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REGULAR FILTERS OF COMMUTATIVE BE-ALGEBRAS

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Abstract. The concept of regular filters is introduced in commutative BE-algebras. The class of all regular filters of a commutative BE-algebra is characterized in terms of dual annihilators. Some equivalent conditions are derived for every filter of a commutative BE-algebra to become a regular filter. Some properties of prime regular filters of a commutative BE-algebra are investigated.

Keywords: Commutative BE-algebra; regular filter; minimal prime filter; prime regular filter.

AMS Subject Classification: 03G25.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras [8]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [5, 6] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. J. Neggers and H. S. Kim [12] introduced the notion of a d-algebra which is a generalization of BCK-algebras, and also they introduced the notion of a B-algebra [13, 14], i.e., (I) $x*x = 0$; (II) $x*0 = x$; (III) $(x*y)*z = x*(z*(0*y))$, for any $x, y, z \in X$, which is equivalent in some sense to the groups. Moreover, Y.B. Jun, E.H. Roh and H.S. Kim [9] introduced a new notion, called an BH-algebra, which is another generalization of $BCH/BCI/BCK$-algebras, i.e., (I); (II) and (IV) $x * y = 0$ and $y * x = 0$ imply $x = y$ for any $x, y \in X$.

The notion of BE-algebras was introduced and extensively studied by H.S. Kim and Y.H. Kim in [10]. These classes of BE-algebras were introduced as a generalization of the class of BCK-algebras of K. Iseki and S. Tanaka [7]. Some properties of filters of

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BE-algebras were studied by S.S. Ahn and Y.H. Kim in [1] and by B.L. Meng in [11]. In [19], A. Walendziak discussed some properties of commutative BE-algebras. He also investigated the relationship between BE-algebras, implicative algebras and J-algebras. In [3], Chajda *et al.*, Characterized the complements and relative complements of the set of all deductive systems as the so-called annihilators of Hilbert algebras. Later, Halaš[4] introduced the concepts of an annihilator and a relative annihilator of a given subset of a BCK-algebra. In 2012, A. Rezaei, and A. Borumand Saeid [15], stated and proved the first, second and third isomorphism theorems in self distributive BE-algebras. Later, these authors [16] introduced the notion of commutative ideals in a BE-algebra. In 2013, A. Rezaei, and A. Borumand Saeid, and R.A. Borzooei [2] extensively studied the properties of some types of filters of BE-algebras. In [18], some properties of dual annihilator filters of commutative BE-algebras are studied. It is proved that the class of all dual annihilator filters of a BE-algebra is a complete Boolean algebra. A set of equivalent conditions is derived for every prime filter of a commutative BE-algebra to become a maximal filter.

In this paper, the notion of regular filters is introduced in commutative BE-algebras. The class of all regular filters of a commutative BE-algebra is then characterized in terms of dual annihilators. Some sufficient conditions are derived for a prime filter of a commutative BE-algebra to become a regular filter. A set of equivalent conditions are derived for every filter of a commutative BE-algebra to become a regular filter. Some properties of the set of all prime regular filters of a commutative BE-algebra are derived.

2. Preliminaries

In this section, we present certain definitions and results which are taken mostly from the papers [1], [10], [17] and [19] for the ready reference of the reader.

Definition 2.1. [10] *An algebra $(X, *, 1)$ of type $(2, 0)$ is called a BE-algebra if it satisfies the following properties:*

(1) $x * x = 1$,
(2) $x * 1 = 1$,
(3) $1 * x = x$,
(4) $x * (y * z) = y * (x * z)$ *for all* $x, y, z \in X$.

A BE-algebra X is called self-distributive if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$. A BE-algebra X is called transitive if $y * z \leq (x * y) * (x * z)$ for all $x, y, z \in X$. Every self-distributive BE-algebra is transitive. A BE-algebra X is called commutative if $(x * y) * y = (y * x) * x$ for all $x, y \in X$. Every commutative BE-algebra is transitive. For any $x, y \in X$, define $x \vee y = (y * x) * x$. If X is commutative then $(X, \vee)$ is a semilattice [19]. We introduce a relation $\leq$ on a BE-algebra X by $x \leq y$ if and only if $x * y = 1$ for all $x, y \in X$. Clearly $\leq$ is reflexive. If X is commutative, then $\leq$ is transitive, anti-symmetric and hence a partial order on X.

Theorem 2.1. [19] *If X is a commutative BE-algebra, then $x * y = 1$ and $y * x = 1$ imply that $x = y$ for all $x, y \in X$.*

Theorem 2.2. [11] *Let X be a transitive BE-algebra and $x, y, z \in X$. Then*

(1) $1 \leq x$ *implies* $x = 1$,
(2) $y \leq z$ *implies* $x * y \leq x * z$ *and* $z * x \leq y * x$.

Definition 2.2. [1] *A non-empty subset F of a BE-algebra X is called a filter of X if, for all $x, y \in X$, it satisfies the following properties:*

(1) $1 \in F$,

(2) $x \in F$ *and* $x * y \in F$ *imply that* $y \in F$.

For any non-empty subset A of a transitive BE-algebra X, the set $\langle A \rangle = \{x \in X \mid a_1 * (a_2 * (\cdots * (a_n * x) \cdots)) = 1$ for some $a_1, a_2, \ldots a_n \in A\}$ is the smallest filter containing A. For any $a \in X, \langle a \rangle = \{x \in X \mid a^n * x = 1$ for some $n \in \mathbb{N}\}$, where $a^n * x = a * (a * (\cdots * (a * x) \cdots))$ with the repetition of a is n times, is called the principal filter generated a. If X is self-distributive, then $\langle a \rangle = \{x \in X \mid a * x = 1\}$. A proper filter P of a BE-algebra is called prime [17] if $\langle x \rangle \cap \langle y \rangle \subseteq P$ implies $x \in P$ or $y \in P$ for any $x, y \in X$. A proper filter M of a transitive BE-algebra X is called maximal [17] if there exists no proper filter Q such that $M \subset Q$. Every maximal filter of a commutative BE-algebra is prime.

