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Coexistence of competing first passage percolation on hyperbolic graphs

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Abstract

We study a natural growth process with competition, which was recently introduced to analyze MDLA, a challenging model for the growth of an aggregate by diffusing particles. The growth process consists of two first-passage percolation processes FPP_1 and FPP_λ , spreading with rates 1 and $\lambda > 0$ respectively, on a graph G . FPP_1 starts from a single vertex at the origin o , while the initial configuration of FPP_λ consists of infinitely many *seeds* distributed according to a product of Bernoulli measures of parameter $\mu > 0$ on $V(G) \setminus \{o\}$. FPP_1 starts spreading from time 0, while each seed of FPP_λ only starts spreading after it has been reached by either FPP_1 or FPP_λ . A fundamental question in this model, and in growth processes with competition in general, is whether the two processes coexist (i.e., both produce infinite clusters) with positive probability. We show that this is the case when G is vertex transitive, non-amenable and hyperbolic, in particular, for any $\lambda > 0$ there is a $\mu_0 = \mu_0(G, \lambda) > 0$ such that for all $\mu \in (0, \mu_0)$ the two processes coexist with positive probability. This is the first non-trivial instance where coexistence is established for this model. We also show that FPP_λ produces an infinite cluster almost surely for any positive λ, μ , establishing fundamental differences with the behavior of such processes on \mathbb{Z}^d .

Keywords: First passage percolation, first passage percolation in hostile environment, hyperbolic graphs, non-amenable graphs, competition, coexistence, two-type Richardson model

1 Introduction

We consider a randomly growing process with competition, which was introduced in [36] under the name of *first passage percolation in a hostile environment* (FPPHE). FPPHE consists of two first passage percolation processes, denoted FPP_1 and FPP_λ , which spread inside an infinite graph G . At time 0, FPP_1 occupies only a single vertex of G , called *the origin* o , whereas FPP_λ starts from countably many “sources”, which we call *seeds* and are distributed according to a product of Bernoulli measures of parameter $\mu \in (0, 1)$ on $V(G) \setminus \{o\}$.

The process evolves from time 0 as follows. FPP_1 starts spreading through the edges of G at rate 1. On the other hand, FPP_λ does not start spreading from time 0, but waits. Whenever a process (either FPP_1 or FPP_λ) attempts to occupy a vertex that hosts a seed of FPP_λ , the attempt fails, the seed is *activated*, and FPP_λ starts spreading from that seed through the edges of G at rate $\lambda > 0$. Seeds that have not been activated remain dormant until they are activated. A vertex that is occupied by either of the processes will remain so forever, and will never be occupied by the other process; hence the two processes compete for space as they grow.

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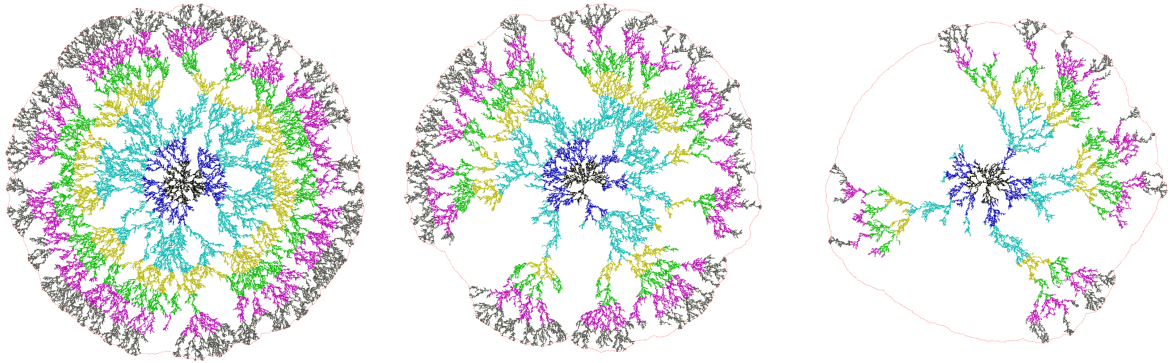


Figure 1: A representation of FPPHE on the lattice \mathbb{Z}^2 with $\lambda = 0.7$ and $\mu = 0.027, 0.029$ and 0.030 , respectively. Colors represent different epochs of the growth of FPP_1 , while the thin curve at the boundary represents the boundary between FPP_λ and vertices that are either unoccupied or host an inactive seed of FPP_λ . The whole white region within this boundary is occupied by activated FPP_λ .

