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Thin sets and Campana points

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Abstract

This thesis concerns two topics on the frontiers of research in arithmetic geometry, namely thin sets and Campana points, and their intersection.

To begin with, we investigate the Hilbert property, a geometric indicator of the abundance of rational points on an algebraic variety. In this area, we prove that certain conic bundles and del Pezzo surfaces (along with their higher-dimensional analogues) satisfy the Hilbert property, meaning that their set of rational points is not thin. These results support a conjecture of Colliot-Thélène that unirational varieties over number fields have the Hilbert property.

We proceed (in joint work with Julian Demeio) to explore the stronger notion of weak approximation for del Pezzo surfaces of low degree. We combine our geometric methods for verifying the Hilbert property with results on arithmetic surjectivity, illustrating the connections not only between geometry and arithmetic, but between the so-called "local" and "global" realms.

We then move on to the study of Campana points, an idea which brings together rational and integral points in one cohesive framework and admits applications to problems of number-theoretic interest. We establish asymptotic formulae for Campana points on certain toric varieties, and in doing so we derive asymptotics for powerful values of norm forms for extensions of number fields.

Finally, in joint work with Masahiro Nakahara, we combine these theories and initiate the study of weak approximation and the Hilbert property for Campana points, both of which are topics of great significance in relation to Manin-type conjectures for Campana points, where one seeks asymptotics outside a thin set.

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Contents

Notation

Algebra

We take $\mathbb{N} = \mathbb{Z}_{\geq 1}$. We denote by R^* the group of units of a ring R. Given a group G, we denote by 1_G the identity element of G, and for any $n \in \mathbb{N}$, we set $G[n] = \{g \in G : g^n = 1_G\}$. For any perfect field F, we fix an algebraic closure $\overline{F}$ and set $G_F = \mathrm{Gal}\big(\overline{F}/F\big)$. Given a topological group G, we denote by $G^\wedge = \mathrm{Hom}\big(G, S^1\big)$ its group of continuous characters. A monomial in the variables $x_1, \ldots, x_n$ is a product $x_1^{a_1} \ldots x_n^{a_n}$, $(a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n$. For any $n \in \mathbb{N}$, we denote by μ_n the group of nth roots of unity and by S_n the symmetric group of order n.

Number theory

Given an extension of number fields L/K with K-basis $\boldsymbol{\omega} = \{\omega_0, \ldots, \omega_{d-1}\}$, we write $N_{\boldsymbol{\omega}}(x_0, \ldots, x_{d-1}) = N_{L/K}(x_0\omega_0 + \cdots + x_{d-1}\omega_{d-1})$ for the associated norm form. We denote by $\mathrm{Val}(K)$ the set of valuations of a number field K, and we denote by S_∞ the set of archimedean valuations. For $v \in \mathrm{Val}(K)$, we denote by $\mathcal{O}_v$ the maximal compact subgroup of K_v. Given a finite set of places S containing S_∞, we denote by $\mathcal{O}_{K,S} = \{\alpha \in K : \alpha \in \mathcal{O}_v \text{ for all } v \notin S\}$ the ring of algebraic S-integers of K (or $\mathcal{O}_S$ if K is clear from context). We write $\mathcal{O}_K = \mathcal{O}_{K,S_\infty}$. For $v \in \mathrm{Val}(K)$ non-archimedean, we denote by $\mathbb{F}_v$ the residue field of $\mathcal{O}_v$. We denote by π_v a uniformiser for $\mathbb{F}_v$, and we set $q_v = \#\mathbb{F}_v$. If $v \mid \infty$, then we set $\log q_v = 1$. For each $v \in \mathrm{Val}(K)$, we choose the absolute value $|x|_v = |N_{K_v/\mathbb{Q}_p}(x)|_p$ for the unique $p \in \mathrm{Val}(\mathbb{Q})$ with $v \mid p$ and the usual absolute value $|\cdot|_p$ on $\mathbb{Q}_p$. We denote by $\mathbb{A}_K = \widehat{\prod}^{\mathcal{O}_v}_{v \in \mathrm{Val}(K)} K_v$ the adele ring of K with the restricted product topology.

Geometry

We denote by $\mathbb{P}^n_R$ the projective n-space over the ring R. We omit the subscript if the ring R is clear. Given a homogeneous polynomial $f \in R[x_0, \ldots, x_n]$, we denote by $Z(f) = \mathrm{Proj}\, R[x_0, \ldots, x_n]/(f)$ the zero locus of f viewed as a closed subscheme of $\mathbb{P}^n_R$. A variety over a field F is an integral separated scheme of finite type over F. Given a variety X defined over F and an extension E/F, we denote by $X_E = V \times_{\mathrm{Spec}\, F} \mathrm{Spec}\, E$ the base change of X over E, and we write $\overline{X} = X \times_{\mathrm{Spec}\, F} \mathrm{Spec}\, \overline{F}$. When $F = K$ and $E = K_v$ for a number field K and a place v of K, we write $X_v = X_{K_v}$. Given a field F, we define $\mathbb{G}_{m,F} = \mathrm{Spec}\, F[x_0, x_1]/(x_0x_1 - 1)$. We omit the subscript F if the field is clear. Given finite sets of places $S \subset T \subset \mathrm{Val}(K)$ and a scheme $\mathcal{X}$ over $\mathrm{Spec}\, \mathcal{O}_S$, we write $\mathcal{X}_T = \mathcal{X} \times_{\mathrm{Spec}\, \mathcal{O}_S} \mathrm{Spec}\, \mathcal{O}_T$. Given a non-archimedean place v and an $\mathcal{O}_v$-scheme $\mathcal{X}$, we write $\mathcal{X}_v = \mathcal{X} \times_{\mathrm{Spec}\, \mathcal{O}_v} \mathrm{Spec}\, \mathbb{F}_v$. Given a morphism of schemes $f : X \to Y$ and a morphism $Z \to Y$, we denote by $f_Z : X_Z \to Z$ the base change $X \times_Y Z \to Z$.

Chapter 1

Introduction

This thesis concerns the study of rational solutions to Diophantine equations, one of the most ancient and storied of all mathematical pursuits. One may think of Diophantine equations, that is to say polynomial equations with integer coefficients, as cutting out geometric spaces, which we call *algebraic varieties*. Under this geometric lens, rational solutions become distinguished points on a variety, appropriately named *rational points*. By exploiting this connection, one may bring to bear tools from algebra, geometry and number theory in order to discover and describe the rational points, and thus the sought rational solutions. It is from this melting pot known as *arithmetic geometry* that some of the most profound and beautiful results in mathematics have emerged, from the work of Diophantus himself over two thousand years ago to Andrew Wiles' celebrated proof of Fermat's Last Theorem in 1995 [91].

Given an algebraic variety, the fundamental question that we may ask of its set of rational points is: is it empty, finite or infinite? To answer this question is often no mean feat; indeed, it is this question precisely which underlies Fermat's Last Theorem. If no rational points exist, one may ask why this is so, and it is this subsequent question that motivated the development of the *Brauer–Manin obstruction*, which we introduce in Chapter 2 and briefly encounter in Chapter 6. When there exist finitely many rational points, the most obvious question to ask is how many there are, either exactly or in terms of optimal bounds. As an object example in this area, we have Faltings' proof of the finiteness of rational points on curves of genus at least two over number fields (also known over the rational numbers as the Mordell Conjecture) in 1983 [33], for which he was awarded the Fields Medal. However, it is primarily the final case, of infinitely many rational points, with which we shall be concerned in this thesis.

When an algebraic variety possesses infinitely many rational points, we may seek to describe further their abundance, and we may also ask how they are distributed on the variety. In order to proceed in either direction, one may employ notions of a quantitative, analytic flavour, or of a qualitative, geometric one.

On the analytic side, one may attach to the rational points a *height function*, which endows each rational point with a "size", and attempt to obtain asymptotics for the number of rational points of bounded height. It is this approach that gave rise to *Manin's conjecture*, formulated by Manin and his collaborators Franke and Tschinkel [34, Conj. 0.2] and developed by Peyre [63] and others, which predicts asymptotics for points of bounded height on Fano varieties away from some exceptional subset. Verification of Manin-type asymptotics for rational points remains one of the most active areas of rational points research, and results in this area demonstrate the power of applying analytic principles, notably the *Hardy-Littlewood circle method* and the *height zeta function method*, to arithmetic problems.

On the geometric side, we have a natural topology on all varieties known as the Zariski topology, and as a first step, we may ask if the rational points are Zariski dense. In Chapter 3, the content of which originally appeared in [82], we will explore a geometric notion of abundance stronger than Zariski density, known as the *Hilbert property*. Informally, rational points on varieties with the Hilbert property cannot be accounted for by a finite collection of "covering" varieties, in the sense that removing the rational points coming from any such collection leaves a dense set of rational points. Returning to the algebraic perspective, this means that the coordinates of the rational points do not, for the most part, come from solving non-linear polynomial equations over the ground field, an occurrence that is taken to be rare over fields of arithmetic interest (e.g. number fields). In more formal terms, this means precisely that the set of rational points is not thin. Further, the study of the Hilbert property and thin sets bears a surprising connection to the *inverse Galois problem*, a major open problem in field theory which asks if every finite group can be realised as the Galois group of an extension of some fixed ground field. After introducing the Hilbert property and describing its connection to Galois theory, we will show that it is satisfied by certain varieties known as *del Pezzo surfaces* and their higher-dimensional analogues, *del Pezzo varieties*.

In Chapter 4, which comes from forthcoming joint work with Julian Demeio [30], we progress from the Hilbert property to the stronger property of *weak approximation*. Given a projective variety X over a number field K, we may naturally embed the set of rational points $X(K)$ into the set of local points $X(K_v)$ at any place $v \in \mathrm{Val}(K)$. Consequently, we obtain a natural "diagonal" embedding into the set of *adelic points* $X(\mathbb{A}_K) = \prod_{v \in \mathrm{Val}(K)} X(K_v)$, equipped with the product topology. A natural question is whether this embedding has dense image. If so, then we say that X satisfies weak approximation, and we may think of the rational points as being well-distributed among the local points. By extending the iterative geometric procedure employed in Chapter 3 to propagate rational points, we will show that *weak weak approximation*, or weak approximation away from finitely many places, is satisfied by general del Pezzo surfaces of degree 1 or 2 over a number field, provided that they possess the structure of a conic bundle.

A natural companion to the theory of rational points is that of *integral points*. Generalising the intuitive notion of integral points as those rational points with integral coordinates, integral points are understood in an algebro-geometric context to be those rational points satisfying certain properties upon reduction modulo primes/places of the underlying number field. In Chapter 5, which comes from [83], we will study a notion encompassing both rational and integral points, namely that of *Campana points*. Arising from the theory of *orbifoldes géométriques* concerning pairs of a variety with a distinguished $\mathbb{Q}$-divisor, which Campana developed across several papers [14], [15], [16], [17], Campana points may be loosely described as integral points with respect to a weighted divisor, with the weights of the divisor dictating certain "intersection multiplicities" upon reduction. Different demands on intersection multiplicity give rise to two notions, christened *weak Campana points* and *Campana points* in recent foundational work of Pieropan, Smeets, Tanimoto and Várilly-Alvarado [66]. In *ibid.*, the authors propose a Manin-type conjecture for Campana points on log Fano orbifolds

[66, Conj. 1.1], which may be thought of as a grand goal of the theory. We will explore both of these notions, with particular focus on the former, which is thought to be challenging to handle in general. We will establish the arithmetic motivation for this choice, which for us will be counting *powerful* values of norm forms; in particular, we will provide an asymptotic for such values by establishing asymptotics in the spirit of [66, Conj. 1.1] for both Campana points and weak Campana points on orbifolds associated to norm forms. While these results offer broad support to the aforementioned conjecture, we will see that they raise important questions about certain aspects of it.

Having studied Campana points in an analytic context in Chapter 5, we will turn in Chapter 6 to a more geometric viewpoint. The content of this chapter originates from joint work with Masahiro Nakahara [59]. Replacing $X(K)$ by the set of Campana points in the set-up of weak approximation, it is easily seen that the intersection multiplicity conditions defining Campana points mean that density within the set of all adelic points is often doomed to failure (at least when the orbifold divisor is non-zero), as these conditions prohibit Campana points from lying within certain regions of local points. Therefore, to study weak approximation in the context of Campana points, which we will call *Campana weak approximation (CWA)*, appropriate notions of local and adelic Campana points are required. We will supply such notions and proceed to prove Campana weak approximation for several orbifold structures on projective space as well as on del Pezzo surfaces, thereby initiating the study of weak approximation for Campana points. Further, we will reproduce, in the Campana context, the result of Colliot-Thélène and Ekedahl that weak weak approximation implies the Hilbert property. That is, if we have *Campana weak weak approximation*, then the set of Campana points is not thin, i.e. the orbifold in question satisfies the *Campana Hilbert property (CHP)*. As a consequence, we will in several cases prove CHP. In addition to verifying the ubiquity of Campana points on certain orbifolds, these ideas and results play an important role in [66, Conj. 1.1]; specifically, CWA allows for explicit calculation of the conjectural leading constant, and the authors of [66] indicate that CHP is a hypothesis of the conjecture to begin with.

Chapter 2

Background

As indicated in the previous chapter, the theory of rational points draws upon ideas and techniques from a broad swathe of mathematics. While the extensive toolkit of arithmetic geometry allows one to tackle challenging problems, it also lends the subject an intimidating reputation. This is reinforced by its dependence on the theory of schemes, infamous for its high level of abstraction.

In this section, we will introduce concepts and tools from algebraic geometry and number theory which we will frequently employ, offering motivation and intuition along the way. We will assume some familiarity with basic algebra, number theory and topology, along with classical algebraic geometry.

2.1 Algebraic number theory

It will be necessary for the comprehension of this thesis to have some familiarity with algebraic number fields, the central objects of study in algebraic number theory. The classic references for this subject are [18] and [60], and we shall draw upon only a small fraction of the deep and powerful results which they expose. The interplay between solubility of an equation over a number field and solubility over its completions is crucial in modern arithmetic geometry. By passing to the local setting, one may apply p-adic methods, e.g. Hensel's lemma, which may shed light on the so-called global setting over the original ground field. This interplay gives rise to the notion of local-global principles in algebraic geometry, which will be explored in Chapter 4 and Chapter 6.

Definition 2.1.1. A *number field* is a finite extension of $\mathbb{Q}$. A *global function field* is a finite extension of $\mathbb{F}_p(t)$ for some prime p. A *global field* is a field isomorphic to either an algebraic number field or a global function field.

Although we will focus on number fields, over which many questions in arithmetic are framed, the study of rational points and number theory over global function fields has the appeal that many problems which cannot be solved over number fields admit solutions over global function fields. Indeed, it was Weil's work on the Riemann hypothesis for zeta functions of smooth curves over finite fields (the function fields of which are precisely the global function fields) that led to the development of many of the underpinning concepts of modern algebraic geometry (see [39, Appendix C] for an overview).

Definition 2.1.2. A valuation (or absolute value) $|\cdot|$ on a field F is *non-archimedean* if $|x+y| \leq \max\{|x|,|y|\}$ for all $x, y \in F$. Otherwise, we say that it is *archimedean.*

Definition 2.1.3. A *place* of a field F is an equivalence class of valuations on F, where we define two valuations to be equivalent if they induce the same completion of F. Given a place v of F, we denote by F_v the completion of F with respect to the metric induced by v.

We will refer to the set of places of F as its set of valuations and denote it by $\mathrm{Val}(F)$, with the understanding that we choose from each equivalence class one representative valuation.

The non-archimedean places of a number field K are in bijection with the prime ideals of its ring of integers, while its archimedean places are in bijection with its real embeddings and conjugate pairs of complex embeddings. Given $v \in \mathrm{Val}(K)$, we denote the associated valuation by $|\cdot|_v$. When v is non-archimedean, we define the *v-adic valuation* on K to be the function v defined by $|\alpha|_v = q_v^{-v(\alpha)}$, and we refer to $|\cdot|_v$ as the v-adic absolute value. In particular, when $K = \mathbb{Q}$, we have exactly one non-archimedean place for each prime number p, represented by the p-adic absolute value $|\cdot|_p : \mathbb{Q} \to \mathbb{R}$ given by $|n|_p = p^{-v_p(n)}$, where $v_p(n)$ is the highest power of p dividing n for $n \in \mathbb{Z}$, along with exactly one archimedean place given by the usual absolute value on $\mathbb{R}$ restricted to $\mathbb{Q}$. The completion of $\mathbb{Q}$ at the place corresponding to p is the field $\mathbb{Q}_p$ of p-adic numbers, while the completion at the unique archimedean place is $\mathbb{R}$.

Definition 2.1.4. A *local field* is the completion of a global field at one of its places.

Thus we see that $\mathbb{Q}_p$ is a local field, and more generally, K_v is a local field for any number field K and any place v of K, hence $\mathbb{R}$ and $\mathbb{C}$ are also local fields.

Given $v \in \mathrm{Val}(K)$ non-archimedean, we define the ring of v-adic integers $\mathcal{O}_v = \{a \in K_v : v(a) \leq 1\}$, a discrete valuation ring. Given a finite subset $S \subset \mathrm{Val}(K)$ containing the set S_∞ of archimedean places, we define the ring of S-integers $\mathcal{O}_{K,S}$ of K to be precisely those $\alpha \in K$ which are v-adic integers for all $v \notin S$. Taking $S = S_\infty$, we recover the ring of integers $\mathcal{O}_K$ of K.

Given a number field K, it is often desirable to work over all of its completions at once, and so the following ring will frequently appear.

Definition 2.1.5. Let K be a number field. We define the ring of *adeles* of K, denoted by $\mathbb{A}_K$, to be the restricted product $\prod'_{v \in \mathrm{Val}(K)} K_v$ with respect to the $\mathcal{O}_v$.

2.1.1 Splitting

It is a key result of algebraic number theory that, while unique factorisation does not hold in the ring of integers of an arbitrary number field K, it holds for prime ideals, i.e. $\mathcal{O}_K$ is a *Dedekind domain.* In particular, given a prime number p, we may decompose the ideal $(p) \subset \mathcal{O}_K$ into a product of prime ideals $\prod_{i=1}^{r} \mathfrak{p}_i^{e_i}$, unique up to reordering.

More generally, given an extension of number fields L/K and a prime ideal $\mathfrak{p} \subset \mathcal{O}_K$, we may consider the prime ideal decomposition

$$\mathfrak{p} \cdot \mathcal{O}_L = \prod_{i=1}^{g} \mathfrak{q}_i^{e_i}. \tag{2.1.1}$$

Definition 2.1.6. Let $\mathfrak{p}$ and $\mathfrak{q}_1, \ldots, \mathfrak{q}_g$ be as in (2.1.1). We call the e_i *ramification indices*, and we say that L/K is *ramified* at $\mathfrak{p}$ if $e_i > 1$ for some $i = 1, \ldots, g$. Otherwise, we say that L/K is *unramified* at $\mathfrak{p}$. We define the *inertia degree* of $\mathfrak{q}_i$ over $\mathfrak{p}$ to be the degree $f_i = [\mathcal{O}_L/\mathfrak{q}_i : \mathcal{O}_K/\mathfrak{p}]$.

Remark 2.1.7. One may also define ramification in the context of morphisms of schemes, and the sharing of this terminology is justified: the inclusion of fields $K \hookrightarrow L$ induces a morphism of affine schemes $\operatorname{Spec} L \to \operatorname{Spec} K$, and the points of $\operatorname{Spec} K$ at which this morphism ramifies correspond to the prime ideals ramified in L/K.

Just as the non-archimedean places of a number field K are intertwined with the prime ideals of $\mathcal{O}_K$, the splitting of prime ideals is intertwined with extensions of valuations. Given an extension L/K of number fields and a non-archimedean place v of K corresponding to a prime ideal $\mathfrak{p}$, the prime ideals of $\mathcal{O}_L$ over $\mathfrak{p}$ (as in (2.1.1)) are in bijection with the extensions of v to L. Central to the study of both sides of this correspondence is the following result, known as the *fundamental identity of valuation theory.*

Proposition 2.1.8. *[60, Prop. II.8.5] Let L/K be an extension of number fields and $\mathfrak{p}$ a prime ideal of $\mathcal{O}_K$. Then, for e_i, f_i and g as in* (2.1.1)*, we have*

$$\sum_{i=1}^{g} e_i f_i = [L : K].$$

Definition 2.1.9. We say that $\mathfrak{p}$ is *totally ramified* (respectively, *totally inert, totally split*) if $g = f_1 = 1$ (respectively, $g = e_1 = 1$, $g = [L : K]$).

When L/K is Galois, the splitting of prime ideals is more controlled. In this situation, the Galois group $\operatorname{Gal}(L/K)$ acts transitively on the $\mathfrak{q}_i$, and it follows that there exist integers e and f such that $e = e_i$ and $f = f_i$ for all $i = 1, \ldots, g$.

2.2 Schemes

In this section we will give a brief and informal introduction to the theory of schemes, the foundation upon which much of modern algebraic geometry (in particular arithmetic geometry) is built. As with algebraic number theory, we will touch only lightly upon the vast expanse of scheme theory, focusing on the areas which will appear most frequently in the subsequent chapters. Our primary reference is Hartshorne's seminal text [39], but some of the more arithmetic content can be found in the books of Liu [53] and Poonen [67]. Much of the intuition which we offer comes from familiarity with the excellent exposition of Vakil's notes [85]. Although we do not reference this text elsewhere in the thesis, we would be remiss to omit it, since it offers arguably the most complete, concise and comprehensible introduction to algebraic geometry available, and its approach not only to algebraic geometry but to mathematics as a whole has proved invaluable.

One issue in the classical theory of varieties is that one cannot distinguish between polynomial systems cutting out the same space. For instance, consider the intersection of the affine plane curves $y = x^2$ and $y = 0$ in $\mathbb{A}^2_{\mathbb{C}}$. This intersection can (and in some sense, ought to) be thought of as being cut out by the equation $x^2 = 0$ on the line $y = 0$. With the fundamental theorem of algebra in mind, we would like to view this as a point with multiplicity two. However, the associated variety coincides with the variety cut out by the system $y = x = 0$. Another issue was encountered by the algebraic geometers of the Italian school, who wished to make precise the concept of a general point of a variety, i.e. a point which behaves in the same way as "most" other points of a variety.

These issues were resolved by Grothendieck (together with Dieudonné) in the foundational treatise *Éléments de géométrie algébrique* [32] by the introduction of schemes. One may think of the first motivation for schemes as coming from Hilbert's Nullstellensatz [39, Thm. I.1.3A], which gives a correspondence between the irreducible closed subsets of a variety over an algebraically closed field and the prime ideals of its coordinate ring. By equipping this set of prime ideals with a topology, we effectively add points to a variety in such a way that every irreducible closed subset gains a unique point dense in the subspace topology. This motivates the following definition.

Definition 2.2.1. Let X be a topological space, and let $\eta \in X$. We say that η is a *generic point* of X if $\overline{\{\eta\}} = X$.

The existence of generic points allows one to make precise the notion of the behaviour of general points on a variety.

We now come to the cornerstone of scheme theory, namely the spectrum of a ring.

Definition 2.2.2. Given a ring R, we define by $\operatorname{Spec} R$ the set of prime ideals of R. The *Zariski topology* on $\operatorname{Spec} R$ is the topology given by declaring the closed sets to be those of the form

$$V(I) = \{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \supset I\}$$

for some ideal I of R.

In order to resolve the remaining issue, which is effectively that of permitting nilpotents, we introduce a further layer of structure to our spectra: that of a *ringed space*.

Definition 2.2.3. A *presheaf of rings* on a topological space X is the data of a ring $\mathcal{O}(U)$ for each open set U of X along with homomorphisms $\rho_{U,V} : \mathcal{O}(U) \to \mathcal{O}(V)$ for each pair of open sets $V \subset U$ (known as *restriction maps*) such that, for all open sets $W \subset V \subset U$, we have $\rho_{U,U} = \mathrm{id}_{\mathcal{O}(U)}$ and $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$. We call elements $s \in \mathcal{O}(U)$ *sections* of $\mathcal{O}$ over U. We call a section of $\mathcal{O}$ over X a *global section*. We sometimes denote $\mathcal{O}(U)$ by $\Gamma(U, \mathcal{O})$.

We call $\mathcal{O}$ a *sheaf of rings* if in addition, it satisfies the following two axioms, given an open set $U \subset X$ and an open cover $U \subset \bigcup_{i \in I} U_i$.

1. Glueability: given sections $s_i \in \mathcal{O}(U_i)$ for each $i \in I$ such that $\rho_{U_i, U_i \cap U_j}(s_i) = \rho_{U_j, U_i \cap U_j}(s_j)$, there exists a section $s \in \mathcal{O}(U)$ such that $\rho_{U,U_i}(s) = s_i$ for all $i \in I$. In other words, we can glue sections on an open cover, provided that they agree on overlaps.

2. Identity: given sections $s, s' \in \mathcal{O}(U)$ such that $\rho_{U,U_i}(s) = \rho_{U,U_i}(s')$ for all $i \in I$, then $s = s'$. In other words, two sections agree on an open cover if and only if they are equal.

(Note that, as a consequence of the identity axiom, the gluing of sections on an open cover as in the gluability axiom is unique.) When $\mathcal{O}$ is a sheaf, we call the pair $(X, \mathcal{O})$ a *ringed space*. We will refer to it simply by X when $\mathcal{O}$ is clear from context. We will think of certain topological spaces as coming equipped with a particular sheaf, which we will refer to as the *structure sheaf*, denoted by $\mathcal{O}_X$.

Those familiar with category theory will recognise a presheaf of rings as a contravariant functor from the category of open sets of X to the category of rings, which can readily be generalised from the category of rings to other categories. Further, one may interpret the identity and gluability axioms as a statement of exactness of certain sequences.

Motivated by the corresponding notion of rings of regular functions in classical algebraic geometry, we think of the sections of the structure sheaf over U as functions defined on U.

The contribution of the structure sheaf is that it captures "infinitesimal" information around a point, which is precisely what we will use to distinguish the schemes $x = y = 0$ and $x^2 = y = 0$ in $\mathbb{A}^2_{\mathbb{C}}$ in our earlier example.

Definition 2.2.4. For each point $P \in X$, we define the *stalk* of $\mathcal{O}$ at P to be the ring $\mathcal{O}_P$ consisting of equivalence classes of pairs $\langle s, U\rangle$ where $P \in U$, $s \in \mathcal{O}(U)$, with $\langle s, U\rangle = \langle t, V\rangle$ iff there exists an open set $W \subset U \cap V$ such that $\rho_{U,W}(s) = \rho_{V,W}(t)$. We call the elements of $\mathcal{O}_P$ the *germs* of $\mathcal{O}$ at P.

Continuing our analogy with classical varieties, we think of germs as "shreds" of function around P.

Definition 2.2.5. We say that the ringed space $(X, \mathcal{O})$ is a *locally ringed space* if for each $P \in X$, the stalk $\mathcal{O}_P$ is a local ring, i.e. it possesses a unique maximal ideal $\mathfrak{m}_P$. We then define the *residue field* of X at P to be $\kappa(P) = \mathcal{O}_P/\mathfrak{m}_P$. Given an irreducible scheme X (Definition 2.2.9) with generic point η, we define the *function field* $k(X)$ of X to be the residue field $\kappa(\eta)$.

Definition 2.2.6. Given a ring R, we define the sheaf $\mathcal{O}_{\operatorname{Spec} R}$ on $\operatorname{Spec} R$ by defining $\mathcal{O}_{\operatorname{Spec} R}(U)$ to be the ring of all functions $s : U \to \coprod_{\mathfrak{p}\in U} R_{\mathfrak{p}}$ such that $s(\mathfrak{p}) \in R_{\mathfrak{p}}$ and s is locally a quotient of elements in R in the sense that, for each $\mathfrak{p} \in U$, there exists an open set $V \subset U$ containing $\mathfrak{p}$ and elements $r, f \in R$ such that for each $\mathfrak{q} \in V$, we have $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = \frac{r}{f}$.

It is easily seen that the stalk $\mathcal{O}_{\operatorname{Spec} R,\mathfrak{p}}$ is a local ring, with maximal ideal generated by the germs of sections vanishing at $\mathfrak{p}$. Thus $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ is a locally ringed space.

Definition 2.2.7. An *affine scheme* is a locally ringed space that is isomorphic (in the category of locally ringed spaces) to the spectrum of a ring. A *scheme* is a locally ringed space that is covered by open sets isomorphic to affine schemes.

Much of algebraic geometry takes place over Noetherian rings (those for which every ascending chain of ideals is eventually stationary, e.g. fields and Dedekind domains), and so we are particularly interested in schemes which look locally like these rings.

Definition 2.2.8. We say that a scheme is *locally Noetherian* if it is covered by open sets isomorphic to the spectra of Noetherian rings, and *Noetherian* if there exists a finite cover of this type.

We introduce the following notions to describe the underlying space of a scheme.

Definition 2.2.9. We say that a non-empty topological space X is *irreducible* (respectively, *connected*) if it cannot be written as the union of two proper closed subsets (respectively, disjoint proper closed subsets). An irreducible (respectively, connected) component is a maximal irreducible (respectively, connected) subset. We say that a scheme is irreducible (or connected) if its underlying topological space possesses this property.

Definition 2.2.10. We define the *dimension* $\dim X$ of a non-empty topological space X to be the supremum of the integers n such that there exists a chain of distinct irreducible closed subsets $\emptyset \neq Z_0 \subset Z_1 \subset \ldots Z_n \subset X$. We define the *dimension* of a scheme to be the dimension of its underlying topological space.

We will also rely upon the notion of reducedness, which takes into account the data from the local rings of a scheme and captures the notion of "multiplicity" at a point.

Definition 2.2.11. We say that a scheme X is *reduced at* $P \in X$ if the local ring $\mathcal{O}_{X,P}$ is reduced, i.e. it contains no non-zero nilpotents. We say that X is *reduced* if it is reduced at all points.

By unravelling the definition, we see that $\mathcal{O}_{\operatorname{Spec} R,\mathfrak{p}} \cong R_{\mathfrak{p}}$. With this observation, we may now differentiate between the schemes in our earlier example.

Example 2.2.12. The scheme $C = \operatorname{Spec} \mathbb{C}[x,y]/\big(y - x^2, y\big) \cong \operatorname{Spec} \mathbb{C}[x]/\big(x^2\big)$ is not reduced at $P = (x, y)$, since $\mathcal{O}_{C,P} \cong \mathbb{C}[x]_{(x)}/\big(x^2\big)$ is not reduced. On the other hand, the scheme $\operatorname{Spec} \mathbb{C}[x,y]/(x,y) \cong \operatorname{Spec} \mathbb{C}$ is reduced.

In addition to multiplicity, the local rings of a scheme may be used to describe how "singular" a scheme is at a point via the following definitions.

Definition 2.2.13. We say that a scheme X is *normal* if all of its local rings are integrally closed. We say that X is *regular* if all of its local rings are regular, i.e. they are Noetherian and the size of a minimal set of generators for the maximal ideal equals the Krull dimension.

Since any regular local ring is integrally closed, all regular schemes are normal. Further, normality (for a scheme with Noetherian local rings) implies regularity in codimension one, i.e. regularity for all local rings of dimension one.

Regularity is closely related to the notion of *smoothness*, which we do not define here. In particular, regularity and smoothness coincide for schemes of finite type over perfect fields, on which we will focus almost exclusively, rarely leaving the characteristic zero realm.

Given a field F (not necessarily algebraically closed), we may associate to an ideal $I \subset F[x_1, \ldots, x_n]$ the affine scheme $\operatorname{Spec} F[x_1, \ldots, x_n]/I$ (as seen in the example above). By functoriality of Spec, we may think of this scheme as living inside the affine n-space $\mathbb{A}^n_F = \operatorname{Spec} F[x_1, \ldots, x_n]$. Over an algebraically closed field, we see that as promised, we have added to each "traditional" affine variety a generic point for each of its irreducible closed subsets.

This construction may be mirrored in order to produce schemes associated to projective varieties. To do so, we define a space $\operatorname{Proj} R$ for each graded ring $R = \bigoplus_{i \geq 0} R_i$

analogous to $\operatorname{Spec} R$ by replacing prime ideals with homogeneous prime ideals not containing the irrelevant ideal $R_+ = \bigoplus_{i>0} R_i$. We define projective n-space $\mathbb{P}^n_R$ over a ring R to be the scheme $\operatorname{Proj} R[x_0, \ldots, x_n]$, and we define a (quasi-)projective R-scheme to be (an open subset of) a scheme X equipped with a closed immersion $X \hookrightarrow \mathbb{P}^n_R$.

The collection of locally ringed spaces naturally forms a category, with morphisms given by continuous maps of the underlying sets together with morphisms of structure sheaves, which one may think of as facilitating the pullback of functions from one scheme to another.

Definition 2.2.14. Given a scheme S, an *S-scheme/scheme over S* is a scheme X together with a morphism $X \to S$, which we call the *structure morphism* of X. Given a ring R, an R-scheme is a scheme over $\operatorname{Spec} R$.

Note that S-schemes form a category upon defining a morphism of S-schemes to be a morphism of schemes commuting with the structure morphisms.

Definition 2.2.15. A *variety* over a field k is an integral separated scheme of finite type over k. A *curve* is a variety of dimension one, and a *surface* is a variety of dimension two.

We say that a scheme is *integral* if it is both reduced and irreducible (since this is equivalent to every ring of sections being an integral domain). Separatedness is the scheme-theoretic analogue of being Hausdorff; note that any non-empty open subset of an irreducible space is dense, hence any irreducible space is not Hausdorff, hence the need for an adaptation of this notion. Finally, the condition that a scheme is of finite type over a field k means roughly that it is covered by finitely many open affine patches on each of which it is cut out by finitely many polynomial equations defined over k.

2.2.1 Rational points

Definition 2.2.16. Given a scheme X over a field k, we define the set $X(k)$ of (k-)rational points of X to be the set of sections (in the category of k-schemes) of the structure morphism $X \to \operatorname{Spec} k$. More generally, given S-schemes X and T, we define the set of T-valued points of X to be $X(T) = \operatorname{Hom}_S(T, X)$.

Note that the spectrum of a field is a point, hence the image of a rational point is, as one would hope, a point of the underlying topological space of X. It is the requirement that the associated section is a k-morphism that encodes the arithmetic property that this point is defined over k. For quasi-projective varieties, this abstract definition coincides with the classical definition of k-rational points as those points with (affine or projective) coordinates in k.

We will work mainly with algebraic varieties X over a number field K. As mentioned in the previous chapter, we have canonical inclusions $X(K) \hookrightarrow X(K_v)$ of the rational points into the so-called local points at each place $v \in \operatorname{Val}(K)$, thus a diagonal embedding of $X(K)$ into $\prod_{v \in \operatorname{Val}(K)} X(K_v)$, which, for X/K projective (or more generally, proper) coincides with the set $X(\mathbb{A}_K)$ of adelic points. This allows us to compare the "global" behaviour of X as a K-variety with its "local" behaviour at each place.

As a first example of this comparison, note that $X(K) \neq \emptyset$ implies that X is *everywhere locally soluble*, i.e. $X(K_v) \neq \emptyset$ for all $v \in \operatorname{Val}(K)$. A natural question to ask is whether, for a given class of varieties, being everywhere locally soluble is enough to guarantee the existence of a rational point.

Definition 2.2.17. We say that a class of varieties over a number field K satisfies the *Hasse (local-global) principle* if for every variety X in the class, $\prod_{v \in \mathrm{Val}(K)} X(K_v) \neq \emptyset \implies X(K) \neq \emptyset$.

The name arises from the *Hasse–Minkowski theorem* [60, Ch. VI, §4], which states that the Hasse principle holds for quadric hypersurfaces (i.e. projective varieties cut out by a single quadratic equation). However, there are many classes of varieties which fail the Hasse principle, with an early example illustrating the failure of the Hasse principle for cubic plane curves being given by Selmer, namely $3x^3 + 4y^3 + 5z^3 = 0$ [73].

One appeal of local arithmetic is that algorithms can be used to determine whether a given variety possesses local points at a place. In particular, over non-archimedean places, we possess an extremely useful tool known as *Hensel's lemma*, which allows us to lift certain solutions over finite fields.

Theorem 2.2.18 (Hensel's lemma)**.** *[67, Thm. 3.5.63] Let X be a scheme over a complete Noetherian local ring A with maximal ideal $\mathfrak{m}$.*

1. *If X is smooth over* $\operatorname{Spec} A$*, then the reduction map $X(A) \to X(A/\mathfrak{m})$ is surjective.*

2. *If X is étale over* $\operatorname{Spec} A$*, then the reduction map $X(A) \to X(A/\mathfrak{m})$ is bijective.*

Smooth and étale morphisms, which we do not define here, may be regarded as the algebraic analogues of smooth submersions and local isomorphisms respectively.

In certain situations, not only does everywhere local solubility guarantee the existence of a rational point, but the rational points are even dense in the local points.

Definition 2.2.19. A variety X over a number field K is said to satisfy *weak approximation* if $X(K)$ is dense in $\prod_{v \in \mathrm{Val}(K)} X(K_v)$. If $X(K)$ is dense in $\prod_{v \notin S} X(K_v)$ for some finite subset $S \subset \mathrm{Val}(K)$, we say that X satisfies *weak approximation away from* S, and we say that X satisfies *weak weak approximation* if it satisfies weak approximation away from some such S. If $X(K)$ is dense in $X(\mathbb{A}_K)$, then we say that X satisfies *strong approximation*. When $\prod_v X(K_v) = \emptyset$, we say that weak approximation holds trivially.

It is a refinement of the Chinese remainder theorem that the affine line $\mathbb{A}^1_K$ satisfies weak approximation. Further, weak approximation is a birational invariant of smooth varieties, essentially as a consequence of the implicit function theorem over the completions of K, hence all rational varieties satisfy weak approximation. See [38] for a comprehensive survey of results in this area.

2.3 Base change

One of the techniques which we will make frequent use of is that of base change. Having defined the notion of schemes over a base S, an obvious question that arises is: given an S-scheme X/S and a morphism $T \to S$, does there exist a scheme Y making the following diagram commute?

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r] & S \end{tikzcd}$$

This is motivated by the case $S = \operatorname{Spec} k$, $T = \operatorname{Spec} L$ for L/k a field extension. Given a k-scheme X, we would like to be able to consider X as a scheme over L. Ideally, we would like not only for such Y to exist, but to be unique, at least up to isomorphism. The reader with experience in category theory will realise that this strongly suggests a definition based on a universal property, which we now give.

Definition 2.3.1. Given morphisms of schemes $f : X \to Z$ and $g : Y \to Z$, we define the *fibre product* of X and Y (over Z) to be the Z-scheme $X \times_Z Y$ satisfying the following universal property: there exist morphisms $\mathrm{pr}_X : X \times_Z Y \to X$ and $\mathrm{pr}_Y : X \times_Z Y \to Y$ such that $f \circ \mathrm{pr}_X = g \circ \mathrm{pr}_Y$, and given any other scheme W with this property, there exists a unique morphism $h : W \to X \times_Z Y$ such that the following diagram commutes.

$$\begin{tikzcd} W \arrow[dr, "h"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \times_Z Y \arrow[r, "\mathrm{pr}_Y"] \arrow[d, "\mathrm{pr}_X"'] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & Z \end{tikzcd}$$

Given a k-scheme X and a field extension L/k, we denote by X_L the base change $X \times_{\operatorname{Spec} k} \operatorname{Spec} L$, which we may also denote by $X \times_k L$. Given a scheme X over a number field K and a place $v \in \operatorname{Val}(K)$, we write $X_v = X \times_K K_v$.

As with any object defined by a universal property, it is first necessary to construct *some* object satisfying the desired property, with uniqueness following thereafter. Thankfully, the locally affine nature of schemes allows one to reduce this construction to the level of affine schemes, where it is readily shown that $\operatorname{Spec} A \times_{\operatorname{Spec} C} \operatorname{Spec} B \cong \operatorname{Spec}(A \otimes_C B)$, with the general construction following by gluing.

Another application of the fibre product of vital importance to our work is the construction of the fibres of a morphism as schemes. Given a scheme Y and a point $P \in Y$, we have a canonical morphism $\operatorname{Spec} \kappa(P) \to Y$ with image P, and so we make the following definition.

Definition 2.3.2. Given a morphism of schemes $f : X \to Y$ and a point $P \in Y$, we define the *fibre* of f over P to be the $\kappa(P)$-scheme $X \times_Y \operatorname{Spec} \kappa(P)$. When Y is irreducible, we define the *generic fibre* of f to be the fibre of f over the generic point of Y.

It follows from elementary topology that any irreducible scheme possesses a unique generic point, hence the generic fibre of a morphism to an irreducible scheme is well-defined.

It is often the case that we are given some scheme or morphism with "nice" properties, and we would like to know whether these properties are preserved upon base change. If this is the case, then we say that the property is *stable under base change*. We will often rely on properties stable under base change; a comprehensive list of such properties can be found in [67, Appendix A].

Similarly, one may ask to go in the other direction, so that properties can be checked after making an appropriate base change. This is the essential idea behind *descent.* Although we will not discuss or employ descent, it plays a key role in powerful methods used to prove the lack of existence of rational points on varieties.

It is a guiding principle of algebraic geometry that "geometry takes place over the algebraic closure". As such, we are led to the following definitions.

Definition 2.3.3. We say that a scheme X over a field k is *geometrically irreducible* (respectively *geometrically reduced, geometrically integral, geometrically connected*) if the base change $X_{\overline{k}}$ is irreducible (respectively reduced, integral, connected).

Example 2.3.4. The affine $\mathbb{Q}$-variety defined by the equation $y^2 - 2x^2 = 0$ in $\mathbb{A}^2_{\mathbb{Q}}$ is irreducible but not geometrically irreducible, since it decomposes into the varieties $y + \sqrt{2}x = 0$ and $y - \sqrt{2}x = 0$ upon base change to $\mathbb{Q}(\sqrt{2}) \subset \overline{\mathbb{Q}}$.

2.4 Divisors

When studying a variety or general scheme, it is natural to seek to break it down into pieces of lower dimension, which may be easier to work with. As such, the study of the *divisors* of a scheme, i.e. subschemes of codimension one, is well-motivated. In this section we introduce Weil divisors and Cartier divisors and elucidate both their connection to each other and to invertible sheaves. We will also indicate the connection to projective morphisms when the underlying scheme is a variety.

2.4.1 Weil divisors

Let X be a Noetherian integral scheme which is *regular in codimension one*, i.e. all of the local rings $\mathcal{O}_P$, $P \in X$ of dimension one are regular.

Definition 2.4.1. A *prime divisor* on X is an integral subscheme of X of codimension one. A *Weil divisor* is an element of the free abelian group $\operatorname{Div} X$ generated by the prime divisors, hence is of the form

$$D = \sum_Z n_Z Z,$$

where Z runs over the prime divisors of X and each $n_Z \in \mathbb{Z}$, with all but finitely many of the n_Z equal to zero. If all $n_Z \geq 0$, we say that D is *effective.* We define the *degree* $\deg D$ of D to be the integer $\sum_Z n_Z$.

Since X is regular in codimension one, the local ring of the generic point of any prime divisor Z is a discrete valuation ring, hence we obtain an associated valuation v_Z. Given $f \in k(X)^*$, we may therefore define the *valuation $v_Z(f)$ of f along Z*. It follows from the Noetherian hypothesis [39, Lem. II.6.1] that $v_Z(f) = 0$ for all but finitely many Z, and so we may make the following definition.

Definition 2.4.2. Given $f \in k(X)^*$, we define the divisor

$$(f) = \sum_Z v_Z(f) Z,$$

where Z runs over all prime divisors of X. We call a Weil divisor of this form a *principal divisor*, and we say that two Weil divisors D and D' are *linearly equivalent* if $D - D'$ is principal.

It follows readily that linear equivalence is an equivalence relation on Weil divisors.

Definition 2.4.3. The *class group* of X is the group $\operatorname{Cl} X$ of linear equivalence classes of Weil divisors under addition.

2.4.2 Cartier divisors

On more general schemes where codimension one subschemes may not be so well-behaved, we instead consider the concept of something which looks locally like the zero locus of a rational function.

Definition 2.4.4. On a scheme X, we define the *sheaf of total quotient rings* $\mathcal{K}$ by taking $\mathcal{K}(U)$ to be the localisation of $\mathcal{O}_X(U)$ at all elements which are not zero-divisors. We then define a *Cartier divisor* to be the data $\{f_i, U_i\}_{i \in I}$ where $X = \bigcup_{i \in I} U_i$ is an open cover of X, $f_i \in \mathcal{K}(U_i)^*$ and $\frac{f_i}{f_j} \in \mathcal{O}(U_i \cap U_j)^*$. We say that a Cartier divisor is *effective* if we may choose each $f_i \in \mathcal{O}(U_i)$. We say that a Cartier divisor is *principal* if it is of the form $\{\rho_{X,U_i}(f), U_i\}$ for some $f \in \mathcal{K}(X)^*$, and we say that two Cartier divisors are *linearly equivalent* if their difference (as global sections of $\mathcal{K}^*/\mathcal{O}^*$) is principal. We call the resulting group of classes the *Cartier class group* $\operatorname{CaCl} X$ of X.

2.4.3 Invertible sheaves

Given a ringed space $(X, \mathcal{O}_X)$, we may define a *sheaf of $\mathcal{O}_X$-modules* to be a sheaf of rings $\mathcal{F}$ such that $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module for every open set $U \subset X$, with the restriction homomorphisms compatible with the module structure. As we associated to $\operatorname{Spec} A$ the sheaf $\mathcal{O}_{\operatorname{Spec} A}$ of certain maps to the stalks, we may similarly associate a sheaf of $\mathcal{O}_{\operatorname{Spec} A}$-modules $\widetilde{M}$ to each A-module M [39, Ch. 2, §5]. Having done so, we make the following definitions.

Definition 2.4.5. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is *quasi-coherent* if there exists an open affine cover $X = \bigcup_{i \in I} U_i$ of X such that the restriction of $\mathcal{F}$ to each $U_i = \operatorname{Spec} A_i$ is of the form $\widetilde{M_i}$ for some A_i-module M_i. We say that $\mathcal{F}$ is *coherent* if in addition each M_i can be taken to be finitely generated over A_i.

Just as we define quasi-coherent sheaves to be sheaves which look locally like the sheaf associated to a module, we may also consider sheaves which look locally like (copies of) the structure sheaf itself.

Definition 2.4.6. We say that a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is *locally free* if it is locally isomorphic to a direct sum of copies of $\mathcal{O}_X$. When X is connected, the number of copies of the structure sheaf needed is constant over open sets and is called the *rank* of $\mathcal{F}$. An *invertible sheaf* (or *line bundle*) is a locally free sheaf of rank one. A *vector bundle* is a locally free sheaf of finite rank.

One may in an obvious way define the tensor product of $\mathcal{O}_X$-modules, and the invertible sheaves, as their name suggests, form a group under this operation.

Definition 2.4.7. The *Picard group* $\operatorname{Pic} X$ of a ringed space X is the group of isomorphism classes of invertible sheaves on X under tensor product.