Theorem 2.3. [17] *Let X be a self-distributive BE-algebra and F a filter of X. Then for any $a, b \in X$, $\langle a \rangle \cap \langle b \rangle \subseteq F$ if and only if $\langle F \cup \{a\} \rangle \cap \langle F \cup \{b\} \rangle = F$.*

For any non-empty subset A of a commutative BE-algebra X, the dual annihilator [18] of A is defined as $A^+ = \{x \in X \mid x \vee a = 1$ for all $a \in A\}$. Clearly A^+ is a filter of X. Obviously $X^+ = \{1\}$ and $\{1\}^+ = X$. For $A = \{a\}$, we simply denote $\{a\}^+$ by $(a)^+$.

Proposition 2.1. [18] *For any two filters F, G of a commutative BE-algebra X, we have*

(1) $F \cap F^+ = \emptyset$,

(2) $F \subseteq F^{++}$,

(3) $F^{+++} = F^+$,

(4) $F \subseteq G$ *implies* $G^+ \subseteq F^+$,

(5) $(F \vee G)^+ = F^+ \cap G^+$,

(6) $(F \cap G)^{++} = F^{++} \cap G^{++}$.

Corollary 2.1. [18] *For any two elements a, b of a commutative BE-algebra X, we have*

(1) $(\langle a \rangle)^+ = (a)^+$,

(2) $\langle a \rangle \subseteq (a)^{++}$,

(3) $a \leq b$ *implies* $(a)^+ \subseteq (b)^+$.

3. Regular filters of BE-algebras

In this section, the notion of regular filters is introduced in commutative BE-algebras. The class of all regular filters of commutative BE-algebras is characterized in terms of dual annihilators. Some equivalent conditions are derived for every filter of a BE-algebra to become a regular filter.

Definition 3.1. *A filter F of a BE-algebra X is called a regular filter if $x \in F$ then $(x)^{++} \subseteq F$ for all $x \in X$.*

Clearly the filters $\{1\}$ and X are regular filters of X.

Example 3.1. *Let $X = \{1, a, b, c\}$ be a set. Define a binary operation $*$ on X as follows:*

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	a	b	1

$\vee$	1	a	b	c
1	1	1	1	1
a	1	a	1	1
b	1	1	b	1
c	1	1	1	c

Then clearly $(X, *, \vee, 1)$ is a commutative BE-algebra. Consider the subset $F = \{1, a, b\}$ of X. Clearly F is a filter of X. Now $(a)^+ = \{1, b, c\}$; $(b)^+ = \{1, a, c\}$ and $(c)^+ = \{1, a, b\}$. Hence $(a)^{++} = \{1, a\} \subset F$; $(b)^{++} = \{1, b\} \subset F$. Therefore F is a regular filter of X.

Lemma 3.1. *Let X be a commutative BE-algebra. Then for any $x, y, a \in X$, we have*
$$(x * y) \vee a \leq (x \vee a) * (y \vee a).$$

Proof. Let $a, x, y \in X$. Since X is commutative, we get

$$\begin{aligned}
((x * y) \vee a) * ((x \vee a) * (y \vee a)) &= ((x * y) \vee a) * (((x * a) * a) * ((y * a) * a)) \\
&= ((x * y) \vee a) * ((y * a) * (((x * a) * a) * a)) \\
&= ((x * y) \vee a) * ((y * a) * (x * a)) \\
&= (((x * y) * a) * a) * ((y * a) * (x * a)) \\
&= (y * a) * ((((x * y) * a) * a) * (x * a)) \\
&= (y * a) * (x * ((((x * y) * a) * a) * a)) \\
&= (y * a) * (x * ((x * y) * a)) \\
&= (y * a) * ((x * y) * (x * a)) \\
&= (x * y) * ((y * a) * (x * a)) \\
&= 1 \qquad \text{since } (x * y) \leq (y * a) * (x * a)
\end{aligned}$$

Therefore $(x * y) \vee a \leq (x \vee a) * (y \vee a)$. □

Theorem 3.1. *Let S be a $\vee$-closed subset of a commutative BE-algebra X. Then the set $F = \{x \in X \mid x \vee s = 1$ for some $s \in S\}$ is a regular filter of X.*

Proof. Clearly $1 \in F$. Let $x \in F$ and $x * y \in F$. Then $x \vee a = 1$ and $(x * y) \vee b = 1$ for some $a, b \in S$. Since S is $\vee$-closed, by the above lemma, we get

$$\begin{aligned}
1 &= (x * y) \vee b \\
&\leq (x * y) \vee (a \vee b) \\
&\leq (x \vee a \vee b) * (y \vee a \vee b) \\
&= (1 \vee b) * (y \vee a \vee b) \\
&= 1 * (y \vee a \vee b) \\
&= y \vee a \vee b
\end{aligned}$$

Since $a \vee b \in S$, we get $y \in F$. Hence F is a filter of X. Let $x \in F$. Then $s \vee x = 1$ for some $s \in S$. Hence $x \in (s)^+$. Thus $(x)^{++} \subseteq (s)^+$. If $t \in (x)^{++} \subseteq (s)^+$, then $s \vee t = 1$ and $s \in S$. Hence $t \in F$ and thus $(x)^{++} \subseteq F$. Therefore F is a regular filter of X. □

Proposition 3.1. *Every dual annihilator filter of a self-distributive and commutative BE-algebra is a regular filter.*

Proof. Let X be a self-distributive and commutative BE-algebra and F a dual annihilator of X. Then $F = S^+$ for some $\emptyset \neq S \subseteq X$. Let $x \in F = S^+$ be an arbitrary element. Then $(x)^{++} \subseteq S^{+++} = S^+ = F$. Therefore $F = S^+$ is a regular filter of X. □

Corollary 3.1. *Each dual annihilator $(a)^+, a \in X$ of a self-distributive and commutative BE-algebra is a regular filter.*

The converse of the above theorem is not true. For consider a proper regular filter $F \neq X$ satisfying the property $F^+ = \{1\}$ is not a dual annihilator because of $F^{++} = (F^+)^+ = (\{1\})^+ = X \neq F$. Hence F is not a dual annihilator. However, in the following, we derive a sufficient condition for a regular filter to become a dual annihilator filter.

