varphi}^\varepsilon \right\rangle = \int_{\hat{\Omega}^\varepsilon} \hat{q}^\varepsilon(t)\hat{\varphi}^\varepsilon \, d\hat{x}^\varepsilon \qquad \forall \hat{\varphi}^\varepsilon \in S(\hat{\Omega}^\varepsilon).$$

Then, it is straightforward to obtain the following variational formulation:

Problem 2. Find a pair $t \mapsto (\hat{\boldsymbol{u}}^\varepsilon(\hat{\boldsymbol{x}}^\varepsilon, t), \hat{\vartheta}^\varepsilon(\hat{\boldsymbol{x}}^\varepsilon, t))$ of $[0, T] \to V(\hat{\Omega}^\varepsilon) \times S(\hat{\Omega}^\varepsilon)$ verifying

$$\hat{\rho}^\varepsilon\left\langle \ddot{\hat{u}}_i^\varepsilon, \hat{v}_i^\varepsilon \right\rangle + a^{V,\varepsilon}(\hat{\boldsymbol{u}}^\varepsilon, \hat{\boldsymbol{v}}^\varepsilon) - c^\varepsilon(\hat{\vartheta}^\varepsilon, \hat{\boldsymbol{v}}^\varepsilon) + \left\langle \hat{P}^\varepsilon(\dot{\hat{\boldsymbol{u}}}^\varepsilon), \hat{\boldsymbol{v}}^\varepsilon \right\rangle = \left\langle \hat{f}^\varepsilon(t), \hat{\boldsymbol{v}}^\varepsilon \right\rangle \ \forall \hat{\boldsymbol{v}}^\varepsilon \in V(\hat{\Omega}^\varepsilon), \ \ a.e. \text{ in } (0, T), \tag{4}$$

$$\hat{\beta}^\varepsilon\left\langle \dot{\hat{\vartheta}}^\varepsilon, \hat{\varphi}^\varepsilon \right\rangle + a^{S,\varepsilon}(\hat{\vartheta}^\varepsilon, \hat{\varphi}^\varepsilon) + c^\varepsilon(\hat{\varphi}^\varepsilon, \dot{\hat{\boldsymbol{u}}}^\varepsilon) = \left\langle \hat{Q}^\varepsilon(t), \hat{\varphi}^\varepsilon \right\rangle \ \forall \hat{\varphi}^\varepsilon \in S(\hat{\Omega}^\varepsilon), \ \ a.e. \text{ in } (0, T), \tag{5}$$

with $\dot{\hat{\boldsymbol{u}}}^\varepsilon(\cdot, 0) = \hat{\boldsymbol{u}}^\varepsilon(\cdot, 0) = 0$ and $\hat{\vartheta}^\varepsilon(\cdot, 0) = 0$.

In favour of simplicity, we are going to assume that the different parameters of the problem (thermal conductivity, thermal dilatation, specific heat coefficient, mass density, Lamé coefficients) are constants.

Theorem 1. *Let us assume that volume forces, tractions and heat source density functions have regularity* $\hat{\boldsymbol{f}}^\varepsilon \in H^1(0, T; [L^2(\hat{\Omega}^\varepsilon)]^3)$, $\hat{\boldsymbol{h}}^\varepsilon \in H^2(0, T; [L^2(\hat{\Gamma}_+^\varepsilon)]^3)$ *and* $\hat{q}^\varepsilon \in H^1(0, T; L^2(\hat{\Omega}^\varepsilon))$, *respectively. Further, assume that* $\hat{\boldsymbol{h}}^\varepsilon(\cdot, 0) = 0$. *Then, there exists a unique pair* $(\hat{\boldsymbol{u}}^\varepsilon(\boldsymbol{x}, t), \hat{\vartheta}^\varepsilon(\hat{\boldsymbol{x}}, t))$ *solution to Problem 2 such that*

$$\begin{cases} \hat{\boldsymbol{u}}^\varepsilon \in L^\infty(0, T; V(\hat{\Omega}^\varepsilon)) \\ \dot{\hat{\boldsymbol{u}}}^\varepsilon \in L^\infty(0, T; [L^2(\hat{\Omega}^\varepsilon)]^3) \cap L^\infty(0, T; V(\hat{\Omega}^\varepsilon)), \\ \ddot{\hat{\boldsymbol{u}}}^\varepsilon \in L^\infty(0, T; V'(\hat{\Omega}^\varepsilon)) \cap L^\infty(0, T; [L^2(\hat{\Omega}^\varepsilon)]^3), \end{cases} \tag{6}$$

$$\begin{cases} \hat{\vartheta}^{\varepsilon} \in L^{\infty}(0,T;L^2(\hat{\Omega}^{\varepsilon})) \cap L^2(0,T;S(\hat{\Omega}^{\varepsilon})), \\ \dot{\hat{\vartheta}}^{\varepsilon} \in L^{\infty}(0,T;L^2(\hat{\Omega}^{\varepsilon})) \cap L^2(0,T;S(\hat{\Omega}^{\varepsilon})). \end{cases} \tag{7}$$

Remark 1. The regularity results in (6 c) and (7 b) imply that the duality products involving $\ddot{\hat{\boldsymbol{u}}}^{\varepsilon}$ and $\dot{\hat{\vartheta}}^{\varepsilon}$ in (4) and (5) can be replaced by the usual inner products in $L^2(\hat{\Omega}^{\varepsilon})$.

We proceed by following the Faedo-Galerkin method. Let $\{\hat{\boldsymbol{w}}_i\}_{i=1}^{\infty}$ be a sequence of functions such that

$$\hat{\boldsymbol{w}}_i \in V(\hat{\Omega}^{\varepsilon}) \quad \forall i, \quad V(\hat{\Omega}^{\varepsilon}) = \overline{\bigcup_{m\geq 1} V_m}, \tag{8}$$

where $V_m = \langle \hat{\boldsymbol{w}}_1, \dots, \hat{\boldsymbol{w}}_m \rangle$ and $\hat{\boldsymbol{w}}_1, \dots, \hat{\boldsymbol{w}}_m$ are orthonormal functions. Similarly, let $\{\hat{s}_i\}_{i=1}^{\infty}$ be a sequence of functions such that

$$\hat{s}_i \in S(\hat{\Omega}^{\varepsilon}) \quad \forall i, \quad S(\hat{\Omega}^{\varepsilon}) = \overline{\bigcup_{m\geq 1} S_m}, \tag{9}$$

where $S_m = \langle \hat{s}_1, \dots, \hat{s}_m \rangle$ and $\hat{s}_1, \dots, \hat{s}_m$ are orthonormal functions for all $m \geq 1$. The approximated solutions $(\hat{\boldsymbol{u}}^m, \hat{\vartheta}^m)$ are defined by the following problem:

Problem 3. Find the functions $\hat{\boldsymbol{u}}^m : [0,T] \to V_m$ and $\hat{\vartheta}^m : [0,T] \to S_m$ in the form

$$\hat{\boldsymbol{u}}^m(\hat{\boldsymbol{x}}, t) = \sum_{i=1}^{m} u_i^m(t) \hat{\boldsymbol{w}}_i(\hat{\boldsymbol{x}}), \quad \hat{\vartheta}^m(\hat{\boldsymbol{x}}, t) = \sum_{i=1}^{m} \vartheta_i^m(t) \hat{s}_i(\hat{\boldsymbol{x}}),$$

such that

$$\hat{\rho}^{\varepsilon}\langle \ddot{\hat{\boldsymbol{u}}}^m, \hat{\boldsymbol{v}}^m \rangle + a^{V,\varepsilon}(\hat{\boldsymbol{u}}^m, \hat{\boldsymbol{v}}^m) - c^{\varepsilon}(\hat{\vartheta}^m, \hat{\boldsymbol{v}}^m) + \left\langle \hat{P}^{\varepsilon}(\dot{\hat{\boldsymbol{u}}}^m), \hat{\boldsymbol{v}}^m \right\rangle = \langle \hat{f}^{\varepsilon}(t), \hat{\boldsymbol{v}}^m \rangle, \ \forall \hat{\boldsymbol{v}}^m \in V_m, \tag{10}$$

$$\hat{\beta}^{\varepsilon}\left\langle \dot{\hat{\vartheta}}^m, \hat{\varphi}^m \right\rangle + a^{S,\varepsilon}(\hat{\vartheta}^m, \hat{\varphi}^m) + c^{\varepsilon}(\hat{\varphi}^m, \dot{\hat{\boldsymbol{u}}}^m) = \langle \hat{Q}^{\varepsilon}(t), \hat{\varphi}^m \rangle, \ \forall \hat{\varphi}^m \in S_m. \tag{11}$$

with the initial conditions

$$\hat{\boldsymbol{u}}^m(0) = \dot{\hat{\boldsymbol{u}}}^m(0) = 0, \quad \hat{\vartheta}^m(0) = 0. \tag{12}$$

Finding a solution for Problem 3 is equivalent to solving a first order differential equation system

$$\dot{\mathbf{Z}}(t) = \mathbf{F}(t, \mathbf{Z}), \quad \mathbf{Z}(0) = \mathbf{0}.$$

where $\mathbf{Z}(t) = (v_1^m(t), \dots, v_m^m(t), u_1^m(t), \dots, u_m^m(t), \vartheta_1^m(t), \dots, \vartheta_m^m(t))$, with $v_j^m(t) = \dot{u}_j^m(t)$. The Picard-Lindeloff theorem gives a unique absolutely continuous solution in an interval $[0, t_m]$ which depends on the supreme of function $\mathbf{F}$ (which does not depend on time). Then, being the functions F_j uniformly Lipschitz in the variable $\mathbf{Z}$, if we prove that the solution $\mathbf{Z}(t)$ is bounded, we can extend the solution to the whole interval $[0, T]$.

Now the goal is to obtain estimations in appropriate normed spaces for $\hat{\boldsymbol{u}}^m$, $\dot{\hat{\boldsymbol{u}}}^m$, $\hat{\vartheta}^m$ and $\dot{\hat{\vartheta}}^m$. To do that, we can take $\hat{\boldsymbol{v}}^m = \dot{\hat{\boldsymbol{u}}}^m \in V_m$ and $\hat{\varphi}^m = \hat{\vartheta}^m \in S_m$ in (10), (11), respectively, and add both equations to have that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\{ \hat{\rho}^{\varepsilon} \left| \dot{\hat{\boldsymbol{u}}}^m(t) \right|_0^2 + a^{V,\varepsilon}(\hat{\boldsymbol{u}}^m(t), \hat{\boldsymbol{u}}^m(t)) + \hat{\beta}^{\varepsilon} \left| \hat{\vartheta}^m(t) \right|_0^2 \right\} + a^{S,\varepsilon}\left(\hat{\vartheta}^m, \hat{\vartheta}^m\right) + \left\langle \hat{P}^{\varepsilon}(\dot{\hat{\boldsymbol{u}}}^m), \dot{\hat{\boldsymbol{u}}}^m \right\rangle \\ &\quad = \left\langle \hat{f}^{\varepsilon}(t), \dot{\hat{\boldsymbol{u}}}^m \right\rangle + \left\langle \hat{Q}^{\varepsilon}(t), \hat{\vartheta}^m \right\rangle. \end{aligned} \tag{13}$$

Notice that we shall use the notation $|\cdot|_0$ for a (vector or scalar) L^2 norm. The same applies for $\|\cdot\|_1$ to denote a H^1 norm. Integrating in $[0,t]$, taking into account (12), the monotonicity of $\hat{p}^{\varepsilon}$, the coercivity of $a^{V,\varepsilon}$, $a^{S,\varepsilon}$, integrating by parts the term in $\hat{\Gamma}_+^{\varepsilon}$ and using Korn's inequality we get

$$\begin{aligned} &\hat{\rho}^{\varepsilon} \left| \dot{\hat{\boldsymbol{u}}}^m(t) \right|_0^2 + C \left\| \hat{\boldsymbol{u}}^m(t) \right\|_V^2 + \hat{\beta}^{\varepsilon} \left| \hat{\vartheta}^m(t) \right|_0^2 + \hat{k}\tilde{C} \int_0^t \left\| \hat{\vartheta}^m(s) \right\|_S^2 ds \\ &\leq \int_0^t \left\{ \left| \hat{\boldsymbol{f}}^{\varepsilon}(s) \right|_0 \left| \dot{\hat{\boldsymbol{u}}}^m(s) \right|_0 + \left| \dot{\hat{\boldsymbol{h}}}^{\varepsilon}(s) \right|_{0,\hat{\Gamma}_+^{\varepsilon}} \left| \hat{\boldsymbol{u}}^m(s) \right|_{0,\hat{\Gamma}_+^{\varepsilon}} + \left| \hat{q}^{\varepsilon}(s) \right|_0 \left| \hat{\vartheta}^m(s) \right|_0 \right\} ds. \end{aligned} \tag{14}$$

Above and in what follows, $C, \tilde{C}$ denote positive constants only depending on data, whose value may change from one equation to other. Next, applying Young's inequality to each term in the right-hand side in (14) and the continuity of the trace operator, yields that

$$\begin{aligned}&\left|\dot{\hat{\boldsymbol{u}}}^m(t)\right|_0^2+\left\|\hat{\boldsymbol{u}}^m(t)\right\|_V^2+\left|\hat{\vartheta}^m(t)\right|_0^2+\textstyle\int_0^t\left\|\hat{\vartheta}^m(s)\right\|_S^2ds\\ &\le C(\hat{\boldsymbol{f}}^\varepsilon,\dot{\hat{\boldsymbol{h}}}^\varepsilon,\hat{q}^\varepsilon)+\tilde{C}\textstyle\int_0^t\left\{\left|\dot{\hat{\boldsymbol{u}}}^m(s)\right|_0^2+\left\|\hat{\boldsymbol{u}}^m(s)\right\|_1^2+\left|\hat{\vartheta}^m(s)\right|_0^2\right\}ds,\end{aligned}\tag{15}$$

which, applying Gronwall's Lemma, gives

$$\left|\dot{\hat{\boldsymbol{u}}}^m(t)\right|_0^2+\left\|\hat{\boldsymbol{u}}^m(t)\right\|_V^2+\left|\hat{\vartheta}^m(t)\right|_0^2\le C(\hat{\boldsymbol{f}}^\varepsilon,\dot{\hat{\boldsymbol{h}}}^\varepsilon,\hat{q}^\varepsilon)+e^{\tilde{C}T},\ \ \forall\,m,\tag{16}$$

from where,

$$\dot{\hat{\boldsymbol{u}}}^m\in L^\infty(0,T;[L^2(\hat{\Omega}^\varepsilon)]^3),\quad \hat{\vartheta}^m\in L^\infty(0,T;L^2(\hat{\Omega}^\varepsilon)),\quad \hat{\boldsymbol{u}}^m\in L^\infty(0,T;V(\hat{\Omega}^\varepsilon)).$$

Further, going back to (15), we have $\hat{\vartheta}^m\in L^2(0,T;S(\hat{\Omega}^\varepsilon))$ and going back to (13), repeating the process, but keeping the term

$$\int_0^t\left\langle\hat{P}^\varepsilon(\dot{\hat{\boldsymbol{u}}}^m),\dot{\hat{\boldsymbol{u}}}^m\right\rangle dr=\int_0^t\hat{\kappa}^\varepsilon(\dot{\hat{u}}_n^m)_+^2dr,$$

we find that

$$(\dot{\hat{u}}_n^m)_+\in L^2(0,T;L^2(\hat{\Gamma}_C^\varepsilon)).\tag{17}$$

Note that all the estimates above are independent of m. Therefore, we can deduce that $\left\{\hat{\boldsymbol{u}}^m\right\}_m$ is a bounded subset of $L^\infty(0,T;V(\hat{\Omega}^\varepsilon))$, $\left\{\dot{\hat{\boldsymbol{u}}}^m\right\}_m$ is a bounded subset of $L^\infty(0,T;[L^2(\hat{\Omega}^\varepsilon)]^3)$, $\left\{\hat{\vartheta}^m\right\}_m$ is a bounded subset of $L^\infty(0,T;L^2(\hat{\Omega}^\varepsilon))$ and $L^2(0,T;S(\hat{\Omega}^\varepsilon))$, and $\left\{(\dot{\hat{u}}_n^m)_+\right\}_m$is a bounded subset of $L^2(0,T;L^2(\hat{\Gamma}_C^\varepsilon))$.