Definition 2.4.8. Given a Cartier divisor $D = \{f_i, U_i\}$ on a scheme X, we define the sheaf $\mathcal{L}(D)$ to be the invertible sub-$\mathcal{O}_X$-module of $\mathcal{K}$ generated by f_i^{-1} on U_i.

Proposition 2.4.9. *[39, Ch. II, §6]*

1. *When X is integral, Noetherian, separated and* locally factorial *(meaning that all of its local rings are UFDs), the map $D = \{f_i, U_i\} \mapsto \sum_Y v_Y(f_i)Y$ (the sum being over prime divisors) is well-defined and constant on linear equivalence classes, and it gives an isomorphism $\mathrm{CaCl}\,X \cong \mathrm{Cl}\,X$.*

2. *The map $D \mapsto \mathcal{L}(D)$ from Cartier divisors to invertible sheaves is constant on linear equivalence classes and defines an injective homomorphism from $\mathrm{CaCl}\,X$ to $\mathrm{Pic}\,X$ which is an isomorphism when X is integral.*

In particular, for X a variety over a field (or more generally, an integral separated scheme of finite type over a Noetherian ring), we obtain $\mathrm{Cl}\,X \cong \mathrm{CaCl}\,X \cong \mathrm{Pic}\,X$, provided that X is locally factorial (in particular, if X is regular).

2.4.4 Projective morphisms

As we shall now see, on varieties, divisors correspond not only to invertible sheaves, but also to morphisms to projective space.

Definition 2.4.10. We say that a sheaf $\mathcal{F}$ on a scheme X is *generated by global sections* if there exist global sections $s_1, \dots, s_r \in \mathcal{F}(X)$ such that, at every point $P \in X$, the stalk $\mathcal{O}_{X,P}$ is generated by the image of the s_i.

Loosely, the fact that the s_i generate every stalk means that they do not simultaneously vanish, and so the following result may not come as too much of a surprise.

Theorem 2.4.11. *[39, Thm. II.7.1] Let X be a scheme over a ring A. Then there exists a bijective correspondence between morphisms $\varphi : X \to \mathbb{P}^n_A = \mathrm{Proj}\,A[x_0, \dots, x_n]$ and invertible sheaves $\mathcal{F}$ generated by global sections, given by associating to φ the sheaf $\varphi^*\mathcal{O}(1)$ (generated by the global sections $s_i = \varphi^*(x_i)$) and to a sheaf $\mathcal{F}$ generated by global sections $s_0, \dots, s_n$ a unique morphism $f : X \to \mathbb{P}^n_A$ with $f^*\mathcal{O}(1) = \mathcal{F}$ and $f^*(x_i) = s_i$.*

By introducing the classical notion of *linear systems*, we obtain a geometric counterpart to the algebraic criterion that the invertible sheaf of a divisor is generated by global sections.

Definition 2.4.12. A *complete linear system* on a variety X is the set of all effective divisors linearly equivalent to a divisor D_0, denoted by $|D_0|$. A *base point* of $|D_0|$ is a point lying on every divisor in $|D_0|$. If $|D_0|$ has no base points, we say that it is *base-point-free*.

Lemma 2.4.13. *[39, Lem. II.7.8] Given a divisor D on a variety X, the invertible sheaf $\mathcal{L}(D)$ is generated by global sections if and only if $|D|$ is base-point-free.*

Before moving on, we define two further notions related to sheaves generated by global sections which we will encounter.

Definition 2.4.14. Let $\mathcal{L}$ be an invertible sheaf on a proper scheme X over a ring A.

1. We say that $\mathcal{L}$ is *very ample* if $\mathcal{L} \cong \varphi^*\mathcal{O}(1)$ for some immersion $\varphi : X \to \mathbb{P}^n_A$.

2. We say that $\mathcal{L}$ is *ample* if there exists $n \in \mathbb{N}$ such that $\mathcal{L}^{\otimes n}$ is very ample.

Note that the existence of an ample line bundle on X implies that X is projective.

2.5 Cohomology

2.5.1 Sheaf cohomology

As we have already seen, the structure sheaf of a scheme encodes a larger amount of information than the space alone. This is entirely in keeping with a central tenet of Grothendieck's formalism, that categorical notions are often best studied as properties of arrows rather than objects, hence an object is best understood via maps to and from it, e.g. sections of a sheaf, which we think of as functions on the underlying space.

We will now outline the theory of sheaf cohomology, the fruits of which are ubiquitous in this thesis and the existence of which underscores the utility of sheaves.

We may define a functor $\Gamma(X, -)$ from abelian sheaves (i.e. sheaves of abelian groups) on a topological space X to abelian groups by taking global sections. Since the source category has enough injectives [39, Cor. III.2.3], we can define a right derived functor with respect to $\Gamma(X, -)$. In other words, we can produce an injective resolution of any abelian sheaf and take its cohomology. We define the ith sheaf cohomology group $H^i(X, \mathscr{F})$ of the abelian sheaf $\mathscr{F}$ to be the ith cohomology group of this injective resolution.

In addition to the structure sheaf, each variety possesses another distinguished sheaf known as the *canonical sheaf.* To a variety X of dimension n we may associate a sheaf of relative differentials $\Omega_{X/k}$, and we define the canonical sheaf ω_X of X to be $\bigwedge^n \Omega_{X/k}$. We define the *canonical (divisor) class* $K_X \in \operatorname{Cl} X$ of X to be the divisor class corresponding to ω_X, and we define the anticanonical class to be $-K_X$.

The cohomology of the structure sheaf and the canonical sheaf encode a great deal of information about the geometry of a variety. In particular, we may generalise the concept of the geometric genus of a complex plane curve (the topological genus of the associated Riemann surface) to produce a birational invariant of smooth varieties that dictates both their geometric and arithmetic properties, as we shall shortly see in the case of curves.

Definition 2.5.1. Let X be a smooth projective geometrically integral variety over a field k. We define the *geometric genus* $g(X)$ of X to be the non-negative integer $\dim_k H^0(X, \omega_X)$, i.e. the dimension of the k-vector space of global sections of ω_X.

There exists another notion of genus for projective schemes known as the *arithmetic genus* $p_a(X)$, which coincides with the geometric genus when X is a smooth projective geometrically integral curve.

2.5.2 Étale cohomology

While sheaf cohomology as defined above works well for coherent sheaves, the Zariski topology is in some sense "too large" for the cohomology of constant sheaves. By making relative the notion of open sets, working with the category of étale morphisms to a space rather than the category of its open sets, one may define a finer (Grothendieck) topology known as the *étale topology* and an accompanying notion of étale sheaves, from which one may proceed to develop the notion of *étale cohomology* $H^i_{\text{ét}}(X, \mathcal{F})$ of an étale sheaf $\mathcal{F}$ on a space X.

Definition 2.5.2. Given a k-scheme X, we define the *(cohomological) Brauer group* of X to be $H^2_{\text{ét}}(X, \mathbb{G}_m)$.

Here, $\mathbb{G}_m$ denotes the abelian sheaf given by $U \mapsto \mathcal{O}_U(U)^*$ (giving invertible sections).

Another possible motivation for the development of étale cohomology is the generalisation of Galois cohomology, i.e. group cohomology for modules equipped with an action of the absolute Galois group $G_k = \mathrm{Gal}(\overline{k}/k)$ for some field k. If so, then this development may be called a success, as we have

$$H^i_{\mathrm{Gal}}(k, \mathbb{G}_m) \cong H^i_{\text{ét}}(\operatorname{Spec} k, \mathbb{G}_m),$$

where $H^i_{\mathrm{Gal}}(k, \mathbb{G}_m) = H^i(G_k, \mathbb{G}_m(\overline{k}))$ denotes the ith cohomology group of G_k with values in $\mathbb{G}_m$.

2.5.3 Brauer–Manin obstruction

One of the most important applications of étale cohomology to arithmetic comes from a construction of Manin [55] to explain the lack of rational points on a variety, which we shall now briefly explain. An excellent treatment of this topic can be found in [80, Ch. 5].

Let X be a variety over a number field K, and let $v \in \mathrm{Val}(K)$. Given $P \in X(K_v)$, we obtain by functoriality a map $\operatorname{Br} X \to \operatorname{Br} K_v$. Hence, to each $\mathcal{A} \in \operatorname{Br} X$, we obtain an associated element $\mathcal{A}(P)$. Shifting perspective, we obtain for each $\mathcal{A} \in \operatorname{Br} X$ a map $\mathrm{ev}_{\mathcal{A}} : X(K_v) \to \operatorname{Br} K_v$, which we call the *evaluation map* associated to $\mathcal{A}$.

Thanks to the existence of local invariant maps $\mathrm{inv}_v : \operatorname{Br} K_v \to \mathbb{Q}/\mathbb{Z}$ for each place v [67, Thm. 1.5.34], we obtain a pairing $\operatorname{Br} X \times X(\mathbb{A}_K) \to \mathbb{Q}/\mathbb{Z}$. It follows from class field theory that $X(K)$ lies in the right kernel of this pairing, i.e. the set

$$X(\mathbb{A}_K)^{\operatorname{Br} X} = \{P \in X(\mathbb{A}_K) : \mathcal{A}(P) = 0 \text{ for all } \mathcal{A} \in \operatorname{Br} X\}.$$

That is, we have inclusions

$$X(K) \subset X(\mathbb{A}_K)^{\operatorname{Br} X} \subset X(\mathbb{A}_K).$$

In particular, we see that $X(\mathbb{A}_K)^{\operatorname{Br} X} = \emptyset \implies X(K) = \emptyset$, and so the emptiness of the former set can be thought of as explaining the lack of rational points. We therefore make the following definition.

Definition 2.5.3. Let X be a variety over a number field K.

1. If $X(\mathbb{A}_K)^{\operatorname{Br} X} = \emptyset$, then we say that there is a *Brauer–Manin obstruction to the Hasse principle* for X.
2. If X is proper (e.g. projective) and $X(\mathbb{A}_K)^{\operatorname{Br} X}$ is not dense in $X(\mathbb{A}_K)$, then we say that there is a *Brauer–Manin obstruction to weak approximation* for X.

Note that $X(\mathbb{A}_K) = \prod_{v \in \mathrm{Val}(K)} X(K_v)$ for X proper, hence the terminology is justified in the latter case.

In certain scenarios, it is possible to show that for a class of varieties, the Brauer–Manin obstruction is the only obstruction to the Hasse principle and/or weak approximation, as made precise in the following definition.

Definition 2.5.4. Given a class of varieties over a number field K, we say that the *Brauer–Manin obstruction is the only one to the Hasse principle* (respectively, *weak approximation*) if $X(\mathbb{A}_K)^{\operatorname{Br} X} \neq \emptyset \implies X(K) \neq \emptyset$ (respectively, if $X(K)$ is dense in $X(\mathbb{A}_K)^{\operatorname{Br} X}$) for every variety X in the class.

2.6 Blowing up

The notion of blowing up a subscheme is fundamental to birational geometry, particularly in the resolution of singularities. We will be primarily interested in the case of blowing up a point on a surface, which one may think of as replacing a point P by a line which separates divisors through the point by their direction. More generally, given a closed subscheme Z on a scheme X, the blowup of X along Z is the "smallest" scheme over X on which Z becomes an effective Cartier divisor. Slightly more concretely, one may make the following definition via a universal property.

Definition 2.6.1. Let Z be a closed subscheme of a scheme X. The *blowup/blow-up of X at/along Z* is a scheme $\widetilde{X}$ together with a morphism $\pi : \widetilde{X} \to X$ satisfying the following universal property: the preimage $\pi^{-1}(Z)$ is an effective Cartier divisor, and any other morphism of schemes $\varphi : Y \to X$ such that $\varphi^{-1}(Z)$ is an effective Cartier divisor on Y factors through $\widetilde{X}$. We call the divisor $\pi^{-1}(Z)$ the *exceptional divisor* of the blowup.

Example 2.6.2. As a first example, let us consider the blowup of the affine plane $\mathbb{A}^2$ at the origin $O = (0, 0)$. Endow $\mathbb{A}^2$ with coordinates x_1, x_2, and endow the projective line $\mathbb{P}^1$ with coordinates y_1, y_2. Consider the subvariety $V \subset \mathbb{A}^2 \times \mathbb{P}^1$ given by the equation $x_1y_2 = x_2y_1$. Letting $\pi : V \to \mathbb{A}^2$ be the projection to the first factor, we see that, over every point $P = (a, b) \in \mathbb{A}^2 \setminus \{O\}$, the fibre $\pi^{-1}(P)$ consists of one point, namely $(P, [a, b])$, while $\pi^{-1}(O) = \{O\} \times \mathbb{P}^1$, a divisor on V.

2.7 Curves

Although we will primarily work on higher-dimensional varieties, we will make use of the properties of (plane) conics, arguably the genesis of the notion of variety itself, and it would seem sacreligious to give any overview of basic algebraic geometry without mentioning the arithmetic classification of curves over number fields by genus.

We begin with one of the most well-known results for curves.

Theorem 2.7.1 (Riemann–Roch for curves)**.** *[39, Thm. IV.1.3] Let C be a smooth projective curve over a field k, and let $D \subset C$ be a divisor on C. Then we have*

$$l(D) - l(K_C - D) = \deg D - g(C) + 1,$$

where $l(D)$ denotes the dimension of the k-vector space of global sections of $\mathcal{L}(D)$.

The Riemann–Roch theorem already demonstrates impressively the interplay between divisors, line bundles and cohomology. We will now put this together with the divisor–projective morphism correspondence to see that a curve of genus zero over k is isomorphic to $\mathbb{P}^1_k$ as soon as it has a k-point.

Proposition 2.7.2. *Let C be a smooth projective geometrically integral curve of geometric genus zero over a field k. If $C(k) \neq \emptyset$, then $C \cong \mathbb{P}^1_k$.*

Proof. Let $P \in C(k)$. Applying Riemann–Roch to the divisor $D = 1 \cdot P$ on C, we have

$$l(D) - l(K_C - D) = 2.$$

On the other hand, applying Riemann–Roch to 0 and K_C, we see that $\deg K_C = -2$, hence $\deg(K_C - D) = -3$. By identifying each global section with its divisor of zeros [39, Ch. 2, §7], it follows that negative-degree line bundles have no global sections, hence $l(K_C - D) = 0$, and so $l(D) = 2$. In particular, it follows that $|D|$ is base-point-free, hence it defines a non-constant morphism $\varphi : C \to \mathbb{P}^1_k$. By compatibility of pulling back sections and taking their divisors of zeros, it follows both that principal divisors on C have degree zero and that degree is constant on divisor classes of C. In particular, we deduce that φ is bijective. $\square$

This result can be seen as part of a trichotomy for the classification of curves in the case when $k = K$ a number field.

Theorem 2.7.3. *Let C be a smooth projective geometrically integral curve over an algebraic number field K.*

1. *If $g(C) = 0$, then $C(K) \neq \emptyset \implies \mathbb{P}^1_K$.*
2. *If $g(C) = 1$ and $C(K) \neq \emptyset$, then $C(K)$ may be endowed with the structure of a finitely generated abelian group.*
3. *If $g(C) \geq 2$, then $C(K)$ is finite.*

In an arithmetic context, one may think of the first statement as saying that, as soon as a curve of genus zero has one rational point, it has many.

The second statement is known as the Mordell–Weil theorem, originally proved by Mordell over $\mathbb{Q}$ [58] and later extended by Weil to abelian varieties over number fields [90]. The geometry and arithmetic of elliptic curves has motivated several of the most striking results in modern number theory, including the proof of the modularity theorem for semistable elliptic curves and thus Fermat's Last Theorem by Wiles [91]. The Birch and Swinnerton–Dyer conjecture [7], one of the seven Millennium prize problems, hypothesises a relationship between the rank of an elliptic curve E and the analytic properties of a certain complex-valued function $L(E, s)$.

The third statement, formerly known over $\mathbb{Q}$ as Mordell's conjecture, is now known as Faltings' theorem thanks to Faltings' proof of a stronger result on the finiteness of isomorphism classes of abelian varieties [33].

2.8 Surfaces

Having established the picture for rational points on algebraic curves, we turn our attention to varieties one dimension up, i.e. surfaces.

2.8.1 Intersection theory

When studying a given variety, the configuration of its divisors (or more generally, closed subschemes) may reveal much about its geometry. This is precisely the spirit behind the notion of a "fibration" of a variety X over a variety Y. Loosely, this means a surjective flat morphism $f : X \to Y$. The surjectivity of f allows us to associate to each $y \in Y$ a non-empty subscheme of X, namely the fibre $f^{-1}(y)$, and the assumption of flatness means that these fibres form a "nicely varying" family. In the right circumstances, we may reduce questions about the variety X to the fibres of f, and the conventional wisdom is that these subschemes will be easier to handle.

In complex geometry we have Bézout's theorem, which tells us that there exists a unique notion of intersection multiplicity between complex projective plane curves such that two curves C and D defined by polynomials of degrees c and d respectively meet in exactly cd points when counted with multiplicity. Generalising this, we have the notion of the *intersection product/pairing* on a smooth projective surface X.

Theorem 2.8.1. *[39, Thm. V.1.1] Let X be a smooth projective geometrically integral surface. There is a unique pairing $\operatorname{Div} X \times \operatorname{Div} X \to \mathbb{Z}$ denoted by $C \cdot D$ for any two divisors C, D such that*

1. *if C and D are non-singular curves meeting transversally, then $C \cdot D = \#(C \cap D)$, the number of points of $C \cap D$,*
2. *it is symmetric: $C \cdot D = D \cdot C$,*
3. *it is additive: $(C_1 + C_2) \cdot D = C_1 \cdot D + C_2 \cdot D$, and*
4. *it depends only on the linear equivalence classes: if $C_1 \sim C_2$ then $C_1 \cdot D = C_2 \cdot D$.*

One interesting consequence of this theorem is that one may compute the *self-intersection* of a divisor $D \subset X$, i.e. the quantity $D \cdot D \in \mathbb{Z}$ (also denoted by D^2). This will feature prominently in our forthcoming discussion of del Pezzo surfaces, on which divisors with self-intersection -1 play a pivotal role. As we shall now see, these so-called (-1)-curves are closely related to the operation of blowing up at a point, also known as a *monoidal transformation.*

Proposition 2.8.2. *[39, Prop. V.3.2] Let X be a smooth projective geometrically integral surface over a field k, and let $P \in X(k)$. Let $\pi : Y = \operatorname{Bl}_P X \to X$ denote the blowup of X at P. Then Y is also a smooth projective geometrically integral surface over k. Let $E \subset \operatorname{Bl}_P X$ denote the exceptional divisor. Then the natural maps $\pi^* : \operatorname{Pic} X \to \operatorname{Pic} Y$ and the map $\mathbb{Z} \to \operatorname{Pic} Y$, $1 \mapsto E$, induce an isomorphism $\operatorname{Pic} Y \cong \operatorname{Pic} X \oplus \mathbb{Z}$. The intersection product on Y is uniquely determined as follows. Let $C_1, C_2 \in \operatorname{Pic} X$.*

1. *$\pi^* C_1 \cdot \pi^* C_2 = C_1 \cdot C_2$.*
2. *$\pi^* C_1 \cdot E = 0$.*
3. *$E^2 = -1$.*

Further, given $D \in \operatorname{Pic} Y$, we have $\pi^* C_1 \cdot D = C_1 \cdot \pi_* D$, where $\pi_* : \operatorname{Pic} Y \to \operatorname{Pic} X$ denotes projection onto the first factor.

In fact, the converse to the above proposition also holds, in the sense that every (-1)-curve isomorphic to $\mathbb{P}^1$ on a smooth projective surface may be viewed as the exceptional divisor for the blowup of another surface.

The utility of intersection theory is further apparent in the following two results, two of the most consequential in the study of algebraic surfaces.

Proposition 2.8.3 (Adjunction formula)**.** *[39, Exercise V.1.3, p. 366] Let D be an effective divisor on a smooth projective surface X over a field. Then*

$$2p_a(D) - 2 = D^2 + D \cdot K_X,$$

where $p_a(D)$ is the arithmetic genus of D. In particular, if D is a smooth projective geometrically integral curve, we have

$$2g(D) - 2 = D^2 + D \cdot K_X.$$

Theorem 2.8.4 (Riemann–Roch for surfaces)**.** *[39, Thm V.1.6] Let D be a divisor on a smooth projective surface X over a field k. Then*

$$l(D) - s(D) + l(K_X - D) = \frac{1}{2} D \cdot (D - K_X) + \chi(X, \mathcal{O}_X),$$

where $s(D) = \dim_k H^1(X, \mathcal{L}(D))$ and $\chi(X, \mathcal{O}_X) = \sum_{i=0}^{2} (-1)^i \dim_k H^i(X, \mathcal{O}_X)$ are the superabundance *of D and the* Euler characteristic *of X respectively.*

2.8.2 Del Pezzo surfaces

Definition 2.8.5. We say that a smooth projective geometrically integral variety X over a field k is *Fano* if the anticanonical divisor $-K_X$ is ample. A *del Pezzo surface* is a Fano variety of dimension 2. Given a del Pezzo surface X, we define the *degree* of X to be $d(X) = K_X^2$.

These surfaces were first introduced by their namesake, Pasquale del Pezzo, who studied those for which the anticanonical divisor is very ample, which are exactly those of degree at least 3.

Over an algebraically closed field, the description of del Pezzo surfaces is relatively simple from the perspective of birational geometry: they are exactly the blowups of $\mathbb{P}^2$ in at most 8 points in general position (along with one extra surface, namely $\mathbb{P}^1 \times \mathbb{P}^1$). Here, general position means that no three of the points lie on a line, no six lie on a conic, and no eight lie on a cubic with a node at one of them.

It follows from Proposition 2.8.2 that the self-intersection of the canonical class decreases by one on blowing up at a point. In particular, we see that for X a del Pezzo surface, we have $d(X) \in \{1, 2, \dots, 9\}$. Each successive blowup introduces new divisors on $X_{\overline{k}}$ with self-intersection -1, which we call *lines* or *exceptional curves* on $X_{\overline{k}}$ (see Section 2.6). Via complex plane geometry, we may compute the number of lines on a del Pezzo surface of given degree. Further, the projective morphism associated to the anticanonical linear system allows us to describe in more classical terms the del Pezzo surfaces of a given degree. We summarise the information on the shape and lines of del Pezzo surfaces in the following theorem.

Theorem 2.8.6. *[56, Ch. IV] Let X be a del Pezzo surface of degree d over a field k.*

1. *If $d = 9$, then $X \cong \mathbb{P}^2_k$, and $X_{\overline{k}}$ contains no exceptional curves. The anticanonical embedding of X is the degree-3 Veronese embedding into $\mathbb{P}^9$.*

2. *If $d = 8$, then X is isomorphic to either $\mathbb{P}^1 \times \mathbb{P}^1$, in which case $X_{\overline{k}}$ has no exceptional curves, or to the blowup of $\mathbb{P}^2$ in one k-point, in which case $X_{\overline{k}}$ has 1 exceptional curve.*

3. *If $d = 7$, then X is isomorphic to the blowup of $\mathbb{P}^2$ in two distinct points, and $X_{\overline{k}}$ has 3 exceptional curves.*

4. *If $d = 6$, then X is isomorphic to the blowup of $\mathbb{P}^2$ in three non-collinear points, and $X_{\overline{k}}$ has 6 exceptional curves.*

5. *If $d = 5$, then X is isomorphic to the blowup of $\mathbb{P}^2$ in four points, no three of which are collinear, and $X_{\overline{k}}$ has 10 exceptional curves.*

6. *If $d = 4$, then the anticanonical embedding of X in $\mathbb{P}^4$ is isomorphic to a smooth complete intersection of two quadrics, and $X_{\overline{k}}$ has 16 exceptional curves.*

7. *If $d = 3$, then the anticanonical embedding of X in $\mathbb{P}^3$ is isomorphic to a cubic surface, and $X_{\overline{k}}$ has 27 exceptional curves.*

8. *If $d = 2$, then the anticanonical class induces a degree-2 morphism to $\mathbb{P}^2$ branched over a quartic plane curve, and $X_{\overline{k}}$ has 56 exceptional curves.*

9. *If $d = 1$, then $|-2K_X|$ induces a degree-2 morphism to a quadratic cone in $\mathbb{P}^3$ branched over a smooth genus four curve cut out by a cubic surface, and $X_{\overline{k}}$ has 240 exceptional curves.*

Note in particular that del Pezzo surfaces are *geometrically rational*, meaning that $X_{\overline{k}}$ is birational to $\mathbb{P}^2_{\overline{k}}$. As such, we think of del Pezzo surfaces as being not too far from rational, and we therefore expect them to have many rational points, provided that they have any to begin with. As evidence in support of this idea, we present the following results.

Theorem 2.8.7. *[67, §9.4] Let X be a del Pezzo surface of degree d over a field k, and suppose that $X(k) \neq \emptyset$.*

1. *If $d \geq 5$, then X is k-rational, i.e. X is birational over k to $\mathbb{P}^2_k$.*

2. *If $d \geq 3$, then X is k-unirational, i.e. there exists a dominant rational map $\mathbb{P}^2_k \dashrightarrow X$.*

In particular, it follows that as soon as a del Pezzo surface of degree at least 3 possesses a rational point, then the rational points are dense on X. Conjecturally however, we expect even more, and on a larger class of varieties.

Definition 2.8.8. Let X be a variety over an uncountable field k. We say that X is *rationally connected* if there exists an open subset $U \subset X \times X$ such that, for $(P, Q) \in U(k)$, there exists a rational curve $C \subset X$ (i.e. the image of a morphism $\mathbb{P}^1 \to X$) through P and Q. We say that X is *geometrically rationally connected* if $X_{\overline{k}}$ is rationally connected.

Since all rational varieties are rationally connected, all del Pezzo surfaces (more generally, all Fano varieties) are geometrically rationally connected, and as such they are included in the following conjecture of Colliot-Thélène.

Conjecture 2.8.9. *[21, p. 174] Let X be a smooth projective geometrically integral surface over a number field K. Suppose that X is geometrically rationally connected. Then the Brauer–Manin obstruction is the only one to weak approximation on X.*

It comes as a consequence of the relationship between schemes over K and over $\mathcal{O}_S$ for some finite subset $S \subset \mathrm{Val}(K)$ that each element of $\mathrm{Br}\, X$ pairs trivially with all local points at all but finitely many places. Further, there exists a canonical injection $\mathrm{Br}\, K \hookrightarrow \mathrm{Br}\, X$ on the image of which the Brauer–Manin pairing is trivial. In particular, when $\mathrm{Br}\, X/\, \mathrm{Br}\, K$ is finite, we may describe the Brauer–Manin obstruction as being supported at finitely many places. In that case, if the Brauer–Manin obstruction is the only one to weak approximation on X, then X satisfies weak weak approximation.

2.8.3 Manin's conjecture

Colliot-Thélène's conjecture may be seen as a prediction of the geometric abundance of rational points on Fano varieties. As an analytic counterpart, we have Manin's conjecture, which in some sense predicts the number of rational points on a Fano variety. More concretely, the theory of adelic metrisations allows one to associate to each very ample line bundle $\mathcal{L}$ a height function $H_{\mathcal{L}} : X(K) \to \mathbb{R}_{>0}$. Taking H to be a height function associated to $-K_X$, we may consider the sets of rational points of bounded height, i.e. for given $B > 0$, the set

$$\{P \in X(K) : H(P) \leq B\}.$$

The asymptotic behaviour of the cardinality of this set as $B \to \infty$ tells us about the analytic abundance of rational points on X. In its most general form, the conjecture is as follows.

Conjecture 2.8.10 (Manin's conjecture)**.** *Let X be a Fano variety over a number field K, and let H be an anticanonical height function. Then there exists an exceptional subset $A \subset X(K)$ such that*

$$\#\{P \in X(K) \setminus A : H(P) \leq B\} \sim cB(\log B)^{\mathrm{rank}\,\mathrm{Pic}\, X - 1}$$

as $B \to \infty$ for some explicit constant c.

The leading constant was subsequently interpreted by Peyre [63] in terms of so-called Tamagawa numbers coming from harmonic analysis over the adelic points.

In Manin's original formulation, the exceptional set was taken to be the set of rational points of some proper closed subset of X, called an *accumulating subvariety*, which one thinks of as contributing a large number of rational points (e.g. a K-rational line on a cubic surface). After the conjecture was shown to be false in this form by Batyrev and Tschinkel [5], the exceptional set was taken instead to be a *thin set*, which we shall encounter in Chapter 3. Counterexamples to this adapted conjecture have yet to be discovered, with support coming first from the work of Le Rudulier [52]. The question of which thin set to remove has motivated recent work of Lehmann, Sengupta and Tanimoto [51], in which the authors provide a conjectural description of the exceptional set related to the constants appearing in Manin's conjecture itself. Manin-type asymptotics have been shown to hold for several families of varieties, including toric varieties. In Chapter 5, we will adapt the methods of Batyrev and Tschinkel in their proof of Manin's conjecture for anisotropic tori to the analytic study of Campana points.

Chapter 3

Hilbert property for double conic bundles and del Pezzo varieties

3.1 Overview

3.1.1 Motivation

In this chapter, we will describe and study the *Hilbert property*: briefly, a variety X/k is said to have the Hilbert property if the set $X(k)$ is not thin (we give the definition of thin sets in Section 3.2). In addition to providing a geometric notion of abundance of rational points on a variety, the Hilbert property is also connected to the so-called *inverse Galois problem*, which is to determine whether every finite group G is realisable as the Galois group of a Galois extension of a given field, by the following conjecture of Colliot-Thélène and Sansuc.

Conjecture 3.1.1. ([24, p. 190]) *Let X be a unirational variety over a number field. Then X has the Hilbert property.*

As noted in the proof of [75, Thm. 3.5.9, p. 30], the truth of this conjecture implies a positive solution to the inverse Galois problem over $\mathbb{Q}$.

The Hilbert property was investigated by Corvaja and Zannier in [26], where the authors proved that the Fermat quartic

$$x^4 + y^4 = z^4 + w^4 \subset \mathbb{P}^3$$

over $\mathbb{Q}$ has the Hilbert property. This surface is not geometrically unirational, hence the converse of Conjecture 3.1.1 is false. Demeio later proved in [29] and [28] that, over a number field K, certain double elliptic surfaces and all cubic hypersurfaces with a K-rational point have the Hilbert property. Further, it follows from the work of Salberger and Skorobogatov in [70] that smooth intersections of two quadrics in $\mathbb{P}^4$ with a rational point over a number field have the Hilbert property. Fibrations play an important role in proving these results, and the role of fibrations in studying the Hilbert property is emphasised by the main theorem of Bary-Soroker, Fehm and Petersen in [3] (Theorem 3.3.1), which allows us to reduce questions about the rational points of a variety to the fibres of some morphism from it. This result is crucial to the methods we shall employ, which make use of conic fibrations. Conic fibrations were also used by Coccia in [20] to prove that affine smooth cubic surfaces over number fields satisfy a version of the Hilbert property for S-integral points after a finite field extension.

3.1.2 Results

The main result that we shall prove is the following theorem.

Theorem 3.1.2. *Let S be a smooth projective surface defined over a Hilbertian field k possessing two conic fibrations $\pi_i : S \to \mathbb{P}^1$, $i = 1, 2$ such that two fibres from different fibrations have non-zero intersection product, and suppose that there exists $P_0 \in S(k)$ lying on a smooth fibre of one of the conic fibrations. Then S has the Hilbert property.*

Remark 3.1.3. We observe that these surfaces are *geometrically rational*, i.e. they are rational surfaces over the algebraic closure of their field of definition.

In particular, we obtain the following result for del Pezzo surfaces.

Theorem 3.1.4. *Let S be a del Pezzo surface of degree d defined over a Hilbertian field k with $S(k) \neq \emptyset$. Suppose that one of the following holds:*

(a) $d \geq 4$*;*

(b) $d = 3$ *and there exists a line L on S (under its anticanonical embedding);*

(c) $d \in \{1, 2\}$ *and S possesses a conic fibration.*

Then S has the Hilbert property.

By work of Manin [56, Thm. 29.4, p. 158, Thm. 30.1, p. 162] and Kollár and Mella [49, Cor. 8, p. 919], all surfaces in Theorem 3.1.4 are unirational.

By induction with Theorem 3.1.4 as the base case, we will also prove the following result for del Pezzo varieties (which we define in Section 3.4).

Theorem 3.1.5. *Let (X, H) be a smooth del Pezzo variety of degree d and dimension $n \geq 2$ defined over a Hilbertian ground field k with $X(k) \neq \emptyset$. Suppose that one of the following holds:*

(a) $d \geq 4$*;*

(b) $d = 3$ *and there exists a line L on X (under its anticanonical embedding).*

Then X has the Hilbert property.

Examples of del Pezzo varieties include cubic hypersurfaces (degree 3) and intersections of two quadric hypersurfaces (degree 4).

3.1.3 Conventions

Throughout this chapter, we will work over an arbitrary ground field k of characteristic 0, and all algebraic varieties shall be understood to be integral over the ground field k and quasi-projective.

We include the following definition for clarity.

Definition 3.1.6. Let S be a smooth projective surface. A *conic fibration* of S is a morphism $\pi : S \to \mathbb{P}^1$ such that all fibres of π are isomorphic to plane conics. We call a surface with a conic fibration a *conic bundle* (over $\mathbb{P}^1$).

Remark 3.1.7. If $\pi : S \to \mathbb{P}^1$ is a conic fibration, then by the adjunction formula (see [35, Lem. 2.1, p. 412]), the fibre $\pi^{-1}(P)$ is reduced for all $P \in \mathbb{P}^1$.

3.2 Thin sets

We now define thin sets and the Hilbert property. We recall our conventions that $\operatorname{char} k = 0$ and that algebraic varieties are integral and quasi-projective. We begin by giving the definition of thin sets from [75, Ch. 3, §1], amending the notation and terminology slightly.

Definition 3.2.1. Let V be a variety over k.

A subset $A \subset V(k)$ is of *type I* if there is a proper Zariski closed subset $W \subset V$ with $A \subset W(k)$.

A subset $A \subset V(k)$ is of *type II* if there is a variety V' with $\dim V' = \dim V$ and a generically finite dominant morphism $\phi : V' \to V$ of degree ≥ 2 with $A \subset \phi(V'(k))$.

A subset $A \subset V(k)$ is *thin* if it is contained in a finite union of subsets of $V(k)$ of types I and II.

Remark 3.2.2. Serre mentions (see [75, p. 20]) that the variety V' in the definition of type II thin sets can be taken to be geometrically irreducible, as otherwise we have $V'(k) \subset W'(k)$ for some Zariski closed proper subset W' of V'.

Remark 3.2.3. Further to the above remark, we can ask that V' be normal and that ϕ be finite. Indeed, given a variety V' and a generically finite dominant morphism $\phi : V' \to V$, the function field $K(V')$ of V' is a finite extension of the function field $K(V)$ of V, and we can consider the *normalisation of* V *in* $K(V')$ (see [53, Def. 4.1.24, p. 120]). This is a normal scheme $\widetilde{V'}$ with $K(\widetilde{V'}) = K(V')$ together with an integral morphism $\widetilde{\phi} : \widetilde{V'} \to V$. By [53, Prop. 4.1.27, p. 121], we have that $\widetilde{\phi}$ is finite and $\widetilde{V'}$ is an algebraic variety. Moreover, by the universal property of normalisation, $\phi : V' \to V$ factors uniquely through $\widetilde{\phi}$:

$$\begin{tikzcd} & \widetilde{V'} \arrow[d, "\widetilde{\phi}"] \\ V' \arrow[ur, dotted, "\exists!"] \arrow[r, "\phi"'] & V. \end{tikzcd}$$

In particular, we see that $\phi(V'(k)) \subset \widetilde{\phi}(\widetilde{V'}(k))$, so replacing the pair (V', ϕ) by $(\widetilde{V'}, \widetilde{\phi})$ allows us to demand these stronger requirements for type II thin sets.

Definition 3.2.4. Let V be an algebraic variety over a field k. We say that V has the *Hilbert property* (over k) if the set $V(k)$ is not thin.

Note 3.2.5. If V has the Hilbert property over k, then $V(k)$ is Zariski dense. Also, the Hilbert property is a birational invariant.

Definition 3.2.6. We say that a field k of characteristic zero is *Hilbertian* if there exists a variety V/k such that V has the Hilbert property over k.

Remark 3.2.7. It can be shown that the field k is Hilbertian if and only if $\mathbb{P}^1$ has the Hilbert property over k (see [75, Exercise 3.1.1, p. 20]).

3.3 Double conic bundles and del Pezzo surfaces

We now prove Theorem 3.1.2. The following result shall be essential to our proof.

Theorem 3.3.1. *[3, Thm. 1.1, p. 1894] Let k be a field and $f : X \to S$ a dominant morphism of k-varieties. Assume that the set of $s \in S(k)$ for which the fibre $f^{-1}(s)$ has the Hilbert property is not thin. Then X has the Hilbert property.*

In particular, we have the following corollary for conic fibrations.

Corollary 3.3.2. *Let k be a Hilbertian field and $\pi : S \to \mathbb{P}^1$ a conic fibration. Assume that the set*

$$\{P \in \mathbb{P}^1(k) : \pi^{-1}(P)(k) \neq \emptyset\}$$

is not thin. Then S has the Hilbert property over k.

Proof. By the Riemann-Roch theorem, any smooth curve of genus zero with a rational point is isomorphic to $\mathbb{P}^1$ (Proposition 2.7.2), which has the Hilbert property (see Remark 3.2.7), and all but finitely many fibres of π are smooth curves of genus zero. □

Then Theorem 3.1.2 follows immediately from the following proposition.

Proposition 3.3.3. *Let S be a smooth projective surface defined over a Hilbertian field k with two distinct conic fibrations $\pi_i : S \to \mathbb{P}^1$, $i = 1, 2$, and suppose that there exists $P_0 \in S(k)$ such that $\pi_1^{-1}(\pi_1(P_0))$ is smooth. For each $i = 1, 2$, the set*

$$A_i = \{P \in \mathbb{P}^1(k) : \pi_i^{-1}(P)(k) \neq \emptyset\}$$

is not thin.

In order to prove Proposition 3.3.3, we will make use of the following well-known result for fibre products of curves over $\mathbb{P}^1$, a proof of which we give for completeness.

Lemma 3.3.4. *Let C and D be two regular, geometrically irreducible curves over a field k equipped with non-constant morphisms $\phi : C \to \mathbb{P}^1$ and $\psi : D \to \mathbb{P}^1$ having disjoint branch loci. Then $C \times_{\mathbb{P}^1} D$ is a regular, geometrically irreducible curve.*

Proof. Since ϕ and ψ have disjoint branch loci, there is an open neighbourhood for any point in $\mathbb{P}^1$ over which one of ϕ and ψ is étale and the other is regular, so the corresponding open set in $C \times_{\mathbb{P}^1} D$ is étale over a regular curve (since the base change of an étale morphism is étale), hence $C \times_{\mathbb{P}^1} D$ is regular.

Since $\overline{k}(C \times_{\mathbb{P}^1} D) = \overline{k}(C) \otimes_{\overline{k}(\mathbb{P}^1)} \overline{k}(D)$, the curve $C \times_{\mathbb{P}^1} D$ is geometrically irreducible if and only if $\overline{k}(C)$ and $\overline{k}(D)$ are linearly disjoint. By [36, Lem. 2.5.3, p. 35], it suffices to check linear disjointness of $\overline{k}(C)$ and $\overline{k}(D^{\mathrm{Gal}})$, where $\psi^{\mathrm{Gal}} : D^{\mathrm{Gal}} \to \mathbb{P}^1$ denotes the Galois closure of $\psi : D \to \mathbb{P}^1$ (see [75, p. 20]), which holds if and only if $\overline{k}(C) \cap \overline{k}(D^{\mathrm{Gal}}) = \overline{k}(\mathbb{P}^1)$. Suppose that $\overline{k}(C)$ and $\overline{k}(D^{\mathrm{Gal}})$ possess a common subextension with corresponding curve and morphism $\theta : E \to \mathbb{P}^1$. Since $\mathbb{P}^1_{\overline{k}}$ is algebraically simply connected, θ is ramified (see [26, §1.2]). Any branch points of θ are common branch points of ϕ and ψ^{Gal}, but the branch points of ψ^{Gal} are the same as those of ψ, hence θ is unramified, contradiction, so $C \times_{\mathbb{P}^1} D$ is geometrically irreducible. □

Before employing this key lemma, we first show that each of the sets A_i is certainly not a type I thin set.

Lemma 3.3.5. *Each of the sets A_i in Proposition 3.3.3 is infinite. That is, each of the conic fibrations has infinitely many fibres which each have a rational point.*

Proof. Since $\pi_1^{-1}(\pi_1(P_0))$ is smooth, $\#\pi_1^{-1}(\pi_1(P_0))(k) = \infty$. Let $\{Q_i\}_{i\in\mathbb{N}}$ be an infinite collection of rational points on $\pi_1^{-1}(\pi_1(P_0))$. Since each smooth fibre of one fibration intersects any fibre from the other fibration in a finite set, $\{\pi_2(Q_i)\} \subset A_2$ is infinite, so A_2 is infinite. Repeating this argument with any one of the fibres $\pi_2^{-1}(\pi_2(Q_i))$ which is smooth, we see that A_1 is also infinite. $\square$

Proof of Proposition 3.3.3. It suffices to prove that A_2 is not thin, since Lemma 3.3.5 implies that π_2 also has a smooth fibre with a rational point. So, assume that $A_2 = \bigcup_{P\in A_1} \pi_2\big(\pi_1^{-1}(P)(k)\big)$ is thin. Then it is easily seen that there exists a finite collection of dominant finite morphisms $\varphi_i : C_i \to \mathbb{P}^1$, $i = 1, ..., r$ where C_i is a normal (hence smooth) geometrically irreducible curve defined over k and $\deg\varphi_i > 1$, such that $\bigcup_{P\in A_1} \pi_2\big(\pi_1^{-1}(P)(k)\big) \subset \bigcup_{i=1}^r \varphi_i(C_i(k))$. Let

$$B = \bigcup_{i=1}^{r} B_{\varphi_i} = \{R_1, \dots, R_m\},$$

where B_{φ_i} denotes the branch locus of the morphism φ_i.

Since each fibre $\pi_2^{-1}(R_i)$ is reduced, the morphism

$$\pi_1|_{\pi_2^{-1}(R_i)} : \pi_2^{-1}(R_i) \to \mathbb{P}^1$$

has finite branch locus for each $i = 1, \dots, m$. Denote the union of these branch loci by B', and denote by A_1' the set of $P \in A_1 \setminus B'$ such that $\pi_1^{-1}(P)$ is smooth, which is non-empty by Lemma 3.3.5. Taking $P \in A_1'$, the morphism $\pi_1|_{\pi_2^{-1}(R_i)} : \pi_2^{-1}(R_i) \to \mathbb{P}^1$ is unramified over P for each i, which is equivalent to $\left(\pi_1|_{\pi_2^{-1}(R_i)}\right)^{-1}(P) = \pi_1^{-1}(P) \cap \pi_2^{-1}(R_i)$ being reduced for each i (see [53, Lem. 4.3.20, p. 139]) and in turn equivalent to $\pi_2|_{\pi_1^{-1}(P)}$ being unramified over every R_i. Then for all $P \in A_1'$, $i = 1, ..., r$, the branch loci of $\pi_2|_{\pi_1^{-1}(P)}$ and C_i are disjoint.

For each $P \in A_1'$, consider the fibre product

$$\begin{array}{ccc} F_i & \xrightarrow{\psi_i} & C_i \\ \big\downarrow{\scriptstyle\theta_i} & & \big\downarrow{\scriptstyle\varphi_i} \\ \pi_1^{-1}(P) & \xrightarrow{\pi_2|_{\pi_1^{-1}(P)}} & \mathbb{P}^1. \end{array}$$

By Lemma 3.3.4, F_i is a smooth, irreducible curve. Since surjectivity is preserved under base change, the maps ψ_i and θ_i are surjective. By comparing ramification indices of points on $\mathbb{P}^1$ for the morphism $\varphi_i \circ \psi_i = \pi_2|_{\pi_1^{-1}(P)} \circ \theta_i : F_i \to \mathbb{P}^1$ and using the compatibility of ramification indices with composition of morphisms, one sees that θ_i is ramified, hence $\deg\theta_i > 1$. Since $\pi_1^{-1}(P)$ has the Hilbert property, the set $\pi_1^{-1}(P)(k) \setminus \bigcup_{i=1}^r \theta_i(F_i(k))$ is infinite.

Let $Q \in \pi_1^{-1}(P)(k) \setminus \bigcup_{i=1}^r \theta_i(F_i(k))$. Then we claim that

$$\pi_2(Q) \in \bigcup_{P\in A_1} \pi_2\big(\pi_1^{-1}(P)(k)\big) \setminus \bigcup_{i=1}^{r} \varphi_i(C_i(k)).$$

Clearly $\pi_2(Q) \in \bigcup_{P\in A_1} \pi_2\big(\pi_1^{-1}(P)(k)\big)$, and if there is $Q' \in C_i(k)$ with $\varphi_i(Q') = \pi_2(Q)$ for some i, then there exists $Q'' \in F_i(k)$ such that $\theta_i(Q'') = Q$, which is impossible since $Q \in \pi_1^{-1}(P)(k)\setminus\theta_i(F_i(k))$. Then we have a contradiction to our original assumption that $\bigcup_{P\in A_1} \pi_2\big(\pi_1^{-1}(P)(k)\big) \subset \bigcup_{i=1}^r \varphi_i(C_i(k))$, and so $\bigcup_{P\in A_1} \pi_2\big(\pi_1^{-1}(P)(k)\big)$ is not thin. $\square$

Proof of Theorem 3.1.2. The result follows from Proposition 3.3.3 and Corollary 3.3.2. □

For the proof of Theorem 3.1.4, we will employ the following fact, which is seen in the proof of [45, Thm. 5, p. 21]. Although the del Pezzo surfaces in the statement there are minimal, minimality is not necessary. We give the proof for completeness.

Lemma 3.3.6. *Let S be a del Pezzo surface of degree $d \in \{1,2,4\}$ with a conic fibration $\pi : S \to \mathbb{P}^1$. Then there exists a second conic fibration $\pi' : S \to \mathbb{P}^1$ such that fibres from distinct fibrations have non-zero intersection product.*

Proof. Let C be a fibre of the conic fibration $\pi : S \to \mathbb{P}^1$, and define

$$D := -\frac{4}{d}K_S - C.$$

We will show that the linear system $|D|$ induces the sought conic fibration π'.