Definition 3.2. *A filter F of a commutative BE-algebra is said to satisfy s-condition if to each $x \notin F$, there exists $y \in F$ such that $(x)^{++} = (y)^+$.*

Example 3.2. *Let $X = \{1, a, b, c\}$ be a set. Define a binary operation $*$ on X as follows:*

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	c
b	1	1	1	c
c	1	a	b	1

$\vee$	1	a	b	c
1	1	1	1	1
a	1	a	a	1
b	1	a	b	1
c	1	1	1	c

Then $(X, *, \vee, 1)$ is a commutative BE-algebra. Observe that $(a)^+ = (b)^+ = \{1, c\}$. and $(c)^+ = \{1, a, b\}$. Again $(a)^{++} = (b)^{++} = \{1, a, b\}$ and $(c)^{++} = \{1, c\} = (a)^+ = (b)^+$. Consider the filter $F = \{1, c\}$ of X. Clearly F is a regular filter of X. It can be easily verified that F is satisfying the s-condition.

Theorem 3.2. *Let F be a non-dense $(F^+ \neq \{1\})$ regular filter of a commutative BE-algebra X. If F satisfies the s-condition, then F is a dual annihilator filter of X.*

Proof. Let F be a non-dense regular filter of X. Assume that F satisfies the s-condition. Clearly $F \subseteq F^{++}$. Conversely, let $x \in F^{++}$. Then $F^+ \subseteq (x)^+$. Suppose $x \notin F$. Then by s-condition, we get $y \in F$ such that $(x)^{++} = (y)^+$. Since $y \in F$ and F is regular, we get $(y)^{++} \subseteq F$. Hence $F^+ \subseteq (x)^+ \subseteq (y)^{++} \subseteq F$. Thus $F^+ = F \cap F^+ = \{1\}$, which is contradiction. Thus $x \in F$, which gives $F^{++} \subseteq F$. Hence $F = F^{++}$. Therefore F is a dual annihilator filter of X. □

Theorem 3.3. *Let P be a non-dense $(P^+ \neq \{1\})$ regular filter of a commutative BE-algebra X. If P is prime, then P is a dual annihilator filter of X.*

Proof. Let P be a non-dense regular filter of X. Assume that P is prime. Clearly $P \subseteq P^{++}$. Conversely, let $x \in P^{++}$. Suppose that $x \notin P$. Since P is prime, we get $(x)^+ \subseteq P$. Since $x \in P^{++}$, we get $P^+ = P^{+++} \subseteq (x)^+ \subseteq P$. Hence $P^+ = P \cap P^+ = \{1\}$, which is a contradiction. Thus $x \in P$, which gives $P^{++} \subseteq P$. Hence $P = P^{++}$. Therefore P is a dual annihilator filter of X. □

Definition 3.3. *A prime filter P of a commutative BE-algebra is called minimal if there exists no prime filter Q of X such that $Q \subset P$.*

Example 3.3. *Let $X = \{1, a, b, c\}$ be a set. Define a binary operation $*$ on X as follows:*

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	c
b	1	1	1	c
c	1	a	b	1

$\vee$	1	a	b	c
1	1	1	1	1
a	1	a	a	1
b	1	a	b	1
c	1	1	1	c

Then $(X, *, \vee, 1)$ is a commutative BE-algebra. Consider the subset $P = \{1, c\}$ of X. Clearly P is a prime filter of X. Observe that the improper filter $\{1\}$ is not prime because of $a \vee c = 1 \in \{1\}$ but neither $a \in \{1\}$ nor $c \in \{1\}$. Therefore P is a minimal prime filter of X. Similarly, we observe that $G = \{1, a, b\}$ is another minimal prime filter of X.

Theorem 3.4. *A prime filter P of a self-distributive and commutative BE-algebra X is minimal if and only if for each $x \in P$ there exists $y \notin P$ such that $x \vee y = 1$.*

Proof. Assume that P is a minimal prime filter of X. Let $x \in P$. Put $S_0 = \{x \vee y \mid y \in X - P\}$. Consider $S = S_0 \cup (X - P)$ and $x \in S$. Then clearly S is a $\vee$-closed subset of X with $X - P \subseteq S$. Suppose $1 \notin S$. Let $\mathfrak{S} = \{M \mid M$ is a proper filter of X and $M \cap S = \emptyset\}$. Clearly $\langle 1 \rangle \in \mathfrak{S}$. Let $\{M\}_\alpha$ be a chain of elements of $\mathfrak{S}$. Clearly $\bigcup\limits_{\alpha} M_\alpha$ is an upper bound of $\{M\}_\alpha$ in $\mathfrak{S}$. Then by the Zorn's lemma, we get a maximal elements of $\mathfrak{S}$, say Q. We now show that Q is prime. Let $a, b \in X$ and suppose $a \notin Q$ and $b \notin Q$. Then $\langle Q \cup \{a\}\rangle = X$ and $\langle Q \cup \{b\}\rangle = X$. Hence $\langle Q \cup \{a\}\rangle \cap \langle Q \cup \{b\}\rangle = X \neq Q$. Then by Theorem 2.3, we get $\langle a\rangle \cap \langle b\rangle \nsubseteq Q$. Thus Q is a prime filter of X such that $Q \cap S = \emptyset$. Hence $Q \cap (X - P) = \emptyset$ and $x \notin Q$ because of $X - P \subseteq S$ and $x \in S$. Thus $Q \subsetneq P$, which is a contradiction to the minimality of P. Therefore $1 \in S = S_0 \cup (X - P)$. Since $1 \notin X - P$, we must have $1 \in S_0$. Hence $x \vee y = 1$ for some $y \in X - P$.

Conversely, assume that P is a prime filter of X satisfying the condition. Suppose Q is a prime filter of X such that $Q \subset P$. Choose $a \in P - Q$. By the condition, there exists $x \notin P$ such that $a \vee x = 1 \in Q$. Since Q is prime and $a \notin Q$, we must have $x \in Q \subset P$. This contradiction proves that P is a minimal prime filter of X. □

We now derive some sufficient conditions for a prime filter of a commutative BE-algebra to become a regular filter. In the following example, we observe that a prime filter of a commutative BE-algebra need not be a regular filter.

Example 3.4. *Let $X = \{1, a, b, c\}$ be a set. Define a binary operation $*$ on X as follows:*

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	b	1	1
c	1	c	c	1

$\vee$	1	a	b	c
1	1	1	1	1
a	1	a	b	c
b	1	b	b	c
c	1	c	c	c

Then clearly $(X, *, \vee, 1)$ is a commutative BE-algebra. Consider the subset $F = \{1, c\}$ of X. Clearly F is a prime filter of X. Now $(c)^+ = \{1\}$ and so $(c)^{++} = (1)^+ = X$. Hence $(c)^{++} \nsubseteq F$. Therefore F is not a regular filter of X.

