TWMS J. App. and Eng. Math. V.11, N.3, 2021, pp. 932-946

METHOD BASED ON QUASI VARIABLE MESH FOR SOLUTION OF SYSTEM OF SECOND ORDER BOUNDARY VALUE PROBLEMS WITH MIXED BOUNDARY CONDITIONS

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Abstract. A new numerical method with third order accuracy is presented for the solution of nonlinear two point boundary value problems(BVPs) with mixed boundary conditions using quasi variable mesh. In case of uniform mesh, method becomes fourth order. The method has been extended to vector form. Error analysis of the proposed scheme using a model problem is discussed. Application to fourth order nonlinear boundary value problem in coupled form is discussed. The proposed method is tested on two examples of linear and nonlinear BVPs and comparison with uniform mesh method has been made to prove the accuracy of the method.

Keywords: Quasi-variable mesh, Mixed boundary conditions, System, Nonlinear.

AMS Subject Classification: 65L10.

1. Introduction

Consider the following system of M nonlinear two point boundary value problems(BVPs)

$$u_{xx}{}^{(i)} = f^{(i)}(x, u^{(1)}, u^{(2)}, .., u^{(i)}, .., u^{(M)}, u_x{}^{(1)}, u_x{}^{(2)}, .., u_x{}^{(i)}, ., u_x{}^{(M)}), \tag{1}$$

subject to mixed boundary conditions

$$\alpha_0^{(i)} u^{(i)}(0) - \alpha_1^{(i)} u_x{}^{(i)}(0) = A_i \tag{2}$$

$$\beta_0^{(i)} u^{(i)}(1) + \beta_1^{(i)} u_x{}^{(i)}(1) = B_i, \tag{3}$$

where $u_x{}^{(i)} = \frac{d}{dx} u^{(i)}, u_{xx}{}^{(i)} = \frac{d^2}{dx^2} u^{(i)}$ and $\alpha_0^{(i)}, \alpha_1^{(i)}, \beta_0^{(i)}, \beta_1^{(i)} > 0, \alpha_0^{(i)} + \alpha_1^{(i)} > 0$, $\beta_0^{(i)} + \beta_1^{(i)} > 0$, $\alpha_0^{(i)} + \beta_0^{(i)} > 0$ for $i = 1(1)M, M \in Z^+$. We assume following conditions

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§ Manuscript received: August 31, 2019; accepted: January 07, 2020.

([9])which assure the existence of unique solution of the system (1) − (3). Let $-\infty < u^{(i)}, {u_x}^{(i)} < \infty$ such that :

$(i) f^{(i)}(x, u^{(1)}, u^{(2)}, ..., u^{(i)}, ..., u^{(M)}, {u_x}^{(1)}, {u_x}^{(2)}, ..., {u_x}^{(i)}, ..., {u_x}^{(M)})$ are continuous;

$(ii) \dfrac{\partial f^{(i)}}{\partial u^{(j)}}$ and $\dfrac{\partial f^{(i)}}{\partial {u_x}^{(j)}}$ exist and are continuous;

$(iii) \displaystyle\sum_{i,j=1}^{n} \frac{\partial f^{(i)}}{\partial u^{(j)}} > 0$ and $|\dfrac{\partial f^{(i)}}{\partial {u_x}^{(j)}}| \le C,$ for some positive constant C and $i, j = 1(1)M$

Many authors have studied the existence of solutions of BVPs with mixed boundary conditions([4],[10],[19]) and simultaneous efforts have been made to solve such problems. Usmani [20] developed a finite difference method of order four to solve second order boundary value problems without significant derivative. Chawla[3]developed a uniform mesh discretization scheme at the boundary as well as interior points of the domain to solve second order BVPs with mixed boundary conditions. Rashidinia et. al. [13] developed non polynomial spline method of order two and four to solve second order nonlinear singular boundary value problems. Also, more recently two parameter alternating group explicit (TAGE) and Newton–TAGE iteration method based on variable mesh was proposed by Mohanty et. al.[12]to solve the nonlinear BVPs with mixed boundary conditions.
In this paper, we have developed a scheme using quasi-variable mesh to solve two point BVPs $(1)-(3)$. We have used quasi variable mesh as truncation error in a finite-difference method depends upon the derivative of the variable as well as mesh size. Thus, in region with large deviated derivatives a fine mesh distributes the truncation error uniformly and accordingly coarse meshes are used for smooth functions[7]. We applied the method on linear as well as nonlinear problems. Several higher order problems which can be decomposed into system $(1) - (3)$ can be efficiently solved by using the proposed scheme. The scheme discretizes the problem at the interior as well as boundary points and resultant linear and nonlinear systems are solved by block Gauss elimination and block Newton's method resp. ([1],[2]). To the best of authors knowledge no such third order variable mesh discretization scheme for solving such a system $(1) - (3)$ is known in the literature so far.

There are six sections in this paper. In section 2, we give derivation of the scheme and in section 3, vector convergence of the proposed scheme is provided. In section 4, we provide its application to a fourth order BVP. Finally in section 5, two examples are considered and numerical results are shown to prove the efficiency of the proposed methods and in section 6, we provide concluding remarks.

2. Derivation of the Scheme

We consider a coupled nonlinear BVP of the following type:

$$u_{xx} = f(x, u(x), v(x), u_x(x), v_x(x)) \tag{4}$$

$$v_{xx} = g(x, u(x), v(x), u_x(x), v_x(x)) \tag{5}$$

subject to :

$$\alpha_0^{(1)} u(0) - \alpha_1^{(1)} u_x(0) = A_1, \beta_0^{(1)} u(1) + \beta_1^{(1)} u_x(1) = B_1 \tag{6}$$

$$\alpha_0^{(2)} v(0) - \alpha_1^{(2)} v_x(0) = A_2, \beta_0^{(2)} v(1) + \beta_1^{(2)} v_x(1) = B_2 \tag{7}$$

