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Апроксимація щільностей потенціалів для плоских в'язкопружних тіл із включеннями, що обмежені кусково-гладкими контурами

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Approximation of density of potentials for the flat viscoelastic bodies with inclusions, bounded by a piecewise smooth contours

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Запропоновано підхід апроксимації невідомих щільностей потенціалів при дослідженні напруженого стану плоского в'язкопружного кусково-однорідного тіла з включеннями, що обмежені кусково-гладкими контурами. Метод базується на побудові системи гранично-часових інтегральних рівнянь для визначення невідомих щільностей потенціалів по контурах включень. Апроксимація невідомих щільностей потенціалів здійснювалася з урахуванням особливості напруженого стану плоского в'язкопружного тіла поблизу кутової точки лінії розділу областей.

Ключові слова: плоске в'язкопружне тіло, в'язкопружні характеристики областей, щільності потенціалів, принцип Вольтерра, резольвентні оператори, гранично-часові інтегральні рівняння, кутова точка, порядок сингулярності.

An approach for approximating unknown densities of potentials in the study of the stressed state of a flat viscoelastic piecewise homogeneous body with inclusions, bounded by piecewise smooth contours, is proposed. The method is based on the construction of a system of boundary-time integral equations to determine the unknown densities of potentials along the contours of the inclusions. The approximation of the unknown densities of potentials was performed taking into account the singularity of the stressed state of a flat viscoelastic body near the angular point of the dividing line of the regions.

Key Words: flat viscoelastic body, viscoelastic characteristics of regions, densities of potentials, Volterra principle, resolvent operators, boundary-time integral equations, angular point, singularity order.

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Wide application of composite materials in various industries and in the construction of structural elements requires the study of their strength properties, reliability and durability. It becomes necessary to solve specific problems of viscoelasticity when viscoelastic environments, consisting of homogeneous and isotropic regions with different values of viscoelastic parameters, are used as models of real objects. A detailed review of the approaches and methods of the theory of linear viscoelasticity is presented in [2].

One of the effective and applied numerical methods for solving boundary value problems of elasticity and viscoelasticity is the method of boundary integral equations, based on the theory of potential and methods of the modern theory of approximation. [5]. When solving the problems of deformation of piecewise homogeneous environments by the method of boundary integral equations, the resolving integral equations are obtained based on the conditions of continuity of movements and the corresponding components of the stress tensor at the contacting boundaries. Essential for solving this class of problems is the use of the apparatus of generalized functions and presentation of the viscoelastic parameters using unit characteristic functions in a form that is uniform for the entire region under consideration.

Let's use the approach proposed to study the stress state of a flat viscoelastic piecewise-homogeneous isotropic body occupying region D, exposed to a known force effects, consisting of a matrix D_0 and a finite number of inclusions D_p of arbitrary shape, bounded by piecewise-smooth contours Γ_0 and Γ_p, respectively ($p = \overline{1, n}$, n is the number of inclusions) [8].

To determine the stress state of the given region, we use the formulation of the second main problem of the theory of elasticity for inhomogeneous bodies in movements [6]:

$$\big(\lambda\, u_{l,l}\big)_{,i} + \Big[\mu\big(u_{i,j} + u_{j,i}\big)\Big]_{,j} + X_i = 0 \quad (x \in D),$$

$$\lambda\, u_{l,l}\, n_i + \mu\big(u_{i,j} + u_{j,i}\big) n_j = g_i \quad (x \in \Gamma), \qquad (1)$$

where $u_i = u_i(x,\, t)$ — components of the motion vector, $X_i = X_i(x,\, t)$ and $g_i = g_i(x,\, t)$, $i = \overline{1, 2}$ — components of the vector of mass forces and the vector of surface forces, respectively; $n_j = \cos\left(\mathbf{n},\, x_j\right)$, $j = \overline{1, 2}$ — components of the unit normal vector $\mathbf{n}$ to the boundary Γ of the region D; $\lambda = \lambda(x,\, t)$, $\mu = \mu(x,\, t)$ — analogs of elastic Lamé constants that depend on time and are piecewise constant functions of coordinates and time. Solution (1) was performed by the integral-operator method using the Volterra principle [2].

