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Умови існування з імовірністю одиниця узагальненого розв'язку задачі Коші для рівняння теплопровідності з випадковою правою частиною

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The conditions of existence with probability one of generalized solutions of Cauchy problem for the heat equation with a random right part

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В роботі досліджуються властивості узагальненого розв'язку задачі Коші для рівняння теплопровідності на прямій, коли права частина є випадковим полем з простору $Sub_\varphi(\Omega)$.

Ключові слова: $Sub_\varphi(\Omega)$ випадкові процеси, рівняння теплопровідності, узагальнений розв'язок.

The subject of this work is at the intersection of two branches of mathematics: mathematical physics and stochastic processes. The influence of random factors should often be taken into account in solving problems of mathematical physics. The heat equation with random conditions is a classical problem of mathematical physics. In this paper we consider a Cauchy problem for the heat equations with a random right part. We study the inhomogeneous heat equation on a line with a random right part. We consider the right part as a random function of the space $Sub_\varphi(\Omega)$. The conditions of existence with probability one generalized solution of the problem are investigated.

Using this results one can construct modeless, which approximate solutions of such equations with given accuracy and reliability in the uniform metric

Key words: $Sub_\varphi(\Omega)$ stochastic processes, heat equation, generalized solution.

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1 Introduction

The heat equation with random conditions is a classical problem of mathematical physics. Recently, a number of have works appeared, which in many ways have investigated this equation according to the type of random initial conditions [1, 2, 3, 4].

The properties of classical solutions of Cauchy problems for the heat equations with a random right part which is random field from the space $Sub_\varphi(\Omega)$, space $L_p(\Omega)$ and from the Orlicz space have been studied in the papers in [5, 6, 7]. Estimations for the distribution of the supremum of solution of such equations have been investigated.

We consider the properties of generalized solutions of Cauchy problem for the heat equations with a random right part. In particular, we give conditions for existence with probability one of the generalized solutions in the case when the right part is a random field, sample continuous with probability one from the $Sub_\varphi(\Omega)$ space.

The paper consists of the introduction and two parts. Section 1 contains necessary definitions and results of the theory of the $Sub_\varphi(\Omega)$ space. In section 2 we consider heat equations with random right part. For such problem conditions of existence with probability one of generalized solution with random right part from the space $Sub_\varphi(\Omega)$ are found.

2 Random processes from $Sub_\varphi(\Omega)$ spaces

Definition 2.1. [8] An even continuous convex function $u(x)$, $x \in R^1$ such that $u(0) = 0$ and $u(x) > 0$ for $x \neq 0$ and

$$\lim_{x\to 0} \frac{u(x)}{x} = 0, \lim_{x\to\infty} \frac{u(x)}{x} = \infty.$$

is called an N-function.

Definition 2.2. [9] We say an N-function u satisfies the q-condition if there exist constants $z_0 >$

$0, \quad k > 0, \quad A > 0$ such that $u(x)\,u(y) \leqslant Au(kxy)$ for all $x > z_0, \quad y > z_0$.

Let $\{\Omega, \Im, P\}$ be a standard probability space.

Definition 2.3. [9] Let $\varphi(x)$ be an N-function for which there exist constants $x_0 > 0$ and $c > 0$ such that $\varphi(x) = cx^2$ for $|x| < x_0$. The set of random variables $\xi(\varpi)$, $\varpi \in \Omega$, is called the space $Sub_\varphi(\Omega)$ generated by the N-function $\varphi(x)$ if $E\xi = 0$ and there exists a constant a_ξ such that

$$E\exp\{\lambda\xi\} \leqslant \exp\{\varphi(\lambda a_\xi)\}$$

for all $\lambda \in R^1$.

Definition 2.4. [8] The stochastic process $X = \{X(t), \ t \in T\}$ belongs to space $Sub_\varphi(\Omega)$, $(X \in Sub_\varphi(\Omega))$ if $X(t) \in Sub_\varphi(\Omega)$ for all $t \in T$.

