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SOME NEW HERMITE–HADAMARD INTEGRAL INEQUALITIES IN MULTIPLICATIVE CALCULUS

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Abstract. In this paper, we tend to establish some new Hermite–Hadamard type integral inequalities for multiplicatively convex function on coordinates and for product of two multiplicatively convex functions on coordinates.

Keywords: Multiplicative double integral – Logarithmically convex functions on coordinates – Hermite-Hadamard inequalities.

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1. Introduction

The class of convex functions is widely known in the literature and is generally defined as:

Definition 1.1. *Let a function $F : \rho \subseteq \mathbb{R} \to \mathbb{R}$, F is called a convex on ρ if we have the following inequality*

$$F(\tau\gamma + (1-\tau)\delta) \leq \tau F(\gamma) + (1-\tau)F(\delta), \ \forall\ \gamma, \delta \in \rho \text{ and } \tau \in [0,1]. \quad (1)$$

Note that F is also called concave if $-F$ is a convex.

Convex functions and their different forms are used to review a large category of problems that arises in varied branches of pure and applied sciences. This theory provides us a natural, unified and general framework to review a large category of unrelated problems. For recent applications, generalizations and alternative aspects of convex functions and

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their different forms, see [21] and the references therein.

The following inequality, named Hermite–Hadamard integral inequality, is one of the most popular inequality within the literature for convex functions.

Theorem 1.1. *Let $F : \rho \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and γ, $\delta \in I$ with $\gamma < \delta$. Then we have the following well known inequality:*

$$F\left(\frac{\gamma+\delta}{2}\right) \leq \frac{1}{\delta-\gamma}\int_{\gamma}^{\delta} F(x)dx \leq \frac{F(\gamma)+F(\delta)}{2}. \tag{2}$$

This inequality (2) is also called trapezium inequality.

The Hermite–Hadamard integral inequality has remained an area of good interest because of its large applications within the field of mathematical analysis. For details readers can read [1, 12, 16, 17, 18, 19] and references theirin.

Definition 1.2. *A positive function $F : \rho \subseteq \mathbb{R} \to (0, +\infty)$ is called logarithmically convex or simply log–convex on ρ, if we have the following inequality:*

$$F(\tau x_1 + (1-\tau)x_2) \leq [F(x_1)]^{\tau}[F(x_2)]^{1-\tau}, \ \forall \ x_1, x_2 \in \rho \ \ and \ \tau \in [0,1]. \tag{3}$$

Note that F is also said to be log–concave if (3) holds in reverse direction.

Definition 1.3. *Let a function $F : \Theta = [\gamma,\delta] \times [\mu, \nu] \subseteq \mathbb{R}^2 \to \mathbb{R}$, F is called convex on coordinates on Θ with $\gamma < \delta$ and $\mu < \nu$, if we have the following inequality:*

$$\begin{aligned} F(\tau x_1 + (1-\tau)z, \vartheta x_2 + (1-\vartheta)w) &\leq t\vartheta F(x_1, x_2) + \tau(1-\vartheta)F(x_1, w) \\ &+ (1-\tau)\vartheta F(z, x_2) + (1-\tau)(1-\vartheta)F(z, w) \end{aligned}$$

for all $\tau, \vartheta \in [0,1]$ and $(x_1, x_2), (z, w) \in \Theta$.

In [3], Alomari and Darus introduced a class of log–convex functions on coordinates as follows.

Definition 1.4. *Let a function $F : \Theta = [\gamma,\delta] \times [\mu, \nu] \subseteq \mathbb{R}^2 \to \mathbb{R}$, F is said to be log–convex on coordinates on Θ with $\gamma < \delta$ and $\mu < \nu$, if we have the following inequality:*

$$\begin{aligned} F(\tau x_1 + (1-\tau)z, \vartheta x_2 + (1-\vartheta)w) &\leq [F(x_1, x_2)]^{\tau\vartheta}[F(x_1, w)]^{\tau(1-\vartheta)} \\ &\times [F(z, x_2)]^{(1-\tau)\vartheta}[F(z, w)]^{(1-\tau)(1-\vartheta)} \end{aligned}$$

for all $\tau, \vartheta \in [0,1]$ and $(x_1, x_2), (z, w) \in \Theta$.