Proposition 3.2. *Every minimal prime filter of a self-distributive and commutative BE-algebra is a regular filter.*

Proof. Let P be a minimal prime filter of a self-distributive and commutative BE-algebra X. Let $x \in P$. Since P is a minimal prime filter, there exists $y \notin P$ such that $x \vee y = 1$. Let $t \in (x)^{++}$. Then we get $(x)^+ \subseteq (t)^+$. Hence $y \in (x)^+ \subseteq (t)^+$. Thus $t \in (t)^{++} \subseteq (y)^+ \subseteq P$. Hence $(x)^{++} \subseteq P$. Therefore P is a regular filter of X. □

The converse of the above proposition is not true. That is, every regular filter of a commutative BE-algebra need not be a minimal prime filter. It can be seen in the following example:

Example 3.5. *Let $X = \{1, a, b, c, d\}$ be a set. Define a binary operation $*$ on X as follows:*

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	a	c	d
b	1	1	1	c	d
c	1	a	b	1	d
d	1	a	b	c	1

$\vee$	1	a	b	c	d
1	1	1	1	1	1
a	1	a	a	1	1
b	1	a	b	1	1
c	1	1	1	c	1
d	1	1	1	1	d

Then clearly $(X, *, \vee, 1)$ is a commutative BE-algebra. Consider the subset $F = \{1, a, b\}$ of X. Clearly F is a filter of X. Now $(a)^+ = \{1, c, d\}; (b)^+ = \{1, c, d\}; (c)^+ = \{1, a, b, d\}$ and $(d)^+ = \{1, a, b, c\}$. Hence $(a)^{++} = (b)^{++} = \{1, a, b\} = F$. Therefore F is a regular of X. Observe that F is not even a prime filter of X because of $c \vee d = 1 \in F$ but neither $c \in F$ nor $d \in F$.

Definition 3.4. *For any prime filter P of a commutative BE-algebra X, define $O(P) = \{x \in X \mid x \vee s = 1$ for some $s \notin P\}$.*

Example 3.6. *Let $X = \{1, a, b, c, d\}$ be a set. Define a binary operation $*$ on X as follows:*

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	d
b	1	c	1	c	d
c	1	b	b	1	d
d	1	a	b	c	1

$\vee$	1	a	b	c	d
1	1	1	1	1	1
a	1	a	b	c	1
b	1	b	b	1	1
c	1	c	1	c	1
d	1	1	1	1	d

Then clearly $(X, *, \vee, 1)$ is a commutative BE-algebra. Consider the subset $P = \{1, a, b, c\}$ of X. Clearly P is a prime filter of X. Now $O(P) = \{x \mid x \vee y = 1$ for some $y \notin P\} = \{1, a, b, c\}$. Now consider the filter $F = \{1, d\}$. Here $O(F) = \{1, b, c, d\}$, which is not a filter of X. Therefore a prime filter P is required to construct a proper regular filter $O(P)$.

Proposition 3.3. *For any prime filter P of a commutative BE-algebra $X, O(P)$ is a regular filter of X.*

Proof. Clearly $1 \in O(P)$. Let $x \in O(P)$ and $x*y \in O(P)$. Then $x \vee a = 1$ and $(x*y) \vee b = 1$ for some $a \notin P$ and $b \notin P$. Since P is prime, we get $a \vee b \notin P$. Then by the argument given in Theorem 3.1, we get $y \vee (a \vee b) = 1$. Since $a \vee b \notin P$, we get $y \in O(P)$. Therefore $O(P)$ is a filter of X. Now, let $x \in O(P)$. Then $x \vee s = 1$ for some $s \notin P$. Let $t \in (x)^{++}$. Then $s \in (x)^{+} \subseteq (t)^{+}$. Hence $t \vee s = 1$ where $s \notin P$. Thus $t \in O(P)$, which infers that $(x)^{++} \subseteq O(P)$. Therefore $O(P)$ is a regular filter of X. ☐

Proposition 3.4. *A prime filter P of a commutative BE-algebra X is a regular filter if it satisfies the property: $P^{+} \neq \{1\}$.*

Proof. Let P be a prime filter of X such that $P^{+} \neq \{1\}$. Let $x \in P$. Since $P^{+} \neq \{1\}$, there exists $1 \neq a \in P^{+}$. Hence $x \vee a = 1$. Since $P \cap P^{+} = \emptyset$, we get $a \notin P$. Now, let $t \in (x)^{++}$. Then $a \in (x)^{+} \subseteq (t)^{+}$, which gives $a \vee t = 1 \in P$. Since P is prime, and $a \notin P$, we must have $t \in P$. Hence $(x)^{++} \subseteq P$. Therefore P is a regular filter of X. ☐

We now derive a characterization theorem for regular filter of self-distributive and commutative BE-algebras. For this purpose, we first observe the following necessary results:

Proposition 3.5. *Let X be a self-distributive and commutative BE-algebra. Then for any $x \in X$, we have*

$$(x)^{+} = \cap\{P \mid P \text{ is a minimal prime filter such that } x \notin P\}$$

Proof. Let $a \in (x)^{+}$ and P be a minimal prime filter such that $x \notin P$. Then $x \vee a = 1 \in P$. Since $x \notin P$, we get $a \in P$ for all minimal prime filters with $x \notin P$. Hence $(x)^{+} \subseteq \cap\{P \mid P$ is a minimal prime filter such that $x \notin P\}$. Conversely, suppose that $t \notin (x)^{+}$. Then $t \vee x \neq 1$. Then there exists a prime filter say P such that $t \vee x \notin P$. Hence $t \notin P$. Let $\Im = \{Q \mid Q$ is a prime filter such that $t \notin Q\}$. Clearly $P \in \Im$. Let $\{Q_i\}_{i\in\Delta}$ be a chain in $\Im$. Clearly $\bigcap\limits_{i\in\Delta} Q_i$ is a lower bound for $\{Q_i\}_{i\in\Delta}$. Therefore by Zorn's Lemma, $\Im$ has a minimal element, say Q_0. Since Q_0 is an element of $\Im$, we get $t \notin Q_0$. Clearly Q_0 is a minimal prime filter such that $t \notin Q_0$. Hence

$$t \notin \bigcap\{P \mid P \text{ is a minimal prime filter such that } x \notin P\}$$

