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Research paper

PDE models for the pricing of a defaultable coupon-bearing bond under an extended JDCEV model

M. Carmen Calvo-Garrido [a], Sidi Diop [b], Andrea Pascucci [b], Carlos Vázquez [a,*]

[a] Department of Mathematics and CITIC, University of A Coruña., Campus Elviña s/n, 15071 A Coruña, Spain
[b] Department of Mathematics, University of Bologna, 40126 Bologna, Italy

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ABSTRACT

We consider a two-factor model for the pricing of a non callable defaultable bond which pays coupons at certain given dates. The model under consideration is the Jump to Default Constant Elasticity of Variance (JDCEV) model. The JDCEV model is an improvement of the reduced form approach, which unifies credit and equity models into a single framework allowing for stochastic and possible negative interest rates. From the mathematical point of view, the valuation involves two partial differential equation (PDE) problems for each coupon. First, we obtain the existence of solution for these PDE problems. In order to solve them, we propose appropriate numerical schemes based on a Crank-Nicolson semi-Lagrangian method for time discretization combined with quadratic Lagrange finite elements for space discretization. Once the numerical solutions of the PDEs are obtained, a post-processing procedure is carried out in order to achieve the value of the bond. This post-processing includes the computation of an integral term which is approximated by using the composite trapezoidal rule. Finally, we present some numerical results for real market bonds issued by different firms in order to illustrate the proper behaviour of the numerical schemes. Moreover, we obtain an agreement between the numerical results from the PDE approach and those ones obtained by applying a Monte Carlo technique and an asymptotic aproximation method.

1. Introduction

Default risk can be modelled by using two different approaches: the structural approach and the reduced-form approach. For more information about the history, advantages and drawbacks of each one of the two approaches we refer the readers to [1,8,9]. In the structural method default occurs when the firm asset value reaches some lower barrier, whereas in the reduced-form approach, the default event is assumed to be unpredictable and governed by a default intensity process that could be either deterministic or stochastic. Moreover, in the reduced-form approach the unexpected default event can occur without any correlation with the firm value. In this work we consider the reduced-form approach.

* Corresponding author.
E-mail addresses: mcalvog@udc.es (M. Carmen Calvo-Garrido), sidy.diop2@unibo.it (S. Diop), andrea.pascucci@unibo.it (A. Pascucci), carlosv@udc.es (C. Vázquez).

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In the literature there are several papers devoted to default risk modelling by using the two different approaches. On one hand, the first approach is taken into consideration in [1] where the author proves an exact formula for the valuation of defaultable coupon bonds by generalizing the one derived in [14]. On the other hand, both approaches are combined in [9] also to price defaultable bonds.

The main objective of this paper is to obtain the price of a defaultable coupon bond under the extended Jump to Default Constant Elasticity of Variance (JDCEV) model proposed in [7], also considering the possibility of incorporating negative interest rates which is introduced in [11]. The JDCEV model was first introduced in [7] as a hybrid credit and equity model, although only taking into account constant positive interest rates. In this work we assume that the interest rates dynamics is governed by a stochastic process which takes into account the possibility of negative interest rates as in [11]. The incorporation of negative interest rates results more realistic according to the current situation of real markets. As it was pointed out before, this model can be set in the framework of the reduced-form approaches. More precisely, in the present paper we define the instantaneous volatility of the stock price as a constant elasticity of variance (CEV) process and we assume that the default intensity is an affine function of the instantaneous variance of the underlying stock.

From the mathematical point of view, the valuation problem of a defaultable coupon bond can be posed in terms of a sequence of partial differential equation (PDE) problems, where the underlying stochastic factors are the interest rates and the stock price. Moreover, the stock price follows a diffusion process interrupted by a possible jump to zero (default), as it is indicated in [7]. In order to compute the value of the bond we need to solve two partial differential equation problems for each coupon, with maturities equal to those coupon payment dates. Concerning the numerical solution of these PDE problems, after a localization procedure to formulate the problems in a bounded domain and the study of the boundaries where boundary conditions are required following the ideas introduced in [18], we propose appropriate numerical schemes based on a Crank-Nicolson semi-Lagrangian method for time discretization combined with quadratic Lagrange finite elements for space discretization. The numerical analysis of this Lagrange-Galerkin method has been addressed in [3,4]. Once the numerical solution of the PDEs is obtained, a kind of post-processing is carried out in order to achieve the value of the bond. This post-processing includes the computation of an integral term which is approximated by using the composite trapezoidal rule.

This paper is organized as follows. In Section 2, we describe the two stochastic factors (i.e. the interest rate and the defaultable stock price) that are involved in the model, and we state the PDE problem that governs the valuation of non callable defaultable coupon bonds. In Section 3 we establish the existence of solution for the PDE problems. In Section 4, we formulate the pricing problem in a bounded domain after a localization procedure and we impose appropriate boundary conditions. Then, we introduce the discretization in time of the problem by using a combination of semi-Lagrangian and Crank-Nicolson methods, and we state the variational formulation of the problem in order to apply finite elements for the discretization in the asset and interest rate variables. In Section 5 we present some numerical results to illustrate the good performance of the models and numerical methods, also including a comparison with the results obtained with an alternative Monte Carlo technique and an asymptotic expansion method. Finally, we finish with some conclusions in Section 6.

2. Mathematical modelling

2.1. Stochastic underlying variables

A non callable defaultable coupon bond is a financial derivative product, the underlying variables of which are the interest rate and the defaultable stock price. The interest rate at time t, r_t, is assumed to be stochastic and its dynamics under a risk neutral probability measure is driven by the Vasicek model [22], in which the spot rate is governed by the Ornstein-Uhlenbeck process:

$$dr_t = \kappa(\theta - r_t)\,dt + \delta\,dW_t^1, \tag{1}$$

where $\kappa > 0$ is the speed of adjustment in the mean reverting process, $\theta > 0$ is the long-term mean of the short-term interest rate, $\delta > 0$ is the interest rate volatility and $W^1 = (W_t^1)_{t\geq 0}$ denotes a standard Wiener process for the interest rate. Negative interest rates are taken into consideration with this model contrary to other interest rate dynamics, such as CIR model [10] under Feller condition and initial positive rate. Note that the possibility of negative interest rates results more realistic according to the current situation of the markets.