An inequality of the Hermite–Hadamard type was established by Alomari and Darus in [3] for log–convex functions on coordinates on a rectangle from the plane $\mathbb{R}^2$, see also [9, 20]. Grossman and Katz in [13] initiated the study of Non-Newtonian calculus and modified the classical calculus [14]. On the other hands, Bashirov et al. in [6] studied the concept of multiplicative calculus and presented a fundamental theorem of multiplicative calculus. Since then a number of interesting results has been obtained in this direction. For more discussion and applications of this discipline, we refer to [2, 4, 6, 7, 10, 22, 23]. Some elements of stochastic multiplicative calculus have been investigated in [11, 15]. Bashirov and Riza in [8] also studied complex multiplicative calculus. Recall that, multiplicative integral called *integral is denoted by $\int_{\gamma}^{\delta}(F(x_1))^{dx_1}$ whereas the ordinary integral is denoted by $\int_{\gamma}^{\delta} F(x_1)dx_1$. It is also known in [6], if F is positive and Riemann integrable on $[\gamma,\delta]$, then it is *integrable on $[\gamma,\delta]$ and

$$\int_{\gamma}^{\delta} (F(x_1))^{dx_1} = e^{\int_{\gamma}^{\delta} \ln(F(x_1))dx_1}.$$

In [5] Bashirov defined double integral that will be very useful to prove our results. Remember that the double multiplicative integral is denoted by

$$\iint_D (F(x,y))^{dA},$$

as long as the ordinary double integral define as

$$\iint_D F(x,y)dxdy.$$

The connection between multiplicative double integral and ordinary double integral is given below:

$$\iint_D (F(x,y))^{dA} = e^{\iint_D \ln(F(x,y))dxdy}.$$

The following results and notations are going to be required within the sequel.

1. $\iint_D ((F(x,y))^p)^{dA} = \left(\iint_D (F(x,y))^{dA}\right)^p$,
2. $\iint_D (F(x,y).g(x,y))^{dA} = \iint_D (F(x,y))^{dA}. \iint_D (g(x,y))^{dA}$,
3. $\iint_D \left(\frac{F(x,y)}{g(x,y)}\right)^{dA} = \frac{\iint_D (F(x,y))^{dA}}{\iint_D (g(x,y))^{dA}}$,
4. $\iint_D (F(x,y))^{dA} = \iint_{D_1} (F(x,y))^{dA}. \iint_{D_2} (F(x,y))^{dA}$, where $D = D_1 + D_2$.

The main objective of our this article is to prove Hermite–Hadamard type integral inequalities for multiplicatively convex function on coordinates and for product of two multiplicatively convex functions on coordinates.

2. Main Results

Theorem 2.1. *Let $F : \Theta = [\gamma,\delta]\times[\mu,\nu] \subseteq \mathbb{R}^2 \to \mathbb{R}_+$ for $\gamma < \delta$ and $\mu < \nu$ be multiplicatively convex function on coordinates Θ. Then following multiplicatively integral inequality hold:*

$$\left(\int_\mu^\nu \int_\gamma^\delta (F(x_1,x_2))^{dx_1dx_2}\right)^{\frac{1}{(\nu-\mu)(\delta-\gamma)}} \leq \frac{1}{2}\left[\left(\int_\gamma^\delta (G(F(x_1,\mu),F(x_1,\nu)))^{dx_1}\right)^{\frac{1}{\delta-\gamma}} + \left(\int_\mu^\nu (G(F(\gamma,x_2),F(\delta,x_2)))^{dx_2}\right)^{\frac{1}{\nu-\mu}}\right], \tag{4}$$

where $G(x_1,x_2)$ is the geometric mean.

Proof. Since $F(x_1, x_2)$ is multiplicatively convex function on coordinates and by setting $x_2 = \tau\mu + (1-\tau)\nu$ for all τ in $[0,1]$, we have

$$\begin{aligned}
&\left(\int_\mu^\nu \int_\gamma^\delta (F(x_1,x_2))^{dx_1dx_2}\right)^{\frac{1}{(\nu-\mu)(\delta-\gamma)}} = e^{\frac{1}{(\nu-\mu)(\delta-\gamma)}\int_\mu^\nu\int_\gamma^\delta \ln(F(x_1,x_2))dx_1dx_2} \\
= \;& e^{\frac{1}{\delta-\gamma}\int_0^1\int_\gamma^\delta \ln(F(x_1,\tau\mu+(1-\tau)\nu))dx_1d\tau} \\
\leq \;& e^{\frac{1}{\delta-\gamma}\int_0^1\int_\gamma^\delta \ln([F(x_1,\mu)]^\tau[F(x_1,\nu)]^{1-\tau})dx_1d\tau} \\
= \;& e^{\frac{1}{\delta-\gamma}\int_0^1\int_\gamma^\delta [\tau\ln(F(x_1,\mu))+(1-\tau)\ln F(x_1,\nu)]dx_1d\tau} \\
= \;& e^{\frac{1}{\delta-\gamma}\int_\gamma^\delta \ln(G(F(x_1,\mu),F(x_1,\nu)))dx_1} \\
= \;& \left(e^{\int_\gamma^\delta \ln(G(F(x_1,\mu),F(x_1,\nu)))dx_1}\right)^{\frac{1}{\delta-\gamma}} \\
= \;& \left(\int_\gamma^\delta (G(F(x_1,\mu),F(x_1,\nu)))^{dx_1}\right)^{\frac{1}{\delta-\gamma}}.
\end{aligned}$$