Definition 2.5. [9] The random variable $\xi \in Sub_\varphi(\Omega)$ is called strongly $Sub_\varphi(\Omega)$, $(SSub_\varphi(\Omega))$ random variable if $\tau_\varphi(\xi) = \left(E\xi^2\right)^{1/2}$.

Properties and applications of $Sub_\varphi(\Omega)$ random variables and stochastic processes from $Sub_\varphi(\Omega)$ can be found in [9].

Definition 2.6. [10] A family Δ of random variables ξ of the space $Sub_\varphi(\Omega)$ is called $SSub_\varphi(\Omega)$ family if

$$\tau_\varphi\left(\sum_{i\in I}\lambda_i\xi_i\right) = \left(E\left(\sum_{i\in I}\lambda_i\xi_i\right)^2\right)^{1/2}$$

for all $\lambda_i \in R^1$, where I is at most countable and $\xi_i \in \Delta_i, \quad i \in I$.

Theorem 1. [*10*] *Let Δ be a strongly $Sub_\varphi(\Omega)$ family of random variables. Then the linear closure $\overline{\Delta}$ of the family Δ in the space $L_2(\Omega)$ and in the mean square sense is a strongly $Sub_\varphi(\Omega)$ family.*

Definition 2.7. [9] The stochastic process $X_i = \{X_i(t), \ t \in T, \ i \in I\}$ is called an $SSub_\varphi(\Omega)$ process if the family of random variables $X_i = \{X_i(t), \ t \in T, \ i \in I\}$ is an $SSub_\varphi(\Omega)$.

Theorem 2. [*10*] *Let $X_i = \{X_i(t), \ t \in T, \ i \in I\}$ be a family of jointly strongly $Sub_\varphi(\Omega)$ stochastic processes. Then (T, O, μ) is a measurable space. If $\left\{\varphi_{k_i}(t), i \in I, k = \overline{1,\infty}\right\}$ is a family of measurable functions in (T, O, μ) and the integral*

$$\xi_{k_i} = \int_T \varphi_k(t)\, X_j(t)\, d\mu(t)$$

is well defined in the mean square sense, than the family of random variables

$$\Delta_\xi = \left\{\xi_{k_i}, \ i \in I, \ k = \overline{1,\infty}\right\}$$

is an $SSub_\varphi(\Omega)$ family.

Theorem 3. [*11*] *Let R^k be the k-dimensional space, $d(t,s) = \max\limits_{1\leqslant i\leqslant k}|t_i - s_i|$, $T = \{0 \leqslant t_i \leqslant T_i, i = 1, 2, ..., k\}$, $T_i > 0. X_n = \{X_n(t), \ t \in T\} \in Sub_\varphi(\Omega)$. Assume that the process $X_n(t)$ is separable and*

$$\sup_{d(t,s)\leqslant h} \tau_\varphi\left(X_n(t) - X_n(s)\right) \leqslant \sigma(h),$$

where $\sigma(h)$ is a monotone increasing continuous function such that $\sigma(h) \to 0$ as $h \to 0$. We also assume that

$$\int_{0+} \Psi\left(\ln\frac{1}{\sigma^{(-1)}(\varepsilon)}\right) d\varepsilon < \infty,$$

where $\Psi(u) = \frac{u}{\varphi^{(-1)}(u)}$ and $\sigma^{(-1)}(\varepsilon)$ is the inverse function to $\sigma(\varepsilon)$. If the processes $X_n(t)$ converge in probability to the process $X(t)$ for all $t \in T$, then $X_n(t)$ converge in probability in the space $C(T)$.

3 The heat equations with random right part

We consider the Cauchy problem for the heat equation

$$\frac{\partial u(x,t)}{\partial t} = a^2\frac{\partial^2 u(x,t)}{\partial x^2} + \xi(x,t), \qquad (1)$$

$$-\infty < x < +\infty, \quad t > 0,$$

subject to the initial condition

$$u(x,0) = 0, \quad -\infty < x < +\infty. \qquad (2)$$

Let the function $\xi(x,t) = \{\xi(x,t), \quad x \in R, \ t > 0\}$ is a random field sample continuity with probability one from the space $Sub_{\varphi(x)}(\Omega)$, such that $E\xi(x,t) = 0$, $E(\xi(x,t))^2 < +\infty$. Let us denote

$$B(x,t,z,s) = E\xi(x,t)\xi(y,s).$$

Let $B(x,t,z,s)$ be a continuous function.