We now sum Eqs. (10) and (11) and write the result at times $t+h$, with $h>0$ and $0\le t\le T-h$, then subtract the resulting equations and next we take $\hat{\boldsymbol{v}}^m=\dot{\hat{\boldsymbol{u}}}^m(t+h)-\dot{\hat{\boldsymbol{u}}}^m(t)\in V_m$ and $\hat{\varphi}^m=\hat{\vartheta}^m(t+h)-\hat{\vartheta}^m(t)\in S_m$ to obtain

$$\begin{aligned}&\hat{\rho}^\varepsilon\left\langle\ddot{\hat{u}}_i^m(t+h)-\ddot{\hat{u}}_i^m(t),\dot{\hat{u}}_i^m(t+h)-\dot{\hat{u}}_i^m(t)\right\rangle+a^{V,\varepsilon}(\hat{\boldsymbol{u}}^m(t+h)-\hat{\boldsymbol{u}}^m(t),\dot{\hat{\boldsymbol{u}}}^m(t+h)-\dot{\hat{\boldsymbol{u}}}^m(t))\\ &\quad+\left\langle\hat{P}^\varepsilon(\dot{\hat{\boldsymbol{u}}}^m(t+h))-\hat{P}^\varepsilon(\dot{\hat{\boldsymbol{u}}}^m(t)),\dot{\hat{\boldsymbol{u}}}^m(t+h)-\dot{\hat{\boldsymbol{u}}}^m(t)\right\rangle\\ &\quad+\hat{\beta}^\varepsilon\left\langle\dot{\hat{\vartheta}}^m(t+h)-\dot{\hat{\vartheta}}^m(t),\hat{\vartheta}^m(t+h)-\hat{\vartheta}^m(t)\right\rangle+a^{S,\varepsilon}(\hat{\vartheta}^m(t+h)-\hat{\vartheta}^m(t),\hat{\vartheta}^m(t+h)-\hat{\vartheta}^m(t))\\ &=\int_{\hat{\Omega}^\varepsilon}(\hat{f}^{i,\varepsilon}(t+h)-\hat{f}^{i,\varepsilon}(t))(\dot{\hat{u}}_i^m(t+h)-\dot{\hat{u}}_i^m(t))\,d\hat{x}^\varepsilon+\int_{\hat{\Gamma}_+^\varepsilon}(\hat{h}^{i,\varepsilon}(t+h)-\hat{h}^{i,\varepsilon}(t))(\dot{\hat{u}}_i^m(t+h)-\dot{\hat{u}}_i^m(t))\,d\hat{\Gamma}^\varepsilon\\ &\quad+\int_{\hat{\Omega}^\varepsilon}(\hat{q}^\varepsilon(t+h)-\hat{q}^\varepsilon(t))(\hat{\vartheta}^m(t+h)-\hat{\vartheta}^m(t))\,d\hat{x}^\varepsilon.\end{aligned}$$

Integrating in time in $[0,t]$, using the monotonicity of $\hat{p}^\varepsilon$, dividing the resulting inequality by h^2 and having in mind (16), we can take limits when $h\to0^+$ to have

$$\begin{aligned}&\frac{1}{2}\hat{\rho}^\varepsilon\left|\ddot{\hat{\boldsymbol{u}}}^m(t)\right|_0^2-\frac{1}{2}\hat{\rho}^\varepsilon\left|\ddot{\hat{\boldsymbol{u}}}^m(0)\right|_0^2+\frac{1}{2}a^{V,\varepsilon}(\dot{\hat{\boldsymbol{u}}}^m(t),\dot{\hat{\boldsymbol{u}}}^m(t))-\frac{1}{2}a^{V,\varepsilon}(\dot{\hat{\boldsymbol{u}}}^m(0),\dot{\hat{\boldsymbol{u}}}^m(0))\\ &\quad+\frac{1}{2}\int_{\hat{\Omega}^\varepsilon}\hat{\beta}^\varepsilon(\dot{\hat{\vartheta}}^m(t))^2d\hat{x}^\varepsilon-\frac{1}{2}\int_{\hat{\Omega}^\varepsilon}\hat{\beta}^\varepsilon(\dot{\hat{\vartheta}}^m(0))^2d\hat{x}^\varepsilon+\int_0^t a^{S,\varepsilon}(\dot{\hat{\vartheta}}^m(r),\dot{\hat{\vartheta}}^m(r))dr\\ &\le\int_0^t\int_{\hat{\Omega}^\varepsilon}\dot{\hat{f}}^{i,\varepsilon}(r)\ddot{\hat{u}}_i^m(r)\,d\hat{x}^\varepsilon dr+\int_0^t\int_{\hat{\Gamma}_+^\varepsilon}\dot{\hat{h}}^{i,\varepsilon}(r)\ddot{\hat{u}}_i^m(r)\,d\hat{\Gamma}^\varepsilon dr+\int_0^t\int_{\hat{\Omega}^\varepsilon}\dot{\hat{q}}^\varepsilon(r)\dot{\hat{\vartheta}}^m(r)\,d\hat{x}^\varepsilon dr.\end{aligned}\tag{18}$$

Integrating by parts the term on $\hat{\Gamma}_+^\varepsilon$ above and applying Young's inequality, we get

$$\begin{aligned}&\hat{\rho}^\varepsilon|\ddot{\hat{\boldsymbol{u}}}^m(t)|_0^2-\hat{\rho}^\varepsilon|\ddot{\hat{\boldsymbol{u}}}^m(0)|_0^2+||\dot{\hat{\boldsymbol{u}}}^m(t)||_V^2+\hat{\beta}^\varepsilon|\dot{\hat{\vartheta}}^m(t)|_0^2-\hat{\beta}^\varepsilon|\dot{\hat{\vartheta}}^m(0)|_0^2+\int_0^t\|\dot{\hat{\vartheta}}^m(r)\|_S^2dr\\ &\le\tilde{C}(\dot{\hat{\boldsymbol{f}}}^\varepsilon,\ddot{\hat{\boldsymbol{h}}}^\varepsilon,\dot{\hat{q}}^\varepsilon)+C\int_0^t\left\{|\ddot{\hat{\boldsymbol{u}}}^m(r)|_0^2+||\dot{\hat{\boldsymbol{u}}}^m(r)||_1^2+|\dot{\hat{\vartheta}}^m(r)|_0^2\right\}dr.\end{aligned}\tag{19}$$

In order to obtain bounds for $|\ddot{\hat{\boldsymbol{u}}}^m(0)|_0^2$ and $|\dot{\hat{\vartheta}}^m(0)|_0^2$ we first notice that Eqs. (10) and (11) hold for $t=0$ due to the compatibility required between initial and boundary conditions. Therefore, taking $t=0$ and $\hat{\boldsymbol{v}}^m=\ddot{\hat{\boldsymbol{u}}}^m(0)\in V_m$ in (10) and $\hat{\varphi}^m=\dot{\hat{\vartheta}}^m(0)\in S_m$ in (11), taking into account the initial conditions, and using Young's inequality, we obtain

$$\hat{\rho}^\varepsilon|\ddot{\hat{\boldsymbol{u}}}^m(0)|_0^2=\textstyle\int_{\hat{\Omega}^\varepsilon}\hat{f}^{i,\varepsilon}(0)\ddot{\hat{u}}_i^m(0)\,d\hat{x}^\varepsilon+\int_{\hat{\Gamma}_+^\varepsilon}\hat{h}^{i,\varepsilon}(0)\ddot{\hat{u}}_i^m(0)\,d\hat{\Gamma}^\varepsilon\le\frac{1}{\delta}C+\delta|\ddot{\hat{\boldsymbol{u}}}^m(0)|_0^2,$$

$$\hat{\beta}^\varepsilon|\dot{\hat{\vartheta}}^m(0)|_0^2=\textstyle\int_{\hat{\Omega}^\varepsilon}\hat{q}^\varepsilon(0)\dot{\hat{\vartheta}}^m(0)\,d\hat{x}^\varepsilon\ \le\frac{1}{\delta}\tilde{C}+\tilde{\delta}|\dot{\hat{\vartheta}}^m(0)|_0^2,$$

where δ, and $\tilde{\delta}$ are sufficiently small positive constants. Next, applying Korn's inequality and Gronwall's lemma in (19) we find

$$|\ddot{\hat{\boldsymbol{u}}}^m(t)|_0^2 + \|\dot{\hat{\boldsymbol{u}}}^m(t)\|_V^2 + |\dot{\hat{\vartheta}}^m(t)|_0^2 \le C.$$

Again, all the estimates are independent of m. Then, we can deduce that $\left\{\dot{\hat{\boldsymbol{u}}}^m\right\}_m$ is a bounded subset of $L^\infty(0,T;V(\hat{\Omega}^\varepsilon))$, $\left\{\ddot{\hat{\boldsymbol{u}}}^m\right\}_m$ is a bounded subset of $L^\infty(0,T;[L^2(\hat{\Omega}^\varepsilon)]^3)$, and $\left\{\dot{\hat{\vartheta}}^m\right\}_m$ is a bounded subset of $L^\infty(0,T;L^2(\hat{\Omega}^\varepsilon))$. Observe that the boundedness of the different sequences above, and earlier in this proof, imply that there exists subsequences of $\hat{\boldsymbol{u}}^m$ and $\hat{\vartheta}^m$, also denoted by $\hat{\boldsymbol{u}}^m$ and $\hat{\vartheta}^m$, and there exist elements $\hat{\boldsymbol{u}}^\varepsilon$, $\dot{\hat{\boldsymbol{u}}}^\varepsilon$, $\ddot{\hat{\boldsymbol{u}}}^\varepsilon$, $\hat{\vartheta}^\varepsilon$, $\dot{\hat{\vartheta}}^\varepsilon$ and χ^ε such that

$$\hat{\boldsymbol{u}}^m \underset{m\to\infty}{\overset{*}{\rightharpoonup}} \hat{\boldsymbol{u}}^\varepsilon \quad \text{in}\, L^\infty\big(0,T;V\big(\hat{\Omega}^\varepsilon\big)\big), \tag{20}$$

$$\dot{\hat{\boldsymbol{u}}}^m \underset{m\to\infty}{\overset{*}{\rightharpoonup}} \dot{\hat{\boldsymbol{u}}}^\varepsilon \quad \text{in}\, L^\infty\Big(0,T;\big[L^2\big(\hat{\Omega}^\varepsilon\big)\big]^3\Big) \cap L^\infty\big(0,T;V\big(\hat{\Omega}^\varepsilon\big)\big), \tag{21}$$

$$\ddot{\hat{\boldsymbol{u}}}^m \underset{m\to\infty}{\overset{*}{\rightharpoonup}} \ddot{\hat{\boldsymbol{u}}}^\varepsilon \quad \text{in}\, L^\infty\Big(0,T;\big[L^2\big(\hat{\Omega}^\varepsilon\big)\big]^3\Big), \tag{22}$$

$$\hat{\vartheta}^m \underset{m\to\infty}{\overset{*}{\rightharpoonup}} \hat{\vartheta}^\varepsilon \quad \text{in}\, L^\infty\big(0,T;L^2\big(\hat{\Omega}^\varepsilon\big)\big) \cap L^\infty\big(0,T;S\big(\hat{\Omega}^\varepsilon\big)\big), \tag{23}$$

$$\dot{\hat{\vartheta}}^m \underset{m\to\infty}{\overset{*}{\rightharpoonup}} \dot{\hat{\vartheta}}^\varepsilon \quad \text{in}\, L^\infty\big(0,T;L^2\big(\hat{\Omega}^\varepsilon\big)\big), \tag{24}$$

$$\left(\dot{\hat{u}}_n^m\right)_+ \underset{m\to\infty}{\rightharpoonup} \chi^\varepsilon \quad \text{in}\, L^2\big(0,T;L^2\big(\hat{\Gamma}_C^\varepsilon\big)\big). \tag{25}$$

In order to show that $\chi^\varepsilon = (\dot{\hat{u}}_n^\varepsilon)_+$, we first observe that (21) and (22) imply that

$$\left\{\dot{\hat{\boldsymbol{u}}}^m\right\}_m \text{is a bounded subset of } [H^1(\hat{\Omega}^\varepsilon \times (0,T))]^3.$$

Since the trace map is a compact operator from $H^1(\hat{\Omega}^\varepsilon \times (0,T))$ to $L^2(\hat{\Gamma}^\varepsilon \times (0,T))$, we can affirm that there exists a subsequence of $\dot{\hat{\boldsymbol{u}}}^m$ (still denoted by $\dot{\hat{\boldsymbol{u}}}^m$) such that

$$\dot{\hat{\boldsymbol{u}}}^m \to \dot{\hat{\boldsymbol{u}}}^\varepsilon, \text{strongly in } [L^2(\hat{\Gamma}_C^\varepsilon \times (0,T))]^3, \quad \text{and then } \dot{\hat{\boldsymbol{u}}}^m(\boldsymbol{y}) \to \dot{\hat{\boldsymbol{u}}}^\varepsilon(\boldsymbol{y})\, a.e. \text{ on } \hat{\Gamma}_C^\varepsilon \times (0,T).$$

Then, being the positive part a continuous function it holds that

$$(\dot{\hat{u}}_n^m)_+ \to (\dot{\hat{u}}_n^\varepsilon)_+ \text{ a.e. on } \hat{\Gamma}_C^\varepsilon \times (0,T). \tag{26}$$

On the other hand, (17) implies that

$$(\dot{\hat{u}}_n^m)_+ \text{ is a bounded subset of } L^2(\hat{\Gamma}_C^\varepsilon \times (0,T)). \tag{27}$$

From (26), (27) and [43, Lemma 1.3] it follows that

$$(\dot{\hat{u}}_n^m)_+ \rightharpoonup (\dot{\hat{u}}_n^\varepsilon)_+ \text{ in } L^2(\hat{\Gamma}_C^\varepsilon \times (0,T)).$$

Since (25) also implies that $(\dot{\hat{u}}_n^m)_+ \rightharpoonup \chi^\varepsilon$ in $L^2(\hat{\Gamma}_C^\varepsilon \times (0,T))$, the uniqueness of weak limits implies that $\chi^\varepsilon = (\dot{\hat{u}}_n^\varepsilon)_+$ and

$$\left(\dot{\hat{u}}_n^m\right)_+ \underset{m\to\infty}{\rightharpoonup} \left(\dot{\hat{u}}_n^\varepsilon\right)_+ \text{ in } L^2\big(0,T;L^2\big(\hat{\Gamma}_C^\varepsilon\big)\big). \tag{28}$$

Consider now $\hat{\boldsymbol{v}}^m = \hat{\boldsymbol{w}}_j$ and $\hat{\varphi}^m = \hat{s}_i$ in Eqs. (10) and (11) fixed, and take $m \to \infty$. Then,

$$\hat{\rho}^\varepsilon(\ddot{\hat{\boldsymbol{u}}}^\varepsilon, \hat{\boldsymbol{w}}_j) + a^{V,\varepsilon}(\hat{\boldsymbol{u}}^\varepsilon, \hat{\boldsymbol{w}}_j) - c^\varepsilon(\hat{\vartheta}^\varepsilon, \hat{\boldsymbol{w}}_j) + \left\langle \hat{P}^\varepsilon(\dot{\hat{\boldsymbol{u}}}^\varepsilon), \hat{\boldsymbol{w}}_j \right\rangle = \left\langle \hat{f}^\varepsilon(t), \hat{\boldsymbol{w}}_j \right\rangle, \tag{29}$$

$$(\dot{\hat{\vartheta}}^\varepsilon, \hat{s}_i) + a^{S,\varepsilon}(\hat{\vartheta}^\varepsilon, \hat{s}_i) + c^\varepsilon(\dot{\hat{\boldsymbol{u}}}^\varepsilon, \hat{s}_i) = \left\langle \hat{Q}^\varepsilon(t), \hat{s}_i \right\rangle, \tag{30}$$

for all $i, j \ge 1$. Next, from (8), (9), (29) and (30) we conclude that (4) holds in $\mathcal{D}'(0,T)$ while (5) holds in $L^\infty(0,T)$. Let us see now that (4) also holds a.e. in $(0,T)$. Indeed, we have that

$$\left\langle \hat{P}^\varepsilon(\dot{\hat{\boldsymbol{u}}}^\varepsilon), \hat{\boldsymbol{w}}_j \right\rangle = -\hat{\rho}^\varepsilon(\ddot{\hat{\boldsymbol{u}}}^\varepsilon, \hat{\boldsymbol{w}}_j) - a^{V,\varepsilon}(\hat{\boldsymbol{u}}^\varepsilon, \hat{\boldsymbol{w}}_j) + c^\varepsilon(\hat{\vartheta}^\varepsilon, \hat{\boldsymbol{w}}_j) + \left\langle \hat{f}^\varepsilon(t), \hat{\boldsymbol{w}}_j \right\rangle, \quad \text{in } \mathcal{D}'(0,T),\ \forall j \ge 1.$$

We observe that the left-hand side is in $\mathcal{D}'(0,T)$, while the right-hand side terms are in $L^\infty(0,T)$, from which we deduce that $\hat{P}^\varepsilon(\dot{\hat{\boldsymbol{u}}}^\varepsilon) \in L^\infty(0,T;V')$ and (4) and (5) hold a.e. in $(0,T)$. Besides, since the initial conditions (12) are null, it is trivial that, when $m \to \infty$, the limit functions have null initial conditions as well, which completes the proof for the existence and regularity of the solutions. The uniqueness follows form the usual argument of taking two solutions, subtracting the respective equations with appropriate test functions and using ellipticity and monotonicity to show that, as a matter of fact, the two solutions are the same one. Details can be found in [44].

3. The three-dimensional shell contact problem

In this section we focus in the particular case when the deformable body is, in fact, a shell. We refer the reader to [9] for a detailed exposition of the notations and preliminary results which are given below in a summarised form.

Let ω be a bounded domain of $\mathbb{R}^2$, with boundary $\gamma = \partial\omega$, assumed to be Lipschitz-continuous and let $\bar{\omega}$ denote its closure. Also, $\boldsymbol{y} = (y_\alpha)$ stands for a point of $\bar{\omega}$ and $S := \boldsymbol{\theta}(\bar{\omega})$ for the middle surface of the shell, being $\boldsymbol{\theta} \in \mathcal{C}^2(\bar{\omega};\mathbb{R}^3)$ an injective mapping. Besides, in order to define covariant basis on the tangent plane, we require that the two vectors $\boldsymbol{a}_\alpha(\boldsymbol{y}) := \partial_\alpha\boldsymbol{\theta}(\boldsymbol{y})$ are linearly independent, for all $\boldsymbol{y} \in \omega$. Recall that in this context it is usual that Greek indices take only the values $\{1,2\}$ while Latin indices range between 1 to 3. We define contravariant basis with the two vectors $\boldsymbol{a}^\alpha(\boldsymbol{y})$ defined by the relations $\boldsymbol{a}^\alpha(\boldsymbol{y}) \cdot \boldsymbol{a}_\beta(\boldsymbol{y}) = \delta^\alpha_\beta$. Further, the unit outward normal vector to S at a point $\boldsymbol{\theta}(\boldsymbol{y})$ is denoted by $\boldsymbol{a}_3(\boldsymbol{y})$, and it is defined as the normalized vector product of $\boldsymbol{a}_1(\boldsymbol{y})$ times $\boldsymbol{a}_2(\boldsymbol{y})$. The metric tensor can be given in covariant components $a_{\alpha\beta} := \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta$ or in contravariant components $a^{\alpha\beta} := \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta$. Notice that the area element along S is $\sqrt{a}dy$ being $a := \det(a_{\alpha\beta})$. Also, the curvature tensor can be given in covariant components $b_{\alpha\beta} := \boldsymbol{a}^3 \cdot \partial_\beta\boldsymbol{a}_\alpha$ or mixed components $b^\beta_\alpha := a^{\beta\sigma} \cdot b_{\sigma\alpha}$. Similarly, the Christoffel symbols of the surface S are defined by $\Gamma^\sigma_{\alpha\beta} := \boldsymbol{a}^\sigma \cdot \partial_\beta\boldsymbol{a}_\alpha$.