where f, g are continuous in $[0,1]$ and $\alpha_0^{(i)}, \alpha_1^{(i)}, \beta_0^{(i)}, \beta_1^{(i)} > 0$, also $\alpha_0^{(1)} + \alpha_1^{(i)} > 0$, $\beta_0^{(i)} + \beta_1^{(i)} > 0$, $\alpha_0^{(i)} + \beta_0^{(i)} > 0, i = 1, 2$. Now, to derive the scheme we first discretise the solution interval [0,1] into N subintervals using nodal points $x_k - x_{k-1} = h_k, k = 1(1)N$ where h_k be the mesh size and the mesh ratio be $\sigma_k = \frac{h_{k+1}}{h_k} > 0, k = 1, 2, 3, ..., N-1$. When $\sigma = 1$, the mesh reduces to a uniform mesh i.e., $h_{k+1} = h_k = h$. In this paper, $\sigma_k \equiv \sigma$ as the considered mesh is quasi variable. Also, assume U_k, V_k and u_k, v_k be the approximate and exact solution of $u(x), v(x)$ for the problem $(4)-(7)$ at the grid points $x_k, k = 0, 1, 2, ..., N$. Then, we follow the scheme [3] and derive discretization schemes at boundary points of the domain. Thus, for the coupled nonlinear BVP $(4)-(7)$, we use the following approximations and schemes to evaluate $u(x)$ and $v(x)$ at the end points x_0 and x_N:

$$u_{x_0} = \frac{\alpha_0^{(1)}}{\alpha_1^{(1)}} u_0 - \frac{A_1}{\alpha_1^{(1)}}, \tag{8}$$

$$v_{x_0} = \frac{\alpha_0^{(2)}}{\alpha_1^{(2)}} v_0 - \frac{A_2}{\alpha_1^{(2)}}, \tag{9}$$

$$f_0 = f(x_0, u_0, v_0, u_{x_0}, v_{x_0}), \tag{10}$$

$$g_0 = g(x_0, u_0, v_0, u_{x_0}, v_{x_0}), \tag{11}$$

$$\bar{u}_{\frac{1}{2}} = u_0 + \frac{h_1}{2} u_{x_0} + \frac{h_1^2}{8} f_0, \tag{12}$$

$$\bar{v}_{\frac{1}{2}} = v_0 + \frac{h_1}{2} v_{x_0} + \frac{h_1^2}{8} g_0, \tag{13}$$

$$\bar{u}_{x_{\frac{1}{2}}} = \frac{3}{4h_1}(u_1 - u_0) + \frac{1}{4} u_{x0} + \frac{h_1}{8} f_0, \tag{14}$$

$$\bar{v}_{x_{\frac{1}{2}}} = \frac{3}{4h_1}(v_1 - v_0) + \frac{1}{4} v_{x0} + \frac{h_1}{8} g_0, \tag{15}$$

$$\bar{f}_{\frac{1}{2}} = f(x_{\frac{1}{2}}, \bar{u}_{\frac{1}{2}}, \bar{v}_{\frac{1}{2}}, \bar{u}_{x_{\frac{1}{2}}}, \bar{v}_{x_{\frac{1}{2}}}), \tag{16}$$

$$\bar{g}_{\frac{1}{2}} = g(x_{\frac{1}{2}}, \bar{u}_{\frac{1}{2}}, \bar{v}_{\frac{1}{2}}, \bar{u}_{x_{\frac{1}{2}}}, \bar{v}_{x_{\frac{1}{2}}}), \tag{17}$$

$$u_1 = u_0 + h_1 u_{x_0} + \frac{h_1^2}{6}\left[f_0 + 2\bar{f}_{\frac{1}{2}}\right] + T_0^3(h_1), \tag{18}$$

$$v_1 = v_0 + h_1 v_{x_0} + \frac{h_1^2}{6}\left[g_0 + 2\bar{g}_{\frac{1}{2}}\right] + T_0^3(h_1), \tag{19}$$

$$u_{x_N} = \frac{B_1}{\beta_1^{(1)}} - \frac{\beta_0^{(1)}}{\beta_1^{(1)}} u_N, \tag{20}$$

$$v_{x_N} = \frac{B_2}{\beta_1^{(2)}} - \frac{\beta_0^{(2)}}{\beta_1^{(2)}} v_N, \tag{21}$$

$$f_N = f(x_N, u_N, v_N, u_{x_N}, v_{x_N}), \tag{22}$$
$$g_N = g(x_N, u_N, v_N, u_{x_N}, v_{x_N}), \tag{23}$$
$$\bar{u}_{N-\frac{1}{2}} = u_N - \frac{h_{N-1}}{2} u_{x_N} + \frac{h_{N-1}^2}{8} f_N, \tag{24}$$
$$\bar{v}_{N-\frac{1}{2}} = v_N - \frac{h_{N-1}}{2} v_{x_N} + \frac{h_{N-1}^2}{8}\ g_N, \tag{25}$$
$$\bar{u}_{x_{N-\frac{1}{2}}} = \frac{3}{4h_{N-1}}(u_N - u_{N-1}) + \frac{1}{4} u_{x_N} - \frac{h_{N-1}}{8} f_N, \tag{26}$$
$$\bar{v}_{x_{N-\frac{1}{2}}} = \frac{3}{4h_{N-1}}(v_N - v_{N-1}) + \frac{1}{4} v_{x_N} - \frac{h_{N-1}}{8} g_N, \tag{27}$$
$$\bar{f}_{N-\frac{1}{2}} = f(x_{N-\frac{1}{2}}, \bar{u}_{N-\frac{1}{2}}, \bar{v}_{N-\frac{1}{2}}, \bar{u}_{x_{N-\frac{1}{2}}}, \bar{v}_{x_{N-\frac{1}{2}}}), \tag{28}$$
$$\bar{g}_{N-\frac{1}{2}} = g(x_{N-\frac{1}{2}}, \bar{u}_{N-\frac{1}{2}}, \bar{v}_{N-\frac{1}{2}}, \bar{u}_{x_{N-\frac{1}{2}}}, \bar{v}_{x_{N-\frac{1}{2}}}), \tag{29}$$
$$u_N = u_{N-1} + h_{N-1} u_{x_N} - \frac{h_{N-1}^2}{6}\left(\bar{f}_N + 2\bar{f}_{N-\frac{1}{2}}\right) + T_N^3(h_{N-1}), \tag{30}$$
$$v_N = v_{N-1} + h_{N-1} v_{x_N} - \frac{h_{N-1}^2}{6}\left(\bar{g}_N + 2\bar{g}_{N-\frac{1}{2}}\right) + T_N^3(h_{N-1}). \tag{31}$$