We say that FPP_1 (resp. FPP_λ) *survives* if in the limit as time goes to infinity we obtain that FPP_1 (resp. FPP_λ) occupies an infinite *connected* region of the graph; otherwise we say that it *dies out*. We stress that the region must be connected for the definition of survival, since almost surely FPP_λ already starts from an infinite set of seeds. In particular, when we say that FPP_λ “dies out”, we mean that it ends up consisting of an infinite collection of connected regions, each of which almost surely of finite size.

FPPHE was introduced in [36] in order to study MDLA, a classical aggregation process showing dendritic growth. Nonetheless, FPPHE is a very interesting process in its own right. In particular, simulations show that FPPHE has a very rich behavior, producing a delicate geometry similar to dendritic growth and undergoing phase transitions. In Figure 1 one can see how, by increasing slightly the density of seeds, the set of vertices occupied by FPP_1 becomes extremely thin. This process is a modified version of the classical FPP model and the Richardson model for competing species with different growth rates. Yet, a fundamental difference is that the Richardson model is a monotone process in the sense that one can easily devise a coupling to show that if one increases the set of sites occupied by one of the types, this type can only occupy a larger portion of the space. On the other hand, FPPHE is not monotone at all: for instance, in some graphs, increasing the initial density μ can increase the probability that FPP_1 survives (see e.g. [11]). Related phenomena for the Richardson model have been investigated in [15].

There are three possible behaviors for FPPHE:

- *Extinction*, where FPP_1 dies out *almost surely*¹,
- *Strong survival*, where with positive probability FPP_1 survives but FPP_λ dies out,
- *Coexistence*, where with positive probability FPP_1 and FPP_λ survive simultaneously.

The above definitions do not exclude that, for some values of μ and λ , FPPHE is both in the strong survival regime and in the coexistence regime, though it is believed that the three regimes are mutually exclusive.

The extinction regime occurs, for example, when $1 - \mu < p_c(G)$, where $p_c(G)$ is the critical probability for site percolation on G (see [20] for a classical reference on percolation and [12, 16]

¹For any $\mu > 0$ and any locally finite G , with positive probability FPP_1 dies out; see Remark 1.2.

for recent results on the non-triviality of p_c on rather general amenable graphs). In fact, when $1 - \mu < p_c(G)$, the set of vertices of $V(G)$ not occupied by seeds of FPP_λ consists only of finite clusters; thus FPP_1 is confined to the finite cluster containing the origin. The other two regimes are much more interesting and less understood. For example, in Figure 1, the leftmost picture seems to be in a regime of strong survival, nonetheless FPP_λ manages to conquer quite large regions, evidencing the long-range dependencies that appear in this process.

It is natural to expect that decreasing μ or λ should favor the survival of FPP_1 . Unfortunately, there is no proof of such monotonicity. In fact, one can even engineer instances of the evolution of the process (that is, instances of passage times and locations of seeds of FPP_λ) such that removing a seed of FPP_λ or slowing down the spread of FPP_λ through a single edge could actually *harm* the survival of FPP_1 . This lack of monotonicity makes this process particularly challenging to analyze.