By the Riemann-Roch theorem for surfaces (Theorem 2.8.4),

$$l(D) - s(D) + l(K_S - D) = \frac{1}{2}D \cdot (D - K_S) + \chi(S, \mathcal{O}_S).$$

Since S is geometrically rational and $-K_S$ is ample, it follows from Serre duality [39, Cor. III.7.7, p. 244] that $\chi(S, \mathcal{O}_S) = 1$. Further, if E is an effective divisor linearly equivalent to $K_S - D$, then the ampleness of $-K_S$ implies that $-K_S \cdot E > 0$, but

$$-K_S \cdot E = -K_S \cdot \left(\left(1 + \frac{4}{d}\right)K_S + C\right) = -\left(1 + \frac{4}{d}\right)d + K_S \cdot C = -(d+2) < 0,$$

where $K_S \cdot C = -2$ by the adjunction formula (Proposition 2.8.3) and the fact that $C^2 = 0$. Thus $l(K_S - D) = 0$. Next, we compute

$$D^2 = \frac{16}{d^2}d + \frac{8}{d}(-2) + 0 = 0,$$

$$-K_S \cdot D = \frac{4}{d}d - 2 = 2.$$

Then the Riemann-Roch theorem applied to D rearranges to give

$$l(D) = 2 + s(D) \geq 2,$$

and so D is effective. By the adjunction formula, D has arithmetic genus zero.

It only remains to verify that $|D|$ is base-point-free. Since $-K_S \cdot D = 2$ and $-K_S$ is ample, any element of $|D|$ has at most 2 irreducible components. Then, for any $E \in |D|$, one of the following possibilities holds:

(a) E is irreducible;

(b) $E = E_1 + E_2$, where $E_1 \neq E_2$, or

(c) $E = 2E_1$

for some irreducible curves E_1, E_2 on S. In (c), we obtain $-K_S \cdot E = 1$, but then the adjunction formula implies that the arithmetic genus of E_1 is not an integer, hence this possibility cannot be realised. In (b), we square to get

$$0 = E_1^2 + E_2^2 + 2E_1 \cdot E_2 \geq E_1^2 + E_2^2.$$

We cannot have $E_i^2 = 0$ for $i = 1, 2$, as then the adjunction formula implies that the arithmetic genus of E_i is not an integer, and on a del Pezzo surface, every effective divisor has self-intersection at least -1, hence $E_1^2 = E_2^2 = -1$. However, there are only finitely many curves on S (namely the exceptional curves) with self-intersection -1, hence all but finitely many curves in $|D|$ are irreducible. Since the intersection product of two irreducible curves is zero if and only if they do not intersect, $|D|$ is base-point-free, hence it gives rise to a morphism $\pi' : S \to \mathbb{P}^1$. Since the arithmetic genus of D is zero, π' is a conic fibration.

Finally, since the intersection product of a fibre from π with a fibre from π' is

$$C \cdot D = C \cdot \left(-\frac{4}{d} K_S - C \right) = \frac{8}{d} > 0,$$

we are done. □

Proof of Theorem 3.1.4. The result for $d \geq 5$ follows immediately from the fact that del Pezzo surfaces of degree $d \geq 5$ are rational, see [67, Thm. 9.4.8, p. 277]. If $d = 4$, then the assumption $S(k) \neq \emptyset$ implies that there exists $P \in S(k)$ not lying on any of the exceptional curves of S, see [56, Thm. 30.1, p. 162]. Blowing up S at P, we obtain a del Pezzo surface $\widetilde{S}$ of degree 3 with a curve L which is a line under the anticanonical embedding, and $L(k) \neq \emptyset$ implies $\widetilde{S}(k) \neq \emptyset$, so part (a) follows from part (b).

For part (b), it can be shown that S has a conic fibration (considering the residual intersection of the hyperplanes containing L), and by [56, Thm. 30.1, p. 162], there exists a rational point $P \in S(k)$ not lying on any of the exceptional curves of S. Blowing up S at P, we obtain a del Pezzo surface of degree 2 with a conic fibration and a rational point, so part (b) follows from part (c).

By [49, Cor. 8, p. 919] and Lemma 3.3.6, a del Pezzo surface S of degree $d \in \{1, 2\}$ with a rational point and a conic fibration (necessarily with $8 - d$ singular fibres) satisfies the hypotheses of Theorem 3.1.2, hence it has the Hilbert property, hence part (c) holds. □

Remark 3.3.7. By a result of Iskovskih (see [44, Thm. 1.6, p. 572]), the canonical class of a relatively minimal double conic bundle has positive self-intersection.

3.4 Del Pezzo varieties

In this section, we introduce smooth del Pezzo varieties and prove Theorem 3.1.5. We draw from the excellent exposition given in [46, Ch. 3].

Definition 3.4.1. A *smooth del Pezzo variety* is a pair (X, H) consisting of a smooth projective algebraic variety X and an ample Cartier divisor H on X such that $-K_X = (n-1)H$, where $n = \dim X$. The *degree* of a smooth del Pezzo variety (X, H) of dimension n is defined by $\deg X := H^n$.

Proposition 3.4.2. *[46, Prop. 3.2.4(ii), p. 54] Let (X, H) be a smooth del Pezzo variety. Put $n := \dim X$ and $d := \deg X$. If $d \geq 3$, then the linear system $|H|$ determines an embedding $\phi_{|H|} : X \hookrightarrow \mathbb{P}^{n+d-2}$.*

Note 3.4.3. This result tells us that, for $d \geq 3$, we can identify the divisor class H with the class of hyperplane sections under the embedding induced by $|H|$.

A complete classification of del Pezzo varieties of dimension $n \geq 3$ arising from the work of Fujita and Iskovskikh can be found in [46, Thm. 3.3.1, p. 55].

Proof of Theorem 3.1.5. We induct on $n = \dim X$. The base case $n = 2$ is Theorem 3.1.4, so assume that $n \geq 3$.

First suppose that $d \geq 4$. Identify X with its image as a projective variety in $\mathbb{P}^{n+d-2}$ under the embedding induced by $|H|$. Let $P \in X(k)$, and denote by $\Lambda(P)$ the linear system of divisors $D \in |H|$ with $P \in D$. This linear system is of dimension $n + d - 3$, corresponding to a hyperplane in $\mathbb{P}(\mathscr{L}(H))$.

Now, an element $D \in \Lambda(P)$ is a hyperplane section of X through P, and by Bertini's theorem, a general element of $\Lambda(P)$ is smooth away from P, hence it is smooth if and only if the corresponding hyperplane does not contain the tangent space of X at P. Since X is smooth, this tangent space has dimension n, so the subspace of $\Lambda(P)$ of hyperplanes containing it has codimension n, hence the general element of $\Lambda(P)$ is smooth. Further, note that, for a smooth divisor $D \in |H|$, the adjunction formula in the form $K_D = D \cdot (K_X + D)$ gives us $K_D = -(n-2)D \cdot D$, so $(D, H|_D)$ is a smooth del Pezzo variety of dimension $n-1$ and degree $(H|_D)^{n-1} = H \cdot H^{n-1} = H^n = d$. Thus the smooth elements of $\Lambda(P)$ are del Pezzo varieties of dimension $n-1$ and degree d.

Choose two smooth elements $D_1, D_2 \in \Lambda(P)$, and denote by $\Pi(P)$ the pencil generated by D_1 and D_2. Let $f : X \dashrightarrow \mathbb{P}^1$ denote the rational map induced by the linear system $\Pi(P)$. Note that the general element of $\Pi(P)$ is smooth, i.e. the pencil has only finitely many singular members, since the smooth divisors in $\Lambda(P)$ correspond to a dense open subset and $\Pi(P)$ corresponds to a line intersecting the aforementioned open subset. By resolution of singularities (see [41, Main Thm. II, p. 142]), we obtain a variety $\widetilde{X}$ birational to X and a morphism $\phi : \widetilde{X} \to \mathbb{P}^1$ extending f, whose smooth fibres are strict transforms of smooth elements of $\Pi(P)$ under the associated blow-up.

Now, a divisor is birational to its strict transform under a blow-up, and the Lang–Nishimura theorem [67, Thm. 3.6.11, p. 92] states that having a smooth k-point is a birational invariant, hence all but finitely many of the fibres of ϕ (i.e. all those fibres corresponding to the strict transform of a smooth divisor of $\Pi(P)$) are birational to a smooth del Pezzo variety of dimension $n-1$ and degree $d \geq 4$ with a rational point. By the inductive hypothesis, these fibres have the Hilbert property, hence, by Theorem 3.3.1, we see that X has the Hilbert property.

It only remains to consider the case where $d = 3$ and there exists a line L on X. Denote by $\Lambda(L)$ the linear system of divisors $D \in |H|$ with $L \subset D$. This is a linear system of dimension $n - 1 \geq 2$, corresponding to a codimension-2 linear variety in $\mathbb{P}(\mathscr{L}(H))$. By Bertini's theorem, a general element of $\Lambda(L)$ is smooth away from L, hence it is smooth if and only if the corresponding hyperplane is not tangent to X at any point $Q \in L$. Since X is smooth, there is a unique hyperplane tangent to X at each $Q \in L$, so the general element of $\Lambda(L)$ is smooth. As above, we conclude that the smooth divisors in $\Lambda(L)$ are del Pezzo varieties of dimension $n-1$ and degree 3 containing the line L, and, defining $\Pi(L)$ to be the pencil generated by two smooth divisors in $\Lambda(L)$, the general element of this pencil is smooth. Further, since $L(k) \neq \emptyset$, the smooth elements have a rational point. Then, proceeding as above, we conclude that X has the Hilbert property. $\square$

Chapter 4

Weak approximation for del Pezzo surfaces of low degree

4.1 Introduction

Questions regarding rational points on del Pezzo surfaces are of long-standing interest in Diophantine geometry. Two guiding principles are that these surfaces should have many rational points (if they have any) and that their complexity should increase as the degree $d := K_X^2 \in \{1, \ldots, 9\}$ decreases. As such, the cases $d = 1, 2$ are thought of as the most challenging, and they will be the focus of this chapter.

One measure of the geometric abundance and distribution of rational points for varieties over a number field is *weak approximation* (Definition 2.2.19). Kresch and Tschinkel [50] and Várilly-Alvarado [88] have provided counterexamples to weak approximation for del Pezzo surfaces of degrees 2 and 1 respectively. However, it is conjectured by Colliot-Thélène that any smooth unirational variety satisfies *weak weak approximation*, or weak approximation away from finitely many places, and all del Pezzo surfaces with a rational point are expected to be unirational (see [67, Rem. 9.4.11]).

In this chapter, we shall prove that del Pezzo surfaces with a rational point over a number field satisfy weak weak approximation, provided we assume the additional structure of two conic fibrations. While this requirement may seem demanding, any conic fibration on a del Pezzo surface of degree $1, 2$ or 4 gives rise to another (Lemma 3.3.6). Further, Iskovskih [45] showed that any geometrically rational surface is birational to either a del Pezzo surface or a conic bundle, so one may think of these surfaces as a special "intersectional" case towards the proof of weak weak approximation for all geometrically rational surfaces. Indeed, del Pezzo surfaces of degree at least 5 are rational and so satisfy the stronger property of weak approximation, while weak weak approximation follows from work of Salberger and Skorobogatov [70] and Swinnerton-Dyer [84] for del Pezzo surfaces of degrees 4 and 3 respectively.

4.1.1 Results

Our main result is essentially a stronger form of Theorem 3.1.2 over number fields, with a caveat on the intersection of singular fibres from distinct fibrations.

Theorem 4.1.1. *Let X be a smooth projective surface over a number field K with two conic fibrations $\pi_i : X \to \mathbb{P}^1$, $i = 1, 2$. Suppose that $X(K) \neq \emptyset$ and that the following* *conditions hold.*

(a) Fibres from distinct fibrations have non-zero intersection product.

(b) Singular fibres from distinct fibrations do not share a singular point.

Then X satisfies weak weak approximation over K.

Note that the hypothesis that X is del Pezzo is not necessary for our methods. However, since weak weak approximation is a birational invariant, one may think of minimal surfaces as the main case of interest, and a minimal surface possesses two conic fibrations if and only if it is del Pezzo (see [44, Thm. 1.6]).

Remark 4.1.2. Since any conic fibration on a del Pezzo surface of degree $1, 2$ or 4 induces a second conic fibration (Lemma 3.3.6), we may think of Theorem 4.1.1 as holding for general del Pezzo surfaces of degrees 1 or 2 with a conic fibration (see Remark 4.5.7).

Remark 4.1.3. By [39, Prop. V.1.4] and [39, Rem. II.7.8.1], the first condition is equivalent to the two conic fibrations not being equal up to an automorphism of $\mathbb{P}^1$, which in turn is equivalent to their sets of fibres not being equal. As such, one may think of it as ensuring that the two fibrations are truly distinct.

In particular, we obtain the following result for del Pezzo surfaces of degree 1 or 2.

Corollary 4.1.4. *Let X be a del Pezzo surface of degree* 1 *or* 2 *with a conic fibration and a rational point. If there exists no* 4*-Eckardt point in $X(\overline{K})$, then X satisfies weak weak approximation.*

For the definition of 4-Eckardt points, see Definition 4.2.2.

By judiciously blowing up, our method gives a new proof of the following result.

Corollary 4.1.5. *Let X be a del Pezzo surface of degree* 4 *containing a rational point. Then X satisfies weak weak approximation.*

The proof of Theorem 4.1.1 builds upon that of Theorem 3.1.2, which uses the two conic fibrations π_1 and π_2 to generate rational points. Starting with a point $P \in X(K)$ on a smooth fibre of π_1 (the existence of which follows, possibly after relabelling, from condition (b)), the curve $\pi_1^{-1}(\pi_1(P))$ is a smooth conic with a rational point, hence it is isomorphic to $\mathbb{P}^1_K$ and therefore has many rational points. By repeating this process with the smooth fibres of π_2 through the rational points of this curve, we produce yet more rational points. As we have seen in Chapter 3, the K-points generated through this method are enough to prove the *Hilbert property* for X, i.e. that $X(K)$ is not thin. In this chapter we prove (under the hypotheses of Theorem 4.1.1) that by taking this iterative process to its fifth step, we may verify weak weak approximation on X, which is stronger than the Hilbert property.

The proof goes by constructing, for each $n \geq 1$, auxiliary *rational* varieties C'_n over K, which are endowed with morphisms $f'_n : C'_n \to X$. Applying a result of Denef [31], we show that f'_n is *arithmetically surjective* for all $n \geq 5$, meaning that for sufficiently large places v of K, $f'_n(C'_n(K_v)) = X(K_v)$. Using the fact that C'_n is a smooth proper *rational* variety, and so satisfies weak approximation, we conclude that weak weak approximation holds on X.

To apply the aforementioned result of Denef, the main difficulty lies in proving that every birational modification $\widetilde{C_5'} \to \widetilde{X}$ of $C_5' \to X$ has split fibres over the codimension-1 points of $\widetilde{X}$, and the establishment of this fact (Proposition 4.5.6) is the technical heart of our proof.

4.2 Background

4.2.1 Weak approximation

Weak approximation (Definition 2.2.19) can be thought of as both an indicator that rational points are well-distributed (among the local points) and that they are numerous (there are enough of them so as to be dense in the local points). As mentioned in Chapter 2, weak approximation is satisfied by $\mathbb{A}^n_K$ and is a birational invariant, hence all rational varieties satisfy weak approximation. Further, it is connected to the notion of thin sets (Definition 3.2.1). Work of Ekedahl and Colliot-Thélène [75, Thm. 3.5.7] shows that weak weak approximation implies that the variety in question possesses the Hilbert property. The main result of Chapter 3 is that the varieties in Theorem 4.1.1 satisfy the Hilbert property not only over number fields, but over any Hilbertian field. It is not known whether every variety over a number field with the Hilbert property satisfies weak weak approximation, although it is suggested by Corvaja and Zannier [26, §1.5] that this may not be the case.

4.2.2 Geometrically rational surfaces

In [45], Iskovskih showed that, over an arbitrary ground field k, any geometrically rational surface is k-birational to either a conic bundle or a del Pezzo surface. Thus, when exploring properties such as weak weak approximation which are invariant under birational transformations, these surfaces are of particular significance.

In this chapter, we focus largely on surfaces lying within the intersection of these two families, i.e. del Pezzo surfaces with a conic fibration. It follows from work of Kollár and Mella [49, Cor. 8] that these surfaces are unirational as soon as they possess a rational point. For our methods, we will require two conic bundle structures, but as we have seen in Lemma 3.3.6, any conic fibration on a del Pezzo surface of degree $d \in \{1, 2, 4\}$ gives rise to another, and fibres from distinct fibrations intersect with multiplicity $\frac{8}{d}$.

Remark 4.2.1. From the above, we see that any del Pezzo surface of degree $d \in \{1, 2, 4\}$ containing a (possibly singular) curve C with $C^2 = 0$ and $-K_X \cdot C = 2$ is endowed with two conic fibrations, and that fibres from the two fibrations have non-zero intersection product (so condition (a) of Theorem 4.1.1 is satisfied). Further, on minimal del Pezzo surfaces, it follows from [45, Thm. 1] that there exist at most two conic fibrations. As such, while the hypotheses of Theorem 4.1.1 ostensibly offer greater flexibility in the choice of the two conic bundles, the scenario in which the two conic bundles are "dual" as in Lemma 3.3.6 should be thought of as the main case.

It follows from the adjunction formula that the singular fibres of a conic fibration of a del Pezzo surface consist of two exceptional curves meeting in a point, and so condition (b) of Theorem 4.1.1 concerns points which lie on four exceptional curves.

Such points play an important role in the arithmetic of del Pezzo surfaces of low degree, as seen in [71], hence we make the following definition.

Definition 4.2.2. Let X be a del Pezzo surface and let n be a positive integer. We say that $P \in X(\overline{K})$ is an *n-Eckardt point* if it lies on at least n exceptional curves of $X_{\overline{K}}$.

The above terminology arises from the study of cubic surfaces (del Pezzo surfaces of degree 3), where points on three exceptional curves are known as Eckardt points. In the degree-2 case, points on four exceptional curves are known as generalised Eckardt points. In each case, such points are distinguished by the fact that they lie on the maximum possible number of exceptional curves. However, on a del Pezzo surface of degree 1 over a field of characteristic zero, the maximum number of concurrent exceptional curves is ten (see [86, Thm. 1.2]). While n-Eckardt points do not necessarily satisfy the "maximality" property of their predecessors, they at least share with them the property of lying on many exceptional curves.

4.3 Auxiliary varieties

Let X be a smooth projective surface over a field k. Let $\pi_1, \pi_2 : X \to \mathbb{P}^1$ be two conic fibrations such that $\pi_1^{-1}(P) \cdot \pi_2^{-1}(Q) > 0$. Suppose that there exists $P_0 \in X(k)$ such that $\pi_1^{-1}(\pi_1(P_0))$ is smooth (as noted earlier, this holds as soon as $X(k) \neq \emptyset$ for X satisfying the hypothesis of Theorem 4.1.1).

We begin by generalising the fibre products used in Chapter 3 to propagate rational points on X, which will be central to our proof.

Definition 4.3.1. Set $C_0 = \{P_0\}$, and denote by $f_0 : C_0 \to X$ the inclusion of C_0 in X. For $n \geq 1$, define C_n to be the fibre product

$$\begin{tikzcd} C_n \arrow[r, "f_n"] \arrow[d, "a_n"] & X \arrow[d, "\pi_i"] \\ C_{n-1} \arrow[r, "\pi_i \circ f_{n-1}"'] & \mathbb{P}^1, \end{tikzcd} \tag{4.3.1}$$

where $i = 1$ if n is odd and $i = 2$ if n is even.

From the rational point C_0, we produce the fibre $C_1 = \pi_1^{-1}(\pi_1(P_0)) \cong \mathbb{P}^1_k$ with infinitely many rational points, and in the next iteration, we produce C_2, whose rational points correspond to pairs (P_1, P_2) with $P_1 \in C_1(k)$, $P_2 \in \pi_2^{-1}(\pi_2(P_1))(k)$. In particular, for each of the infinitely rational points $P \in C_1(k)$ such that $\pi_2^{-1}(\pi_2(P_1))$ is smooth, we have infinitely many rational points on X lying on this fibre. Hence we may think of these fibre products as giving rise to many rational points on X. We will show that, upon further iterating this process, the same is true for local points. In particular, we will show that the task of proving weak weak approximation on X can be translated to C_n.

Proposition 4.3.2. *Let C_n and X be defined as above.*

(i) For $n \geq 2$, the morphism $C_n \to X$ is surjective.

(ii) For $n \geq 1$, the scheme C_n is geometrically integral of dimension n.

(iii) For $n \geq 0$, C_n is a rational variety.

Proof. (i) Let us first prove that $\pi_2 \circ f_1 : C_1 \to \mathbb{P}^1$ is surjective. Indeed, the morphism $\pi_2 \circ f_1$ is either surjective or constant. In the latter case, we would have that C_1 is contained in a fibre of π_2, which would imply that $C_1 \cdot \pi_2^{-1}(P) = 0$ for every $P \in \mathbb{P}^1$. By hypothesis (a) of Theorem 4.1.1, this cannot hold. Since surjectivity is preserved under base change and composition, the claim follows by induction.

(ii) Since the formation of the C_n commutes with base change of the field, we may assume that k is algebraically closed, hence we need only prove that C_n is integral. We prove this by induction on n.

$(n = 1)$: Since C_1 is smooth and connected, it is integral.

$(n - 1 \Rightarrow n, n \geq 2)$: Assume that C_{n-1} is integral. Note that each π_i is flat with geometrically integral generic fibre. These two properties are preserved under surjective base change, so $a_n : C_n \to C_{n-1}$ satisfies them as well. Applying [53, Prop. 4.3.8] to a_n, we deduce that C_n is integral.

(iii) Again, we induct on n. Note that the cases $n = 0$ and $n = 1$ are trivial.

Assume that C_{n-1} is rational. Note that the morphism $\pi_i \circ f_{n-1} : C_{n-1} \to \mathbb{P}^1$ is surjective for every $n \geq 2$, hence it sends the generic point of C_{n-1} to the generic point of $\mathbb{P}^1$. It follows that the geometric generic fibre of $C_n \to C_{n-1}$, being a base change of the geometric generic fibre of $\pi_i : X \to \mathbb{P}^1$, is isomorphic to $\mathbb{P}^1$. Note, moreover, that the morphism $C_n \to C_{n-1}$ has a natural section induced by $\mathrm{id}_{C_{n-1}}$ and f_{n-1}. Therefore the generic fibre of $C_n \to C_{n-1}$ is a form of $\mathbb{P}^1$ with a rational point, hence it is isomorphic to $\mathbb{P}^1$. Since C_{n-1} is rational and C_n is integral, this implies that C_n is rational as well. $\square$

Now let $k = K$, a number field.

Let $C'_n \to C_n$ be a desingularisation of C_n, and f'_n be the composition $C'_n \to C_n \xrightarrow{f_n} X$. Note that C'_n is smooth and rational, hence it satisfies weak approximation. Therefore, in order to verify that X satisfies weak weak approximation, it suffices to show that for all but finitely many places $v \in \mathrm{Val}(K)$, the map $f'_n : C'_n(K_v) \to X(K_v)$ is surjective.

Lemma 4.3.3. *For $n \geq 3$, the generic fibre of f_n is geometrically integral, hence so is that of the composition $C'_n \to C_n \xrightarrow{f_n} X$.*

Proof. First, observe that f_n is flat for all $n \geq 2$. Indeed, $\pi_2 \circ f_1 : C_1 \to \mathbb{P}^1$ is a finite morphism of smooth curves, hence flat [53, Rem. 4.3.11]. Since flatness is preserved under base change, flatness of f_2 follows from that of $\pi_2 \circ f_1$. Moreover, flatness of $f_{n+1}, n \geq 2$ follows from flatness of f_n since flatness is preserved under base change and composition. For flat proper morphisms, having geometrically integral fibres is an open condition [37, Thm. 12.2.4(viii)], hence it suffices to show that, for $n \geq 3$, there exists $P \in X$ such that $f_n^{-1}(P)$ is geometrically integral.

Let us first consider the case $n = 3$. Given $P \in X$, one may identify $f_3^{-1}(P)$ with the fibre of $\pi_1 \circ f_2 : C_2 \to \mathbb{P}^1$ over $\pi_1(P)$, and in turn with the fibre product of $\pi_1^{-1}(\pi_1(P))$ and $\pi_1^{-1}(\pi_1(P_0))$ mapping to $\mathbb{P}^1$ under π_2. A sufficient condition for (geometric) integrality of this fibre product is that $\pi_1^{-1}(\pi_1(P))$ is geometrically irreducible (which is true for P not lying on any of the singular fibres of π_1) and the two morphisms to $\mathbb{P}^1$ have disjoint branch loci (see Lemma 3.3.4). On the other hand, $Q \in \mathbb{P}^1$ is a common branch point if and only if $\pi_2^{-1}(Q)$ intersects both $\pi_1^{-1}(\pi_1(P))$ and $\pi_1^{-1}(\pi_1(P_0))$ non-transversally. Let r be the intersection product of fibres from distinct fibrations. Since all fibres of the π_i are reduced, it follows from the Riemann–Hurwitz formula [39, Cor. IV.2.4] that each fibre of π_2 intersects at most $2r-2$ fibres of π_1 non-transversally. Then, for P chosen outside some finite union of fibres of π_1, this fibre product is integral. We deduce the result for $n = 3$.

Now we establish the induction step. Let F_n be the generic fibre of f_n, and assume that it is geometrically integral. Let E_n be the generic fibre of $\pi_i \circ f_n : C_n \to \mathbb{P}^1$. By combining Cartesian squares, we see that $F_{n+1} \cong E_n \times_{k(\mathbb{P}^1)} k(X)$, hence it suffices to verify that E_n is geometrically integral. In turn, letting D_i be the generic fibre of π_i, a smooth conic over $k(\mathbb{P}^1)$, we have $E_n \cong C_n \times_X D_i$, and the generic fibre of $E_n \to D_i$ is isomorphic to F_n. Since F_n and D_i are geometrically integral and the morphism $E_n \to D_i$ is flat as the base change of the flat morphism $f_n : C_n \to X$, it follows from [53, Prop. 4.3.8] that E_n is geometrically integral, hence F_{n+1} is also geometrically integral. $\square$

4.4 Splitness

In this section we give a notion of "split reduction" for surjective proper morphisms between k-varieties $f : Y \to X$, stemming from the following definition which first appeared in [79, Def. 0.1].

Definition 4.4.1. We say that a scheme X of finite type over a perfect field F is *split* if there exists a geometrically integral open subscheme $U \subset X$.

4.4.1 Split schemes over DVRs

For a field K and a subfield $k \subset K$, we define the following set of discrete valuation rings (DVRs):

$$\mathrm{DVR}(K,k) := \{\text{DVRs } R \mid k \subseteq R \subseteq K,\ \mathrm{Frac}(R) = K\}.$$

If K is a finitely generated field over k, we say that $R \in \mathrm{DVR}(K,k)$ is *divisorial* if there exists a normal k-variety X such that its function field $k(X)$ is isomorphic to K, and a codimension-1 point $\eta \in X$ such that R is the image of the DVR $\mathcal{O}_{X,\eta} \subseteq k(X)$ under the isomorphism $k(X) \cong K$. We denote the set of divisorial DVRs $R \in \mathrm{DVR}(K,k)$ by $\mathrm{DVR}'(K,k)$. Finally, for a k-variety X, we use the following notation:

$$\mathrm{DVR}(X) := \left\{ R \in \mathrm{DVR}'(k(X),k) \;\middle|\; \begin{array}{c} \text{There exists a commutative diagram} \\ \begin{tikzcd} & \operatorname{Spec} R \arrow[dr] & \\ \operatorname{Spec} k(X) \arrow[ur] \arrow[rr] & & X \end{tikzcd} \end{array} \right\}.$$

Note that, when X is proper, we have $\mathrm{DVR}(X) = \mathrm{DVR}'(k(X),k)$.

The following definition seems to have never appeared explicitly in the literature, although it is implicit in several works (see e.g. [79], [22, §3], [31] or [25, Ch. 9]).

Definition 4.4.2. Let K be a finitely generated field over a field k of characteristic zero. Let Y be a K-variety, and let $R \in \mathrm{DVR}(K,k)$. We say that Y has *split reduction at R* if for some regular integral proper R-scheme $\mathcal{Y}$ with generic fibre smooth and birational to Y, the special fibre of $\mathcal{Y}$ is split.

We say that a proper surjective morphism $f : Y \to X$ between K-varieties has split reduction at $R \in \mathrm{DVR}(k(X),k)$ if its generic fibre does.

Note 4.4.3. By standard desingularisation results, an R-scheme $\mathcal{Y}$ as in Definition 4.4.2 exists for R divisorial. Further, for X and Y smooth, $f : Y \to X$ has split reduction at $R \in \mathrm{DVR}(X)$ if and only if f has split fibre over the associated codimension-1 point. By [79, Lem. 1.1], the special fibre of $\mathcal{Y}$ is split if and only if there exists a residually closed local flat extension of DVRs $i : R \to R'$ of ramification index one such that the generic fibre of $\mathcal{Y}$ has a $\mathrm{Frac}(R')$-point. By the Lang–Nishimura theorem, we see that the above definition is independent of the choice of R-model $\mathcal{Y}$.

4.4.2 Split fibres

Proposition 4.4.4. *Let k be a field with $\mathrm{char}\, k = 0$, and $f : Y \to X$ be a proper morphism of k-varieties. Assume that there exists an open subscheme $U \subseteq Y$ such that $f|_U : U \to X$ is smooth and has split fibres. Then, for every $R \in \mathrm{DVR}(f(U)) \subseteq \mathrm{DVR}(X)$, f has split reduction at R.*

Proof. Since $f|_U$ is smooth, it is flat, hence $f(U)$ is open. In particular, after restricting X to $f(U)$, we may assume that f is surjective. Let $R \in \mathrm{DVR}(X)$ and $\xi \in \mathrm{Spec}\, R$ be the special point. We recall that $R \in \mathrm{DVR}(X)$ means that R is a divisorial DVR containing k, whose fraction field is $k(X)$ and such that there exists a (necessarily unique) morphism $\phi : \mathrm{Spec}\, R \to X$ whose restriction to the generic point of $\mathrm{Spec}\, R$ is the natural morphism $\mathrm{Spec}\, k(X) \to X$.

We have an open embedding $U \times_X \mathrm{Spec}\, R \subset Y \times_X \mathrm{Spec}\, R$. Note that $U \times_X \mathrm{Spec}\, R$, being smooth over the regular ring R, is regular, hence there exists a desingularisation $Y' \to Y \times_X \mathrm{Spec}\, R$ that is an isomorphism over $U \times_X \mathrm{Spec}\, R$. In particular, the special fibre $Y' \times_R \xi$ contains the open subscheme $U' := U \times_X \xi$, which is non-empty since $\phi(\xi) \in f(U)$. Moreover, since $U' = U \times_X \xi = U_{\phi(\xi)} \times_{\phi(\xi)} \xi$, and the property of being a split F-scheme is invariant by extension of the base field F, U' is a split scheme over $k(\xi)$. Therefore $Y' \times_X \xi$ is split, i.e. f has split reduction at R. □

4.5 Proof of main theorem

In order to employ Denef's result on arithmetic surjectivity, it suffices to show that some f'_n has geometrically integral generic fibre (Lemma 4.3.3) and has split reduction at every $R \in \mathrm{DVR}(X)$. Indeed, in the language of [31], the fibre of a modification of f'_n over a codimension-1 point is split if and only if f'_n has split reduction at its local ring. Establishing this split reduction will be our main focus in this section.

We denote by $\mathrm{Bad}(\pi_i) \subset X, i = 1, 2$ the set of points $x \in X$ where the morphism $\pi_i : X \to \mathbb{P}^1$ is not smooth. This coincides with the set of singular points on the singular fibres of π_i, a finite set of closed points. We endow $\mathrm{Bad}(\pi_i) \subset X$ with the natural reduced scheme structure.

Lemma 4.5.1. *Let U, U', W, W' be k-varieties and let*

$$\begin{tikzcd} W \arrow[r] \arrow[d] & W' \arrow[d, "f"] \\ U \arrow[r, "\phi"] & U', \end{tikzcd} \tag{4.5.1}$$

be a Cartesian diagram, where $\phi : U \to U'$ is smooth. Let S be a subset of U' and, for each $u \in S$, L_u be a finite field extension of $\kappa(u)$, such that:

(i) the fibres of ϕ at points $u \notin S$ are split;

(ii) for each $u \in S$, the base-changed fibre $\phi^{-1}(u)_{L_u}$ is a split L_u-scheme;

(iii) for every $w \in W'$ such that $f(w) \in S$, there exists an embedding of $\kappa(f(w))$-field extensions $L_{f(w)} \hookrightarrow \kappa(w)$.

Then $W \to W'$ is smooth with split fibres.

Proof. This is immediate. □

Lemma 4.5.2. *Let k be a field, X be a split finite type k-scheme and $f : Y \to X$ be a flat morphism with split generic fibres. Then Y is a split k-scheme.*

Proof. Let $U \subset X$ be a geometrically integral open subscheme, and let $\eta \in U$ be the generic point. Let V_0 be a geometrically integral open $\kappa(\eta)$-subscheme of $f^{-1}(\eta)$, and let $V \subset Y$ be an open subscheme such that $V \cap f^{-1}(\eta) = V_0$. Note that $f|_V : V \to U$ is a flat morphism with geometrically integral generic fibre over a geometrically integral base. Hence the base change $V \times_k \overline{k} \to U \times_k \overline{k}$ satisfies the same properties. Applying [53, Prop. 4.3.8] to this last morphism, we deduce that V is geometrically integral. Since $V \subset Y$ is open, Y is split. □

Proposition 4.5.3. *Let $n \geq 4$. Assume that there exists a non-empty open subscheme $U \subseteq C_{n-1}$ such that $f_{n-1}|_U : U \to X$ is smooth with split fibres. Then, letting $V := f_{n-1}(U)$, there exists an open subscheme $W \subseteq C_n$ such that $f_n|_W : W \to X$ has image $\pi_i^{-1}(\pi_i(V)) \setminus \mathrm{Bad}(\pi_i)$ (where $i = 1$ if n is odd and $i = 2$ if n is even) and is smooth with split fibres.*

Proof. After restricting V (resp. U) to $V \setminus \mathrm{Bad}(\pi_i)$ (resp. $f_{n-1}^{-1}(V \setminus \mathrm{Bad}(\pi_i)) \cap U$) and noting that $\pi_i^{-1}(\pi_i(V)) = \pi_i^{-1}(\pi_i(V \setminus \mathrm{Bad}(\pi_i)))$, we may assume that $\mathrm{Bad}(\pi_i) \cap V = \emptyset$. Then, letting $U' := \pi_i(V) \subset \mathbb{P}^1$ (which is open as π_i is flat [39, Exercise III.9.1]), we have that $V \to U'$ is smooth.

Let $W' := \pi_i^{-1}(U') \setminus \mathrm{Bad}(\pi_i)$. Note that the composition $U \to V \to U'$ is smooth as it is a composition of smooth morphisms. Let $W := U \times_{U'} W'$. Since $U \to U'$ is surjective, so is $W \to W'$. Note that we have a natural open embedding $W = U \times_{U'} W' \subseteq C_{n-1} \times_{\mathbb{P}^1} X = C_n$.

In order to illustrate the relationship between the various morphisms introduced thus far, we include the following diagram.

$$\begin{tikzcd}
 & C_n \arrow[rr, "f_n"] \arrow[d, "a_n"] & & X \arrow[d, "\pi_i"'] & \\
 & C_{n-1} \arrow[r, "f_{n-1}"] & X \arrow[r, "\pi_i"] & \mathbb{P}^1_K & \\
W \arrow[uur, "\subseteq"] \arrow[rrrr, "f_n|_W"] \arrow[d, "a_n|_W"'] & & & & W' \arrow[uul, "\subseteq"'] \arrow[d, "\pi_i|_{W'}"] \\
U \arrow[uur, "\subseteq"] \arrow[rr, "f_{n-1}|_U"] & & V \arrow[uu, "\subseteq"'] \arrow[rr, "\pi_i|_V"] & & U' \arrow[uul, "\subseteq"']
\end{tikzcd} \tag{4.5.2}$$

All of the depth-oriented morphisms are open embeddings and the front and back squares are Cartesian by definition.

We claim that $f_n|_W : W \to W' \subset X$ has split fibres. To show this, we use Lemma 4.5.1 on the front square of Diagram (4.5.2). We must define S and the extensions L_u, and verify that the assumptions of the lemma hold.

Let $S \subset \mathbb{P}^1_K$ be the (finite) subset over which π_i has a singular fibre. For each $u \in S$, let $\pi_i^{-1}(u)^{reg}$ be the regular locus of $\pi_i^{-1}(u)$ and $\pi_i^{-1}(u)^{reg} \to \operatorname{Spec} L'_u \to \operatorname{Spec} k(u)$ be the Stein decomposition of $\pi_i^{-1}(u)^{reg} \to \operatorname{Spec} k(u)$. Since all fibres of π_i are conics, L'_u is a quadratic extension of $k(u)$, possibly split. We define $L_u := L'_u$ when L'_u is a field, and $L_u := k(u)$ when $L'_u \cong k(u)^{\oplus 2}$. Note that all irreducible components of $\pi_i^{-1}(u)^{reg}$ (i.e. two affine lines if $L_u = k(u)$ and all of $\pi_i^{-1}(u)^{reg}$ if L_u is a field) are geometrically integral L_u-schemes, and that $\pi_i^{-1}(u)$ is a geometrically integral (hence split) $k(u)$-scheme for $u \notin S$.

For each $u \in U'$, let W'_u (resp. V_u, U_u) be the fibre of $W' \to U'$ (resp. $V \to U'$, $U \to U'$) at u. Note that, for each $u \in S$, W'_u and V_u are open subschemes of $\pi_i^{-1}(u)^{reg}$. In particular:

1. $W'_u \to \operatorname{Spec} k(u)$ factors as $W'_u \to \operatorname{Spec} L_u \to \operatorname{Spec} k(u)$ (hence assumption (iii) holds);

2. $V_u \times_{k(u)} L_u$ is a split L_u-scheme (if $L'_u \cong k(u)^{\oplus 2}$ this is clear, otherwise note that $\pi_i^{-1}(u)^{reg}$ is irreducible, hence V_u is dense in it, and $V_u \times_{k(u)} L_u$ is dense in the split L_u-scheme $\pi_i^{-1}(u)^{reg} \times_{k(u)} L_u$).

Note that, for each $u \in U'$ (resp. $u \in S$), the morphism $U_u \to V_u$ (resp. $U_u \times_{k(u)} L_u \to V_u \times_{k(u)} L_u$) is surjective, smooth and has split fibres, as all of these properties are invariant under base change and they are satisfied by the morphism $U \to V$.

Applying, for each $u \notin S$ (resp. $u \in S$), Lemma 4.5.2 to the morphism $U_u \to V_u$ (resp. $U_u \times_{k(u)} L_u \to V_u \times_{k(u)} L_u$), we deduce that U_u (resp. $U_u \times_{k(u)} L_u$) is a split $k(u)$ (resp. L_u)-scheme, i.e. assumptions (i) and (ii) hold.

We deduce from Lemma 4.5.1 that the morphism $W \to W'$ is smooth with split fibres, hence we have proved our claim.

Remembering that $W' = \pi_i^{-1}(\pi_i(V)) \setminus \operatorname{Bad}(\pi_i)$ and noting that W is an open subscheme of C_n, this proves the proposition. $\square$

Proposition 4.5.4. *Let $n \geq 4$. Let $R \in \operatorname{DVR}(X)$ be such that $f'_{n-1} : C'_{n-1} \to X$ has split reduction at R. Then $f'_n : C'_n \to X$ has split reduction at R.*

Proof. Recall that Diagram (4.3.1) implies the existence of a section $\sigma_n : C_{n-1} \to C_n$ (commuting with projection to X) to the morphism $a_n : C_n \to C_{n-1}$. Being the dominant base change of a generically smooth morphism, a_n is generically smooth, i.e. there exists an open subscheme $U \subseteq C_{n-1}$ such that $a_n^{-1}(U)$ is a smooth open subscheme of C_n. In particular, the image $\sigma_n(\eta(C_{n-1}))$ of the generic point $\eta(C_{n-1})$ of C_{n-1} is a smooth point of the K-variety C_n. Therefore, if $C'_{n-1} \to C_{n-1}$ is a desingularisation of C_{n-1} and $C'_n \to C_n$ is a desingularisation of C_n, then there exists a rational map $\sigma'_n : C'_{n-1} \dashrightarrow C'_n$, commuting with projection to X. Denoting by $(C'_{n-1})_{k(X)}$ and $(C'_n)_{k(X)}$ the generic fibres of f'_{n-1} and f'_n, we have that σ'_n induces a rational map $(C'_{n-1})_{k(X)} \dashrightarrow (C'_n)_{k(X)}$ of $k(X)$-varieties. Since $(C'_{n-1})_{k(X)}$ has split reduction at R (see Definition 4.4.2), by [79, Lem. 1.1(a)] there exists a residually closed local flat extension of DVRs $i : R \to R'$ such that, denoting by $k(R')$ the fraction field of R', $(C'_{n-1})_{k(X)}$ has a $k(R')$-point. By the Lang–Nishimura theorem and the existence of a rational map $(C'_{n-1})_{k(X)} \dashrightarrow (C'_n)_{k(X)}$, we deduce that $(C'_n)_{k(X)}$ has a $k(R')$-point. This last condition implies, by [79, Lem. 1.1(b)], that $(C'_n)_{k(X)}$ has split reduction at R, thus concluding the proof of the proposition. □

Corollary 4.5.5. *If* $\mathrm{Bad}(\pi_1) \cap \mathrm{Bad}(\pi_2) = \emptyset$*, then, for every* $R \in \mathrm{DVR}(X)$*, the morphism* $f'_5 : C'_5 \to X$ *has split reduction at* R*.*

Proof. By Lemma 4.3.3, the generic fibre of $\pi_1 \circ f_2 : C_2 \to \mathbb{P}^1$ is geometrically integral. Hence, for a sufficiently small neighbourhood V of the generic point of C_2, we may assume that $\pi_1 \circ f_2|_V : V \to \mathbb{P}^1$ is smooth with geometrically integral fibres. We let $U := \pi_1(f_2(V))$.

For $n = 3, 4, 5$, we define U_n as $\pi_1^{-1}(U)$ for $n = 3$ and, iteratively, as $\pi_i^{-1}(\pi_i(U_{n-1})) \setminus \mathrm{Bad}(\pi_i)$ (where $i = 1$ if n is odd and $i = 2$ if n is even) for $n = 4, 5$.

Note that $\pi_1^{-1}(U)$ contains the generic fibre $\pi_1^{-1}(\eta(\mathbb{P}^1))$, where $\eta(\mathbb{P}^1) \in \mathbb{P}^1$ denotes the generic point. In particular, $\pi_2(U_3) = \pi_2(\pi_1^{-1}(\eta(\mathbb{P}^1))) = \mathbb{P}^1$, hence $U_4 = X \setminus \mathrm{Bad}(\pi_2)$. Analogously, $U_5 = X \setminus \mathrm{Bad}(\pi_1)$.

We claim that, for $n = 3, 4, 5$, there exists an open subscheme $V_n \subseteq C_n$ such that $f_n(V_n) = U_n$ and $f_n|_{V_n} : V_n \to X$ is smooth with split fibres. Indeed, for $n = 3$, remembering that the properties of being smooth and having split fibres are invariant under base change, this holds with $V_3 := V \times_{\mathbb{P}^1, \pi_1} X \subseteq C_3$; while for $n = 4, 5$ this follows (by induction) applying Proposition 4.5.3 with $U = V_{n-1}$ and letting $V_n := W$.

Let, for $i = 1, 2$, $X_i := X \setminus \mathrm{Bad}(\pi_i)$. We deduce from Proposition 4.4.4 applied to $f = f_4$, (resp. $f = f_5$) and $U = V_4$ (resp. $U = V_5$), that f'_4 (resp. f'_5) has split reduction for all $R \in \mathrm{DVR}(X_1)$ (resp. for all $R \in \mathrm{DVR}(X_2)$). Since f'_4 has split reduction for all $R \in \mathrm{DVR}(X_1)$, it follows by Proposition 4.5.4 that the same holds for f'_5.

The assumption that $\mathrm{Bad}(\pi_1) \cap \mathrm{Bad}(\pi_2) = \emptyset$ implies that $\mathrm{DVR}(X_1) \cup \mathrm{DVR}(X_2) = \mathrm{DVR}(X)$. Therefore, we deduce that $f'_5 : C'_5 \to X$, which has split reduction for all $R \in \mathrm{DVR}(X_1) \cup \mathrm{DVR}(X_2)$, has split reduction for all $R \in \mathrm{DVR}(X)$, as wished. □

We now prove the following proposition, the heart of the proof of Theorem 4.1.1.

Proposition 4.5.6. *If* $\mathrm{Bad}(\pi_1) \cap \mathrm{Bad}(\pi_2) = \emptyset$*, then* $f'_5 : C'_5 \to X$ *is arithmetically surjective. That is,* $f'_5(C'_5(K_v)) = X(K_v)$ *for all but finitely many places* $v \in M_K$*.*

Proof. The proposition is an application of [31, Thm. 1.2] to the morphism $f'_5 : C'_5 \to X$. We need only verify that f'_5 has geometrically integral generic fibre (Lemma 4.3.3) and that, using the terminology of [31], for every birational modification $\widetilde{f_5'} : \widetilde{C_5'} \to \widetilde{X}$ of $f_5' : C_5' \to X$ and every divisor $D \in \widetilde{X}^{(1)}$, the fibre $\widetilde{f_5'}^{-1}(D)$ is split. With our definitions, this last condition (that $\widetilde{f_5'}^{-1}(D)$ is split) means precisely that $f_5' : C_5' \to X$ has split reduction at the DVR $\mathcal{O}_{\widetilde{X},D} \subset \mathrm{DVR}(X)$, which follows from Corollary 4.5.5. □

Proof of Theorem 4.1.1. Let M_K be the set of places of K and let $S \subset M_K$ be a finite subset such that $f_5'(C_5'(K_v)) \to X(K_v)$ is surjective for all $v \notin S$. We will show that X satisfies weak approximation off S. Let $(P_v)_{v \notin S} \in \prod_{v \notin S} X(K_v)$ be a collection of local points. By Proposition 4.5.6, there exists $(\widetilde{P}_v)_{v \notin S} \in \prod_{v \notin S} C_5'(K_v)$ such that $f_5'(\widetilde{P}_v) = P_v$ for all $v \notin S$.