Now we follow Mohanty et.al[11] and derive the discretization scheme for $(4)-(7)$ at the interior points $x_k, k = 1(1)N-1$. The approximations used to evaluate $v(x)$ as well as $u(x)$ are as follows:

$$S = \sigma(\sigma+1), \tag{32}$$
$$P = \sigma^2 + \sigma - 1, \tag{33}$$
$$Q = (1+\sigma)(\sigma^2 + 3\sigma + 1), \tag{34}$$
$$R = \sigma(1 + \sigma - \sigma^2), \tag{35}$$
$$\bar{v}_{x_k} = \frac{v_{k+1} + (\sigma^2 - 1)v_k - \sigma^2 v_{k-1}}{h_k S}, \tag{36}$$
$$\bar{f}_k = f(x_k, u_k, v_k, \bar{u}_{x_k}, \bar{v}_{x_k}), \tag{37}$$
$$\bar{g}_k = g(x_k, u_k, v_k, \bar{u}_{x_k}, \bar{v}_{x_k}), \tag{38}$$
$$\bar{v}_{x_{k-1}} = \frac{-v_{k+1} + (1+\sigma)^2 v_k - \sigma(2+\sigma)v_{k-1}}{h_k S}, \tag{39}$$
$$\bar{v}_{x_{k+1}} = \frac{(1+2\sigma)v_{k+1} - (1+\sigma)^2 v_k + \sigma^2 v_{k-1}}{h_k S}, \tag{40}$$
$$\bar{f}_{k\pm1} = f(x_{k\pm1}, u_{k\pm1}, v_{k\pm1}, \bar{u}_{x_{k\pm1}}, \bar{v}_{x_{k\pm1}}), \tag{41}$$
$$\bar{g}_{k\pm1} = g(x_{k\pm1}, u_{k\pm1}, v_{k\pm1}, \bar{u}_{x_{k\pm1}}, \bar{v}_{x_{k\pm1}}), \tag{42}$$
$$\hat{u}_{x_k} = \bar{u}_{x_k} + \mu_1 h_k[\bar{f}_{k+1} - \bar{f}_{k-1}], \tag{43}$$
$$\hat{v}_{x_k} = \bar{v}_{x_k} + \mu_2 h_k[\bar{g}_{k+1} - \bar{g}_{k-1}],\ \mu_1, \mu_2 \text{ are to be determined}, \tag{44}$$
$$\hat{f}_k = f(x_k, u_k, v_k, \hat{u}_{x_k}, \hat{v}_{x_k}), \tag{45}$$

$$\hat{g}_k = f(x_k, u_k, v_k, \hat{u}_{x_k}, \hat{v}_{x_k}), \tag{46}$$

$$\sigma u_{k-1} - (1+\sigma)u_k + u_{k+1} = \frac{h_k^2}{12}(P\bar{f}_{k+1} + Q\hat{f}_k + R\bar{f}_{k-1}) + T_k^3(h_k), \tag{47}$$

$$\sigma v_{k-1} - (1+\sigma)v_k + v_{k+1} = \frac{h_k^2}{12}(P\bar{g}_{k+1} + Q\hat{g}_k + R\bar{g}_{k+1}) + T_k^3(h_k). \tag{48}$$

Simplifying the approximations $(36), (39)-(40), (42), (44), (46)$ for v , we get

$$\bar{v}_{x_k} = v_{x_k} + \frac{1}{6}\sigma h_k^2 v_{xxx_k} + O(h_k^3), \tag{49}$$

$$\bar{v}_{x_{k+1}} = v_{x_{k+1}} - \frac{1}{6}\sigma(1+\sigma)h_k^2 v_{xxx_k} + O(h_k^3), \tag{50}$$

$$\bar{v}_{x_{k-1}} = v_{x_{k-1}} - \frac{1}{6}(1+\sigma)h_k^2 v_{xxx_k} + O(h_k^3), \tag{51}$$

$$\bar{g}_{k+1} = g_{k+1} - \frac{1}{6}\sigma(1+\sigma)h_k^2 v_{xxx_k} G + O(h_k^3), \tag{52}$$

$$\bar{g}_{k-1} = g_{k-1} - \frac{1}{6}(1+\sigma)h_k^2 v_{xxx_k} G + O(h_k^3), \tag{53}$$

$$\hat{v}_{x_k} = v_{x_k} + [\frac{1}{6}\sigma + \mu_2(1+\sigma)]h_k^2 v_{xxx_k} + O(h_k^3), \tag{54}$$

$$\hat{g}_k = g_k - [\frac{1}{6}\sigma + \mu_2(1+\sigma)]h_k^2 v_{xxx_k} G + O(h_k^3), \text{where } G = \frac{\partial g}{\partial v_{x_k}}, \tag{55}$$

Using $(52)-(53), (55)$ in (48), we get:

$$\sigma v_{k-1} - (1+\sigma)v_k + v_{k+1} = \frac{h_k^2}{12}(P\bar{g}_{k+1} + Q\hat{g}_k + R\bar{g}_{k-1})$$
$$+ \frac{h_k^4}{72}[\sigma(1+\sigma)P - (\sigma + \mu_2(1+\sigma))Q + (1+\sigma)R]v_{xxx_k} G + T_k^3(h_k). \tag{56}$$

To make the proposed equation(56) of $O(h_k^5)$, the coefficients of h_k^4 is equated to zero, hence we get $\mu_2 = \frac{-\sigma(1+\sigma+\sigma^2)}{6Q}$ and the local truncation error becomes $T_k^3(h_k) = T_k^3 = O(h_k^5)$. Hence, we get the following two equations:

$$\sigma u_{k-1} - (1+\sigma)u_k + u_{k+1} = \frac{h_k^2}{12}(P\bar{f}_{k+1} + Q\hat{f}_k + R\bar{f}_{k-1}) + O(h_k^5), \tag{57}$$

$$\sigma v_{k-1} - (1+\sigma)v_k + v_{k+1} = \frac{h_k^2}{12}(P\bar{g}_{k+1} + Q\hat{g}_k + R\bar{g}_{k-1}) + O(h_k^5). \tag{58}$$