Problem when the function $\xi(x,t)$ nonrandom has been seen in [12]. Consider

$$G(y,t) = \frac{1}{\sqrt{2\pi}} \int\limits_0^t e^{-a^2y^2(t-\tau}\tilde{\xi}(y,\tau)d\tau,$$

$$\tilde{\xi}(y,\tau) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \cos yx\xi(x,\tau)dx,$$

and

$$u(x,t) = \int\limits_{-\infty}^{+\infty} \cos yxG(y,t)dy. \quad (3)$$

Let $D = \{(x,t): \ x \in [-A, \ A], \ t \in [0, \ T]$ and

$$u_n(x,t) = \int\limits_{-a_n}^{+a_n} \cos yxG(y,t)dy, \quad (4)$$

where $a_n \to \infty$ for $n \to \infty$.

Definition 3.1. The solution $u(x,t)$ which is represented in the form (3) is called a generalized solution of problem (1)—(2) in the domain D if sequence (4) converges uniformly in probability and satisfies the condition (2).

Lemma 1. *[5] Let $\xi(x,t)$ be a random field, sample continuity for each $t > 0$ with probability one, there is a continuous derivative $\frac{\partial\xi(x,t)}{\partial x}$ for $x \in R$ and satisfy condition*

$$\int\limits_R \sqrt{E(\xi^2(x,t))}dx < \infty. \quad (5)$$

Then for the function $\xi(x,t)$ for each $t > 0$ the integral Fourier transform

$$\tilde{\xi}(y,\tau) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \cos yx\xi(x,\tau)dx$$

exist and

$$\xi(x,t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \cos yx\tilde{\xi}(y, \ t)dy.$$

Lemma 2. *Let $\xi(x,t)$ be a random field, sample continuity from the space $Sub_\varphi(\Omega)$. Let $B(x,t,v,s)$ be the correlation function of the field $\xi(x,t)$. For all $t > 0, \ s > 0$ assume that* $\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} |B(x,t,v,s)|\, dxdx \leqslant B < \infty.$

Then Lebesgue integrals

$$\int\limits_{-\infty}^{+\infty} \cos yxG(y,t)dy$$

exist with probability one.

Proof. We shall prove the existence of the integral

$$\int\limits_{-\infty}^{+\infty} \cos yxG(y,t)dy.$$

For existence of this integral with probability one it is enough to prove that there exists following integral

$$\int\limits_{-\infty}^{+\infty} E|G(y,t)|dy.$$

There is an inequality

$$\int\limits_{-\infty}^{+\infty} E|G(y,t)|dy \leqslant \int\limits_{-\infty}^{+\infty} \sqrt{E(G(y,t))^2}dy.$$

Consider

$$E(G(y,t))^2 = \frac{1}{2\pi} \int\limits_0^t \int\limits_0^t e^{-a^2y^2(t-\tau)}e^{-a^2y^2(t-s)} \times$$
$$E(\tilde{\xi}(y,\tau)\tilde{\xi}(y,s))d\tau ds.$$

$$E(\tilde{\xi}(y,\tau)\tilde{\xi}(y,s)) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \cos yx \cos yv \times$$
$$E(\xi(x,\tau)\xi(v,s))dxdv =$$
$$\frac{1}{2\pi} \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \cos yx \cos yv B(x,\tau,v,s)dxdv.$$

We obtain

$$|E(\tilde{\xi}(y,\tau)\tilde{\xi}(y,s))| \leqslant$$
$$\frac{1}{2\pi} \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} |B(x,t,v,s)|\, dxdv \leqslant \frac{1}{2\pi} \cdot B.$$