Hence

$$\left(\int_\mu^\nu \int_\gamma^\delta (F(x_1,x_2))^{dxdy}\right)^{\frac{1}{(\nu-\mu)(\delta-\gamma)}} \leq \left(\int_\gamma^\delta (G(F(x_1,\mu),F(x_1,\nu)))^{dx_1}\right)^{\frac{1}{\delta-\gamma}}. \tag{5}$$

Now, similarly by setting $x_1 = \tau\gamma + (1-\tau)\delta$, we have

$$\left(\int_\mu^\nu \int_\gamma^\delta (F(x_1,x_2))^{dxdy}\right)^{\frac{1}{(\nu-\mu)(\delta-\gamma)}} \leq \left(\int_\mu^\nu (G(F(\gamma,x_2),F(\delta,x_2)))^{dx_{12}}\right)^{\frac{1}{\nu-\mu}}. \tag{6}$$

By adding (5) and (6), we have inequality (4). □

Theorem 2.2. *Let $F : \Theta = [\gamma,\delta] \times [\mu,\nu] \subseteq \mathbb{R}^2 \to \mathbb{R}_+$ with $\gamma < \delta$ and $\mu < \nu$ be multiplicatively convex function on coordinates on Θ. Then we have the following multiplicative integral inequalities:*

$$\begin{aligned}
&F\left(\frac{\gamma+\delta}{2}, \frac{\mu+\nu}{2}\right) \\
\leq \;& \frac{1}{2}\left[\left(\int_\gamma^\delta F\left(x_1, \frac{\mu+\nu}{2}\right)^{dx_1}\right)^{\frac{1}{\delta-\gamma}} + \left(\int_\mu^\nu F\left(\frac{\gamma+\delta}{2}, x_2\right)^{dx_2}\right)^{\frac{1}{\nu-\mu}}\right] \\
\leq \;& \left(\int_\mu^\nu \int_\gamma^\delta (F(x_1,x_2))^{dx_1dx_2}\right)^{\frac{1}{(\delta-\gamma)(\nu-\mu)}}.
\end{aligned} \tag{7}$$

Proof. Since F is multiplicatively convex function, we have

$$\begin{aligned}
&F\left(\frac{\gamma+\delta}{2}, \frac{\mu+\nu}{2}\right) \\
= \;& F\left(\frac{1}{2}(\tau\gamma + (1-\tau)\delta + (1-\tau)\gamma + \tau\delta), \frac{1}{2}\left(\frac{\mu+\nu}{2} + \frac{\mu+\nu}{2}\right)\right) \\
\leq \;& \left[F\left(\tau\gamma + (1-\tau)\delta, \frac{\mu+\nu}{2}\right) F\left((1-\tau)\gamma + \tau\delta, \frac{\mu+\nu}{2}\right)\right]^{\frac{1}{2}}.
\end{aligned} \tag{8}$$

Integrating (8) w.r.t. τ on $[0,1]$, we have

$$\begin{aligned}
&F\left(\frac{\gamma+\delta}{2},\frac{\mu+\nu}{2}\right)\\
\leq\ & \int_0^1\left(\left[F\left(\tau\gamma+(1-\tau)\delta,\frac{\mu+\nu}{2}\right)F\left((1-\tau)\gamma+\tau\delta,\frac{\mu+\nu}{2}\right)\right]^{\frac{1}{2}}\right)^{d\tau}\\
=\ & e^{\int_0^1 \ln\left[F\left(\tau\gamma+(1-\tau)\delta,\frac{\mu+\nu}{2}\right)F\left((1-\tau)\gamma+\tau\delta,\frac{\mu+\nu}{2}\right)\right]^{\frac{1}{2}}d\tau}\\
=\ & e^{\int_0^1\left[\frac{1}{2}\ln(F(\tau\gamma+(1-\tau)\delta,\frac{\mu+\nu}{2}))+\frac{1}{2}\ln(F((1-\tau)\gamma+\tau\delta,\frac{\mu+\nu}{2}))\right]d\tau}\\
=\ & e^{\frac{1}{2}\int_0^1\ln(F(\tau\gamma+(1-\tau)\delta,\frac{\mu+\nu}{2}))d\tau+\frac{1}{2}\int_0^1\ln(F((1-\tau)\gamma+\tau\delta,\frac{\mu+\nu}{2}))d\tau}\\
=\ & e^{\frac{1}{\delta-\gamma}\int_\gamma^\delta \ln F(x_1,\frac{\mu+\nu}{2})dx_1}\\
=\ & \left(\int_\gamma^\delta\left(F\left(x_1,\frac{\mu+\nu}{2}\right)\right)^{dx_1}\right)^{\frac{1}{\delta-\gamma}}.
\end{aligned}$$

