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Альтернативна оцінка ймовірності виходу за криву траєкторії субгауссового випадкового процесу

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Alternative estimate of curve exceeding probability of sub-Gaussian random process

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У даній роботі було отримано альтернативну оцінку зверху ймовірності виходу траєкторій субгауссового процесу, заданого на компакті, за криву, зокрема за лінійну функцію . Дослідження базується на результатах, отриманих у роботі [3]. Субгауссові випадкові процеси представляють значний інтерес для дослідження, оскільки отримані результати також можна застосовувати до гауссових процесів.
Ключові слова: субгауссовий процес, метрична ентропія, розподіл супремуму, траєкторія випадкового процесу.

Investigation of sub-gaussian random processes are of special interest since obtained results can be applied to Gaussian processes. In this article the properties of trajectories of a sub-Gaussian process drifted by a curve a studied. The following functionals of extremal type from stochastic process are studied: $\sup_{t\in B}(X(t)-f(t))$, $\inf t\in B(X(t)-f(t))$ *and* $\sup_{t\in B}|X(t)-f(t)|$. *An alternative estimate of exceeding by sub-Gaussian process a level, given by a continuous linear curve is obtained. The research is based on the results obtained in the work [3]. The results can be applied to such problems of queuing theory and financial mathematics as estimation of buffer overflow probability and bankruptcy probability.*
Key Words: sub-Gaussian process, metric entropy, supremum distribution, trajectory of random process.

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1 Introduction

The paper continues cycle of works studying properties of sub-Gaussian random processes and their generalizations from the class $V(\varphi,\psi)$. Recall, that sub-gaussian random variables are majorized in distribution by Gaussian random variables, and thus are of particular interest as their natural generalization. Here we look at the probability of exceeding by a sub-Gaussian random process a level given by linear function form another view and obtain an alternative estimate. Such estimates can be useful in queuing theory, financial mathematics and in some engineering problems. For example, to support operations of electronic devices the electric current must be in between maximum permissible amperage and permissible voltage drop. The estimate $\mathbf{P}\{\sup_{t\in B}(X(t)-f(t))>x\}$ determines the probability that the current exceeds the upper bound $f(t)$, $\mathbf{P}\{\inf_{t\in B}(X(t)-f(t))>-x\}$ determines the probability that the current fails below the lower bound $f(t)$.

The next part of this section contains some basic notions of random variables and processes from the spaces $Sub_\varphi(\Omega)$ and the class $V(\varphi,\psi)$. More details can be found in monographs [1, 2].

Let $(\Omega,\mathcal{B},P)$ be a standard probability space and T be some parametrical space.

Definition 1.1. [1] Metric entropy with regard to pseudometric (metric) ρ, or just metric entropy is a function $H_T(u)=H(u)=\begin{cases}\log N_{(T,\rho)}(u), & \text{if } N_{(T,\rho)}(u)<+\infty\\ +\infty, & \text{if } N_{(T,\rho)}(u)=+\infty\end{cases}$, where $N_T(u)=N(u)$ denotes the least number of closed ρ-balls with radius u covering space (T,ρ).

Definition 1.2. [1] Let φ be such an Orlicz N-function, that $\varphi(x)=cx^2$ as $|x|\leqslant x_0$ for some $c>0$; $x_0>0$. Centered random variable ξ belongs to the space $Sub_\varphi(\Omega)$, if for all $\lambda \in \mathbb{R}$ there exists a constant $a \geqslant 0$, which satisfies the following inequality

$$\mathsf{E}\exp(\lambda\xi) \leqslant \exp(\varphi(a\lambda)).$$

Definition 1.3. [3] N-function φ is subordinated by an Orlicz N-function ψ ($\varphi \prec \psi$) if there are exist such numbers $x_0 > 0$ and $k > 0$ that $\varphi(x) < \psi(kx)$ for $x > x_0$.

Definition 1.4. [3] Let $\varphi \prec \psi$ are two Orlicz N-functions. Random process $X = \{X(t), t \in T\}$ belongs to class $V(\varphi,\psi)$ if for all $t \in T$ the random variable $X(t)$ is from $Sub_\psi(\Omega)$ and for all $s,t \in T$ increments $(X(t) - X(s))$ belong to the family $Sub_\varphi(\Omega)$.

2 Main result

Let (T,ρ) be a pseudometrical (metrical) compact space with pseudometric (metric) ρ and $X = \{X(t), t \in T\}$ be a separable random process from the class $V(\varphi,\psi)$.