As noted in Section 4.3, C_n' satisfies weak approximation for every $n \geq 0$. In particular, there exists $\widetilde{P} \in C_5'(K)$ that is arbitrarily close (in $\prod_{v \notin S} C_5'(K_v)$) to $(\widetilde{P}_v)_{v \notin S}$, hence $f_5'(\widetilde{P}) \in X(K)$ is arbitrarily close to $(P_v)_{v \notin S}$. □

Remark 4.5.7. The assumption $\mathrm{Bad}(\pi_1) \cap \mathrm{Bad}(\pi_2) = \emptyset$ is not satisfied by all double conic bundles on del Pezzo surfaces of degree 1 or 2. For example, consider the del Pezzo surface X of degree 2 given by

$$X : w^2 = x^4 + 4x^2y^2 + z^4 \subset \mathbb{P}(1,1,1,2).$$

Note that, after rearranging the equation, we may factorise both sides to obtain

$$X : g_1 g_2 = h_1 h_2,$$

where

$$\begin{aligned} g_1(x,y,z,w) &= w - \left(x^2 + 2y^2\right), \\ g_2(x,y,z,w) &= w + \left(x^2 + 2y^2\right), \\ h_1(x,y,z,w) &= z^2 - 2y^2, \\ h_2(x,y,z,w) &= z^2 + 2y^2. \end{aligned}$$

From this factorisation we obtain the conic bundles

$$\begin{aligned} \pi_1 &: X \to \mathbb{P}^1, \quad [x:y:z:w] \mapsto [g_1 : h_1] \quad (\text{or } [h_2 : g_2]), \\ \pi_2 &: X \to \mathbb{P}^1, \quad [x:y:z:w] \mapsto [-g_2 : h_1] \quad (\text{or } [-h_2 : g_1]). \end{aligned}$$

It is easily seen that the points $[1,0,0,\pm 1]$ belong to $\mathrm{Bad}(\pi_1) \cap \mathrm{Bad}(\pi_2)$. Further, none of the exceptional curves meeting at these points is rational, so one cannot simply blow one of them down to obtain a cubic surface.

On the other hand, a general del Pezzo surface of degree 2 possesses no point at which four exceptional curves meet. Indeed, such a point exists if and only if the plane quartic over which the anticanonical model ramifies has an involution, and the generic plane quartic has trivial automorphism group. Consequently, one may think of examples with $\mathrm{Bad}(\pi_1) \cap \mathrm{Bad}(\pi_2) \neq \emptyset$ as rare.

Proof of Corollary 4.1.5. Since X contains a rational point, it is unirational, hence $X(K)$ is dense. Blowing up any two points $P_1, P_2 \in X(K)$ not on exceptional curves and not both on a curve of self-intersection zero, we obtain a del Pezzo surface Z of degree 2. Note that, blowing up first at P_1, we obtain a del Pezzo surface Y_1 of degree 3 with a K-rational exceptional curve E_1. The class $-K_{Y_1} - E_1$ gives rise to a conic fibration on Y_1, and the pullback of this class gives rise to a conic fibration with class C_1 on Z. Similarly, we may first blow up at P_2 to produce a surface Y_2 with exceptional divisor E_2 and pull back the class $-K_{Y_2} - E_2$ to a class C_2 on Z giving a conic fibration. It follows from [39, Prop. V.3.2] that

$$C_1 = -K_Z - L_1 + L_2, \quad C_2 = -K_Z + L_1 - L_2,$$

where L_1 and L_2 are the pullbacks of E_1 and E_2 respectively.

In order to apply Theorem 4.1.1, it remains to show that we can choose P_1 and P_2 so that the singular fibres of these conic fibrations on Z share no singular point.

Suppose that there exists a bad point Q on Z (i.e. a shared singular point of fibres from the two conic fibrations), and let $M_{i,j}$, $i, j \in \{1, 2\}$ be the exceptional curves meeting at Q so that $C_i = M_{i,1} + M_{i,2}$. It follows from elementary intersection multiplicity calculations and the adjunction formula that the projection of each $M_{i.j}$ onto X gives a class $D_{i,j}$ whose smooth members are conics on X. It is well-known that there are ten one-dimensional families of conics on X (defined over $\overline{K}$), and that these split into pairs whose sum (as classes) gives $-K_X$. It follows that $D_{i,1} + D_{i,2} = -K_X$. Since $M_{i,j} \cdot L_i = 2$ and $M_{i,j} \cdot L_{3-i} = 0$, it follows that $D_{i,1}$ and $D_{i,2}$ are two conics on X meeting precisely at P_i and Q. So, it suffices to show that we can choose P_1 and P_2 so that, picking any of the five dual pairs of conics through P_1 and any of the four remaining four dual pairs of conics through P_2, these four conics do not all meet in one point.

Choose $P_1 \in X(K)$ to be any rational point not on an exceptional curve of X. Denote by $(F_i, -K_X - F_i)$, $i = 1, \dots, 5$ the five pairs of dual conic classes on X. For each pair, consider the unique pair of representative curves through P_1. Note that $F_i \cdot (-K_X - F_i) = 2$, hence these representative curves intersect in at most one other point Q_i. Now consider the unique pair of dual conics from the four remaining pairs through each Q_i. Again, each such pair intersects in at most one other point $R_{i,j}$. There are at most twenty points of the form $R_{i,j}$. Choosing P_2 outside of this finite set and not on any exceptional curve of X or the ten conics containing P_1, we deduce that the surface Z_{P_1,P_2} obtained by blowing up X at P_1 and P_2 is a del Pezzo surface of degree 2 satisfying the hypotheses of Theorem 4.1.1, hence it satisfies weak weak approximation, and therefore X also satisfies weak weak approximation. $\square$

Chapter 5

Campana points and powerful values of norm forms

5.1 Introduction

The theory of Campana points is of growing interest in arithmetic geometry due to its ability to interpolate between rational and integral points. Two competing notions of Campana points can be found in the literature, both extending a definition of "orbifold rational points" for curves within Campana's theory of "orbifoldes géométriques" in [14], [15], [16] and [17]. They capture the idea of rational points which are *integral with respect to a weighted boundary divisor*. These two notions have been termed *Campana points* and *weak Campana points* in the recent paper [66] of Pieropan, Smeets, Tanimoto and Várilly-Alvarado, in which the authors initiate a systematic quantitative study of points of the former type on smooth Campana orbifolds and prove a logarithmic version of Manin's conjecture for Campana points on vector group compactifications. The only other quantitative results in the literature are found in [12], [87], [13], [65] and [95] (along with results in [76] and [77] appearing after the completion of [83]), all of which indicate the close relationship between Campana points and m-full solutions of equations. (Given $m \in \mathbb{Z}_{\geq 2}$, we say that $n \in \mathbb{Z} \setminus \{0\}$ is *m-full* if all primes in the prime decomposition of n have multiplicity at least m.)

In this chapter, we bring together the perspectives in the above papers and provide the first result for Campana points on singular orbifolds. As observed in [66, §1.1], the study of weak Campana points of bounded height is challenging and requires new ideas for the regularisation of certain Fourier transforms, and these ideas for the orbifolds in consideration are the main innovation of this work. We adopt a height zeta function approach, similar to the one employed in [66] and modelled on the work of Loughran in [54] and Batyrev and Tschinkel in [4] on toric varieties, in order to prove log Manin conjecture-type results for both types of Campana points on $\left(\mathbb{P}_K^{d-1}, \left(1 - \frac{1}{m}\right) Z(N_{\boldsymbol{\omega}})\right)$, where $N_{\boldsymbol{\omega}}$ is a norm form associated to a K-basis $\boldsymbol{\omega}$ of a Galois extension of number fields E/K of degree $d \geq 2$ coprime to $m \in \mathbb{Z}_{\geq 2}$ if d is not prime. When $K = \mathbb{Q}$, we derive from the result for weak Campana points an asymptotic for the number of elements of E of bounded height with m-full norm over $\mathbb{Q}$. We compare the result for Campana points to a conjecture of Pieropan, Smeets, Tanimoto and Várilly-Alvarado [66, Conj. 1.1, p. 3].

5.1.1 Results

Theorem 5.1.1. *Let E/K be a Galois extension of number fields of degree $d \geq 2$, and let $m \geq 2$ be an integer which is coprime to d if d is not prime. Let $\boldsymbol{\omega}$ be a K-basis of E. Denote by $\Delta_m^{\boldsymbol{\omega}}$ the $\mathbb{Q}$-divisor $\left(1-\frac{1}{m}\right)Z(N_{\boldsymbol{\omega}})$ of $\mathbb{P}_K^{d-1}$ for $N_{\boldsymbol{\omega}}$ the norm form corresponding to $\boldsymbol{\omega}$. Let H denote the anticanonical height function on $\mathbb{P}_K^{d-1}$ from Definition 5.4.5. Then there exists an explicit finite set $S(\boldsymbol{\omega}) \subset \mathrm{Val}(K)$ such that, for any finite set of places $S \supset S(\boldsymbol{\omega})$, the number $N\left(\left(\mathbb{P}_K^{d-1}, \Delta_m^{\boldsymbol{\omega}}\right), H, B, S\right)$ of weak Campana $\mathcal{O}_{K,S}$-points of height at most B on the orbifold $\left(\mathbb{P}_K^{d-1}, \Delta_m^{\boldsymbol{\omega}}\right)$ with respect to the model $\mathbb{P}_{\mathcal{O}_{K,S}}^{d-1}$ of $\mathbb{P}_K^{d-1}$ has the asymptotic formula*

$$N\left(\left(\mathbb{P}_K^{d-1}, \Delta_m^{\boldsymbol{\omega}}\right), H, B, S\right) \sim c(\boldsymbol{\omega}, m, S) B^{\frac{1}{m}} (\log B)^{b(d,m)-1}$$

for some explicit positive constant $c(\boldsymbol{\omega}, m, S)$, where

$$b(d,m) = \frac{1}{d}\left(\binom{d+m-1}{d-1} - \binom{m-1}{d-1}\right).$$

Note 5.1.2. If $\boldsymbol{\omega}$ is a relative integral basis of E/K, then $S(\boldsymbol{\omega}) = S_\infty$, the set of archimedean places of K, in Theorem 5.1.1 (see Remark 5.4.4).

Each rational point $P \in \mathbb{P}^{d-1}(\mathbb{Q})$ possesses precisely two sets of coordinates in $\mathbb{Z}_{\text{prim}}^d = \{(x_0, \ldots, x_{d-1}) \in \mathbb{Z}^d : \gcd(x_0, \ldots, x_{d-1}) = 1\}$. Interpreting H and $N_{\boldsymbol{\omega}}$ as functions on this set, we immediately obtain the following result.

Corollary 5.1.3. *Taking $K = \mathbb{Q}$ and letting $\boldsymbol{\omega}$ be an integral basis with the notation and hypotheses of Theorem 5.1.1, we have*

$$\#\{x \in \mathbb{Z}_{prim}^d : H(x) \leq B,\ N_{\boldsymbol{\omega}}(x) \text{ is } m\text{-full}\} \sim 2c(\boldsymbol{\omega}, m, S_\infty) B^{\frac{1}{m}} (\log B)^{b(d,m)-1}.$$

Arithmetically special (e.g. prime, square-free) values of norm forms are a topic of long-standing interest in number theory (see e.g. [27], [57]).

Campana points are only defined and studied for smooth orbifolds (i.e. smooth varieties for which the orbifold divisor has strict normal crossings support) in [66]. In order to study the Campana points of $\left(\mathbb{P}_K^{d-1}, \Delta_m^{\boldsymbol{\omega}}\right)$, which is smooth only when $d = 2$, we must first generalise the definition of Campana points, which we do in Section 5.2.1. Using the same strategy employed in the proof of Theorem 5.1.1, we then derive an asymptotic for the number of Campana points on $\left(\mathbb{P}_K^{d-1}, \Delta_m^{\boldsymbol{\omega}}\right)$.

Theorem 5.1.4. *In the setting of Theorem 5.1.1, denote by $\widetilde{N}\left(\left(\mathbb{P}_K^{d-1}, \Delta_m^{\boldsymbol{\omega}}\right), H, B, S\right)$ the number of Campana $\mathcal{O}_{K,S}$-points on $\left(\mathbb{P}_K^{d-1}, \Delta_m^{\boldsymbol{\omega}}\right)$ of height at most B with respect to H for some finite set of places $S \supset S(\boldsymbol{\omega})$. Then there exists an explicit positive constant $\widetilde{c}(\boldsymbol{\omega}, m, S)$ such that*

$$\widetilde{N}\left(\left(\mathbb{P}_K^{d-1}, \Delta_m^{\boldsymbol{\omega}}\right), H, B, S\right) \sim \widetilde{c}(\boldsymbol{\omega}, m, S) B^{\frac{1}{m}}.$$

Remark 5.1.5. It is not clear if the exponent of the logarithm in Theorem 5.1.1 admits a geometric interpretation as it does in Theorem 5.1.4 (cf. [66, Conj. 1.1, p. 3]).

5.2 Background

5.2.1 Campana points

In this section we define Campana orbifolds, Campana points and weak Campana points, generalising the definitions in [66, §3.2] in such a way that the exponents in Theorem 5.1.4 match those in [66, Conj. 1.1, p. 3].

Definition 5.2.1. A *Campana orbifold* over a field F is a pair (X, D_ϵ) consisting of a proper, normal variety X over F and an effective Cartier $\mathbb{Q}$-divisor

$$D_\epsilon = \sum_{\alpha \in \mathcal{A}} \epsilon_\alpha D_\alpha$$

on X, where the D_α are prime divisors and $\epsilon_\alpha = 1 - \frac{1}{m_\alpha}$ for some $m_\alpha \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$ (by convention, we take $\frac{1}{\infty} = 0$). We define the *support* of the $\mathbb{Q}$-divisor D_ϵ to be

$$D_{\text{red}} = \sum_{\alpha \in \mathcal{A}} D_\alpha.$$

We say that (X, D_ϵ) is *smooth* if X is smooth and D_{red} has strict normal crossings.

Let (X, D_ϵ) be a Campana orbifold over a number field K. Let $S \subset \operatorname{Val}(K)$ be a finite set containing S_∞.

Definition 5.2.2. A *model* of (X, D_ϵ) over $\mathcal{O}_{K,S}$ is a pair $(\mathcal{X}, \mathcal{D}_\epsilon)$, where $\mathcal{X}$ is a flat proper model of X over $\mathcal{O}_{K,S}$ (i.e. a flat proper $\mathcal{O}_{K,S}$-scheme with $\mathcal{X}_{(0)} \cong X$) and $\mathcal{D}_\epsilon = \sum_{\alpha \in \mathcal{A}} \epsilon_\alpha \mathcal{D}_\alpha$ for $\mathcal{D}_\alpha$ the Zariski closure of D_α in $\mathcal{X}$.

Define $\mathcal{D}_{\text{red}} = \sum_{\alpha \in \mathcal{A}} \mathcal{D}_\alpha$. Denote by $\mathcal{D}_{\alpha_v}$, $\alpha_v \in \mathcal{A}_v$ the irreducible components of $\mathcal{D}_{\text{red}}$ over $\operatorname{Spec} \mathcal{O}_v$. We write $\alpha_v \mid \alpha$ if $\mathcal{D}_{\alpha_v} \subset \mathcal{D}_\alpha$.

Let $(\mathcal{X}, \mathcal{D}_\epsilon)$ be a model for (X, D_ϵ) over $\mathcal{O}_{K,S}$. Given $P \in X(K_v)$ at a place $v \notin S$, we get an induced point $\mathcal{P}_v \in \mathcal{X}(\mathcal{O}_v)$ by the valuative criterion of properness [39, Thm. II.4.7, p. 101].

Definition 5.2.3. Let $P \in X(K_v)$ for some $v \notin S$. For each $\alpha_v \in \mathcal{A}_v$, we define the *sub-local intersection multiplicity* $n_v(\mathcal{D}_{\alpha_v}, P)$ of $\mathcal{D}_{\alpha_v}$ and P at v to be ∞ if $\mathcal{P}_v \subset \mathcal{D}_{\alpha_v}$ and the colength of the ideal $\mathcal{P}_v^* \mathcal{D}_{\alpha_v} \subset \mathcal{O}_v$ otherwise. We then define for each $\alpha \in \mathcal{A}$ the *local intersection multiplicity* $n_v(\mathcal{D}_\alpha, P) = \sum_{\alpha_v \mid \alpha} n_v(\mathcal{D}_{\alpha_v}, P)$, and we define the *total intersection multiplicity* $n_v(\mathcal{D}_\epsilon, P) = \sum_{\alpha \in \mathcal{A}} \epsilon_\alpha n_v(\mathcal{D}_\alpha, P)$.

We are now ready to define the notions of Campana points and weak Campana points, the latter of which comes from [1] and [2]. Although we will primarily be concerned with these notions in the context of rational points in this chapter, the definitions are naturally suited to local points, and so we will define them first in the local setting.

Definition 5.2.4. Let $v \notin S$, and let $P \in X(K_v)$.

(1) We say that P is a *weak Campana $\mathcal{O}_v$-point* of $(\mathcal{X}, \mathcal{D}_\epsilon)$ if the following implications hold for all $\alpha \in \mathcal{A}$.

(i) If $\epsilon_\alpha = 1$ (meaning $m_\alpha = \infty$), then $n_v(\mathcal{D}_\alpha, P) = 0$.

(ii) If $n_v(\mathcal{D}_\epsilon, P) > 0$, then

$$\sum_{\alpha \in \mathcal{A}} \frac{1}{m_\alpha} n_v(\mathcal{D}_\alpha, P) \geq 1.$$

(2) We say that P is a *Campana $\mathcal{O}_v$-point* of $(\mathcal{X}, \mathcal{D}_\epsilon)$ if the following implications hold for all $\alpha \in \mathcal{A}$.

(i) If $\epsilon_\alpha = 1$ (meaning $m_\alpha = \infty$), then $n_v(\mathcal{D}_\alpha, P) = 0$.

(ii) If $\epsilon_\alpha \neq 1$ and $n_v(\mathcal{D}_{\alpha_v}, P) > 0$ for some $\alpha_v \mid \alpha$, then

$$n_v(\mathcal{D}_{\alpha_v}, P) \geq \frac{1}{1 - \epsilon_\alpha}, \text{ i.e. } n_v(\mathcal{D}_{\alpha_v}, P) \geq m_\alpha.$$

We denote the sets of weak Campana $\mathcal{O}_v$-points and Campana $\mathcal{O}_v$-points of $(\mathcal{X}, \mathcal{D}_\epsilon)$ by $(\mathcal{X}, \mathcal{D}_\epsilon)_{\mathbf{w}}(\mathcal{O}_v)$ and $(\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_v)$ respectively. We set $(\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{v'}) = (\mathcal{X}, \mathcal{D}_\epsilon)_{\mathbf{w}}(\mathcal{O}_{v'}) = X(K_{v'})$ for any place $v' \in S$.

We will now see as promised above that Campana points and weak Campana points are truly defined on the level of local points, and rational points are Campana/weak Campana points precisely when they are so in the local setting.

Definition 5.2.5. We say that $P \in X(K)$ is a *weak Campana $\mathcal{O}_S$-point* (resp. a *Campana $\mathcal{O}_S$-point*) of $(\mathcal{X}, \mathcal{D}_\epsilon)$ if its image in $X(K_v)$ belongs to $(\mathcal{X}, \mathcal{D}_\epsilon)_{\mathbf{w}}(\mathcal{O}_v)$ (resp. to $(\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_v)$) for all $v \notin S$.

Having defined local Campana and weak Campana points at each place above (taking all local points at places over which the model is not defined), we are led to natural definitions of corresponding adelic points.

Definition 5.2.6. For any finite set of places $T \subset \mathrm{Val}(K)$, we define the set of *Campana $\mathbb{A}_K^T$-points* to be $(\mathcal{X}, \mathcal{D}_\epsilon)(\mathbb{A}_K^T) = \prod_{v \notin T} (\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_v)$, i.e.

$$(\mathcal{X}, \mathcal{D}_\epsilon)(\mathbb{A}_K^T) = \prod_{v \notin S \cup T} (\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_v) \times \prod_{v \in S \setminus T} X(K_v),$$

and we define the set of *adelic Campana points* of $(\mathcal{X}, \mathcal{D}_\epsilon)$ to be

$$(\mathcal{X}, \mathcal{D}_\epsilon)(\mathbb{A}_K) = \prod_{v \notin S} (\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_v) \times \prod_{v \in S} X(K_v).$$

Both $(\mathcal{X}, \mathcal{D}_\epsilon)(\mathbb{A}_K^T)$ and $(\mathcal{X}, \mathcal{D}_\epsilon)(\mathbb{A}_K)$ are equipped with the subspace topology inherited from $X(\mathbb{A}_K^T)$ and $X(\mathbb{A}_K)$. We define the sets $(\mathcal{X}, \mathcal{D}_\epsilon)_{\mathbf{w}}(\mathbb{A}_K^T)$ of T-adelic weak Campana points and $(\mathcal{X}, \mathcal{D}_\epsilon)_{\mathbf{w}}(\mathbb{A}_K)$ of adelic weak Campana points analogously.

Note 5.2.7. As observed in [66, §6.2], intersection multiplicity is locally constant on $(X \setminus D)(K_v)$ for all places $v \notin S$, whence it follows that $(\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_v)$ is both closed and open in $X(K_v)$ for all $v \in \mathrm{Val}(K)$.

Remark 5.2.8. The rational points of X are precisely the Campana points when D_{red} is the zero divisor, and integral points (with respect to D_{red}) correspond to every coefficient of the orbifold divisor being 1. Informally, one can think of weak Campana points as those rational points $P \in X(K)$ avoiding $\cup_{\epsilon_\alpha=1}\mathcal{D}_\alpha$ such that, upon reduction modulo any place $v \notin S$, the point P either either does not lie on D_{red} or lies on D_α with multiplicity at least m_α on average over α. Similarly, Campana points are rational points $P \in X(K)$ avoiding $\cup_{\epsilon_\alpha=1}\mathcal{D}_\alpha$ which, upon reduction modulo any place $v \notin S$, either do not lie on D_{red} or lie on each v-adic irreducible component of each D_α with multiplicity either 0 or at least m_α.

Remark 5.2.9. Our definition of Campana points differs from the one in [66, §3.2], in which one requires simply that $n_v(\mathcal{D}_\alpha, P) \geq m_\alpha$ instead of $n_v(\mathcal{D}_{\alpha_v}, P) \geq m_\alpha$ for all $\alpha_v \mid \alpha$ in the second implication. If one were to apply this definition to the orbifold $\left(\mathbb{P}^{d-1}_K, \Delta^{\boldsymbol{\omega}}_m\right)$ of Theorem 5.1.1, which is singular for all $d \geq 3$, then the weak Campana points and the Campana points would be the same, but the asymptotic of Theorem 5.1.1 differs to [66, Conj. 1.1, p. 3] for $d \geq 3$ (at least if one takes the thin set there to be the empty set). Using the definitions above, we obtain the asymptotic for Campana points in Theorem 5.1.4, whose exponents match this conjecture.

Lemma 5.2.10. *Let (X, D_ϵ) be a smooth Campana orbifold over a number field K, and let $(\mathcal{X}, \mathcal{D}_\epsilon)$ be a model of (X, D_ϵ) over $\mathcal{O}_{K,S}$ with $\mathcal{X}$ smooth over $\mathcal{O}_{K,S}$ and $\mathcal{D}_{red}$ a relative strict normal crossings divisor in $\mathcal{X}/\mathcal{O}_{K,S}$ as defined in [43, §2]. Then the definition of Campana points on $(\mathcal{X}, \mathcal{D}_\epsilon)$ above coincides with the one in [66, §3.2].*

Proof. Since $\mathcal{D}_{\text{red}}$ is a relative strict normal crossings divisor, each irreducible component $\mathcal{D}_\alpha$ is smooth over $\mathcal{O}_{K,S}$. In particular, its base change over $\operatorname{Spec}\mathcal{O}_v$ is smooth for any $v \notin S$, so the divisors $\mathcal{D}_{\alpha_v}$, $\alpha_v \mid \alpha$ are disjoint. Then, for any rational point $P \in X(K)$, the reduction of P at the place v can lie on at most one of the divisors $\mathcal{D}_{\alpha_v}$, $\alpha_v \mid \alpha$, so $n_v(\mathcal{D}_\alpha, P) = \sum_{\alpha_v \mid \alpha} n_v(\mathcal{D}_{\alpha_v}, P)$ is either 0 or at least m_α if and only if each $n_v(\mathcal{D}_{\alpha_v}, P)$, $\alpha_v \mid \alpha$, is either 0 or at least m_α. □

5.2.2 Toric varieties

Definition 5.2.11. An *(algebraic) torus* over a field F is an algebraic group T over F such that $\overline{T} \cong \mathbb{G}_m^n$ for some $n \in \mathbb{N}$. The *splitting field* of a torus T over a field F is defined to be the smallest Galois field extension E of F for which $T_E \cong \mathbb{G}_m^n$.

Definition 5.2.12. A *toric variety* is a smooth projective variety X equipped with a faithful action of an algebraic torus T such that there is an open dense orbit containing a rational point.

Definition 5.2.13. Let T be a torus over a field F. The *character group* of T is $X^*\big(\overline{T}\big) = \operatorname{Hom}\big(\overline{T}, \mathbb{G}_m\big)$, and we have $X^*(T) = X^*\big(\overline{T}\big)^{G_F}$. The *cocharacter group* of T is $X_*\big(\overline{T}\big) = \operatorname{Hom}\big(X^*\big(\overline{T}\big), \mathbb{Z}\big)$, and we have $X_*(T) = X_*\big(\overline{T}\big)^{G_F}$. We let $X^*(T)_{\mathbb{R}} = X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ and $X_*(T)_{\mathbb{R}} = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition 5.2.14. An algebraic torus T over a field F is *anisotropic* if it has trivial character group over F, i.e. $X^*(T) = 0$.

Let T be a torus over a number field K with splitting field E. Set $T_\infty = \prod_{v \mid \infty} T_v$. For $v \in \operatorname{Val}(K)$, let $T(\mathcal{O}_v)$ denote the maximal compact subgroup of $T(K_v)$.

Definition 5.2.15. Let $v \in \mathrm{Val}(K)$ and $w \in \mathrm{Val}(E)$ with $w \mid v$.

For $v \nmid \infty$ with ramification degree e_v in E/K, define the maps

$$\deg_{T,v} : T(K_v) \to X_*(T_v), \quad t_v \mapsto [\chi_v \mapsto v(\chi_v(t_v))]$$

and $\deg_{T,E,v} = e_v \deg_{T,v}$.

For $v \mid \infty$, define the maps

$$\deg_{T,v} : T(K_v) \to X_*(T_v)_{\mathbb{R}}, \quad t_v \mapsto [\chi_v \mapsto \log|\chi_v(t_v)|_v]$$

and $\deg_{T,E,v} = [E_w : K_v] \deg_{T,v}$.

Finally, define the maps

$$\deg_T = \sum_{v \in \mathrm{Val}(K)} (\log q_v) \deg_{T,v}, \quad \deg_{T,E} = \sum_{v \in \mathrm{Val}(K)} (\log q_w) \deg_{T,E,v}\,.$$

Lemma 5.2.16. *[8, §2.2] Let $v \in \mathrm{Val}(K)$, and let f be either $\deg_{T,v}$ or $\deg_{T,E,v}$.*

(i) If v is non-archimedean, then we have the exact sequence

$$0 \to T(\mathcal{O}_v) \to T(K_v) \xrightarrow{f} X_*(T_v).$$

The image of f is open and of finite index. Further, if v is unramified in E, then f is surjective.

(ii) If v is archimedean, then we have the short exact sequence

$$0 \to T(\mathcal{O}_v) \to T(K_v) \xrightarrow{f} X_*(T_v)_{\mathbb{R}} \to 0.$$

Further, f admits a canonical section.

(iii) Letting g be either $\deg_T$ or $\deg_{T,E}$ and denoting its kernel by $T(\mathbb{A}_K)^1$, we have the split short exact sequence

$$0 \to T(\mathbb{A}_K)^1 \to T(\mathbb{A}_K) \xrightarrow{g} X_*(T)_{\mathbb{R}} \to 0,$$

hence we have an isomorphism

$$T(\mathbb{A}_K) \cong T(\mathbb{A}_K)^1 \times X_*(T)_{\mathbb{R}}. \tag{5.2.1}$$

Definition 5.2.17. Let χ be a character of $T(\mathbb{A}_K)$. We say that χ is *automorphic* if it is trivial on $T(K)$. We say that χ is *unramified at* $v \in \mathrm{Val}(K)$ if χ_v is trivial on $T(\mathcal{O}_v)$, and we say that it is *unramified* if it is unramified at every $v \in \mathrm{Val}(K)$.

The canonical sections of the maps $T(K_v) \xrightarrow{\deg_{T,v}} X_*(T_v)_{\mathbb{R}}$ for each $v \mid \infty$ from Lemma 5.2.16(ii) induce a canonical section of the composition

$$T(\mathbb{A}_K) \to \prod_{v \mid \infty} T(K_v) \to X_*(T_\infty)_{\mathbb{R}},$$

which in turn induces a "type at infinity map"

$$T(\mathbb{A}_K)^\wedge \to X^*(T_\infty)_{\mathbb{R}}, \quad \chi \mapsto \chi_\infty. \tag{5.2.2}$$

Defining $\mathsf{K}_T = \prod_v T(\mathcal{O}_v)$, the splitting (5.2.1) for $g = \deg_T$ induces a map

$$\Big(T(\mathbb{A}_K)^1/T(K)\mathsf{K}_T\Big) \to X^*(T_\infty)_{\mathbb{R}}$$

with finite kernel and image a codimension-rank $X^*(T)$ lattice (see [8, Lem. 4.52, p. 96]).

Note 5.2.18. When T is anisotropic, we have $T(\mathbb{A}_K)^1 = T(\mathbb{A}_K)$ by Lemma 5.2.16(iii), and then we see from the above that there is a map

$$(T(\mathbb{A}_K)/T(K)\mathsf{K}_T) \to X^*(T_\infty)_\mathbb{R}$$

with finite kernel and image a lattice of full rank.

5.2.3 Hecke characters

Definition 5.2.19. A *Hecke character* for K is an automorphic character of $\mathbb{G}_{m,K}$.

Each Hecke character χ has a *conductor* $q(\chi) \in \mathbb{N}$ (see [47, §5.10]), which measures the ramification of χ at the non-archimedean places of K.

Definition 5.2.20. A Hecke character is *principal* if it is trivial on $\mathbb{G}_{m,K}(\mathbb{A}_K)^1$.

By Lemma 5.2.16(iii), χ is principal if and only if $\chi = \|\cdot\|^{i\theta}$ for some $\theta \in \mathbb{R}$, where $\|\cdot\|$ denotes the adelic norm map, i.e.

$$\|\cdot\| : \mathbb{A}_K^* \to S^1, \quad (x_v)_v \mapsto \prod_{v \in \mathrm{Val}(K)} |x_v|_v.$$

Definition 5.2.21. The *(Hecke) L-function* $L(\chi, s)$ of a Hecke character χ is

$$L(\chi, s) = \prod_v \left(1 - \frac{\chi_v(\pi_v)}{q_v^s}\right)^{-1},$$

where the product is taken over all places $v \nmid \infty$ at which χ is unramified.

The *Dedekind zeta function* of K is

$$\zeta_K(s) = L(1, s).$$

Given a Hecke character χ for a number field L and $w \in \mathrm{Val}(L)$, we denote by $L_w(\chi, s)$ the local factor at w for the Euler product of $L(\chi, s)$, i.e.

$$L_w(\chi, s) = \begin{cases} \left(1 - \frac{\chi_w(\pi_w)}{q_w^s}\right)^{-1} & \text{if } w \nmid \infty \text{ and } \chi \text{ is unramified at } w, \\ 1 & \text{otherwise.} \end{cases}$$

When working over the field $L \supset K$, we define $L_v(\chi, s)$ for each $v \in \mathrm{Val}(K)$ by

$$L_v(\chi, s) = \prod_{w \mid v} L_w(\chi, s).$$

Theorem 5.2.22. *[40, §6] The L-function of a Hecke character χ admits a meromorphic continuation to $\mathbb{C}$. If $\chi = \|\cdot\|^{i\theta}$ for some $\theta \in \mathbb{R}$, then this continuation admits a single pole of order 1 at $s = 1 + i\theta$. Otherwise, it is holomorphic.*

Definition 5.2.23. Let ψ be a character of $\prod_{v \mid \infty} K_v^*$. The restriction of ψ to each $\mathbb{R}_{>0} \subset K_v^*$ is of the form $x \mapsto |x|^{i\kappa_v}$ for some $\kappa_v \in \mathbb{R}$. We define

$$\|\psi\| = \max_{v \mid \infty} |\kappa_v|.$$

Lemma 5.2.24. *[54, Lem. 3.1, p. 2561] Let χ be a non-principal Hecke character of K, let C be a compact subset of* $\operatorname{Re} s \geq 1$ *and let $\varepsilon > 0$. Then*

$$L(\chi, s) \ll_{\varepsilon,C} q(\chi)^{\varepsilon}(1 + \|\chi_\infty\|)^{\varepsilon}, \quad (s-1)\zeta_K(s) \ll_{\varepsilon,C} 1, \quad s \in C.$$

Definition 5.2.25. Let E/K be Galois, let χ be a Hecke character for E and let $g \in \operatorname{Gal}(E/K)$. We define the *(Galois) twist of χ by g* to be the character

$$\chi^g : \mathbb{A}_E^* \to S^1, \quad (t_w)_w \mapsto \chi\Big((g_w(t_w))_{gw}\Big).$$

Here, gw denotes the place of E obtained by the action of g on $\operatorname{Val}(E)$, and $g_w : E_w \to E_{gw}$ is the induced map on completions. One may easily verify that χ^g is trivial on E^*, hence it is also a Hecke character for E.

5.3 The norm torus

In this section, we fix an extension of number fields L/K of degree $d \geq 2$ with Galois closure E and a K-basis $\boldsymbol{\omega} = \{\omega_0, \dots, \omega_{d-1}\}$. We write $N_{\boldsymbol{\omega}}(x_0, \dots, x_{d-1})$ for the norm form corresponding to $\boldsymbol{\omega}$, and $G = \operatorname{Gal}(E/K)$. From the equality

$$N_{\boldsymbol{\omega}}(x_0, \dots, x_{d-1}) = \prod_{g \in G/\operatorname{Gal}(E/L)} (x_0 g(\omega_0) + \cdots + x_{d-1} g(\omega_{d-1})), \tag{5.3.1}$$

we see that $N_{\boldsymbol{\omega}}$ is irreducible over K and has splitting field E. We denote by T the norm torus $T_{\boldsymbol{\omega}} = \mathbb{P}_K^{d-1} \setminus Z(N_{\boldsymbol{\omega}})$. As noted in [54, §1.2], $\mathbb{P}_K^{d-1}$ is a toric variety with respect to T, and $T \cong R_{L/K}\mathbb{G}_m/\mathbb{G}_m$ is anisotropic. Since its boundary is $Z(N_{\boldsymbol{\omega}})$, its splitting field is E. We have the short exact sequence

$$0 \to \mathbb{G}_m \to R_{L/K}\mathbb{G}_m \to T \to 0. \tag{5.3.2}$$

Note 5.3.1. The isomorphisms $T(\mathbb{A}_K) \cong \mathbb{A}_L^*/\mathbb{A}_K^*$ and $T(K) \cong L^*/K^*$ follow from Hilbert's Theorem 90 [67, Prop. 1.3.15(ii), p. 11] by applying Galois cohomology to (5.3.2). They allow us to interpret an automorphic character χ of T as a Hecke character for L, and we will do so frequently. In fact, distinct automorphic characters of T correspond to distinct Hecke characters of L by [4, Cor. 1.4.16, p. 606] and [4, Thm. 3.1.1, p. 619]. Since $T(K_v) \cong \left(\prod_{w|v} L_w^*\right)/K_v^*$ for each $v \in \operatorname{Val}(K)$, we see that, if χ is unramified at v, then it is unramified as a Hecke character at all $w \mid v$. In particular, if χ is unramified at v and v is unramified in L/K, then $\prod_{w|v} \chi_w(\pi_w) = 1$, since π_v is a uniformiser for L_w for each $w \mid v$.

5.3.1 Geometry

In this section we study fan-theoretic objects related to T. We begin by describing the fan $\Sigma \subset X_*(\overline{T})_{\mathbb{R}}$ associated to the equivariant compactification $\mathbb{P}_K^{d-1}$ of T and the associated piecewise-linear function φ_Σ (see [4, §1.2]) used to define the Batyrev–Tschinkel height function.

Denoting by $l_0(x), \dots, l_{d-1}(x) \in E[x]$ the E-linear factors of $N_{\boldsymbol{\omega}}(x)$, we have the E-isomorphism

$$\Phi : \overline{T} = \mathbb{P}^{d-1} \setminus \bigcup_{i=0}^{d-1} Z(l_i) \xrightarrow{\sim} \mathbb{G}_m^{d-1} = \mathbb{P}^{d-1} \setminus \bigcup_{j=0}^{d-1} Z(x_j),$$
$$[x_0, \dots, x_{d-1}] \mapsto [l_0(x), \dots, l_{d-1}(x)].$$

By [42, §1.1], the fan associated to $\mathbb{P}_E^{d-1}$ as a compactification of $\mathbb{G}_{m,E}^{d-1}$ is the fan whose r-dimensional cones are generated by the r-fold subsets of $\{e_0', \dots, e_{d-1}'\}$ for $0 \leq r \leq d-1$, where $e_i' \in X_*\big(\mathbb{G}_m^{d-1}\big) \cong \operatorname{Hom}\big(\mathbb{G}_m, \mathbb{G}_m^{d-1}\big)$ is defined by

$$e_i' : \mathbb{G}_m \to \mathbb{G}_m^{d-1}, \quad t \mapsto [x_{0,i}(t), \dots, x_{d-1,i}(t)],$$

where in turn

$$x_{j,i}(t) = \begin{cases} t \text{ if } i = j, \\ 1 \text{ otherwise.} \end{cases}$$

Definition 5.3.2. Set $e_i = \Phi^{-1} \circ e_i'$ for $i = 0, \dots, d-1$, and define Σ to be the fan whose r-dimensional cones are generated by the r-fold subsets of $\{e_0, \dots, e_{d-1}\}$ for $0 \leq r \leq d-1$.

It follows that Σ is the fan associated to $\mathbb{P}_E^{d-1}$ as a compactification of T_E. Also, we see that $\sum_{i=0}^{d-1} e_i = 0$ and that $\{e_1, \dots, e_{d-1}\}$ is the dual of the basis $\{m_1, \dots, m_{d-1}\}$ of $X^*\big(\overline{T}\big)$, where $m_i(x) = \frac{l_i(x)}{l_0(x)}$ for $i = 1, \dots, d-1$. By [42, §1.2], Σ is the fan associated to the compactification $\mathbb{P}_K^{d-1} = \mathbb{P}_E^{d-1}/G$ of T over K.

By [4, Prop. 1.2.12, p. 597], the line bundle $L(\varphi_\Sigma)$ associated to the piecewise-linear function $\varphi_\Sigma : X_*\big(\overline{T}\big)_{\mathbb{R}} \to \mathbb{R}$ (see [4, Prop. 1.2.9, p. 597]) defined by $\varphi_\Sigma(e_i) = 1$ for all $i = 0, \dots, d-1$ is the anticanonical bundle $-K_{\mathbb{P}^{d-1}}$.

By [4, Prop. 1.3.11, p. 601], G acts transitively on $\Sigma(1) = \{\langle e_0 \rangle, \dots, \langle e_{d-1} \rangle\}$ (since $\operatorname{Pic} \mathbb{P}_K^{d-1} \cong \mathbb{Z}$). For $v \in \operatorname{Val}(K)$ non-archimedean, let G_v denote the associated decomposition subgroup of G. By the proof of [4, Thm. 3.1.3, p. 619], the G_v-orbits of $\Sigma(1)$ are in bijection with the places of L over v, and the length of the G_v-orbit corresponding to a place $w \mid v$ is its inertia degree.

We now show that the action of G on $\Sigma(1)$ is compatible with its action on the E-linear factors of $N_{\boldsymbol{\omega}}$. Denote by $*$ the action of G, and set $l_{g(i)} = g * l_i$.

Lemma 5.3.3. *For all $g \in G$ and $i = 0, \dots, d-1$, we have*

$$g * e_i = e_{g(i)}.$$

Proof. Let $g \in G$. It suffices to show that

$$(g * e_i)(m_j) = e_{g(i)}(m_j) \tag{5.3.3}$$

for all $i \in \{0, \dots, d-1\}$ and $j \in \{1, \dots, d-1\}$. Note that, for any $i, j, k \in \{0, \dots, d-1\}$, we have

$$e_i\left(\frac{l_j}{l_k}\right) = \delta_{ij} - \delta_{ik}, \tag{5.3.4}$$

where δ_{ij} is the Kronecker delta symbol, defined by

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ otherwise.} \end{cases}$$

Then (5.3.3) becomes

$$\delta_{ig^{-1}(j)} - \delta_{ig^{-1}(0)} = \delta_{g(i)j} - \delta_{g(i)0},$$

which clearly holds. □

Proposition 5.3.4. *Let $v \in \mathrm{Val}(K)$ be non-archimedean with ramification degree e_v in E/K, and let*

$$\Sigma(1) = \bigcup_{w|v} \Sigma_w(1)$$

denote the decomposition of $\Sigma(1) = \{\langle e_0 \rangle, \dots, \langle e_{d-1} \rangle\}$ into G_v-orbits. For each $w \mid v$, let n_w be the sum of the elements of $\Sigma_w(1)$ and let $f_w(x)$ be the product of the linear factors in the G_v-orbit of $\{l_0, \dots, l_{d-1}\}$ corresponding to $\Sigma_w(1)$ by Lemma 5.3.3. Then the map $\deg_{T,E,v} : T(K_v) \to X_*(T_v)$ is given by

$$t_v \mapsto e_v \sum_{w|v} \frac{v(f_w(t_v))}{\deg f_w} n_w.$$

Proof. The image of t_v in $X_*(T_v) \cong X_*\big(\overline{T}\big)^{G_v}$ under $\deg_{T,E,v}$ is the cocharacter

$$\varphi_{t_v} : X^*(T_v) \to \mathbb{Z}, \quad \lambda \mapsto e_v v(\lambda(t_v)).$$

We first show that $\{n_w : w \mid v\}$ spans $X_*\big(\overline{T}\big)^{G_v}$. Given $g \in G$ and $\sigma = \sum_{i=0}^{d-1} a_i e_i$, we have $g * \sigma = \sum_{i=0}^{d-1} a_{g^{-1}(i)} e_i$, so $g * \sigma = \sigma$ if and only if there exists $r_g \in \mathbb{Z}$ such that $a_i = a_{g^{-1}(i)} + r_g$ for all $i \in \{0, \dots, d-1\}$. Setting $s = \#G$, we have

$$a_i = a_{g^s(i)} = a_{g^{s-1}(i)} + r_g = \cdots = a_i + s r_g,$$

hence $r_g = 0$. We deduce that $\sigma \in \Sigma^{G_v}$ if and only if $a_i = a_j$ for all e_i, e_j in the same G_v-orbit, so the result follows. Moreover, we observe that $\sum_{w|v} a_w n_w = \sum_{w|v} b_w n_w$ if and only if there exists $r \in \mathbb{Z}$ such that $b_w = a_w + r$ for all $w \mid v$.

Now, write

$$\varphi_{t_v} = \sum_{w|v} \alpha_w n_w.$$

Define $\mu_i \in X^*\big(\overline{T}\big)$ and $\lambda_w \in X^*\big(\overline{T}\big)^{G_v}$ for all $i \in \{0, \dots, d-1\}$ and all $w \mid v$ by

$$\mu_i(x) = \frac{l_i(x)^d}{N_{\boldsymbol{\omega}}(x)}, \quad \lambda_w(x) = \prod_{e_i \in \Sigma_w(1)} \mu_i(x) = \frac{f_w(x)^d}{N_{\boldsymbol{\omega}}(x)^{\deg f_w}}.$$

By (5.3.4), we have

$$e_i(\mu_j) = \begin{cases} d-1 \text{ if } i = j, \\ -1 \text{ otherwise.} \end{cases}$$

Then, setting $d_w = \deg f_w$, we see that

$$n_w(\lambda_{w'}) = \begin{cases} dd_w - d_w^2 \text{ if } w = w', \\ -d_w d_{w'} \text{ otherwise,} \end{cases}$$

so we deduce that

$$e_v v(\lambda_w(t_v)) = dd_w \alpha_w - d_w \sum_{w'|v} d_{w'} \alpha_{w'} \tag{5.3.5}$$

for all $w \mid v$. On the other hand, we have

$$e_v v(\lambda_w(t_v)) = e_v d v(f_w(t_v)) - e_v d_w \sum_{w'|v} v(f_{w'}(t_v)). \tag{5.3.6}$$

Set $\beta_w = d_w \alpha_w - e_v v(f_w(t_v))$. Combining (5.3.5) and (5.3.6), we obtain

$$d\beta_w = d_w \sum_{w'|v} \beta_{w'},$$

hence $\beta_{w'} = \frac{d_{w'}}{d_w}\beta_w$ for all $w \mid v$, $w' \mid v$. Since $K_v \cong E_w^{G_v}$ for any $w \mid v$, it follows that $d_w \mid v(f_w(t_v))$, so $\beta_w \in d_w\mathbb{Z}$ for all $w \mid v$. We deduce that there exists an integer $n \in \mathbb{Z}$ such that, for all $w \mid v$, we have $\beta_w = d_w n$, hence

$$\alpha_w = e_v \frac{v(f_w(t_v))}{\deg f_w} + n.$$

Since $\sum_{w|v} n_w = \sum_i e_i = 0$, we conclude that

$$\varphi_{t_v} = e_v \sum_{w|v} \frac{v(f_w(t_v))}{\deg f_w} n_w. \qquad \square$$

We now study polynomials introduced by Batyrev and Tschinkel in [4, §2.2], which play a key role in the analysis of local Fourier transforms in Section 5.5.