Therefore $\bigcap\{P \mid P$ is a minimal prime filter such that $x \notin P\} \subseteq (x)^{+}$. ☐

If X is a self-distributive BE-algebra, then clearly $\langle a\rangle = \{x \in X \mid a * x = 1\}$. Hence we have

Lemma 3.2. *If X is a self-distributive and commutative BE-algebra, then $\langle a\rangle \cap \langle b\rangle = \langle a \vee b\rangle$ for all $a, b \in X$.*

Proof. Clearly $\langle a \vee b\rangle \subseteq \langle a\rangle \cap \langle b\rangle$. Conversely, let $x \in \langle a\rangle \cap \langle b\rangle$. Then $a * x = 1$ and $b * x = 1$. Hence $a \leq x$ and $b \leq x$, which gives $a \vee b \leq x$. Thus $x \in \langle a \vee b\rangle$, which means $\langle a\rangle \cap \langle b\rangle \subseteq \langle a \vee b\rangle$. ☐

Lemma 3.3. *Let X be a self-distributive and commutative BE-algebra. Then for all $a, b \in X$*

$$(a \vee b)^{++} = (a)^{++} \cap (b)^{++}.$$

Proof. Let $a, b \in X$. Observed that $(a)^{++} \cap (b)^{++} = \langle a\rangle^{++} \cap \langle b\rangle^{++} = (\langle a\rangle \cap \langle b\rangle)^{++} = (\langle a \vee b\rangle)^{++} = (a \vee b)^{++}$. ☐

As a consequence of these results, a characterization theorem of regular filters is now derived in the following:

Theorem 3.5. *Let X be a self-distributive and commutative BE-algebra and F a filter of X. Then the following conditions are equivalent.*

(1) *F is a regular filter;*
(2) *For $x, y \in X, (x)^+ = (y)^+$ and $x \in F$ imply that $y \in F$;*
(3) *$F = \bigcup_{x \in F} (x)^{++}$;*
(4) *For $x, y \in X, h(x) = h(y)$ and $x \in F$ imply that $y \in F$ where $h(x) = \{P \in Minp(X) \mid x \in P\}$, $Minp(X)$ is the class of all minimal prime filters of X.*

Proof. (1) $\Rightarrow$ (2): Assume that F is a regular filter of X. Let $x, y \in X$ be such that $(x)^+ = (y)^+$ and $x \in F$. Since F is a regular filter, we get that $y \in (y)^{++} = (x)^{++} \subseteq F$.
(2) $\Rightarrow$ (3): Assume the condition (2). Let $x \in X$. Then we have $\langle x\rangle \subseteq (x)^{++}$. Hence $F = \bigcup_{x \in F} \langle x\rangle \subseteq \bigcup_{x \in F} (x)^{++}$. Conversely, let $a \in \bigcup_{x \in F} (x)^{++}$. Then $(a)^{++} \subseteq (x)^{++}$ for some $x \in F$. Hence $(a)^{++} = (a)^{++} \cap (x)^{++} = (a \vee x)^{++}$. Hence $(a)^+ = (a)^{+++} = (a \vee x)^{+++} = (a \vee x)^+$ and $a \vee x \in F$. Therefore by condition (2), it yields that $a \in F$.
(3) $\Rightarrow$ (1): Assume the condition (3). Let $a \in F$. Then $a \in (t)^{++}$ for some $t \in F$. Hence $(a)^{++} \subseteq (t)^{++} \subseteq \bigcup_{x \in F} (x)^{++} = F$. Therefore F is a regular filter of X.
(2) $\Leftrightarrow$ (4): It follows from the observation that $h(x) = h(y) \Leftrightarrow (x)^+ = (y)^+$. Assume that $h(x) = h(y)$. Then

$$\begin{aligned} a \in (x)^+ &\Leftrightarrow a \in P \text{ for all } P \in Minp(X) - h(x) \\ &\Leftrightarrow a \in P \text{ for all } P \in Minp(X) - h(y) \\ &\Leftrightarrow a \in (y)^+ \end{aligned}$$

Therefore we obtain that $(x)^+ = (y)^+$. Conversely, assume that $(x)^+ = (y)^+$. Then

$$\begin{aligned} P \in Minp(X) - h(x) &\Leftrightarrow P \notin h(x) \\ &\Leftrightarrow x \notin P \\ &\Leftrightarrow (x)^+ \subseteq P \\ &\Leftrightarrow (y)^+ \subseteq P \\ &\Leftrightarrow y \notin P \qquad \text{since } P \text{ is minimal} \\ &\Leftrightarrow P \in Minp(X) - h(y) \end{aligned}$$

Therefore $h(x) = h(y)$. Hence the theorem is proved. □

In [15], A. Rezaei and A.B. Saeid studied about the homomorphic images and inverse images of filters of a BE-algebras. In general, the inverse image of a regular filter of a commutative BE-algebras need not be a regular filter. However, in the following, we observe a necessary and sufficient condition for the inverse image of a regular filter of a commutative BE-algebra to become again a regular filter.

Theorem 3.6. *Let f be a homomorphism of a commutative BE-algebra X into another BE-algebra X'. If G is a regular filter of X' and f^{-1} exists, then $f^{-1}(G)$ is a regular filter of X if and only if $f^{-1}((x)^+)$ is a regular filter of X for each $x \in X'$.*

Proof. Assume that $f^{-1}(G)$ is a regular filter of X for each regular filter G of X'. Since $(x)^+$ is a regular filter of X' for each $x \in X'$, we get that $f^{-1}((x)^+)$ is a regular filter of X. Conversely, assume that $f^{-1}((x)^+)$ is a regular filter of X for each $x \in X'$. Let G be a regular filter of X'. Then clearly $f^{-1}(G)$ is a filter of X. Let $x, y \in X$ be such that $(x)^+ = (y)^+$ and $x \in f^{-1}(G)$. Then $f(x) \in G$. For any $a \in X'$, we get

$$\begin{aligned} a \in (f(x))^+ &\Leftrightarrow f(x) \in (a)^+ \\ &\Leftrightarrow x \in f^{-1}((a)^+) \\ &\Leftrightarrow y \in f^{-1}((a)^+) \quad f^{-1}((x)^+) \text{ is a regular filter of X} \\ &\Leftrightarrow f(y) \in (a)^+ \\ &\Leftrightarrow a \in (f(y))^+ \end{aligned}$$