In the JDCEV model we assume that the local volatility of the stock is given by

$$\sigma(t, S_t) = a(t)\,S_t^{\beta},$$

where S_t denotes the stock price at time t, $\beta < 0$ is the elasticity parameter and $a(t) > 0$ is the time dependent volatility scale function. Moreover, the default intensity can be written in terms of the stock volatility and the stock price in the following way

$$\lambda(t, S_t) = b(t) + c\,\sigma(t, S_t)^2 = b(t) + c\,a(t)^2\,S_t^{2\beta}, \tag{2}$$

where $b(t) \geq 0$ is a deterministic non-negative function of time and $c > 0$ governs the sensitivity of the default intensity with respect to the volatility, as it is indicated in [7]. The other source of uncertainty, the predefaultable stock price at time t, S_t, under a risk neutral probability measure satisfies the following stochastic differential equation:

$$dS_t \;=\; (r_t + \lambda(t, S_t))S_t\, dt \;+\; \sigma(t, S_t)\, S_t\, dW_t^2. \tag{3}$$

where $W^2 = (W_t^2)_{t\geq 0}$ is a standard Wiener process for the stock price. Both Wiener processes, W^1 and W^2 can be correlated according to $dW_t^1 dW_t^2 = \rho dt$, where ρ denotes the instantaneous correlation coefficient such that $|\rho| < 1$.

Moreover, as it is pointed in Carr-Linetsky [7], in this model the time of default ξ has two parts. On one hand, as we do not assume that σ and λ are bounded when $S \to 0$, then the process S_t may hit zero. As indicated in [7], this may happen when $c \in (0, 1/2)$ in expression (2), as it is chosen in the numerical examples. So, we assume that the process is killed at the first hitting time $\xi_0 = \inf\{t \geq 0 : S_t = 0\}$ and remains death forever. On the other hand, the process can jump to default (and remain there forever) at a certain random time $\tilde{\xi}$, which is given by the following expression:

$$\tilde{\xi} = \inf\left\{t \geq 0 : \int_0^t \lambda(u, S_u)du \geq e\right\}, \tag{4}$$

where e is the exponential random variable $e \sim Exp(1)$. Thus, we assume that default can happen either at time ξ_0 via diffusion to zero or at time $\tilde{\xi}$ via jump to default, whichever comes first. So, the time of default is defined as $\xi = \xi_0 \wedge \tilde{\xi}$.

Finally, the defautable stock price is given by $\bar{S}_t = S_t \mathbb{1}_{\{\xi > t\}}$.

2.2. PDE formulation

In this work we consider the valuation of a non callable defaultable bond with the possibility of paying a series of coupons at given dates t_m, for $m = 1, \ldots, M$, where M is the number of coupons and $T = t_M$ is the maturity of the bond. At maturity, the bond holder receives the face value (FV) plus the last coupon. Thus, let us denote by cp_m the coupon rate paid at coupon payment date t_m. Thus, the amount of the coupon payment is computed by multiplying the coupon rate, the time interval between coupon payments and the face value. Having this in view, the value of a non callable defaultable coupon bearing bond at time $t = t_0 = 0$ for the spot values S_0 and r_0, $V(0, S_0, r_0; T)$, which is understood as the discounted value of the future coupon payments and the face value of the bond, is given by

$$\begin{aligned} V(0, S_0, r_0; T) = FV\Bigg[&\sum_{m=1}^{M} cp_m\, \mathbb{E}\left[\exp\left(-\int_0^{t_m}(r_u + \lambda(u, S_u))du\right)\right] + \mathbb{E}\left[\exp\left(-\int_0^{T}(r_u + \lambda(u, S_u))du\right)\right] \\ &+ \eta\left(1 - \mathbb{E}\left[\exp\left(-\int_0^{T}(r_u + \lambda(u, S_u))du\right)\right] - \int_0^T \mathbb{E}\left[\exp\left(-\int_0^{\tau_1}(r_u + \lambda(u, S_u))du\right) r_{\tau_1}\right] d\tau_1\right)\Bigg], \end{aligned} \tag{5}$$

where η is the recovery rate in case of default. An analysis of the recovery rate by industrial sector and debt seniority is carried out in [2]. For the derivation of the expression of the recovery part of formula (5) and the forthcoming formula (6), i.e. the term in their respective second lines, we refer to Proposition 2.1 in [11].

Next, if we denote by

$$\begin{aligned} u_1(0, S_0, r_0; t_m) &= \mathbb{E}\left[\exp\left(-\int_0^{t_m}(r_u + \lambda(u, S_u))du\right)\right],\; m = 1, \ldots, M, \\ u_1(0, S_0, r_0; T) &= \mathbb{E}\left[\exp\left(-\int_0^{T}(r_u + \lambda(u, S_u))du\right)\right], \\ u_2(0, S_0, r_0; \tau_1) &= \mathbb{E}\left[\exp\left(-\int_0^{\tau_1}(r_u + \lambda(u, S_u))du\right) r_{\tau_1}\right],\; \tau_1 \in [0, T], \end{aligned}$$

then the expression of the bond value (5) can be written equivalently as

$$V(0, S_0, r_0; T) = FV\left[\sum_{m=1}^{M} cp_m\, u_1(0, S_0, r_0; t_m) + u_1(0, S_0, r_0; T) + \eta\left(1 - u_1(0, S_0, r_0; T) - \int_0^T u_2(0, S_0, r_0; \tau_1)\, d\tau_1\right)\right]. \tag{6}$$

Moreover, by applying the Feynman-Kac formula (see [21], for example) and using the change of variable $y_t = r_t \exp(\kappa t)$, we consider the function u_{km} as a solution of the Cauchy problem

$$\begin{cases} \bar{\mathcal{L}}[u_{km}(t, S, y)] = 0, & t < t_m,\; (S, y) \in (0, \infty) \times (-\infty, \infty), \\ u_{km}(t_m, S, y) = h_{km}(S, y), & (S, y) \in (0, \infty) \times (-\infty, \infty), \end{cases} \tag{7}$$

for $k = 1, 2$ and $m = 1, \dots, M$, with $h_{km}(S, y) = (y\exp(-\kappa t_m))^{k-1}$. Moreover, the operator $\overline{\mathcal{L}}$ is defined as follows for a function u:

$$\overline{\mathcal{L}}[u] = \partial_t u + \frac{1}{2}\sigma^2(t,S)S^2\,\partial_{SS}u + \rho\delta\sigma(t,S)\exp(\kappa t)S\,\partial_{Sy}u + \frac{1}{2}\delta^2\exp(2\kappa t)\,\partial_{yy}u + (\exp(-\kappa t)y + \lambda(t,S))S\,\partial_S u$$
$$+\kappa\theta\exp(\kappa t)\,\partial_y u - (\exp(-\kappa t)y + \lambda(t,S))u. \tag{8}$$

From the solution u_{km} of the Cauchy problem for the different final conditions at different final times, we can recover the terms involved in expression (6). Thus, we have:

$$\begin{aligned} u_1(0,S_0,r_0;t_m) &= u_{1m}(0,S_0,r_0),\ m = 1,\dots,M;\\ u_1(0,S_0,r_0;T) &= u_{1M}(0,S_0,r_0);\\ u_2(0,S_0,r_0;t_m) &= u_{2m}(0,S_0,r_0),\ m = 1,\dots,M. \end{aligned} \tag{9}$$

Although in last term of (6), the value $u_2(0, S_0, r_0; \tau_1)$ for $\tau_1 \in [0, T]$ appears as the integrand, we will discretize the integral with a composed trapezoidal formula using the coupon payment dates as quadrature nodes in the approximation. Therefore, in practice we only use the values $u_2(0, S_0, r_0; t_m)$.