Definition 5.3.5. Let $v \in \mathrm{Val}(K)$ be non-archimedean, and let $\Sigma(1) = \bigcup_{i=1}^{l} \Sigma_i(1)$ be the decomposition of $\Sigma(1)$ into G_v-orbits. Let d_i be the cardinality of $\Sigma_i(1)$. For each $\Sigma_i(1)$, define an independent variable u_i. Let $\sigma \in \Sigma^{G_v}$, and let $\Sigma_{i_1}(1) \cup \cdots \cup \Sigma_{i_k}(1)$ be the set of 1-dimensional faces of σ. We define the rational function

$$R_{\sigma,v}(u_1, \dots, u_l) = \frac{u_{i_1}^{d_{i_1}} \dots u_{i_k}^{d_{i_k}}}{\left(1 - u_{i_1}^{d_{i_1}}\right) \dots \left(1 - u_{i_k}^{d_{i_k}}\right)},$$

and we define the polynomial $Q_{\Sigma,v}(u_1, \dots, u_l)$ by

$$\frac{Q_{\Sigma,v}(u_1, \dots, u_l)}{\left(1 - u_1^{d_1}\right) \dots \left(1 - u_l^{d_l}\right)} = \sum_{\sigma \in \Sigma^{G_v}} R_{\sigma,v}(u_1, \dots, u_l).$$

Proposition 5.3.6. *For all non-archimedean valuations $v \in \mathrm{Val}(K)$, we have*

$$Q_{\Sigma,v}(u_1, \dots, u_l) = 1 - u_1^{d_1} \dots u_l^{d_l}.$$

Proof. Observe that the G_v-invariant cones in Σ are precisely those cones generated by a set of 1-dimensional cones of the form $\Sigma_{i_1}(1)\cup\dots\cup\Sigma_{i_k}(1)$ for some $i_1,\dots,i_k\in\{1,\dots,l\}$ pairwise distinct with $k<l$. From this observation, we deduce that

$$\frac{Q_{\Sigma,v}(u_1,\dots,u_l)}{\left(1-u_1^{d_1}\right)\dots\left(1-u_l^{d_l}\right)}=\sum_{k=1}^{l-1}\sum_{\substack{i_1,\dots,i_k\in\{1,\dots,l\}\\ \text{pairwise distinct}}}\frac{u_{i_1}^{d_{i_1}}\dots u_{i_k}^{d_{i_k}}}{\left(1-u_{i_1}^{d_{i_1}}\right)\dots\left(1-u_{i_k}^{d_{i_k}}\right)}.$$

In particular, we see that

$$Q_{\Sigma,v}(u_1,\dots,u_l)=\sum_{(t_1,\dots,t_l)\in\{0,1\}^l}\prod_{i=1}^{l}\left(t_i+(1-2t_i)u_i^{d_i}\right)-u_1^{d_1}\dots u_l^{d_l},$$

so it suffices to prove that

$$\sum_{(t_1,\dots,t_l)\in\{0,1\}^l}\prod_{i=1}^{l}\left(t_i+(1-2t_i)u_i^{d_i}\right)=1. \tag{5.3.7}$$

Splitting the sum into two smaller sums for $t_1=0$ and $t_1=1$, we obtain

$$\begin{aligned}&\sum_{(t_1,\dots,t_l)\in\{0,1\}^l}\prod_{i=1}^{l}\left(t_i+(1-2t_i)u_i^{d_i}\right)\\ =&\left(u_1^{d_1}+\left(1-u_1^{d_1}\right)\right)\sum_{(t_2,\dots,t_n)\in\{0,1\}^{l-1}}\prod_{i=2}^{l}\left(t_i+(1-2t_i)u_i^{d_i}\right)\\ =&\sum_{(t_2,\dots,t_n)\in\{0,1\}^{l-1}}\prod_{i=2}^{l}\left(t_i+(1-2t_i)u_i^{d_i}\right).\end{aligned}$$

Repeating this process for each variable $t_2,\dots,t_l$, we deduce (5.3.7). ☐

5.3.2 Haar measures and volume

Let ω be an invariant d-form on T. By a classical construction (see [19, §2.1.7]), ω gives rise to a Haar measure $|\omega|_v$ on $T(K_v)$ for each $v\in\mathrm{Val}(K)$. In [61, §3.3], Ono constructs the convergence factors

$$c_v=\begin{cases}L_v\left(X^*(\overline{T}),1\right)^{-1} \text{ if } v\nmid\infty,\\ 1 \text{ if } v\mid\infty.\end{cases}$$

Here, $L_v\left(X^*(\overline{T}),s\right)$ is the local factor at v of the Artin L-function $L\left(X^*(\overline{T}),s\right)$. Defining $\mu_v=c_v^{-1}|\omega|_v$, the product of the μ_v converges to give a Haar measure μ on $T(\mathbb{A}_K)$, which is independent of ω by the product formula.

Note 5.3.7. From the short exact sequence (5.3.2), we obtain

$$L\left(X^*(\overline{T}),s\right)=\frac{\zeta_L(s)}{\zeta_K(s)}.$$

Lemma 5.3.8. *With respect to the Haar measure μ, we have*

$$\operatorname{vol}(T(\mathbb{A}_K)/T(K)) = d\frac{\operatorname{Res}_{s=1}\zeta_L(s)}{\operatorname{Res}_{s=1}\zeta_K(s)}.$$

Proof. By [61, §3.5] and [62, Main Thm., p. 68], we have

$$\operatorname{vol}\Big(T(\mathbb{A}_K)^1/T(K)\Big) = \frac{|\operatorname{Pic} T|}{|\text{Ш}(T)|}L\big(X^*(\overline{T}),1\big),$$

where $\text{Ш}(T) = \ker\Big(H^1_{\text{ét}}(K,T) \to \prod_{v\in\operatorname{Val}(K)} H^1_{\text{ét}}(K_v,T)\Big)$ is the Tate-Shafarevich group of T. By [72, Prop. 8.3, p. 58] and [54, Cor. 4.6, p. 2568], the rationality of T implies that $\text{Ш}(T)$ is trivial. Further, we have $\operatorname{Pic} T \cong \mathbb{Z}/d\mathbb{Z}$ (see [39, Prop. II.6.5(c), p. 133]). Since $\zeta_K(s)$ and $\zeta_L(s)$ both have a simple pole at $s=1$, we have $L\big(X^*(\overline{T}),1\big) = \frac{\operatorname{Res}_{s=1}\zeta_L(s)}{\operatorname{Res}_{s=1}\zeta_K(s)}$. Finally, as T is anisotropic, we have $T(\mathbb{A}_K)^1 = T(\mathbb{A}_K)$. □

5.4 Heights and indicator functions

In this section we define functions which allow us to use harmonic analysis to study weak Campana points. Let L/K be an extension of number fields with K-basis $\boldsymbol{\omega} = \{\omega_0,\dots,\omega_{d-1}\}$ and Galois closure E/K. When $L=E$, for any $i,j\in\{0,\dots,d-1\}$ and $g\in G=\operatorname{Gal}(E/K)$, write

$$\omega_i\cdot\omega_j = \sum_{k=0}^{d-1} a_k^{ij}\omega_k, \quad g(\omega_i) = \sum_{k=0}^{d-1} b_k^g\omega_k, \quad 1 = \sum_{k=0}^{d-1} c_k\omega_k.$$

Definition 5.4.1. When $L=E$, we define $S(\boldsymbol{\omega})$ to be the minimal subset of $\operatorname{Val}(K)$ containing S_∞ such that $a_k^{ij}, b_k^g, c_k \in \mathcal{O}_v$ for all $v\notin S(\boldsymbol{\omega})$, $i,j,k\in\{0,\dots,d-1\}$ and $g\in G$. Otherwise, we define $S(\boldsymbol{\omega})$ to be the minimal subset of $\operatorname{Val}(K)$ containing S_∞ such that $N_{\boldsymbol{\omega}}$ is an irreducible polynomial over $\mathcal{O}_{K,S(\boldsymbol{\omega})}$.

Remark 5.4.2. When $L=E$, by (5.3.1), the $S(\boldsymbol{\omega})$-integrality of the a_k^{ij} and b_k^g implies that $N_{\boldsymbol{\omega}}$ is defined over $\mathcal{O}_{K,S(\boldsymbol{\omega})}$, while the $S(\boldsymbol{\omega})$-integrality of the c_k implies that the coefficients of $N_{\boldsymbol{\omega}}$ are not all divisible by some $\alpha\in\mathcal{O}_{K,S(\boldsymbol{\omega})}\setminus\mathcal{O}^*_{K,S(\boldsymbol{\omega})}$. Since $N_{\boldsymbol{\omega}}$ is irreducible over K, we deduce that $N_{\boldsymbol{\omega}}$ is irreducible over $\mathcal{O}_{K,S(\boldsymbol{\omega})}$, hence, for any L, the Zariski closure of $Z(N_{\boldsymbol{\omega}})$ in $\mathbb{P}^{d-1}_{\mathcal{O}_{K,S(\boldsymbol{\omega})}}$ is $\operatorname{Proj}\mathcal{O}_{K,S(\boldsymbol{\omega})}[x_0,\dots,x_{d-1}]/(N_{\boldsymbol{\omega}})$.

From now on, we fix the model $\Big(\mathbb{P}^{d-1}_{\mathcal{O}_{K,S(\boldsymbol{\omega})}},\mathcal{D}^{\boldsymbol{\omega}}_m\Big)$ for $\Big(\mathbb{P}^{d-1}_K,\Delta^{\boldsymbol{\omega}}_m\Big)$, where $\mathcal{D}^{\boldsymbol{\omega}}_m = \big(1-\frac{1}{m}\big)\operatorname{Proj}\mathcal{O}_{K,S(\boldsymbol{\omega})}[x_0,\dots,x_{d-1}]/(N_{\boldsymbol{\omega}})$. We denote by $\mathcal{D}^{\boldsymbol{\omega}}_{\text{red}}$ the support of $\mathcal{D}^{\boldsymbol{\omega}}_m$.

Note 5.4.3. In both the Galois and non-Galois cases, the conditions on $S(\boldsymbol{\omega})$ ensure that we may take the "obvious" model above. The potentially stronger conditions in the Galois case (in which we obtain our results) ensure a certain compatibility between intersection multiplicity and toric multiplication, as we shall see in Section 5.4.2.

Remark 5.4.4. When $L=E$ and $\boldsymbol{\omega}$ is a relative integral basis, it is clear that $S(\boldsymbol{\omega}) = S_\infty$, since every algebraic integer is expressible as an $\mathcal{O}_K$-linear combination of elements of a relative integral basis, and $\mathcal{O}_L$ is closed under multiplication and conjugation.

5.4.1 Definitions

Definition 5.4.5. [4, §2.1] For each place v of K, we define the *local height function*

$$H_v : T(K_v) \to \mathbb{R}_{>0}, \quad t_v \mapsto e^{\varphi_\Sigma\left(\deg_{T,E,v}(t_v)\right)\log q_v}.$$

We then define the *global height function*

$$H : T(\mathbb{A}_K) \to \mathbb{R}_{>0}, \quad (t_v)_v \mapsto \prod_{v\in\mathrm{Val}(K)} H_v(t_v).$$

Definition 5.4.6. For each place $v \notin S(\boldsymbol{\omega})$, define the function

$$H'_v : T(K_v) \to \mathbb{R}_{>0}, \quad x \mapsto \frac{\max\{|x_i|_v^d\}}{|N_{\boldsymbol{\omega}}(x)|_v}.$$

Remark 5.4.7. Note that $H'_v(x) \geq 1$ for all $x \in T(K_v)$. Indeed, one may always select v-adic coordinates x_i such that $\max\{|x_i|_v\} = 1$, and $N_{\boldsymbol{\omega}}$ has coefficients in $\mathcal{O}_v$ by Remark 5.4.2, so, by the strong triangle inequality, we have $|N_{\boldsymbol{\omega}}(x)|_v \leq 1$.

Lemma 5.4.8. *For all but finitely many places $v \notin S(\boldsymbol{\omega})$, we have $H'_v = H_v$.*

Proof. Note that H'_v is the local Weil function associated to the basis of global sections of $-K_{\mathbb{P}^{d-1}}$ consisting of all monomials of degree d in [4, Def. 2.1.1, p. 606]. It is well-known (see [19, §2.2.3]) that two height functions corresponding to adelic metrisations of the same line bundle are equal over all but finitely many places. □

Definition 5.4.9. We define the finite set

$$S'(\boldsymbol{\omega}) = S(\boldsymbol{\omega}) \cup \{v \notin S(\boldsymbol{\omega}) : H'_v \neq H_v\} \cup \{v \in \mathrm{Val}(K) : E/K \text{ is ramified at } v\}.$$

Definition 5.4.10. For each place $v \notin S(\boldsymbol{\omega})$, define the *local indicator function*

$$\phi_{m,v} : T(K_v) \to \{0,1\}, \quad t_v \mapsto \begin{cases} 1 \text{ if } H'_v(t_v) = 1 \text{ or } H'_v(t_v) \geq q_v^m, \\ 0 \text{ otherwise.} \end{cases}$$

Setting $\phi_{m,v} = 1$ for $v \in S(\boldsymbol{\omega})$, we then define the *global indicator function*

$$\phi_m : T(\mathbb{A}_K) \to \{0,1\}, \quad (t_v)_v \mapsto \prod_{v\in\mathrm{Val}(K)} \phi_{m,v}(t_v).$$

Remark 5.4.11. Let $v \notin S(\boldsymbol{\omega})$ be a non-archimedean place of K. Since H'_v is continuous with discrete image in $\mathbb{R}_{>0}$, its level sets are clopen. It follows that $\phi_{m,v}$ is continuous for all $v \in \mathrm{Val}(K)$. Also, since $\phi_{m,v}(T(\mathcal{O}_v)) = 1$ for all $v \notin S'(\boldsymbol{\omega})$ by Lemma 5.2.16(i), we see that ϕ_m is well-defined and continuous on $T(\mathbb{A}_K)$.

Lemma 5.4.12. *The weak Campana $\mathcal{O}_{K,S(\boldsymbol{\omega})}$-points of $\left(\mathbb{P}_K^{d-1}, \Delta_m^{\boldsymbol{\omega}}\right)$ are precisely the rational points $t \in T(K)$ such that $\phi_m(t) = 1$.*

Proof. Take $v \notin S(\boldsymbol{\omega})$, and let $t_0, \ldots, t_{d-1}$ be a set of $\mathcal{O}_v$-coordinates for $t \in T(K)$ with at least one $t_i \in \mathcal{O}_v^*$. Then we have

$$H'_v(t) = \frac{1}{|N_{\boldsymbol{\omega}}(t_0,\ldots,t_{d-1})|_v} = q_v^{v(N_{\boldsymbol{\omega}}(t_0,\ldots,t_{d-1}))} = q_v^{n_v\left(\mathcal{D}_{\mathrm{red}}^{\boldsymbol{\omega}},t\right)}.$$ □

5.4.2 Invariant subgroups

For this section, let $L = E$ be Galois over K.

Lemma 5.4.13. *For all $v \notin S(\boldsymbol{\omega})$ and $x, y \in T(K_v)$, we have*

$$H'_v(x \cdot y) \leq H'_v(x) H'_v(y).$$

Proof. Choose sets of projective coordinates $\{x_0, \dots, x_{d-1}\}$ and $\{y_0, \dots, y_{d-1}\}$ for x and y respectively. Note that

$$(x_0\omega_0 + \cdots + x_{d-1}\omega_{d-1}) \cdot (y_0\omega_0 + \cdots + y_{d-1}\omega_{d-1}) = (z_0\omega_0 + \cdots + z_{d-1}\omega_{d-1}),$$

where, for $a_k^{ij} \in \mathcal{O}_v$ as in Definition 5.4.1, we have

$$z_k = \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} a_k^{ij} x_i y_j.$$

Using $N_{\boldsymbol{\omega}}(x \cdot y) = N_{\boldsymbol{\omega}}(x) N_{\boldsymbol{\omega}}(y)$ and the strong triangle inequality, we deduce that

$$\begin{aligned} H'_v(x \cdot y) &= \frac{\max\{|\sum_{i=0}^{d-1} \sum_{j=0}^{d-1} a_k^{ij} x_i y_j|_v^d\}}{|N_{\boldsymbol{\omega}}(x \cdot y)|_v} \\ &\leq \frac{1}{|N_{\boldsymbol{\omega}}(x)|_v} \frac{1}{|N_{\boldsymbol{\omega}}(y)|_v} \max\{|a_k^{ij}|_v^d\} \max\{|x_i|_v^d\} \max\{|y_j|_v^d\} \\ &\leq H'_v(x) H'_v(y). \end{aligned}$$

□

Lemma 5.4.14. *For any place $v \notin S(\boldsymbol{\omega})$, the level set*

$$\mathcal{K}_v = \{t_v \in T(K_v) : H'_v(t_v) = 1\}$$

is a subgroup of $T(\mathcal{O}_v)$.

Proof. From Proposition 5.3.4 and Lemma 5.2.16(i), it is clear that $H'_v(t_v) = 1$ implies $t_v \in T(\mathcal{O}_v)$, so $\mathcal{K}_v \subset T(\mathcal{O}_v)$. It is also clear that $H'_v(1) = 1$, and closure under multiplication follows from Lemma 5.4.13 and Remark 5.4.7. It only remains to verify that $x \in \mathcal{K}_v$ implies $x^{-1} \in \mathcal{K}_v$. Let $x \in \mathcal{K}_v$, and choose coordinates $x_0, \dots, x_{d-1}$ with $\max\{|x_i|_v^d\} = 1$. Since $H'_v(x) = 1$, we must have $|N_{\boldsymbol{\omega}}(x)|_v = 1$. Note that

$$(x_0\omega_0 + \cdots + x_{d-1}\omega_{d-1})^{-1} = \frac{1}{N_{\boldsymbol{\omega}}(x_0, \dots, x_{d-1})} \prod_{\substack{g \in G \\ g \neq 1_G}} (x_0 g(\omega_0) + \cdots + x_0 g(\omega_0)).$$

Recursively applying Lemma 5.4.13, we obtain

$$H'_v(x^{-1}) \leq \prod_{\substack{g \in G \\ g \neq 1_G}} H'_v(g(x)).$$

By Remark 5.4.7, it suffices to show that, for any $g \in G$, we have $H'_v(g(x)) = 1$. Since $N_{\boldsymbol{\omega}}(g(x)) = N_{\boldsymbol{\omega}}(x)$, it suffices by Remark 5.4.7 to show that $\max\{|g(x)_i|_v\} \leq 1$. This follows from the fact that $b_k^g \in \mathcal{O}_v$ since $v \notin S(\boldsymbol{\omega})$, see Definition 5.4.1. □

Corollary 5.4.15. *For every place $v \notin S(\boldsymbol{\omega})$, the function H'_v is $\mathcal{K}_v$-invariant.*

Proof. Take $x \in \mathcal{K}_v$, and let $y \in T(K_v)$. Then by Lemma 5.4.13, we have

$$H'_v(x \cdot y) \leq H'_v(x) H'_v(y) = H'_v(y),$$

while on the other hand, since $x^{-1} \in \mathcal{K}_v$ by Lemma 5.4.14, we have

$$H'_v(y) = H'_v\big(x^{-1} \cdot (x \cdot y)\big) \leq H'_v\big(x^{-1}\big) H'_v(x \cdot y) = H'_v(x \cdot y),$$

so we conclude that $H'_v(x \cdot y) = H'_v(y)$. □

Lemma 5.4.16. *For each place $v \notin S(\boldsymbol{\omega})$, the functions H_v and $\phi_{m,v}$ are both $\mathcal{K}_v$-invariant and 1 on $\mathcal{K}_v$. Further, $\mathcal{K}_v$ is compact, open and of finite index in $T(\mathcal{O}_v)$. Moreover, when $v \notin S'(\boldsymbol{\omega})$, we have $\mathcal{K}_v = T(\mathcal{O}_v)$.*

Proof. Let $v \notin S(\boldsymbol{\omega})$. By [4, Thm. 2.1.6(i), p. 608], H_v is $T(\mathcal{O}_v)$-invariant, hence trivial and invariant on all of $T(\mathcal{O}_v)$. By Corollary 5.4.15, the function $\phi_{m,v}$ is $\mathcal{K}_v$-invariant; since $\phi_{m,v}(1) = 1$, it is also trivial on $\mathcal{K}_v$.

Now, since $\mathcal{K}_v = \big(H'_v|_{T(\mathcal{O}_v)}\big)^{-1}(\{1\})$, it is open. Since the cosets of an open subgroup form an open cover of a topological group, any open subgroup of a compact topological group is closed and of finite index. Then $\mathcal{K}_v \subset T(\mathcal{O}_v)$ is closed, hence compact, and of finite index. Finally, when $v \notin S'(\boldsymbol{\omega})$, we have $H'_v = H_v$, and $H_v^{-1}(\{1\}) = T(\mathcal{O}_v)$ by Lemma 5.2.16(i) and the equality $\varphi_\Sigma^{-1}(\{0\}) = \{0\}$, so $\mathcal{K}_v = T(\mathcal{O}_v)$. □

Definition 5.4.17. For each $v \in S(\boldsymbol{\omega})$, set $\mathcal{K}_v = T(\mathcal{O}_v)$. Let $\mathcal{K} = \prod_{v \in \mathrm{Val}(K)} \mathcal{K}_v$, and let $\mathcal{U}$ be the group of automorphic characters of T which are trivial on $\mathcal{K}$.

5.4.3 Height zeta function and Fourier transforms

Definition 5.4.18. For $\operatorname{Re} s \gg 0$, we define the *height zeta function*

$$Z_m : \mathbb{C} \to \mathbb{C}, \quad s \mapsto \sum_{x \in \mathbb{P}^{d-1}(K)} \frac{\phi_m(x)}{H(x)^s}.$$

Definition 5.4.19. Let $f : T(\mathbb{A}_K) \to \mathbb{C}$ be a continuous function given as a product of local factors $f_v : T(K_v) \to \mathbb{C}$ such that $f_v(T(\mathcal{O}_v)) = 1$ for all but finitely many places $v \in \mathrm{Val}(K)$. For each place $v \in \mathrm{Val}(K)$ and each character χ of $T(\mathbb{A}_K)$, we define the *local Fourier transform of χ_v with respect to f_v* to be

$$\widehat{H}_v(f_v, \chi_v; -s) = \int_{T(K_v)} \frac{f_v(t_v)\chi_v(t_v)}{H_v(t_v)^s} d\mu_v$$

for all $s \in \mathbb{C}$ for which the integral exists. We then define the *global Fourier transform of χ with respect to f* to be

$$\widehat{H}(f, \chi; -s) = \prod_{v \in \mathrm{Val}(K)} \widehat{H}_v(f_v, \chi_v; -s) = \int_{T(\mathbb{A}_K)} \frac{f(t)\chi(t)}{H(t)^s} d\mu.$$

5.5 Weak Campana points

In this section we prove Theorem 5.1.1. Fix an extension of number fields L/K of degree d with K-basis $\boldsymbol{\omega}$, set $T = T_{\boldsymbol{\omega}}$ as in Section 5.3 and let $m \in \mathbb{Z}_{\geq 2}$.

5.5.1 Strategy

Following [54] and [4], we will apply a Tauberian theorem [4, Thm. 3.3.2, p. 624] to our height zeta function $Z_m(s)$ in order to find an asymptotic for the number of weak Campana points of bounded height. By *loc. cit.*, it suffices to show that $Z_m(s)$ is absolutely convergent for $\operatorname{Re} s > \frac{1}{m}$ and that $Z_m(s)\left(s - \frac{1}{m}\right)^{b(d,m)}$ admits an extension to a holomorphic function on $\operatorname{Re} s \geq \frac{1}{m}$ which is not zero at $s = \frac{1}{m}$. In order to do this, we will apply the version of the Poisson summation formula given by Bourqui [8, Thm. 3.35, p. 64]. Formally applying this version with $\mathcal{G} = T(\mathbb{A}_K)$, $\mathcal{H} = T(K)$, $dg = d\mu$, dh the discrete measure on $T(K)$ and $F(t) = \frac{\phi_m(t)}{H(t)^s}$ for some $s \in \mathbb{C}$ with $\operatorname{Re} s > \frac{1}{m}$ gives

$$Z_m(s) = \frac{1}{\operatorname{vol}(T(\mathbb{A}_K)/T(K))} \sum_{\chi \in (T(\mathbb{A}_K)/T(K))^\wedge} \widehat{H}(\phi_m, \chi; -s). \tag{5.5.1}$$

5.5.2 Analytic properties of Fourier transforms

Lemma 5.5.1. *For any place $v \in \operatorname{Val}(K)$, any character χ_v of $T(K_v)$ and any $\epsilon > 0$, the local Fourier transform $\widehat{H}_v(\phi_{m,v}, \chi_v; -s)$ is absolutely convergent and is bounded uniformly (in terms of ϵ and v) on $\operatorname{Re} s \geq \epsilon$.*

Proof. Let $\operatorname{Re} s \geq \epsilon$. Since

$$|\widehat{H}_v(\phi_{m,v}, \chi_v; -s)| \leq \int_{T(K_v)} \left| \frac{\phi_{m,v}(t_v)\chi_v(t_v)}{H_v(t_v)^s} \right| d\mu_v \leq \widehat{H}_v(1, 1; -\epsilon),$$

it suffices to prove that $\widehat{H}_v(1, 1; -\epsilon)$ is convergent. For $v \mid \infty$, this follows from [4, Prop. 2.3.2, p. 614], so assume that $v \nmid \infty$. The following argument is essentially the one in [4, Rem. 2.2.8, p. 613], but we fill in the details for the sake of clarity.

Since H_v and $d\mu_v$ are $T(\mathcal{O}_v)$-invariant and $\int_{T(\mathcal{O}_v)} d\mu_v = 1$, we have

$$\widehat{H}_v(1, 1; -\epsilon) = \int_{T(K_v)} \frac{1}{H_v(t_v)^\epsilon} d\mu_v = \sum_{\overline{t_v} \in T(K_v)/T(\mathcal{O}_v)} \frac{1}{H_v(\overline{t_v})^\epsilon}.$$

Now, by Lemma 5.2.16(i), $T(K_v)/T(\mathcal{O}_v)$ can be identified with a sublattice of finite index in $X_*(T_v)$, and this sublattice coincides with $X_*(T_v)$ when v is unramified in L/K. Then we see that, interpreting H_v as a function on $X_*(T_v)$, we have

$$\sum_{\overline{t_v} \in T(K_v)/T(\mathcal{O}_v)} \frac{1}{H_v(\overline{t_v})^\epsilon} \leq \sum_{n_v \in X_*(T_v)} \frac{1}{H_v(n_v)^\epsilon},$$

and the proof of [4, Thm. 2.2.6, p. 611] and Proposition 5.3.6 give

$$\sum_{n_v \in X_*(T_v)} \frac{1}{H_v(n_v)^\epsilon} = \left(1 - \frac{1}{q_v^{d\epsilon}}\right) \prod_{w \mid v} \left(1 - \frac{1}{q_w^\epsilon}\right)^{-1},$$

so we deduce that $\widehat{H}_v(1, 1; -\epsilon)$ is convergent, and this concludes the proof. ∎

Lemma 5.5.2. *For any $v \in \operatorname{Val}(K)$, the local Fourier transform $\widehat{H}_v(\phi_{m,v}, 1; -s)$ is non-zero for all $s \in \mathbb{R}_{>0}$.*

Proof. The proof is analogous to the proof of [54, Lem. 5.1, p. 2575]. □

Lemma 5.5.3. *Let $L = E$ be a Galois extension of K. For any place $v \in \mathrm{Val}(K)$, let χ_v be a character of $T(K_v)$ which is non-trivial on $\mathcal{K}_v$. Then*

$$\widehat{H}_v(\phi_{m,v}, \chi_v; -s) = 0.$$

Proof. Since $\phi_{m,v}$ and H_v are $\mathcal{K}_v$-invariant, the result follows by character orthogonality. □

Corollary 5.5.4. *Let $L = E$ be a Galois extension of K, and let χ be an automorphic character of T. If $\chi \notin \mathcal{U}$ for $\mathcal{U}$ as in Definition 5.4.17, then*

$$\widehat{H}(\phi_m, \chi; -s) = 0.$$

Lemma 5.5.5. *Let $v \nmid \infty$ be a non-archimedean place of K unramified in L/K, and let χ be an automorphic character of T which is unramified at v. Then we have*

$$\widehat{H}_v(1, \chi_v; -s) = \left(1 - \frac{1}{q_v^{ds}}\right) \prod_{w|v} \left(1 - \frac{\chi_w(\pi_w)}{q_w^s}\right)^{-1} = L_v(\chi, s)\zeta_{K,v}(ds)^{-1}.$$

Proof. The result follows from [4, Thm. 2.2.6, p. 611] and Proposition 5.3.6. □

Definition 5.5.6. Given a vector $u = (u_1, \dots, u_r) \in \mathbb{N}^r$, define $f_{r,n,u}(x_1 \dots, x_r)$ to be the sum of all monomials of degree n in the r variables $x_1, \dots, x_r$ with respect to the weighting induced by u, i.e.

$$f_{r,n,u}(x_1, \dots, x_r) = \sum_{\substack{\sum_{i=1}^r u_i a_i = n \\ a_i \in \mathbb{Z}_{\geq 0} \, \forall i}} x_1^{a_1} \dots x_r^{a_r}.$$

Set

$$f_{r,n}(x_1, \dots, x_r) = f_{r,n,u}(x_1, \dots, x_r)$$

for $u = (1, \dots, 1) \in \mathbb{N}^r$.

Proposition 5.5.7. *Let $v \notin S'(\boldsymbol{\omega})$ be a non-archimedean place of K, and let $\chi \in \mathcal{U}$. Let $w_1, \dots, w_r \in \mathrm{Val}(L)$ be the places of L over v. Let u_i be the inertia degree of w_i over v for each $i = 1, \dots, r$. Set*

$$c_{\chi,v,n} = f_{r,n,u}(\chi_{w_1}(\pi_{w_1}), \dots, \chi_{w_r}(\pi_{w_r})).$$

Then, for $\mathrm{Re}\, s > 0$*, we have*

$$\widehat{H}_v(\phi_{m,v}; \chi_v; -s) = 1 + \sum_{n=m}^{\infty} \frac{c_{\chi,v,n} - c_{\chi,v,n-d}}{q_v^{ns}}.$$

Proof. Let $s \in \mathbb{C}$ with $\mathrm{Re}\, s > 0$. As $\chi \in \mathcal{U}$ and $v \notin S'(\boldsymbol{\omega})$, it follows that χ is unramified at v. Then, expanding geometric series, we have

$$L_v(\chi, s) = \prod_{i=1}^r \left(1 - \frac{\chi_{w_i}(\pi_{w_i})}{q_v^{u_i s}}\right)^{-1} = 1 + \sum_{n=1}^{\infty} \frac{c_{\chi,v,n}}{q_v^{ns}},$$

so, by Lemma 5.5.5, we obtain

$$\widehat{H}_v(1, \chi_v; -s) = 1 + \sum_{n=1}^{\infty} \frac{c_{\chi,v,n} - c_{\chi,v,n-d}}{q_v^{ns}}.$$

On the other hand, we may write

$$\widehat{H}_v(1, \chi_v; -s) = \int_{T(K_v)} \frac{\chi_v(t_v)}{H_v(t_v)^s} d\mu_v = \sum_{n=0}^{\infty} \frac{1}{q_v^{ns}} \int_{H_v(t_v)=q_v^n} \chi_v(t_v) d\mu_v,$$

so, comparing these expressions, we see for $n \geq 1$ that

$$c_{\chi,v,n} - c_{\chi,v,n-d} = \int_{H_v(t_v)=q_v^n} \chi_v(t_v) d\mu_v.$$

Since $v \notin S'(\boldsymbol{\omega})$, we have $\phi_{m,v}(t_v) = 1$ if and only if $H_v(t_v) = 1$ or $H_v(t_v) \geq q_v^m$, so the result follows. $\square$

5.5.3 Regularisation

Now that we have expressions for the local Fourier transforms at all but finitely many places, our goal is to find "regularisations" for the global Fourier transforms, i.e. functions expressible as Euler products whose convergence is well-understood and whose local factors approximate the local Fourier transforms well (as expansions in q_v) at all but finitely many places. As in [4], [54] and [66], we will construct our regularisations from L-functions.

Proposition 5.5.8. *Let G be a subgroup of S_d which acts freely and transitively on $\{1, \ldots, d\}$, and let $m \geq 2$ be a positive integer. Let S_m act upon G^m by permutation of coordinates, and let G act on G^m/S_m by right multiplication of every element of a representative m-tuple. Set $S(G, m) = (G^m/S_m)/G$.*

(i) If d is coprime to m, then we have

$$f_{d,m}(x_1, \ldots, x_d) = \sum_{\overline{(g_1,\ldots,g_m)} \in S(G,m)} \sum_{i=1}^{d} x_{g_1(i)} \ldots x_{g_m(i)}. \tag{5.5.2}$$

(ii) If d is prime and $m = kd$, then we have

$$f_{d,m}(x_1, \ldots, x_d) + (d-1)x_1^k \ldots x_d^k = \sum_{\overline{(g_1,\ldots,g_m)} \in S(G,m)} \sum_{i=1}^{d} x_{g_1(i)} \ldots x_{g_m(i)}. \tag{5.5.3}$$

Proof. Let d and m be coprime with $m \geq 2$. For $(g_1, \ldots, g_m) \in G^m$, set

$$\phi_{(g_1,\ldots,g_m)}(x_1, \ldots, x_d) = \sum_{i=1}^{d} x_{g_1(i)} \ldots x_{g_m(i)}.$$

First, we claim that, if $\overline{(g_1, \ldots, g_m)} = \overline{(h_1, \ldots, h_m)}$ in $S(G, m)$, then

$$\phi_{(g_1,\ldots,g_m)}(x_1, \ldots, x_d) = \phi_{(h_1,\ldots,h_m)}(x_1, \ldots, x_d).$$

From this, it will follow that the sum on the right-hand side of (5.5.2) is well-defined and contains every monomial of degree m at least once. Note that $\overline{(g_1, \dots, g_m)} = \overline{(h_1, \dots, h_m)}$ if and only if $\{h_1, \dots, h_m\} = \{g_1 g, \dots, g g_m\}$ as multisets for some $g \in G$. If the coordinates of $(h_1, \dots, h_m) \in G^m$ are a permutation of those of $(g_1, \dots, g_m) \in G^m$, then

$$x_{h_1(i)} \dots x_{h_m(i)} = x_{g_1(i)} \dots x_{g_m(i)}$$

for any $i \in \{1, \dots, d\}$, thus

$$\phi_{(g_1,\dots,g_m)}(x_1, \dots, x_d) = \phi_{(h_1,\dots,h_m)}(x_1, \dots, x_d).$$

If $(h_1, \dots, h_m) = (g_1 g, \dots, g_m g)$ for some $g \in G$, then for any $i \in \{1, \dots, d\}$, we have

$$x_{g_1 g(i)} \dots x_{g_m g(i)} = x_{g_1(j)} \dots x_{g_m(j)}$$

for the unique $j \in \{1, \dots, d\}$ with $g(i) = j$, so

$$\phi_{(g_1,\dots,g_m)}(x_1, \dots, x_d) = \phi_{(g_1 g,\dots,g_m g)}(x_1, \dots, x_d) = \phi_{(h_1,\dots,h_m)}(x_1, \dots, x_d).$$

The claim now follows. It now suffices to prove that the sums $\phi_{(g_1,\dots,g_m)}(x_1, \dots, x_d)$ and $\phi_{(h_1,\dots,h_m)}(x_1, \dots, x_d)$ share no common summand if $\overline{(g_1, \dots, g_m)} \neq \overline{(h_1, \dots, h_m)}$, and that no monomial appears twice in any $\phi_{(g_1,\dots,g_m)}(x_1, \dots, x_d)$, as then every degree-$m$ monomial appears at most once on the right-hand side of (5.5.2).

Since each $\phi_{(g_1,\dots,g_m)}(x_1, \dots, x_d)$ is invariant under right $G-$action on $(g_1, \dots, g_m)$ and reordering of the g_i, we may take $g_1 = 1_G$ without loss of generality. Suppose that $\phi_{(1_G,g_2,\dots,g_m)}(x_1, \dots, x_d)$ and $\phi_{(1_G,h_2,\dots,h_m)}(x_1, \dots, x_d)$ share a common summand, i.e.

$$x_i x_{g_2(i)} \dots x_{g_m(i)} = x_j x_{h_2(j)} \dots x_{h_m(j)}$$

for some $i, j \in \{1, \dots, d\}$. This is equivalent to the equality of multisets

$$\{i, g_2(i), \dots, g_m(i)\} = \{j, h_2(j), \dots, h_m(j)\}.$$

If $i = j$, we have

$$\{g_2(i), \dots, g_m(i)\} = \{h_2(j), \dots, h_m(j)\} = \{h_2(i), \dots, h_m(i)\},$$

and by the freeness of the action of G on $\{1, \dots, d\}$, we have $(g_2, \dots, g_m) = (h_2, \dots, h_m)$ up to reordering, i.e. the m-tuple $(1_G, h_2, \dots, h_m)$ is a permutation of the m-tuple $(1_G, g_2, \dots, g_m)$. If $i \neq j$, we may take $g_2(i) = j$ without loss of generality (note that $g_2 \neq 1_G$ in this case), and we obtain the equality of multisets

$$\{j, h_2(j), \dots, h_m(j)\} = \{g_2(i), h_2 g_2(i), \dots, h_m g_2(i)\}.$$

Once again, by freeness of the action of G on $\{1, \dots, d\}$, we get that

$$\{1_G, g_2, \dots, g_m\} = \{g_2, h_2 g_2, \dots, h_m g_2\}$$

as multisets, but then we see that, up to permuting coordinates, we have

$$(1_G, g_2, \dots, g_m) = (1_G g_2, h_2 g_2, \dots, h_m g_2),$$

i.e. the m-tuples belong to the same orbit under the right action of G. It follows that $\phi_{\overline{(g_1,\dots,g_m)}}(x_1,\dots,x_d)$ and $\phi_{\overline{(h_1,\dots,h_m)}}(x_1,\dots,x_d)$ share no common summand if $\overline{(g_1,\dots,g_m)} \neq \overline{(h_1,\dots,h_m)}$ and that a monomial appears twice in $\phi_{\overline{(g_1,\dots,g_m)}}(x_1,\dots,x_d)$ if and only if we have

$$\{1_G, g_2, \dots, g_m\} = \{1_G g_r, g_2 g_r, \dots, g_m g_r\} \tag{5.5.4}$$

as multisets for some $r \in \{2,\dots,m\}$ with $g_r \neq 1_G$. Without loss of generality we may take $r = 2$ in (5.5.4), so $g_2 \neq 1_G$ and it becomes

$$\{1_G, g_2, \dots, g_m\} = \{g_2, g_2^2, g_3 g_2, \dots, g_m g_2\}.$$

This is equivalent to the multiset $\{1_G, g_2, \dots, g_m\}$ being closed under right multiplication by g_2. In particular, it must contain all powers of g_2. Let n be the order of g_2. Then $n \mid d$ by Lagrange's theorem, and the multiset $\{1_G, g_2, \dots, g_m\}$ contains the set $\{1_G, g_2, g_2^2, \dots, g_2^{n-1}\}$. Further, it must contain all of the sets $\{g_s, g_s g_2, g_s g_2^2 \dots, g_s g_2^{n-1}\}$ for $s = 3, \dots, m$. The sets corresponding to g_{s_1} and g_{s_2} are not disjoint if and only if $g_{s_2} = g_{s_1} g_2^t$ for some $t \in \mathbb{N}$, in which case they are equal, so $\{1_G, g_2, \dots, g_m\}$ as a multiset can be written as a disjoint union of sets of size n. Then $n \mid m$, but $n \mid d$ and d and m are coprime, hence $g_2 = 1_G$, contradicting the assumption $g_2 \neq 1_G$. We conclude that no monomial appears twice in $\phi_{\overline{(g_1,\dots,g_m)}}(x_1,\dots,x_d)$. Hence we have proved the first result.

Let now $d = p$ a prime, and let $m = kp$ for some $k \in \mathbb{N}$. By the above, the sum on the right-hand side of (5.5.3) contains every degree-kp monomial in the variables $x_1, \dots, x_p$ at least once, no two summands of the outer sum share a common summand and a monomial appears twice in $\phi_{\overline{(1_G, g_2 \dots, g_{kp})}}(x_1, \dots, x_p)$ if and only if

$$\{1_G, g_2, \dots, g_{kp}\} = \bigcup_{i=1}^{r} \{h_i, h_i g, \dots, h_i g^{n-1}\}$$

as sets for some elements $h_1, \dots, h_r$ of G and some non-identity element $g \in G$. Since G is of prime order, such an element g generates G, so

$$\overline{(g_1, \dots, g_{kp})} = \overline{(1_G, g, \dots, g^{p-1}, \dots, 1_G, g, \dots, g^{p-1})}. \tag{5.5.5}$$

Further, the right-hand side of (5.5.5) is independent of the choice of g. Letting g be a generator of G, we conclude that $\phi_{\overline{(1_G, g, \dots, g^{p-1}, \dots, 1_G, g, \dots, g^{p-1})}}(x_1, \dots, x_p)$ is the only one of the polynomials $\phi_{\overline{(g_1,\dots,g_{kp})}}(x_1, \dots, x_p)$, $\overline{(g_1, \dots, g_{kp})} \in S(G, kp)$ in which a monomial appears twice. In this polynomial, the only monomial which appears is $x_1^k \dots x_p^k$, and so it appears p times. $\square$

Remark 5.5.9. It follows immediately from Proposition 5.5.8 that

$$\#S(G,m) = \begin{cases} \frac{1}{d}\binom{d+m-1}{d-1} \text{ if } d \text{ and } m \text{ are coprime,} \\ \frac{1}{d}\left(\binom{d+m-1}{d-1} - 1\right) + 1 \text{ if } d \text{ is prime and } d \text{ divides } m, \end{cases}$$

since the number of monomials of degree m in d variables is $\binom{d+m-1}{d-1}$.

For the rest of this section, let $L = E$ be Galois over K with Galois group G, and assume that m is coprime to d if d is not prime.

Lemma 5.5.10. *Let $v \notin S'(\boldsymbol{\omega})$ be a non-archimedean place which is totally split in E/K, let $\chi \in \mathcal{U}$ and define*

$$F'_{m,\chi,v}(s) = \frac{\prod_{\overline{(g_1,\dots,g_m)}\in S(G,m)} L_v(\chi^{g_1}\dots\chi^{g_m}, ms)}{\prod_{\overline{(h_1,\dots,h_{m-d})}\in S(G,m-d)} L_v\big(\chi^{h_1}\dots\chi^{h_{m-d}}, ms\big)}.$$

Then we have

$$F'_{m,\chi,v}(s) = \prod_{\overline{(g_1,\dots,g_m)}\in S'(G,m)} L_v(\chi^{g_1}\dots\chi^{g_m}, ms),$$

where

$$S'(G,m) = \Big\{\overline{(g_1,\dots,g_m)} \in S(G,m) : \#\{g_1,\dots,g_m\} \le d-1\Big\}.$$

.

Proof. Let $G = \{g_1,\dots,g_d\}$. First, we show that every factor of the denominator appears on the numerator. Let $\overline{(h_1,\dots,h_{m-d})} \in S(G,m-d)$. Then we claim that

$$L_v\Big(\chi^{h_1}\dots\chi^{h_{m-d}}, ms\Big) = L_v\Big(\chi^{h_1}\dots\chi^{h_{m-d}}\chi^{g_1}\dots\chi^{g_d}, ms\Big).$$

Since G acts freely and transitively on the places $w_1,\dots,w_d$ of E over v, we have

$$\{\chi_{w_i}^{g_1}(\pi_{w_i}),\dots,\chi_{w_i}^{g_d}(\pi_{w_i})\} = \{\chi_{w_1}(\pi_{w_1}),\dots,\chi_{w_d}(\pi_{w_d})\}$$

for any $i = 1,\dots,d$. Since χ_v is trivial on $\mathcal{K}_v = T(\mathcal{O}_v)$, we have from Note 5.3.1 that $\prod_{i=1}^d \chi_{w_i}(\pi_{w_i}) = \chi_v(1) = 1$, so $(\chi^{g_1}\dots\chi^{g_d})_w(\pi_w) = 1$ for all $w \mid v$. Then the equality follows.

It now suffices to show that, for $\overline{(h_1,\dots,h_{m-d})} \neq \overline{(h'_1,\dots,h'_{m-d})}$, we have

$$\overline{(h_1,\dots,h_{m-d},g_1,\dots,g_d)} \neq \overline{(h'_1,\dots,h'_{m-d},g_1,\dots,g_d)}.$$

If not, then $\{h'_1,\dots,h'_{m-d},g_1,\dots,g_d\} = \{gh_1,\dots,gh_{m-d},gg_1,\dots,gg_d\}$ as multisets for some $g \in G$. Since we have $\{gg_1,\dots,gg_d\} = \{g_1,\dots,g_d\}$, this implies that $\{h'_1,\dots,h'_{m-d}\} = \{gh_1,\dots,gh_{m-d}\}$ as multisets, but then we have $\overline{(h_1,\dots,h_{m-d})} = \overline{(h'_1,\dots,h'_{m-d})}$, which is false. ☐

Remark 5.5.11. From the proof of Lemma 5.5.10, we obtain $\#S'(G,m) = S(G,m) - S(G,m-d)$. Combining this with Remark 5.5.9, we obtain

$$\#S'(G,m) = \frac{1}{d}\left(\binom{d+m-1}{d-1} - \binom{m-1}{d-1}\right) = b(d,m).$$

Note that the term $\frac{1}{d}\binom{m-1}{d-1}$ only appears when $d \le m$.

Definition 5.5.12. For all $\chi \in \mathcal{U}$, $\operatorname{Re} s > 0$ and non-archimedean places $v \nmid \infty$, set

$$F_{m,\chi,v}(s) = \prod_{\overline{(g_1,\dots,g_m)}\in S'(G,m)} L_v(\chi^{g_1}\dots\chi^{g_m}, ms), \quad G_{m,\chi,v}(s) = \frac{\widehat{H}_v(\phi_{m,v},\chi_v;-s)}{F_{m,\chi,v}(s)},$$

and define

$$F_{m,\chi}(s) = \prod_{v\nmid\infty} F_{m,\chi,v}(s) = \prod_{(g_1,\ldots,g_m)\in S'(G,m)} L(\chi^{g_1}\ldots\chi^{g_m}, ms), \tag{5.5.6}$$

$$G_{m,\chi}(s) = \prod_{v\nmid\infty} G_{m,\chi,v}(s).$$

For any non-archimedean place $v \nmid \infty$, write

$$\widehat{H}_v(\phi_{m,v},\chi_v;-s) = \sum_{n=0}^{\infty} \frac{a_{\chi,v,n}}{q_v^{ns}},$$

where $a_{\chi,v,n} = \int_{H_v(t_v)=q_v^n} \phi_{m,v}(t_v)\chi_v(t_v)d\mu_v$, and write

$$F_{m,\chi,v}(s) = 1 + \sum_{n=1}^{\infty} \frac{b_{\chi,v,mn}}{q_v^{mns}}$$

for the expansion of $F_{m,\chi,v}(s)$ as a multidimensional geometric series in q_v^{ms}, so

$$G_{m,\chi,v}(s) = \frac{\sum_{n=0}^{\infty} \frac{a_{\chi,v,n}}{q_v^{ns}}}{1+\sum_{n=1}^{\infty} \frac{b_{\chi,v,mn}}{q_v^{mns}}} = \sum_{n=0}^{\infty} \frac{d_{\chi,v,n}}{q_v^{ns}},$$

where $d_{\chi,v,n}$ is defined for all $n \geq 0$ by the iterative formula

$$d_{\chi,v,n} = a_{\chi,v,n} - \sum_{r=1}^{\lfloor \frac{n}{m} \rfloor} b_{\chi,v,mr} d_{\chi,v,n-mr}. \tag{5.5.7}$$

In particular, we have $d_{\chi,v,n} = a_{\chi,v,n}$ for $0 \leq n \leq m-1$.