Hence $(f(x))^+ = (f(y))^+$. Since $f(x) \in G$ and G is a regular filter, we get $f(y) \in G$. Thus $y \in f^{-1}(G)$. Therefore $f^{-1}(G)$ is a regular filter in X. □

Theorem 3.7. *Let X be a self-distributive and commutative BE-algebra X. Then the following conditions are equivalent.*

(1) *every filter is a regular filter;*
(2) *every principal filter is a regular filter;*
(3) *every prime filter is a regular filter;*
(4) *for $a, b \in X, (a)^+ = (b)^+$ implies $\langle a \rangle = \langle b \rangle$.*

Proof. (1) ⇒ (2): It is clear.
(2) ⇒ (3): Assume that every principal filter is a regular filter. Let P be a prime filter of X. Suppose $(a)^+ = (b)^+$ and $a \in P$. Then clearly $\langle a \rangle \subseteq P$. Since $a \in \langle a \rangle$ and $\langle a \rangle$ is a regular filter, we get that $(a)^{++} \subseteq \langle a \rangle \subseteq P$. Hence $b \in \langle b \rangle \subseteq (b)^{++} = (a)^{++} \subseteq P$. Therefore P is a regular filter.
(3) ⇒ (4): Assume that every prime filter of X is a regular filter. Let $a, b \in X$ such that $(a)^+ = (b)^+$. Suppose $\langle a \rangle \neq \langle b \rangle$. Without loss of generality assume that $\langle a \rangle \nsubseteq \langle b \rangle$. Consider $\Sigma = \{ F \in \mathcal{F}(X) \mid a \vee b \in F \text{ and } a \notin F \}$. Then clearly $\langle a \vee b \rangle \in \Sigma$. Let $\{F_i\}_{i \in \Delta}$ be a chain in Σ. Then clearly $\bigcup_{i \in \Delta} F_i$ is a filter, $a \vee b \in \bigcup_{i \in \Delta} F_i$ and $a \notin \bigcup_{i \in \Delta} F_i$. Hence $\bigcup_{i \in \Delta} F_i$ is an upper bound for $\{F_i\}_{i \in \Delta}$ in Σ. Therefore, by Zorn's Lemma, Σ has a maximal element, say P. We now prove that P is prime. Let $x, y \in X$ be such that $x \notin P$ and $y \notin P$. Hence $P \subset \langle P \cup \{x\} \rangle$ and $P \subset \langle P \cup \{y\} \rangle$. Therefore by the maximality of P, $\langle P \cup \{x\} \rangle$ and $\langle P \cup \{y\} \rangle$ are not in Σ. Hence $a \in \langle P \cup \{x\} \rangle$ and $a \in \langle P \cup \{y\} \rangle$. Therefore

$$a \in \langle P \cup \{x\} \rangle \cap \langle P \cup \{y\} \rangle$$

Since $a \notin P$, we get $\langle P \cup \{x\}\rangle \cap \langle P \cup \{x\}\rangle \neq P$. By Theorem 2.3, we get $\langle x \vee y\rangle = \langle x\rangle \cap \langle y\rangle \nsubseteq P$. Hence P is a prime filter. Therefore by condition (3), we get that P is a regular filter of X. Since $P \in \Sigma$, we get that $a \vee b \in P$ and $a \notin P$. Since P is prime, we get $b \in P$. Since $b \in P$ and P is a regular filter, we get $a \in P$, which is a contradiction to $a \notin P$. Therefore $\langle a\rangle = \langle b\rangle$.

(4) $\Rightarrow$ (1): Assume the condition (4). Let F be a filter of X. Suppose $a, b \in X$ be such that $(a)^+ = (b)^+$ and $a \in F$. Then $b \in \langle b\rangle = \langle a\rangle \subseteq F$. Hence F is a regular filter of X. □

4. Properties of prime regular filters

In this section, we derive some properties of the class of all prime regular filters of a commutative BE-algebra. A necessary and sufficient condition is derived for every prime regular filter to become maximal.

Let X be a commutative BE-algebra and $\mathcal{P}(X)$ denote the set of all prime regular filters of X. For any $S \subseteq X$, let $K(S) = \{P \in \mathcal{P}(X) \mid S \nsubseteq P\}$ and for any $x \in X; K(x) = K(\{x\})$. Then we have the following observations which can be verified directly.

Lemma 4.1. *Let X be a self-distributive and commutative BE-algebra and $x, y \in X$. Then the following conditions hold:*

(1) $\bigcup_{x \in X} K(x) = \mathcal{P}(X)$,

(2) $K(x) \cap K(y) = K(x \vee y)$,

(3) $K(x) = \emptyset \Leftrightarrow x = 1$,

(4) $K(x) = \mathcal{P}(X)$ *if and only if* $(x)^+ = \{1\}$.

Proposition 4.1. *Let X be a self-distributive and commutative BE-algebra and $x \in X$. Then the following conditions hold:*

(1) $K(\langle x\rangle) = K(x) = K((x)^{++})$

(2) $K((x)^{++}) \subseteq \mathcal{P}(X) - K((x)^+)$

Proof. (1). Let $P \in \mathcal{P}(X)$ be such that $P \in K(\langle x\rangle)$. Then we get $\langle x\rangle \nsubseteq P$. Hence there exists $y \in \langle x\rangle$ such that $y \notin P$. Hence $x \leq y$, which implies $x \vee y = y$. Suppose $x \in P$. Then we get $y = y \vee x \in P$, which is a contradiction. Thus $P \in K(x)$. Hence $K(\langle x\rangle) \subseteq K(x)$. Conversely, let $P \in \mathcal{P}(X)$ be such that $P \in K(x)$. Hence $x \notin P$. Thus $\langle x\rangle \nsubseteq P$. Therefore $P \in K(\langle x\rangle)$. Hence $K(\langle x\rangle) = K(x)$. We now prove that $K(\langle x\rangle) = K((x)^{++})$. Let $P \in \mathcal{P}(X)$ be such that $P \in K(\langle x\rangle)$. Then $\langle x\rangle \nsubseteq P$ and hence $(x)^{++} \nsubseteq P$ because of $\langle x\rangle \subseteq (x)^{++}$. Thus $P \in K((x)^{++})$. Therefore $K(\langle x\rangle) \subseteq K((x)^{++})$. Conversely, choose that $P \in \mathcal{P}(X)$ such that $P \in K((x)^{++})$. Hence $(x)^{++} \nsubseteq P$. Since P is a regular filter, we get $x \notin P$. Thus $P \in K(x)$. Therefore $K((x)^{++}) \subseteq K(x)$.