Condition Σ. Suppose there exists such a continuous monotonically increasing function $\sigma = \{\sigma(h), h > 0\}$, that $\sigma(h) \to 0$, as $h \to 0$, and the following inequality for increments of the process is true

$$\sup_{\rho(t,s)\leqslant h} \tau_\varphi(X(t) - X(s)) \leqslant \sigma(h). \tag{1}$$

Put $\gamma(u) = \tau_\psi(X(u)) < \infty$, and $\beta > 0$, be such a number that $\beta \leqslant \sigma(\inf_{s\in B}\sup_{t\in B}\rho(t,s))$.

The following lemma contains conditions on boundness of supremum of the drifted random process $X_f(t) = X(t) - f(t)$ and estimates of its exponential moments.

Lemma 1. *[3] Let $X = \{X(t), t \in T\}$ be a separable random process from the class $V(\phi,\psi)$ satisfying Condition Σ and let $f = \{f(t), t \in T\}$ be a continuous function such that $|f(u) - f(v)| \leqslant \delta(\rho(u,v))$, where where $\delta = \{\delta(s), s > 0\}$ is some monotonically increasing nonnegative function. Let $\{q_k, k = 1,2,...\}$ be a sequence such that $q_k > 1$ and $\sum_{k=1}^{\infty}\frac{1}{q_k} \leqslant 1$. Then for all $\lambda > 0$ and $p \in (0,1)$ the following inequality holds true*

$$E\exp\{\lambda\sup_{t\in B} X_f(t)\} \leqslant \prod_{k=1}^{\infty}(N_B(\sigma^{(-1)}(\beta p^k)))^{\frac{1}{q_k}} \times$$
$$\times\exp\left\{\frac{1}{q_1}\sup_{u\in B}(\psi(\lambda q_1\gamma(u)) - \lambda q_1 f(u))\right\}\times$$
$$\times\prod_{k=1}^{\infty}\exp\left\{\frac{1}{q_k}\varphi(\lambda q_k\beta p^{k-1}) + \lambda\delta(\sigma^{(-1)}(\beta p^{k-1}))\right\}. \tag{2}$$

Different kinds of sequences q_k were studied previously (see [2]), for example, $q_k^{'} = ((1-p)p^{k-1})^{-1}$, and $q_1^{''} = v$, where v is such a number that $v \geqslant \frac{1}{1-p}$ and $q_k^{''} = \frac{1}{\lambda\beta p^{k-1}}\varphi^{(-1)}\left(\varphi\left(\frac{\lambda\beta}{1-p}\right) + H(\varepsilon_k)\right)$, $k = 2,3\ldots$.

Consider another sequence $q_k = 2^k, k \geqslant 1$. As a example, let's apply Lemma 1 to to a sub-Gaussian random process X, for which $\sigma(h) = Ch^\alpha$, $C > 0$, drifted by the function $f(t) = at$, $a > 0$.

Theorem 2.1. *Let $X = (X(t), t \in B)$ be a sub-Gaussian random process then the following equations hold true for $p \in (0,1)$ and $x > 0$.*

$$\mathbf{P}\left\{\sup_{t\in B}(X(t) - at) > x\right\} \leqslant Z(\lambda,p,\beta,x),$$

$$\mathbf{P}\left\{\inf_{t\in B}(X(t) - at) < -x\right\} \leqslant Z(\lambda,p,\beta,x),$$

$$\mathbf{P}\left\{\sup_{t\in B}|X(t) - at| > x\right\} \leqslant 2Z(\lambda,p,\beta,x),$$

where $Z(\lambda,p,\beta,x) =$

$$r^{(-1)}\left(2\int_0^{\frac{1}{2}} r\left(N(\sigma^{(-1)}(\beta pu^{log_{\frac{1}{2}}p}))\right)\mathsf{d}u\right)\times$$

$$\exp\left\{\frac{-\left(2au + x - (\frac{\beta}{C})^{(\frac{1}{\alpha})}\cdot\frac{p^{\frac{1}{\alpha}}}{1-p^{\frac{1}{\alpha}}}\right)^2(1-2p^2)}{2((1-2p^2)\sup_{u\in B}[2\gamma^2(u)] + 2\beta^2 2p^2)}\right\}.$$