Corollary 5.5.13. *For $v \notin S'(\boldsymbol{\omega})$ a non-archimedean place, we have $d_{\chi,v,0} = 1$ and $d_{\chi,v,n} = 0$ for all $n \in \{1,\ldots,m\}$.*

Proof. Let $c_{\chi,v,n}$ be as defined in Proposition 5.5.7. Since $v \notin S'(\boldsymbol{\omega})$, we have from *loc. cit.* that $a_{\chi,v,0} = 1$, $a_{\chi,v,n} = 0$ for $n \in \{1,\ldots,m-1\}$ and $a_{\chi,v,m} = c_{\chi,v,m} - c_{\chi,v,m-d}$. Then, by (5.5.7), we see that $d_{\chi,v,0} = a_{\chi,v,0} = 1$ and $d_{\chi,v,n} = a_{\chi,v,n} = 0$ for $1 \leq n \leq m-1$. Further, we obtain

$$d_{\chi,v,m} = a_{\chi,v,m} - b_{\chi,v,m}d_{\chi,v,0} = c_{\chi,v,m} - c_{\chi,v,m-d} - b_{\chi,v,m},$$

so, to complete the proof, it suffices to show that $b_{\chi,v,m} = c_{\chi,v,m} - c_{\chi,v,m-d}$.
Since E/K is Galois, all of the places $w_1,\ldots,w_r$ of E over v share a common inertia degree d_v. Since $\chi_v(T(\mathcal{O}_v)) = 1$, it is unramified as a Hecke character at all of the w_i (see Note 5.3.1), and for any $g_1,\ldots,g_m \in G$, so is $\chi^{g_1}\ldots\chi^{g_m}$. Then

$$\begin{aligned} L_v(\chi^{g_1}\ldots\chi^{g_m}, ms) &= \prod_{i=1}^{r}\left(1 - \frac{(\chi^{g_1}\ldots\chi^{g_m})_{w_i}(\pi_{w_i})}{q_v^{d_v ms}}\right)^{-1} \\ &= 1 + \frac{1}{q_v^{d_v ms}}\sum_{i=1}^{r}(\chi^{g_1}\ldots\chi^{g_m})_{w_i}(\pi_{w_i}) + O\left(\frac{1}{q_v^{(d_v m+1)s}}\right). \end{aligned} \tag{5.5.8}$$

First, suppose that v is totally split in E/K. Then (5.5.8) gives

$$L_v(\chi^{g_1}\dots\chi^{g_m}, ms) = 1 + \frac{\phi_{(g_1,\dots,g_m)}(\chi_{w_1}(\pi_{w_1}),\dots,\chi_{w_d}(\pi_{w_d}))}{q_v^{ms}} + O\left(\frac{1}{q_v^{(m+1)s}}\right).$$

Since G acts freely and transitively on the w_i, it follows from Proposition 5.5.8 and Lemma 5.5.10 that $b_{\chi,v,m} = c_{\chi,v,m} - c_{\chi,v,m-d}$, and so $d_{\chi,v,m} = 0$.

Now assume that v is not totally split in E/K. If $\gcd(d,m) = 1$, then $c_{\chi,v,m} = c_{\chi,v,m-d} = 0$ as $d_v \mid d$ implies $d_v \nmid m$. If d is prime, then v is inert and we have $\widehat{H}_v(\phi_{m,v}, \chi_v; -s) = 1$ since $T(K_v) = T(\mathcal{O}_v)$. Then, in either case, we have $c_{\chi,v,m} - c_{\chi,v,m-d} = 0$, and (5.5.8) implies that $b_{\chi,v,m} = 0$, hence $d_{\chi,v,m} = 0$. □

Corollary 5.5.14. *For any $\chi \in \mathcal{U}$, we have*

$$\widehat{H}(\phi_m;\chi;-s) = \prod_{v\mid\infty} \widehat{H}_v(1,\chi_v;-s)F_{m,\chi}(s)G_{m,\chi}(s),$$

where $G_{m,\chi}(s)$ is holomorphic and uniformly bounded with respect to χ for $\operatorname{Re} s \geq \frac{1}{m}$ and $G_{m,1}(\frac{1}{m}) \neq 0$. In particular, $\widehat{H}(\phi_m,\chi;-s)$ possesses a holomorphic continuation to the line $\operatorname{Re} s = \frac{1}{m}$, apart from possibly at $s = \frac{1}{m}$. When $\chi = 1$, the right-hand side has a pole of order $b(d,m)$ at $s = \frac{1}{m}$.

Proof. By construction, $\widehat{H}(\phi_m;\chi;-s) = \prod_{v\mid\infty} \widehat{H}_v(1,\chi_v;-s)F_{m,\chi}(s)G_{m,\chi}(s)$. We will prove the stronger result that $G_{m,\chi}(s)$ is holomorphic on $\operatorname{Re} s > \frac{1}{m+1}$ and uniformly bounded with respect to both χ and ϵ on $\operatorname{Re} s \geq \frac{1}{m+1} + \epsilon$ for all $\epsilon > 0$.

For a place $v \nmid \infty$ and $s \in \mathbb{C}$ with $\operatorname{Re} s = \sigma \geq \epsilon$ for some $\epsilon > 0$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\left|\frac{a_{\chi,v,n}}{q_v^{ns}}\right| &= \sum_{n=0}^{\infty}\frac{1}{q_v^{n\sigma}}\left|\int_{H_v(t_v)=q_v^n}\phi_{m,v}(t_v)\chi_v(t_v)d\mu_v\right| \\
&\leq \sum_{n=0}^{\infty}\frac{1}{q_v^{n\sigma}}\int_{H_v(t_v)=q_v^n}|\phi_{m,v}(t_v)\chi_v(t_v)|d\mu_v \\
&= \int_{T(K_v)}\left|\frac{\phi_{m,v}(t_v)\chi_v(t_v)}{H_v(t_v)^s}\right|d\mu_v,
\end{aligned}$$

so, by Lemma 5.5.1, the series $\sum_{n=0}^{\infty}\frac{a_{\chi,v,n}}{q_v^{ns}}$ is absolutely convergent and bounded by a constant depending only on ϵ and v. Now, for any $N \in \mathbb{N}$, we have

$$\left|\sum_{n=0}^{\infty}\frac{a_{\chi,v,n}}{q_v^{ns}} - \sum_{n=0}^{N}\frac{a_{\chi,v,n}}{q_v^{ns}}\right| = \sum_{n=N+1}^{\infty}\left|\frac{a_{\chi,v,n}}{q_v^{ns}}\right| \leq \sum_{n=0}^{\infty}\frac{|a_{\chi,v,n}|}{q_v^{n\epsilon}},$$

from which it follows that $\sum_{n=0}^{\infty}\frac{a_{\chi,v,n}}{q_v^{ns}}$ is also uniformly convergent, hence the function $\widehat{H}_v(\phi_{m,v},\chi_v;-s)$ is holomorphic on $\operatorname{Re} s > 0$. Then, we note that $F_{m,\chi,v}(s)$ is clearly holomorphic on $\operatorname{Re} s > 0$, and we have

$$\frac{1}{|F_{m,\chi,v}(s)|} = \prod_{(g_1,\dots,g_m)\in S'(G,m)}\left|L_v(\chi^{g_1}\dots\chi^{g_m},ms)^{-1}\right| \leq \left(1+\frac{1}{q_v^{m\epsilon}}\right)^{db(d,m)},$$

hence $G_{m,\chi,v}(s)$ is holomorphic on $\operatorname{Re} s > 0$ and is bounded uniformly in terms of ϵ and v on $\operatorname{Re} s \geq \epsilon$.

To conclude the result, it suffices to prove that there exists $N \in \mathbb{N}$ such that

$$\prod_{q_v > N} G_{m,\chi,v}(s)$$

is holomorphic and uniformly bounded with respect to χ on $\operatorname{Re} s \geq \frac{1}{m+1} + \epsilon$ for all $\epsilon > 0$. Let $v \notin S'(\boldsymbol{\omega})$ be non-archimedean, and let $\operatorname{Re} s = \sigma \geq \frac{1}{m+1} + \epsilon$. From

$$\widehat{H}_v(\phi_{m,v}, \chi_v; -s) = \left(1 - \frac{1}{q_v^{ds}}\right) L_v(\chi, s),$$

and the definition of $F_{m,\chi,v}(s)$, we have

$$|a_{\chi,v,n}| \leq 2d^n, \quad |b_{\chi,v,n}| \leq (b(d,m)d)^n.$$

Then, by (5.5.7), it follows inductively that we have

$$|d_{\chi,v,n}| \leq 2^n (b(d,m)d)^n = (2b(d,m)d)^n. \tag{5.5.9}$$

Choose $N > (2b(d,m)d)^{\frac{1}{\sigma}}$ so that, for all places $v \nmid \infty$ with $q_v > N$, we have $v \notin S'(\boldsymbol{\omega})$. Now, any normally convergent infinite product is holomorphic (see [69, §2]), and $\prod_{q_v > N} G_{m,\chi,v}(s)$ converges normally if and only if

$$\sum_{q_v > N} \sum_{n=m+1}^{\infty} \frac{|d_{\chi,v,n}|}{q_v^{n\sigma}}$$

converges, so, by (5.5.9), we need only check convergence of

$$\sum_{q_v > N} \frac{1}{q_v^{(m+1)\sigma}},$$

which is clear. Then $G_{m,\chi}(s)$ is holomorphic on $\operatorname{Re} s > \frac{1}{m+1}$. Further, for $\operatorname{Re} s \geq \frac{1}{m+1} + \epsilon$, we have the bound

$$\left| \prod_{q_v > N} G_{m,\chi,v}(s) \right| \leq \prod_{q_v > N} \left(1 + \sum_{n=m+1}^{\infty} \left(\frac{2b(d,m)d}{q_v^{\frac{1}{m+1}+\epsilon}} \right)^n \right),$$

which is uniform with respect to χ. Now, as a convergent infinite product, $G_{m,1}\left(\frac{1}{m}\right)$ is zero if and only if $G_{m,1,v}\left(\frac{1}{m}\right) = \frac{\widehat{H}_v\left(\phi_{m,v}, 1; -\frac{1}{m}\right)}{F_{m,1,v}\left(\frac{1}{m}\right)} = 0$ for some place $v \nmid \infty$. However, $\widehat{H}_v\left(\phi_{m,v}, 1; -\frac{1}{m}\right) \neq 0$ by Lemma 5.5.2, so we conclude that $G_{m,1}\left(\frac{1}{m}\right) \neq 0$. The order of the pole of the right-hand side being $b(d,m)$ follows from Theorem 5.2.22, since

$$F_{m,1}(s) = \zeta_E(ms)^{b(d,m)}. \qquad \square$$

Note 5.5.15. In constructing the regularisation $F_{m,\chi}(s)$, one must ensure that

$$\frac{\widehat{H}_v(\phi_{m,v}, \chi_v; -s)}{F_{m,\chi,v}(s)} = 1 + O\left(\frac{1}{q_v^{(m+1)s}}\right)$$

for all non-archimedean places v with q_v is sufficiently large. As seen above, the restrictions on d, m and E ensure that this is automatic for all such places which are not totally split, i.e. we only need to approximate the local Fourier transform at totally split places not in $S'(\boldsymbol{\omega})$. Without these restrictions, one might have to approximate the local Fourier transform at places of more than one splitting type simultaneously, and to do this would require a new approach.

Before applying our key theorems, we give one more result, which will be used in order to move from the Poisson summation formula to the Tauberian theorem.

Lemma 5.5.16. *[54, Lem. 5.9, p. 2577] Choose an $\mathbb{R}$-vector space norm $\|\cdot\|$ on $X^*(T_\infty)_\mathbb{R}$ and let $\mathcal{L} \subset X^*(T_\infty)_\mathbb{R}$ be a lattice. Let C be a compact subset of $\operatorname{Re} s \geq \frac{1}{m}$ and let $g : X^*(T_\infty)_\mathbb{R} \times C \to \mathbb{C}$ be a function. If there exists $0 \leq \delta < \frac{1}{d-1}$ such that*

$$|g(\psi, s)| \ll_C (1 + \|\psi\|)^\delta$$

for all $\psi \in X^*(T_\infty)_\mathbb{R}$ and $s \in C$, then the sum

$$\sum_{\psi \in \mathcal{L}} g(\psi, s) \prod_{v \mid \infty} \widehat{H}_v(1, \psi; -s)$$

is absolutely and uniformly convergent on C.

Theorem 5.5.17. *Let*

$$\Omega_m(s) = Z_m(s)\left(s - \frac{1}{m}\right)^{b(d,m)}.$$

Then $\Omega_m(s)$ admits an extension to a holomorphic function on $\operatorname{Re} s \geq \frac{1}{m}$.

Proof. Let $s \in \mathbb{C}$ with $\operatorname{Re} s > \frac{1}{m}$. Combining the formal application (5.5.1) of the Poisson summation formula with Lemma 5.3.8 and Corollary 5.5.4 gives

$$Z_m(s) = \frac{\operatorname{Res}_{s=1} \zeta_K(s)}{d \operatorname{Res}_{s=1} \zeta_E(s)} \sum_{\chi \in \mathcal{U}} \widehat{H}(\phi_m, \chi; -s). \tag{5.5.10}$$

By Corollary 5.5.14, the function $\frac{\phi_m(t)}{H(t)^s}$ is L^1 for $\operatorname{Re} s > \frac{1}{m}$. To show that this application is valid, we apply Bourqui's criterion [8, Cor. 3.36, p. 64], by which it suffices to show that the right-hand side of (5.5.10) is absolutely convergent, $\frac{\phi_m(t)}{H(t)^s}$ is continuous and there exists an open neighbourhood $U \subset T(\mathbb{A}_K)$ of the origin and strictly positive constants C_1 and C_2 such that for all $u \in U$ and all $t \in T(\mathbb{A}_K)$, we have

$$C_1 \left|\frac{\phi_m(t)}{H(t)^s}\right| \leq \left|\frac{\phi_m(ut)}{H(ut)^s}\right| \leq C_2 \left|\frac{\phi_m(t)}{H(t)^s}\right|.$$

We may take $U = \mathcal{K}$, and continuity is clear. It only remains to prove the absolute convergence. We will prove the stronger result that

$$\sum_{\chi \in \mathcal{U}} \widehat{H}(\phi_m, \chi; -s)\left(s - \frac{1}{m}\right)^{b(d,m)}$$

is absolutely and uniformly convergent on any compact subset C of the half-plane $\operatorname{Re} s \geq \frac{1}{m}$, which will both verify validity of the application and prove the theorem.

Since $\mathcal{K} \subset \mathsf{K}_T$ is of finite index, the map (5.2.2) yields a homomorphism

$$\mathcal{U} \to X^*(T_\infty)_{\mathbb{R}}, \quad \chi \mapsto \chi_\infty,$$

with finite kernel $\mathcal{N}$ and image $\mathcal{L}$ a lattice of full rank. We obtain

$$\begin{aligned} &\sum_{\chi \in \mathcal{U}} \widehat{H}(\phi_m, \chi; -s)\left(s - \frac{1}{m}\right)^{b(d,m)} \\ &= \sum_{\psi \in \mathcal{L}} \prod_{v \mid \infty} \widehat{H}_v(1, \psi; -s) \sum_{\substack{\chi \in \mathcal{U} \\ \chi_\infty = \psi}} \prod_{v \nmid \infty} \widehat{H}_v(\phi_{m,v}, \chi_v; -s)\left(s - \frac{1}{m}\right)^{b(d,m)}, \end{aligned}$$

where the inner sum is finite. Then, for $s \in C$, we have

$$\begin{aligned} &\sum_{\chi \in \mathcal{U}} \widehat{H}(\phi_m, \chi; -s)\left(s - \frac{1}{m}\right)^{b(d,m)} \\ &\ll \sum_{\psi \in \mathcal{L}} \prod_{v \mid \infty} \left|\widehat{H}_v(1, \psi; -s)\right| \sum_{\substack{\chi \in \mathcal{U} \\ \chi_\infty = \psi}} \left| \prod_{v \nmid \infty} \widehat{H}_v(\phi_{m,v}, \chi_v; -s)\left(s - \frac{1}{m}\right)^{b(d,m)} \right|. \end{aligned}$$

Now, for $\chi \in \mathcal{U}$, we deduce from the proof of Corollary 5.5.14 that

$$\left| \prod_{v \nmid \infty} \widehat{H}_v(\phi_{m,v}, \chi_v; -s)\left(s - \frac{1}{m}\right)^{b(d,m)} \right| \ll_C \left| F_{m,\chi}(s)\left(s - \frac{1}{m}\right)^{b(d,m)} \right|.$$

In order to deduce the result from Lemma 5.5.16, it suffices to prove that, for each $\psi \in \mathcal{L}$ and some constant $0 \leq \delta < \frac{1}{d-1}$, we have

$$\left| \sum_{\substack{\chi \in \mathcal{U} \\ \chi_\infty = \psi}} F_{m,\chi}(s)\left(s - \frac{1}{m}\right)^{b(d,m)} \right| \ll_C (1 + \|\psi\|)^\delta$$

for $\|\cdot\|$ as in Definition 5.2.23. As $\mathcal{K} \subset \mathsf{K}_T$ is of finite index, there exists a constant $Q > 0$ such that $q(\chi) < Q$ for all $\chi \in \mathcal{U}$. Since $F_{m,\chi}(s)$ is a product of $b(d,m)$ L-functions of Hecke characters evaluated at ms, it follows from Lemma 5.2.24 that

$$\left| \sum_{\substack{\chi \in \mathcal{U} \\ \chi_\infty = \psi}} F_{m,\chi}(s)\left(s - \frac{1}{m}\right)^{b(d,m)} \right| \ll_{\varepsilon,C} |\mathcal{N}| \cdot Q^\varepsilon (1 + \|\psi\|)^\varepsilon$$

for all for all $\varepsilon > 0$ and $s \in C$. The result now follows from Lemma 5.5.16. $\square$

5.5.4 The leading constant

In order to apply [4, Thm. 3.3.2, p. 624] and deduce Theorem 5.1.1 from Theorem 5.5.17, it only remains to show that $\Omega_m\left(\frac{1}{m}\right) \neq 0$.

Definition 5.5.18. Let $\mathcal{U}^G$ be the subgroup of G-invariant elements of $\mathcal{U}$, and set

$$\mathcal{U}_0 = \begin{cases} \mathcal{U}[m] \text{ if } d = 2, \\ \mathcal{U}^G \cap \mathcal{U}[m] \text{ otherwise.} \end{cases}$$

Lemma 5.5.19. *For any Galois extension of number fields E/K, the subgroup $\mathcal{U}[m] \leq (T(\mathbb{A}_K)/T(K))^\wedge$ is finite. In particular, $\mathcal{U}_0$ is a finite subgroup of $\mathcal{U}$.*

Proof. By class field theory [60, Ch. VI, §6, Ch. VII, §6], $\mathcal{U}$ may be interpreted as a subset of $\mathrm{Gal}\left(E^{\mathrm{ab}}_{S'(\boldsymbol{\omega})}/E\right)^\wedge$, where $E^{\mathrm{ab}}_{S'(\boldsymbol{\omega})}$ is the maximal $S'(\boldsymbol{\omega})$-unramified abelian extension of E, hence $\mathcal{U}[m]$ is in bijection with a subset of $\mathrm{Hom}\left(\mathrm{Gal}\left(E^{\mathrm{ab}}_{S'(\boldsymbol{\omega})}/E\right), \mu_m\right)$, which is finite. □

Lemma 5.5.20.

(i) The characters $\chi \in \mathcal{U}$ contributing to the pole of $Z_m(s)$ of order $b(d,m)$ at $s = \frac{1}{m}$ are precisely the characters $\chi \in \mathcal{U}_0$ such that $\prod_{v \mid \infty} \widehat{H}_v\left(1, \chi_v; -\frac{1}{m}\right) G_{m,\chi}\left(\frac{1}{m}\right) \neq 0$.

(ii) Suppose that $d \neq 2$. If d and m are coprime, then $\mathcal{U}_0 = \{1\}$. If d is prime and m is a multiple of d, then $\mathcal{U}_0 = \{\chi \in \mathcal{U} : \chi^d = 1\}$.

Proof. From Theorem 5.2.22 and Corollary 5.5.14, $\chi \in \mathcal{U}$ contributes to the pole of $Z_m(s)$ at $s = \frac{1}{m}$ if and only if each factor of $F_{m,\chi}(s)$ in (5.5.6) equals $\zeta_E(ms)$ and $\prod_{v \mid \infty} \widehat{H}_v\left(1, \chi_v; -\frac{1}{m}\right) G_{m,\chi}\left(\frac{1}{m}\right) \neq 0$. Denoting by ψ the Hecke character associated to χ, this means precisely that $\prod_{v \mid \infty} \widehat{H}_v\left(1, \chi_v; -\frac{1}{m}\right) G_{m,\chi}\left(\frac{1}{m}\right) \neq 0$ and, for each $\overline{(g_1, \dots, g_m)} \in S'(G, m)$, we have $\left(\psi^{g_1} \dots \psi^{g_m}\right)_v = 1$ for all $v \nmid \infty$, which is equivalent by strong approximation [18, Thm., p. 67] to $\psi^{g_1} \dots \psi^{g_m} = 1$. By Note 5.3.1, this holds if and only if

$$\chi^{g_1} \dots \chi^{g_m} = 1 \text{ for all } \overline{(g_1, \dots, g_m)} \in S'(G, m). \tag{5.5.11}$$

To conclude the first part, it only remains to show that (5.5.11) holds if and only if $\chi \in \mathcal{U}_0$. Taking $\overline{(g_1, \dots, g_m)} = \overline{(1, \dots, 1)}$ in (5.5.11), we obtain $\chi^m = 1$. If $d = 2$, then $S'(G, m) = \left\{\overline{(1, \dots, 1)}\right\}$, and we are done. Otherwise, taking $\overline{(g_1, \dots, g_m)} = \overline{(g, 1, \dots, 1)}$ for any $g \in G$, we obtain $\chi^{m-1}\chi^g = 1$, so $\chi^m = 1$ and $\chi = \chi^g$ for all $g \in G$. Conversely, if $\chi^m = 1$ and $\chi = \chi^g$ for all $g \in G$, then (5.5.11) holds.

Let now $d \neq 2$, $\chi \in \mathcal{U}_0$, $v \notin S'(\boldsymbol{\omega})$ and $w \mid v$. We have $\psi_w(\pi_w) = \psi_w^g(\pi_w) = \psi_{gw}(\pi_{gw})$ for all $g \in G$. Since $\prod_{w \mid v} \psi_w(\pi_w) = 1$ (see Note 5.3.1) and G acts transitively on the places of E over v, we obtain $\psi_w^d(\pi_w) = 1$, hence $\chi^d = 1$ by strong approximation. On the other hand, $\chi^m = 1$. For d and m coprime, we conclude that $\chi = 1$. □

Proposition 5.5.21. *The limit*

$$\Omega_m\left(\frac{1}{m}\right) = \lim_{s \to \frac{1}{m}} \left(s - \frac{1}{m}\right)^{b(d,m)} \sum_{\chi \in \mathcal{U}_0} \widehat{H}(\phi_m, \chi; -s)$$

is non-zero.

Proof. We have

$$\sum_{\chi\in\mathcal{U}_0}\widehat{H}(\phi_m,\chi;-s)=\sum_{\chi\in\mathcal{U}_0}\int_{T(\mathbb{A}_K)}\frac{\phi_m(t)\chi(t)}{H(t)^s}d\mu=\int_{T(\mathbb{A}_K)}\frac{\phi_m(t)}{H(t)^s}\sum_{\chi\in\mathcal{U}_0}\chi(t)d\mu.$$

Let $t\in T(\mathbb{A}_K)$. Note that, if there exists $\chi'\in\mathcal{U}_0$ with $\chi'(t)\neq 1$, then

$$\sum_{\chi\in\mathcal{U}_0}\chi(t)=\sum_{\chi\in\mathcal{U}_0}\chi\chi'(t)=\chi'(t)\sum_{\chi\in\mathcal{U}_0}\chi(t),$$

so $\sum_{\chi\in\mathcal{U}_0}\chi(t)=0$. Then we have

$$\sum_{\chi\in\mathcal{U}_0}\widehat{H}(\phi_m,\chi;-s)=|\mathcal{U}_0|\int_{T(\mathbb{A}_K)^{\mathcal{U}_0,\phi_m}}\frac{1}{H(t)^s}d\mu,$$

where

$$T(\mathbb{A}_K)^{\mathcal{U}_0,\phi_m}=\{t\in T(\mathbb{A}_K):\phi_m(t)=\chi(t)=1\text{ for all }\chi\in\mathcal{U}_0\}.$$

For any $\chi\in\mathcal{U}_0$ and non-archimedean place $v\notin S'(\boldsymbol{\omega})$, comparing the series expressions of $\widehat{H}_v(\phi_{m,v},\chi_v;-s)$ and $F_{m,\chi,v}(s)=F_{m,1,v}(s)$ in Definition 5.5.12, we see that

$$\int_{H_v(t_v)=1}\chi_v(t_v)d\mu_v=\int_{H_v(t_v)=1}d\mu_v,\quad\int_{H_v(t_v)=q_v^m}\chi_v(t_v)d\mu_v=\int_{H_v(t_v)=q_v^m}d\mu_v,$$

so $\chi_v(t_v)=1$ for all $\chi\in\mathcal{U}_0$ whenever $H_v(t_v)=1$ or $H_v(t_v)=q_v^m$.

For each place $v\notin S'(\boldsymbol{\omega})$, define the continuous function

$$\theta_{m,v}:T(K_v)\to\{0,1\},\quad t_v\mapsto\begin{cases}1\text{ if }H_v'(t_v)=1\text{ or }H_v'(t_v)=q_v^m,\\0\text{ otherwise.}\end{cases}$$

Letting $\theta_{m,v}$ be the indicator function of $\mathcal{K}_v$ for $v\in S'(\boldsymbol{\omega})$, we define the function

$$\theta_m:T(\mathbb{A}_K)\to\{0,1\},\quad\theta_m((t_v)_v)=\prod_{v\in\mathrm{Val}(K)}\theta_{m,v}(t_v).$$

By the above, we deduce that $T(\mathbb{A}_K)^{\theta_m}\subset T(\mathbb{A}_K)^{\mathcal{U}_0,\phi_m}$, where

$$T(\mathbb{A}_K)^{\theta_m}=\{t\in T(\mathbb{A}_K):\theta_m(t)=1\}.$$

Then, by comparing limits along the real line, we see that it suffices to prove that

$$\lim_{s\to\frac{1}{m}}\left(s-\frac{1}{m}\right)^{b(d,m)}\widehat{H}(\theta_m,1;-s)\neq 0.$$

It is easily seen that for any non-archimedean place $v\notin S'(\boldsymbol{\omega})$, we have

$$\widehat{H}_v(\theta_{m,v},1;-s)=1+\frac{a_{1,v,m}}{q_v^{ms}}$$

for $a_{\chi,v,n}$ as in Definition 5.5.12, so, as in Corollary 5.5.14, we may deduce that

$$\widehat{H}(\theta_m,1;-s)=\zeta_E(ms)^{b(d,m)}G_m(s)$$

for $G_m(s)$ a function holomorphic on $\mathrm{Re}\,s\geq\frac{1}{m}$. It also follows that $G_m\big(\frac{1}{m}\big)\neq 0$, since $\widehat{H}_v\big(\theta_{m,v},1;-\frac{1}{m}\big)\neq 0$ analogously to Lemma 5.5.2. Then the result follows. $\square$

Corollary 5.5.22. *We have*

$$\Omega_m\left(\frac{1}{m}\right) = \frac{\operatorname{Res}_{s=1}\zeta_K(s)}{d\operatorname{Res}_{s=1}\zeta_E(s)} \lim_{s\to\frac{1}{m}}\left(s-\frac{1}{m}\right)^{b(d,m)} \sum_{\chi\in\mathcal{U}_0} \widehat{H}(\phi_m,\chi;-s) \neq 0.$$

Proof of Theorem 5.1.1. Since $\Omega_m\left(\frac{1}{m}\right) \neq 0$, the result for $S = S(\boldsymbol{\omega})$ follows from [4, Thm. 3.3.2, p. 624] and Theorem 5.5.17, taking $c(\boldsymbol{\omega}, m, S(\boldsymbol{\omega}))$ to be

$$\frac{m\operatorname{Res}_{s=1}\zeta_K(s)}{(b(d,m)-1)!\,d\operatorname{Res}_{s=1}\zeta_E(s)} \lim_{s\to\frac{1}{m}}\left(s-\frac{1}{m}\right)^{b(d,m)} \sum_{\chi\in\mathcal{U}_0} \widehat{H}(\phi_m,\chi;-s).$$

The result for $S \supset S(\boldsymbol{\omega})$ follows analogously upon redefining $\phi_{m,v}$ to be identically 1 for each $v \in S \setminus S(\boldsymbol{\omega})$ in Definition 5.4.10. $\square$

5.6 Campana points

In this section we prove Theorem 5.1.4. We will be brief when the argument is largely similar to the case of weak Campana points, emphasising only the key differences. Fix a Galois extension E/K of number fields with K-basis $\boldsymbol{\omega} = \{\omega_0, \dots, \omega_{d-1}\}$, let $m \in \mathbb{Z}_{\geq 2}$ and set $T = T_{\boldsymbol{\omega}}$ as in Section 5.3.

Definition 5.6.1. For each non-archimedean place $v \notin S(\boldsymbol{\omega})$, let

$$N_{\boldsymbol{\omega}}(x) = \prod_{w|v} f_w(x)$$

denote the v-adic decomposition of the norm form $N_{\boldsymbol{\omega}}$ associated to $\boldsymbol{\omega}$ into irreducible polynomials $f_w(x) \in \mathcal{O}_v[x]$. For each $w \mid v$, we define the functions

$$\widetilde{H}_w : T(K_v) \to \mathbb{R}_{>0}, \quad x \mapsto \frac{\max\{|x_i|_v^{\deg f_w}\}}{|f_w(x)|_v},$$

$$\psi_{m,w} : T(K_v) \to \{0,1\}, \quad t_v \mapsto \begin{cases} 1 \text{ if } \widetilde{H}_w(t_v) = 1 \text{ or } \widetilde{H}_w(t_v) \geq q_v^m, \\ 0 \text{ otherwise.} \end{cases}$$

We then define the *Campana local indicator function*

$$\psi_{m,v} : T(K_v) \to \{0,1\}, \quad t_v \mapsto \prod_{w|v} \psi_{m,w}(t_v).$$

Setting $\psi_{m,v} = 1$ for $v \in S(\boldsymbol{\omega})$, we then define the *Campana indicator function*

$$\psi_m : T(\mathbb{A}_K) \to \{0,1\}, \quad (t_v)_v \mapsto \prod_{v\in\mathrm{Val}(K)} \psi_{m,v}(t_v).$$

If $v \notin S'(\boldsymbol{\omega})$, then for each $w \mid v$, we also define the function

$$\sigma_{m,w} : T(K_v) \to \{0,1\}, \quad t_v \mapsto \begin{cases} 1 \text{ if } \widetilde{H}_w(t_v) = 1 \text{ or } \widetilde{H}_w(t_v) = q_v^m, \\ 0 \text{ otherwise} \end{cases}$$

and we define

$$\sigma_{m,v} : T(K_v) \to \{0,1\}, \quad t_v \mapsto \prod_{w|v} \sigma_{m,w}(t_v).$$

Letting $\sigma_{m,v}$ be the indicator function for $\mathcal{K}_v$ for $v \in S'(\boldsymbol{\omega})$, we define the function

$$\sigma_m : T(\mathbb{A}_K) \to \{0,1\}, \quad (t_v)_v \mapsto \prod_{v \in \mathrm{Val}(K)} \sigma_{m,v}(t_v).$$

Lemma 5.6.2. *The Campana $\mathcal{O}_{K,S(\boldsymbol{\omega})}$-points of $\left(\mathbb{P}^{d-1}_K, \Delta^{\boldsymbol{\omega}}_m\right)$ are precisely the rational points $t \in T(K)$ such that $\psi_m(t) = 1$.*

Proof. Taking coordinates $t_0, \ldots, t_{d-1}$ as in the proof of Lemma 5.4.12, we have

$$\widetilde{H}_w(t) = \frac{1}{|f_w(t_0, \ldots, t_{d-1})|_v} = q_v^{v(f_w(t_0,\ldots,t_{d-1}))} = q_v^{n_{\alpha_w}(\mathcal{Z}(f_w),t)}$$

for all non-archimedean places $v \notin S(\boldsymbol{\omega})$ and places $w \mid v$, where $\mathcal{Z}(f_w)$ denotes the Zariski closure of $Z(f_w)$ in $\mathbb{P}^{d-1}_{\mathcal{O}_{K,S(\boldsymbol{\omega})}}$. □

Lemma 5.6.3. *For all $v \in \mathrm{Val}(K)$, the function $\psi_{m,v}$ is $\mathcal{K}_v$-invariant and 1 on $\mathcal{K}_v$.*

Proof. For $v \in S(\boldsymbol{\omega})$ the result is trivial, so let $v \notin S(\boldsymbol{\omega})$ and $w \mid v$. Since $f_w(x \cdot y) = f_w(x) f_w(y)$ for $x, y \in L$, it follows as in the proof of Lemma 5.4.13 that $\widetilde{H}_w(x \cdot y) \leq \widetilde{H}_w(x)\widetilde{H}_w(y)$ for all $x, y \in T(K_v)$. Since $H'_v = \prod_{w|v} \widetilde{H}_w$, we have $\widetilde{H}_w(\mathcal{K}_v) = 1$ for all $w \mid v$, hence it follows as in the proof of Corollary 5.4.15 that $\widetilde{H}_w$ and $\psi_{m,w}$ are $\mathcal{K}_v$-invariant. Since $\psi_{m,v}(1) = 1$, we deduce that $\psi_{m,v}(\mathcal{K}_v) = 1$. □

Proposition 5.6.4. *Given $v \notin S'(\boldsymbol{\omega})$ and $t_v \in T(K_v)$, the image of t_v in $X_*(T_v)$ is*

$$\sum_{w|v} \frac{\log_{q_v}\left(\widetilde{H}_w(t_v)\right)}{\deg f_w} n_w,$$

with n_w defined as in Proposition 5.3.4.

Proof. Follows from Proposition 5.3.4. □

Corollary 5.6.5. *For $v \notin S'(\boldsymbol{\omega})$ non-archimedean with q_v sufficiently large, χ an automorphic character of T unramified at v and $s \in \mathbb{C}$ with $\mathrm{Re}\, s > 0$, we have*

$$\widehat{H}_v(\psi_{m,v}, \chi_v; -s) = 1 + \frac{1}{q_v^{ms}} \sum_{\substack{w|v \\ \deg f_w | m}} \chi_w(\pi_w)^m + O\left(\frac{1}{q_v^{(m+1)s}}\right).$$

Proof. Since χ_v, H_v and $\psi_{m,v}$ are $T(\mathcal{O}_v)$-invariant and $v \notin S'(\boldsymbol{\omega})$, we have

$$\widehat{H}_v(\psi_{m,v}, \chi_v; -s) = \int_{T(K_v)} \frac{\psi_{m,v}(t_v)\chi_v(t_v)}{H_v(t_v)^s} d\mu_v = \sum_{\overline{t_v} \in T(K_v)/T(\mathcal{O}_v)} \frac{\psi_{m,v}(\overline{t_v})\chi_v(\overline{t_v})}{H_v(\overline{t_v})^s}$$

$$= \sum_{n_v \in X_*(T_v)} \frac{\psi_{m,v}(n_v)\chi_v(n_v)}{e^{\varphi_\Sigma(n_v) s \log q_v}} = \sum_{r=0}^{\infty} \frac{\gamma_{\chi,v,r}}{q_v^{rs}},$$

where

$$\gamma_{\chi,v,r} = \sum_{\substack{n_v \in X_*(T_v) \\ H_v(n_v) = q_v^r}} \psi_{m,v}(n_v)\chi_v(n_v).$$

Put $d_w = \deg f_w$ and let $n_v = \sum_{w|v} \alpha_w n_w \in X_*(T_v)$ with $\min_w\{\alpha_w\} = 0$. By Proposition 5.3.4 and Note 5.3.1, we have

$$\log_{q_v} H_v(n_v) = \sum_{w|v} d_w \alpha_w, \quad \chi_v(n_v) = \prod_{w|v} \chi_w(\pi_w)^{d_w \alpha_w},$$

$$\psi_{m,v}(n_v) = \begin{cases} 1 \text{ if } \alpha_w = 0 \text{ or } \alpha_w \geq \frac{m}{d_w} \text{ for all } w \mid v, \\ 0 \text{ otherwise.} \end{cases}$$

In particular, $\psi_{m,v}(n_v) = 0$ whenever $q_v \leq H_v(n_v) \leq q_v^{m-1}$, hence $\gamma_{\chi,v,r} = 0$ for $1 \leq r \leq m-1$. Further, we see that $\psi_{m,v}(n_v) = 1$ and $H_v(n_v) = q_v^m$ if and only if there is exactly one place $w_0 \mid v$ such that $\alpha_{w_0} = \frac{m}{d_{w_0}}$ and $\alpha_w = 0$ for $w \neq w_0$, so

$$\gamma_{\chi,v,m} = \sum_{\substack{w|v \\ \deg f_w | m}} \chi_w(\pi_w)^m.$$

Since $|\psi_{m,v}(n_v)\chi_v(n_v)| \leq 1$, we deduce that

$$|\gamma_{\chi,v,r}| \leq \#\left\{\beta_1, \dots, \beta_d \in \mathbb{Z}_{\geq 0} : \sum_{i=1}^{d} \beta_i = r\right\} \leq d^r.$$

Analogously to the proof of Corollary 5.5.14, we deduce for q_v sufficiently large that

$$\sum_{r=m+1}^{\infty} \frac{\gamma_{\chi,v,r}}{q_v^{rs}} = O\left(\frac{1}{q_v^{(m+1)s}}\right). \qquad \square$$

Proposition 5.6.6. *For all places $v \notin S'(\boldsymbol{\omega})$ with q_v sufficiently large, we have*

$$\widehat{H}_v(\psi_{m,v}, \chi_v; -s) = L_v(\chi^m, ms)\left(1 + O\left(1 + \frac{1}{q_v^{(m+1)s}}\right)\right), \quad \operatorname{Re} s > 0.$$

Proof. Let $v \notin S'(\boldsymbol{\omega})$ with q_v sufficiently large as in Corollary 5.6.5. If v is totally split in E/K, then $\deg f_w = 1$ for all $w \mid v$, so Corollary 5.6.5 gives

$$\widehat{H}_v(\psi_{m,v}, \chi_v; -s) = 1 + \frac{1}{q_v^{ms}} \sum_{w|v} \chi_w^m(\pi_w) + O\left(\frac{1}{q_v^{(m+1)s}}\right),$$

and so we deduce the equality, since

$$L_v(\chi^m, ms) = \prod_{w|v}\left(1 - \frac{\chi_w^m(\pi_w)}{q_v^{ms}}\right)^{-1} = 1 + \frac{1}{q_v^{ms}} \sum_{w|v} \chi_w^m(\pi_w) + O\left(\frac{1}{q_v^{(m+1)s}}\right).$$

Now let v have inertia degree $d_v > 1$ in E/K. Then $\deg f_w = d_v \mid d$ for all $w \mid v$. If d and m are coprime, then $d_v \nmid m$, hence $\gamma_{\chi,v,m} = 0$ and the result follows from

$$L_v(\chi^m, ms) = \prod_{w|v}\left(1 - \frac{\chi_w^m(\pi_w)}{q_v^{d_v ms}}\right)^{-1} = 1 + O\left(\frac{1}{q_v^{d_v ms}}\right).$$

If d is prime, then v is inert, so $T(\mathcal{O}_v) = T(K_v)$, $\widehat{H}_v(\psi_{m,v}, \chi_v; -s) = 1$, and

$$L_v(\chi^m, ms) = 1 - \frac{1}{q_v^{dms}} = 1 + O\left(\frac{1}{q_v^{(m+1)s}}\right). \qquad \square$$

Proposition 5.6.7. *For any $\chi \in \mathcal{U}$, we have*

$$\widehat{H}(\psi_m, \chi; -s) = \prod_{v \mid \infty} \widehat{H}_v(1, \chi_v; -s) L(\chi^m, ms) \widetilde{G}_{m,\chi}(s),$$

where $\widetilde{G}_{m,\chi}(s)$ is a function which is holomorphic on $\operatorname{Re} s \geq \frac{1}{m}$, $\widetilde{G}_{m,1}(\frac{1}{m}) \neq 0$ and $\widehat{H}(\psi_m, 1; -s)$ has a simple pole at $s = \frac{1}{m}$.

Proof. Defining $\widetilde{G}_{m,\chi,v}(s) = \frac{\widehat{H}_v(\psi_{m,v}, \chi_v; -s)}{L_v(\chi^m, ms)}$ for each place $v \nmid \infty$, it follows as in the proof of Corollary 5.5.14 that $\widetilde{G}_{m,\chi,v}(s)$ is holomorphic and bounded uniformly in terms of ϵ and v on $\operatorname{Re} s \geq \epsilon$ for all $\epsilon > 0$. Since Proposition 5.6.6 gives

$$\widetilde{G}_{m,\chi,v}(s) = 1 + O\left(\frac{1}{q_v^{(m+1)s}}\right),$$

it follows as in the proof of Corollary 5.5.14 that $\widetilde{G}_{m,\chi}(s)$ is holomorphic and uniformly bounded with respect to χ for $\operatorname{Re} s \geq \frac{1}{m}$ with $\widetilde{G}_{m,1}(\frac{1}{m}) \neq 0$. Then, since

$$L(1, ms) = \zeta_E(ms),$$

we conclude from Theorem 5.2.22 that $\widehat{H}(\psi_m, 1; -s)$ has a simple pole at $s = \frac{1}{m}$. $\square$

Definition 5.6.8. For $\operatorname{Re} s \gg 0$, define the functions

$$\widetilde{Z}_m : \mathbb{C} \to \mathbb{C}, \quad s \mapsto \sum_{x \in \mathbb{P}^{d-1}(K)} \frac{\psi_m(x)}{H(x)^s}, \quad \widetilde{\Omega}_m = \widetilde{Z}_m(s)\left(s - \frac{1}{m}\right).$$

The proofs of the following two results are analogous to those of their weak Campana counterparts, namely Theorem 5.5.17 and Lemma 5.5.20 respectively.

Theorem 5.6.9. *The function $\widetilde{\Omega}_m(s)$ admits a holomorphic extension to $\operatorname{Re} s \geq \frac{1}{m}$.*

Lemma 5.6.10. *The characters $\chi \in \mathcal{U}$ contributing to the simple pole of $\widetilde{Z}_m(s)$ at $s = \frac{1}{m}$ are precisely the characters $\chi \in \mathcal{U}[m]$ such that $\widetilde{G}_{m,\chi}(\frac{1}{m}) \neq 0$.*

Proposition 5.6.11. *The limit*

$$\lim_{s \to \frac{1}{m}} \left(s - \frac{1}{m}\right) \sum_{\chi \in \mathcal{U}[m]} \widehat{H}(\psi_m, \chi; -s)$$

is non-zero.

Proof. By the same reasoning as in the proof of Proposition 5.5.21, we have

$$\sum_{\chi \in \mathcal{U}[m]} \widehat{H}(\psi_m, \chi; -s) = \sum_{\chi \in \mathcal{U}[m]} \int_{T(\mathbb{A}_K)} \frac{\psi_m(t)\chi(t)}{H(t)^s} d\mu = |\mathcal{U}[m]| \int_{T(\mathbb{A}_K)^{\mathcal{U}[m],\psi_m}} \frac{1}{H(t)^s} d\mu,$$

where

$$T(\mathbb{A}_K)^{\mathcal{U}[m],\psi_m} = \{t \in T(\mathbb{A}_K) : \psi_m(t) = \chi(t) = 1 \text{ for all } \chi \in \mathcal{U}[m]\}.$$

Now, take $\chi \in \mathcal{U}[m]$, $v \notin S'(\boldsymbol{\omega})$ non-archimedean. If $\sigma_v(t_v) = 1$ for some $t_v \in T(K_v)$, then the image of t_v in $X_*(T_v)$ is of the form $\sum_{w|v} \alpha_w n_w$, where each α_w is either 0 or $\frac{m}{d_v}$ for d_v the common inertia degree of the places of E over v, so

$$\chi_v(t_v) = \prod_{w|v} \chi_w(\pi_w)^{d_v \alpha_w} = 1,$$

since each $d_v\alpha_w$ is 0 or m and $\chi^0 = \chi^m = 1$. Then $\chi_v(t_v) = 1$ for all $\chi \in \mathcal{U}[m]$. In particular, we deduce that $T(\mathbb{A}_K)^{\sigma_m} \subset T(\mathbb{A}_K)^{\mathcal{U}[m],\psi_m}$, where

$$T(\mathbb{A}_K)^{\sigma_m} = \{t \in T(\mathbb{A}_K) : \sigma_m(t) = 1\}.$$

Then it suffices to prove that

$$\lim_{s \to \frac{1}{m}} \left(s - \frac{1}{m}\right) \widehat{H}(\sigma_m, 1; -s) \neq 0.$$

Analogously to the proof of Proposition 5.6.7, we may deduce that

$$\widehat{H}(\sigma_m, 1; -s) = \zeta_E(ms)\widetilde{G}_m(s)$$

for $\widetilde{G}_m(s)$ a function holomorphic on $\operatorname{Re} s \geq \frac{1}{m}$ with $\widetilde{G}_m\left(\frac{1}{m}\right) \neq 0$, so the result follows. □

Proof of Theorem 5.1.4. Since $\widetilde{\Omega}_m\left(\frac{1}{m}\right) \neq 0$ by Proposition 5.6.11, the result for $S = S(\boldsymbol{\omega})$ now follows from [4, Thm. 3.3.2, p. 624] and Theorem 5.6.9, taking

$$\widetilde{c}(\boldsymbol{\omega}, m, S(\boldsymbol{\omega})) = \frac{m \operatorname{Res}_{s=1} \zeta_K(s)}{d \operatorname{Res}_{s=1} \zeta_E(s)} \lim_{s \to \frac{1}{m}} \left(s - \frac{1}{m}\right) \sum_{\chi \in \mathcal{U}[m]} \widehat{H}(\psi_m, \chi; -s).$$

The result for $S \supset S(\boldsymbol{\omega})$ follows analogously upon redefining $\psi_{m,v}$ to be identically 1 for each $v \in S \setminus S(\boldsymbol{\omega})$. □

5.7 Comparison to Manin-type conjecture

In this section we compare the leading constant in Theorem 5.1.4 with the Manin–Peyre constant in the conjecture of Pieropan, Smeets, Tanimoto and Várilly-Alvarado.

5.7.1 Statement of the conjecture

Let (X, D_ϵ) be a smooth Campana orbifold over a number field K which is *klt* (i.e. $\epsilon_\alpha < 1$ for all $\alpha \in \mathcal{A}$) and *(log) Fano* (i.e. $-(K_X + D_\epsilon)$ is ample). Let $(\mathcal{X}, \mathcal{D}_\epsilon)$ be a regular $\mathcal{O}_{K,S}$-model of (X, D_ϵ) for some finite set $S \subset \mathrm{Val}(K)$ containing S_∞ (i.e. $\mathcal{X}$ regular over $\mathcal{O}_{K,S}$). Let $\mathcal{L} = (L, \|\cdot\|)$ be an adelically metrised big line bundle with associated height function $H_{\mathcal{L}} : X(K) \to \mathbb{R}_{>0}$ (see [63, §1.3]). For any subset $U \subset X(K)$ and any $B \in \mathbb{R}_{>0}$, we define

$$N(U, \mathcal{L}, B) = \#\{P \in U : H_{\mathcal{L}}(P) \le B\}.$$

We are now ready to give the statement of the conjecture.