We define now $\Omega^\varepsilon := \omega \times (-\varepsilon,\varepsilon)$. The boundary $\Gamma^\varepsilon = \partial\Omega^\varepsilon$ is divided into three disjoint parts, an upper face $\Gamma^\varepsilon_+ := \omega \times \{\varepsilon\}$, a lower face $\Gamma^\varepsilon_C := \omega \times \{-\varepsilon\}$, and a lateral face which contains $\Gamma^\varepsilon_0 := \gamma_0 \times [-\varepsilon,\varepsilon]$, where $\gamma_0 \subseteq \gamma$. In what follows, $\boldsymbol{x}^\varepsilon = (x^\varepsilon_i)$ denotes a point of $\bar{\Omega}^\varepsilon$ and ∂^ε_i denotes the partial derivative with respect to x^ε_i. Notice that we can identify $x^\varepsilon_\alpha = y_\alpha$ and $\partial^\varepsilon_\alpha = \partial_\alpha$. To describe a three-dimensional shell with S as middle surface, we introduce the mapping $\boldsymbol{\Theta} : \bar{\Omega}^\varepsilon \to \mathbb{R}^3$, defined by

$$\boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon) := \boldsymbol{\theta}(\boldsymbol{y}) + x^\varepsilon_3\boldsymbol{a}_3(\boldsymbol{y}) \ \forall \boldsymbol{x}^\varepsilon = (\boldsymbol{y}, x^\varepsilon_3) = (y_1, y_2, x^\varepsilon_3) \in \bar{\Omega}^\varepsilon, \tag{31}$$

and identify $\hat{\Omega}^\varepsilon = \boldsymbol{\Theta}(\Omega^\varepsilon)$. This way, the parts of the boundary of the shell $\hat{\Gamma}^\varepsilon = \boldsymbol{\Theta}(\Gamma^\varepsilon)$ are defined as well, like for example the part of the lateral face where the Dirichlet conditions are to be implemented: $\hat{\Gamma}^\varepsilon_0 = \boldsymbol{\Theta}(\Gamma^\varepsilon_0)$, etc. This way, we may cast this setting as a particular case of the more general framework of the preceding section. Following [9], Th. 3.1-1, if $\boldsymbol{\theta} : \bar{\omega} \to \mathbb{R}^3$ is injective and sufficiently smooth, then $\boldsymbol{\Theta} : \bar{\Omega}^\varepsilon \to \mathbb{R}^3$ is also injective provided that $\varepsilon > 0$ is small enough, and the vectors $\boldsymbol{g}^\varepsilon_i(\boldsymbol{x}^\varepsilon) := \partial^\varepsilon_i\boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon)$ are linearly independent. Thus, under these hypotheses, with the three vectors $\boldsymbol{g}^\varepsilon_i(\boldsymbol{x}^\varepsilon)$ we can build the covariant basis at $\boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon)$. Also, we can build the vectors of the contravariant basis $\boldsymbol{g}^{i,\varepsilon}(\boldsymbol{x}^\varepsilon)$, defined by the condition $\boldsymbol{g}^{i,\varepsilon} \cdot \boldsymbol{g}^\varepsilon_j = \delta^i_j$. Then, we can define the metric tensor in covariant components, $g^\varepsilon_{ij} := \boldsymbol{g}^\varepsilon_i \cdot \boldsymbol{g}^\varepsilon_j$. Notice that the volume element in the set $\boldsymbol{\Theta}(\bar{\Omega}^\varepsilon)$ is $\sqrt{g^\varepsilon}dx^\varepsilon$ and the surface element in $\boldsymbol{\Theta}(\Gamma^\varepsilon)$ is $\sqrt{g^\varepsilon}d\Gamma^\varepsilon$ where $g^\varepsilon := \det(g^\varepsilon_{ij})$. The metric tensor in contravariant components is given by $g^{ij,\varepsilon} := \boldsymbol{g}^{i,\varepsilon} \cdot \boldsymbol{g}^{j,\varepsilon}$ and the Christoffel symbols by $\Gamma^{p,\varepsilon}_{ij} := \boldsymbol{g}^{p,\varepsilon} \cdot \partial^\varepsilon_i\boldsymbol{g}^\varepsilon_j$.

The expression of the normal components of any vector on $\boldsymbol{\Theta}(\Gamma^\varepsilon_C)$ is of particular interest for this problem. Recall that the unit outward normal vector on $\boldsymbol{x}^\varepsilon \in \Gamma^\varepsilon$ is denoted by $\boldsymbol{n}^\varepsilon(\boldsymbol{x}^\varepsilon)$ while on $\hat{\boldsymbol{x}}^\varepsilon = \boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon) \in \boldsymbol{\Theta}(\Gamma^\varepsilon)$ it is denoted by $\hat{\boldsymbol{n}}^\varepsilon(\hat{\boldsymbol{x}}^\varepsilon)$. Observe that on Γ^ε_C, the normal vector takes the form $\boldsymbol{n}^\varepsilon = (0,0,-1)$. Besides, from (31) one can deduce that $\boldsymbol{g}^\varepsilon_3 = \boldsymbol{g}^{3,\varepsilon} = \boldsymbol{a}_3$ and therefore $g^{33,\varepsilon} = |\boldsymbol{g}^{3,\varepsilon}| = 1$. By using the expression of $\hat{\boldsymbol{n}}^\varepsilon$ in terms of $\boldsymbol{n}^\varepsilon$ (see, for example [45, p. 41]), we deduce that $\hat{\boldsymbol{n}}^\varepsilon(\hat{\boldsymbol{x}}^\varepsilon) = -\boldsymbol{g}_3(\boldsymbol{x}^\varepsilon) = -\boldsymbol{a}_3(\boldsymbol{y})$, where $\hat{\boldsymbol{x}}^\varepsilon = \boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon)$ and $\boldsymbol{x}^\varepsilon = (\boldsymbol{y},-\varepsilon) \in \Gamma^\varepsilon_C$. Now, if we denote by $\{\hat{\boldsymbol{e}}^i\}^3_{i=1}$ the cartesian basis on $\boldsymbol{\Theta}(\bar{\Omega}^\varepsilon)$, given a field $\hat{\boldsymbol{v}}^\varepsilon$, its covariant curvilinear coordinates (v^ε_i) in $\bar{\Omega}^\varepsilon$ are defined as $\hat{\boldsymbol{v}}^\varepsilon(\hat{\boldsymbol{x}}^\varepsilon) = \hat{v}^\varepsilon_i(\hat{\boldsymbol{x}}^\varepsilon)\hat{\boldsymbol{e}}^i =: v^\varepsilon_i(\boldsymbol{x}^\varepsilon)\boldsymbol{g}^{i,\varepsilon}(\boldsymbol{x}^\varepsilon)$ with $\hat{\boldsymbol{x}}^\varepsilon = \boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon)$. Therefore, on Γ^ε_C, we have

$$\hat{v}_n := \hat{\boldsymbol{v}}^\varepsilon \cdot \hat{\boldsymbol{n}}^\varepsilon = (\hat{v}^\varepsilon_i\hat{n}^{i,\varepsilon}) = (\hat{v}^\varepsilon_i\hat{\boldsymbol{e}}^i) \cdot (-\boldsymbol{g}_3) = (v^\varepsilon_i\boldsymbol{g}^{i,\varepsilon}) \cdot (-\boldsymbol{g}_3) = -v^\varepsilon_3.$$

Also, since $v^\varepsilon_i n^{i,\varepsilon} = -v^\varepsilon_3$ on Γ^ε_C, it is verified in particular that $\hat{v}_n = (\hat{v}^\varepsilon_i\hat{n}^{i,\varepsilon}) = v^\varepsilon_i n^{i,\varepsilon} = -v^\varepsilon_3$. We now focus on the applied forces densities, whose contravariant components in curvilinear coordinates are defined as:

$$\hat{f}^{i,\varepsilon}(\hat{\boldsymbol{x}}^\varepsilon)\hat{\boldsymbol{e}}_i\,d\hat{x}^\varepsilon =: f^{i,\varepsilon}(\boldsymbol{x}^\varepsilon)\boldsymbol{g}^\varepsilon_i(\boldsymbol{x}^\varepsilon)\sqrt{g^\varepsilon(\boldsymbol{x}^\varepsilon)}\,dx^\varepsilon, \quad \hat{h}^{i,\varepsilon}(\hat{\boldsymbol{x}}^\varepsilon)\hat{\boldsymbol{e}}_i d\hat{\Gamma}^\varepsilon =: h^{i,\varepsilon}(\boldsymbol{x}^\varepsilon)\boldsymbol{g}^\varepsilon_i(\boldsymbol{x}^\varepsilon)\sqrt{g^\varepsilon(\boldsymbol{x}^\varepsilon)}d\Gamma^\varepsilon,$$

while for the displacements field, the covariant components in curvilinear coordinates are given by: $\hat{\boldsymbol{u}}^\varepsilon(\hat{\boldsymbol{x}}^\varepsilon) = \hat{u}^\varepsilon_i(\hat{\boldsymbol{x}}^\varepsilon)\hat{\boldsymbol{e}}^i =: u^\varepsilon_i(\boldsymbol{x}^\varepsilon)\boldsymbol{g}^{i,\varepsilon}(\boldsymbol{x}^\varepsilon)$, with $\hat{\boldsymbol{x}}^\varepsilon = \boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon)$. Notice that forces and unknowns above depend also on the time variable $t \in [0,T]$, but we decided to keep it implicit for the sake of readiness, since the subject of the change of variable is the spatial component. The same comment applies in a number of situations below. We also define $\vartheta^\varepsilon(\boldsymbol{x}^\varepsilon) := \hat{\vartheta}^\varepsilon(\hat{\boldsymbol{x}}^\varepsilon)$ and $q^\varepsilon(\boldsymbol{x}^\varepsilon) := \hat{q}^\varepsilon(\hat{\boldsymbol{x}}^\varepsilon)$. Regarding the normal damped response function, we define $p^\varepsilon(r^\varepsilon) := \hat{p}^\varepsilon(r^\varepsilon)$. Let us define the spaces,

$$V(\Omega^\varepsilon) = \{\boldsymbol{v}^\varepsilon = (v^\varepsilon_i) \in [H^1(\Omega^\varepsilon)]^3; \boldsymbol{v}^\varepsilon = \boldsymbol{0} \text{ on } \Gamma^\varepsilon_0\}, \qquad S(\Omega^\varepsilon) = \{\varphi^\varepsilon \in H^1(\Omega^\varepsilon); \varphi^\varepsilon = 0 \text{ on } \Gamma^\varepsilon_0\}.$$

Both are real Hilbert spaces with the induced inner product of $[H^1(\Omega^\varepsilon)]^d$, $d \in \{1, 3\}$, and we denote by $\|\cdot\|_{1,\Omega^\varepsilon}$ the corresponding norm in both cases, since no confusion is possible. Using these definitions, the following variational problem can be derived straightforwardly from Problem 2 (follow similar arguments to those in [9] for the linear elastic case):

Problem 4. Find a pair $t \mapsto (\boldsymbol{u}^\varepsilon(\boldsymbol{x}^\varepsilon, t), \vartheta^\varepsilon(\boldsymbol{x}^\varepsilon, t))$ of $[0, T] \to V(\Omega^\varepsilon) \times S(\Omega^\varepsilon)$ verifying

$$\begin{aligned}
&\int_{\Omega^\varepsilon} \rho^\varepsilon (\ddot{u}^\varepsilon_\alpha g^{\alpha\beta,\varepsilon} v^\varepsilon_\beta + \ddot{u}^\varepsilon_3 v^\varepsilon_3)\sqrt{g^\varepsilon}\, dx^\varepsilon + \int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e^\varepsilon_{k\|l}(\boldsymbol{u}^\varepsilon) e^\varepsilon_{i\|j}(\boldsymbol{v}^\varepsilon)\sqrt{g^\varepsilon} dx^\varepsilon \\
&\quad - \int_{\Omega^\varepsilon} \alpha^\varepsilon_T (3\lambda^\varepsilon + 2\mu^\varepsilon)\vartheta^\varepsilon (e^\varepsilon_{\alpha\|\beta}(\boldsymbol{v}^\varepsilon) g^{\alpha\beta,\varepsilon} + e^\varepsilon_{3\|3}(\boldsymbol{v}^\varepsilon))\sqrt{g^\varepsilon} dx^\varepsilon - \int_{\Gamma^\varepsilon_C} p^\varepsilon(-\dot{u}^\varepsilon_3) v^\varepsilon_3 \sqrt{g^\varepsilon} d\Gamma^\varepsilon \\
&= \int_{\Omega^\varepsilon} f^{i,\varepsilon} v^\varepsilon_i \sqrt{g^\varepsilon} dx^\varepsilon + \int_{\Gamma^\varepsilon_+} h^{i,\varepsilon} v^\varepsilon_i \sqrt{g^\varepsilon} d\Gamma^\varepsilon \quad \forall \boldsymbol{v}^\varepsilon \in V(\Omega^\varepsilon),\ a.e. \text{ in } (0, T), \\
&\int_{\Omega^\varepsilon} \beta^\varepsilon \dot{\vartheta}^\varepsilon \varphi^\varepsilon \sqrt{g^\varepsilon} dx^\varepsilon + \int_{\Omega^\varepsilon} k^\varepsilon (\partial^\varepsilon_\alpha \vartheta^\varepsilon g^{\alpha\beta,\varepsilon} \partial^\varepsilon_\beta \varphi^\varepsilon + \partial^\varepsilon_3 \vartheta^\varepsilon \partial^\varepsilon_3 \varphi^\varepsilon)\sqrt{g^\varepsilon} dx^\varepsilon \\
&\quad + \int_{\Omega^\varepsilon} \alpha^\varepsilon_T (3\lambda^\varepsilon + 2\mu^\varepsilon)\varphi^\varepsilon (e^\varepsilon_{\alpha\|\beta}(\dot{\boldsymbol{u}}^\varepsilon) g^{\alpha\beta,\varepsilon} + e^\varepsilon_{3\|3}(\dot{\boldsymbol{u}}^\varepsilon))\sqrt{g^\varepsilon} dx^\varepsilon \\
&= \int_{\Omega^\varepsilon} q^\varepsilon \varphi^\varepsilon \sqrt{g^\varepsilon} dx^\varepsilon \quad \forall \varphi^\varepsilon \in S(\Omega^\varepsilon),\ a.e. \text{ in } (0, T),
\end{aligned}$$

with $\dot{\boldsymbol{u}}^\varepsilon(\cdot, 0) = \boldsymbol{u}^\varepsilon(\cdot, 0) = 0$ and $\vartheta^\varepsilon(\cdot, 0) = 0$.

Above, $A^{ijkl,\varepsilon} = A^{jikl,\varepsilon} = A^{klij,\varepsilon} \in \mathcal{C}^1(\bar{\Omega}^\varepsilon)$, defined by

$$A^{ijkl,\varepsilon} := \lambda g^{ij,\varepsilon} g^{kl,\varepsilon} + \mu (g^{ik,\varepsilon} g^{jl,\varepsilon} + g^{il,\varepsilon} g^{jk,\varepsilon}), \tag{32}$$

stands for the contravariant components of the three-dimensional elasticity tensor, and the functions $e^\varepsilon_{i\|j}(\boldsymbol{v}^\varepsilon) = e^\varepsilon_{j\|i}(\boldsymbol{v}^\varepsilon) \in L^2(\Omega^\varepsilon)$, representing the covariant components of the linearized change of metric tensor, are defined by

$$e^\varepsilon_{i\|j}(\boldsymbol{v}^\varepsilon) := \frac{1}{2}(\partial^\varepsilon_j v^\varepsilon_i + \partial^\varepsilon_i v^\varepsilon_j) - \Gamma^{p,\varepsilon}_{ij} v^\varepsilon_p,$$

for all $\boldsymbol{v}^\varepsilon \in [H^1(\Omega^\varepsilon)]^3$. Also note that, as a consequence of (31), the following simplifications are verified:

$$\Gamma^{3,\varepsilon}_{\alpha 3} = \Gamma^{p,\varepsilon}_{33} = 0 \text{ in } \bar{\Omega}^\varepsilon, \quad A^{\alpha\beta\sigma 3,\varepsilon} = A^{\alpha 333,\varepsilon} = 0 \text{ in } \bar{\Omega}^\varepsilon. \tag{33}$$

Moreover, in [9], Theorem 1.8-1 it is shown that for $A^{ijkl,\varepsilon}$ defined as in (32) and $\varepsilon > 0$ small enough, there exists a constant $C_e > 0$, independent of ε, such that,

$$\sum_{i,j} |t_{ij}|^2 \leq C_e A^{ijkl,\varepsilon}(\boldsymbol{x}^\varepsilon) t_{kl} t_{ij}, \tag{34}$$

for all $\boldsymbol{x}^\varepsilon \in \bar{\Omega}^\varepsilon$ and all $\boldsymbol{t} = (t_{ij}) \in \mathbb{S}^3$ (vector space of 3×3 real symmetric matrices).

Remark 2. We recall that the vector field $\boldsymbol{u}^\varepsilon = (u^\varepsilon_i) : \Omega^\varepsilon \times [0, T] \to \mathbb{R}^3$ solution of Problem 4 needs to be interpreted properly. The functions $u^\varepsilon_i : \bar{\Omega}^\varepsilon \times [0, T] \to \mathbb{R}^3$ are the covariant, time dependent, components of the "true" displacements field $\mathcal{U}^\varepsilon := u^\varepsilon_i \boldsymbol{g}^{i,\varepsilon} : \bar{\Omega}^\varepsilon \times [0, T] \to \mathbb{R}^3$.

Next, we consider a scaled domain $\Omega := \omega \times (-1, 1)$ which is independent of the small parameter ε and denote by $\Gamma = \partial\Omega$ its boundary where we distinguish three parts: $\Gamma_+ := \omega \times \{1\}$, $\Gamma_C := \omega \times \{-1\}$ and $\Gamma_0 := \gamma_0 \times [-1, 1]$. A point in $\bar{\Omega}$ is denoted by $\boldsymbol{x} = (x_1, x_2, x_3)$ and ∂_i denotes the $i-th$ partial derivative. A projection map $\pi^\varepsilon : \bar{\Omega} \to \bar{\Omega}^\varepsilon$, verifying that $\pi^\varepsilon(\boldsymbol{x}) = \boldsymbol{x}^\varepsilon = (x^\varepsilon_i) = (x^\varepsilon_1, x^\varepsilon_2, x^\varepsilon_3) = (x_1, x_2, \varepsilon x_3) \in \bar{\Omega}^\varepsilon$, is considered, hence, $\partial^\varepsilon_\alpha = \partial_\alpha$ and $\partial^\varepsilon_3 = \frac{1}{\varepsilon}\partial_3$. The scaled displacements $\boldsymbol{u}(\varepsilon) = (u_i(\varepsilon)) : \bar{\Omega} \times [0, T] \to \mathbb{R}^3$ and vector fields $\boldsymbol{v} = (v_i) : \bar{\Omega} \to \mathbb{R}^3$ are defined as $u^\varepsilon_i(\boldsymbol{x}^\varepsilon) =: u_i(\varepsilon)(\boldsymbol{x})$ and $v^\varepsilon_i(\boldsymbol{x}^\varepsilon) =: v_i(\boldsymbol{x})$ respectively, for all $\boldsymbol{x} \in \bar{\Omega}$, $\boldsymbol{x}^\varepsilon = \pi^\varepsilon(\boldsymbol{x}) \in \bar{\Omega}^\varepsilon$. Besides, we define the scaled temperature $\vartheta(\varepsilon) : \bar{\Omega} \times [0, T] \to \mathbb{R}$ defined as $\vartheta(\varepsilon)(\boldsymbol{x}) := \vartheta^\varepsilon(\boldsymbol{x}^\varepsilon)$ for all $\boldsymbol{x} \in \Omega$ where $\boldsymbol{x}^\varepsilon = \pi^\varepsilon(\boldsymbol{x}) \in \Omega^\varepsilon$.

For the sake of simplicity, from now on, we are going to assume that the different parameters of the problem (thermal conductivity, thermal dilatation, specific heat coefficient, mass density, Lamé coefficients) are all independent of ε.