Similarily, we can derive $\mu_1 = \frac{-\sigma(1+\sigma+\sigma^2)}{6Q}$. The same truncation error for uniform mesh becomes $O(h^6)$. Moreover, a condition required for convergence of the scheme[6] is satisfied as the coefficients P, Q, R are positive for $\frac{(\sqrt{5}-1)}{2} < \sigma < \frac{(\sqrt{5}+1)}{2}$. Also, since U_k, V_k are the approximate solution of $(4)-(7)$ respectively, using $(18)-(19), (30)-(31), (57)-(58)$ we get the following three point discretization scheme:

$$U_1 = U_0 + h_1 U_{x_0} + \frac{h_1^2}{6}\big[f_0 + 2\bar{f}_{\frac{1}{2}}\big], \tag{59}$$

$$V_1 = V_0 + h_1 V_{x_0} + \frac{h_1^2}{6}\big[g_0 + 2\bar{g}_{\frac{1}{2}}\big], \tag{60}$$

$$\sigma U_{k-1} - (1+\sigma)U_k + U_{k+1} = \frac{h_k^2}{12}(P\bar{f}_{k+1} + Q\hat{f}_k + R\bar{f}_{k-1}), \tag{61}$$

$$\sigma V_{k-1} - (1+\sigma)V_k + V_{k+1} = \frac{h_k^2}{12}(P\bar{g}_{k+1} + Q\hat{g}_k + R\bar{g}_{k-1}), \tag{62}$$

$$U_N = U_{N-1} + h_{N-1}U_{x_N} - \frac{h_{N-1}^2}{6}\big(\bar{f}_N + 2\bar{f}_{N-\frac{1}{2}}\big), \tag{63}$$

$$V_N = V_{N-1} + h_{N-1}V_{x_N} - \frac{h_{N-1}^2}{6}\big(\bar{g}_N + 2\bar{g}_{N-\frac{1}{2}}\big). \tag{64}$$

Now, further if $\alpha_1^{(1)}$ or $\alpha_1^{(2)} = 0$ then we use $u_0 = \frac{A_1}{\alpha_0^{(1)}}, v_0 = \frac{A_2}{\alpha_0^{(2)}}$ in (18) – (19), else if both are non zero then we substitute $u_{x_0} = \frac{\alpha_0^{(1)}u_0 - A_1}{\alpha_1^{(1)}}, v_{x_0} = \frac{\alpha_0^{(2)}v_0 - A_2}{\alpha_1^{(2)}}$ respectively. Similarly, if $\beta_1^{(1)}$ or $\beta_1^{(2)} = 0$ then $u_N = \frac{B_1}{\beta_0^{(1)}}, v_N = \frac{B_2}{\beta_0^{(2)}}$ in (30) – (31) or else if both are non zero then we substitute $u_{x_N} = \frac{B_1 - \beta_0^{(1)}u_N}{\beta_1^{(1)}}, v_{x_N} = \frac{B_2 - \beta_0^{(2)}v_N}{\beta_1^{(2)}}$. Thus, depending on the values of $\alpha_1^{(1)}, \alpha_1^{(2)}, \beta_1^{(1)}$ and $\beta_1^{(2)}$ whether all or either of them or none of them are zero we solve a $N-1 \times N-1$ or $N \times N$ or $N+1 \times N+1$ tri-diagonal system [29].

3. Application of the scheme

We consider a fourth order nonlinear BVP of the following type:

$$u_{xxxx} = f(x, u(x), u_x(x), u_{xx}(x), u_{xxx}(x)), \tag{65}$$

subject to boundary conditions:

$$\alpha_0^{(1)}u(0) - \alpha_1^{(1)}u_x(0) = A_1, \beta_0^{(1)}u(1) + \beta_1^{(1)}u_x(1) = B_1, \tag{66}$$

$$\alpha_0^{(2)}u_{xx}(0) - \alpha_1^{(2)}u_{xxx}(0) = A_2, \beta_0^{(2)}u_{xx}(1) + \beta_1^{(2)}u_{xxx}(1) = B_2. \tag{67}$$

where f is continuous in $[0,1]$ and $\alpha_0^{(i)}, \alpha_1^{(i)}, \beta_0^{(i)}, \beta_1^{(i)} > 0, \alpha_0^{(i)} + \alpha_1^{(i)} > 0$, $\beta_0^{(i)} + \beta_1^{(i)} > 0$, $\alpha_0^{(i)} + \beta_0^{(i)} > 0, i = 1,2$ We decompose the problem (65) into a system of second order BVPs

$$\frac{d^2}{dx^2}u(x) = v(x) \equiv f^{(1)}(x, u^{(1)}, u^{(2)}, u_x^{(1)}, u_x^{(2)}), \tag{68}$$

$$\frac{d^2}{dx^2}v(x) = f(x, u(x), v(x), u_x(x), v_x(x)) \equiv f^{(2)}(x, u^{(1)}, u^{(2)}, u_x^{(1)}, u_x^{(2)}), \tag{69}$$

subject to accordingly modified mixed boundary condition,

$$\alpha_0^{(1)}u(0) - \alpha_1^{(1)}u_x(0) = A_1, \beta_0^{(1)}u(1) + \beta_1^{(1)}u_x(1) = B_1, \tag{70}$$

$$\alpha_0^{(2)}v(0) - \alpha_1^{(2)}v_x(0) = A_2, \beta_0^{(2)}v(1) + \beta_1^{(2)}v_x(1) = B_2. \tag{71}$$

Now as discussed in section 2, we get the following schemes

$$U_1 = U_0 + h_1 U_{x_0} + \frac{h_1^2}{6}\big[\bar{V}_0 + 2\bar{V}_{\frac{1}{2}}\big], \tag{72}$$

$$V_1 = V_0 + h_1 V_{x_0} + \frac{h_1^2}{6}\big[\bar{f}_0 + 2\bar{f}_{\frac{1}{2}}\big], \tag{73}$$

$$\sigma U_{k-1} - (1+\sigma)U_k + U_{k+1} = h_k^2(PV_{k+1} + QV_k + RV_{k-1}), \tag{74}$$

$$\sigma V_{k-1} - (1+\sigma)V_k + V_{k+1} = h_k^2(P\bar{f}_{k+1} + Q\hat{f}_k + R\bar{f}_{k-1}), k = 1(1)N-1, \tag{75}$$

$$U_N = U_{N-1} + h_{N-1}U_{x_N} - \frac{h_{N-1}^2}{6}\big(\bar{V}_N + 2\bar{V}_{N-\frac{1}{2}}\big), \tag{76}$$

$$V_N = V_{N-1} + h_{N-1}V_{x_N} - \frac{h_{N-1}^2}{6}\big(\bar{f}_N + 2\bar{f}_{N-\frac{1}{2}}\big). \tag{77}$$

4. Convergence Analysis

For simplicity, we provide the convergence analysis for $M = 2$ i.e., a coupled nonlinear BVP in case of both $\alpha_1^{(1)}, \alpha_1^{(2)}$ and $\beta_1^{(1)}, \beta_1^{(2)}$ being non zero. As when both are zero, we get the usual Dirichlet boundary condition which has been discussed by other authors. In case of either of them is non zero, such case can be discussed with similar arguments discussed in this section.