Conjecture 5.7.1. *[66, Conj. 1.1, p. 3] Suppose that L is nef and $(\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{K,S})$ is not thin. Then there exists a thin set $Z \subset (\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{K,S})$ and explicit positive constants $a = a((X, D_\epsilon), L)$, $b = b(K, (X, D_\epsilon), L)$ and $c = c(K, S, (\mathcal{X}, \mathcal{D}_\epsilon), \mathcal{L}, Z)$ such that, as $B \to \infty$, we have*

$$N((\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{K,S}) \setminus Z, \mathcal{L}, B) \sim cB^a (\log B)^{b-1}.$$

5.7.2 Interpretation for norm orbifolds

The orbifold $\left(\mathbb{P}^{d-1}_K, \Delta^{\boldsymbol{\omega}}_m\right)$ in Theorem 5.1.4 is klt and Fano. It is smooth precisely when $d = 2$. The $\mathcal{O}_{K,S(\boldsymbol{\omega})}$-model $\left(\mathbb{P}^{d-1}_{\mathcal{O}_{K,S(\boldsymbol{\omega})}}, \mathcal{D}^{\boldsymbol{\omega}}_m\right)$ is regular. The Batyrev–Tschinkel height arises from an adelic metrisation $\mathcal{L}$ of $-K_{\mathbb{P}^{d-1}} = \mathcal{O}(d)$. According to [66, §3.3], we have

$$c\Big(K, S, \left(\mathbb{P}^{d-1}_{\mathcal{O}_{K,S(\boldsymbol{\omega})}}, \mathcal{D}^{\boldsymbol{\omega}}_m\right), \mathcal{L}, Z\Big) = \frac{1}{d}\tau\Big(K, S(\boldsymbol{\omega}), \left(\mathbb{P}^{d-1}_K, \Delta^{\boldsymbol{\omega}}_m\right), \mathcal{L}, Z\Big),$$

where

$$\tau\Big(K, S(\boldsymbol{\omega}), \left(\mathbb{P}^{d-1}_K, \Delta^{\boldsymbol{\omega}}_m\right), \mathcal{L}, Z\Big) = \int_{\overline{\mathbb{P}^{d-1}(K)}_\epsilon} H(t)^{1-\frac{1}{m}} d\tau_{\mathbb{P}^{d-1}}.$$

Here, $\tau_{\mathbb{P}^{d-1}}$ is the Tamagawa measure defined in [54, Def. 2.8, p. 372], and $\overline{\mathbb{P}^{d-1}(K)}_\epsilon$ denotes the topological closure of the Campana points inside $\mathbb{P}^{d-1}(\mathbb{A}_K)$. If one assumes that weak approximation for Campana points holds for this orbifold (see [66, Question 3.9, p. 13]), it follows from the definition of $\tau_{\mathbb{P}^{d-1}}$ (cf. [66, §9]) that

$$\tau = m\frac{\mathrm{Res}_{s=1}\zeta_K(s)}{\mathrm{Res}_{s=1}\zeta_E(s)} \lim_{s\to\frac{1}{m}} \left(s - \frac{1}{m}\right)\widehat{H}(\psi_m, 1; -s).$$

Given the assumption on weak approximation, the conjectural leading constant is

$$c\Big(K, S, \left(\mathbb{P}^{d-1}_{\mathcal{O}_{K,S(\boldsymbol{\omega})}}, \mathcal{D}^{\boldsymbol{\omega}}_m\right), \mathcal{L}, Z\Big) = \frac{m\,\mathrm{Res}_{s=1}\zeta_K(s)}{d\,\mathrm{Res}_{s=1}\zeta_E(s)} \lim_{s\to\frac{1}{m}} \left(s - \frac{1}{m}\right)\widehat{H}(\psi_m, 1; -s).$$

On the other hand, the leading constant given by Theorem 5.1.4 in this case is

$$\widetilde{c}(\boldsymbol{\omega}, m, S(\boldsymbol{\omega})) = \frac{m\,\mathrm{Res}_{s=1}\zeta_K(s)}{d\,\mathrm{Res}_{s=1}\zeta_E(s)} \lim_{s\to\frac{1}{m}} \left(s - \frac{1}{m}\right) \sum_{\chi\in\mathcal{U}[m]} \widehat{H}(\psi_m, \chi; -s).$$

We observe that our constant differs from the conjectural one in the potential inclusion of non-trivial characters in the limit.

5.7.3 The quadratic case

We now consider the case $d = 2$, in which the orbifold in Theorem 5.1.4 is smooth. We will show later on (Proposition 6.3.12) that in this setting, weak approximation for Campana points holds for any $m \in \mathbb{Z}_{\geq 2}$.

In Theorem 5.1.4, it is not clear that there are non-trivial characters contributing to the leading constant and whether their contribution is positive if so. However, we now exhibit an extension for which a non-trivial character contributes positively to the leading constant, and all contributing characters do so positively.

Proposition 5.7.2. *Let $K = \mathbb{Q}\big(\sqrt{-39}\big)$, $E = \mathbb{Q}\big(\sqrt{-3}, \sqrt{13}\big)$ and $m = 2$. Choose the K-basis $\boldsymbol{\omega} = \{1, \frac{1+\sqrt{-3}}{2}\}$ of E. Then $S(\boldsymbol{\omega}) = S_\infty$, $\#\mathcal{U}[2] > 1$, and for every $\chi \in \mathcal{U}[2]$, we have $\lim_{s\to\frac{1}{m}} \widehat{H}(\psi_m, \chi; -s) > 0$.*

Proof. Writing $a \cdot 1 + b \cdot \frac{1+\sqrt{-3}}{2}$ as (a, b) and $G = \mathrm{Gal}(E/K)$ as $\{1_G, g\}$, we have

$$(1, 0)^2 = (1, 0),\ (1, 0) \cdot (0, 1) = (0, 1),\ (0, 1)^2 = (-1, 1),$$

$$1_G((1, 0)) = (1, 0),\ 1_G((0, 1)) = (0, 1),\ g((1, 0)) = (1, 0),\ g((0, 1)) = (1, -1),$$

and clearly $1 = (1, 0)$, hence $S(\boldsymbol{\omega}) = S_\infty$. Note that $N_{\boldsymbol{\omega}}(x, y) = x^2 + xy + y^2$.

Since $\mathrm{Cl}(E) \cong \mathbb{Z}/2\mathbb{Z}$, the Hilbert class field M of E is quadratic over E. We obtain the unramified Hecke character χ_M of E, which is defined for all split $w \in \mathrm{Val}(E)$ by $\chi_{M,w}(\pi_w) = -1$ and is trivial at all other places. Since χ_M is trivial on $\mathbb{A}_K^*$, it may be viewed inside $\mathcal{U}[2]$, hence $\#\mathcal{U}[2] > 1$.

Let $\chi \in \mathcal{U}[2]$. To show that $\lim_{s\to\frac{1}{2}} \big(s - \frac{1}{2}\big)\widehat{H}(\psi_2, \chi; -s) > 0$, it suffices to show that $\widehat{H}_v\big(\psi_{2,v}, \chi_v; -\frac{1}{2}\big) > 0$ for all $v \in \mathrm{Val}(K)$. If $v \mid \infty$, then $\widehat{H}_v\big(\psi_{2,v}, \chi_v; -\frac{1}{2}\big) = \widehat{H}_v\big(1, 1; -\frac{1}{2}\big) > 0$, as χ_v gives a continuous homomorphism from $T(K_v)/T(\mathcal{O}_v) \cong \mathbb{R}_{>0}$ to μ_2, and $\mathbb{R}_{>0}$ has no proper open subgroups. If v is inert, then we have $\widehat{H}_v\big(\psi_{2,v}, \chi_v; -\frac{1}{2}\big) = 1$. If v is not inert, then $N_{\boldsymbol{\omega}}(x, y) = (x + \theta_1 y)(x + \theta_2 y)$ for $\theta_1, \theta_2 \in \mathcal{O}_v$ roots of $z^2 - z + 1$. By Proposition 5.6.4, we have $H_v = H_v'$ if and only if there are no $(a, b) \in (K_v^*)^2$ with $\min\{v(a), v(b)\} = 0$ such that $v(a + \theta_1 b), v(a + \theta_2 b) \geq 1$. If $v(a + \theta_1 b), v(a + \theta_2 b) \geq 1$, then we deduce from the equalities

$$\theta_1(a + \theta_2 b) - \theta_2(a + \theta_1 b) = (\theta_1 - \theta_2)a, \quad (a + \theta_1 b) - (a + \theta_2 b) = (\theta_1 - \theta_2)b,$$

that $v(a), v(b) \geq 1 - v(\theta_1 - \theta_2)$. Since $\min\{v(a), v(b)\} = 0$, we have $H_v' \neq H_v$ if and only if $v(\theta_1 - \theta_2) \geq 1$. Since $(\theta_1 - \theta_2)^2 = -3$, the only such place is the unique place v_0 of K above 3, and $v_0(\theta_1 - \theta_2) = 1$.

For any split place $v \neq v_0$, we have $\mathcal{K}_v = T(\mathcal{O}_v)$ and $\psi_{2,v} = \phi_{2,v}$, so

$$\widehat{H}_v\left(\psi_{2,v}, \chi_v; -\frac{1}{2}\right) = 1 + \sum_{n=2}^{\infty} \frac{c_{\chi,v,n} - c_{\chi,v,n-2}}{q_v^{\frac{n}{2}}}$$

by Proposition 5.5.7. In fact, for w_1 and w_2 the places of E over v, we have $\chi_{w_2}(\pi_{w_2}) = \chi_{w_1}(\pi_{w_1})^{-1} \in \{1, -1\}$, hence $c_{\chi,v,n} - c_{\chi,v,n-2} = 2\chi_{w_1}(\pi_{w_1})^n$, so

$$\widehat{H}_v\left(\psi_{2,v}, \chi_v; -\frac{1}{2}\right) = 1 + \sum_{n=2}^{\infty} \frac{2\chi_{w_1}(\pi_{w_1})^n}{q_v^{\frac{n}{2}}} = 1 + \frac{2}{q_v}\left(\frac{1}{1 - q_v^{-\frac{1}{2}}\chi_{w_1}(\pi_{w_1})}\right) > 0.$$

It only remains to check that $\widehat{H}_{v_0}\left(\psi_{2,v_0}, \chi_{v_0}; -\frac{1}{2}\right) > 0$. We will make use of the following property of valuations:

$$v_0(\alpha+\beta) \geq \min\{v_0(\alpha), v_0(\beta)\}, \text{ with equality if } v_0(\alpha) \neq v_0(\beta). \tag{5.7.1}$$

Assume that, for $a, b \in \left(K_{v_0}^*\right)^2$ as above, we have $v_0(a+\theta_2 b) \geq 2$. We claim that $v_0(a+\theta_1 b) = 1$. First, we deduce from (5.7.1) that

$$\begin{aligned} 2 \leq v_0(a+\theta_2 b) &= v_0((a+\theta_1 b) + (\theta_1 - \theta_2)b) \\ &\geq \min\{v_0(a+\theta_1 b), v_0((\theta_1-\theta_2)b)\}, \end{aligned}$$

with equality if $v_0(a+\theta_1 b) \neq v_0((\theta_1-\theta_2)b)$. Since $v_0\Big((2\theta_i - 1)^2\Big) = v_0(-3) = 2$, we have $v_0(2\theta_i - 1) = 1$, so $v_0(\theta_i) = v_0(2\theta_i) = v_0(1) = 0$ by (5.7.1), so $\min\{v_0(a), v_0(\theta_2 b)\} = \min\{v_0(a), v_0(b)\} = 0$. Then, since $v_0(a+\theta_2 b) \geq 2$, it follows that $v_0(a) = v_0(\theta_2 b)$. We deduce that $v_0(a) = v_0(\theta_2 b) = v_0(b)$, so $v_0(a) = v_0(b) = \min\{v_0(a), v_0(b)\} = 0$. Since $v_0(\theta_1 - \theta_2) = 1$, we have $v_0((\theta_1-\theta_2)b) = 1$. It follows that $v_0(a+\theta_1 b) = 1$ by (5.7.1).

We deduce that $\psi_{2,v_0}(t_{v_0}) = 1$ if and only if $t_{v_0} \in \mathcal{K}_{v_0}$, hence

$$\widehat{H}_{v_0}\left(\psi_{2,v_0}, \chi_{v_0}; -\frac{1}{2}\right) = \int_{\mathcal{K}_{v_0}} d\mu_{v_0} > 0;$$

positivity follows since $\mathcal{K}_v \subset T(\mathcal{O}_v)$ is of finite index for all $v \in \mathrm{Val}(K)$. □

5.7.4 Possible thin sets

Assuming the truth of Conjecture 5.7.1, the question arises of which thin set Z should be removed in the setting of Proposition 5.7.2. Informally, its removal should remove the contribution of all non-trivial characters $\chi \in \mathcal{U}[2]$. One might therefore postulate that, for each non-trivial character $\chi \in \mathcal{U}[2]$, there is a finite morphism $\varphi_\chi : C_\chi \to \mathbb{P}^1_K$, where C_χ is a smooth projective curve, and $Z = \bigcup_{\chi \in \mathcal{U}[2]} \varphi_\chi(C_\chi(K))$. By the height bounds in [78, §9.7], we would have $C_\chi \cong \mathbb{P}^1_K$ and $\deg(\varphi_\chi) = 2$, making the morphisms φ_χ degree-two endomorphisms of $\mathbb{P}^1_K$. However, it is not clear how one should construct such endomorphisms. This may be an interesting direction to pursue in future work. Indeed, since the completion of this work, the removal of thin sets in Conjecture 5.7.1 was explored in detail by Shute in [77].

Chapter 6

Weak approximation and the Hilbert property for Campana points

6.1 Introduction

Despite the arithmetic appeal of Campana points, questions regarding their geometric abundance and distribution remain largely unanswered outside of the case of curves ([15, 48], [81, Appendix]). In particular, the question of whether the Campana points of an orbifold are thin is open even for the most elementary orbifold structures on projective space. Thin sets play an important role in modern refinements of Manin's conjecture originally due to Peyre [64], in which the exceptional set is thin (see [52, Thm. 1.3] and [11, Thm. 1.1] for examples and [51, §5] for an overview). The exceptional set in the Manin-type conjecture for Campana points of Pieropan, Smeets, Tanimoto and Várilly-Alvarado [66, Conj. 1.1] is thin, and so the study of thin sets of Campana points is well-motivated.

In this chapter, we take a step in this direction by studying local and adelic Campana points and their relationship to their rational counterparts. In particular, we focus on *Campana weak weak approximation* (Definition 6.2.1). This property is interesting in its own right, since it tells us whether the Campana points of an orbifold are "equidistributed" in some sense. Further, it relates, as it does for rational points, to the Hilbert property. We show that Campana weak weak approximation implies that the set of Campana points is not thin, i.e. that the orbifold in question satisfies the *Campana Hilbert property* (Definition 6.2.3). We exploit this connection in order to verify the Campana Hilbert property for various orbifolds. We also generalise a version of the fibration method used in the study of the Hilbert property for surfaces in order to verify the Campana Hilbert property for del Pezzo surfaces with a rational line.

6.1.1 Results

Let (X, D) be a Campana orbifold over a number field K (in this chapter, we will suppress the subscript indicating the coefficients of D). Recall that D is a $\mathbb{Q}$-divisor on X which we write as

$$D = \sum_{\alpha \in \mathcal{A}} \epsilon_\alpha D_\alpha,$$

where the D_α are prime divisors and $\epsilon_\alpha = 1 - \frac{1}{m_\alpha}$ for some $m_\alpha \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$ (taking $\frac{1}{\infty} = 0$).

Let S be a finite subset of places of K containing all of the archimedean places. Let $\mathcal{X}$ be a flat proper $\mathcal{O}_S$-model for X and let $\mathcal{D}$ be the closure of D in $\mathcal{X}$. We denote the set of Campana points of $(\mathcal{X}, \mathcal{D})$ by $(\mathcal{X}, \mathcal{D})(\mathcal{O}_S)$. This is a subset of $X(K)$, and one can ask whether it is thin. If not, we say that $(\mathcal{X}, \mathcal{D})$ satisfies the Campana Hilbert property.

Recall that a variety Y over K satisfies *weak weak approximation* (WWA) if there exists a finite subset of places S such that $Y(K)$ is dense in $\prod_{v \notin S} Y(K_v)$, and that it was shown by Colliot-Thélène and Ekedahl that weak weak approximation implies the Hilbert property [75, Thm. 3.5.7]. Our first result is that the corresponding statement holds for Campana points.

Theorem 6.1.1. *Let (X, D) be a Campana orbifold over a number field K, let S be a finite subset of places of K containing all archimedean places and let $(\mathcal{X}, \mathcal{D})$ be an $\mathcal{O}_S$-model of (X, D). If $(\mathcal{X}, \mathcal{D})$ satisfies Campana weak weak approximation and $(\mathcal{X}, \mathcal{D})(\mathcal{O}_S) \neq \emptyset$, then it has the Campana Hilbert property.*

For the definition of Campana weak weak approximation, see Definition 6.2.1. With this relation in mind, we devote the rest of the chapter to Campana weak weak approximation and the stronger property of Campana weak approximation.

We first turn to the case when $X = \mathbb{P}^n_K$ for some number field K, $\mathcal{A} = \{0, \ldots, r\}$ and the D_i are hyperplanes. We say that an orbifold (X, D) is log Fano when $-(K_X + D)$ is ample. Then the orbifold $(\mathbb{P}^n, D)$ is log Fano precisely when

$$\sum_{i=0}^{r} \left(1 - \frac{1}{m_i}\right) < n + 1. \tag{6.1.1}$$

A variant of the following question was asked in [66, Question 3.8].

Question 6.1.2. Suppose that $(\mathbb{P}^n, D)$ is log Fano. Given a finite set of places $S \subset \mathrm{Val}(K)$ containing all archimedean places, is the set of Campana points $(\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{D})(\mathcal{O}_S)$ non-thin?

Asymptotics for Campana points of bounded height on such orbifolds over $\mathbb{Q}$ were studied by Browning and Yamagishi [13]. They considered the case where $r = n + 1$ (the first non-trivial case for counting) and D_i is the coordinate hyperplane $x_i = 0$ for $0 \leq i \leq n$, while D_{n+1} is given by $\sum_i x_i = 0$. Using their asymptotics, they showed that a certain collection of thin sets are not enough to cover $(\mathbb{P}^n_{\mathbb{Z}}, \mathcal{D})(\mathbb{Z})$ [13, Thm. 1.3] for n large relative to the m_i. In other words, the number of Campana points grows faster than those in these thin sets.

Our approach to answer Question 6.1.2 is to study the distribution of Campana points through Campana weak weak approximation. We give an affirmative answer when $r \leq n$ and $H_0, \ldots, H_r$ are in general linear position (Definition 6.1.8) provided that there exist local Campana points over each place $v \notin S$. Note that in this case, (6.1.1) is satisfied for all choices of m_i. In fact, we prove the stronger result that *Campana weak approximation* (Definition 6.2.1) holds. We also prove a partial result when $r = n+1$, conditional on the conjecture of Colliot-Thélène that the Brauer–Manin obstruction is the only obstruction to weak approximation for geometrically rationally connected varieties (Conjecture 2.8.9), which needs only to be assumed for certain Fano diagonal hypersurfaces.

Theorem 6.1.3. *Let $(\mathbb{P}^n, D)$ be log Fano as above, and suppose that $H_0, \ldots, H_r$ are in general linear position.*

(i) If $r \leq n$, then $\left(\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{D}\right)$ satisfies Campana weak approximation.

(ii) Let $r = n+1$. Assume that all m_i are equal to some integer m. If weak approximation with Brauer–Manin obstruction holds for smooth diagonal hypersurfaces of degree m in $\mathbb{P}^{n+1}$, then $\left(\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{D}\right)$ satisfies Campana weak weak approximation. Further, if $(m, n) \neq (3, 2)$, then $\left(\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{D}\right)$ satisfies Campana weak approximation.

Using the fact that weak approximation with Brauer–Manin obstruction holds for quadrics [38, Thm. 2.2.1] and a result of Skinner on weak approximation for complete intersections [78, Thm.], one may remove the assumption of weak approximation with Brauer–Manin obstruction in the second part of Theorem 6.1.3 when $m = 2$ and for $n \geq (m-1)2^m - 2$ when $m \geq 3$. However, Skinner's exponential bound can be improved upon for diagonal hypersurfaces. Over a general number field K, results of Wooley [94, Thm. 15.6] and of Birch [6, Thm. 3] collectively establish weak approximation for diagonal hypersurfaces of degree m in $\mathbb{P}^{n+1}$ for $n \geq \min\{m(m+1) - 1, 2^m - 1\}$, which improves upon Skinner's result for all $m \geq 3$. This bound can be improved to $n \geq \min\{m(m+1) - 2, 2^m - 1\}$ for $K = \mathbb{Q}$ using [13, Thm. 1.4], and one can make a further small improvement using methods from [92] and [93]. We therefore deduce the following result.

Corollary 6.1.4. *Let $(\mathbb{P}^n, D)$ be as in the second part of Theorem 6.1.3. Campana weak approximation for $\left(\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{D}\right)$ holds in the following cases.*

(i) $m = 2$.

(ii) $n \geq \min\{m(m+1) - 1, 2^m - 1\}$.

(iii) $K = \mathbb{Q}$ *and* $n \geq \min\{m(m+1) - 2, 2^m - 1\}$.

Note 6.1.5. One may view Corollary 6.1.4 as a generalisation of [13, Thm. 1.3] in the case where all of the m_i are equal. Indeed, *loc. cit.* shows that certain thin sets are not enough to cover the Campana points when $n \geq m(m+1) - 2$, whereas our results show that no collection of thin sets can do this.

Next we develop a fibration result for Campana points, based on the fibration result of Bary-Soroker, Fehm and Petersen (Theorem 3.3.1). As an application, we obtain the following result for del Pezzo surfaces.

Theorem 6.1.6. *Let X be a del Pezzo surface of degree d over a number field K, let $L \subset X$ be a rational line on X, and let $m \geq 2$ be an integer. Put $D = \left(1 - \frac{1}{m}\right)L$. Let $(\mathcal{X}, \mathcal{D})$ be an $\mathcal{O}_S$-model of (X, D) for some finite set of places S of K containing all archimedean places. Suppose that one of the following holds.*

(a) $d = 4$ *and X has a conic fibration.*

(b) $d = 3$.

(c) $d = 2$ *and X has a conic fibration.*

Then $(\mathcal{X}, \mathcal{D})$ *has the Campana Hilbert property.*

The analogous result for rational points is Theorem 3.1.4. To prove Theorem 6.1.6, we borrow ideas from Chapter 3 by using the induced double conic bundle structure. We first establish a lemma relating non-thin sets of Campana points and blowing up. This allows us to reduce, as in *ibid.*, to case (c), with extra difficulty coming from the fact that we must consider Campana points on $\mathbb{P}^1$ with a degree-2 point as the orbifold divisor. Hence, the proof relies on the application of a crucial result (Proposition 6.3.12) establishing Campana Hilbert property for such orbifolds in Section 6.3.

6.1.2 Conventions

Definition 6.1.7. Let X be a del Pezzo surface. A *line* on X is an irreducible curve $E \subset X$ such that $E^2 = E \cdot K_X = -1$.

Definition 6.1.8. Given a hyperplane $H : \sum_{i=0}^{n} a_i x_i = 0$ in $\mathbb{P}^n$, we call the point $[a_0, \ldots, a_n]$ the *normal* to H. We say that the hyperplanes $H_0, \ldots, H_r$ are in *general linear position* if their normals are in general linear position, i.e. no k of them lie in a $(k-2)$-dimensional linear subspace for $k = 2, 3, \ldots, n+1$.

6.2 CWWA and CHP

In this section we define Campana weak approximation and the Campana Hilbert property, and we prove Theorem 6.1.1. Let (X, D) be a Campana orbifold over a number field K, and let S be a finite set of places of K containing S_∞. Recall the definitions given in Section 5.2.1 of Campana points on the local, adelic and rational levels.

Definition 6.2.1. We say that the model $(\mathcal{X}, \mathcal{D})$ satisfies *Campana weak weak approximation (CWWA)* if there exists a finite set T of places of K such that $(\mathcal{X}, \mathcal{D})(\mathcal{O}_S)$ is dense in $(\mathcal{X}, \mathcal{D})\big(\mathbb{A}_K^T\big)$. We say that $(\mathcal{X}, \mathcal{D})$ satisfies *Campana weak approximation (CWA)* if $(\mathcal{X}, \mathcal{D})(\mathcal{O}_S)$ is dense in $(\mathcal{X}, \mathcal{D})(\mathbb{A}_K)$.

Observe that CWA implies CWWA (taking $T = \emptyset$), and note that one recovers the definitions of weak approximation and weak weak approximation on taking D to be the zero divisor. The following lemma tells us that CWA is independent of the choice of model, provided that it is preserved upon enlarging S.

Lemma 6.2.2. *Let K be a number field and let S, T be finite subsets of places of K where S contains S_∞. Let (X, D) be a Campana orbifold over K with $\mathcal{O}_S$-models $(\mathcal{X}, \mathcal{D})$ and $(\mathcal{Y}, \mathcal{E})$. Suppose that for all finite subsets $S \subset S' \subset \mathrm{Val}(K)$, the set $(\mathcal{X}_{S'}, \mathcal{D}_{S'})(\mathcal{O}_{S'})$ is dense in $(\mathcal{X}_{S'}, \mathcal{D}_{S'})(\mathbb{A}_K^T)$. Then $(\mathcal{Y}, \mathcal{E})(\mathcal{O}_S)$ is dense in $(\mathcal{Y}, \mathcal{E})(\mathbb{A}_K^T)$.*

Proof. By spreading out (see [67, §3.2]), there exists a finite set of places S' containing S such that

$$(\mathcal{X}_{S'}, \mathcal{D}_{S'}) \simeq (\mathcal{Y}_{S'}, \mathcal{E}_{S'})$$

over $\operatorname{Spec} \mathcal{O}_{S'}$. The hypothesis on $(\mathcal{X}, \mathcal{D})$ and the above isomorphism imply that $(\mathcal{Y}_{S'}, \mathcal{E}_{S'})(\mathcal{O}_{S'})$ is dense in $(\mathcal{Y}_{S'}, \mathcal{E}_{S'})(\mathbb{A}_K^T)$. Since the local Campana points are open at each place (see Note 5.2.7), $(\mathcal{Y}, \mathcal{E})(\mathbb{A}_K^T)$ is open inside $(\mathcal{Y}_{S'}, \mathcal{E}_{S'})(\mathbb{A}_K^T)$. Hence, we see that $(\mathcal{Y}, \mathcal{E})(\mathcal{O}_S) = (\mathcal{Y}_{S'}, \mathcal{E}_{S'})(\mathcal{O}_{S'}) \cap (\mathcal{Y}, \mathcal{E})(\mathbb{A}_K^T)$ is dense in $(\mathcal{Y}, \mathcal{E})(\mathbb{A}_K^T)$. ∎

6.2.1 Campana Hilbert property

In this section we introduce the Hilbert property in the context of Campana points and we prove Theorem 6.1.1. As mentioned in the introduction, the motivation for studying thin sets of Campana points is twofold: to understand from a geometric perspective their ubiquity, and to better understand the exceptional set in [66, Conj. 1.1]. An assumption forced by this conjecture (see [66, §3.4]) is that the set of Campana points itself (on the orbifold of interest) is not thin. However, little was known prior to the completion of this work regarding thin sets of Campana points and integral points aside from the result of Browning–Yamagishi [13, Thm. 1.3] mentioned earlier and work of Coccia on integral points on affine cubic surfaces [20]. As mentioned at the end of Chapter 5, recent work of Shute [77] explores thin sets of Campana points in the context of Conjecture 5.7.1.

Recall that, given a variety V over a field F of characteristic zero, we say that V satisfies the *Hilbert property* if $V(F)$ is not thin. Motivated by this, we define the following analogue for Campana points. Let (X, D) be a Campana orbifold over a number field K with $\mathcal{O}_S$-model $(\mathcal{X}, \mathcal{D})$ for some finite set of places S of K containing S_∞.

Definition 6.2.3. We say that (X, D) satisfies the *Campana Hilbert property (CHP)* if the set $(\mathcal{X}, \mathcal{D})(\mathcal{O}_S) \subset X(K)$ is not thin.

6.2.2 Proof of Theorem 6.1.1

We now prove that CWWA implies CHP, provided that the set of Campana points is non-empty. The strategy is very similar to Serre's original proof for rational points [75, Thm. 3.5.3].

Proof of Theorem 6.1.1. We prove the following statement: if $A \subset (\mathcal{X}, \mathcal{D})(\mathcal{O}_S)$ is thin and $T_0 \subset \mathrm{Val}(K)$ is finite, then there is another finite subset $T \subset \mathrm{Val}(K)$ disjoint with T_0 such that A is not dense in $\prod_{v \in T}(\mathcal{X}, \mathcal{D})(\mathcal{O}_v)$. Note that, if this statement is true for the thin subsets A_1 and A_2, then it is also true for $A_1 \cup A_2$ (cf. [75, Proof of Thm. 3.5.3]). Then it suffices to prove that the statement is true when A is type I and when A is type II. Denote by Y the proper closed subset of singular points of X.

Let $A \subset W(K) \cap (\mathcal{X}, \mathcal{D})(\mathcal{O}_S)$ for $W \subset X$ a proper closed subset of X. Since $W(K_v) \subset X(K_v)$ is closed, it suffices to find $v \in \mathrm{Val}(K) \setminus T_0$ such that $(\mathcal{X}, \mathcal{D})(\mathcal{O}_v) \setminus W(K_v) \neq \emptyset$. Taking q_v sufficiently large and applying the Lang–Weil estimate [75, Thm. 3.6.1] (which we may do as X is normal with a rational point, hence geometrically irreducible) along with generic smoothness of $X \setminus Y$ shows that there exists a smooth point $Q \in \mathcal{X}_v(\mathbb{F}_v) \setminus (\mathcal{D}_v(\mathbb{F}_v) \cup \mathcal{W}_v(\mathbb{F}_v) \cup \mathcal{Y}_v(\mathbb{F}_v))$ lifting to some $P \in (\mathcal{X} \setminus \mathcal{D})(\mathcal{O}_v) \setminus W(K_v) \subset (\mathcal{X}, \mathcal{D})(\mathcal{O}_v) \setminus W(K_v)$.

Now let $A \subset \pi(W(K)) \cap (\mathcal{X}, \mathcal{D})(\mathcal{O}_S)$ for W a normal K-variety and $\pi \colon W \to X$ a finite morphism of degree ≥ 2. Since π is finite, we deduce that $\pi(W(K_v)) \subset X(K_v)$ is closed, so it suffices to find $v \in \mathrm{Val}(K)$ such that $(\mathcal{X}, \mathcal{D})(\mathcal{O}_v) \setminus \pi(W(K_v)) \neq \emptyset$. By combining [75, Thm. 3.6.2] and another application of the Lang–Weil estimate along with generic smoothness of $X \setminus Y$, there exists a smooth point $Q \in \mathcal{X}_v(\mathbb{F}_v) \setminus (\mathcal{D}_v(\mathbb{F}_v) \cup \pi(\mathcal{W}_v(\mathbb{F}_v)) \cup \mathcal{Y}_v(\mathbb{F}_v))$ for all v totally split in the algebraic closure of K in the Galois closure $K(W)^{\mathrm{gal}}/K(V)$ with q_v sufficiently large. Taking such v and Q, we may lift Q to a point $P \in (\mathcal{X} \setminus \mathcal{D})(\mathcal{O}_v) \setminus \pi(W(K_v)) \subset (\mathcal{X}, \mathcal{D})(\mathcal{O}_v) \setminus \pi(W(K_v))$. □

Remark 6.2.4. The above proof actually gives a slightly stronger result: given a thin set $A \subset X(K)$ and a finite subset $T_0 \subset \mathrm{Val}(K)$, there exists a finite set of places $T \subset \mathrm{Val}(K)$ disjoint with T_0 such that the image of A in $\prod_{v\in T}(\mathcal{X} \setminus \mathcal{D})(\mathcal{O}_v)$ is not dense. In particular, in order to verify that a Campana orbifold has CHP, it suffices to show that, for all finite sets of places $T_0 \supset S$, the closure of $(\mathcal{X}, \mathcal{D})(\mathcal{O}_S)$ in $\prod_{v\notin T_0}(\mathcal{X}, \mathcal{D})(\mathcal{O}_v)$ contains $\prod_{v\notin T_0}(\mathcal{X} \setminus \mathcal{D})(\mathcal{O}_v)$.

6.3 Projective space

Let K be a number field, and let $S \subset \mathrm{Val}(K)$ be a finite set containing S_∞. In this section we consider approximation properties for Campana $\mathcal{O}_S$-points on orbifolds $(\mathbb{P}^n, D)$, where

$$D = \sum_{i=0}^{r}\left(1 - \frac{1}{m_i}\right)D_i$$

for $D_0, \ldots, D_r$ hyperplanes in general linear position and $\{m_0, \ldots, m_r\} \subset \mathbb{Z}_{\geq 2}$.

6.3.1 Reduction to coordinate hyperplanes

Assume $r \leq n + 1$. Define the hyperplanes

$$H_i : x_i = 0 \quad \text{for } 0 \leq i \leq n, \quad H_{n+1}\colon \sum_{i=0}^{n} x_i = 0,$$

and define the $\mathbb{Q}$-divisor

$$H = \sum_{i=0}^{r}\left(1 - \frac{1}{m_i}\right)H_i. \tag{6.3.1}$$

Let $D_0, \ldots, D_r \subset \mathbb{P}^n$ denote arbitrary hyperplanes in general linear position. By applying a projective transformation, we obtain an isomorphism

$$f\colon (\mathbb{P}^n, D) \xrightarrow{\sim} (\mathbb{P}^n, H).$$

Denote by $\mathcal{H}$ the closure of H in $\mathbb{P}^n_{\mathcal{O}_S}$. By Lemma 6.2.2, if $(\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{H})$ satisfies CWA (or more generally $(\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{H})(\mathcal{O}_S) \subset (\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{H})(\mathbb{A}_K^T)$ is dense) for arbitrary S, then $(\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{D})$ also satisfies CWA (resp. $(\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{D})(\mathcal{O}_S) \subset (\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{D})(\mathbb{A}_K^T)$ is dense). Hence, when proving CWA or CWWA for arbitrary S, we can reduce to the case $D = H$. In this case, we have the following concrete description for the set of Campana points. Write $P = [x_0, \ldots, x_n]$ and set $x_{n+1} = \sum x_i$. For any $v \notin S$,

$$(\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{H})(\mathcal{O}_v) = \{P \in \mathbb{P}^n(K_v) : v(x_i) - \min_{0\leq j\leq r}\{v(x_j)\} \in \mathbb{Z}_{\geq m_i} \cup \{0, \infty\}, 0 \leq i \leq r\}, \tag{6.3.2}$$

and

$$(\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{H})(\mathcal{O}_S) = \mathbb{P}^n(K) \cap \left(\prod_{v\notin S}(\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{D})(\mathcal{O}_v) \times \prod_{v\in S}\mathbb{P}^n(K_v)\right). \tag{6.3.3}$$

6.3.2 Local solubility

Before investigating CWA, let us consider the question of local solubility for Campana points on $(\mathbb{P}^n, D)$, i.e. whether $(\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{D})(\mathbb{A}_K)$ is non-empty. If there are no adelic Campana points, then CWA holds trivially, but CHP clearly fails. In fact, this question is more delicate than one might expect. Let us give two illustrative examples.

Example 6.3.1. Suppose $D = H$, with H as in (6.3.1). Then it is clear that $[1, \dots, 1] \in \mathbb{P}^n(K)$ is a Campana $\mathcal{O}_K$-point according to (6.3.2) and (6.3.3).

Example 6.3.2. Consider the hyperplanes $D_i \subset \mathbb{P}^5$ defined by $f_i = 0$, where

$$\begin{aligned} f_0 &= x_0 + 2x_1 + 4g_0, & f_1 &= 5x_0 + 4x_1 + 4g_1, \\ f_2 &= 2x_0 + x_1 + 4g_2, & f_3 &= 4x_0 + 5x_1 + 4g_3, \\ f_4 &= x_0 + x_1 + 4g_4, & f_5 &= x_0 + 3x_1 + 4g_5, \end{aligned}$$

and $g_i \in \mathbb{Z}[x_2, x_3, x_4, x_5]$ are linear forms chosen so that the f_i are linearly independent. Set $m_i = m \geq 2$ for each $i = 0, \dots, r$. Since there is no $\mathbb{Q}_2$-point lying on every D_i, we may choose m large enough so that $f_0 = \cdots = f_5 = 0$ has no solution in $\mathbb{P}^5(\mathbb{Z}/2^m\mathbb{Z})$. Choosing such m, we will show that $\big(\mathbb{P}^5_{\mathbb{Z}}, \mathcal{D}\big)(\mathbb{Z}_2) = \emptyset$.

Suppose that $P = [a_0, \dots, a_5] \in \big(\mathbb{P}^5_{\mathbb{Z}}, \mathcal{D}\big)(\mathbb{Z}_2)$, with $a_0, \dots, a_5$ coprime 2-adic integers. Note that modulo 2, each hyperplane D_i reduces to one of the following:

$$x_0 = 0, \quad x_1 = 0, \quad x_0 + x_1 = 0.$$

Clearly any point in $\mathbb{P}^5_{\mathbb{F}_2}$ must lie in one of the three hyperplanes above. Let P_2 denote the reduction of P modulo 2. Suppose that P_2 lies on $x_0 = 0$, i.e. that $2 \mid a_0$. According to the equations, this means $n_2(\mathcal{D}_0, P), n_2(\mathcal{D}_1, P) > 0$. Since $P \in \big(\mathbb{P}^5_{\mathbb{Z}}, \mathcal{D}\big)(\mathbb{Z}_2)$, we must then have $n_2(\mathcal{D}_0, P), n_2(\mathcal{D}_1, P) \geq m \geq 2$. Since $f_1 - f_0 \equiv 2x_1 \pmod 4$, this means that $2 \mid a_1$ as well, i.e. P_2 lies on $x_1 = 0$, so $n_2(\mathcal{D}_2, P), n_2(\mathcal{D}_3, P) > 0$. Since P_2 lies on $x_0 = 0$ and $x_1 = 0$, it also lies on $x_0 + x_1$, hence $n_2(\mathcal{D}_4, P), n_2(\mathcal{D}_5, P) > 0$. We conclude that $n_2(\mathcal{D}_i, P) > 0$ for all i, hence we must have $n_2(\mathcal{D}_i, P) \geq m$ for all i. However, since no point in $\mathbb{P}^5(\mathbb{Z}_2)$ lies on all hyperplanes D_i modulo 2^m, this is impossible. We conclude that P is not a Campana point. The same argument works if P_2 were to lie on either $x_1 = 0$ or $x_0 + x_1 = 0$, and so we conclude that $\big(\mathbb{P}^5_{\mathbb{Z}}, \mathcal{D}\big)(\mathbb{Z}_2) = \emptyset$.

Example 6.3.2 shows how things can go wrong when q_v is small compared to n. We now give some sufficient conditions for the existence of a v-adic Campana point.

Corollary 6.3.3. *Let $D_0, \dots, D_r \subset \mathbb{P}^n$ be hyperplanes in general linear position. Then $(\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{D})(\mathcal{O}_v) \neq \emptyset$ if one of the following conditions are satisfied:*

1. *$r < n$;*
2. *The reductions $(\mathcal{D}_0)_v, \dots, (\mathcal{D}_r)_v$ are linearly independent;*
3. *$q_v \geq n$.*

Proof. If $r < n$, then choose any $P \in D_0 \cap \cdots \cap D_r(K_v)$. Then $P \in (\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{D})(\mathcal{O}_v)$ by definition. For the rest of the proof, we assume that $r = n$.

If $(\mathcal{D}_0)_v, \dots, (\mathcal{D}_n)_v$ are linearly independent, then choose a point $P \in D_0 \cap \cdots \cap D_{n-1}(K_v)$. By assumption, the reduction of P modulo v does not lie on $(\mathcal{D}_n)_v$. Hence $n_v(D_i, P) = \infty$ if $i \leq n-1$ and 0 if $i = n$. It follows that $P \in (\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{D})(\mathcal{O}_v)$.

Now assume $q_v \geq n$. If $(\mathcal{D}_0)_v = (\mathcal{D}_i)_v$ for some $i \geq 1$, then

$$|\mathbb{P}^n(\mathbb{F}_v)| = \frac{q_v^{n+1}-1}{q_v-1} > n\frac{q_v^n-1}{q_v-1} > |(\mathcal{D}_1 \cup \cdots \cup \mathcal{D}_n)(\mathbb{F}_v)| = |(\mathcal{D}_0 \cup \cdots \cup \mathcal{D}_n)(\mathbb{F}_v)|,$$

hence there exists $P_v \in \mathbb{P}^n(\mathbb{F}_v) \setminus \bigcup_{i=0}^n \mathcal{D}_i(\mathbb{F}_v)$. Lifting to $P \in \mathbb{P}^n(K_v)$ via Hensel's lemma gives a local Campana point. Otherwise, we have

$$|\mathcal{D}_0(\mathbb{F}_v)| = \frac{q_v^n-1}{q_v-1} > n\frac{q_v^{n-1}-1}{q_v-1} \geq |(\mathcal{D}_0 \cap (\mathcal{D}_1 \cup \cdots \cup \mathcal{D}_n))(\mathbb{F}_v)|,$$

hence there exists $P_v \in \mathcal{D}_0(\mathbb{F}_v) \setminus \bigcup_{i=1}^n \mathcal{D}_i(\mathbb{F}_v)$. Using Hensel's lemma, choose a lift $P \in D_0(K_v)$. Then $n_v(D_i, P) = \infty$ for $i = 0$ and 0 for all $i > 0$. It follows that $P \in (\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{D})(\mathcal{O}_v)$. □

Remark 6.3.4. It would be of interest to determine easily checkable necessary conditions for the existence of local Campana points at a place.

6.3.3 Independent hyperplanes

Let us now investigate CWA in the case $r \leq n$.

Proposition 6.3.5. *Let $D_0, \ldots, D_r \subset \mathbb{P}^n$ be hyperplanes in general linear position. Then the set of Campana points $(\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{D})(\mathcal{O}_S)$ satisfies CWA.*

Proof. By Section 6.3.1, we may reduce to the case $D = H$, with H as defined in (6.3.1). Let $T \subset \mathrm{Val}(K)$ be a finite set of places. Write $T = T_f \cup T_\infty$ where $T_\infty = T \cap S_\infty$ and $T_f = T \setminus T_\infty$. For each $v \in T$, let $Q_v = [y_{v,0}, \ldots, y_{v,n}] \in (\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{H})(\mathcal{O}_v)$. Our goal is to find $P = [x_0, \ldots, x_n] \in \left(\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{H}\right)(\mathcal{O}_S)$ simultaneously approximating each Q_v. Without loss of generality, we assume that $T_f \neq \emptyset$. Further, we may assume without loss of generality that $y_{v,i} \neq 0$ for all $v \in T_f$ and $i \in \{0, \ldots, n\}$, since $\left(\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{H}\right)(\mathcal{O}_v) \setminus H(K_v)$ is dense in $\left(\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{H}\right)(\mathcal{O}_v)$ for all $v \in T_f$, so we may choose the coordinates $y_{v,i}$ such that $\min_i\{v(y_{v,i})\} = 0$. By iteratively applying strong approximation [18, II.15], for each $v \in T_f$, we may choose a uniformiser $\pi_v \in \mathcal{O}_S$ for $\mathcal{O}_v$ such that $\min(u(\pi_v), u(\pi_w)) = 0$ for all $v, w \in T_f$ and all places $u \notin S_\infty$. For each $v \in T_f$, write $y_{v,i} = u_{v,i}\pi_v^{e_{v,i}}$, where $u_{v,i} \in \mathcal{O}_v^\times$. Note that $e_{v,i} \geq m_i$ or $e_{v,i} = 0$ if $v \in T_f \setminus S$ and $0 \leq i \leq r$, since $Q_v \in (\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{H})(\mathcal{O}_v)$. Let N be a large positive integer. For each $v \in T_f$, set $d_v = |(\mathcal{O}_v/\pi_v^N)^\times|$ and $d = 1 + \prod_{v \in T_f} d_v$. By again iteratively applying strong approximation, there exist $\alpha_0, \ldots, \alpha_n \in \mathcal{O}_S$ such that for any i, j, we have

$$v\left(\alpha_i - u_{v,i}\prod_{w \in T_f, w \neq v} \pi_w^{-e_{w,i}}\right) \geq N, \quad v \in T_f,$$

$$\left|\alpha_i^d \prod_{w \in T_f} \pi_w^{e_{w,i}} - y_{v,i}\right|_v \leq 1/N, \quad v \in T_\infty,$$

and

$$v(\alpha_i)v(\pi_w) = v(\alpha_i)v(\alpha_j) = 0, \quad v \notin T \cup S_\infty, w \in T_f.$$

For each $i \in \{0, \dots, n\}$, define the S-integer

$$x_i = \alpha_i^d \prod_{v \in T_f} \pi_v^{e_{v,i}}.$$

Then, for each $v \in T_f$, we have

$$\begin{aligned} v(x_i - y_{v,i}) &\geq \min\left\{ v\left(\left(\alpha_i^d - \alpha_i\right) \prod_{w \in T_f} \pi_w^{e_{w,i}}\right), v\left(\alpha_i \prod_{w \in T_f} \pi_w^{e_{w,i}} - u_{v,i}\pi_v^{e_{v,i}}\right)\right\} \\ &\geq \min\left\{ v\left(\alpha_i^d - \alpha_i\right) + e_{v,i}, e_{v,i} + N\right\} = e_{v,i} + N. \end{aligned} \tag{6.3.4}$$

Hence, by choosing N large enough, x_i simultaneously approximates each $y_{v,i}$ arbitrarily well, so $P = [x_0, \dots, x_n]$ approximates each Q_v. To show that P is a Campana point, it suffices by (6.3.2) to show that, for all $v \notin S$, we have

$$v(x_i) - \min_{0 \leq i \leq r} \{v(x_j)\} \in \mathbb{Z}_{\geq m_i} \cup \{0\}. \tag{6.3.5}$$

For $v \in T \setminus S$, this follows from (6.3.4) since $Q_v \in (\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{H})(\mathcal{O}_v)$. Now let $v \notin S \cup T$. If $v(\pi_w) = 0$ for each $w \in T_f$, then $v(x_i) = dv(\alpha_i)$ for all i, so (6.3.5) follows since $d \geq \max_{0 \leq i \leq r}\{m_i\}$ for N sufficiently large. If $v(\pi_w) = a > 0$, then $v(\pi_{w'}) = 0$ for any $w' \in T_f, w' \neq w$, and $v(\alpha_i) = 0$ for all i, hence $v(x_i) = ae_{w,i}$ for all i and (6.3.5) is satisfied. □

6.3.4 An extra hyperplane

We now consider the case $r = n + 1$. We restrict to the case where the orbifold coefficients m_i are all equal to some m. Consider the family of diagonal hypersurfaces of degree m in $\mathbb{P}^{n+1}$,

$$\mathfrak{X}\colon a_0 x_0^m + \dots + a_{n+1} x_{n+1}^m = 0 \subset \mathbb{P}^{n+1} \times \mathbb{P}^{n+1}, \tag{6.3.6}$$

together with its projection $\pi\colon \mathfrak{X} \to \mathbb{P}^{n+1}_{a_0, \dots, a_{n+1}}$ to the first factor. For $P \in \mathbb{P}^{n+1}(K)$, let $\mathfrak{X}_P$ denote the fibre $\pi^{-1}(P)$. For a finite set $T \subset \mathrm{Val}(K)$, define

$$V_T = \left\{P \in \pi(\mathfrak{X}(K)) : \mathfrak{X}_P(K) \text{ is dense in } \mathfrak{X}_P(\mathbb{A}_K^T)\right\}.$$

Lastly, define the rational map

$$\rho\colon \mathfrak{X} \dashrightarrow \mathbb{P}^n, \quad ([a_0, \dots, a_{n+1}], [x_0, \dots, x_{n+1}]) \mapsto [a_0 x_0^m, \dots, a_n x_n^m].$$

Lemma 6.3.6. *Let X be a smooth hypersurface in $\mathbb{P}^{n+1}_K$ defined by*

$$a_0 x_0^m + \dots + a_{n+1} x_{n+1}^m = 0.$$

Let $v \in \mathrm{Val}(K)$ and assume that $X(K_v) \neq \emptyset$.