Also, let the functions, $\Gamma^{p,\varepsilon}_{ij}, g^\varepsilon, A^{ijkl,\varepsilon}$ be associated with the functions $\Gamma^p_{ij}(\varepsilon)$, $g(\varepsilon)$, $A^{ijkl}(\varepsilon)$, defined by $\Gamma^p_{ij}(\varepsilon)(\boldsymbol{x}) := \Gamma^{p,\varepsilon}_{ij}(\boldsymbol{x}^\varepsilon)$, $g(\varepsilon)(\boldsymbol{x}) := g^\varepsilon(\boldsymbol{x}^\varepsilon)$ and $A^{ijkl}(\varepsilon)(\boldsymbol{x}) := A^{ijkl,\varepsilon}(\boldsymbol{x}^\varepsilon)$ for all $\boldsymbol{x} \in \bar{\Omega}$, $\boldsymbol{x}^\varepsilon = \pi^\varepsilon(\boldsymbol{x}) \in \bar{\Omega}^\varepsilon$. For all $\boldsymbol{v} = (v_i) \in [H^1(\Omega)]^3$, the scaled linearized strains, denoted as $(e_{i\|j}(\varepsilon)(\boldsymbol{v})) \in [L^2(\Omega)]^{3\times 3}_{sym}$ or $(e_{i\|j}(\varepsilon; \boldsymbol{v}))$, are defined by

$$e_{\alpha\|\beta}(\varepsilon; \boldsymbol{v}) := \frac{1}{2}(\partial_\beta v_\alpha + \partial_\alpha v_\beta) - \Gamma^p_{\alpha\beta}(\varepsilon) v_p, \tag{35}$$

$$e_{\alpha\|3}(\varepsilon; \boldsymbol{v}) := \frac{1}{2}(\frac{1}{\varepsilon}\partial_3 v_\alpha + \partial_\alpha v_3) - \Gamma^p_{\alpha 3}(\varepsilon) v_p, \tag{36}$$

$$e_{3||3}(\varepsilon;\boldsymbol{v}) := \frac{1}{\varepsilon}\partial_3 v_3. \tag{37}$$

Notice that from these definitions one can easily check that $e^{\varepsilon}_{i||j}(\boldsymbol{v}^{\varepsilon})(\pi^{\varepsilon}(\boldsymbol{x})) = e_{i||j}(\varepsilon;\boldsymbol{v})(\boldsymbol{x})$ for all $\boldsymbol{x}\in\Omega$.

Remark 3. The functions $\Gamma^p_{ij}(\varepsilon), g(\varepsilon), A^{ijkl}(\varepsilon)$ converge in $\mathcal{C}^0(\bar{\Omega})$ when ε tends to zero. Also, when we consider $\varepsilon = 0$ the functions will be defined with respect to $\boldsymbol{y}\in\bar{\omega}$. Note that (36) and (37) present a singularity if $\varepsilon = 0$. In the notation for the three-dimensional Christoffel symbols we will explicit the dependence on ε as $\Gamma^{\sigma}_{\alpha\beta}(\varepsilon)$ in order to distinguish them from the two-dimensional ones associated to S denoted by $\Gamma^{\sigma}_{\alpha\beta}$.

Another important result that can be found in [9, Theorem 3.3-2] states that under suitable regularity conditions, take for example $\boldsymbol{\theta}\in\mathcal{C}^2(\bar{\omega};\mathbb{R}^3)$, there exists an $\varepsilon_0 > 0$ such that $A^{ijkl}(\varepsilon)$ is positive-definite, uniformly with respect to $\boldsymbol{x}\in\bar{\Omega}$ and ε, provided that $0<\varepsilon\leq\varepsilon_0$. Besides, it shows that the asymptotic behavior of $A^{ijkl}(\varepsilon)$ is the following:

$$A^{ijkl}(\varepsilon) = A^{ijkl}(0) + \mathcal{O}(\varepsilon) \text{ and } A^{\alpha\beta\sigma 3,\varepsilon} = A^{\alpha 333\varepsilon} = 0,$$

for all ε, $0<\varepsilon\leq\varepsilon_0$, and

$$A^{\alpha\beta\sigma\tau}(0) = \lambda a^{\alpha\beta}a^{\sigma\tau} + \mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}), \quad A^{\alpha\beta 33}(0) = \lambda a^{\alpha\beta}, \tag{38}$$

$$A^{\alpha 3\sigma 3}(0) = \mu a^{\alpha\sigma}, \quad A^{3333}(0) = \lambda + 2\mu, \quad A^{\alpha\beta\sigma 3}(0) = A^{\alpha 333}(0) = 0. \tag{39}$$

Moreover, and related with (34), there exists a constant $C_e > 0$, independent of the variables and ε, such that

$$\sum_{i,j}|t_{ij}|^2 \leq C_e A^{ijkl}(\varepsilon)(\boldsymbol{x})t_{kl}t_{ij}, \tag{40}$$

for all ε, $0<\varepsilon\leq\varepsilon_0$, for all $\boldsymbol{x}\in\bar{\Omega}$ and all $\boldsymbol{t} = (t_{ij})\in\mathbb{S}^3$.

Notice that the limits are functions of $\boldsymbol{y}\in\bar{\omega}$ only, that is, independent of the transversal variable x_3. We also recall [9, Theorem 3.3-1], which provides the asymptotic behavior of Christoffel's symbols $\Gamma^p_{ij}(\varepsilon)$, $g^{ij}(\varepsilon)$ and $g(\varepsilon)$. Indeed, if $\boldsymbol{\theta}\in\mathcal{C}^3(\bar{\omega};\mathbb{R}^3)$, then

$$\Gamma^{\sigma}_{\alpha\beta}(\varepsilon) = \Gamma^{\sigma}_{\alpha\beta} - \varepsilon x_3 b^{\sigma}_{\beta}|_{\alpha} + \mathcal{O}(\varepsilon^2), \quad \partial_3\Gamma^p_{\alpha\beta}(\varepsilon) = \mathcal{O}(\varepsilon), \quad \Gamma^3_{\alpha 3}(\varepsilon) = \Gamma^p_{33}(\varepsilon) = 0, \tag{41}$$

$$\Gamma^3_{\alpha\beta}(\varepsilon) = b_{\alpha\beta} - \varepsilon x_3 b^{\sigma}_{\alpha}b_{\sigma\beta}, \quad \Gamma^{\sigma}_{\alpha 3}(\varepsilon) = -b^{\sigma}_{\alpha} - \varepsilon x_3 b^{\tau}_{\alpha}b^{\sigma}_{\tau} + \mathcal{O}(\varepsilon^2), \tag{42}$$

$$g^{\alpha\beta}(\varepsilon) = a^{\alpha\beta} + 2\varepsilon x_3 a^{\alpha\sigma}b^{\beta}_{\sigma} + \mathcal{O}(\varepsilon^2), \quad g^{i3}(\varepsilon) = \delta^{i3}, \quad g(\varepsilon) = a + \mathcal{O}(\varepsilon), \tag{43}$$

for all ε, $0<\varepsilon\leq\varepsilon_0$, where the order symbols $\mathcal{O}(\varepsilon)$ and $\mathcal{O}(\varepsilon^2)$ are meant with respect to the norm $\|\cdot\|_{0,\infty,\bar{\Omega}}$ defined by $\|w\|_{0,\infty,\bar{\Omega}} = \sup\{|w(\boldsymbol{x})|; \boldsymbol{x}\in\bar{\Omega}\}$, and the covariant derivatives $b^{\sigma}_{\beta}|_{\alpha}$ are defined by $b^{\sigma}_{\beta}|_{\alpha} := \partial_{\alpha}b^{\sigma}_{\beta} + \Gamma^{\sigma}_{\alpha\tau}b^{\tau}_{\beta} - \Gamma^{\tau}_{\alpha\beta}b^{\sigma}_{\tau}$. The functions a, $b_{\alpha\beta}, b^{\sigma}_{\alpha}, \Gamma^{\sigma}_{\alpha\beta}$ and $b^{\sigma}_{\beta}|_{\alpha}$ are identified with functions in $\mathcal{C}^0(\bar{\Omega})$. Further, there exist constants a_0, g_0 and g_1 such that

$$\begin{aligned} &0 < a_0 \leq a(\boldsymbol{y}) \ \forall \boldsymbol{y}\in\bar{\omega},\\ &0 < g_0 \leq g(\varepsilon)(\boldsymbol{x}) \leq g_1 \ \forall \boldsymbol{x}\in\bar{\Omega} \text{ and } \forall\ \varepsilon, 0<\varepsilon\leq\varepsilon_0. \end{aligned} \tag{44}$$

Let the scaled heat source $q(\varepsilon): \Omega\times(0,T)\to\mathbb{R}$ and scaled applied forces $\boldsymbol{f}(\varepsilon): \Omega\times(0,T)\to\mathbb{R}^3$ and $\boldsymbol{h}(\varepsilon): \Gamma_+\times(0,T)\to\mathbb{R}^3$ be defined by

$$\begin{aligned} &q^{\varepsilon}(\boldsymbol{x}^{\varepsilon}) =: q(\varepsilon)(\boldsymbol{x}) \quad \forall \boldsymbol{x}\in\Omega, \text{ where } \boldsymbol{x}^{\varepsilon} = \pi^{\varepsilon}(\boldsymbol{x})\in\Omega^{\varepsilon},\\ &\boldsymbol{f}^{\varepsilon} = (f^{i,\varepsilon})(\boldsymbol{x}^{\varepsilon}) =: \boldsymbol{f}(\varepsilon) = (f^i(\varepsilon))(\boldsymbol{x}) \quad \forall \boldsymbol{x}\in\Omega, \text{ where } \boldsymbol{x}^{\varepsilon} = \pi^{\varepsilon}(\boldsymbol{x})\in\Omega^{\varepsilon},\\ &\boldsymbol{h}^{\varepsilon} = (h^{i,\varepsilon})(\boldsymbol{x}^{\varepsilon}) =: \boldsymbol{h}(\varepsilon) = (h^i(\varepsilon))(\boldsymbol{x}) \quad \forall \boldsymbol{x}\in\Gamma_+, \text{ where } \boldsymbol{x}^{\varepsilon} = \pi^{\varepsilon}(\boldsymbol{x})\in\Gamma^{\varepsilon}_+. \end{aligned}$$

With regard to the normal damped response function, we define $p(\varepsilon)(r(\varepsilon)) := p^{\varepsilon}(r^{\varepsilon})$. Also, we define the spaces

$$V(\Omega) = \{\boldsymbol{v} = (v_i)\in[H^1(\Omega)]^3; \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_0\}, \qquad S(\Omega) = \{\varphi\in H^1(\Omega); \varphi = 0 \text{ on } \Gamma_0\},$$

which are Hilbert spaces, with associated norms denoted by $\|\cdot\|_{1,\Omega}$. Then, the scaled variational problem can be written as follows:

Problem 5. Find a pair $t\mapsto(\boldsymbol{u}(\varepsilon)(\boldsymbol{x},t), \vartheta(\varepsilon)(\boldsymbol{x},t))$ of $[0,T]\to V(\Omega)\times S(\Omega)$ verifying

$$\int_{\Omega}\rho(\ddot{u}_{\alpha}(\varepsilon)g^{\alpha\beta}(\varepsilon)v_{\beta} + \ddot{u}_3(\varepsilon)v_3)\sqrt{g(\varepsilon)}\,dx + \int_{\Omega}A^{ijkl}(\varepsilon)e_{k||l}(\varepsilon;\boldsymbol{u}(\varepsilon))e_{i||j}(\varepsilon;\boldsymbol{v})\sqrt{g(\varepsilon)}dx$$

$$- \int_\Omega \alpha_T(3\lambda+2\mu)\vartheta(\varepsilon)(e_{\alpha||\beta}(\varepsilon;\boldsymbol{v})g^{\alpha\beta}(\varepsilon)+e_{3||3}(\varepsilon;\boldsymbol{v}))\sqrt{g(\varepsilon)}dx - \frac{1}{\varepsilon}\int_{\Gamma_C} p(\varepsilon)(-\dot{u}_3(\varepsilon))v_3\sqrt{g(\varepsilon)}d\Gamma$$
$$= \int_\Omega f^i(\varepsilon)v_i\sqrt{g(\varepsilon)}dx + \frac{1}{\varepsilon}\int_{\Gamma_+} h^i(\varepsilon)v_i\sqrt{g(\varepsilon)}d\Gamma \ \forall \boldsymbol{v}\in V(\Omega),\ a.e. \text{ in } (0,T), \tag{45}$$

$$\int_\Omega \beta\dot{\vartheta}(\varepsilon)\varphi\sqrt{g(\varepsilon)}dx + \int_\Omega k(\partial_\alpha\vartheta(\varepsilon)g^{\alpha\beta}(\varepsilon)\partial_\beta\varphi + \frac{1}{\varepsilon^2}\partial_3\vartheta(\varepsilon)\partial_3\varphi)\sqrt{g(\varepsilon)}dx$$
$$+ \int_\Omega \alpha_T(3\lambda+2\mu)\varphi(e_{\alpha||\beta}(\varepsilon;\dot{\boldsymbol{u}}(\varepsilon))g^{\alpha\beta}(\varepsilon)+e_{3||3}(\varepsilon;\dot{\boldsymbol{u}}(\varepsilon)))\sqrt{g(\varepsilon)}dx$$
$$= \int_\Omega q(\varepsilon)\varphi\sqrt{g(\varepsilon)}dx \quad \forall\varphi\in S(\Omega),\ a.e. \text{ in } (0,T), \tag{46}$$

with $\dot{\boldsymbol{u}}(\varepsilon)(\cdot,0)=\boldsymbol{u}(\varepsilon)(\cdot,0)=0$ and $\vartheta(\varepsilon)(\cdot,0)=0$.

Remark 4. Notice that the time-dependent version of the linearized strain tensor above is well posed when we define

$$e_{i||j}(\varepsilon;\boldsymbol{u}(\varepsilon))(t) := e_{i||j}(\varepsilon;\boldsymbol{u}(\varepsilon)(t)).$$

See for example [46]. Further, as commented earlier, we usually omit the explicit time dependence for the sake of a shorter notation.

Remark 5. The uniqueness of solution for Problem 5 provided that $\varepsilon>0$ is small enough is similar to Problem 4 and the regularity obtained for the solutions is analogue. In particular, we find $\dot{\boldsymbol{u}}(\varepsilon)(\cdot,t)\in V(\Omega)$ and $\dot{\vartheta}(\varepsilon)(\cdot,t)\in S(\Omega)$ *a.e.* in $(0,T)$.

4. Formal asymptotic analysis

In order to identify possible two-dimensonal limit problems, we are going to follow the general procedure described in [9], which has been already used in the framework of contact problems for shells in [37] and [41]. As we shall see, it depends on the geometry of the middle surface, on the set where the boundary conditions are imposed, and on the order of the different functions involved. Particularly interesting for us in the present problem is the case of the normal damped response function. We consider scaled applied forces and heat source of the form

$$\boldsymbol{f}(\varepsilon)(\boldsymbol{x}) = \varepsilon^m\boldsymbol{f}^m(\boldsymbol{x}),\ q(\varepsilon)(\boldsymbol{x})=\varepsilon^m q^m(\boldsymbol{x})\ \forall\boldsymbol{x}\in\Omega, \quad \boldsymbol{h}(\varepsilon)(\boldsymbol{x})=\varepsilon^{m+1}\boldsymbol{h}^{m+1}(\boldsymbol{x})\ \forall\boldsymbol{x}\in\Gamma_+,$$

where m is an integer that will give the order of the respective forces. We also define the scaled normal damped response function $p(\varepsilon)(r(\varepsilon))=\varepsilon^{m+1}p^{m+1}(r(\varepsilon))$. Substituting in (45) we obtain the following scaled problem:

Problem 6. Find a pair $t\mapsto(\boldsymbol{u}(\varepsilon)(\boldsymbol{x},t),\vartheta(\varepsilon)(\boldsymbol{x},t))$ of $[0,T]\to V(\Omega)\times S(\Omega)$ verifying

$$\int_\Omega \rho(\ddot{u}_\alpha(\varepsilon)g^{\alpha\beta}(\varepsilon)v_\beta+\ddot{u}_3(\varepsilon)v_3)\sqrt{g(\varepsilon)}\,dx + \int_\Omega A^{ijkl}(\varepsilon)e_{k||l}(\varepsilon;\boldsymbol{u}(\varepsilon))e_{i||j}(\varepsilon;\boldsymbol{v})\sqrt{g(\varepsilon)}dx$$
$$- \int_\Omega \alpha_T(3\lambda+2\mu)\vartheta(\varepsilon)(e_{\alpha||\beta}(\varepsilon;\boldsymbol{v})g^{\alpha\beta}(\varepsilon)+e_{3||3}(\varepsilon;\boldsymbol{v}))\sqrt{g(\varepsilon)}dx - \int_{\Gamma_C}\varepsilon^m p^{m+1}(-\dot{u}_3(\varepsilon))v_3\sqrt{g(\varepsilon)}d\Gamma$$
$$= \int_\Omega \varepsilon^m f^{i,m}v_i\sqrt{g(\varepsilon)}dx + \int_{\Gamma_+}\varepsilon^m h^{i,m+1}v_i\sqrt{g(\varepsilon)}d\Gamma \ \forall\boldsymbol{v}\in V(\Omega),\ a.e. \text{ in } (0,T), \tag{47}$$

$$\int_\Omega \beta\dot{\vartheta}(\varepsilon)\varphi\sqrt{g(\varepsilon)}dx + \int_\Omega k(\partial_\alpha\vartheta(\varepsilon)g^{\alpha\beta}(\varepsilon)\partial_\beta\varphi + \frac{1}{\varepsilon^2}\partial_3\vartheta(\varepsilon)\partial_3\varphi)\sqrt{g(\varepsilon)}dx$$
$$+ \int_\Omega \alpha_T(3\lambda+2\mu)\varphi(e_{\alpha||\beta}(\varepsilon;\dot{\boldsymbol{u}}(\varepsilon))g^{\alpha\beta}(\varepsilon)+e_{3||3}(\varepsilon;\dot{\boldsymbol{u}}(\varepsilon)))\sqrt{g(\varepsilon)}dx$$
$$= \int_\Omega \varepsilon^m q^m\varphi\sqrt{g(\varepsilon)}dx \quad \forall\varphi\in S(\Omega),\ a.e. \text{ in } (0,T), \tag{48}$$

with $\dot{\boldsymbol{u}}(\varepsilon)(\cdot,0)=\boldsymbol{u}(\varepsilon)(\cdot,0)=0$ and $\vartheta(\varepsilon)(\cdot,0)=0$.