3. Existence of solution

In this section, we aim to prove the existence of a "local" solution to the Cauchy problem (7). For this purpose, let X be the logarithm of the pre-defaultable stock price, i.e. $X_t = \log S_t$, $t \in [0, T]$. Our model becomes

$$\begin{cases} \overline{S}_t = S_0\exp(X_t)\mathbb{1}_{\{\xi>t\}}, & S_0 > 0,\\ dr_t = \kappa(\theta - r_t)dt + \delta dW_t^1,\\ dX_t = (r_t - \frac{1}{2}\sigma(t,X_t)^2 + \lambda(t,X_t))dt + \sigma(t,X_t)dW_t^2,\\ dW_t^1 dW_t^2 = \rho dt, \end{cases} \tag{10}$$

with $\sigma(t,x) = a(t)e^{\beta x}$ and $\lambda(t,x) = b(t) + c\sigma(t,x)^2$. The corresponding infinitesimal generator is the operator $\mathcal{L}$ defined as

$$\begin{aligned}\mathcal{L} &= \partial_t + \frac{1}{2}\sigma^2(t,x)\,\partial_{xx} + \rho\delta\sigma(t,x)\,\partial_{xr} + \frac{1}{2}\delta^2\,\partial_{rr} + \left(r - \frac{1}{2}\sigma^2(t,x) + \lambda(t,x)\right)\partial_x + \kappa(\theta - r)\,\partial_r - (r + \lambda(t,x))\\ &= \partial_t + \frac{1}{2}\langle\Sigma\nabla,\nabla\rangle + \langle\mu,\nabla\rangle + \gamma, \end{aligned} \tag{11}$$

where

$$\begin{aligned}\Sigma(t,x,r) &= \begin{pmatrix} \sigma^2(t,x) & \rho\delta\sigma(t,x)\\ \rho\delta\sigma(t,x) & \delta^2 \end{pmatrix}, \quad \mu(t,x,r) = \begin{pmatrix} r - \frac{1}{2}\sigma^2(t,x) + \lambda(t,x)\\ \kappa(\theta - r) \end{pmatrix},\\ \gamma(t,x,r) &= -(r + \lambda(t,x)). \end{aligned} \tag{12}$$

Operator $\mathcal{L}$ is *only locally* uniformly parabolic in the sense that, for any ball with radius $R > 0$,

$$\mathcal{O}_R := \left\{(x,r) \in \mathbb{R}^2 \mid |(x,r)| < R\right\},$$

the coefficients of $\mathcal{L}$ satisfy the following conditions:

(H1) The matrix $\Sigma(t,x,r)$ is positive definite, uniformly with respect to $(t,x,r) \in (0,T] \times \mathcal{O}_R$.

(H2) The coefficients Σ, μ, γ are bounded and α-Hölder-continuous on $(0,T] \times \mathcal{O}_R$ for some $\alpha \in (0,1)$.

Under these conditions we can resort on the recent results in [20], Theor. 2.6, or [15], Theor.1.5, about the existence of a *local density* for the process (X, r).

Theorem 3.1. *For any $R > 0$, the process (X, r) has a local transition density on $\mathcal{O}_R$, that is a non-negative measurable function $\Gamma = \Gamma(t,x,r;T,z,s)$ defined for any $0 < t < T$, $(x,r) \in \mathbb{R}^2$ and $(z,s) \in \mathcal{O}_R$ such that, for any continuous function $h = h(x,r)$ with compact support in $\mathcal{O}_R$, we have*

$$u(t,x,r) := \mathbb{E}[h(X_T,r_T) \mid X_t = x, r_t = r] = \int_{\mathcal{O}_R}\Gamma(t,x,r;T,z,s)h(z,s)dzds$$

and u satisfies

$$\begin{cases} \mathcal{L}u(t,x,r) = 0, & (t,x,r) \in [0,T) \times \mathcal{O}_R,\\ u(T,x,r) = h(x,r) & (x,r) \in \mathcal{O}_R. \end{cases} \tag{13}$$

Problem (13) can be used for numerical approximation purposes. However, notice that (13) does not have a unique solution due to the lack of lateral boundary conditions. Nevertheless, numerical schemes can be implemented imposing artificial boundary conditions and the result which guarantees the validity of such approximations is the so-called *principle of not feeling the boundary*. A rigorous statement of this result can be found in [13], Appendix A, or in [20], Lemma 4.11.

4. Numerical methods

In order to compute the price of a non callable defaultable coupon bond by using the previously proposed PDEs models, we must address the numerical solution of the Cauchy problem (7) for $k = 1$ and 2, with maturities $T_1 = t_m$ for $m = 1, \dots, M$. Thus, problem (7) has to be solved at each coupon payment date for both $k = 1$ and 2. Once the corresponding solutions have been approximated by the numerical method, the value of the bond is given by expression (6), where an integral term appears. That integral term will be approximated by means of the classical composite trapezoidal rule. In order to solve numerically the Cauchy problem (7), we first pose its formulation in a bounded domain and next we consider a full discretization with a Lagrange-Galerkin method. More precisely, we use a second order Crank-Nicolson characteristics scheme for the time discretization and a piecewise quadratic Lagrange finite element method for the spatial approximation. The convergence properties of this Lagrange-Galerkin method have been mathematically analyzed in [3,4] for time and space discretization. More recently, this combination of numerical techniques has been applied to the valuation of pension plans without and with early retirement in [6] and [5], respectively.