Hence

$$F\left(\frac{\gamma+\delta}{2},\frac{\mu+\nu}{2}\right)\leq\left(\int_\gamma^\delta\left(F\left(x_1,\frac{\mu+\nu}{2}\right)\right)^{dx_1}\right)^{\frac{1}{\delta-\gamma}}. \tag{9}$$

Similarly we can prove

$$F\left(\frac{\gamma+\delta}{2},\frac{\mu+\nu}{2}\right)\leq\left(\int_\mu^\nu\left(F\left(\frac{\gamma+\delta}{2},x_2\right)\right)^{dx_2}\right)^{\frac{1}{\nu-\mu}}. \tag{10}$$

Summing equation (9) and (10), we have the left hand inequality of (7).
Now we have to prove that the right hand inequality of (7). Since F is multiplicatively convex function, we have

$$F\left(x_1,\frac{\mu+\nu}{2}\right)\leq[F(x_1,\tau\mu+(1-\tau)\nu)F(x_1,(1-\tau)\gamma+\tau\delta)]^{\frac{1}{2}}. \tag{11}$$

By integrating (11) w. r. t. (x_1,τ) on $[\gamma,\delta]\times[0,1]$, we get

$$\begin{aligned}
&\left(\int_\gamma^\delta\left(F\left(x_1,\frac{\mu+\nu}{2}\right)\right)^{dx_1}\right)^{\frac{1}{\delta-\gamma}}\\
\leq\ & \left(\int_0^1\int_\gamma^\delta\left([F(x_1,\tau\mu+(1-\tau)\nu)F(x_1,(1-\tau)\gamma+\tau\delta)]^{\frac{1}{2}}\right)^{dx_1d\tau}\right)^{\frac{1}{\delta-\gamma}}\\
=\ & e^{\frac{1}{\delta-\gamma}\int_0^1\int_\gamma^\delta\ln\left([F(x_1,\tau\mu+(1-\tau)\nu)F(x_1,(1-\tau)\gamma+\tau\delta)]^{\frac{1}{2}}\right)dx_1d\tau}\\
=\ & e^{\frac{1}{2(\delta-\gamma)}\int_0^1\int_\gamma^\delta\ln(F(x_1,\tau\mu+(1-\tau)\nu))dx_1d\tau+\frac{1}{2(\delta-\gamma)}\int_0^1\int_\gamma^\delta\ln(F(x_1,(1-\tau)\gamma+\tau\delta))dx_1d\tau}\\
=\ & e^{\frac{1}{(\delta-\gamma)(\nu-\mu)}\int_\mu^\nu\int_\gamma^\delta\ln(F(x_1,x_2))dx_1dx_2}\\
=\ & \left(\int_\mu^\nu\int_\gamma^\delta(F(x_1,x_2))^{dx_1dx_2}\right)^{\frac{1}{(\delta-\gamma)(\nu-\mu)}}.
\end{aligned}$$

Hence

$$\left(\int_\gamma^\delta\left(F\left(x_1,\frac{\mu+\nu}{2}\right)\right)^{dx_1}\right)^{\frac{1}{\delta-\gamma}}\leq\left(\int_\mu^\nu\int_\gamma^\delta(F(x_1,x_2))^{dx_1dx_2}\right)^{\frac{1}{(\delta-\gamma)(\nu-\mu)}}. \tag{12}$$

Similarly we have

$$\left(\int_{\mu}^{\nu}\left(F\left(\frac{\gamma+\delta}{2},x_2\right)\right)^{dx_2}\right)^{\frac{1}{\nu-\mu}} \leq \left(\int_{\mu}^{\nu}\int_{\gamma}^{\delta}(F(x_1,x_2))^{dx_1dx_2}\right)^{\frac{1}{(\delta-\gamma)(\nu-\mu)}}. \qquad (13)$$

Adding (12) and (13) and using the resulting inequality in the left hand inequality of (7), then we have the right hand inequality of (7) that is desired inequality. □

Theorem 2.3. *Let $F,\ h:\Theta=[\gamma,\delta]\times[\mu,\nu]\subseteq\mathbb{R}^2\to\mathbb{R}_+$ for $\gamma<\delta$ and $\mu<\nu$ are multiplicatively convex functions on coordinates Θ. Then following multiplicatively integral inequality hold:*

$$\begin{aligned}&\left(\int_{\mu}^{\nu}\int_{\gamma}^{\delta}(F(x_1,x_2)h(x_1,x_2))^{dx_1dx_2}\right)^{\frac{1}{(\nu-\mu)(\delta-\gamma)}}\\ \leq\ &\frac{1}{2}\left[\left(\int_{\gamma}^{\delta}(G(F(x_1,\mu)h(x_1,\mu),F(x_1,\nu)h(x_1,\nu)))^{dx_1}\right)^{\frac{1}{\delta-\gamma}}\right.\\ &\left.+\left(\int_{\mu}^{\nu}(G(F(\gamma,x_2)h(\gamma,x_2),F(\delta,x_2)h(\delta,x_2)))^{dx_2}\right)^{\frac{1}{\nu-\mu}}\right],\end{aligned} \qquad (14)$$

where $G(x_1,x_2)$ is the geometric mean.