Assume that $\boldsymbol{\theta}\in\mathcal{C}^3(\bar{\omega};\mathbb{R}^3)$ and that the scaled unknowns $\boldsymbol{u}(\varepsilon)$, $\vartheta(\varepsilon)$ admit asymptotic expansions taking the following form:

$$\boldsymbol{u}(\varepsilon) = \boldsymbol{u}^0+\varepsilon\boldsymbol{u}^1+\varepsilon^2\boldsymbol{u}^2+\ldots, \tag{49}$$
$$\vartheta(\varepsilon) = \vartheta^0+\varepsilon\vartheta^1+\varepsilon^2\vartheta^2+\ldots$$

where $\boldsymbol{u}^0\in V(\Omega)$, $\boldsymbol{u}^j\in[H^1(\Omega)]^3$, $\vartheta^0\in S(\Omega)$, $\vartheta^j\in H^1(\Omega)$, $j\geq 1$. Assumption (49) implies the following asymptotic expansion of the scaled linear strain:

$$e_{i||j}(\varepsilon) \equiv e_{i||j}(\varepsilon;\boldsymbol{u}(\varepsilon)) = \frac{1}{\varepsilon}e^{-1}_{i||j}+e^0_{i||j}+\varepsilon e^1_{i||j}+\varepsilon^2 e^2_{i||j}+\varepsilon^3 e^3_{i||j}+\ldots$$

The interested reader can check the expression of $e^m_{i||j}$ in terms of u^m_k and the geometry in [9], and as an extension of what is featured in this paper, in [44].

The same applies to the expansion of $e_{i||j}(\varepsilon;\boldsymbol{v})$ for an arbitrary $\boldsymbol{v}\in V(\Omega)$, of the form

$$e_{i||j}(\varepsilon;\boldsymbol{v}) = \frac{1}{\varepsilon}e^{-1}_{i||j}(\boldsymbol{v}) + e^{0}_{i||j}(\boldsymbol{v}) + \varepsilon e^{1}_{i||j}(\boldsymbol{v}) + \dots$$

Upon substitution on (47) and (48), we can characterize the terms involved in the asymptotic expansions by giving values for m and grouping terms of the same order. In this way, taking in (47) the order $m=-2$ and particular cases of test functions, we reason that $\boldsymbol{f}^{-2}=\boldsymbol{h}^{-1}=0$ and $p^{-1}=0$, which leads to $\partial_3\boldsymbol{u}^0=0$. From (48), we reason that $q^{-2}=0$ and find that $\partial_3\vartheta^0=0$. Thus the zeroth order terms of both unknowns would be independent of the transversal variable x_3. Particularly, $\boldsymbol{u}^0$ can be identified with a function $\boldsymbol{\xi}^0\in V(\omega)$, and ϑ^0 can be identified with a function $\zeta^0\in S(\omega)$ where

$$V(\omega) := \{\boldsymbol{\eta}=(\eta_i)\in[H^1(\omega)]^3;\ \eta_i=0 \text{ on } \gamma_0\},\quad S(\omega):=\{\varphi\in H^1(\omega);\ \varphi=0 \text{ on } \gamma_0\}.$$

Taking $m=-1$, and using particular cases of test functions, we reason that $\boldsymbol{f}^{-1}=\boldsymbol{h}^0=0$ and $p^0=0$ and we find that

$$e^0_{\alpha||3}=0,\quad \lambda a^{\alpha\beta}e^0_{\alpha||\beta}+(\lambda+2\mu)e^0_{3||3}=\alpha_T(3\lambda+2\mu)\vartheta^0,\quad e^0_{\alpha||\beta}=\gamma_{\alpha\beta}(\boldsymbol{\xi}^0),$$

where

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{2}(\partial_\beta\eta_\alpha+\partial_\alpha\eta_\beta)-\Gamma^\sigma_{\alpha\beta}\eta_\sigma-b_{\alpha\beta}\eta_3, \tag{50}$$

denote the covariant components of the linearized change of metric tensor associated with a displacement field $\eta_i\boldsymbol{a}^i$ of the surface S. From (48) we reason that $q^{-1}=0$ and find that $\partial_3\vartheta^1=0$.

With these results in mind, for $m=0$, expanding $A^{ijkl}(0)$ and taking $\boldsymbol{v}=\boldsymbol{\eta}\in V(\omega)$ and $\varphi\in S(\omega)$ leads to a set of two-dimensional equations which will be presented and analyzed in the following section, in the framework of the elliptic membrane shells, where it is well posed.

5. Elliptic membrane case. Convergence

Guided by the formal asymptotic analysis developed in the previous section, we need now a functional framework in which the limit problem is well posed and we can find rigorous convergence results. To do that, we now focus in the particular case in which the middle surface, S, is uniformly elliptic and, further, it is clamped on the whole lateral face, that is $\gamma_0=\gamma$. These kind of shells are known as elliptic membrane shells.

Further, we assume the hypotheses that emerged from the formal asymptotic analysis, specifically

$$\boldsymbol{f}(\varepsilon)(\boldsymbol{x})=\boldsymbol{f}^0(\boldsymbol{x}),\ q(\varepsilon)(\boldsymbol{x})=q^0(\boldsymbol{x})\ \forall\boldsymbol{x}\in\Omega,\ \boldsymbol{h}(\varepsilon)(\boldsymbol{x})=\varepsilon\boldsymbol{h}^1(\boldsymbol{x})\ \forall\boldsymbol{x}\in\Gamma_+,$$
$$p(\varepsilon)(r(\varepsilon))=\varepsilon p^1(r).$$

Since there is no room for confusion, in the following we omit the superindices indicating the order of the functions involved.

In [9, Theorem 2.7-3] we are provided with a two dimensional Korn's inequality for the case of elliptic membrane shells. Thus, there exists a constant $c_M=c_M(\omega,\boldsymbol{\theta})>0$ such that

$$\left(\sum_\alpha\|\eta_\alpha\|^2_{1,\omega}+|\eta_3|^2_{0,\omega}\right)^{1/2}\le c_M\left(\sum_{\alpha,\beta}|\gamma_{\alpha\beta}(\boldsymbol{\eta})|^2_{0,\omega}\right)^{1/2}\quad\forall\boldsymbol{\eta}\in V_M(\omega), \tag{51}$$

where $V_M(\omega):=H^1_0(\omega)\times H^1_0(\omega)\times L^2(\omega)$ is the appropriate space for guarantying the well-posedness of Problem 7 below. In this section and in the sequel, C represents a positive constant which is independent of ε and the unknowns and whose specific value may change from one equation to other. Besides, for the sake of simplicity, we assume that all the parameters involved are constant. Also, the notation $\bar{\boldsymbol{v}}$ stands for the average on x_3, i.e., $\bar{\boldsymbol{v}}:=\frac{1}{2}\int_{-1}^{1}\boldsymbol{v}(x_3)dx_3$.

It is in this context that the limit two-dimensional equations found following the formal asymptotic analysis of the previous section are well posed, as we shall see.

Problem 7. Find a pair $t\mapsto(\boldsymbol{\xi}(\boldsymbol{y},t),\zeta(\boldsymbol{y},t))$ of $[0,T]\to V_M(\omega)\times H^1_0(\omega)$ verifying

$$2\int_\omega\rho(\ddot{\xi}_\alpha a^{\alpha\beta}\eta_\beta+\ddot{\xi}_3\eta_3)\sqrt{a}dy+\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\xi})\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy-4\int_\omega\frac{\alpha_T\mu(3\lambda+2\mu)}{\lambda+2\mu}\zeta a^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy$$
$$-\int_{\Gamma_C}p(-\dot{\xi}_3)\eta_3\sqrt{a}d\Gamma=\int_\omega F^i\eta_i\sqrt{a}dy\quad\forall\boldsymbol{\eta}=(\eta_i)\in V_M(\omega),\ a.e.\text{ in }(0,T), \tag{52}$$

$$2\int_\omega\left(\beta+\frac{\alpha_T^2(3\lambda+2\mu)^2}{\lambda+2\mu}\right)\dot{\zeta}\varphi\sqrt{a}dy+2\int_\omega k\partial_\alpha\zeta a^{\alpha\beta}\partial_\beta\varphi\sqrt{a}dy$$

$$+4\int_\omega \frac{\alpha_T\mu(3\lambda+2\mu)}{\lambda+2\mu}\varphi a^{\alpha\beta}\gamma_{\alpha\beta}(\dot{\boldsymbol{\xi}})\sqrt{a}dy = \int_\omega Q\varphi\sqrt{a}dy \quad \forall\varphi\in H_0^1(\omega),\ a.e.\ \text{in } (0,T), \tag{53}$$

with $\dot{\boldsymbol{\xi}}(\cdot,0)=\boldsymbol{\xi}(\cdot,0)=0$ and $\zeta(\cdot,0)=0$.

Above, we have used $F^i := \int_{-1}^1 f^i dx_3 + h_+^i$ with $h_+^i(\cdot)=h^i(\cdot,+1)$ and $Q := \int_{-1}^1 q dx_3$. Also, $a^{\alpha\beta\sigma\tau}$ denotes the contravariant components of the fourth order two-dimensional elasticity tensor, defined as follows:

$$a^{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda+2\mu}a^{\alpha\beta}a^{\sigma\tau} + 2\mu(a^{\alpha\sigma}a^{\beta\tau}+a^{\alpha\tau}a^{\beta\sigma}). \tag{54}$$

Notice that there exists a constant $c_e>0$ independent of the variables and ε, such that

$$\sum_{\alpha,\beta}|t_{\alpha\beta}|^2 \le c_e a^{\alpha\beta\sigma\tau}(\boldsymbol{y})t_{\sigma\tau}t_{\alpha\beta}, \tag{55}$$

for all $\boldsymbol{y}\in\bar{\omega}$ and all $\boldsymbol{t}=(t_{\alpha\beta})\in\mathbb{S}^2$ (vector space of 2×2 real symmetric matrices). The following result shows that there exists a unique solution for this problem.

Theorem 2. *Let ω be a bounded domain in $\mathbb{R}^2$, let $\boldsymbol{\theta}\in\mathcal{C}^2(\bar{\omega};\mathbb{R}^3)$ be an injective mapping such that the two vectors $\boldsymbol{a}_\alpha=\partial_\alpha\boldsymbol{\theta}$ are linearly independent at all points of $\bar{\omega}$. Let f^i and $q\in H^1(0,T;L^2(\Omega))$, $h^i\in H^2(0,T;L^2(\Gamma_+))$ and assume (3). Then the Problem 7, has a unique solution $(\boldsymbol{\xi},\zeta)$ such that*

$$\boldsymbol{\xi}\in L^\infty(0,T;V_M(\omega)),\ \dot{\boldsymbol{\xi}}\in L^\infty(0,T;[L^2(\omega)]^3)\cap L^\infty(0,T;V_M(\omega)),\ \ddot{\boldsymbol{\xi}}\in L^\infty(0,T;[L^2(\omega)]^3),$$
$$\zeta\in L^\infty(0,T;L^2(\omega))\cap L^2(0,T;H_0^1(\omega)),\ \dot{\zeta}\in L^\infty(0,T;L^2(\omega))\cap L^2(0,T;H_0^1(\omega)).$$

Proof. Like in Theorem 1, we will use a Faedo-Galerkin approach to prove the existence part. Then, a proof by contradiction will show uniqueness.

Existence: Since $V_M(\omega)$ is a separable space, there exists a countable base $\{\boldsymbol{v}^m\}\subset V_M(\omega)$ such that

$$V_M(\omega)=\overline{\bigcup_{m\ge 1}V_m}, \quad \text{where } V_m=\text{Span}\{\boldsymbol{v}^1,\boldsymbol{v}^2,\dots,\boldsymbol{v}^m\}.$$

Similarly, there exists a countable base $\{\chi^m\}\subset H_0^1(\omega)$ such that

$$H_0^1(\omega)=\overline{\bigcup_{m\ge 1}S_m}, \quad \text{where } S_m=\text{Span}\{\chi^1,\chi^2,\dots,\chi^m\}.$$

We now formulate Problem 7 for the finite dimensional subspaces:

Problem 8. Find a pair $t\mapsto(\boldsymbol{\xi}^m(\boldsymbol{y},t),\zeta^m(\boldsymbol{y},t))$ of $[0,T]\to V_m\times S_m$ verifying

$$2\int_\omega\rho(\ddot{\xi}_\alpha^m a^{\alpha\beta}\eta_\beta^m+\ddot{\xi}_3^m\eta_3^m)\sqrt{a}dy+\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\xi}^m)\gamma_{\alpha\beta}(\boldsymbol{\eta}^m)\sqrt{a}dy-4\int_\omega\frac{\alpha_T\mu(3\lambda+2\mu)}{\lambda+2\mu}\zeta^m a^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{\eta}^m)\sqrt{a}dy$$
$$-\int_{\Gamma_C}p(-\dot{\xi}_3^m)\eta_3^m\sqrt{a}d\Gamma=\int_\omega F^i\eta_i^m\sqrt{a}dy \quad \forall\boldsymbol{\eta}^m=(\eta_i^m)\in V_m,\ \forall\, t\in[0,T], \tag{56}$$

$$2\int_\omega\left(\beta+\frac{\alpha_T^2(3\lambda+2\mu)^2}{\lambda+2\mu}\right)\dot{\zeta}^m\varphi^m\sqrt{a}dy+2\int_\omega k\partial_\alpha\zeta^m a^{\alpha\beta}\partial_\beta\varphi^m\sqrt{a}dy$$
$$+4\int_\omega\frac{\alpha_T\mu(3\lambda+2\mu)}{\lambda+2\mu}\varphi^m a^{\alpha\beta}\gamma_{\alpha\beta}(\dot{\boldsymbol{\xi}}^m)\sqrt{a}dy=\int_\omega Q\varphi^m\sqrt{a}dy \quad \forall\varphi^m\in S_m,\ \forall\, t\in[0,T], \tag{57}$$

with $\dot{\boldsymbol{\xi}}^m(\cdot,0)=\boldsymbol{\xi}^m(\cdot,0)=0$ and $\zeta^m(\cdot,0)=0$.

Now, the classical theory of systems of ordinary differential equations guarantees the existence and uniqueness of solution for Problem 8. Taking $\boldsymbol{\eta}^m=\dot{\boldsymbol{\xi}}^m$ in (56) and $\varphi^m=\zeta^m$ in (57), adding both expressions and integrating the time variable in $[0,t]$ gives

$$\rho|\dot{\boldsymbol{\xi}}^m(t)|_{a,\omega}^2+\frac{1}{2}\|\boldsymbol{\xi}^m(t)\|_{a,\omega}^2+\left(\beta+\frac{\alpha_T^2(3\lambda+2\mu)^2}{\lambda+2\mu}\right)|\zeta^m(t)|_{0,\omega}^2+2k\int_0^t\||\zeta^m(r)\||_{a,\omega}^2\,dr \tag{58}$$
$$-\int_0^t\int_{\Gamma_C}p\big(-\dot{\xi}_3^m(r)\big)\dot{\xi}_3^m(r)\sqrt{a}d\Gamma\,dr=\int_0^t\int_\omega Q(r)\zeta^m\sqrt{a}dy\,dr \tag{58}$$
$$+\int_0^t\int_\omega\int_{-1}^1 f^i(r)dx_3\dot{\xi}_i^m(r)\sqrt{a}dy\,dr+\int_0^t\int_{\Gamma_+}h^i(r)\dot{\xi}_i^m(r)\sqrt{a}d\Gamma\,dr, \tag{58}$$

where we have introduced the following norms:

$$|\boldsymbol{\eta}|^2_{a,\omega} := \int_\omega (\eta_\alpha a^{\alpha\beta}\eta_\beta + (\eta_3)^2)\sqrt{a}dy \ \forall \boldsymbol{\eta} \in [L^2(\omega)]^3,$$

which is equivalent to the usual norm $|\cdot|_{0,\omega}$ because of the ellipticity of $(a^{\alpha\beta})$ and the regularity of $\boldsymbol{\theta}$. Also,

$$\|\boldsymbol{\eta}\|^2_{a,\omega} := \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\eta})\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy \ \forall \boldsymbol{\eta} \in V_M(\omega),$$

which is a norm in $V_M(\omega)$ because of the Korn inequality (51) and the ellipticity of $a^{\alpha\beta\sigma\tau}$ (see (55)). Finally,

$$|||\varphi|||^2_{a,\omega} := \int_\omega \partial_\alpha\varphi a^{\alpha\beta}\partial_\beta\varphi\sqrt{a}dy,$$

which is a norm in $H^1_0(\omega)$ equivalent to the usual $\|\cdot\|_{1,\omega}$ because of the ellipticity of $(a^{\alpha\beta})$, the regularity of $\boldsymbol{\theta}$ and the Poincaré inequality. Further, in [18, Theorem 3] we included, as a standalone theorem, a result that can be found inside the proof of [9, Theorem 4.4-1], which shows that we can define a kind of trace operator on upper and lower faces, continuous, in $X(\Omega) := \{v \in L^2(\Omega);\ \partial_3 v \in L^2(\Omega)\}$. Moreover, there exists a constant $c_1 > 0$ such that

$$\|v\|_{L^2(\Gamma_+\cup\Gamma_C)} \le c_1\left(|v|^2_{0,\Omega} + |\partial_3 v|^2_{0,\Omega}\right)^{1/2}$$

for all $\boldsymbol{v} \in X(\Omega)$, which also implies that there exists a constant $c_2 > 0$ such that

$$\|v_3\|_{L^2(\Gamma_+\cup\Gamma_C)} \le c_2\left(\sum_{i,j}|e_{i\|j}(\varepsilon;\boldsymbol{v})|^2_{0,\Omega}\right)^{1/2} \quad \forall \boldsymbol{v} \in V(\Omega). \tag{59}$$

By using the monotonicity of p, the Hölder inequality in the right-hand side terms of (58) and using [18, Theorem 3] for the terms on Γ_+, followed by the use of Gronwall inequality, we obtain that the following weak convergences take place for subsequences indexed by m as well:

$$\boldsymbol{\xi}^m \underset{m\to\infty}{\overset{*}{\rightharpoonup}} \boldsymbol{\xi} \text{ in } L^\infty(0,T;V_M(\omega)), \quad \dot{\boldsymbol{\xi}}^m \underset{m\to\infty}{\overset{*}{\rightharpoonup}} \dot{\boldsymbol{\xi}} \text{ in } L^\infty\left(0,T;\left[L^2(\omega)\right]^3\right), \tag{60}$$

$$\zeta^m \underset{m\to\infty}{\overset{*}{\rightharpoonup}} \zeta \text{ in } L^\infty\left(0,T;L^2(\omega)\right), \quad \zeta^m \underset{m\to\infty}{\rightharpoonup} \zeta \text{ in } L^2\left(0,T;H^1_0(\omega)\right), \tag{61}$$

$$p\left(-\dot{\xi}^m_3\right) \underset{m\to\infty}{\overset{*}{\rightharpoonup}} \chi \text{ in } L^\infty\left(0,T;L^2(\omega)\right). \tag{62}$$

Notice that (62) is a consequence of the Lipschitz continuity of p, the fact that $p(0)=0$, and the boundedness of its argument. Using these convergences back in (56)–(57), we can formulate the following limit problem:

Problem 9. Find a pair $t \mapsto (\boldsymbol{\xi}(\boldsymbol{y},t), \zeta(\boldsymbol{y},t))$ of $[0,T] \to V_M(\omega)\times H^1_0(\omega)$ verifying

$$\begin{aligned}
&2\int_\omega \rho(\ddot{\xi}_\alpha a^{\alpha\beta}\eta_\beta + \ddot{\xi}_3\eta_3)\sqrt{a}dy + \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\xi})\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy - 4\int_\omega \frac{\alpha_T\mu(3\lambda+2\mu)}{\lambda+2\mu}\zeta a^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy \\
&\quad - \int_{\Gamma_C}\chi\eta_3\sqrt{a}d\Gamma = \int_\omega F^i\eta_i\sqrt{a}dy \quad \forall \boldsymbol{\eta} = (\eta_i) \in V_M(\omega),\ a.e. \text{ in } (0,T), \\
&2\int_\omega\left(\beta + \frac{\alpha_T^2(3\lambda+2\mu)^2}{\lambda+2\mu}\right)\dot{\zeta}\varphi\sqrt{a}dy + 2\int_\omega k\partial_\alpha\zeta a^{\alpha\beta}\partial_\beta\varphi\sqrt{a}dy \\
&\quad + 4\int_\omega \frac{\alpha_T\mu(3\lambda+2\mu)}{\lambda+2\mu}\varphi a^{\alpha\beta}\gamma_{\alpha\beta}(\dot{\boldsymbol{\xi}})\sqrt{a}dy = \int_\omega Q\varphi\sqrt{a}dy \quad \forall \varphi \in H^1_0(\omega),\ a.e. \text{ in } (0,T),
\end{aligned}$$

with $\dot{\boldsymbol{\xi}}(\cdot,0) = \boldsymbol{\xi}(\cdot,0) = 0$ and $\zeta(\cdot,0) = 0$.