(2). Let $P \in \mathcal{P}(X)$ be such that $P \in K((x)^{++})$. Then we get $(x)^{++} \nsubseteq P$. Since P is a regular filter, we get that $x \notin P$. Hence $(x)^+ \subseteq P$, which infers that $P \notin K((x)^+)$. Thus it yields that $P \in \mathcal{P}(X) - K((x)^+)$. Therefore $K((x)^{++}) \subseteq \mathcal{P}(X) - K((x)^+)$. □

Definition 4.1. *A proper regular filter M of a commutative BE-algebra is called a maximal regular filter if there exists no proper regular filter M_0 such that $M \subset M_0$.*

Example 4.1. *In Example 3.1, the filter $F = \{1, a, b\}$ is a regular filter which maximal because of there exists no other proper filter which properly contains the regular filter F.*

Theorem 4.1. *Let X be a self-distributive and commutative BE-algebra. Then for any $P, Q \in \mathcal{P}(X)$ there exists $a, b \in X$ such that $P \in K(a) - K(b)$ and $Q \in K(b) - K(a)$ if and only if every prime regular filter is maximal.*

Proof. Assume that for any $P, Q \in \mathcal{P}(X)$ there exists $a, b \in X$ such that $P \in K(a) - K(b)$ and $Q \in K(b) - K(a)$. Let P_1 be a prime regular filter of X. Suppose there exists a proper regular filter P_2 of X such that $P_1 \subset P_2$. Since by the assumption, there exists two basic open sets $K(x)$ and $K(y)$ such that $P_1 \in K(x) - K(y)$ and $P_2 \in K(y) - K(x)$. Since $P_1 \notin K(y)$, we get $y \in P_1 \subset P_2$, which is a contradiction to that $P_2 \in K(y)$. Hence P_1 is a maximal regular filter.

Conversely, assume that every prime regular filter is a maximal regular filter. Let P and Q be two distinct elements of $\mathcal{P}(X)$. Hence by the assumption, both P and Q are maximal regular filters in X. Hence $P \nsubseteq Q$ and $Q \nsubseteq P$. Then there exists $a, b \in X$ such that $a \in P - Q$ and $b \in Q - P$. Hence $P \in K(b) - K(a)$ and $Q \in K(a) - K(b)$. Hence the proof is completed $\square$

Proposition 4.2. *Let X be a commutative BE-algebra. Then each dual annihilator $(x)^+$ is a direct factor of X if and only if $(x)^+ \vee (x)^{++} = X$ for all $x \in X$.*

Proof. Assume that each dual annihilator is a direct factor of X. Let $x \in X$. Then $(x)^+$ is a direct factor of X. Then there exists a filter G such that $(x)^+ \cap G = \{1\}$ and $(x)^+ \vee G = X$. Since $(x)^+ \cap G = \{1\}$, we get $G \subseteq (x)^{++}$. Hence $X = (x)^+ \vee G \subseteq (x)^+ \vee (x)^{++}$. Conversely, assume the condition. Let $x \in X$. Clearly $(x)^+ \cap (x)^{++} = \{1\}$ and by the assumption $(x)^+ \vee (x)^{++} = X$. Therefore $(x)^+$ is a direct factor of X. $\square$

Theorem 4.2. *Let X be a self-distributive and commutative BE-algebra such that each dual annihilator is a direct factor of X. Then for any two $P, Q \in \mathcal{P}(X)$ with $P \neq Q$, there exists $a, b \in X$ such that $P \in K(a)$ and $Q \in K(b)$ with $K(a) \cap K(b) = \emptyset$ if and only if for any two distinct prime regular filters P', Q' in X, there exists $a, b \in X$ such that $(a)^+ \subseteq P'$ and $(b)^+ \subseteq Q'$ and there does not exist any $R \in \mathcal{P}(X)$ such that $a \vee b \notin R$.*

Proof. Assume that for any two $P, Q \in \mathcal{P}(X)$ with $P \neq Q$, there exists $a, b \in X$ such that $P \in K(a)$ and $Q \in K(b)$ with $K(a) \cap K(b) = \emptyset$. Let P', Q' be two distinct prime regular filters of X. Then there exists two open sets $K(a)$ and $K(b)$ such that $P' \in K(a)$ and $Q' \in K(b)$ and $K(a) \cap K(b) = \emptyset$. Since $P' \in K(a) = K((a)^{++})$, we get $(a)^{++} \nsubseteq P'$. Choose $x \in (a)^{++}$ and $x \notin P'$. Hence $(a)^+ \subseteq (x)^+$ and $(x)^+ \subseteq P'$. Hence $(a)^+ \subseteq P$. Similarly, we obtain $(b)^+ \subseteq Q'$. Suppose there exists a prime regular filter R such that $a \vee b \notin R$. Then $R \in K(a \vee b) = K(a) \cap K(b)$, which is a contradiction to that $K(a) \cap K(b) = \emptyset$.

Conversely, assume the condition. Let P, Q be two distinct elements of $\mathcal{P}(X)$. Then by the assumption, there exists $a, b \in X$ such that $(a)^+ \subseteq P$ and $(b)^+ \subseteq Q$. Hence by the assumption, we get $(a)^+$ and $(b)^+$ are direct factors of X. Therefore by the above proposition, we get $(a)^+ \vee (a)^{++} = X = (b)^+ \vee (b)^{++}$. Suppose $a \in P$. Since P is a regular filter, we get $(a)^{++} \subseteq P$. Hence $X = (a)^+ \vee (a)^{++} \subseteq P$, which is a contradiction. Hence $a \notin P$. Similarly, we get $b \notin Q$. Hence $P \in K(a)$ and $Q \in K(b)$. Suppose $K(a) \cap K(b) \neq \emptyset$. Then there exists a prime regular filter R such that $R \in K(a) \cap K(b) = K(a \vee b)$. Hence $a \vee b \notin R$, which is a contradiction. Thus the proof is completed. $\square$