4.1. Localization procedure and formulation in a bounded domain

As the Cauchy problem (7) is posed in an unbounded domain in the plane (S, y) and the Lagrange-Galerkin method requires the use of a bounded domain in spatial variables, we truncate the domain to a bounded one. For this purpose, we also introduce the notation

$$x_0 = t, \quad \tilde{x}_1 = S \quad \text{and} \quad \tilde{x}_2 = y. \tag{14}$$

and consider large enough values of $\tilde{x}_1^\infty$ and $\tilde{x}_2^\infty$ as the upper limits of variables $\tilde{x}_1$ and $\tilde{x}_2$, respectively, so that the region of financial interest is not affected by the truncation of the domain. Moreover, we consider $x_0^\infty = T_1$ and the bounded domain

$$\Omega_b^* = (0, x_0^\infty) \times \left(\frac{1}{\tilde{x}_1^\infty}, \tilde{x}_1^\infty\right) \times (-\tilde{x}_2^\infty, \tilde{x}_2^\infty)$$

Additionally, we make the changes of variables

$$x_1 = \tilde{x}_1 - \frac{1}{\tilde{x}_1^\infty}, \quad x_2 = \tilde{x}_2 + \tilde{x}_2^\infty, \tag{15}$$

so that the bounded domain Ω_b^* is transformed into

$$\tilde{\Omega} = (0, x_0^\infty) \times (0, x_1^\infty) \times (0, x_2^\infty)$$

with $x_1^\infty = \tilde{x}_1^\infty - \frac{1}{\tilde{x}_1^\infty}$ and $x_2^\infty = 2\tilde{x}_2^\infty$.

Next, we can decompose the boundary of $\tilde{\Gamma} = \partial\tilde{\Omega}$ as the union of the six flat boundaries, i.e. $\tilde{\Gamma} = \bigcup_{i=0}^{2}(\tilde{\Gamma}_i^- \cup \tilde{\Gamma}_i^+)$, with

$$\tilde{\Gamma}_i^- = \{(x_0, x_1, x_2) \in \tilde{\Gamma} \mid x_i = 0\}, \quad \tilde{\Gamma}_i^+ = \{(x_0, x_1, x_2) \in \tilde{\Gamma} \mid x_i = x_i^\infty\}, \quad i = 0, 1, 2.$$

Moreover, we can write the differential operator (8) in the new variables as

$$\overline{\mathcal{L}}[u] = \sum_{i,j=0}^{2} a_{ij} \frac{\partial^2 u}{\partial x_i x_j} + \sum_{j=0}^{2} a_j \frac{\partial u}{\partial x_j} + a_0 u, \tag{16}$$

where the functional expression of the coefficients is defined by

$$\mathbf{A} = (a_{ij}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} a^2(t) \left(x_1 + \frac{1}{\tilde{x}_1^\infty}\right)^{2\beta+2} & \frac{1}{2}\rho\delta a(t)\left(x_1 + \frac{1}{\tilde{x}_1^\infty}\right)^{\beta+1} \exp(\kappa t) \\ 0 & \frac{1}{2}\rho\delta a(t)\left(x_1 + \frac{1}{\tilde{x}_1^\infty}\right)^{\beta+1} \exp(\kappa t) & \frac{1}{2}\delta^2 \exp(2\kappa t) \end{pmatrix} \tag{17}$$

$$\mathbf{a} = (a_j) = \begin{pmatrix} 1 \\ \left(\exp(-\kappa t)(x_2 - \tilde{x}_2^\infty) + \lambda\left(t, x_1 + \frac{1}{\tilde{x}_1^\infty}\right)\right)\left(x_1 + \frac{1}{\tilde{x}_1^\infty}\right) \\ \kappa\theta \exp(\kappa t) \end{pmatrix} \tag{18}$$

$$a_0 = -\left(\exp(-\kappa t)(x_2 - \tilde{x}_2^\infty) + \lambda\left(t, x_1 + \frac{1}{\tilde{x}_1^\infty}\right)\right). \tag{19}$$

As we aim to formulate a well posed Cauchy-boundary value problem in the bounded domain $\tilde{\Omega}$, we need to determine the boundaries that require imposed boundary conditions for PDE problem. For this purpose we follow [18] (which is based on Fichera theory [12]). So, we introduce the normal vector to the boundary pointing inward $\tilde{\Omega}$, $\mathbf{n} = (n_0, n_1, n_2)$, and the sets at the boundary:

$$\Sigma^0 = \left\{ x \in \tilde{\Gamma} / \sum_{i,j=0}^{2} a_{ij} n_i n_j = 0 \right\}, \quad \Sigma^1 = \tilde{\Gamma} - \Sigma^0,$$

$$\Sigma^2 = \left\{ x \in \Sigma^0 / \sum_{i=0}^{2} \left(a_i - \sum_{j=0}^{2} \frac{\partial a_{ij}}{\partial x_j} \right) n_i < 0 \right\}.$$

As stated in [18], for the PDE problem associated with (16), only boundary conditions on $\Sigma^1 \bigcup \Sigma^2$ are required. As $\Sigma^1 = \tilde{\Gamma}_1^- \bigcup \tilde{\Gamma}_1^+ \bigcup \tilde{\Gamma}_2^- \bigcup \tilde{\Gamma}_2^+$ and $\Sigma^2 = \tilde{\Gamma}_0^+$, then we must define boundary conditions on $\tilde{\Gamma}_1^-$, $\tilde{\Gamma}_1^+$, $\tilde{\Gamma}_2^-$ and $\tilde{\Gamma}_2^+$, in addition to the final condition at time $t = T_1$. Therefore, we will consider Dirichlet boundary conditions on $\tilde{\Gamma}_1^-$, $\tilde{\Gamma}_2^-$ and $\tilde{\Gamma}_2^+$, while a homogeneous Neumann condition is imposed on $\tilde{\Gamma}_1^+$.

Next, we introduce the new time variable $\tau = T_1 - t$ and consider the spatial variables defined in (15). For notational purposes, we consider the bounded spatial domain $\Omega = (0, x_1^\infty) \times (0, x_2^\infty)$ and decompose its boundary $\partial\Omega = \bigcup_{i=1}^{2} (\Gamma_i^- \cup \Gamma_i^+)$ as follows:

$$\Gamma_i^- = \{(x_1, x_2) \in \Gamma \mid x_i = 0\}, \quad \Gamma_i^+ = \{(x_1, x_2) \in \Gamma \mid x_i = x_i^\infty\}, \quad i = 1, 2.$$

Thus, the initial boundary value problem (IBVP) (7) can be written in the equivalent divergence follows:

Find the function $u_{km} : [0, T_1] \times \Omega \to \mathbb{R}$ for $k = 1$ and 2, such that

$$\partial_\tau u_{km} - Div(\mathbf{B}\nabla u_{km}) + \mathbf{v} \cdot \nabla u_{km} + l u_{km} = 0 \ \text{ in } \ (0, T_1) \times \Omega\,, \tag{20}$$

$$u_{km}(0, .) = g_{km} \ \text{ in } \Omega, \tag{21}$$

$$u_{km} = f \ \text{ on } \ (0, T_1) \times \left(\Gamma_1^- \cup \Gamma_2^- \cup \Gamma_2^+\right), \tag{22}$$

$$\frac{\partial u_{km}}{\partial x_1} = 0 \ \text{ on } \ (0, T_1) \times \Gamma_1^+. \tag{23}$$