It can be also said that, we use a fourth order BVP to provide the convergence analysis as, such problems can be written as coupled BVPs. Many authors have provided convergence analysis in case of higher or fourth order BVP. To name a few, Rashidinia et. al. ([14],[15],[16]), Sharifi et. al.([17],[18]), Usmani ([20], [21], [22],[24],[27]) and Usmani et.al. ([23], [25], [26]). But in the aforesaid, particular boundary conditions has been dealt with(see Gupta [5]). Moreover, significant derivative has not been considered in the BVP discussed. Whereas, we have used a more general boundary conditions and not only considered significant derivatives but also nonlinear singular BVPs.

Convergence analysis of the scheme (59) – (64) at the interior points has been given by Mohanty et. al.[11]. Also, for BVP with mixed boundary conditions, Usmani [20] has provided a order four convergence for the scheme. The problems considered were without significant derivatives. Also, Rashidinia et. al. [13] has provided order four convergence for the scheme they developed using separately derived boundary conditions. They used quasilinearization to linearize the nonlinear terms and used second order approximations for the significant derivatives. In all the said convergence analysis, the schemes were based on uniform mesh. Whereas in this paper, we have provided the convergence analysis using fourth order BVP based on quasivariable mesh and Newton's Method has been used for nonlinearity. Also, we provide separate discretized schemes at the interior as well exterior points of the domain using both nodal and mid-points on the mesh.

In this section, we use the following coupled boundary value problem to verify the accuracy of the scheme:

$$u_{xx}(x) = v(x), \tag{78}$$

$$v_{xx} = a(x)u(x) + f(x). \tag{79}$$

subject to modified conditions:

$$\alpha_0^{(1)}u(0) - \alpha_1^{(1)}u_x(0) = A_1, \beta_0^{(1)}u(1) + \beta_1^{(1)}u_x(1) = B_1 \tag{80}$$

$$\alpha_0^{(2)}v(0) - \alpha_1^{(2)}v_x(0) = A_2, \beta_0^{(2)}v(1) + \beta_1^{(2)}v_x(1) = B_2 \tag{81}$$

where $u(x), v(x), f(x)$ are continuous on $[0,1]$ and $\alpha_0^{(i)}, \alpha_1^{(i)}, \beta_0^{(i)}, \beta_1^{(i)} > 0, \alpha_0^{(i)} + \alpha_1^{(i)} > 0$, $\beta_0^{(i)} + \beta_1^{(i)} > 0$, $\alpha_0^{(i)} + \beta_0^{(i)} > 0, i = 1,2$ are real constants. Now,as discussed in section 2 we use the approximate solutions U_k, V_k, the following approximations and the schemes as discussed in section 3:

$$\bar{U}_{\frac{1}{2}} = U_0 + \frac{h_1}{2}U_{x_0} + \frac{h_1^2}{8}\bar{V}_0, \tag{82}$$

$$\bar{U}_{x_{\frac{1}{2}}} = \frac{3}{4h_1}(U_1 - U_0) + \frac{1}{4}U_{x_0} + \frac{h_1}{8}\bar{V}_0 \tag{83}$$

$$U_{x_0} = \frac{\alpha_0^{(1)}}{\alpha_1^{(1)}}U_0 - \frac{A_1}{\alpha_1^{(1)}}, \tag{84}$$

$$h_1 U_{x_0} = (U_1 - U_0) - \frac{h_1^2}{6}\big(\bar{V}_0 + 2\bar{V}_{\frac{1}{2}}\big) \tag{85}$$

$$\bar{U}_{N-\frac{1}{2}} = U_N - \frac{h_{N-1}}{2}U_{x_N} + \frac{h_{N-1}^2}{8}\ \bar{V}_N, \tag{86}$$

$$\bar{U}_{x_{N-\frac{1}{2}}} = \frac{3}{4h_{N-1}}(U_N - U_{N-1}) + \frac{1}{4}U_{x_N} - \frac{h_{N-1}}{8}\bar{V}_N \tag{87}$$

$$U_{x_N} = \frac{B_1}{\beta_1^{(1)}} - \frac{\beta_0^{(1)}}{\beta_1^{(1)}}U_N, \tag{88}$$

$$h_{N-1}U_{x_N} = (U_N - U_{N-1}) + \frac{h_{N-1}^2}{6}\big(\bar{V}_N + 2\bar{V}_{N-\frac{1}{2}}\big) \tag{89}$$

$$\bar{V}_{\frac{1}{2}} = V_0 + \frac{h_1}{2}V_{x_0} + \frac{h_1^2}{8}\bar{F}_0, \tag{90}$$

$$\bar{V}_{x_{\frac{1}{2}}} = \frac{3}{4h_1}(V_1 - V_0) + \frac{1}{4}V_{x_0} + \frac{h_1}{8}\bar{F}_0 \tag{91}$$

$$V_{x_0} = \frac{\alpha_0^{(1)}}{\alpha_1^{(1)}}V_0 - \frac{A_1}{\alpha_1^{(1)}}, \tag{92}$$

$$h_1 V_{x_0} = (V_1 - V_0) - \frac{h_1^2}{6}\big(\bar{F}_0 + 2\bar{F}_{\frac{1}{2}}\big), \tag{93}$$

$$\bar{V}_{N-\frac{1}{2}} = V_N - \frac{h_{N-1}}{2}V_{x_N} + \frac{h_{N-1}^2}{8}\ \bar{F}_N, \tag{94}$$

$$\bar{V}_{x_{N-\frac{1}{2}}} = \frac{3}{4h_{N-1}}(V_N - V_{N-1}) + \frac{1}{4}V_{x_N} - \frac{h_{N-1}}{8}\bar{F}_N, \tag{95}$$

$$V_{x_N} = \frac{B_1}{\beta_1^{(1)}} - \frac{\beta_0^{(1)}}{\beta_1^{(1)}}V_N, \tag{96}$$

$$h_{N-1}V_{x_N} = (V_N - V_{N-1}) + \frac{h_{N-1}^2}{6}\big(\bar{F}_N + 2\bar{F}_{N-\frac{1}{2}}\big). \tag{97}$$

where $V_r \approx v(x_r), U_r \approx u(x_r), F_r = a_r U_r + f_r, r = 0, \frac{1}{2}, N - \frac{1}{2}$. Thereafter, substituting $(82)-(84), (86)-(88)$ and $(90)-(92), (94)-(96)$ in $(85), (89)$ and $(93), (97)$ respectively. We get the following equations evaluating $u(x)$ and $v(x)$ at the boundary points

x_0 and x_N.