Theorem 4.3. *Let Y be a non-empty subset of $\mathcal{P}(X)$ such that $\bigcap_{P \in Y} P = \{1\}$. Then for any two $P, Q \in Y$ with $P \neq Q$, there exists $a, b \in X$ such that $P \in K(a)$ and $Q \in K(b)$ with $K(a) \cap K(b) = \emptyset$ if and only if for each $P \in Y, P$ is the unique member in Y that containing $O(P)$.*

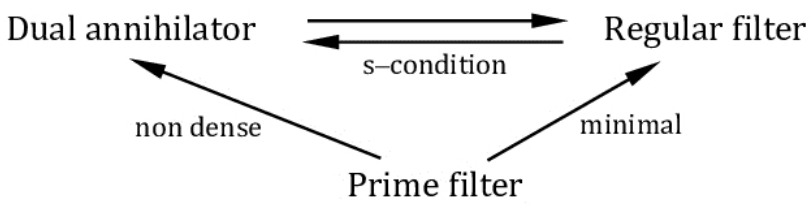
Proof. Assume that for any two $P, Q \in Y$ with $P \neq Q$, there exists $a, b \in X$ such that $P \in K(a)$ and $Q \in K(b)$ with $K(a) \cap K(b) = \emptyset$. Let $P \in Y$. Clearly $O(P) \subseteq P$. Suppose $Q \in Y$ such that $O(P) \subseteq Q$ and $P \neq Q$. Then there exist $a, b \in X$ such that $P \in K(a)$ and $Q \in K(b)$ and $K(a \vee b) = K(a) \cap K(b) \cap Y = \emptyset$. Hence $a \vee b \in R$ for all $R \in Y$.

Since $\bigcap_{P\in Y} P = \{1\}$, we get $a \vee b = 1$. Since $a \notin P$, we get $b \in O(P) \subseteq Q$, which is a contradiction to that $b \notin Q$. Hence P is the unique member in Y such that $O(P) \subseteq P$.

Conversely, assume that P is the unique member in Y such that $O(P) \subseteq P$. Let P_1 and P_2 be two distinct elements in Y. Hence by hypothesis, we get $O(P_1) \nsubseteq P_2$. Then choose $x \in O(P_1)$ such that $x \notin P_2$. Since $x \in O(P_1)$, there exists $y \notin P_1$ such that $x \vee y = 1$. Thus $P_1 \in K(y)$ and $P_2 \in K(x)$. Then clearly $K(x) \cap K(y) \cap Y = K(x \vee y) \cap Y = \emptyset$. $\square$

Conclusion: In this paper, we introduced the notion of regular filters of self-distributive and commutative BE-algebras and obtained some sufficient conditions for a prime filter to become a regular filter. In addition, we have established some equivalent conditions for a filter of a self-distributive and commutative BE-algebra to become a regular filter. We think such results are very useful for the further characterization of prime regular filters in terms of congruences of this structure.

Now, in the following diagram we summarize the results of this paper and the past results in this field and we give the relations among prime filters, dual annihilator filters, minimal prime filters. The mark $A \to B$ means that A implies B. A condition with the mark $A \to B$ indicates that A conclude B with the condition.



For the future research, we investigate some new filters of commutative BE-algebras with the help of dual annihilator filter and regular filters.

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References

[1] Ahn, S. S., Kim, Y. H. and Ko, J.M., (2012), Filters in commutative BE-algebras, Commun. Korean. Math. Soc., 27, no.2, pp. 233-242.

[2] Borumand Saeid, A., Rezaei, A., Borzooei, R. A., (2013), Some types of filters in BE-algebras, Math. Comput. Sci., 7, pp. 341-352.

[3] Chajda, I., Halaš, R. and Jun, Y. B., (2002), Annihilators and deductive systems in commutative Hilbert algebras, Comment. Math. Univ. Carol, 43 (3) pp. 407-417.

[4] Halaš, R., (2003), Annihilators in BCK-algebras,*Czech. Math.J.*, 53, (128), pp. 1001-1007.

[5] Hu, Q. P. and Li, X., (1983), On BCH-algebras, Math. Sem., Notes Kobe Univ., 11, no. 2, part 2, pp. 313-320.

[6] Hu, Q. P. and Li, X., (1985) On proper BCH-algebras, Math. Japon., 30, no. 4, pp. 659-661.

[7] Iseki, K. and Tanaka, S., (1979), An introduction to the theory of BCK-algebras, Math. Japon., 23, no.1, pp. 1-26.

[8] Iseki, K., (1980), On BCI-algebras, Math. Sem. Notes, Kobe Univ. 8, pp. 125-130.

[9] Jun, Y. B., Roh, E. H. and Kim, H. S., (1998) On BH-algebras, Sci. Math. Jpn., 1, no.3, pp. 347-354.

[10] Kim, H. S. and Kim, Y. H., (2006), On BE-algebras, Sci. Math. Japon. Online, pp. 1299-1302.

[11] Meng, B. L., (2010), On filters in BE-algebras, Sci. Math. Japon. Online, pp. 105-111.

[12] Neggers, J. and Kim, H. S., (1999), On d-algebras, Math. Slovaca, 49, no. 1 , pp. 19-26.

[13] Neggers, J. and Kim, H. S., (2002) On B-algebras, Mat. Vesnik, 54, no. 1-2, pp. 21-29.

[14] Neggers, J. and Kim, H. S., (2002) A fundamental theorem of B-homomorphism for B-algebras, Int. Math. J., 2, no.2, pp. 215-219.

[15] Rezaei, A. and Borumand Saeid, A., (2012), Some results in BE-algebras, Analele Universitatii Oradea Fasc. Matematica, Tom XIX, pp. 33-44.
[16] Rezaei, A. and Borumand Saeid, A., (2012), Commutative ideals in BE-algebras, Kyungpook Math. J., 52, pp. 483-494.
[17] Sambasiva Rao, M., (2012), Prime filters of commutative BE-algebras, J. Appl. Math. & Informatics, 33, No. 5-6, pp. 579-591
[18] Venkata Kumar, V. and Sambasiva Rao, M., (2017), Dual annihilator filters of commutative BE-algebras, Asian-European j. Math. 10, no.1, 1750013(11 pages)
[19] Walendziak, A., (2008), On commutative BE-algebras, Sci. Math. Japon. Online, pp. 585-588.

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