Доведення. Let's consider each element of (2). Note that $p^k = (\frac{1}{2})^{k\log_{\frac{1}{2}}p}$. Since

$$\int_{\frac{1}{2^{k+1}}}^{\frac{1}{2^k}} r(N(\sigma^{(-1)}(\beta u^{\log_{\frac{1}{2}}p^k})))\,\mathsf{d}u$$

$$\geqslant r(N(\sigma^{(-1)}(\beta p^k)))(\frac{1}{2^k} - \frac{1}{2^{k+1}}),$$

we have that

$$\prod_{k=1}^{\infty}(N_B(\epsilon_k))^{\frac{1}{q_k}})$$

$$\leqslant r^{(-1)}\left(2\sum_{k=1}^{\infty}\int_{\frac{1}{2^{k+1}}}^{\frac{1}{2^k}} r(N(\sigma^{(-1)}(\beta u^{log_{\frac{1}{2}}p})))\,\mathsf{d}u\right)$$

$$= r^{(-1)}\left(2\int_{0}^{\frac{1}{2}} r(N(\sigma^{(-1)}(\beta p u^{log_{\frac{1}{2}}p})))\,\mathsf{d}u\right) \quad (3)$$

Let's consider other elements in (2). Since $\varphi(x) = \psi(x) = \frac{x^2}{2}$, $\delta(u) = a|u|$ and $\sigma^{(-1)}(h) = (\frac{h}{C})^{\frac{1}{\alpha}}$, we have that

$$\exp\left\{\frac{1}{q_1}\sup_{u\in B}(\psi(\lambda q_1\gamma(u)) - \lambda q_1 f(u)) - \lambda x\right\}$$

$$= \exp\left\{\frac{1}{2}\sup_{u\in B}\left[\frac{(2\lambda\gamma(u))^2}{2}\right] - 2\lambda a u - \lambda x\right\}.$$

Next,

$$\prod_{k=1}^{\infty}\exp\left\{\frac{1}{q_k}\varphi(\lambda q_k\beta p^{k-1})\right\}$$

$$= \prod_{k=1}^{\infty}\exp\left\{\frac{1}{2^k}2\lambda^2 2^{2k}\beta^2 p^{2k-2}\right\}$$

$$= \exp\left\{\frac{\lambda\beta^2}{2}\sum_{k=2}^{\infty}2^k p^{2k-2}\right\} = \exp\left\{\frac{\lambda\beta^2 2p^2}{1-2p^2}\right\},$$

and

$$\prod_{k=1}^{\infty}\exp\left\{\lambda\delta(\sigma^{(-1)}(\beta p^{k-1}))\right\}$$

$$= \prod_{k=1}^{\infty}\exp\left\{\lambda\delta|\frac{\beta p^{k-1}}{C}|^{\frac{1}{\alpha}}\right\}$$

$$= \exp\left\{(\frac{\beta}{C})^{\frac{1}{\alpha}}\lambda\sum_{k=1}^{\infty}(p^{\frac{1}{\alpha}})^{k-1}\right\}$$

$$= \exp\left\{\lambda(\frac{\beta}{C})^{\frac{1}{\alpha}}\cdot\frac{p^{\frac{1}{\alpha}}}{1-p^{\frac{1}{\alpha}}}\right\}.$$

Combining the above, consider only the exponential part, i.e.

$$\frac{1}{2}\sup_{u\in B}\left[\frac{(2\lambda\gamma(u))^2}{2}\right] - 2\lambda a u - \lambda x +$$

$$+\frac{\lambda^2\beta^2 2p^2}{1-2p^2} + \lambda(\frac{\beta}{C})^{\frac{1}{\alpha}}\cdot\frac{p^{\frac{1}{\alpha}}}{1-p^{\frac{1}{\alpha}}} \quad (4)$$

Then we can minimize this expression with respect to λ. Equating its partial derivative to zero, the following equation is obtained.

$$\lambda\sup_{u\in B}[(2\gamma(u))^2] - 2au - x +$$

$$+\frac{2\lambda\beta^2 2p^2}{1-2p^2} + \lambda(\frac{\beta}{C})^{\frac{1}{\alpha}}\cdot\frac{p^{\frac{1}{\alpha}}}{1-p^{\frac{1}{\alpha}}} = 0$$

Its solution is

$$\lambda = \frac{2au + x - (\frac{\beta}{C})^{\frac{1}{\alpha}}\cdot\frac{p^{\frac{1}{\alpha}}}{1-p^{\frac{1}{\alpha}}}}{\sup_{u\in B}[(2\gamma(u))^2] + \frac{2\beta^2 2p^2}{1-2p^2}}$$

$$= \frac{\left(2au + x - (\frac{\beta}{C})^{\frac{1}{\alpha}}\cdot\frac{p^{\frac{1}{\alpha}}}{1-p^{\frac{1}{\alpha}}}\right)(1-2p^2)}{(1-2p^2)\sup_{u\in B}[(2\gamma(u))^2] + 2\beta^2 2p^2}.$$

After substituting this λ in (4), the assertion of the Theorem follows from (3) and Lemma 1. ☐

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