Proof. Since F,h are multiplicatively convex functions, we have

$$\left(\int_{\mu}^{\nu}\int_{\gamma}^{\delta}(F(x_1,x_2)h(x_1,x_2))^{dx_1dx_2}\right)^{\frac{1}{(\nu-\mu)(\delta-\gamma)}} = e^{\frac{1}{(\nu-\mu)(\delta-\gamma)}\int_{\mu}^{\nu}\int_{\gamma}^{\delta}\ln(F(x_1,x_2)h(x_1,x_2))dx_1dx_2}, \qquad (15)$$

by setting $x_2=\tau\mu+(1-\tau)\nu$ in (15), we get

$$\begin{aligned}&\left(\int_{\mu}^{\nu}\int_{\gamma}^{\delta}(F(x_1,x_2)h(x_1,x_2))^{dx_1dx_2}\right)^{\frac{1}{(\nu-\mu)(\delta-\gamma)}}\\ =\ &e^{\frac{1}{(\delta-\gamma)}\int_0^1\int_{\gamma}^{\delta}\ln(F(x_1,\tau\mu+(1-\tau)\nu)h(x_1,\tau\mu+(1-\tau)\nu))dx_1d\tau}\\ \leq\ &e^{\frac{1}{\delta-\gamma}\int_0^1\int_{\gamma}^{\delta}\ln(F((x_1,\mu))^{\tau}(F(x_1,\nu))^{1-\tau})+\ln(h((x_1,\mu))^{\tau}(h(x_1,\nu))^{1-\tau})dx_1d\tau}\\ =\ &\exp\left[\frac{1}{\delta-\gamma}\int_0^1\int_{\gamma}^{\delta}\tau\ln F(x_1,\mu)+(1-\tau)\ln F(x_1,\nu)dx_1d\tau\right.\\ &\left.+\frac{1}{\delta-\gamma}\int_0^1\int_{\gamma}^{\delta}\tau\ln h(x_1,\mu)+(1-\tau)\ln h(x_1,\nu)dx_1d\tau\right]\\ =\ &e^{\frac{1}{(\delta-\gamma)}\int_{\gamma}^{\delta}(\ln\hbar(F(x_1,\mu)h(x_1,\mu),F(x_1,\nu)h(x_1,\nu)))dx_1}\\ =\ &\left(\int_{\gamma}^{\delta}(G(F(x_1,\mu)h(x_1,\mu),F(x_1,\nu)h(x_1,\nu)))^{dx_1}\right)^{\frac{1}{(\delta-\gamma)}}.\end{aligned}$$

Hence

$$\begin{aligned}&\left(\int_{\mu}^{\nu}\int_{\gamma}^{\delta}(F(x_1,x_2)h(x_1,x_2))^{dx_1dx_2}\right)^{\frac{1}{(\nu-\mu)(\delta-\gamma)}}\\ \leq\ &\left(\int_{\gamma}^{\delta}(G(F(x_1,\mu)h(x_1,\mu),F(x_1,\nu)h(x_1,\nu)))^{dx_1}\right)^{\frac{1}{(\delta-\gamma)}}.\end{aligned} \qquad (16)$$

Similarly by setting $x_1 = \tau\gamma + (1-\tau)\delta$ in (15), we get

$$\begin{aligned} &\left(\int_\mu^\nu \int_\gamma^\delta (F(x_1,x_2)h(x_1,x_2))^{dx_1dx_2}\right)^{\frac{1}{(\nu-\mu)(\delta-\gamma)}} \\ \leq\ & \left(\int_\mu^\nu (\hbar(F(\gamma,x_2)h(\gamma,x_2),F(\delta,x_2)h(\delta,x_2)))^{dx_2}\right)^{\frac{1}{(\delta-\gamma)}}. \end{aligned} \tag{17}$$