In the absence of mass forces, and also when the unknown densities of potentials are the stresses on the contours of inclusions, based on the properties of the generalized functions $S\big(D_p\big)$ and $\delta(x,\, \xi)$, and due to Maxwell's theorem [6], we have

$$u_k(x,\, t) = \int_{\Gamma_0} g_i(\xi,\, t)\, U_k^i(x,\, \xi,\, t)\, d\gamma(\xi) - \\ - \frac{\overline{E}_p - \overline{E}_0}{\overline{E}_p} \int_{\Gamma_p} \sigma_{ij}^{(p)}(\xi,\, t)\, n_j(\xi)\, U_k^i(x,\, \xi,\, t)\, d\gamma(\xi). \qquad (2)$$

Here

$$U_k^i\,(x,\, \xi,\, t) = \dot{U}_k^i\,(x,\, \xi,\, t) + \widehat{U}_k^i\,(x,\, \xi,\, t)\,,$$

where $\dot{U}_k^i\,(x,\, \xi,\, t)$ is the fundamental solution for problems of two-dimensional flat-strain state of a viscoelastic infinite body; $\widehat{U}_k^i\,(x,\, \xi,\, t)$ — an additional term, which provides that the condition $g_k^i\,(x,\, \xi,\, t) = 0$ is satisfied for all $x \in \Gamma$ and $\xi \in D$ [8]; $\overline{E}_0$ and $\overline{E}_p$ — viscoelastic operators of homogeneous regions belonging to the class of resolvent operators [7]. The expression for $\sigma_{ij}^{(p)}(\xi,\, t)$ has the form

$$\sigma_{ij}^{(p)}(\xi,\, t) = \lambda_p(t)\, u_{l,l}(\xi,\, t)\, \delta_{ij} + \\ + \mu_p(t)\cdot\big(u_{i,j}(\xi,\, t) + u_{j,i}(\xi,\, t)\big),$$

where $\lambda_0(t)$, $\mu_0(t)$, $\lambda_p(t)$, $\mu_p(t)$ — viscoelastic characteristics of the regions D_0 and D_p, respectively.

Using the Cauchy relations, Hooke's law, and formulas (2), we write the expression for the stress tensor components in the following form:

$$\sigma_{ij}(x,\, t) = \left(1 + \frac{\overline{E}_q - \overline{E}_0}{\overline{E}_0} S\big(D_q\big)\right) \times \\ \times \Bigg[\int_{\Gamma_0} g_k(\xi, t)\, U_{ij}^k(x, \xi, t)\, d\gamma(\xi) - \\ - \frac{\overline{E}_p - \overline{E}_0}{\overline{E}_p} \int_{\Gamma_p} \sigma_{kl}^{(p)}(\xi, t)\, n_l(\xi)\, U_{ij}^k(x, \xi, t)\, d\gamma(\xi)\Bigg], \qquad (3)$$

where $U_{ij}^k(x, \xi, t)$ are stresses arising in a viscoelastic homogeneous body occupying region D under the action of unit concentrated forces at the point $\xi \in D$.

It follows from (3) that the stresses at any point of the considered region at an arbitrary time instant can be determined through the viscoelastic potentials of the double layer along the contour Γ_0 with a given density $g_k(\xi, t)$ and along the contours Γ_p with unknown densities $\sigma_{kl}^{(p)}(\xi, t)\, n_l(\xi)$. To determine the unknown densities, let us tend the point x to each of the contours Γ_p from inside the region D_p. Multiply both sides of (3) by $n_j(x)$ and obtain the system of boundary-time integral equations:

$$\sigma_{ij}^{(q)}(x,\, t)\, n_j(x) = \frac{2\overline{E}_q}{\overline{E}_q + \overline{E}_0} \times \\ \times \Bigg[\int_{\Gamma_0} g_k(\xi, t)\, U_{ij}^k(x, \xi, t)\, n_j(x)\, d\gamma(\xi) - \\ - \frac{\overline{E}_p - \overline{E}_0}{\overline{E}_p} \times \\ \times \int_{\Gamma_p} \sigma_{kl}^{(p)}(\xi, t)\, n_l(\xi)\, U_{ij}^k(x, \xi, t)\, n_j(x)\, d\gamma(\xi)\Bigg]. \qquad (4)$$

For the numerical solution of system (4), let us discretize the contours of the inclusions by linear elements, which are characterized by the coordinates of their midpoints.