$$U_0[1+h_1\frac{\alpha_0^{(1)}}{\alpha_1^{(1)}}+\frac{h_1^4}{24}a_0]+U_1(-1)+V_0\frac{h_1^2}{2}[1+\frac{h_1\alpha_0^{(2)}}{3\alpha_1^{(2)}}]+V_1(0)=\Psi_0^1, \tag{98}$$

$$U_0[\frac{h_1^2}{2}a_0+\frac{h_1^3}{6}(a_{x_0}+\frac{a_0\alpha_0^{(1)}}{\alpha_1^{(1)}})+h_1^4(\frac{\alpha_0^{(1)}a_{x_0}}{12\alpha_1^{(1)}}+\frac{a_{xx_0}}{24})]+U_1(0)+V_0(1+\frac{h_1\alpha_0^{(2)}}{\alpha_1^{(2)}})$$
$$+V_1(-1)=\Psi_0^2, \tag{99}$$

$$U_N[1+h_{N-1}\frac{\beta_0^{(1)}}{\beta_1^{(1)}}+\frac{h_{N-1}^4}{24}a_N]+U_{N-1}(-1)+V_N\frac{h_{N-1}^2}{2}[1+\frac{h_{N-1}\beta_0^{(2)}}{\beta_1^{(2)}}]+V_{N-1}(0)=\Psi_N^1, \tag{100}$$

$$U_N[\frac{h_{N-1}^2}{2}a_N-\frac{h_{N-1}^3}{6}(a_{x_N}+\frac{a_N\beta_0^{(1)}}{\beta_1^{(1)}})+h_{N-1}^4(\frac{\beta_0^{(1)}a_{x_N}}{12\beta_1^{(1)}}+\frac{a_{xxN}}{24})]$$
$$+V_N(1+\frac{h_{N-1}\beta_0^{(2)}}{\beta_1^{(2)}})+V_{N-1}(-1)=\Psi_N^2. \tag{101}$$

where

$$\Psi_0^1=h_1\frac{A_1}{\alpha_1^{(1)}}+\frac{h_1^3A_2}{6\alpha_1^{(2)}}-\frac{h_1^4f_0}{24}, \tag{102}$$

$$\Psi_0^2=h_1\frac{A_2}{\alpha_1^{(2)}}+\frac{h_1^2}{6}[-3f_0+h_1(\frac{a_0A_1}{\alpha_1^{(1)}}-f_{x_0})+h_1^2(-\frac{f_{xx_0}}{4}+\frac{A_1a_{x_0}}{2\alpha_1^{(1)}})], \tag{103}$$

$$\Psi_N^1=h_{N-1}\frac{B_1}{\beta_1^{(1)}}+\frac{h_{N-1}^3B_2}{6\beta_1^{(2)}}-\frac{h_{N-1}^4f_N}{24}, \tag{104}$$

$$\Psi_N^2=h_{N-1}\frac{B_2}{\beta_1^{(2)}}+\frac{h_{N-1}^2}{6}[-3f_N-h_{N-1}(a_N\frac{B_1}{\beta_1^{(1)}}-f_{x_N})+h_{N-1}^2(\frac{f_{xx_N}}{4}+\frac{B_1a_{x_N}}{2\beta_1^{(1)}})]. \tag{105}$$

Next, we define following approximation for $a_{k\pm1}, f_{k\pm1}, k=1(1)N-1$ in the derived equations:

$$a_{k-1} = a_k-h_ka_{xk}+\frac{h_k^2}{2}a_{xx_k}, \tag{106}$$

$$a_{k+1} = a_k+h_{k+1}a_{xk}+\frac{h_{k+1}^2}{2}a_{xx_k}. \tag{107}$$

Similar, approximations can be defined for $f_{k\pm1}$. Thus, we obtain the vector difference equation of BVP using the approximations (106) – (107) in equation (61) – (62) at interior points i.e. for $k=1,...,N-1$:

$$\begin{bmatrix} BD_k^{11} & BD_k^{12}\\ BD_k^{21} & BD_k^{22}\end{bmatrix}\begin{bmatrix}U_{k-1}\\ V_{k-1}\end{bmatrix}+\begin{bmatrix} D_k^{11} & D_k^{12}\\ D_k^{21} & D_k^{22}\end{bmatrix}\begin{bmatrix}U_k\\ V_k\end{bmatrix}+\begin{bmatrix} AD_k^{11} & AD_k^{12}\\ AD_k^{21} & AD_k^{22}\end{bmatrix}\begin{bmatrix}U_{k+1}\\ V_{k+1}\end{bmatrix}=\begin{bmatrix}\Psi_k^1\\ \Psi_k^2\end{bmatrix} \tag{108}$$

where

$$D_k^{11} = (1+\sigma), D_k^{12} = \frac{h_k^2 Q}{12}, \tag{109}$$

$$D_k^{21} = h_k^2 Q a_k, D_k^{22} = 1+\sigma, \tag{110}$$

$$BD_k^{11} = -\sigma, BD_k^{12} = \frac{h_k^2 R}{12}, \tag{111}$$

$$BD_k^{21} = h_k^2 R(a_k - h_k a_{x_k} + h_k^2 \frac{a_{xx_k}}{2}), BD_k^{22} = -\sigma, \tag{112}$$

$$AD_k^{11} = -1, AD_k^{12} = \frac{h_k^2 P}{12}, \tag{113}$$

$$AD_k^{21} = h_k^2 P(a_k + h_k \sigma a_{x_k} + h_k^2 \frac{\sigma^2 a_{xx_k}}{2}), AD_k^{22} = -1, \tag{114}$$

$$\Psi_k^1 = 0, \tag{115}$$

$$\Psi_k^2 = \frac{h_k^2}{2}\left(f_k \sigma(1+\sigma) + h_k f_{x_k} \sigma(\sigma^2 - 1) + 2h_k^2 f_{xx_k} \sigma(\sigma^3 + 1)\right), k = 1(1)N-1, \tag{116}$$

The schemes $(98)-(101),(108)$ evaluating $u(x), v(x)$ at the boundary as well as interior points can be written as the following matrix form:

$$L\hat{U} + \hat{\Psi} = \begin{bmatrix} sub & diag & sup \end{bmatrix} \begin{bmatrix} \hat{U}_{k-1} \\ \hat{U}_k \\ \hat{U}_{k+1} \end{bmatrix} + \hat{\Psi} = \hat{0} \tag{117}$$

where $k = 1(1)N-1$, L is a tridiagonal matrix of order $N+1$ consisting of the following:

(i) off-diagonal block elements sub with components $[U_{N-1}, V_{N-1}]^T$ and $BD_k^{i,j}, k = 1(1)N-1; i,j = 1,2$,