By adding inequalities (16) and (17) we have the desired inequality (14). □

Theorem 2.4. *Let $F,\ h : \Theta = [\gamma,\delta] \times [\mu,\nu] \subseteq \mathbb{R}^2 \to \mathbb{R}_+$ with $\gamma < \delta$ and $\mu < \nu$ are multiplicatively convex functions on coordinates on Θ. Then we have the following multiplicative integral inequalities:*

$$\begin{aligned} & F\left(\frac{\gamma+\delta}{2},\frac{\mu+\nu}{2}\right) h\left(\frac{\gamma+\delta}{2},\frac{\mu+\nu}{2}\right) \\ \leq\ & \frac{1}{2}\left[\left(\int_\gamma^\delta \left(F\left(x_1,\frac{\mu+\nu}{2}\right) h\left(x_1,\frac{\mu+\nu}{2}\right)\right)^{dx_1}\right)^{\frac{1}{\delta-\gamma}}\right. \\ & \left. + \left(\int_\mu^\nu \left(F\left(\frac{\gamma+\delta}{2},x_2\right) h\left(\frac{\gamma+\delta}{2},x_2\right)\right)^{dx_1}\right)^{\frac{1}{\nu-\mu}}\right] \\ \leq\ & \left(\int_\mu^\nu \int_\gamma^\delta (F(x_1,x_2)h(x_1,x_2))^{dx_1dx_2}\right)^{\frac{1}{(\delta-\gamma)(\nu-\mu)}}. \end{aligned} \tag{18}$$

Proof. Since F and h are multiplicatively convex functions and by using the definition of multiplicatively convex function we have

$$\begin{aligned} & F\left(\frac{\gamma+\delta}{2},\frac{\mu+\nu}{2}\right) h\left(\frac{\gamma+\delta}{2},\frac{\mu+\nu}{2}\right) \\ =\ & F\left(\frac{1}{2}(\tau\gamma+(1-\tau)\delta+(1-\tau)\gamma+\tau\delta),\frac{1}{2}\left(\frac{\mu+\nu}{2}+\frac{\mu+\nu}{2}\right)\right) \\ \times\ & h\left(\frac{1}{2}(\tau\gamma+(1-\tau)\delta+(1-\tau)\gamma+\tau\delta),\frac{1}{2}\left(\frac{\mu+\nu}{2}+\frac{\mu+\nu}{2}\right)\right) \\ \leq\ & \left[F\left(\tau\gamma+(1-\tau)\delta,\frac{\mu+\nu}{2}\right) F\left((1-\tau)\gamma+\tau\delta,\frac{\mu+\nu}{2}\right)\right]^{\frac{1}{2}} \\ \times\ & \left[h\left(\tau\gamma+(1-\tau)\delta,\frac{\mu+\nu}{2}\right) h\left((1-\tau)\gamma+\tau\delta,\frac{\mu+\nu}{2}\right)\right]^{\frac{1}{2}}. \end{aligned} \tag{19}$$

Integrating (19) w.r.t. τ on $[0,1]$, we have

$$\begin{aligned}
& F\left(\frac{\gamma+\delta}{2},\frac{\mu+\nu}{2}\right)h\left(\frac{\gamma+\delta}{2},\frac{\mu+\nu}{2}\right)\\
\leq\ & \int_0^1\left(\left[F\left(\tau\gamma+(1-\tau)\delta,\frac{\mu+\nu}{2}\right)F\left((1-\tau)\gamma+\tau\delta,\frac{\mu+\nu}{2}\right)\right]^{\frac{1}{2}}\right)^{d\tau}\\
& \times\int_0^1\left(\left[h\left(\tau\gamma+(1-\tau)\delta,\frac{\mu+\nu}{2}\right)h\left((1-\tau)\gamma+\tau\delta,\frac{\mu+\nu}{2}\right)\right]^{\frac{1}{2}}\right)^{d\tau}\\
=\ & \exp\left[\int_0^1\ln\left(\left[F\left(\tau\gamma+(1-\tau)\delta,\frac{\mu+\nu}{2}\right)F\left((1-\tau)\gamma+\tau\delta,\frac{\mu+\nu}{2}\right)\right]^{\frac{1}{2}}\right.\right.\\
& \left.\left.\times\left[h\left(\tau\gamma+(1-\tau)\delta,\frac{\mu+\nu}{2}\right)h\left((1-\tau)\gamma+\tau\delta,\frac{\mu+\nu}{2}\right)\right]^{\frac{1}{2}}\right)d\tau\right]\\
=\ & \exp\left[\int_0^1\left[\frac{1}{2}\ln(F(\tau\gamma+(1-\tau)\delta,\frac{\mu+\nu}{2}))+\frac{1}{2}\ln(F((1-\tau)\gamma+\tau\delta,\frac{\mu+\nu}{2}))\right]d\tau\right.\\
& \left.+\int_0^1\left[\frac{1}{2}\ln(h(\tau\gamma+(1-\tau)\delta,\frac{\mu+\nu}{2}))+\frac{1}{2}\ln(h((1-\tau)\gamma+\tau\delta,\frac{\mu+\nu}{2}))\right]d\tau\right]\\
=\ & \exp\left[\frac{1}{2}\int_0^1\ln(F(\tau\gamma+(1-\tau)\delta,\frac{\mu+\nu}{2}))d\tau+\frac{1}{2}\int_0^1\ln(F((1-\tau)\gamma+\tau\delta,\frac{\mu+\nu}{2}))d\tau\right.\\
& \left.+\frac{1}{2}\int_0^1\ln(h(\tau\gamma+(1-\tau)\delta,\frac{\mu+\nu}{2}))d\tau+\frac{1}{2}\int_0^1\ln(h((1-\tau)\gamma+\tau\delta,\frac{\mu+\nu}{2}))d\tau\right]\\
=\ & e^{\frac{1}{\delta-\gamma}\int_\gamma^\delta\ln F(x_1,\frac{\mu+\nu}{2})h(x_1,\frac{\mu+\nu}{2})dx_1}\\
=\ & \left(\int_\gamma^\delta\left(F\left(x_1,\frac{\mu+\nu}{2}\right)h\left(x_1,\frac{\mu+\nu}{2}\right)\right)^{dx_1}\right)^{\frac{1}{\delta-\gamma}}.
\end{aligned}$$