For problem (20)-(23), the coefficients of the diffusion matrix $\mathbf{B} = (b_{ij})$ are given by

$$\begin{aligned}
b_{11} &= \frac{1}{2} a^2 (T_1 - \tau) \left(x_1 + \frac{1}{\tilde{x}_1^\infty} \right)^{2\beta+2}, \\
b_{12} &= \frac{1}{2} \rho\delta a (T_1 - \tau) \left(x_1 + \frac{1}{\tilde{x}_1^\infty} \right)^{\beta+1} \exp(\kappa (T_1 - \tau))\,, \\
b_{21} &= \frac{1}{2} \rho\delta a (T_1 - \tau) \left(x_1 + \frac{1}{\tilde{x}_1^\infty} \right)^{\beta+1} \exp(\kappa (T_1 - \tau))\,, \\
b_{22} &= \frac{1}{2} \delta^2 \exp(2\kappa (T_1 - \tau))\,,
\end{aligned}$$

and the components of velocity field $\mathbf{v} = (v_i)$ are given by

$$\begin{aligned}
v_1 &= \frac{1}{2} a^2 (T_1 - \tau)(2\beta + 2) \left(x_1 + \frac{1}{\tilde{x}_1^\infty} \right)^{2\beta+1} - \left(\exp(-\kappa (T_1 - \tau))(x_2 - \tilde{x}_2^\infty) + \lambda \left(T_1 - \tau, x_1 + \frac{1}{\tilde{x}_1^\infty} \right) \right) \left(x_1 + \frac{1}{\tilde{x}_1^\infty} \right) \\
v_2 &= \frac{1}{2} \rho\delta a (T_1 - \tau)(\beta + 1) \left(x_1 + \frac{1}{\tilde{x}_1^\infty} \right)^{\beta} \exp(\kappa (T_1 - \tau)) - \kappa\theta \exp(\kappa (T_1 - \tau)),
\end{aligned}$$

while the reaction function l, the initial conditions g_k for $k = 1, 2$ and the function f for the Dirichlet boundary conditions are defined as follows:

$$l(x_1, x_2) = \exp(-\kappa (T_1 - \tau))(x_2 - \tilde{x}_2^\infty) + \lambda \left(T_1 - \tau, x_1 + \frac{1}{\tilde{x}_1^\infty} \right),$$

$$g_{km}(x_1,x_2)=h_{km}\left(x_1+\frac{1}{\tilde{x}_1^{\infty}},x_2-\tilde{x}_2^{\infty}\right),$$

$$f(\tau,x_1,x_2)=\exp\left(-\int_{T_1-\tau}^{T_1}\left(\exp(-\kappa\tilde{u})(x_2-\tilde{x}_2^{\infty})+\lambda\left(\tilde{u},x_1+\frac{1}{\tilde{x}_1^{\infty}}\right)\right)d\tilde{u}\right)g(x_1,x_2).$$

4.2. Method of characteristics for the semidiscretization in time

The method of characteristics is based on a finite differences scheme for the discretization of the material derivative, i.e., the time derivative along the characteristic lines of the convective part of the Eq. (20). The material derivative operator associated to PDE (20) is given by

$$\frac{D}{D\tau}=\partial_\tau+\mathbf{v}\cdot\nabla.$$

For a brief description of the method, we first define the characteristics curve through $\mathbf{x}=(x_1,x_2)$ at time $\bar{\tau}$, $X(\mathbf{x},\bar{\tau};s)$, which satisfies the first order ordinary differential equation problem:

$$\frac{\partial}{\partial s}X(\mathbf{x},\bar{\tau};s)=\mathbf{v}(X(\mathbf{x},\bar{\tau};s)),\quad X(\mathbf{x},\bar{\tau};\bar{\tau})=\mathbf{x}. \tag{24}$$

Note that $X(\mathbf{x},\bar{\tau};s)$ represents the position at time s of a point that is located at position $\mathbf{x}$ at time $\bar{\tau}$ in the trajectory associated to the velocity field $\mathbf{v}$. Moreover, $X(\mathbf{x},\bar{\tau};s)$ is defined for $s\in(\bar{\tau}-\delta,\bar{\tau}+\delta)$, i.e., in a large enough neighborhood of $\bar{\tau}$.

For the time discretization of Eq. (20), we choose a positive integer number $N>1$ to define the constant time step $\Delta\tau=T_1/N$ and the time discretization points $\tau^n=n\Delta\tau$, for $n=0,\frac{1}{2},1,\frac{3}{2},\ldots,N$.

At each time mesh point we approximate the material derivative with the method of characteristics by using the following upwinded finite differences scheme:

$$\frac{Du_{km}}{D\tau}\approx\frac{u_{km}^{n+1}-u_{km}^{n}\circ X^n}{\Delta\tau},$$

where "$\circ$" denotes the composition operation, $u_{km}^n=u_{km}(\tau^n,\cdot)$ and $X^n(\mathbf{x})=X(\mathbf{x},\tau^{n+1};\tau^n)$ is the position at time τ^n of a point placed at time τ^{n+1} in $\mathbf{x}$, so that $X^n(\mathbf{x})$ is computed by solving (24). In this case, the solution of (24) is not computed analytically. Instead, we consider numerical ODE solvers to approximate the characteristics curves (see [3], for example). More precisely, in this work we employ the explicit second order Runge-Kutta method to approximate the values of $X^n(\mathbf{x})$.

Moreover, in order to approximate the other terms in (20), we choose a second order Crank-Nicolson scheme around the point $\left(X(\mathbf{x},\tau^{n+1};\tau),\tau\right)$ for $\tau=\tau^{n+\frac{1}{2}}$. After that, we obtain the following semidiscretized problem at time τ^{n+1}:

Find u_{km}^{n+1} such that:

$$\frac{u_{km}^{n+1}(\mathbf{x})-u_{km}^{n}(X^n(\mathbf{x}))}{\Delta\tau}-\frac{1}{2}Div(\mathbf{B}\nabla u_{km}^{n+1})(\mathbf{x})-\frac{1}{2}Div(\mathbf{B}\nabla u_{km}^{n})(X^n(\mathbf{x}))+\frac{1}{2}(l\,u_{km}^{n+1})(\mathbf{x})+\frac{1}{2}(l\,u_{km}^{n})(X^n(\mathbf{x}))=0. \tag{25}$$