$(ii)sup$ with components $AD_k^{i,j}, k = 1(1)N-1; i,j = 1,2$ and $[U_1, V_1]^T$ and similarily

(iii)diagonal block elements $diag$ with components $[U_0, V_0]^T, D_k^{i,j}, k = 1(1)N-1, i,j = 1,2$ and $[U_N, V_N]^T$ also

$\hat{U} = [\hat{U}_0, \hat{U}_1, ..., \hat{U}_k, ...\hat{U}_N]^T$, where $\hat{U}_k = [U_k, V_k]^T$

$\hat{\Psi} = [[\Psi_0^1, \Psi_0^2]^T, ..., [\Psi_k^1, \Psi_k^2]^T, ..., [\Psi_N^1, \Psi_N^2]^T]^T$, which is a constant vector,

$\hat{T}_k^3 = [\ [T_0^3, T_0^3]^T,\ [T_1^3, T_1^3]^T, ..., [T_k^3, T_k^3]^T, ...,\ [T_N^3, T_N^3\ ]^T]^T$

$\hat{0} = [[0,0]^T, [0,0]^T, ..., [0,0]^T]^T$

Let $\hat{u} = [[u_0, v_0]^T, [u_1, v_1]^T, [u_2, v_2]^T ......, [u_k, v_k]^T, ...[u_N, v_N]^T]^T \cong \hat{u}$ satisfy

$$L\hat{u} + \hat{\Psi} + \hat{T}_k^3 = 0, \text{ where L is defined in (117)} \tag{118}$$

Let $\hat{e_k} = [U_k - u_k, V_k - v_k]^T \equiv [e_{k_u}, e_{k_v}]^T$ be the discretization error, then $\hat{U} - \hat{u} = E = [\hat{e_0}, \hat{e_1}, ..., \hat{e_N}]^T$. Also, subtracting equation(118) from (117), we obtain the error equation as follows

$$LE = \hat{T}_k^3, \tag{119}$$

Let $|a_k| \le K_1, |a_{x_k}| \le K_2, |a_{xx_k}| \le K_3$, where $k = 0(1)N$, K_i are positive constants. Then, using $(98)-(101),(111)-(114)$ and for $0 \le k \le N$, we get:

$$\|sup\|_\infty \le 1, \tag{120}$$

$$\|sub\|_\infty \le \sigma. \tag{121}$$

Thus, again using $(98)-(101),(111)-(114)$ and for sufficiently small h_k, we can say that the off-diagonal block elements are non zero. Hence, L is irreducible[28].

Let sum_{rowl} be the sum of elements of l_{th} row of L, then

$$sum_{rowl} = \begin{cases} h_1\frac{\alpha_0^{(1)}}{\alpha_1^{(1)}} + \frac{h_1^2}{6}(3 + h_1\frac{\alpha_0^{(2)}}{\alpha_1^{(2)}} + h_1^2\frac{a_0}{4}), l = 1 \\ h_1\frac{\alpha_0^{(2)}}{\alpha_1^{(2)}} + \frac{h_1^2}{6}(3a_0 + h_1(a_0\frac{\alpha_0^{(1)}}{\alpha_1^{(1)}} + a_{x0}) + h_1^2(\frac{a_{xx0}}{4} + \frac{\alpha_0^{(1)}a_{x0}}{2\alpha_1^{(1)}})), l = 2 \end{cases} \tag{122}$$

$$sum_{rowl} = \begin{cases} \frac{h_k^2 S}{2}, l = 3, 5, 7, ..., N-2 \\ \frac{h_k^2 S a_k}{2} + h_k^3 a_{x_k}\frac{\sigma(\sigma^2-2)}{12} + h_k^4 a_{xx_k}\frac{(\sigma^4+\sigma)}{24}, l = 4, 6, 8, ..., N-1 \end{cases} \tag{123}$$

$$sum_{rowl} = \begin{cases} h_{N-1}\frac{\beta_0^{(1)}}{\beta_1^{(1)}} + \frac{h_{N-1}^2}{6}(3 + h_{N-1}(\frac{\beta_0^{(2)}}{\beta_1^{(2)}} + h_{N-1}^2\frac{a_N}{4})), l = N \\ h_{N-1}\frac{\beta_0^{(2)}}{\beta_1^{(2)}} + \frac{h_{N-1}^2}{6}(3a_N - h_{N-1}(a_N\frac{\beta_0^{(1)}}{\beta_1^{(1)}} + a_{x_N}) + \\ h_{N-1}^2(\frac{a_{xx_N}}{4} + \frac{\beta_0^{(1)}a_{x_N}}{2\beta_1^{(1)}})), l = N+1 \end{cases} \tag{124}$$

Now,let $0 < K_{min} \le min(K_1, K_2, K_3) \le K_{max}$, where K_{min}, K_{max} are positive numbers. Using $(122) - (124)$ and for sufficiently small h_k, we can easily prove that L is Monotone[28]. Therefore, L^{-1} exist and $L^{-1} \ge 0$. Hence by (119) we have,

$$||E|| = ||L^{-1}||||\hat{T}_k^3|| \tag{125}$$

Now for sufficiently small h_k and $(122) - (124)$, we can say that:

$$sum_{rowl} > \begin{cases} \frac{h_1^2 K_{min}}{2},\ l = 1 \end{cases} \tag{126}$$

$$sum_{rowl} \ge \begin{cases} \frac{h_k^2}{2}K_{min}S,\ l = 2, 3...N \text{ and } k = 1, 2, ..., N-1 \end{cases} \tag{127}$$

$$sum_{rowl} > \begin{cases} \frac{h_{N-1}^2 K_{min}}{2},\ l = N+1 \end{cases} \tag{128}$$

Let ${L_{i,l}}^{-1}$ be the $(i,l)^{th}$ element of L^{-1} , then by theory of matrices for $i = 1(1)N+1$,

$${L_{i,l}}^{-1} \le \frac{1}{sum_{rowl}} \tag{129}$$

Hence using $(126) - (128)$, we have

$${L_{i,l}}^{-1} \le \begin{cases} \frac{2}{h_1^2 K_{min}},\ l = 1 \\ \frac{2}{h_k^2 S K_{min}},\ l = 2, 3, ..., N \text{and } k = 1, 2, ..., N-1 \\ \frac{2}{h_{N-1}^2 K_{min}},\ l = N+1 \end{cases} \tag{130}$$

Now let us define,

$$\parallel L^{-1} \parallel = \max_{1\le i\le N+1}\sum_{l=1}^{N+1} \mid {L_{i,l}}^{-1} \mid, \parallel T \parallel = \max_{0\le k\le N} \mid \hat{T}_k^3 \mid \tag{131}$$

As $T_0^3 = O(h_1^5), T_N^3 = O(h_{N-1}^5), T_k^3 = O(h_k^5), k = 1(1)N-1$ and by $(119), (125) - (131)$ we get,

$$\parallel E \parallel \le \frac{2}{K_{min}}\left(\frac{1}{h_1^2}O(h_1^5) + \frac{1}{min_{(1\le k\le N-1)}h_k^2\ S}O(h_k^5) + \frac{1}{h_{N-1}^2}O(h_{N-1}^5)\right) = O(h_k^3). \tag{132}$$

Hence, the third order vector convergence of the proposed scheme $(59) - (64)$ for BVPs of the type $(4) - (7)$ follows.