Hence

$$F\left(\frac{\gamma+\delta}{2},\frac{\mu+\nu}{2}\right)h\left(\frac{\gamma+\delta}{2},\frac{\mu+\nu}{2}\right)\leq\left(\int_\gamma^\delta\left(F\left(x_1,\frac{\mu+\nu}{2}\right)h\left(x_1,\frac{\mu+\nu}{2}\right)\right)^{dx_1}\right)^{\frac{1}{\delta-\gamma}}. \tag{20}$$

Similarly we can prove

$$F\left(\frac{\gamma+\delta}{2},\frac{\mu+\nu}{2}\right)h\left(\frac{\gamma+\delta}{2},\frac{\mu+\nu}{2}\right)\leq\left(\int_\mu^\nu\left(F\left(\frac{\gamma+\delta}{2},x_2\right)h\left(\frac{\gamma+\delta}{2},x_2\right)\right)^{dx_2}\right)^{\frac{1}{\nu-\mu}}. \tag{21}$$

By adding (20) and (21), we have left hand inequality of (18).
Now we have to prove the right hand inequality of (18). Since F,h are multiplicatively convex functions, we get

$$\begin{aligned}
& F\left(x_1,\frac{\mu+\nu}{2}\right)h\left(x_1,\frac{\mu+\nu}{2}\right)\\
\leq\ & [F(x_1,\tau\mu+(1-\tau)\nu)F(x_1,(1-\tau)\gamma+\tau\delta)]^{\frac{1}{2}}\\
\times\ & [h(x_1,\tau\mu+(1-\tau)\nu)h(x_1,(1-\tau)\gamma+\tau\delta)]^{\frac{1}{2}}.
\end{aligned} \tag{22}$$

By integrating (22) w. r. t. (x_1, τ) on $[\gamma,\delta] \times [0, 1]$, we get

$$\begin{aligned}
&\left(\int_\gamma^\delta \left(F\left(x_1, \frac{\mu+\nu}{2}\right)\right)^{dx_1}\right)^{\frac{1}{\delta-\gamma}} \\
\leq\ & \int_0^1 \int_\gamma^\delta \left(\left([F(x_1, \tau\mu + (1-\tau)\nu)F(x_1, (1-\tau)\gamma + \tau\delta)]^{\frac{1}{2}}\right)^{dx_1 d\tau}\right)^{\frac{1}{\delta-\gamma}} \\
& \int_0^1 \int_\gamma^\delta \left(\left([h(x_1, \tau\mu + (1-\tau)\nu)h(x_1, (1-\tau)\gamma + \tau\delta)]^{\frac{1}{2}}\right)^{dx_1 d\tau}\right)^{\frac{1}{\delta-\gamma}} \\
=\ & \exp\left[\frac{1}{\delta-\gamma}\int_0^1 \int_\gamma^\delta \ln\left([F(x_1, \tau\mu + (1-\tau)\nu)F(x_1, (1-\tau)\gamma + \tau\delta)]^{\frac{1}{2}}\, dx_1 d\tau\right)\right. \\
& \left.\frac{1}{\delta-\gamma}\int_0^1 \int_\gamma^\delta \left([h(x_1, \tau\mu + (1-\tau)\nu)h(x_1, (1-\tau)\gamma + \tau\delta)]^{\frac{1}{2}}\, dx_1 d\tau\right)\right] \\
=\ & \left[\frac{1}{2(\delta-\gamma)}\int_0^1 \int_\gamma^\delta \ln(F(x_1, \tau\mu + (1-\tau)\nu)h(x_1, \tau\mu + (1-\tau)\nu))dxdt\right. \\
& \left. + \frac{1}{2(\delta-\gamma)}\int_0^1 \int_\gamma^\delta \ln(F(x_1, (1-\tau)\gamma + \tau\delta)h(x_1, (1-\tau)\gamma + \tau\delta))dx_1 d\tau\right] \\
=\ & e^{\frac{1}{(\delta-\gamma)(\nu-\mu)}\int_\mu^\nu \int_\gamma^\delta \ln(F(x_1,x_2)h(x_1,x_2))dx_1 dx_2} \\
=\ & \left(\int_\mu^\nu \int_\gamma^\delta (F(x_1, x_2)h(x_1, x_2))^{dx_1 dx_2}\right)^{\frac{1}{(\delta-\gamma)(\nu-\mu)}}.
\end{aligned}$$