As we aim to discretize in space Eq. (25) with a finite element method, we previously state the variational formulation of this equation. For this purpose, we multiply (25) by a function belonging to the functional space of test functions:

$$V=\{\varphi\in H^1(\Omega)/\varphi=0\text{ on }\Gamma_1^-\cup\Gamma_2^-\cup\Gamma_2^+\},$$

and integrate the resulting product in the domain Ω. Next, we apply the classical Green integration formula and the following one obtained in [17]:

$$\begin{aligned}\int_\Omega Div(\mathbf{B}\nabla u_{km}^n)(X^n(\mathbf{x}))\varphi(\mathbf{x})d\mathbf{x}=&\int_\Gamma(\nabla X^n)^{-T}(\mathbf{x})\mathbf{n}(x)\cdot(\mathbf{B}\nabla u_{km}^n)(X^n(\mathbf{x}))\varphi(\mathbf{x})dA_{\mathbf{x}}\\&-\int_\Omega(\nabla X^n)^{-1}(\mathbf{x})(\mathbf{B}\nabla u_{km}^n)(X^n(\mathbf{x}))\cdot\nabla\varphi(\mathbf{x})d\mathbf{x}\\&-\int_\Omega Div((\nabla X^n)^{-T}(\mathbf{x}))\cdot(\mathbf{B}\nabla u_{km}^n)(X^n(\mathbf{x}))\varphi(\mathbf{x})d\mathbf{x}.\end{aligned} \tag{26}$$

Note that, as the characteristics curves cannot be obtained analytically, the terms $(\nabla X^n)^{-1}(\mathbf{x})$ and $Div((\nabla X^n)^{-T}(\mathbf{x}))$ in (26) are replaced by the following approximations (see [3] for more details):

$$(\nabla X^n)^{-1}(\mathbf{x})=\mathbf{I}(\mathbf{x})+\Delta\tau\nabla\mathbf{v}^n(X^n(\mathbf{x}))+O(\Delta\tau^2),$$

$$Div((\nabla X^n)^{-T}(\mathbf{x}))=\Delta\tau\,\nabla\,Div\,(\mathbf{v}^n(X^n(\mathbf{x})))+O(\Delta\tau^2).$$

From the previous arguments and computations, a variational formulation for the time discretized problem is given by:

Find $u_{km}^{n+1} \in H^1(\Omega)$ satisfying the Dirichlet boundary condition (21), such that:

$$\begin{aligned}
\frac{1}{\Delta\tau}\int_\Omega u_{km}^{n+1}(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x} + \frac{1}{2}\int_\Omega(\mathbf{B}\nabla u_{km}^{n+1})(\mathbf{x})\cdot\nabla\varphi(\mathbf{x})d\mathbf{x} + \frac{1}{2}\int_\Omega(lu_{km}^{n+1})(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x} \\
= \frac{1}{\Delta\tau}\int_\Omega u_{km}^{n}(X^n(\mathbf{x}))\varphi(\mathbf{x})d\mathbf{x} - \frac{1}{2}\int_\Omega(\mathbf{B}\nabla u_{km}^{n})(X^n(\mathbf{x}))\cdot\nabla\varphi(\mathbf{x})d\mathbf{x} \\
-\frac{\Delta\tau}{2}\int_\Omega \nabla\mathbf{v}^n(X^n(\mathbf{x}))(\mathbf{B}\nabla u_{km}^{n})(X^n(\mathbf{x}))\cdot\nabla\varphi(\mathbf{x})d\mathbf{x} - \frac{1}{2}\int_\Omega(lu_{km}^{n})(X^n(\mathbf{x}))\varphi(\mathbf{x})d\mathbf{x} \\
-\frac{\Delta\tau}{2}\int_\Omega \nabla\, Div\,(\mathbf{v}^n(X^n(\mathbf{x})))\cdot(\mathbf{B}\nabla u_{km}^{n})(X^n(\mathbf{x}))\varphi(\mathbf{x})d\mathbf{x} \\
+\frac{1}{2}\int_\Gamma(\mathbf{I}(\mathbf{x}) + \Delta\tau\nabla\mathbf{v}^n(X^n(\mathbf{x})))^T\mathbf{n}(x)\cdot(\mathbf{B}\nabla u_{km}^{n})(X^n(\mathbf{x}))\varphi(\mathbf{x})dA_{\mathbf{x}} \\
+\frac{1}{2}\int_{\Gamma_1^+} b_{12}(\mathbf{x})\frac{\partial u_{km}^n}{\partial x_2}(\mathbf{x})\varphi(\mathbf{x})dA_{\mathbf{x}},
\end{aligned} \tag{27}$$

for all $\varphi \in V$. In the last term, b_{12} is the corresponding coefficient of the matrix $\mathbf{B}$. Note that dA_x denotes the integration measure on the boundary Γ.

4.3. Finite elements for discretization in space

In order to complete the full discretization of the PDE problem, we consider a finite elements method for the discretization in the spatial variables. For this purpose, we use a family of quadrangular meshes of the domain Ω depending on a parameter h, each one of them denoted by $\{\tau_h\}$. Associated to each mesh we consider the globally continuous and piecewise quadratic Lagrangian finite elements $(K, \mathcal{Q}^2, \Sigma_K)$, with $\mathcal{Q}^2$ denoting the space of polynomials that are quadratic in each spatial variable defined in the rectangle $K \in \tau_h$ and Σ_K being the set of nine nodes of the rectangle K. We choose quadratic finite elements so that the approximation error is of order two in the spatial discretization, which is in agreement with the second order approximation of the characteristics Crank-Nicolson scheme for the time discretization.

4.4. Composite trapezoidal rule

In order to obtain the value of the bond at the origin, i.e. $V(0, S_0, r_0; T)$, by means of expression (6) the computation of an integral term is required. The approximation of this integral is carried out by using a suitable numerical integration procedure. More precisely, we employ the classical composite trapezoidal rule with $M+1$ points corresponding to the coupon payment dates plus the initial date, in the following way:

$$\int_0^T u_2(0, S_0, r_0; \tau_1)d\tau_1 \approx \frac{h}{2}\left[u_2(0, S_0, r_0; 0) + 2\sum_{m=1}^{M-1} u_2(0, S_0, r_0; t_m) + u_2(0, S_0, r_0; T)\right] \tag{28}$$

where $h = \frac{T}{M}$ and $u_2(0, S_0, r_0; 0) = r_0$. Note that the terms in (28) can be obtained from the previously computed numerical solutions u_{2m} as indicated in (9). The methodology can be easily adapted to alternative choices of the integration nodes.

5. Numerical results

In order to show the good performance of the model and numerical methods explained in Section 4, we present some numerical results. In the following examples, the value of some of the parameters involved in the underlying factors are taken from [11], where the authors calibrate the model parameters to market data, specifically for UBS and BNP Paribas CDS spreads. The calibration in [11] is based on a two-step procedure. Firstly, the interest rate model is calibrated separately to daily yields curves for zero-coupon bonds (ZCB), generated using Libor swap curve. Subsequently, CDS contracts with different maturity dates are considered: the calibration is based on approximation formulas for the CDS spreads and survival probabilities derived using the recent methodology introduced in [16,19], which consists in an asymptotic expansion of the solution to the pricing partial differential equation.