Then

$$E(G(y,t))^2 = \left(\frac{1}{2\pi}\right)^2 \cdot B \times$$
$$\int\limits_0^t \int\limits_0^t e^{-a^2y^2(t-\tau)}e^{-a^2y^2(t-s)}d\tau ds =$$

$$\left(\frac{1}{2\pi}\right)^2 \cdot B \cdot (1 - e^{a^2y^2t})^2.$$

Therefore

$$\int\limits_{-\infty}^{+\infty} \sqrt{E(G(y,t))^2}dy \leqslant \frac{\sqrt{B}}{2\pi a^2} \int\limits_{-\infty}^{+\infty} \frac{\left(1 - e^{-a^2y^2t}\right)}{y^2}dy$$

for $y \neq 0$. □

Твердження 1. *[5]*

$$\left|e^{-a^2y^2(t-t_1)} - 1\right| \leqslant$$

$$\max(1, a^2)\frac{(\ln(y^2 + e^\delta))^\delta}{\left(\ln\left(\frac{1}{|t-t_1|} + e^\delta\right)\right)^\delta}, \quad (6)$$

$$|\cos yx - \cos yx_1| \leqslant \frac{(\ln(|y| + e^\delta))^\delta}{\left(\ln\left(\frac{1}{|x-x_1|} + e^\delta\right)\right)^\delta} \quad (7)$$

for some $\delta > 0$.

Let

$$u_n(x,t) = \int\limits_{-a_n}^{a_n} \cos yxG(y,t)dy.$$

Theorem 4. *Let $\xi(x,t)$ be a random field, sample continuous with probability one from the $Sub_\varphi(\Omega)$ and the conditions of lemma 1 and lemma 2 hold,*

$$\sup_{\substack{|x-x_i|\leqslant h\\ |t-t_1|\leqslant h}} \tau_\varphi\left(u_n(x,t) - u_n(x_1,t_1)\right) \leqslant \sigma(h),$$

where $\sigma(h)$ is a monotone increasing continuous function such that $\sigma(h) \to 0$ as $h \to 0$, moreover,

$$\int\limits_{0+} \Psi\left(\ln \frac{1}{\sigma^{(-1)}(\varepsilon)}\right) d\varepsilon < \infty, \quad (8)$$

where $\Psi(u) = \frac{u}{\varphi^{(-1)}(u)}$, and $\sigma^{(-1)}(\varepsilon)$ is the inverse function to $\sigma(\varepsilon)$.

Then the function $u(x,t)$ which is represented in the form (3) is a generalized solution to the problem (1)–(2).

Proof. This theorem follows from Theorem 3. □

Example 1. Let $\varphi(x)$ be a function such that $\varphi(x) = |x|^p$, for some $p > 1$ and all $|x| > 1$. Then $\Psi(x) = x^{1-\frac{1}{p}}$ for $x > 1$ and condition (8) holds for all $\varepsilon > 0$

$$\int\limits_{0+} \left(\ln \frac{1}{\sigma^{(-1)}(u)}\right)^{1-\frac{1}{p}} du < \infty. \quad (9)$$

Condition (9) holds if $\sigma(h) = \frac{C}{|\ln|h||^\delta}$, for $\delta > 1 - \frac{1}{p}$, $C > 0$. In this case the condition of Theorem (4) is satisfied if there exist constant C such that

$$\left(E\left|u_n(x,t) - u_n(x_1,t_1)\right|^2\right)^{1/2} \leqslant \frac{C}{|\ln|h||^\delta}, \quad (10)$$

for $\delta > 1 - \frac{1}{p}$, all $n = 1, 2, ...,$, and sufficiently small $|h|$.

Theorem 5. *Let $\xi(x,t)$ be a random field, sample continuous with probability one from the space $SSub_\varphi(\Omega)$, where $\varphi(x)$ is a function such that $\varphi(x) = |x|^p$ for some $p > 1$ and all $|x| > 1$ and the conditions of Lemma 1 and lemma 2 hold and*

$$\int\limits_{-\infty}^{+\infty} \left(E\left|\xi(x,\tau)\right|^2\right)^{\frac{1}{2}} dx < \Theta$$

for some $\Theta > 0$. Then the function $u(x,t)$ which is represented in the form (3) is generalized solution to the problem (1)–(2).