(i) If $v(m) = 0$, then there exists $P \in X(K_v)$ such that $\rho(P) \in (\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{H})(\mathcal{O}_v)$.

(ii) If $v(a_i) = v(a_j)$ for all i, j, then $\rho(X(K_v)) \subset (\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{H})(\mathcal{O}_v)$.

Proof. Let $P = [x_0, \ldots, x_{n+1}] \in X(K_v)$. Note that $a_i \neq 0$ for all i since X is smooth. Let $e = \min_i\{v(a_i x_i^m)\}$ and set

$$e_i = v(a_i x_i^m) - e,$$

taking $e_i = \infty$ if $x_i = 0$. Let $E = \{i : e_i = 0\}$. Consider the equation

$$\sum_{i \in E} \frac{a_i x_i^m}{\pi^e} y_i^m \equiv 0 \pmod{v}, \tag{6.3.7}$$

where $\pi \in \mathcal{O}_v$ is a uniformiser. There is the obvious solution $y_i \equiv 1 \pmod{v}$. For $i \notin E$, set $x_i' = \pi^{e+|v(a_i)|+1}$. Now consider the equation

$$\sum_{i \in E} \frac{a_i x_i^m}{\pi^e} y_i^m + \sum_{i \notin E} \frac{a_i x_i'^m}{\pi^e} = 0. \tag{6.3.8}$$

The reduction modulo v is precisely (6.3.7), and the solution $y_i \equiv 1 \pmod{v}$ can be lifted (as $v(m) = 0$) to a solution of (6.3.8). Choose such a solution $\{y_i\}$ to (6.3.8) and let $x_i' = x_i y_i$ for $i \in E$. Observe that $P' = [x_0', \ldots, x_{n+1}'] \in X(K_v)$. Set

$$e_i' = v(a_i x_i'^m) - \min_j\{v(a_j x_j'^m)\}.$$

We have $e_i' = 0$ for $i \in E$ and $e_i' \geq m$ for $i \notin E$, hence $\rho(P') \in (\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{H})(\mathcal{O}_v)$.

If $v(a_i) = v(a_j)$ for all i, j, then $m \mid e_i$ for each i, hence $\rho(P) \in (\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{H})(\mathcal{O}_v)$. □

Proposition 6.3.7. *Suppose that the m_i are all equal to some $m \geq 2$. If V_T is dense in $\pi(\mathfrak{X}(\mathbb{A}_K))$, then $(\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{H})(\mathcal{O}_S)$ is dense in $(\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{H})(\mathbb{A}_K^T)$.*

Proof. Let T' be any finite set of places of K disjoint from T. For each $v \in T'$, let $Q_v \in (\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{H})(\mathcal{O}_v)$ if $v \notin S$ and $Q_v \in \mathbb{P}^n(K_v)$ otherwise. It is easily seen that the restriction of ρ to the locus of points on smooth hypersurfaces is surjective onto $\mathbb{P}^n$, so we may choose points $P_v \in \mathfrak{X}(K_v)$ such that $\rho(P_v) = Q_v$ and $\mathfrak{X}_{\pi(P_v)}$ is smooth. We will construct a point $M \in \mathfrak{X}(K)$ which is v-adically close to P_v for each $v \in T'$ and $\rho(M) \in (\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{H})(\mathcal{O}_S)$.

Let $R_v = \pi(P_v) \in \mathbb{P}^{n+1}(K_v)$. Using the implicit function theorem, let $U_v \subset \mathbb{P}^{n+1}(K_v)$ be a small open neighborhood around R_v with a section $\sigma_v \colon U_v \to \mathfrak{X}(K_v)$ to π with $\sigma_v(R_v) = P_v$. Set $R_0 = [-1, 1, \ldots, 1] \in \pi(\mathfrak{X}(K))$. As V_T is dense in $\pi(\mathfrak{X}(\mathbb{A}_K))$, there is a point $R = [a_0, \ldots, a_{n+1}] \in V_T$, which

1. lies in U_v for each $v \in T'$, and
2. approximates R_0 in $\mathbb{P}^{n+1}(K_v)$ for each $v \in T$ and each $v \notin T'$ dividing m.

We now construct a point $\{M_v\} \in \mathfrak{X}_R(\mathbb{A}_K)$ as follows:

(a) For each $v \in T'$, let $M_v = \sigma_v(R)$.

(b) For each $v \notin T' \cup S$, choose any point $M_v \in \mathfrak{X}_R(K_v)$ such that $\rho(M_v) \in (\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{H})(\mathcal{O}_v)$ using Lemma 6.3.6 and (2).

(c) For each $v \in S \setminus T'$ choose any point $M_v \in \mathfrak{X}_R(K_v)$.

Let $E \subset \mathrm{Val}(K)$ be the finite subset of valuations v such that $v(a_i) \neq v(a_j)$ for some i, j. Note that by (2), we have $E \cap T = \emptyset$.

Since $\mathfrak{X}_R(K)$ is dense in $\mathfrak{X}_R(\mathbb{A}_K^T)$, there exists $M \in \mathfrak{X}_R(K)$ approximating M_v in $\mathfrak{X}_R(K_v)$ for each $v \in T' \cup E$. Note that, for $v \in T'$, the point M is v-adically close to P_v, hence $\rho(M)$ is v-adically close to Q_v. Indeed, M is v-adically close to $M_v = \sigma_v(R)$, which lies near P_v as the open neighbourhood U_v is small. To see that $\rho(M) \in (\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{H})(\mathcal{O}_S)$, first note that $\rho(M) \in (\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{H})(\mathcal{O}_v)$ for $v \in (T' \cup E) \setminus S$, since M approximates M_v and $\rho(M_v) \in (\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{H})(\mathcal{O}_v)$, since all points sufficiently close to Q_v are local Campana points. For any other $v \notin S$, we have $v(a_i) = v(a_j)$ for all i, j, so we can apply Lemma 6.3.6. Finally, for all $v \in S \cap T'$, $\rho(M)$ approximates $\rho(M_v)$ which in turn approximates Q_v, hence $\rho(M) \in (\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{H})(\mathcal{O}_v)$, and for $v \in S \setminus T'$, there is no local Campana condition. □

Corollary 6.3.8. *Let $D_0, \dots, D_{n+1} \subset \mathbb{P}^n$ be hyperplanes in general linear position. Suppose m_i are all equal to some $m \geq 2$. If all smooth degree-m hypersurfaces in $\mathbb{P}^{n+1}$ satisfy weak approximation, then $\left(\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{D}\right)$ satisfies CWA.*

Proof. By Section 6.3.1, we can assume that $D = H$ as in (6.3.1). Let $\mathfrak{X}_0 \subset \mathfrak{X}$ denote the open locus consisting of points lying on smooth hypersurfaces. Since $\pi(\mathfrak{X}_0(K)) \subset V_\emptyset$ by assumption, it suffices to show that $\pi(\mathfrak{X}(K))$ is dense in $\pi(\mathfrak{X}(\mathbb{A}_K))$ by Proposition 6.3.7. To see this, note that the projection of $\mathfrak{X}$ to the second factor $\mathbb{P}^{n+1}_{x_0,\dots,x_{n+1}}$ in (6.3.6) realises it as a $\mathbb{P}^n$-bundle over $\mathbb{P}^{n+1}$. Hence, $\mathfrak{X}$ is rational and smooth, so it satisfies weak approximation. Now $\mathfrak{X}(K)$ being dense in $\mathfrak{X}(\mathbb{A}_K)$ implies $\pi(\mathfrak{X}(K))$ is dense in $\pi(\mathfrak{X}(\mathbb{A}_K))$. □

Remark 6.3.9. By (6.1.1), the log Fano condition for $(\mathbb{P}^n, D)$ is $m \leq n+1$, which is precisely the condition for the smooth fibres of π, which are hypersurfaces of degree m in $\mathbb{P}^{n+1}$, to be Fano. Taking $n = 2$, this becomes $m \leq 3$. When $m = 3$, then $\mathfrak{X}$ is the family of diagonal cubic surfaces. Work of Bright–Browning–Loughran [10, Thm. 1.6] implies that $V_\emptyset$ contains 0% of the points in $\mathbb{P}^3(K)$ when ordered by height. However, this does not imply that $V_\emptyset$ cannot be dense in $\mathbb{P}^3(\mathbb{A}_K)$.

We now show that CWA almost always holds if we assume Colliot-Thélène's conjecture that the Brauer–Manin obstruction is the only one to weak approximation for geometrically rationally connected varieties (Conjecture 2.8.9).

Corollary 6.3.10. *Let $D_0, \dots, D_{n+1} \subset \mathbb{P}^n$ be hyperplanes in general linear position. Suppose m_i are all equal to some $m \geq 2$. Assume Conjecture 2.8.9 holds. Suppose that $(\mathbb{P}^n, D)$ is log Fano.*

(i) If $(n, m) = (2, 3)$, then for some place v_0, we have $(\mathbb{P}^2_{\mathcal{O}_S}, \mathcal{D})(\mathcal{O}_S)$ is dense in $(\mathbb{P}^2_{\mathcal{O}_S}, \mathcal{D})(\mathbb{A}_K^{\{v_0\}})$.

(ii) Otherwise, $(\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{D})(\mathcal{O}_S)$ satisfies CWA.

We first present a lemma which will be used in the proof. Given a variety X over a field k, we denote by $\mathrm{Br}_0(X)$ the subgroup $\mathrm{im}(\mathrm{Br}\, k \to \mathrm{Br}\, X)$ of $\mathrm{Br}\, X$, where $\mathrm{Br}\, k \to \mathrm{Br}\, X$ is the map arising from the structure morphism of X.

Lemma 6.3.11. *There exists a constant B such that the following statement is true: let X be any diagonal cubic surface*

$$a_0x_0^3 + a_1x_1^3 + a_2x_2^3 + a_3x_3^3 = 0$$

such that there exists a finite place v where $v(a_0) = 1$ and $v(a_i) = 0$ for all $i \neq 0$. Assume that $q_v > B$. Let $C \subset X$ be the curve given by $x_0 = 0$. Let $\mathcal{A} \in \mathrm{Br}(X)/\mathrm{Br}_0(X)$ be a class of order 3. Then the restricted evaluation map

$$\mathrm{ev}_{\mathcal{A}}|_C : C(K_v) \hookrightarrow X(K_v) \to \mathrm{Br}(K_v)[3]$$

is surjective.

Proof. We first show that, for q_v large enough, the evaluation map

$$\mathrm{ev}_{\mathcal{A}} : X(K_v) \to \mathrm{Br}(K_v)[3]$$

is surjective. Taking natural $\mathcal{O}_v$-models $\mathcal{X}$ and $\mathcal{C}$ for X and C respectively, note that $\mathcal{X}_v$ is a projective cone over the cubic curve $\mathcal{C}_v$. Then the result follows from [9, Thm. 6.5] and [23, §5]. The proof goes roughly as follows. Denote by $\mathcal{X}_v^{sm}$ the smooth locus of the special fibre of $\mathcal{X}$. Consider the natural map $\delta\colon \mathrm{Br}(X) \to \mathrm{H}^1_{\text{ét}}(\mathcal{X}_v^{sm}, \mathbb{Q}/\mathbb{Z})$ associating $\mathcal{A}$ to an étale cover of $\mathcal{X}_v^{sm}$. Since $\mathcal{X}_v$ is a cone over the curve $\mathcal{C}_v$, the image $\delta(\mathcal{A})$ has order 3 and thus corresponds to a degree-3 étale cover $Y \to \mathcal{X}_v^{sm}$ [9, Lem. 6.3]. Suppose $\beta \in \mathrm{Br}(K_v)[3]$ is such that the twisted cover Y^β has an $\mathbb{F}_v$-point mapping to a point $P_v \in \mathcal{X}_v(\mathbb{F}_v)$. Then for any lift $P \in \mathcal{X}(\mathcal{O}_v)$ of P_v, we have $\langle \mathcal{A}, P \rangle = \beta$ [9, Lem. 5.12]. By the Lang–Weil estimate, the existence of such P is ensured for q_v sufficiently large.

Now let $\mathcal{C}$ be the natural $\mathcal{O}_v$-model for C defined by the same equation. Since $\mathcal{X}_v^{sm}$ is isomorphic to $\mathcal{C}_v \times \mathbb{A}^1$, there exists a degree-3 étale cover $D \to \mathcal{C}_v$ such that $Y^\beta = D \times_{\mathcal{C}_v} \mathcal{X}_v^{sm}$. Together with the inclusion $\mathcal{C}_v \subset \mathcal{X}_v^{sm}$, we have the diagram

$$\begin{array}{ccccc} D & \longrightarrow & D \times \mathcal{X}_v^{sm} & \longrightarrow & D \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{C}_v & \longrightarrow & \mathcal{X}_v^{sm} & \longrightarrow & \mathcal{C}_v \end{array}$$

where the composition of the horizontal arrows are isomorphisms. If q_v is large enough, then D has an $\mathbb{F}_v$-point by the Hasse–Weil bound [67, Cor. 7.2.1] (note that D is a curve of genus one). Lifting the image of such a point in $\mathcal{C}_v$ to a point $P \in \mathcal{C}(\mathcal{O}_v)$, we conclude using the last paragraph that $\langle \mathcal{A}, P \rangle = \beta$. □

Proof of Corollary 6.3.10. By Section 6.3.1, we can assume that $D = H$ as in (6.3.1).

Let us first consider the case $n = 2$. Then $(\mathbb{P}^2, D)$ is log Fano if and only if $m \leq 3$. The result for $m = 2$ follows from Corollary 6.3.8, since weak approximation with Brauer–Manin obstruction holds for quadrics, so let $m = 3$. Fix a place v_0 such that $q_{v_0} > B$ for B as in Lemma 6.3.11. We show that $V_{\{v_0\}}$ is dense in $\pi(\mathfrak{X}(\mathbb{A}_K))$. Let T' be any finite set of places not including v_0. Define R_v, σ_v, U_v for $v \in T'$ and R_0 as in the proof of Proposition 6.3.7. As shown in the proof of Corollary 6.3.8, $\pi(\mathfrak{X}(K))$ is dense in $\pi(\mathfrak{X}(\mathbb{A}_K))$, so there exists $R = [a_0, \dots, a_{n+1}] \in \pi(\mathfrak{X}(K))$ that

1. lies in U_v for each $v \in T'$,

2. approximates R_0 in $\mathbb{P}^{n+1}(K_v)$ for each $v \notin T' \cup \{v_0\}$ dividing 3, and

3. satisfies $v_0(a_0) = 1$ and $v_0(a_i) = 0$ for all other $i > 0$.

The cubic surface $X := \mathfrak{X}_R$ given by

$$a_0x_0^3 + a_1x_1^3 + a_2x_2^3 + a_3x_3^3 = 0$$

contains a K-point since $R \in \pi(\mathfrak{X}(K))$. The above conditions imply $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ (see [23, §1]). Let $\mathcal{A} \in \mathrm{Br}(X)$ be a generator for this quotient. Let $C \subset X$ be the curve given by $x_0 = 0$.

By Lemma 6.3.11, the restricted evaluation map

$$\mathrm{ev}_{\mathcal{A}} \colon C(K_{v_0}) \hookrightarrow X(K_{v_0}) \to \mathbb{Z}/3\mathbb{Z}$$

is surjective. (The reason for restricting to C instead of X is to guarantee that we obtain a Campana point, as we will see later.) We now construct a point $\{M_v\} \in X(\mathbb{A}_K)$ as follows:

(a) For each $v \in T'$, let $M_v = \sigma_v(R)$.

(b) For each $v \notin T' \cup S \cup \{v_0\}$ choose any point $M_v \in X(K_v)$ such that $\rho(M_v) \in (\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{D})(\mathcal{O}_v)$ using Lemma 6.3.6.

(c) For each $v \in S \setminus T'$ choose any point $M_v \in X(K_v)$.

Then there exists a point $M_{v_0} = [0, b_1, \ldots, b_3] \in C(K_{v_0})$ such that

$$\sum_{v \neq v_0} \mathrm{inv}_v \, \mathcal{A}(M_v) + \mathrm{inv}_{v_0} \mathcal{A}(M_{v_0}) = 0.$$

The adelic point $\{M_v\} \in X(\mathbb{A}_K)$ is then orthogonal to $\mathrm{Br}(X)$. Since Conjecture 2.8.9 is assumed to hold, there exists $M \in X(K)$ approximating M_v for each (i) $v \in T'$ and (ii) v such that $v(a_i) \neq v(a_j)$ for some i, j. Similarly to the proof of Proposition 6.3.7, we have $\rho(M) \in (\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{D})(\mathcal{O}_S)$ using openness of Campana points and (b) above, together with Lemma 6.3.6 (note that $\rho(M_{v_0}) = [0, a_1b_1^3, a_2b_2^3, a_3b_3^3] \in (\mathbb{P}^3_{\mathcal{O}_S}, \mathcal{H})(\mathcal{O}_{v_0})$).

Suppose now that $n \geq 3$ and $(\mathbb{P}^n, D)$ is log Fano. Then for any $P \in \mathbb{P}^{n+1}(K)$ such that the fibre $\mathfrak{X}_P$ of π is smooth, we have $\mathrm{Br}(\mathfrak{X}_P) = \iota^* \mathrm{Br}(K)$, where $\iota \colon \mathfrak{X}_P \to \mathrm{Spec}\, k$ is the structure morphism (see [68, Prop. A.1]). Moreover, as $\mathfrak{X}_P$ is Fano, it is geometrically rationally connected. Hence Conjecture 2.8.9 together with $\mathrm{Br}(\mathfrak{X}_P) = \iota^* \mathrm{Br}(K)$ imply that $\mathfrak{X}_P(K)$ is dense in $\mathfrak{X}_P(\mathbb{A}_K)$. Thus, $\mathfrak{X}_P$ satisfies weak approximation. We conclude that the hypotheses of Proposition 6.3.7 are satisfied with $T = \emptyset$, and so $(\mathbb{P}^n_{\mathcal{O}_S}, \mathcal{D})$ satisfies CWA. □

6.3.5 Quadratic point on the projective line

We conclude this section with a case where the orbifold divisor is not a collection of hyperplanes. This will be used in Section 6.4 for applications to conic bundles.

Recall that, given a curve C over a number field K and a closed point $P \in C$, we define the *degree* of P to be $\deg(P) = [K(P) : K]$.

Proposition 6.3.12. *Let $P \in \mathbb{P}^1$ be a closed point of degree 2, and let $D = (1 - 1/m)P$ for some integer $m \geq 2$. Then $(\mathbb{P}^1_{\mathcal{O}_S}, \mathcal{D})(\mathcal{O}_S)$ satisfies CWA.*

Proof. Since S is arbitrary, we may assume by Lemma 6.2.2 (by applying a suitable automorphism of $\mathbb{P}^1$) that P is given by $x_0^2 - ax_1^2 = 0$, where $a \in \mathcal{O}_S$ is square-free. Note that, at any place $v \notin S$, the special fibre of $\mathcal{D}$ consists of two distinct $\mathbb{F}_v$-points of degree 1, an $\mathbb{F}_v$-point of degree 2 or a double $\mathbb{F}_v$-point. In the former two cases, we see that $[x_0, x_1]$ is a local Campana point if and only if

$$v\big(x_0^2 - ax_1^2\big) - \min_i\{v\big(x_i^2\big)\} \in \mathbb{Z}_{\geq m} \cup \{0, \infty\}, \tag{6.3.9}$$

while in the third case, the local Campana condition is

$$v\big(x_0^2 - ax_1^2\big) - \min_i\{v\big(x_i^2\big)\} \in \mathbb{Z}_{\geq 2m} \cup \{0, \infty\}. \tag{6.3.10}$$

Since m is arbitrary, it suffices to approximate any finite collection of local Campana points by a rational point satisfying (6.3.9) at all places $v \notin S$.

Let T be a finite set of places. Let $Q_v \in (\mathbb{P}^1_{\mathcal{O}_S}, \mathcal{D})(\mathcal{O}_v)$ for $v \in T\backslash S$ and $Q_v \in \mathbb{P}^1(K_v)$ for $v \in T \cap S$. Our goal is to find $[x_0, x_1] \in (\mathbb{P}^1_{\mathcal{O}_S}, \mathcal{D})(\mathcal{O}_S)$ approximating all Q_v simultaneously. By slightly perturbing the Q_v if need be, we may assume that they do not coincide with P.

Write $Q_v = [y_{v,0}, y_{v,1}]$ where $y_{v,i} \in \mathcal{O}_v$. Set

$$z_{v,0} = \frac{y_{v,0}}{y_{v,0}^2 - ay_{v,1}^2}, \quad z_{v,1} = \frac{-y_{v,1}}{y_{v,0}^2 - ay_{v,1}^2}.$$

Let $A = K[t_0, t_1, t_2, t_3]$ and $x_0, x_1 \in A$ be defined by

$$(t_0 + \sqrt{a}t_1)^m(t_2 + \sqrt{a}t_3)^{m+1} = x_0 + \sqrt{a}x_1.$$

Let $Z \subset \mathbb{A}^4 = \operatorname{Spec} A$ be defined by $x_0 = x_1 = 0$. We claim that Z is of codimension 2 in $\mathbb{A}^4$. Indeed, note that, if x_0 and x_1 have a common factor, then

$$(t_0 + \sqrt{a}t_1)^m(t_2 + \sqrt{a}t_3)^{m+1} \quad \text{and} \quad (t_0 - \sqrt{a}t_1)^m(t_2 - \sqrt{a}t_3)^{m+1}$$

must also have a common factor. This is the case if and only if

$$(t_0 + \sqrt{a}t_1)(t_2 + \sqrt{a}t_3) \quad \text{and} \quad (t_0 - \sqrt{a}t_1)(t_2 - \sqrt{a}t_3)$$

have a common factor, which is clearly false. By [89, Lem. 1.1], the open set $U = \mathbb{A}^4 \backslash Z$ satisfies strong approximation off of a place v_0, so the image

$$U(K) \to \prod_{w \in S \cup T \backslash \{v_0\}} U(K_w) \times \prod_{w \notin S \cup T \cup \{v_0\}} U(\mathcal{O}_w)$$

is dense. Choose $v_0 \notin T$ to be a non-archimedean place with $a \notin K_{v_0}^2$. Then there exists $R = (z_0', z_1', y_0', y_1') \in U(K)$ that approximates $z_{v,i}, y_{v,i}$ at every $v \in T$, and lies in $U(\mathcal{O}_w)$ for all non-archimedean places $w \notin S \cup T \cup \{v_0\}$. Let $x_i = x_i(R)$. The latter condition tells us $\min(w(x_0), w(x_1)) = 0$.

Since

$$(z_0' + \sqrt{a}z_1')(y_0' + \sqrt{a}y_1') = z_0'y_0' + az_1'y_1' + \sqrt{a}(z_0'y_1' + z_1'y_0')$$

and

$$z_{v,0}y_{v,0} + az_{v,1}y_{v,1} = 1, \quad z_{v,0}y_{v,1} + z_{v,1}y_{v,0} = 0,$$

we see that x_i approximates $y_{v,i}$ in K_v. Observe that

$$\begin{aligned} x_0^2 - ax_1^2 &= (x_0 + \sqrt{a}x_1)(x_0 - \sqrt{a}x_1) \\ &= (z_0' + \sqrt{a}z_1')^m(z_0' - \sqrt{a}z_1')^m(y_0' + \sqrt{a}y_1')^{m+1}(y_0' - \sqrt{a}y_1')^{m+1} \\ &= (z'^2_0 - az'^2_1)^m(y'^2_0 - ay'^2_1)^{m+1} \end{aligned}$$

Since $\min(w(x_0), w(x_1)) = 0$ for any place $w \notin S \cup T \cup \{v_0\}$, we have

$$w(x_0^2 - ax_1^2) - \min\{w(x_i^2)\} = \begin{cases} 0 \text{ or at least } m & \text{if } w \notin S \cup \{v_0\}, \\ 0 & \text{if } w = v_0. \end{cases}$$

The case for $w = v_0$ above follows from the fact that $\sqrt{a} \notin K_{v_0}$. Hence, the Campana conditions are satisfied and we have $[x_0, x_1] \in (\mathbb{P}^1_{\mathcal{O}_S}, \mathcal{D})(\mathcal{O}_S)$. □

Remark 6.3.13. Let $D = (1-1/m)P$ be as in Proposition 6.3.12. While Lemma 6.2.2 and Proposition 6.3.12 guarantee that any model $(\mathcal{X}, \mathcal{D})$ of $(\mathbb{P}^1, D)$ satisfies CWA, it is possible that $(\mathcal{X}, \mathcal{D})$ does not have Campana local points. To construct such an example, one can take a model where $\mathcal{X}_v(\mathbb{F}_v) = \mathcal{P}_v(\mathbb{F}_v)$. Then for any $R \in X(K_v)$, we have $n_v(\mathcal{P}, R) > 0$. However, if $\mathcal{P}(\mathcal{O}_v) = \emptyset$, then it is impossible for $n_v(\mathcal{P}, R)$ to be arbitrarily large by Hensel's lemma. We have the following concrete example.

Let $\mathcal{C} \subset \mathbb{P}^2_{\mathbb{Z}}$ be the conic given by $x^2 + y^2 - 4z^2 = 0$. Let $\mathcal{D} = (1 - \frac{1}{3})\mathcal{P}$ with $\mathcal{P} \subset \mathcal{C}$ given by $x^2 + y^2 = 0$. For any $Q = [x, y, z] \in \mathcal{C}(\mathbb{Z}_2)$ with x, y and z coprime 2-adic integers, it follows from modular arithmetic that we have

$$\min\{v_2(x), v_2(y)\} = v_2(z) + 1,$$

hence $\min\{v_2(x), v_2(y), v_2(z)\} = v_2(z) = 0$, and

$$n_2(\mathcal{P}, Q) = v_2\big(x^2 + y^2\big) = v_2\big(4z^2\big) = 2,$$

hence $(\mathcal{C}, \mathcal{D})(\mathbb{Z}_2) = \emptyset$.

6.4 Fibrations and blowups

6.4.1 CHP for fibrations

In this section we explore the links between fibrations and the Campana Hilbert property.

The following result is a slight variant of Theorem 3.3.1.

Theorem 6.4.1. *Let $f\colon X \to S$ be a dominant morphism of varieties over K. Let $B \subset S(K)$, $A \subset X(K)$. Suppose that the set $\{s \in B : f^{-1}(s)(K) \cap A \text{ is not thin}\}$ is not thin. Then A is not thin.*

Proof. The result follows upon replacing $X(K)$ by A and Σ by $\{s \in B : f^{-1}(s)(K) \cap A \text{ is not thin}\}$ in the proofs of [3, Thm. 1.1] and [3, Lem. 3.2]. □

Lemma 6.4.2. *Let (X, D) be a Campana orbifold over a number field K with associated $\mathcal{O}_S$-model $(\mathcal{X}, \mathcal{D})$ for some finite set of places S of K containing S_∞. Let Y be a proper normal K-variety with $\mathcal{O}_S$-model $\mathcal{Y}$. Let $\phi : X \to Y$ be a dominant morphism that* *extends to* $\Phi : \mathcal{X} \to \mathcal{Y}$*. Write* $D = \sum_{\alpha \in \mathcal{A}} \epsilon_\alpha D_\alpha$*, where each* D_α *is a prime divisor on* X*. Assume that each* $\mathcal{D}_\alpha$ *is flat over* $\mathcal{Y}$*.*

Let $P \in Y(K)$ *with* $Z = \phi^{-1}(P)$ *normal. Denote by* $\mathcal{Z}$ *the Zariski closure of* Z *in* $\mathcal{X}$*. Then* $Z \cap D$ *is an orbifold divisor on* Z*, the pair* $(\mathcal{Z}, \mathcal{Z} \cap \mathcal{D})$ *is an* $\mathcal{O}_S$*-model for* $(Z, Z \cap D)$*, and we have*

$$(\mathcal{Z}, \mathcal{Z} \cap \mathcal{D})(\mathcal{O}_S) = (\mathcal{X}, \mathcal{D})(\mathcal{O}_S) \cap Z(K).$$

Proof. We first show that $(\mathcal{Z}, \mathcal{Z} \cap \mathcal{D})$ is an $\mathcal{O}_S$-model for $(Z, Z \cap D)$. Flatness and properness of $\mathcal{Z}$ are clear. Consider the following diagram, where the three vertical squares away from the front one are fibre products.

$$\begin{tikzcd}
& D_\alpha \arrow[rr] \arrow[dd] & & \mathcal{D}_\alpha \arrow[dd] \\
Z \cap D_\alpha \arrow[rr] \arrow[ur] \arrow[dd] & & \mathcal{Z} \cap \mathcal{D}_\alpha \arrow[ur] \arrow[dd] & \\
& Y \arrow[rr] & & \mathcal{Y} \\
\operatorname{Spec} K \arrow[rr] \arrow[ur] & & \operatorname{Spec} \mathcal{O}_S \arrow[ur] &
\end{tikzcd} \tag{6.4.1}$$

It follows that the front square is also a fibre product. Flatness of $\phi|_{D_\alpha}$ implies that $Z \cap D_\alpha$ is a divisor on Z. To show that $\mathcal{Z} \cap \mathcal{D}_\alpha$ has dense generic fibre, it suffices to show that each irreducible component of $\mathcal{Z} \cap \mathcal{D}_\alpha$ dominates $\operatorname{Spec} \mathcal{O}_S$, since then [53, Exercise 3.1.3] implies that the generic fibre of each irreducible component is dense. Since $\operatorname{Spec} \mathcal{O}_S$ is a Dedekind scheme, [53, Prop. 4.3.9] shows that this is equivalent to flatness of $\mathcal{Z} \cap \mathcal{D}_\alpha$ over $\operatorname{Spec} \mathcal{O}_S$. Since $\mathcal{D}_\alpha$ is flat over $\mathcal{Y}$, applying preservation of flatness under base change to the rightmost face of the cube on the right of (6.4.1) shows that $\mathcal{Z} \cap \mathcal{D}_\alpha$ is flat over $\operatorname{Spec} \mathcal{O}_S$.

It only remains to check that, for any $Q \in (Z \setminus D_{\mathrm{red}})(K)$ and $\alpha_v \in \mathcal{A}_v$, we have

$$n_v(\mathcal{Z} \cap \mathcal{D}_{\alpha_v}, Q) = n_v(\mathcal{D}_{\alpha_v}, Q). \tag{6.4.2}$$

This holds if and only if $\mathcal{Q}_v^* \mathcal{D}_{\alpha_v} = \mathcal{Q}_v^*(\mathcal{Z} \cap \mathcal{D}_{\alpha_v})$, which is clear since $\mathcal{Q} \subset \mathcal{Z}$. □

6.4.2 Blowing up

In this section we study the relationship between the Campana Hilbert property and blowing up.

Lemma 6.4.3. *Let* $\varphi \colon X \dashrightarrow Y$ *be a birational map of varieties over a field* F*. Let* $B \subset X(F)$ *be a non-thin subset. Then* $\varphi(B) \subset Y(F)$ *is not thin.*

Proof. Denote by $U \subset X$ an open set on which φ is an isomorphism, and suppose that $\varphi(B) \subset Y(F)$ is thin. Then $\varphi(B \cap U) \subset \varphi(B)$ is also thin, so there exists a finite collection of generically finite dominant morphisms of varieties $f_i : W_i \to Y$, $i = 1, \dots, r$ of degree ≥ 2 such that $\varphi(B \cap U) \setminus \bigcup_{i=1}^r f_i(W_i(F))$ is not dense in Y. Then $\varphi(B \cap U) \setminus \bigcup_{i=1}^r f_i(W_i(F))$ is not dense in $\varphi(U) \cong U$. Let $W_i' = f_i^{-1}(\varphi(U))$ and $f_i' = f_i|_{W_i'}$, so that $\varphi^{-1} \circ f_i' \colon W_i' \to X$ are generically finite dominant morphisms of degree ≥ 2. Then $(B \cap U) \setminus \bigcup_{i=1}^r \big(\varphi^{-1} \circ f_i'\big)(W_i'(F))$ is not dense in U, which implies that B is thin, a contradiction. □

Definition 6.4.4. Given a Campana orbifold (X, D) with $\mathcal{O}_S$-model $(\mathcal{X}, \mathcal{D})$, we say that the set of Campana points $(\mathcal{X}, \mathcal{D})(\mathcal{O}_S)$ is *locally not thin* if for all $P \in (\mathcal{X}, \mathcal{D})(\mathcal{O}_S)$ and open subsets $U \subset (\mathcal{X}, \mathcal{D})(\mathbb{A}_K^S)$ containing P, the set $(\mathcal{X}, \mathcal{D})(\mathcal{O}_S) \cap U \subset X(K)$ is not thin.

Lemma 6.4.5. *Let (X, D) be a Campana orbifold with $\mathcal{O}_S$-model $(\mathcal{X}, \mathcal{D})$. Let $P_0 \in (\mathcal{X}, \mathcal{D})(\mathcal{O}_S)$ and $P \in (X \setminus D_{red})(K)$ be two distinct points. Define*

$$Y = \mathrm{Bl}_P X \xrightarrow{\rho} X, \quad \mathcal{Y} = \mathrm{Bl}_{\mathcal{P}} \mathcal{X} \xrightarrow{\widetilde{\rho}} \mathcal{X},$$

$$F = \rho^* D, \quad \mathcal{F} = \widetilde{\rho}^* \mathcal{D}.$$

Then $(\mathcal{Y}, \mathcal{F})$ is an $\mathcal{O}_S$-model for (Y, F). Suppose that for all finite subsets $S \subset T \subset \mathrm{Val}(K)$ large enough, the set of $\mathcal{O}_T$-Campana points $(\mathcal{Y}_T, \mathcal{F}_T)(\mathcal{O}_T)$ is locally not thin. Then for any open subset $U \subset (\mathcal{X}, \mathcal{D})(\mathbb{A}_K^S)$ containing P_0, the set $(\mathcal{X}, \mathcal{D})(\mathcal{O}_S) \cap U \subset X(K)$ is not thin.

Proof. The fact that $(\mathcal{Y}, \mathcal{F})$ is an $\mathcal{O}_S$-model for (Y, F) follows from commutativity of blowups and flat base change [53, Prop. 8.1.12(c)]. Let $U \subset (\mathcal{X}, \mathcal{D})(\mathbb{A}_K^S)$ be an open neighborhood of P_0. We have $\widetilde{\rho}^{-1}(\mathcal{D}) = \widetilde{\rho}^* \mathcal{D} + \mathcal{E}$, where $\mathcal{E} \subset \mathcal{Y}$ is supported over the set of places T over which $\mathcal{P}$ and $\mathcal{D}$ intersect, which is finite since $P \notin D_{\mathrm{red}}$. Note that $\rho^{-1}(P_0) \in (\mathcal{Y}_T, \mathcal{F}_T)(\mathcal{O}_T)$ since for each $v \notin T$, the point $\mathcal{P}_v$ and $\mathcal{D}_v$ are disjoint.

For each $v \in T \setminus S$, there exists an open neighbourhood $U'_v \subset X(K_v)$ of P_0 contained in $(\mathcal{X}, \mathcal{D})(\mathcal{O}_v)$. Shrinking U if necessary, we can assume that its image under $X(\mathbb{A}_K^S) \to X(K_v)$ is contained in U'_v. Let B be the preimage of U under

$$(\mathcal{Y}_T, \mathcal{F}_T)(\mathcal{O}_T) \hookrightarrow Y(\mathbb{A}_K^S) \to X(\mathbb{A}_K^S).$$

Enlarging T if necessary, we can assume that B is not thin by the hypothesis. We claim that $\rho(B) \subset (\mathcal{X}, \mathcal{D})(\mathcal{O}_S)$. Let $Q \in B$. The places where we must check that $\rho(Q)$ is a local Campana point are those where $\mathcal{Q}$ meets $\mathcal{E}$. These places are contained in T, and for $v \in T$, we have $\rho(Q) \in U'_v \subset (\mathcal{X}, \mathcal{D})(\mathcal{O}_v)$. Hence, $\rho(B) \subset (\mathcal{X}, \mathcal{D})(\mathcal{O}_S)$, and the result follows as $\rho(B)$ is not thin by Lemma 6.4.3. $\square$

6.4.3 Proof of Theorem 6.1.6

We are now ready to prove Theorem 6.1.6. In fact, we will prove the stronger result that the Campana points are locally not thin.

Theorem 6.4.6. *Let X be a del Pezzo surface of degree d over a number field K and let $L \subset X$ be a rational line. Let $D = \left(1 - \frac{1}{m}\right)L$ for some integer $m \geq 2$. Let $(\mathcal{X}, \mathcal{D})$ be an $\mathcal{O}_S$-model of (X, D). Suppose that one of the following holds:*

(a) $d = 4$ and X has a conic fibration.

(b) $d = 3$.

(c) $d = 2$ and X has a conic fibration.

Then $(\mathcal{X}, \mathcal{D})(\mathcal{O}_S)$ is locally not thin. In particular, since $L(K) \subset (\mathcal{X}, \mathcal{D})(\mathcal{O}_S)$, $(\mathcal{X}, \mathcal{D})$ satisfies CHP.

Proof. Let $A = (\mathcal{X}, \mathcal{D})(\mathcal{O}_S)$ and $P_0 \in A$. It suffices to show that for any open set $P_0 \in U \subset (\mathcal{X}, \mathcal{D})(\mathbb{A}_K^S)$, the intersection $A \cap U$ is not thin.

In case (b), the divisor class $-K_X - L$ gives rise to a conic fibration of X, hence in all cases, X has a conic fibration. Let C be the divisor class of the fibres of the conic fibration, and let $U \subset X(\mathbb{A}_K^S)$ be an open subset containing P_0. Set $A^\circ = A \cap U$.

(a),(b): Since X contains a rational point and $d \geq 3$, it is unirational (see [67, Rem. 9.4.11]). Then $X(K)$ is dense, and we can blow up $Y \xrightarrow{\rho} X$ away from P_0 and D_{red} so that Y is a del Pezzo surface over K of degree 2. Let $\mathcal{Y}$ be the $\mathcal{O}_S$-model for Y obtained by blowing up $\mathcal{X}$ at the corresponding $\mathcal{O}_S$-points. Note that the composition $Y \to X \to \mathbb{P}^1$ of ρ with the conic fibration of X induces a conic fibration of Y. Since Y is a del Pezzo surface of degree 2, it suffices by Lemma 6.4.5 to consider the case (c).

(c) The classes C and $-2K_X - C$ give rise to two conic fibrations $\pi_i \colon X \to \mathbb{P}^1$, $i = 1, 2$ respectively, with $\pi_1^{-1}(P) \cdot \pi_2^{-1}(Q) = 4$ (as shown in Lemma 3.3.6) for any $P, Q \in \mathbb{P}^1(K)$. Without loss of generality, we have either (i) $L \cdot C = 1$ or (ii) $L \cdot C = 2$ by the adjunction formula [39, Exercise V.1.3]. By spreading out, there is a finite subset $S \subset T \subset \operatorname{Val}(K)$ such that each π_i extends over $\operatorname{Spec} \mathcal{O}_T$ to a flat morphism $\Pi_i \colon \mathcal{X}_T \to \mathbb{P}^1_{\mathcal{O}_T}$. By enlarging T if necessary, we may assume that $\mathcal{D}_{\text{red}}$ is flat over $\operatorname{Spec} \mathcal{O}_T$.

Using weak approximation on $\mathbb{P}^1$, choose a point $P \in \mathbb{P}^1(K) \cap \pi_1(U)$ such that $\pi_1^{-1}(P)$ is smooth. Set $C_1 = \pi_1^{-1}(P)$, and denote by $\mathcal{C}_1$ the closure of C_1 in $\mathcal{X}_T$. Further enlarging T if necessary, we may assume that $\mathcal{D}_{\text{red}}$ is flat relative to Π_1, and that it is flat relative also to Π_2 in case (i). Shrinking U if necessary, we may assume that for each $v \in T \setminus S$, the image of the projection $U \to X(K_v)$ lies in $(\mathcal{X}, \mathcal{D})(\mathcal{O}_v)$. In other words, any Campana $\mathcal{O}_T$-point lying inside U is automatically a Campana $\mathcal{O}_S$-point, so

$$(\mathcal{X}_T, \mathcal{D}_T)(\mathcal{O}_T) \cap U = A^\circ. \tag{6.4.3}$$

Let

$$Z_1 = \{P \in \mathbb{P}^1(K) : \pi_1^{-1}(P) \text{ smooth and } \pi_1^{-1}(P)(K) \cap A^\circ \neq \emptyset\},$$
$$Z_2 = \{P \in \mathbb{P}^1(K) : \pi_1^{-1}(P) \text{ smooth and } \pi_1^{-1}(P)(K) \cap A^\circ \text{ is not thin in } \pi_1^{-1}(P)\}.$$

It suffices to show that Z_2 is not thin by Theorem 6.4.1. If $P \in Z_1$, then by Lemma 6.4.2, $\pi_1^{-1}(P) \cap A^\circ$ is the same as the set of Campana $\mathcal{O}_T$-points on a model of $(\pi_1^{-1}(P), \pi_1^{-1}(P) \cap D)$ intersected with U. Note that Proposition 6.3.5 and Proposition 6.3.12 imply that the latter intersection is not thin in $\pi_1^{-1}(P)$ (the choice of model here does not matter by Lemma 6.2.2). Hence, $\pi_1^{-1}(P) \cap A^\circ$ is not thin as well, and so $Z_1 = Z_2$, so it suffices to show that Z_1 is not thin.

By Lemma 6.4.2 and the above, $(\mathcal{C}_1, \mathcal{C}_1 \cap \mathcal{D}_T)$ is an $\mathcal{O}_T$-model for $(C_1, C_1 \cap D_T)$. Then Lemma 6.2.2 and Proposition 6.3.5 (resp. Proposition 6.3.12) show that $(\mathcal{C}_1, \mathcal{C}_1 \cap \mathcal{D}_T)$ satisfies CWA in case (i) (resp. case (ii)). By Lemma 6.4.2, $(\mathcal{C}_1, \mathcal{C}_1 \cap \mathcal{D}_T)(\mathcal{O}_T) = (\mathcal{X}_T, \mathcal{D}_T)(\mathcal{O}_T) \cap C_1(K)$, giving infinitely many Campana $\mathcal{O}_T$-points lying on U. By (6.4.3), this gives infinitely many points in $\pi_2(A^\circ) \subset \mathbb{P}^1(\mathbb{A}_K^S)$.

Let $B \subset \mathbb{P}^1(K)$ be any thin subset. We adapt the method in the proof of Proposition 3.3.3 to construct a fibre C_2 of π_2 such that $\pi_1(C_2(K) \cap A^\circ) \not\subset B$. From this, we can deduce that Z_1 is not contained in B, and so Z_1 is not thin. We can exclude type I thin sets from B since any type I thin set is contained in a type II thin set.

Write $B = \cup_{i=1}^n f_i(Y_i(K))$ for $f_i \colon Y_i \to \mathbb{P}^1$ finite covers of degree ≥ 2. Let $\Sigma_i \subset \mathbb{P}^1$ denote the finitely many branch points of f_i, and let Λ denote the union of the branch loci of the morphisms

$$\pi_2|_{\pi_1^{-1}(P)}\colon \pi_1^{-1}(P) \to \mathbb{P}^1,\ P \in \bigcup_{i=1}^{n} \Sigma_i,$$

which is a finite set. Thus, we can take $Q \in \pi_2(A^\circ) \setminus \Lambda$ such that $\pi_2^{-1}(Q)$ is smooth, and set $C_2 = \pi_2^{-1}(Q)$. The proof of Proposition 3.3.3 shows that the branch locus of $\pi_1|_{C_2}\colon C_2 \to \mathbb{P}^1$ is disjoint with the branch locus of each of the morphisms $f_i\colon Y_i \to \mathbb{P}^1$, hence the fibre products $C_2 \times_{\mathbb{P}^1} Y_i$ are all covers of degree ≥ 2 of C_2. Since the rational points of the fibre product are in bijection with the rational points of both curves which map to the same point on $\mathbb{P}^1$, and since C_2 has the Hilbert property, this implies that $\{P \in C_2(K) \mid \pi_1(P) \in B\} \subset C_2(K)$ is thin. It now suffices to show that $C_2(K) \cap A^\circ$ is not thin. Let $\mathcal{C}_2$ denote the closure of C_2 in $\mathcal{X}_T$.

In case (i), we have

$$C_2(K) \cap A^\circ = C_2(K) \cap (\mathcal{X}_T, \mathcal{D}_T)(\mathcal{O}_T) \cap U = (\mathcal{C}_2, \mathcal{C}_2 \cap \mathcal{D}_T)(\mathcal{O}_T) \cap U,$$

where the first equality is by (6.4.3) and the second equality is by Lemma 6.4.2, recalling that $\mathcal{D}_{\mathrm{red}}$ may be assumed flat relative to Π_2. Note also that this set is non-empty as it contains Q. Lemma 6.2.2 implies that $(\mathcal{C}_2, \mathcal{C}_2 \cap \mathcal{D}_T)(\mathcal{O}_T)$ satisfies CWA, so it follows that $(\mathcal{C}_2, \mathcal{C}_2 \cap \mathcal{D}_T)(\mathcal{O}_T) \cap U$ is not thin.

In case (ii), since C_2 does not meet L, we may assume upon enlarging T that $(\mathcal{C}_2 \cap \mathcal{D}_{\mathrm{red}})_T = \emptyset$. Hence, $C_2(K) \cap A^\circ = C_2(K) \cap U$. Since C_2 satisfies weak approximation, $C_2(K) \cap U$ is not thin in C_2. $\square$

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