Now, to identify the term on Γ_C, we will use an argument of monotonicity (see, for example, [33]). We first define for any given $\phi \in H^1(0,T;L^2(\omega))$, with $\phi(0) = 0$, the following quantity:

$$X^m = -\int_0^t\int_{\Gamma_C}\left(p(-\dot{\xi}^m_3(r)) - p(-\dot{\phi}(r))\right)(\dot{\xi}^m_3(r) - \dot{\phi}(r))\sqrt{a}d\Gamma\, dr \ge 0.$$

From (58) we find that

$$\begin{aligned}
X^m = &\int_0^t\int_\omega F^i(r)\dot{\xi}^m_i(r)\sqrt{a}dy\, dr - \rho|\dot{\boldsymbol{\xi}}^m(t)|^2_{a,\omega} - \frac{1}{2}\|\boldsymbol{\xi}^m(t)\|^2_{a,\omega} \\
&+ \int_0^t\int_\omega Q(r)\zeta^m\sqrt{a}dy\, dr - \left(\beta + \frac{\alpha_T^2(3\lambda+2\mu)^2}{\lambda+2\mu}\right)|\zeta^m(t)|^2_{0,\omega} - 2k\int_0^t |||\zeta^m(r)|||^2_{a,\omega}\, dr \\
&- \int_0^t\int_{\Gamma_C} p\left(-\dot{\xi}^m_3(r)\right)\left(-\dot{\phi}(r)\right)\sqrt{a}d\Gamma\, dr - \int_0^t\int_{\Gamma_C} -p\left(-\dot{\phi}(r)\right)\left(\dot{\xi}^m_3(r) - \dot{\phi}(r)\right)\sqrt{a}d\Gamma\, dr.
\end{aligned}$$

Thus, on one hand

$$0 \le \limsup_{m\to\infty} X^m \le \int_0^t \int_\omega F^i(r)\dot{\xi}_i(r)\sqrt{a}dy\,dr - \rho|\dot{\boldsymbol{\xi}}(t)|^2_{a,\omega} - \frac{1}{2}\|\boldsymbol{\xi}(t)\|^2_{a,\omega}$$
$$+ \int_0^t \int_\omega Q(r)\zeta\sqrt{a}dy\,dr - \left(\beta + \frac{\alpha_T^2(3\lambda+2\mu)^2}{\lambda+2\mu}\right)|\zeta(t)|^2_{0,\omega} - 2k\int_0^t |||\zeta(r)|||^2_{a,\omega}\,dr$$
$$- \int_0^t \int_{\Gamma_C} \chi(r)(-\dot{\phi}(r))\sqrt{a}d\Gamma\,dr - \int_0^t \int_{\Gamma_C} -p(-\dot{\phi}(r))(\dot{\xi}_3(r) - \dot{\phi}(r))\sqrt{a}d\Gamma\,dr,$$

where he have used the weak upper semicontinuity of various terms. On the other hand, doing in Problem 9 the substitutions $\boldsymbol{\eta} = \dot{\boldsymbol{\xi}}$, $\varphi = \zeta$, then the summation of both equations, followed by the integration in $[0, t]$, and using the resulting identity into the inequality above, we find that

$$0 \le -\int_0^t \int_{\Gamma_C} \chi(r)\dot{\xi}_3(r)\sqrt{a}d\Gamma\,dr - \int_0^t \int_{\Gamma_C} \chi(r)(-\dot{\phi}(r))\sqrt{a}d\Gamma\,dr - \int_0^t \int_{\Gamma_C} -p(-\dot{\phi}(r))(\dot{\xi}_3(r) - \dot{\phi}(r))\sqrt{a}d\Gamma\,dr$$
$$= -\int_0^t \int_{\Gamma_C} (\chi(r) - p(-\dot{\phi}(r)))(\dot{\xi}_3(r) - \dot{\phi}(r))\sqrt{a}d\Gamma\,dr.$$

Therefore, by using arguments adapted from those in [19, p. 55], we deduce that $\chi = p(-\dot{\xi}_3)$. Indeed, this is because we can always take $\phi = \xi_3 - \varsigma\varphi$ with $\varsigma > 0$ and $\varphi \in H^1(0, T; L^2(\omega))$, with $\varphi(0) = 0$, to find

$$0 \le -\int_0^t \int_{\Gamma_C} (\chi(r) - p(-\dot{\xi}_3(r) + \varsigma\dot{\varphi}(r)))\dot{\varphi}(r)\sqrt{a}d\Gamma\,dr,$$

and take $\varsigma \to 0$, from where $\chi = p(-\dot{\xi}_3)$. Therefore, we find that Problem 9 is indeed the same as Problem 7.

We find now additional regularity for $\dot{\boldsymbol{\xi}}$, $\ddot{\boldsymbol{\xi}}$ and $\dot{\zeta}$. The process is similar to what we have done above, in the proof of Theorem 1, so we omit it. The interested reader can consult the details in [44].

Particularly, we find that

$$|\ddot{\boldsymbol{\xi}}^m(t)|^2_{0,\omega} + |\dot{\zeta}^m(t)|^2_{0,\omega} \le C, \quad \forall t \in [0, T],$$

and further

$$\|\dot{\boldsymbol{\xi}}^m(t)\|^2_{a,\omega} + 2k\int_0^t |||\dot{\zeta}^m(r)|||^2_{a,\omega}dr \le C \quad \forall t \in [0, T].$$

Therefore, the following weak convergences take place for subsequences still indexed by m.

$$\dot{\boldsymbol{\xi}}^m \underset{m\to\infty}{\overset{*}{\rightharpoonup}} \dot{\boldsymbol{\xi}} \text{ in } L^\infty(0, T; V_M(\omega)), \quad \ddot{\boldsymbol{\xi}}^m \underset{m\to\infty}{\overset{*}{\rightharpoonup}} \ddot{\boldsymbol{\xi}} \text{ in } L^\infty\left(0, T; \left[L^2(\omega)\right]^3\right), \tag{63}$$

$$\dot{\zeta}^m \underset{m\to\infty}{\overset{*}{\rightharpoonup}} \dot{\zeta} \text{ in } L^\infty\left(0, T; L^2(\omega)\right), \quad \dot{\zeta}^m \underset{m\to\infty}{\rightharpoonup} \dot{\zeta} \text{ in } L^2\left(0, T; H_0^1(\omega)\right). \tag{64}$$

Uniqueness: We proceed by contradiction. We first assume that there exist two solutions $(\boldsymbol{\xi}^1, \zeta^1)$ and $(\boldsymbol{\xi}^2, \zeta^2)$. Define $\bar{\boldsymbol{\xi}} = \boldsymbol{\xi}^1 - \boldsymbol{\xi}^2$ and $\bar{\zeta} = \zeta^1 - \zeta^2$. Now, take $\boldsymbol{\eta} = \dot{\bar{\boldsymbol{\xi}}}$ in the version of (52) for $\boldsymbol{\xi}^1$ and $\boldsymbol{\eta} = -\dot{\bar{\boldsymbol{\xi}}}$ in the version of (52) for $\boldsymbol{\xi}^2$. We then sum both expresions to find that

$$2\int_\omega \rho(\ddot{\bar{\xi}}_\alpha a^{\alpha\beta}\dot{\bar{\xi}}_\beta + \ddot{\bar{\xi}}_3\dot{\bar{\xi}}_3)\sqrt{a}dy + \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\bar{\boldsymbol{\xi}})\gamma_{\alpha\beta}(\dot{\bar{\boldsymbol{\xi}}})\sqrt{a}dy - 4\int_\omega \frac{\alpha_T\mu(3\lambda+2\mu)}{\lambda+2\mu}\bar{\zeta}a^{\alpha\beta}\gamma_{\alpha\beta}(\dot{\bar{\boldsymbol{\xi}}})\sqrt{a}dy$$
$$- \int_{\Gamma_C} (p(-\dot{\xi}_3^1) - p(-\dot{\xi}_3^2))\dot{\bar{\xi}}_3\sqrt{a}d\Gamma = 0.$$

Similarly, take $\varphi = \bar{\zeta}$ in the version of (53) for ζ^1 and $\varphi = -\bar{\zeta}$ in the version of (53) for ζ^2. Then, we sum both expresions to find that

$$2\int_\omega \left(\beta + \frac{\alpha_T^2(3\lambda+2\mu)^2}{\lambda+2\mu}\right)\dot{\bar{\zeta}}\bar{\zeta}\sqrt{a}dy + 2\int_\omega k\partial_\alpha\bar{\zeta}a^{\alpha\beta}\partial_\beta\bar{\zeta}\sqrt{a}dy + 4\int_\omega \frac{\alpha_T\mu(3\lambda+2\mu)}{\lambda+2\mu}\bar{\zeta}a^{\alpha\beta}\gamma_{\alpha\beta}(\dot{\bar{\boldsymbol{\xi}}})\sqrt{a}dy = 0.$$

Then, we add both expressions above and integrate with respect to the time variable in $[0, t]$, to find

$$\rho|\dot{\bar{\boldsymbol{\xi}}}(t)|^2_{a,\omega} + \frac{1}{2}\|\bar{\boldsymbol{\xi}}(t)\|^2_{a,\omega} + \left(\beta + \frac{\alpha_T^2(3\lambda+2\mu)^2}{\lambda+2\mu}\right)|\bar{\zeta}(t)|^2_{0,\omega} + 2k\int_0^t |||\bar{\zeta}(r)|||^2_{a,\omega}\,dr$$
$$= \int_0^t \int_{\Gamma_C} \left(p\left(-\dot{\xi}_3^1(r)\right) - p\left(-\dot{\xi}_3^2(r)\right)\right)\left(\dot{\xi}_3^1(r) - \dot{\xi}_3^2(r)\right)\sqrt{a}d\Gamma\,dr \le 0, \tag{65}$$

where we have used the monotonicity of p. We deduce from (65) that $\bar{\boldsymbol{\xi}} = 0$ and $\bar{\zeta} = 0$, thus showing uniqueness.

In what follows, and for the sake of simplicity, we assume that for each $\varepsilon > 0$ the initial condition for the scaled linear strain is

$$e_{i\|j}(\varepsilon)(0,\cdot) = 0, \tag{66}$$

this is, the domain is on its natural state with no strains on it at the beginning of the period of observation.

Theorem 3. *For the case of elliptic membrane shells, let $(\boldsymbol{u}(\varepsilon), \vartheta(\varepsilon))$ denote the solution of the three-dimensional scaled Problem 6 (for $m = 0$). Then, under the assumption that $\boldsymbol{\theta} \in \mathcal{C}^3(\bar{\omega}; \mathbb{R}^3)$, and hypotheses (3) and (66), there exist functions $\vartheta, u_\alpha \in H^1(\Omega)$ a.e. in $(0,T)$ satisfying $\vartheta = 0$, $u_\alpha = 0$ on $\gamma \times [-1,1]$ and a function $u_3 \in L^2(\Omega)$ a.e. in $(0,T)$, such that*

(a) *$\vartheta(\varepsilon) \to \vartheta$, $u_\alpha(\varepsilon) \to u_\alpha$ in $H^1(\Omega)$ and $u_3(\varepsilon) \to u_3$ in $L^2(\Omega)$ a.e. in (0,T), when $\varepsilon \to 0$,*
(b) *ϑ and $\boldsymbol{u} = (u_i)$ are independent of the transversal variable x_3.*

Moreover, the pair $(\boldsymbol{u}, \vartheta)$ is indeed the solution of Problem 7.

Proof. The proof has a similar structure to the one given in [9, Theorem 4.4-1] for the elastic elliptic membrane shells case. Therefore, some details will be omitted on those steps that can be proved similarly to what is done in there. Below, all references to (47) or (48) have to be thought of by taking $m = 0$ and omit the superindices. The proof consists of six parts, numbered from (i) to (vi).

(i) A priori boundedness and extraction of weak convergent sequences. For $\varepsilon > 0$ sufficiently small, there exist bounded sequences, also indexed by ε, and weak limits as specified below:

$$\begin{aligned}
&u_\alpha(\varepsilon) \underset{\varepsilon\to 0}{\overset{*}{\rightharpoonup}} u_\alpha \text{ in } L^\infty\big(0,T;H^1(\Omega)\big), \quad u_3(\varepsilon) \underset{\varepsilon\to 0}{\overset{*}{\rightharpoonup}} u_3 \text{ in } L^\infty\big(0,T;L^2(\Omega)\big),\\
&\dot{\boldsymbol{u}}(\varepsilon) \underset{\varepsilon\to 0}{\overset{*}{\rightharpoonup}} \dot{\boldsymbol{u}} \text{ in } L^\infty\Big(0,T;\big[L^2(\Omega)\big]^3\Big), \quad e_{i\|j}(\varepsilon) \underset{\varepsilon\to 0}{\overset{*}{\rightharpoonup}} e_{i\|j} \text{ in } L^\infty\big(0,T;L^2(\Omega)\big)\\
&\vartheta(\varepsilon) \underset{\varepsilon\to 0}{\overset{*}{\rightharpoonup}} \vartheta \text{ in } L^\infty\big(0,T;L^2(\Omega)\big), \quad \partial_\alpha\vartheta(\varepsilon) \underset{\varepsilon\to 0}{\rightharpoonup} \vartheta_\alpha \text{ in } L^2\big(0,T;L^2(\Omega)\big),\\
&\qquad \varepsilon^{-1}\partial_3\vartheta(\varepsilon) \underset{\varepsilon\to 0}{\rightharpoonup} \vartheta_{3,-1} \text{ in } L^2\big(0,T;L^2(\Omega)\big).
\end{aligned}$$

Moreover, $\vartheta, u_\alpha = 0$ on $\Gamma_0 \times [0,T]$.

Above and below, $e_{i\|j}(\varepsilon;\boldsymbol{u}(\varepsilon))$ will be shortened to $e_{i\|j}(\varepsilon) := e_{i\|j}(\varepsilon;\boldsymbol{u}(\varepsilon))$ for the sake of readibility. In this step we take $\boldsymbol{v} = \dot{\boldsymbol{u}}(\varepsilon)$ in (47) (see Remark 5) and $\varphi = \vartheta(\varepsilon)$ in (48) and sum both expressions to find

$$\begin{aligned}
&\int_\Omega \rho(\ddot{u}_\alpha(\varepsilon)g^{\alpha\beta}(\varepsilon)\dot{u}_\beta(\varepsilon) + \ddot{u}_3(\varepsilon)\dot{u}_3(\varepsilon))\sqrt{g(\varepsilon)}\,dx + \int_\Omega A^{ijkl}(\varepsilon)e_{k\|l}(\varepsilon)\dot{e}_{i\|j}(\varepsilon)\sqrt{g(\varepsilon)}dx\\
&\quad + \int_\Omega \beta\dot{\vartheta}(\varepsilon)\vartheta(\varepsilon)\sqrt{g(\varepsilon)}dx + \int_\Omega k(\partial_\alpha\vartheta(\varepsilon)g^{\alpha\beta}(\varepsilon)\partial_\beta\vartheta(\varepsilon) + \frac{1}{\varepsilon^2}\partial_3\vartheta(\varepsilon)\partial_3\vartheta(\varepsilon))\sqrt{g(\varepsilon)}dx\\
&\quad - \int_{\Gamma_C} p(-\dot{u}_3(\varepsilon))\dot{u}_3(\varepsilon)\sqrt{g(\varepsilon)}d\Gamma\\
&\quad = \int_\Omega f^i\dot{u}_i(\varepsilon)\sqrt{g(\varepsilon)}dx + \int_{\Gamma_+} h^i\dot{u}_i(\varepsilon)\sqrt{g(\varepsilon)}d\Gamma + \int_\Omega q\vartheta(\varepsilon)\sqrt{g(\varepsilon)}dx.
\end{aligned}\tag{67}$$