Proof. It follows from Lemma 2 that there exist integral with probability one

$$\int\limits_{-\infty}^{+\infty} y\cos yxG(y,t)dy.$$

To make the function $u(x,t)$ which is represented in the form (3) to be the generalized solution of problem (1)–(2) it is sufficient to prove that integral (4) converge uniformly in probability in $|x| \leqslant A$, $0 \leqslant t \leqslant T$ to the integral

$$\int\limits_{-\infty}^{+\infty} \cos yxG(y,t)dy.$$

for any $A > 0$, $T > 0$.

According to Theorem 4, using the example 1, to make integral (4) converge in probability in $C(\widetilde{T})$ the following conditions must hold

$$\left(E\left|u_n(x,t) - u_n(x_1,t_1)\right|^2\right)^{\frac{1}{2}} \leqslant \frac{C}{|\ln|h||^\delta}.$$

Using generalized Minkovskoho inequality we obtain

$$\left(E\left|u_n(x,t) - u_n(x_1,t_1)\right|^2\right)^{\frac{1}{2}} =$$

$$\left(E\left|\int_{-a_n}^{a_n}\cos yxG(y,t)dy-\int_{-a_n}^{a_n}\cos yx_1G(y,t_1)dy\right|^2\right)^{\frac{1}{2}}=$$

$$\left(E\left|\int_{-a_n}^{a_n}[\cos yxG(y,t)-\cos yx_1G(y,t_1)]\,dy|^2\right)^{\frac{1}{2}}=$$

$$\left(E\left|\int_{-a_n}^{a_n}[(\cos yx-\cos yx_1)G(y,t_1)+(G(y,t)-G(y,t_1))\cos yx]\,dy|^2\right)^{\frac{1}{2}}\leqslant$$

$$\int_{-\infty}^{\infty}\Big[|\cos yx-\cos yx_1|\left(|G(y,t_1)|^2\right)^{\frac{1}{2}}+\left(E|G(y,t)-G(y,t_1)|^2\right)^{\frac{1}{2}}\Big]\,dy. \qquad (11)$$

Let $|x-x_1|\leqslant h$ and for sufficiently small $|h|$, using the inequality (7), we have

$$|\cos yx-\cos yx_1|\leqslant 2\left|\sin\frac{y(x-x_1)}{2}\right|\leqslant\frac{(\ln(|y|+e^{\delta}))^{\delta}}{(|\ln|h||)^{\delta}}. \qquad (12)$$

Consider

$$\left(E|G(y,t_1)|^2\right)^{\frac{1}{2}}=\frac{1}{\sqrt{2\pi}}\left(E\left|\int_0^{t_1}e^{-a^2y^2(t_1-\tau)}\tilde{\xi}(y,\tau)d\tau\right|^2\right)^{\frac{1}{2}}\leqslant$$

$$\frac{1}{\sqrt{2\pi}}\int_0^{t_1}e^{-a^2y^2(t_1-\tau)}\left(E|\tilde{\xi}(y,\tau)|^2\right)^{\frac{1}{2}}d\tau.$$

It follows from Lemma 1 that

$$\left(E|\tilde{\xi}(y,\tau)|^2\right)^{\frac{1}{2}}=\frac{1}{\sqrt{2\pi}}\left(E\left|\int_{-\infty}^{+\infty}\cos yx\xi(x,\tau)dx\right|^2\right)^{\frac{1}{2}}\leqslant$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(E\,|\xi(x,\tau)|^2\right)^{\frac{1}{2}}dx<\frac{1}{\sqrt{2\pi}}\Theta.$$

Therefore

$$\left(E|G(y,t_1)|^2\right)^{\frac{1}{2}}\leqslant\frac{1}{2\pi}\int_0^{t_1}\Theta e^{-a^2y^2(t_1-\tau)}d\tau\leqslant\frac{1}{2\pi}\Theta\frac{1}{a^2y^2}\left|1-e^{-a^2y^2t_1}\right|. \qquad (13)$$