In both examples, the number of elements and nodes of the finite element meshes employed in the numerical solution of the problems are shown in Table 2.

5.1. Example 1

First, we consider the simple case of the valuation of default-free zero-coupon bonds with different maturities. In this setting, the valuation problem is reduced to a one-factor model. The purpose of this example is to compare the value of the bonds we obtain with the market zero-coupon curve. In order to obtain the value of the bonds, we solve the IBVP (20)-(22) with initial condition $g_1(x_1, x_2) = 1$ (corresponding to $h_1(S, r) = 1$) and only taking into account as underlying factor

Table 1
Parameters of the model for the UBS bond.

Parameters of the defaultable stock price model
$a_1 = 0.0337851$, $a_2 = 0.0523625$ $b_1 = 0.0026639$, $b_2 = 0.0027968$ $c = 0.0435673$, $\beta = -0.268496$
Parameters of the interest rate model
$\kappa = 0.04520533766268042$ $\delta = 0.02146900332086033$ $\theta = 0.10334921942765922$
Correlation coefficient
$\rho = 0.0$
Initial conditions
$S_0 = 1.0$, $r_0 = -0.009159871729892612$

Table 2
Different finite element meshes with their respective number of elements (NEL) and nodes (NNOD).

	Mesh1	Mesh2	Mesh3	Mesh4
NEL	16	64	256	1024
NNOD	81	289	1089	4225

Table 3
Market and model values of the ZCB.

Maturity (years)	Market ZCB	Model ZCB
1	1.00229	1.006751
2	1.00372	1.009062
3	1.00333	1.007495
4	1.00099	1.002601
5	0.995825	0.994902
6	0.987805	0.984889
7	0.976833	0.973024
8	0.963223	0.959738
9	0.947687	0.945429
10	0.932845	0.930463

Table 4
Value of the UBS bond for different meshes and time steps.

Time steps per year	Mesh1	Mesh2	Mesh3	Mesh4
90	102.499496	102.603837	102.616859	102.619028
180	102.499599	102.605953	102.617976	102.619681
360	102.500454	102.606351	102.618421	102.620069

the interest rate. The parameters values involved in the interest rate model are collected in Table 1. In Table 3 we present the market prices of the zero-coupon curve and the ones obtained by solving the here proposed model. For computing the numerical solution of Table 3 we consider Mesh4 from Table 2 (the finest one) and the time step $\Delta\tau = \frac{1}{360}$ (one day).

5.2. Example 2

Next, we consider the valuation of two real defaultable bonds traded in the market and issued by different firms. For this purpose, as in [11], we assume that the coefficients $a(t)$ and $b(t)$ in (2) are linearly dependent on time and their expressions are given by $a(t) = a_1 t + a_2$, $b(t) = b_1 t + b_2$, where a_1, a_2, b_1 and b_2 are constants.

On one hand, we take into account the pricing of a bond from UBS with maturity 5 years and a face value of 100. The bond pays annually coupon rates of 1.25 basis points and the recovery rate at the event of default is 40%. The model parameter values for this example are collected in Table 1. In this case the correlation coefficient ρ is assumed to be zero. Next, in Table 4 we present the value of the bond for different meshes and time steps. In this case, we can observe that the price of the bond converges to 102.62.

Table 5
Parameters of the model for the JP Morgan bond.

Parameters of the defaultable stock price model
$a_1 = 0.0312763$, $a_2 = 0.0356952$ $b_1 = 0.00038362$, $b_2 = 0.00172115$ $c = 0.346622$, $\beta = -0.223027$
Parameters of the interest rate model
$\kappa = 0.14485883018483803$ $\delta = 0.01330207057173363$ $\theta = 0.03467342840511061$
Correlation coefficient
$\rho = 0.497108$
Initial conditions
$S_0 = 1.0$, $r_0 = 0.01469383913023823$

Table 6
Value of the JP Morgan bond for different meshes and time steps.

Time steps per year	Mesh1	Mesh2	Mesh3	Mesh4
90	103.725041	103.596891	103.572191	103.570225
180	103.841153	103.605155	103.575270	103.572747
360	103.794567	103.602389	103.576483	103.574147

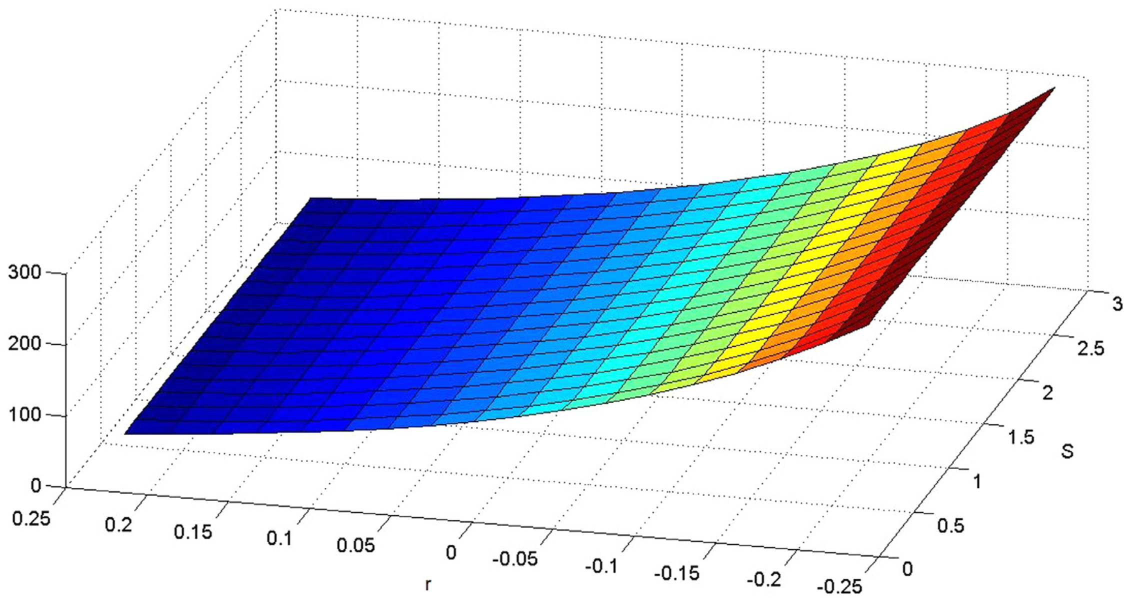

Fig. 1. Value of the UBS bond.