Theorem:
The solution of BVPs $(1) - (3)$ be sufficiently smooth such that the required higher order derivatives of $u(x)$ exist in the solution domain. Then, the scheme $(59) - (64)$ with sufficiently small $h_k, 0 < \sigma < 1$ and $\frac{(\sqrt{5}-1)}{2} < \sigma < \frac{(\sqrt{5}+1)}{2}$ has third order convergence.

5. Numerical Illustration

In this section, we have solved two BVPs and present their numerical results. The numerical results are tabulated in the Tables 1-2. The first mesh width is $h_1 = \frac{(\sigma-1)}{(\sigma^N-1)}, \sigma \neq 1$. Therefore,the rest of the h_k 's can be obtained as $h_{k+1} = \sigma h_k, k = 1(1)N$. The computational order of convergence (COC)[8] is also given for fourth order uniform mesh method. All calculations have been done in Matlab 07. In the following problems $u^i(x)$ means i^{th} derivative of $u(x)$. We have used root mean square errors(e_{rms})[11] in case of quasi-variable mesh and maximum absolute error (e_{max}) for uniform mesh. e_{rms}, e_{max}, COC are defined as follows:

$$e_{rms}(k) = (\frac{1}{N+1}\sum_{k=0}^{N}(U_k - u(x_k))^2)^{\frac{1}{2}}$$

$$e_{max}(k) = \max_{0\leq k\leq N} |U_k - u(x_k)|$$

$$COC = log_2 \frac{e_{max}(k)}{e_{max}(k+1)}$$

Example 5.1 Consider the following fourth order linear BVP of the form:

$$u^4(x) = u(x) + 12\exp(x) + 8x\exp(x), 0 \leq x \leq 1$$
$$u(0) - u^1(0) = 0, u(1) + u^1(1) = 10.873$$
$$u^2(0) - u^3(0) = -4, u^2(1) + u^3(1) = 54.366$$

with exact solution $u(x) = x^2e^x$.

Table 1. Example 5.1

	e_{rms}	e_{max}	COC
N	$O({h_k}^3)$ method	$O(h^4)$ method	
16	2.2847e-04	2.1913e-05	-
32	1.0828e-04	1.4369e-06	3.9307
64	3.8898e-05	9.1941e-08	3.9661
128	8.9433e-06	5.4697e-09	4.0065

Example 5.2 Consider the fourth order nonlinear BVP of the form:

$$u^4(x) + u(x) = u(x)u^3(x) + \sinh(x)(2 - \cosh(x))$$
$$u(0) - 2u^1(0) = -2, u(1) + 2u^1(1) = 4.2614$$
$$u^2(0) - 2u^3(0) = -2, u^2(1) + 2u^3(1) = 4.2614$$

with exact solution $u(x) = \sinh x$.

Table 2. Example 5.2

	e_{rms}	e_{max}	COC
N	$O({h_k}^3)$ method	$O(h^4)$ method	
16	1.1115e-04	4.7092e-05	-
32	2.7478e-05	3.4495e-06	3.7710
64	1.3248e-05	2.3197e-07	3.8944
128	4.0344e-06	1.4949e-08	3.9558

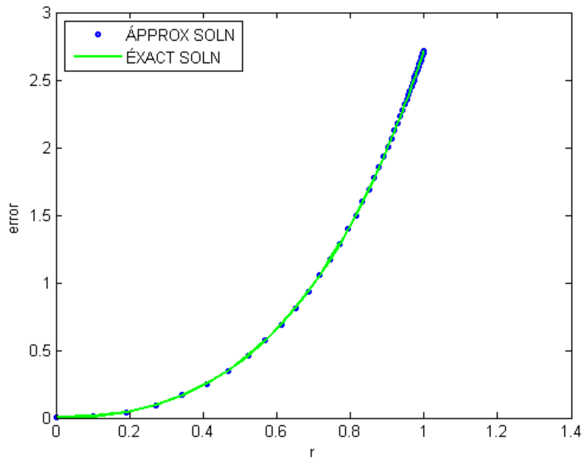

FIGURE 1. Exact versus the approximate solution in third order method for N = 64 and $\sigma = 0.9$ in Example 5.1

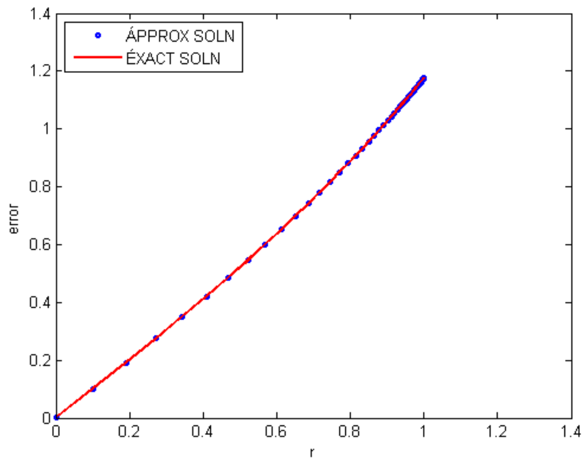

FIGURE 2. Exact versus the approximate solution in third order method for N = 64 and $\sigma = 0.9$ in Example 5.2

6. Conclusion

The numerical results for our method verify the fourth-order convergence in case of uniform mesh whereas third order convergence is proved analytically in case of quasi-variable mesh. Our method works efficiently for higher order linear and nonlinear BVPs with mixed boundary conditions, which can be decomposed into system of second order BVPs. As an experiment, only fourth order BVPs are considered whereas the method can be also applied for higher even order nonlinear and linear BVPs. Also an important consequence of using quasi variable mesh is that even higher order singularly perturbed BVPs can be solved easily.

Acknowledgement

The authors are thankful to the referees for their valuable suggestions which improved the quality of the paper.

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