Let $t_1<t$ then

$$\left(E|G(y,t)-G(y,t_1)|^2\right)^{\frac{1}{2}}=\frac{1}{\sqrt{2\pi}}\left(E\left|\int_0^{t}e^{-a^2y^2(t-\tau)}\tilde{\xi}(y,\tau)d\tau-\int_0^{t_1}e^{-a^2y^2(t_1-\tau)}\tilde{\xi}(y,\tau)d\tau\right|^2\right)^{\frac{1}{2}}=$$

$$\frac{1}{\sqrt{2\pi}}\left(E\left|\int_0^{t_1}\left[e^{-a^2y^2(t-\tau)}-e^{-a^2y^2(t_1-\tau)}\right]\times\tilde{\xi}(y,\tau)d\tau+\int_{t_1}^{t}e^{-a^2y^2(t-\tau)}\tilde{\xi}(y,\tau)d\tau\right|^2\right)^{\frac{1}{2}}=$$

$$\frac{1}{\sqrt{2\pi}}\left(\int_0^{t_1}\left[\left|e^{-a^2y^2(t-\tau)}-e^{-a^2y^2(t_1-\tau)}\right|\times\left(E|\tilde{\xi}(y,\tau)|^2\right)^{\frac{1}{2}}\right]d\tau+\int_{t_1}^{t}e^{-a^2y^2(t-\tau)}\left(E|\tilde{\xi}(y,\tau)|^2\right)^{\frac{1}{2}}d\tau\right).$$

Let $|t-t_1|\leqslant h$ and for sufficiently small $|h|$, using the inequality (6), we have

$$\left|e^{-a^2y^2(t-\tau)}-e^{-a^2y^2(t_1-\tau)}\right|=\left|e^{-a^2y^2(t_1-\tau)}\right|\left|e^{-a^2y^2(t-t_1)}-1\right|\leqslant e^{-a^2y^2(t_1-\tau)}\max(1,a^2)\frac{(\ln(y^2+e^{\delta}))^{\delta}}{(|\ln|h||)^{\delta}}.$$

Therefore

$$\left(E|G(y,t)-G(y,t_1)|^2\right)^{\frac{1}{2}}\leqslant$$

$$\frac{1}{2\pi}\left(\int_0^t e^{-a^2y^2(t_1-\tau)}\max(1,a^2)\times\right.$$

$$\left.\frac{(\ln(y^2+e^\delta))^\delta}{(|\ln|h||)^\delta}\Theta d\tau+\int_{t_1}^t e^{-a^2y^2(t-\tau)}\Theta d\tau\right)=$$

$$\frac{\Theta}{2\pi}\left(\max(1,a^2)\frac{(\ln(y^2+e^\delta))^\delta}{(|\ln|h||)^\delta}\frac{1}{a^2y^2}\times\right.$$

$$\left.\left|1-e^{-a^2y^2t_1}\right|+\int_{t_1}^t e^{-a^2y^2(t-\tau)}d\tau\right). \qquad (14)$$

Thus we obtain from (11), (12), (13) and (14) that

$$\left\|u_{a_n}^{(0)}(x,t)-u_{a_n}^{(0)}(x_1,t_1)\right\|_p\leqslant$$

$$\frac{\Theta}{2\pi}\int_{-\infty}^{+\infty}\left[\frac{(\ln(|y|+e^\delta))^\delta}{(|\ln|h||)^\delta}\frac{1}{a^2y^2}\left|1-e^{-a^2y^2t_1}\right|+\right.$$

$$\max(1,a^2)\frac{(\ln(y^2+e^\delta))^\delta}{(|\ln|h||)^\delta}\frac{1}{a^2y^2}\left|1-e^{-a^2y^2t_1}\right|+$$

$$\left.\int_{t_1}^t e^{-a^2y^2(t-\tau)}d\tau\right]dy=$$

$$\frac{\Theta}{\pi}\int_0^{+\infty}\left[\frac{(\ln(y+e^\delta))^\delta}{(|\ln|h||)^\delta}\frac{1}{a^2y^2}\left|1-e^{-a^2y^2t_1}\right|+\right.$$