We now introduce the following norms:

$$|\boldsymbol{v}|^2_{g(\varepsilon),\Omega} := \int_\Omega (v_\alpha g^{\alpha\beta}(\varepsilon)v_\beta + (v_3)^2)\sqrt{g(\varepsilon)}dx \ \forall \boldsymbol{v} \in [L^2(\Omega)]^3,$$

which is equivalent to the usual norm $|\cdot|_{0,\Omega}$ because of the ellipticity of $(g^{\alpha\beta}(\varepsilon))$ and the regularity of $\boldsymbol{\Theta}$. Also,

$$\|\boldsymbol{v}\|^2_{A(\varepsilon),\Omega} := \int_\Omega A^{ijkl}(\varepsilon)e_{k\|l}(\varepsilon;\boldsymbol{v})e_{i\|j}(\varepsilon;\boldsymbol{v})\sqrt{g(\varepsilon)}dx \ \forall \boldsymbol{v} \in V(\Omega),$$

which is a norm in $V(\Omega)$ because of the Korn inequality (see [9, Theorem 4.4-1]) and the ellipticity of $A^{ijkl}(\varepsilon)$. Finally,

$$|||\varphi|||_{g(\varepsilon),\Omega} := \int_\Omega \partial_\alpha\varphi g^{\alpha\beta}(\varepsilon)\partial_\beta\varphi\sqrt{g(\varepsilon)}dx,$$

which is a seminorm in $S(\Omega)$. Because of the uniform ellipticity of the tensors and matrices involved, and the properties of $g(\varepsilon)$, we are going to be able to use constants independent of ε in the estimates below. Indeed, going back to (67), we obtain

$$\begin{aligned}
&\frac{\rho}{2}\frac{d}{dt}\{|\dot{\boldsymbol{u}}(\varepsilon)|^2_{g(\varepsilon),\Omega}\} + \frac{1}{2}\frac{d}{dt}\{\|\boldsymbol{u}(\varepsilon)\|^2_{A(\varepsilon),\Omega}\} + \frac{\beta}{2}\frac{d}{dt}\{|\vartheta(\varepsilon)|^2_{0,\Omega}\} + k|||\vartheta(\varepsilon)|||^2_{g(\varepsilon),\Omega} + \frac{k}{\varepsilon^2}|\partial_3\vartheta(\varepsilon)|^2_{0,\Omega}\\
&\quad = \int_{\Gamma_C} p(-\dot{u}_3(\varepsilon))\dot{u}_3(\varepsilon)\sqrt{g(\varepsilon)}d\Gamma + \int_\Omega f^i\dot{u}_i(\varepsilon)\sqrt{g(\varepsilon)}dx + \int_{\Gamma_+} h^i\dot{u}_i(\varepsilon)\sqrt{g(\varepsilon)}d\Gamma + \int_\Omega q\vartheta(\varepsilon)\sqrt{g(\varepsilon)}dx.
\end{aligned}$$

Integrating in $[0,t]$ with respect to the time variable, using the equivalences mentioned above, together with the uniformity with respect to ε of the constants involved in those equivalences, integrating by parts the term with the tractions h^i, using [18, Theorem 3] and Young's inequality, we find that there exist a constant $C>0$ independent of ε such that

$$|\dot{\boldsymbol{u}}(\varepsilon)(t)|_{0,\Omega}^2+|e_{i\|j}(\varepsilon)(t)|_{0,\Omega}^2+|\vartheta(\varepsilon)(t)|_{0,\Omega}^2+\int_0^t(|\partial_\alpha\vartheta(\varepsilon)(r)|_{0,\Omega}^2+\frac{1}{\varepsilon^2}|\partial_3\vartheta(\varepsilon)(r)|_{0,\Omega}^2)dr$$
$$-\int_0^t\int_{\Gamma_C}p(-\dot{u}_3(\varepsilon)(r))\dot{u}_3(\varepsilon)(r)\sqrt{g(\varepsilon)}d\Gamma\,dr\le C\Big(\int_0^t|\dot{\boldsymbol{u}}(\varepsilon)(r)|_{0,\Omega}^2dr+\int_0^t|\vartheta(\varepsilon)(r)|_{0,\Omega}^2dr+\int_0^t|e_{i\|j}(\varepsilon)(r)|_{0,\Omega}^2dr$$
$$+\int_0^t|\boldsymbol{f}(r)|_{0,\Omega}^2dr+\int_0^t|q(r)|_{0,\Omega}^2dr+\int_0^t|\dot{\boldsymbol{h}}(r)|_{0,\Gamma_+}^2dr+|\dot{\boldsymbol{h}}(t)|_{0,\Gamma_+}^2\Big)$$

Hence, by using Gronwall's inequality and the three-dimensional Korn's inequality that can be found in [9, Theorem 4.3-1], all the assertions of (i) follow.

(ii) The limits of the scaled unknowns, u_i, ϑ found in Step (i) are independent of x_3.

The part corresponding to u_i is similar to Step (ii) in [9, Theorem 4.4-1], so we will omit it. Regarding ϑ, its independence on x_3 is a consequence of the boundedness of $\{\varepsilon^{-1}\partial_3\vartheta(\varepsilon)\}$.

(iii) Extraction of weakly convergence subsequences on the contact boundary. The norms $|u_3(\varepsilon)|_{0,\Gamma_C}$, $|\dot{u}_3(\varepsilon)|_{0,\Gamma_C}$ are bounded independently of $\varepsilon, 0<\varepsilon\le\varepsilon_1$ almost everywhere in $(0,T)$. Moreover, there exist subsequences, also denoted by $(u_3(\varepsilon))_{\varepsilon>0}$ and $(\dot{u}_3(\varepsilon))_{\varepsilon>0}$ such that $u_3(\varepsilon)\overset{*}{\rightharpoonup}u_3$ and $\dot{u}_3(\varepsilon)\overset{*}{\rightharpoonup}\dot{u}_3$ in $L^\infty(0,T;L^2(\Gamma_C))$.

The first part is an straightforward consequence of Step (i) and (59). For $v=u_3(\varepsilon)$ we obtain that

$$|u_3(\varepsilon)|_{0,\Gamma_C}\le C|e_{i\|j}(\varepsilon)|_{0,\Omega}\quad a.e.\text{ in }(0,T).$$

Then, there exists $\psi\in L^\infty(0,T;L^2(\Gamma_C))$ such that for a subsequence keeping the same notation, it holds $u_3(\varepsilon)\overset{*}{\rightharpoonup}\psi$ in $L^\infty(0,T;L^2(\Gamma_C))$. Since the conditions of [41, Theorem 3.6] hold, we can identify $\psi=u_3$.

For the second part, we first recall that $\dot{\boldsymbol{u}}(\varepsilon)\in V(\Omega)$ and $\dot{\vartheta}(\varepsilon)\in S(\Omega)$ (see Remark 5). Next, we use the technique of incremental coefficients in the time variable, then integrate on $[0,t]$ to obtain the expression similar to (18) in the scaled framework and without tractions. Indeed,

$$\frac12\rho|\ddot{\boldsymbol{u}}(\varepsilon)(t)|_0^2-\frac12\rho|\ddot{\boldsymbol{u}}(\varepsilon)(0)|_0^2+\frac12a^V(\dot{\boldsymbol{u}}(\varepsilon)(t),\dot{\boldsymbol{u}}(\varepsilon)(t))-\frac12a^V(\dot{\boldsymbol{u}}(\varepsilon)(0),\dot{\boldsymbol{u}}(\varepsilon)(0))$$
$$+\frac12\int_\Omega\beta(\dot{\vartheta}(\varepsilon)(t))^2dx-\frac12\int_\Omega\beta(\dot{\vartheta}(\varepsilon)(0))^2dx+\int_0^ta^S(\dot{\vartheta}(\varepsilon)(r),\dot{\vartheta}(\varepsilon)(r))dr$$
$$\le\int_0^t\int_\Omega\dot{f}^i(r)\ddot{u}(\varepsilon)_i(r)\,dxdr+\int_0^t\int_\Omega\dot{q}(r)\dot{\vartheta}(\varepsilon)(r)\,dxdr.\tag{68}$$

Then, we use Korn's inequality on the left-hand side and apply Gronwall's inequality to obtain that $|e_{i\|j}(\dot{\boldsymbol{u}})(\varepsilon)|_{0,\Omega}^2$ is bounded independently of ε. Then we can proceed like in the first part using (59) for $v=\dot{u}_3(\varepsilon)$ to prove that $\dot{u}_3(\varepsilon)\overset{*}{\rightharpoonup}\dot{u}_3$ in $L^\infty(0,T;L^2(\Gamma_C))$.

(iv) The limits $e_{i\|j}$ found in (i) are independent of the variable x_3. Moreover, their relations with the limits $\boldsymbol{u}:=(u_i)$ and ϑ are the following:

$$e_{\alpha\|\beta}=\gamma_{\alpha\beta}(\boldsymbol{u}):=\frac12(\partial_\alpha u_\beta+\partial_\beta u_\alpha)-\Gamma_{\alpha\beta}^\sigma u_\sigma-b_{\alpha\beta}u_3,$$
$$e_{\alpha\|3}=0,\tag{69}$$

$$e_{3\|3}=\frac{\alpha_T(3\lambda+2\mu)}{\lambda+2\mu}\vartheta-\frac{\lambda}{\lambda+2\mu}a^{\alpha\beta}e_{\alpha\|\beta}.\tag{70}$$

Indeed, first taking $\boldsymbol{v}=\boldsymbol{u}(\varepsilon)$ in (35) and $\boldsymbol{\eta}=\boldsymbol{u}$ in (50) (*par abus de langage*, since $\boldsymbol{u}$ is independent of x_3, but actually $\boldsymbol{u}\in[H^1(\Omega)]^2\times L^2(\Omega)$), taking into account the results in Step (i) and the convergences $\Gamma_{\alpha\beta}^\sigma(\varepsilon)\to\Gamma_{\alpha\beta}^\sigma$ and $\Gamma_{\alpha\beta}^3(\varepsilon)\to b_{\alpha\beta}$ in $\mathcal{C}^0(\bar{\Omega})$ given by (41)–(43), we have that

$$e_{\alpha\|\beta}(\varepsilon)=\frac12(\partial_\beta u_\alpha(\varepsilon)+\partial_\alpha u_\beta(\varepsilon))-\Gamma_{\alpha\beta}^p(\varepsilon)u_p(\varepsilon)\rightharpoonup e_{\alpha\|\beta}=\gamma_{\alpha\beta}(\boldsymbol{u})\text{ in }L^2(\Omega)\ a.e.\text{ in }(0,T).$$

Moreover, $e_{\alpha\|\beta}$ are independent of x_3, as a straightforward consequence of u_i being independent on x_3 (Step (ii)). Besides, let $\boldsymbol{v}\in V(\Omega)$. From the definition of the scaled strains in (35)–(37), we get

$$\varepsilon e_{\alpha\|\beta}(\varepsilon;\boldsymbol{v})\to0\text{ in }L^2(\Omega),\quad\varepsilon e_{\alpha\|3}(\varepsilon;\boldsymbol{v})\to\frac12\partial_3v_\alpha\text{ in }L^2(\Omega),$$
$$\varepsilon e_{3\|3}(\varepsilon;\boldsymbol{v})=\partial_3v_3\text{ in }L^2(\Omega),\ \text{ for all }\varepsilon>0.$$

Now, we can take as test function $\varepsilon\boldsymbol{v} \in V(\Omega)$ in (47). Then, taking into account (33), we have

$$\varepsilon \int_\Omega \rho(\ddot{u}_\alpha(\varepsilon) g^{\alpha\beta}(\varepsilon) v_\beta + \ddot{u}_3(\varepsilon) v_3)\sqrt{g(\varepsilon)}\,dx + \varepsilon \int_\Omega A^{ijkl}(\varepsilon) e_{k||l}(\varepsilon) e_{i||j}(\varepsilon;\boldsymbol{v})\sqrt{g(\varepsilon)}dx$$
$$- \int_\Omega \alpha_T(3\lambda+2\mu)\vartheta(\varepsilon)(\varepsilon e_{\alpha||\beta}(\varepsilon;\boldsymbol{v})g^{\alpha\beta}(\varepsilon) + \varepsilon e_{3||3}(\varepsilon;\boldsymbol{v}))\sqrt{g(\varepsilon)}dx - \varepsilon\int_{\Gamma_C} p(-\dot{u}_3(\varepsilon))v_3\sqrt{g(\varepsilon)}d\Gamma$$
$$= \varepsilon \int_\Omega f^i v_i \sqrt{g(\varepsilon)}dx.$$

Passing to the limit as $\varepsilon \to 0$, decomposing $A^{ijkl}(\varepsilon)$ into the components with different asymptotic behaviour (see (38)–(39)), the properties of $g(\varepsilon)$ (see (44)) and the convergences in Step (i), we obtain the following equality:

$$\int_\Omega \big(2\mu a^{\alpha\sigma} e_{\alpha||3}\partial_3 v_\sigma + (\lambda+2\mu)e_{3||3}\partial_3 v_3\big)\sqrt{a}dx + \int_\Omega \lambda a^{\alpha\beta} e_{\alpha||\beta}\partial_3 v_3\sqrt{a}dx$$
$$= \int_\Omega \alpha_T(3\lambda+2\mu)\vartheta\,\partial_3 v_3\sqrt{a}dx \quad \forall \boldsymbol{v}\in V(\Omega),\ a.e.\ \text{in}\ (0,T). \tag{71}$$

By taking particular test functions and using [9, Theorem 3.4-1], we deduce (69). Then, we go back to (71) and use again [9, Theorem 3.4-1] to deduce (70). The independence of $e_{3||3}$ on x_3 is a consequence of this relation, as well.

(v) We find a limit two-dimensional problem verified by functions $\bar{\boldsymbol{u}} = (\bar{\boldsymbol{u}}_i)$ and $\bar{\vartheta}$. In particular, since the solution of this problem is unique, the convergences on Step (i) are verified for the whole families $(\boldsymbol{u}(\varepsilon))_{\varepsilon>0}$ and $(\vartheta(\varepsilon))_{\varepsilon>0}$. We have that $\bar{\boldsymbol{u}}(t) = (\bar{u}_i(t)) \in V_M(\omega)$ and $\bar{\vartheta}(t) \in S(\omega)$ a.e. in (0,T).

By using [9, Theorem 4.2-1] (parts (a) and (b)), and Step (*ii*) we obtain that $\bar{u}_\alpha \in H_0^1(\omega)$ and $\bar{\vartheta} \in H_0^1(\omega)$. Therefore, $\bar{\boldsymbol{u}} \in V_M(\omega)$ a.e. in (0,T). Now, let $\boldsymbol{v} = (v_i) \in V(\Omega)$ be independent of the variable x_3. Then, from the asymptotic behaviour of $\Gamma^p_{\alpha\beta}(\varepsilon)$ and $\Gamma^\sigma_{\alpha 3}(\varepsilon)$ (see (41)–(43)) we find the following convergences when $\varepsilon \to 0$ (see (35)–(37)):

$$e_{\alpha||\beta}(\varepsilon;\boldsymbol{v}) \to \gamma_{\alpha\beta}(\boldsymbol{v}) := \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \Gamma^\sigma_{\alpha\beta}v_\sigma - b_{\alpha\beta}v_3 \ \text{in}\ L^2(\Omega), \tag{72}$$

$$e_{\alpha||3}(\varepsilon;\boldsymbol{v}) \to \frac{1}{2}\partial_\alpha v_3 + b^\sigma_\alpha v_\sigma \ \text{in}\ L^2(\Omega), \quad e_{3||3}(\varepsilon;\boldsymbol{v}) = 0. \tag{73}$$

Taking this into account, let us take now $\boldsymbol{v} = (v_i) \in V(\Omega)$ independent of x_3 in (47) and pass to the limit when $\varepsilon \to 0$. If we use the asymptotic behaviour of $A^{ijkl}(\varepsilon)$ (see (38)–(39)) and $g(\varepsilon)$ (see (44)), take into account the weak convergences $e_{i||j}(\varepsilon) \overset{*}{\rightharpoonup} e_{i||j}$ in $L^\infty(0,T;L^2(\Omega))$, simplify by using (69) and consider the precise limits of the functions $e_{i||j}(\varepsilon;\boldsymbol{v})$ in (72)–(73) we obtain the following equality

$$2\int_\omega \rho(\ddot{\bar{u}}_\alpha a^{\alpha\beta}\bar{v}_\beta + \ddot{\bar{u}}_3\bar{v}_3)\sqrt{a}dy + \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\bar{\boldsymbol{u}})\gamma_{\alpha\beta}(\bar{\boldsymbol{v}})\sqrt{a}dy$$
$$-4\int_\omega \frac{\alpha_T\mu(3\lambda+2\mu)}{\lambda+2\mu}\bar{\vartheta}a^{\alpha\beta}\gamma_{\alpha\beta}(\bar{\boldsymbol{v}})\sqrt{a}dy - \int_{\Gamma_C}\chi\bar{v}_3\sqrt{a}d\Gamma$$
$$= \int_\omega\left(\int_{-1}^1 f^i dx_3\right)\bar{v}_i\sqrt{a}dy + \int_{\Gamma_+} h^i\bar{v}_i\sqrt{a}d\Gamma,\ a.e.\ \text{in}\ (0,T), \tag{74}$$

where we used Step (*iii*) and (3) to find that there exists $\chi \in L^\infty(0,T;L^2(\Gamma_C))$ such that $p(-\dot{u}_3(\varepsilon)) \overset{*}{\rightharpoonup} \chi$. We also used (70) and, since $\boldsymbol{u}$, $\boldsymbol{v}$ and ϑ are all independent of x_3 (see Step (*ii*)), we identified them with their averages. Above, the components $a^{\alpha\beta\sigma\tau}$ are defined as in (54). Notice that we can formulate the equation above in terms of test functions $\boldsymbol{\eta} = (\eta_i) \in [H_0^1(\omega)]^3$. To do that we just have to take $\boldsymbol{v}$ independent of x_3 and consider $\boldsymbol{\eta}(\boldsymbol{y}) = \boldsymbol{v}(\boldsymbol{y},x_3)$ for all $(\boldsymbol{y},x_3) \in \Omega$. Further, since both sides of the equation are continuous linear forms with respect to $\bar{v}_3 = \eta_3 \in L^2(\omega)$, and given that $H_0^1(\omega)$ is dense in $L^2(\omega)$, we find that the problem can be formulated for test functions in $\boldsymbol{\eta} = (\eta_i) \in V_M(\omega)$, instead.

Similarly, we now consider $\varphi \in S(\Omega)$ independent of x_3 in (48) and pass to the limit when $\varepsilon \to 0$. Again, if we take into account the weak convergences in Step (i) and simplify by using the time derivative of (70) we find the equality

$$2\int_\omega\left(\beta + \frac{\alpha_T^2(3\lambda+2\mu)^2}{\lambda+2\mu}\right)\dot{\bar{\vartheta}}\varphi\sqrt{a}dy + 2\int_\omega k\partial_\alpha\bar{\vartheta}a^{\alpha\beta}\partial_\beta\varphi\sqrt{a}dy$$
$$+4\int_\omega \frac{\alpha_T\mu(3\lambda+2\mu)}{\lambda+2\mu}\varphi a^{\alpha\beta}\gamma_{\alpha\beta}(\dot{\bar{\boldsymbol{u}}})\sqrt{a}dy = \int_\omega Q\varphi\sqrt{a}dy \quad \forall\varphi\in H_0^1(\omega)\ a.e.\ \text{in}\ (0,T), \tag{75}$$

hence obtaining (53), with ζ identified with $\bar{\vartheta}$.

(vi) The weak convergences are, in fact, strong.