Let's approximate the unknown densities of potentials in expression (4). We introduce the function $f\left(x_n^{(p)},\, \xi,\, t\right)$, that approximates the stresses at the n-th boundary element of the p-th inclusion with nodal point $x_n^{(p)}$ and depends on the time t. Then the unknown densities of potentials for an arbitrary, for example, n-th boundary element of the p-th inclusion can be represented as follows:

$$\sigma_{kl}^{(p)}(\xi,\, t)\, n_l(\xi) = \\ = A_{kl}\left(x_n^{(p)},\, t\right) f\left(x_n^{(p)},\, \xi,\, t\right) n_l\left(x_n^{(p)}\right), \quad (5)$$

where $A_{kl}\left(x_n^{(p)},\, t\right)$ are unknown constants determined from discrete analogs of boundary-time integral equations (4); $n_l\left(x_n^{(p)}\right)$ — components of the outward normal vector to the contour of the inclusion at the point $x_n^{(p)}$.

For inclusions bounded by contours that do not contain angular points, the density of potential along each boundary element is assumed to be constant. In this case, the function $f\left(x_n^{(p)},\, \xi\right)$, which approximates the stresses at each boundary element, does not depend on the time t and is equal to 1. Then expression (5) takes the form

$$\sigma_{kl}^{(p)}(\xi,\, t)\, n_l(\xi) = A_{kl}\left(x_n^{(p)},\, t\right) n_l\left(x_n^{(p)}\right).$$

To take into account the influence of concentrators such as angular points on the stress-strain state in a viscoelastic piecewise homogeneous body, we use the approach proposed in [1]. Investigation of the singularity of the stress state in the flat problem for a compound viscoelastic body near the angular point of the dividing line of the regions leads to the solution of the transcendental equation [3, 4] depending on the viscoelastic parameters of the regions $\overline{E}$, $\overline{\nu}$, as well as the opening angles θ of these regions.

Solving this transcendental equation, we obtain the value of $s(t)$, which is its root with the least positive real part. In this case, the order of the singularity equals $\left|\operatorname{Re} s(t) - 1\right|$. Thus, the nature of the stress state at the vertex of the corner of a compound body is determined by the type of boundary conditions, viscoelastic characteristics of the material, geometry of the regions and does not depend on the type of loading.

For the angular boundary elements, we choose the approximation of the unknown densities of potentials corresponding to the peculiarity of the stress-strain state near the given angular point.

The densities of potentials in formula (4) are stresses, which in the vicinity of the angular point can be represented as

$$\sigma_{ij}(x,\, t) = o\left(r^{s_1(t)-1}\right), \quad (6)$$

where $s_1(t) = \operatorname{Re} s(t)$, $0 < s(t) < 1$, $s(t)$ is the root with the least positive real part of the transcendental equation [4].

Based on the above, we can conclude that in the case of a flat viscoelastic piecewise-homogeneous body consisting of a matrix D_0 and a finite number of inclusions D_p of arbitrary shape, bounded by piecewise-smooth contours Γ_0 and Γ_p, $p = \overline{1,\, n}$, respectively, the unknown densities of potentials for an arbitrary n-th boundary element of the p-th inclusion can be represented in the form (5).

Moreover, the function $f\left(x_n^{(p)},\, \xi,\, t\right)$ in this formula has the following properties:

- $f\left(x_n^{(p)},\, \xi,\, t\right) = 1$, if the nodal point $x_n^{(p)}$ belongs to an ordinary boundary element;
- $f\left(x_n^{(p)},\, \xi,\, t\right) = \left(\dfrac{r}{d}\right)^{s_1(t)-1}$, if the nodal point $x_n^{(p)}$ belongs to an angular boundary element, and d is the length of this element; a_i, $i = \overline{1,\, 2}$ are the coordinates of the angular point, $r = \left[(\xi_i - a_i)\,(\xi_i - a_i)\right]^{\frac{1}{2}}$.

The advantage of the approach proposed in the paper is that for a given representation of unknown densities of potentials, there are no difficulties associated with aligning the nodal points with the angular points of the conjugated contours, since the values of the unknown densities are determined in the middle of the boundary elements.

The study of the singularities of the type of angular points shows that the influence of these singularities on the stress state in a viscoelastic piecewise-homogeneous body affects only in small neighborhoods of the angular points and practically does not affect the nature of the stress change in time. This is explained by the order of the degree of the leading term of the asymptotics when solving the problem of the stress state of a compound region at certain values of the viscoelastic and geometric parameters.

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