$$\max(1,a^2)\frac{(\ln(y^2+e^\delta))^\delta}{(|\ln|h||)^\delta}\frac{1}{a^2y^2}\left|1-e^{-a^2y^2t_1}\right|+$$

$$\left.\int_{t_1}^t e^{-a^2y^2(t-\tau)}d\tau\right]dy=$$

$$\frac{\Theta}{\pi}\left\{\int_0^1\left[\frac{(\ln(y+e^\delta))^\delta}{(|\ln|h||)^\delta}\frac{1}{a^2y^2}\left|1-e^{-a^2y^2t_1}\right|+\right.\right.$$

$$\max(1,a^2)\frac{(\ln(y^2+e^\delta))^\delta}{(|\ln|h||)^\delta}\frac{1}{a^2y^2}\left|1-e^{-a^2y^2t_1}\right|$$

$$\left.+\int_{t_1}^t e^{-a^2y^2(t-\tau)}d\tau\right]dy+$$

$$\int_1^{+\infty}\left[\frac{(\ln(y+e^\delta))^\delta}{(|\ln|h||)^\delta}\frac{1}{a^2y^2}\left|1-e^{-a^2y^2t_1}\right|+\right.$$

$$\max(1,a^2)\frac{(\ln(y^2+e^\delta))^\delta}{(|\ln|h||)^\delta}\frac{1}{a^2y^2}\left|1-e^{-a^2y^2t_1}\right|+$$

$$\left.\left.\int_{t_1}^t e^{-a^2y^2(t-\tau)}d\tau\right]dy\right\}=\frac{\Theta}{\pi}\left(I_1+I_2\right).$$

Consider

$$I_1=\int_0^1\left[\frac{(\ln(y+e^\delta))^\delta}{(|\ln|h||)^\delta}\frac{1}{a^2y^2}\left|1-e^{-a^2y^2t_1}\right|+\right.$$

$$\max(1,a^2)\frac{(\ln(y^2+e^\delta))^\delta}{(|\ln|h||)^\delta}\frac{1}{a^2y^2}\left|1-e^{-a^2y^2t_1}\right|$$

$$\left.+\int_{t_1}^t e^{-a^2y^2(t-\tau)}d\tau\right]dy=$$

$$\frac{1}{a^2(|\ln|h||)^\delta}\int_0^1\frac{(\ln(y+e^\delta))^\delta}{y^2}\left|1-e^{-a^2y^2t_1}\right|dy+$$

$$\frac{\max(1,a^2)}{a^2(|\ln|h||)^\delta}\int_0^1\frac{(\ln(y^2+e^\delta))^\delta}{y^2}\left|1-e^{-a^2y^2t_1}\right|dy+$$

$$\int_0^1\left(\int_{t_1}^t e^{-a^2y^2(t-\tau)}d\tau\right)dy=$$

$$\frac{1}{a^2(|\ln|h||)^\delta}I_{11}+\frac{\max(1,a^2)}{a^2(|\ln|h||)^\delta}I_{12}+I_{13}.$$

Since $\left|1-e^{-a^2y^2t_1}\right|\leqslant a^2y^2t_1\leqslant a^2y^2T$, we have

$$I_{11}=\int_0^1\frac{(\ln(y+e^\delta))^\delta}{y^2}\left|1-e^{-a^2y^2t_1}\right|dy\leqslant$$

$$a^2T\int_0^1(\ln(y+e^\delta))^\delta dy=a^2TC_{11}.$$

$$I_{12}=\int_0^1\frac{\ln(y^2+e^\delta))^\delta}{y^2}\left|1-e^{-a^2y^2t_1}\right|dy\leqslant$$

$$a^2T\int_0^1(\ln(y^2+e^\delta))^\delta dy=a^2TC_{12}.$$

Using that $e^{-a^2y^2(t-\tau)}\leqslant 1$ i $t-t_1\leqslant h$, then the $\delta>0$ and for sufficiently small h, we have