For this step we first consider a case without tractions, that is, we take $\boldsymbol{h} = 0$. Then we will show the changes to be made for the case with tractions. In both cases we are using a monotonicity argument. We define the quantity:

$$\Lambda(\varepsilon) := \int_\Omega \rho\big((\ddot{u}_\alpha(\varepsilon) - \ddot{u}_\alpha)g^{\alpha\beta}(\varepsilon)(\dot{u}_\beta(\varepsilon) - \dot{u}_\beta) + (\ddot{u}_3(\varepsilon) - \ddot{u}_3)(\dot{u}_3(\varepsilon) - \dot{u}_3)\big)\sqrt{g(\varepsilon)}\,dx$$

$$+\int_\Omega A^{ijkl}(\varepsilon)(e_{k||l}(\varepsilon)-e_{k||l})(\dot e_{i||j}(\varepsilon)-\dot e_{i||j})\sqrt{g(\varepsilon)}dx$$
$$-\int_{\Gamma_C}(p(-\dot u_3(\varepsilon))-p(-\dot u_3))(\dot u_3(\varepsilon)-\dot u_3)\sqrt{g(\varepsilon)}d\Gamma$$
$$+\int_\Omega \beta(\dot\vartheta(\varepsilon)-\dot\vartheta)(\vartheta(\varepsilon)-\vartheta)\sqrt{g(\varepsilon)}dx$$
$$+\int_\Omega k\{\partial_\alpha(\vartheta(\varepsilon)-\vartheta)g^{\alpha\beta}(\varepsilon)\partial_\beta(\vartheta(\varepsilon)-\vartheta)+\frac{1}{\varepsilon^2}(\partial_3(\vartheta(\varepsilon)-\vartheta))^2\}\sqrt{g(\varepsilon)}dx.$$

On one hand, we integrate with respect to the time variable in $[0,t]$ and take into account (66) and the initial conditions in Problem 6 to obtain

$$2\int_0^t\Lambda(\varepsilon)dr=\int_\Omega\rho\Big((\dot u_\alpha(\varepsilon)-\dot u_\alpha)g^{\alpha\beta}(\varepsilon)(\dot u_\beta(\varepsilon)-\dot u_\beta)+(\dot u_3(\varepsilon)-\dot u_3)^2\Big)\sqrt{g(\varepsilon)}\,dx$$
$$+\int_\Omega A^{ijkl}(\varepsilon)(e_{k||l}(\varepsilon)-e_{k||l})(e_{i||j}(\varepsilon)-e_{i||j})\sqrt{g(\varepsilon)}dx$$
$$+2\int_0^t\int_{\Gamma_C}(p(-\dot u_3(\varepsilon))-p(-\dot u_3))(-\dot u_3(\varepsilon)+\dot u_3)\sqrt{g(\varepsilon)}d\Gamma\,dr+\int_\Omega\beta(\vartheta(\varepsilon)-\vartheta)^2\sqrt{g(\varepsilon)}dx$$
$$+2\int_0^t\int_\Omega k\{\partial_\alpha(\vartheta(\varepsilon)-\vartheta)g^{\alpha\beta}(\varepsilon)\partial_\beta(\vartheta(\varepsilon)-\vartheta)+\frac{1}{\varepsilon^2}(\partial_3(\vartheta(\varepsilon)-\vartheta))^2\}\sqrt{g(\varepsilon)}dxdr, \tag{76}$$

and as consequence of the monotonicity of p, (40) and (44), we find

$$\int_0^t\Lambda(\varepsilon)ds\geq C\Bigg(|\dot{\boldsymbol u}(\varepsilon)-\dot{\boldsymbol u}|^2_{0,\Omega}+|e_{i||j}(\varepsilon)-e_{i||j}|^2_{0,\Omega}+|\vartheta(\varepsilon)-\vartheta|^2_{0,\Omega}$$
$$+\int_0^t|\partial_\alpha\vartheta(\varepsilon)-\partial_\alpha\vartheta|^2_{0,\Omega}ds+\frac{1}{\varepsilon^2}\int_0^t|\partial_3\vartheta(\varepsilon)-\partial_3\vartheta|^2_{0,\Omega}ds\Bigg). \tag{77}$$

On the other hand, from the expression of $\Lambda(\varepsilon)$ and making use of (47)–(48) for $\boldsymbol v=\dot{\boldsymbol u}(\varepsilon)$ and $\varphi=\vartheta(\varepsilon)$, we deduce that

$$\Lambda(\varepsilon)=\int_\Omega f^i\dot u_i(\varepsilon)\sqrt{g(\varepsilon)}dx-\frac{d}{dt}\int_\Omega A^{ijkl}(\varepsilon)e_{k||l}(\varepsilon)e_{i||j}\sqrt{g(\varepsilon)}dx+\int_\Omega A^{ijkl}(\varepsilon)e_{k||l}\dot e_{i||j}\sqrt{g(\varepsilon)}dx$$
$$-\frac{d}{dt}\int_\Omega\rho\dot u_\alpha(\varepsilon)g^{\alpha\beta}(\varepsilon)\dot u_\beta\sqrt{g(\varepsilon)}\,dx+\int_\Omega\rho\ddot u_\alpha g^{\alpha\beta}(\varepsilon)\dot u_\beta\sqrt{g(\varepsilon)}\,dx$$
$$-\frac{d}{dt}\int_\Omega\rho\dot u_3(\varepsilon)\dot u_3\sqrt{g(\varepsilon)}\,dx+\int_\Omega\rho\ddot u_3\dot u_3\sqrt{g(\varepsilon)}\,dx$$
$$+\int_{\Gamma_C}p(-\dot u_3)(\dot u_3(\varepsilon)-\dot u_3)\sqrt{g(\varepsilon)}d\Gamma+\int_{\Gamma_C}p(-\dot u_3(\varepsilon))\dot u_3\sqrt{g(\varepsilon)}d\Gamma$$
$$+\int_\Omega q\vartheta(\varepsilon)\sqrt{g(\varepsilon)}dx-\frac{d}{dt}\int_\Omega\beta\vartheta(\varepsilon)\vartheta\sqrt{g(\varepsilon)}dx+\int_\Omega\beta\dot\vartheta\vartheta\sqrt{g(\varepsilon)}dx$$
$$-\int_\Omega k\partial_\alpha\vartheta g^{\alpha\beta}(\varepsilon)\partial_\beta(\vartheta(\varepsilon)-\vartheta)\sqrt{g(\varepsilon)}dx-\int_\Omega k\partial_\alpha\vartheta(\varepsilon)g^{\alpha\beta}(\varepsilon)\partial_\beta\vartheta\sqrt{g(\varepsilon)}dx$$
$$-\frac{1}{\varepsilon^2}\int_\Omega k\partial_3\vartheta\partial_3(\vartheta(\varepsilon)-\vartheta)\sqrt{g(\varepsilon)}dx-\frac{1}{\varepsilon^2}\int_\Omega k\partial_3\vartheta(\varepsilon)\partial_3\vartheta\sqrt{g(\varepsilon)}dx. \tag{78}$$

Next, integrate with respect to the time variable in $[0,t]$, take into account the initial conditions given in Problem 6 and (66), use that $\partial_3\vartheta=0$, and let $\varepsilon\to 0$. Because of the weak convergences studied in steps (i), (iii) and (v), the asymptotic behaviour of the functions $A^{ijkl}(\varepsilon)$ and $g(\varepsilon)$ (see (38)–(39) and (44)) and by using the Lebesgue dominated convergence theorem, we find that

$$\lim_{\varepsilon\to0}\int_0^t\Lambda(\varepsilon)dr=\int_0^t\int_\Omega f^i\dot u_i\sqrt a dxdr-\int_0^t\int_\Omega\rho\ddot u_\alpha a^{\alpha\beta}\dot u_\beta\sqrt a dxdr-\int_0^t\int_\Omega\rho\ddot u_3\dot u_3\sqrt a dxdr$$
$$-\int_0^t\int_\Omega A^{ijkl}(0)e_{k||l}\dot e_{i||j}\sqrt a dxdr+\int_0^t\int_{\Gamma_C}\chi\dot u_3\sqrt a d\Gamma dr+\int_0^t\int_\Omega q\vartheta\sqrt a dxdr$$
$$-\int_0^t\int_\Omega\beta\dot\vartheta\vartheta\sqrt a dxdr-\int_0^t\int_\Omega k\partial_\alpha\vartheta a^{\alpha\beta}\partial_\beta\vartheta\sqrt a dxdr. \tag{79}$$

Moreover, by the expressions of $A^{ijkl}(0)$ (see (38)–(39)) and using (69) we have

$$\int_\Omega A^{ijkl}(0)e_{k||l}\dot e_{i||j}\sqrt a dx=\int_\Omega\Big(\lambda a^{\alpha\beta}a^{\sigma\tau}+\mu(a^{\alpha\sigma}a^{\beta\tau}+a^{\alpha\tau}a^{\beta\sigma})\Big)e_{\sigma||\tau}\dot e_{\alpha||\beta}\sqrt a dx$$

$$+\int_{\Omega}\lambda a^{\alpha\beta}e_{3||3}\dot{e}_{\alpha||\beta}\sqrt{a}dx+\int_{\Omega}\big(\lambda a^{\sigma\tau}e_{\sigma||\tau}+(\lambda+2\mu)e_{3||3}\big)\dot{e}_{3||3}\sqrt{a}dx.$$

Then, using (70), we find that (79) is actually null, since its expression above coincides with the result of adding (74) for $\bar{\boldsymbol{v}}=\dot{\boldsymbol{u}}$ to (75) for $\varphi=\vartheta$ (both integrated in $[0,t]$). Indeed,

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_0^t\Lambda(\varepsilon)dr=&\int_0^t\Big(\int_{\Omega}f^i\dot{u}_i\sqrt{a}dx-\int_{\Omega}\rho\ddot{u}_\alpha a^{\alpha\beta}\dot{u}_\beta\sqrt{a}dx-\int_{\Omega}\rho\ddot{u}_3\dot{u}_3\sqrt{a}dx\\&-\frac{1}{2}\int_{\Omega}a^{\alpha\beta\sigma\tau}e_{\sigma||\tau}\dot{e}_{\alpha||\beta}\sqrt{a}dx+\int_{\Gamma_C}\chi\dot{u}_3\sqrt{a}d\Gamma+\int_{\Omega}q\vartheta\sqrt{a}dx\\&-\int_{\Omega}\Big(\beta+\frac{\alpha_T^2(3\lambda+2\mu)^2}{\lambda+2\mu}\Big)\dot{\vartheta}\vartheta\sqrt{a}dx-\int_{\Omega}k\partial_\alpha\vartheta a^{\alpha\beta}\partial_\beta\vartheta\sqrt{a}dx\Big)dr=0.\end{aligned}\tag{80}$$

Now, for the case where tractions are not null, in (78) we have an additonal term

$$\int_{\Gamma_+}h^i\dot{u}_i(\varepsilon)\sqrt{g(\varepsilon)}d\Gamma.$$

When passing to the limit $\varepsilon\to 0$, the terms with $u_\alpha(\varepsilon)$ above converge by using compactness arguments, since $u_\alpha(\varepsilon)\in H^1(\Omega\times(0,T))$ and the trace into $L^2(\Gamma\times(0,T))$ is a compact operator (see [33, p. 416]). For the term with $\dot{u}_3(\varepsilon)$, we have to combine integration by parts and arguments similar to those in Step (iii) with Γ_C replaced by Γ_+ to find that $\dot{u}_3(\varepsilon)\overset{*}{\rightharpoonup}\dot{u}_3$ in $L^\infty(0,T;L^2(\Gamma_+))$. We refer the interested reader to [44].

The strong convergences $e_{i||j}(\varepsilon)\to e_{i||j}$ in $L^\infty(0,T;L^2(\Omega))$ also imply the strong convergences for $u_i(\varepsilon)$, by following arguments not depending on the particular set of equations, but on the same arguments of differential geometry and functional analysis used in [9, Theorem 4.4-1]. Therefore, we just omit them here and refer the interested reader to said reference.

It only remains to show that $\chi=p(-\dot{u}_3)$. To do that we can reason like in Step (x) in [41, Theorem 5.3].

Remark 6. Notice that unlike what happens in the references [33,36], cited several times in this work, we cannot use compactness arguments for the convergence of all the contact boundary terms, since in our functional framework (that of linearly elliptic membrane shells) we do not have enough regularity to conclude that $u_3(\varepsilon)\in H^1(\Omega\times(0,T))$. Indeed, we have found no uniform upper bounds for $\partial_\alpha u_3(\varepsilon)$. Furthermore, the trace defined in [47, Theorem 3] is not a compact operator.

We still have to provide convergence results in terms of de-scaled unknowns. To do that, we first formulate the limit problem in a de-scaled form. From the scalings in Section 3 we deduce the de-scalings $\xi_i^\varepsilon(\boldsymbol{y})=\xi_i(\boldsymbol{y})$ and $\zeta^\varepsilon(\boldsymbol{y})=\zeta(\boldsymbol{y})$ for all $\boldsymbol{y}\in\bar{\omega}$. Therefore, from Problem 7 we obtain

Problem 10. Find a pair $t\mapsto(\boldsymbol{\xi}^\varepsilon(\boldsymbol{y},t),\zeta^\varepsilon(\boldsymbol{y},t))$ of $[0,T]\to V_M(\omega)\times H_0^1(\omega)$ verifying

$$\begin{aligned}&2\varepsilon\int_{\omega}\rho(\ddot{\xi}_\alpha^\varepsilon a^{\alpha\beta}\eta_\beta+\ddot{\xi}_3^\varepsilon\eta_3)\sqrt{a}dy+\varepsilon\int_{\omega}a^{\alpha\beta\sigma\tau,\varepsilon}\gamma_{\sigma\tau}(\boldsymbol{\xi}^\varepsilon)\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy-\int_{\Gamma_C^\varepsilon}p^\varepsilon(-\dot{\xi}_3^\varepsilon)\eta_3^\varepsilon\sqrt{a}d\Gamma\\&\quad-4\varepsilon\int_{\omega}\frac{\alpha_T\mu(3\lambda+2\mu)}{\lambda+2\mu}\zeta^\varepsilon a^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy=\int_{\omega}F^{i,\varepsilon}\eta_i\sqrt{a}dy\ \forall\boldsymbol{\eta}=(\eta_i)\in V_M(\omega),\\&2\varepsilon\int_{\omega}\Big(\beta+\frac{\alpha_T^2(3\lambda+2\mu)^2}{\lambda+2\mu}\Big)\dot{\zeta}^\varepsilon\varphi\sqrt{a}dy+2\varepsilon\int_{\omega}k\partial_\alpha^\varepsilon\zeta^\varepsilon a^{\alpha\beta}\partial_\beta^\varepsilon\varphi\sqrt{a}dy\\&\quad+4\varepsilon\int_{\omega}\frac{\alpha_T\mu(3\lambda+2\mu)}{\lambda+2\mu}\varphi a^{\alpha\beta}\gamma_{\alpha\beta}(\dot{\boldsymbol{\xi}}^\varepsilon)\sqrt{a}dy=\int_{\omega}Q^\varepsilon\varphi\sqrt{a}dy\quad\forall\varphi\in H_0^1(\omega),\end{aligned}$$

with $\dot{\boldsymbol{\xi}}^\varepsilon(\cdot,0)=\boldsymbol{\xi}^\varepsilon(\cdot,0)=0$ and $\zeta^\varepsilon(\cdot,0)=0$,

where we have used $F^{i,\varepsilon}:=\int_{-\varepsilon}^{\varepsilon}f^{i,\varepsilon}dx_3^\varepsilon+h_+^{i,\varepsilon}$, with $h_+^{i,\varepsilon}(\cdot)=h^{i,\varepsilon}(\cdot,\varepsilon)$, and $Q^\varepsilon=\int_{-\varepsilon}^{\varepsilon}q^\varepsilon dx_3^\varepsilon$.

A brief inspection of the two-dimensional problem above shows that it keeps the structure of its three-dimensional counterpart, Problem 4, with some terms changing its relative weight due to their new coefficients. That is the case of the thermal dilatation term in the first equation and more noticeable, the thermal conductivity term in the second one. We found no record in the existing engineering literature about this contribution of Lamé's coefficients to these terms. Experimental results should provide some insight in this direction, but this exceeds the scope of this paper. Note also that the three-dimensional information is being taken into account in this two-dimensional problem. Indeed, the ε coefficient in most of the left-hand side terms accounts for half the thickness of the three-dimensional shells, while both right-hand sides include averages in the transversal direction. Moreover, the contact term is posed on a two-dimensional domain, but it is originated in the boundary of the three-dimensional setting (that is the reason why we keep the notation Γ_C^ε). Finally, note that these terms are of lower order (zero-th order) in terms of ε, thus showing that membranes are prone to deform under external forces and when entering in (fast) contact with a foundation.

Finally, regarding convergence, by combining the results in Theorem 3 and [9, Theorem 4.2-1] we can show that

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_\alpha^\varepsilon dx_3^\varepsilon \to \xi_\alpha \text{ in } H^1(\Omega), \quad \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_3^\varepsilon dx_3^\varepsilon \to \xi_3 \text{ in } L^2(\Omega),$$
$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \zeta^\varepsilon dx_3^\varepsilon \to \zeta \text{ in } L^2(\Omega) \; a.e. \text{ in } (0,T).$$

Further, following the same arguments as in [9, Theorem 4.6-1], we can show that the averages of the tangential and normal components of the three-dimensional displacement vector field converge, as well.

6. Conclusions and outlook

For the particular case of the so-called elliptic membrane shells, we have obtained and mathematically justified a two-dimensional limit model for a three-dimensional contact problem in thermoelasticity. The contact is modeled by using a normal damped response function. In the process we used the insight provided by the asymptotic expansion method and we have justified this approach by obtaining convergence theorems. Also, we have proved existence, uniqueness and regularity results for both three and two-dimensional problems by combining Faedo-Galerking techniques, monotonicity and compacity arguments.

There are many problems yet to be studied in this field. To begin with, the contact model considered in this paper is frictionless. But one can easily think of many real life applications where friction can not be neglected. Further, friction may be coupled with other tribological effects such as wear or adhesion. Besides, many geometries such as cylinders or cones are beyond the elliptic membrane framework. Thus, our future work will be devoted to the study of the asymptotic limits of alternative shell models, such as flexural shells or Koiter shells, when there is contact on a part of their boundary and where friction is taken into account.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

M.T. Cao-Rial: Conceptualization, Methodology, Writing – original draft, Investigation, Supervision, Writing – review & editing. **G. Castiñeira:** Conceptualization, Methodology, Writing – original draft, Investigation, Supervision, Writing – review & editing. **Á. Rodríguez-Arós:** Conceptualization, Methodology, Writing – original draft, Investigation, Supervision, Writing – review & editing. **S. Roscani:** Conceptualization, Methodology, Writing – original draft, Investigation, Supervision, Writing – review & editing.

Acknowledgement

This project has received funding from the European Union Horizon 2020 Research and Innovation Programme under the Marie Sklodowska-Curie Grant Agreement No 823731 CONMECH and grant MTM2016-78718-P by Ministerio de Economía Industria y Competitividad of Spain with the participation of FEDER.

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