Hence

$$\left(\int_\gamma^\delta \left(F\left(x_1, \frac{\mu+\nu}{2}\right)\right)^{dx_1}\right)^{\frac{1}{\delta-\gamma}} \leq \left(\int_\mu^\nu \int_\gamma^\delta (F(x_1, x_2)h(x_1, x_2))^{dx_1 dx_2}\right)^{\frac{1}{(\delta-\gamma)(\nu-\mu)}}. \quad (23)$$

In similar way we can prove

$$\left(\int_\gamma^\delta \left(F\left(\frac{\gamma+\delta}{2}, x_2\right)\right)^{dx_2}\right)^{\frac{1}{\nu-\mu}} \leq \left(\int_\mu^\nu \int_\gamma^\delta (F(x_1, x_2)h(x_1, x_2))^{dx_1 dx_2}\right)^{\frac{1}{(\delta-\gamma)(\nu-\mu)}}. \quad (24)$$

By adding inequalities (23) and (24), we have the right hand inequality of (18). □

Theorem 2.5. *Let $F : \Theta = [\gamma,\delta]\times[\mu, \nu] \subseteq \mathbb{R}^2 \to \mathbb{R}_+$ for $\gamma < \delta$ and $\mu < \nu$ be multiplicatively convex function on coordinates Θ. Then the following multiplicatively integral inequality hold:*

$$\begin{aligned}
&\left(\int_\mu^\nu \int_\gamma^\delta \left(\frac{F(x_1, x_2)}{h(x_1, x_2)}\right)^{dx_1 dx_2}\right)^{\frac{1}{(\nu-\mu)(\delta-\gamma)}} \leq \frac{1}{2}\left[\left(\int_\gamma^\delta \left(\frac{G(F(x_1, \mu), F(x_1, \nu))}{G(h(x_1, \mu), h(x_1, \nu))}\right)^{dx_1}\right)^{\frac{1}{\delta-\gamma}}\right. \\
&\left. + \left(\int_\mu^\nu \left(\frac{G(F(\gamma, x_2), F(\delta, x_2))}{G(h(\gamma, x_2), h(\delta, x_2))}\right)^{dx_2}\right)^{\frac{1}{\nu-\mu}}\right], \quad (25)
\end{aligned}$$

where $G(x_1, x_2)$ is the geometric mean.

Proof. We can easily prove our this result by using the idea of Theorem 2.3. □

Theorem 2.6. *Let $F,\ h:\Theta=[\gamma,\delta]\times[\mu,\nu]\subseteq\mathbb{R}^2\to\mathbb{R}_+$ with $\gamma<\delta$ and $\mu<\nu$ are multiplicatively convex functions on coordinates on Θ. Then we have the following multiplicative integral inequalities:*

$$\begin{aligned}
&\frac{F\left(\frac{\gamma+\delta}{2},\frac{\mu+\nu}{2}\right)}{h\left(\frac{\gamma+\delta}{2},\frac{\mu+\nu}{2}\right)}\\
\leq\ &\frac{1}{2}\left[\left(\int_\gamma^\delta\left(\frac{F\left(x_1,\frac{\mu+\nu}{2}\right)}{h\left(x_1,\frac{\mu+\nu}{2}\right)}\right)^{dx_1}\right)^{\frac{1}{\delta-\gamma}}+\left(\int_\mu^\nu\left(\frac{F\left(\frac{\gamma+\delta}{2},x_2\right)}{h\left(\frac{\gamma+\delta}{2},x_2\right)}\right)^{dx_2}\right)^{\frac{1}{\nu-\mu}}\right]\\
\leq\ &\left(\int_\mu^\nu\int_\gamma^\delta\left(\frac{F(x_1,x_2)}{h(x_1,x_2)}\right)^{dx_1dx_2}\right)^{\frac{1}{(\delta-\gamma)(\nu-\mu)}}. \qquad (26)
\end{aligned}$$

Proof. We can easily prove our this result by using the idea of Theorem 2.4. □

3. Conclusion

In this paper, we established some new Hermite–Hadamard integral inequalities in multiplicative calculus. Interested reader can derive more inequalities of this type in multiplicative calculus by using different convexities and approaches.

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