$$I_{13}=\int_0^1\left(\int_{t_1}^t e^{-a^2y^2(t-\tau)}d\tau\right)dy\leqslant$$

$$\int_0^1(t-t_1)dy\leqslant|h|\leqslant\frac{1}{|\ln|h||^\delta}.$$

So we have

$$I_1 \leqslant \frac{1}{|\ln|h||^\delta}\left(TC_{11}+\max(1,a^2)TC_{22}+1\right).$$

$$I_2=\int_1^\infty \left[\frac{(\ln(y+e^\delta))^\delta}{(|\ln|h||)^\delta}\frac{1}{a^2y^2}\left|1-e^{-a^2y^2t_1}\right|+\right.$$

$$\max(1,a^2)\frac{(\ln(y^2+e^\delta))^\delta}{(|\ln|h||)^\delta}\frac{1}{a^2y^2}\left|1-e^{-a^2y^2t_1}\right|$$

$$\left.+\int_{t_1}^t e^{-a^2y^2(t-\tau)}d\tau\right]dy=$$

$$\frac{1}{a^2(|\ln|h||)^\delta}\times\int_1^\infty \frac{\ln(y+e^\delta))^\delta}{y^2}\left|1-e^{-a^2y^2t_1}\right|dy+$$

$$\frac{\max(1,a^2)}{a^2(|\ln|h||)^\delta}\times$$

$$\int_1^\infty \frac{\ln(y^2+e^\delta))^\delta}{y^2}\left|1-e^{-a^2y^2t_1}\right|dy+$$

$$\int_1^\infty\left(\int_{t_1}^t e^{-a^2y^2(t-\tau)}d\tau\right)dy=$$

$$\frac{1}{a^2(|\ln|h||)^\delta}I_{21}+\frac{\max(1,a^2)}{a^2(|\ln|h||)^\delta}I_{22}+I_{23}.$$

$$I_{21}=\int_1^{+\infty}\frac{(\ln(y+e^\delta))^\delta}{y^2}\left|1-e^{-a^2y^2t_1}\right|dy\leqslant$$

$$\int_1^{+\infty}\frac{(\ln(y+e^\delta))^\delta}{y^2}dy=C_{21}.$$

$$I_{22}=\int_1^{+\infty}\frac{(\ln(y^2+e^\delta))^\delta}{y^2}\left|1-e^{-a^2y^2t_1}\right|dy\leqslant$$

$$\int_1^{+\infty}\frac{(\ln(y^2+e^\delta))^\delta}{y^2}dy=C_{22}.$$

$$I_{23}=\int_1^{+\infty}\left(\int_{t_1}^t e^{-a^2y^2(t-\tau)}d\tau\right)dy=$$

$$\frac{1}{a^2}\int_1^{+\infty}\frac{1}{y^2}\left(1-e^{-a^2y^2(t-t_1)}\right)dy\leqslant$$

$$\frac{\max(1,a^2)}{a^2|(\ln|h||)^\delta}\int_1^{+\infty}\frac{(\ln(y^2+e^\delta))^\delta}{y^2}dy=$$

$$\frac{\max(1,a^2)}{a^2(|\ln|h||)^\delta}C_{23}.$$

Therefore

$$I_2\leqslant\frac{1}{a^2(|\ln|h||)^\delta}\left(C_{21}+\max(1,a^2)(C_{22}+C_{23})\right).$$

Then for $\delta>1-\frac{1}{p}$, we have

$$\left(E\left|u_n(x,t)-u_n(x_1,t_1)\right|^2\right)^{1/2}\leqslant\frac{C}{|\ln|h||^\delta},$$

where

$$C=\frac{\Theta}{\pi}\left(TC_{11}+\max(1,a^2)TC_{22}+1+\right.$$

$$\left.\frac{1}{a^2}\left(C_{21}+\max(1,a^2)(C_{22}+C_{23})\right)\right),$$

$C_{ij}, i=1,2,\ j=1,2,3$ are some constants.

□

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