On the other hand, we present a correlated case. More precisely, we address the valuation of a bond from JP Morgan with maturity 5 years and a face value of 100. The bond pays annually coupon rates of 3.25 basis points and the recovery rate at the event of default is again 40%. For this second example the parameter values of the model are the ones which appear in Table 5. As we have just pointed out the correlation coefficient ρ is different from zero. Finally, the value of this bond is shown in Table 6. In this case, the value of the bond converges to 103.57.

In both cases, in order to obtain the value of the bond we need to solve for each coupon payment date the IBVP (20)-(22) with maturity equal to those dates and with initial condition $g_1(x_1, x_2) = 1$ (corresponding to $h_1(S, r) = 1$) or $g_2(x_1, x_2) = \exp(-\kappa T)(x_2 - \tilde{x}_2^{\infty})$ (corresponding to $h_2(S, r) = r$). More precisely, in both examples the maturity of the bond is 5 years and the frequency of coupon payments is annually, thus to obtain the value of the bond we need to solve the problem (20)-(22) 10 times, i.e. 5 times with h_1 as initial condition to obtain the value of u_1 and 5 times with h_2 as initial condition to obtain the value of u_2.

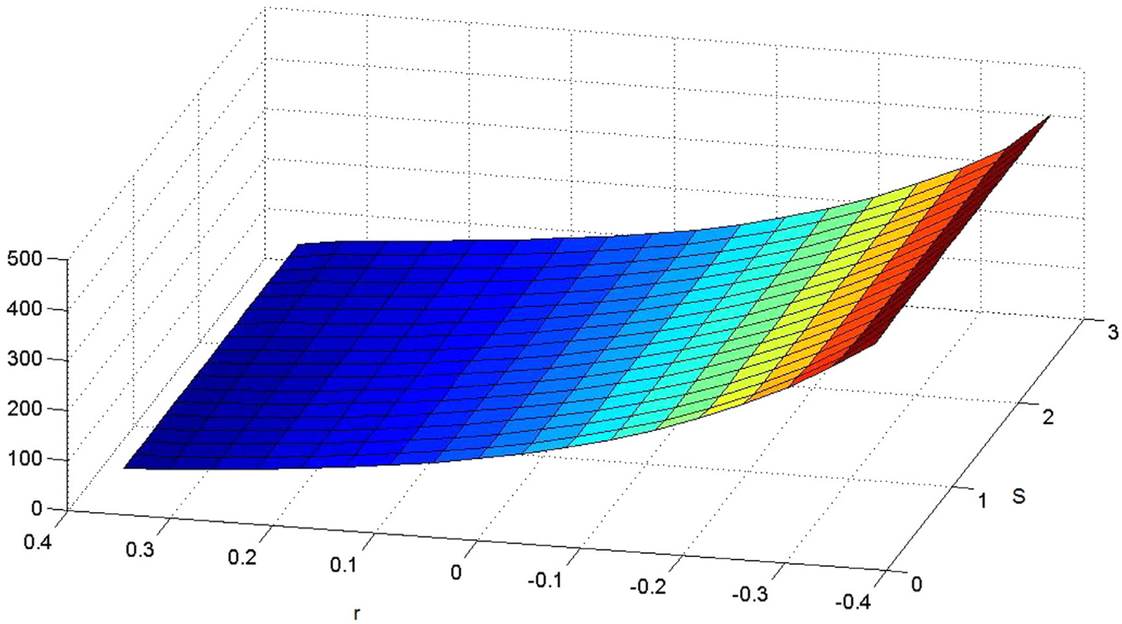

Fig. 2. Value of the JP Morgan bond.

Table 7
Comparison of PDE numerical solution with 360 time steps and Mesh4 with respect to Monte Carlo 95% confidence interval with 100,000 simulations and 2nd-order asymptotic approximation. Note that the real market prices of the bonds of the UBS and JPM are respectively 102.62 and 103.57.

	PDE solution	MC confidence interval	Asymptotic approximation
UBS	102.620069	[102.52477413, 102.72696887]	102.566
JPM	103.574147	[103.55570424, 103.65668368]	103.584

Next, in Figs. 1 and 2 we show the mesh value of the UBS bond and the JP Morgan bond, respectively. Both figures are obtained with the finest mesh and with time step $\Delta\tau = \frac{1}{360}$ (one day).

Finally, for UBS and JPM bonds the Table 7 shows a comparison between the results obtained with the proposed numerical method for the PDE model, a crude Monte Carlo technique and the asymptotic approximation method introduced in [16,19]. More precisely, for the PDE numerical results we consider Mesh 32 with 360 time steps while we use 100000 simulations in Monte Carlo and we show the 95% confidence interval.

In the fourth column, we present the prices of the UBS and JPM bonds computed with the asymptotic approximation method. This method consists of approximating the solution u of the parabolic PDE (13) by applying Theorem 3.2 in [16]. Note that the obtained value from the PDE numerical methods belongs to the confidence interval and is more accurate than the asymptotic approximation when compared with the real market prices.

Concerning computational time, the asymptotic method takes around 0.047 seconds to obtain one value and Monte Carlo simulation takes 74 seconds for UBS bond and 45 seconds for JPM bond to obtain one value, while the numerical solution of the PDE takes around 440 seconds to obtain the bond values at all the 4225 mesh nodes.

6. Conclusions

In this paper we have considered a new methodology for the valuation of a non callable defaultable coupon bond where the underlying stochastic factors are the interest rate and the defaultable stock price. The pricing problem involves the solution of a set of IBVPs. More precisely, for each coupon payment date two PDE problems with different initial conditions and with the coupon payment date as maturity need to be solved. Once the numerical solution of these problems is carried out, the value of the bond is computed by means of an integral expression. For the numerical solution of PDE problems, we combine a Crank-Nicolson semi-Lagrangian scheme with quadratic Lagrange finite elements. Moreover, the integral term is approximated by means of a composed trapezoidal rule. Finally, in order to illustrate the performance of the proposed methodology, in two examples we compare the results with those ones obtained with an alternative Monte Carlo technique and an asymptotic approximation method. From these comparisons, we conclude that results of the proposed method are in agreement with alternative ones. Moreover, bond values for a large number of mesh nodes can be obtained in a competitive computational time, when compared with alternative methods that compute just the bond value for a single mesh point each time. Note that the possible parallelization of the proposed algorithm has not been addressed yet and it clearly can take advantage of the simultaneous solution of the different PDE problems associated to coupon payment dates, thus leading to an important speed up of computations.

Credit Author Statement

Andrea Pascucci and Sidi Diop mainly proposed the model and obtained the theoretical results related to the mathematical analysis of the model. M.Carmen Calvo-Garrido, and Carlos Vázquez proposed and developed the numerical methods to solve the model. All authors contributed to the validation of the numerical results and models through appropriate test examples and data.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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