In [36] FPPHE was analyzed with G being the lattice \mathbb{Z}^d . They developed a very involved proof, based on a multi-scale analysis, to show that there exists strong survival for $d \geq 2$. More precisely, in [36, Theorem 1.2], it is shown that for *any* $\lambda < 1$, there exists $\mu_0 > 0$ such that if $\mu \in (0, \mu_0)$ then there is strong survival. Unfortunately, [36] gives no information regarding whether FPPHE has a coexistence regime in \mathbb{Z}^d , $d \geq 2$, which remains a fascinating open problem. In general, establishing coexistence is even more challenging than establishing strong survival. The main reason is that, in the strong survival regime, each seed of FPP_λ that gets activated will produce a finite cluster. Thus, after some finite time, FPP_1 will go around that seed and encapsulate its cluster inside a finite region. At that moment, the presence of that seed ceases to interfere with the spread of FPP_1 . On the other hand, in a coexistence regime, dependencies are even stronger (in the sense that their range is unbounded) since there will be seeds of FPP_λ that will have an everlasting effect in the evolution of FPP_1 . Our main contribution is to establish for the first time (with the exception of the trivial cases below) a regime of coexistence for FPPHE .

When $d = 1$, the strong survival regime and the coexistence regime do not exist. More generally, when G is a tree, or even a free product (see Section 8 for a definition and a more thorough discussion), the regime of strong survival cannot exist since FPP_1 is not able to “go around and block the spread” of an activated seed of FPP_λ . When G is a tree, survival of FPP_1 translates to the origin being in an infinite cluster of the graph obtained from G by removing the vertices occupied by seeds of FPP_λ . Consequently, on trees, coexistence occurs whenever $1 - \mu > p_c(G)$; an analogous result can be derived for free products.

1.1 Our results

We will consider the case of G being *hyperbolic* and *non-amenable* (see Section 2.1 for a rigorous definition). The main goals of this paper are to establish a regime of coexistence for FPPHE , and to show that FPPHE on such graphs has a substantially different behavior than on \mathbb{Z}^d .

Roughly speaking, hyperbolicity means that the sides of any geodesic triangle are “close” to one another, and non-amenability means that each finite subset has a large boundary. Hyperbolic and non-amenable graphs are classical and important classes of graphs, for example, hyperbolic graphs were introduced by Gromov [21] and can be seen as a discrete analogue of manifolds with negative curvature. We will further consider natural assumptions on G , such as vertex transitivity and boundedness of degrees.

In the first theorem, we show that FPP_1 survives with positive probability for any value of λ , provided that μ is small enough. This theorem does not require G to be transitive.

Theorem 1.1 (Survival of FPP_1). *Let G be a hyperbolic, non-amenable graph of bounded degree. For any growth rate $\lambda > 0$ of FPP_λ , there is a value $\mu_0 = \mu_0(G, \lambda) > 0$ such that whenever*

$\mu \in (0, \mu_0)$, FPP_1 survives with positive probability.

A trivial and immediate (therefore not very interesting) example of a graph that satisfies the assumptions of Theorem 1.1 is an infinite tree of bounded degree. Less trivial examples are given by the graphs produced by regular tessellations of the hyperbolic plane.

Remark 1.2. FPP_1 surviving with positive probability is the best one can hope for since, for any $\mu > 0$ and any $\lambda > 0$, with positive probability FPP_1 dies out: for example, because all neighbors of o are occupied by seeds of FPP_λ . Furthermore, non-amenability is essential in Theorem 1.1 since on \mathbb{Z} (which is hyperbolic but amenable) we obtain that FPP_1 dies out for all $\mu, \lambda > 0$.

Theorem 1.1 resembles the result of [36, Theorem 1.2] on \mathbb{Z}^d . The main difference is that, on \mathbb{Z}^d , FPP_1 can only survive if $\lambda < 1$. On hyperbolic, non-amenable graphs, FPP_1 can survive even if λ is arbitrarily large; that is, if FPP_λ spreads at a much faster rate than FPP_1 .

Our next result establishes that, regardless of the values of μ and λ , FPP_λ survives almost surely; this does not even require non-amenability. This is a striking difference with the behavior on \mathbb{Z}^d : for example, it shows that there is no strong survival on hyperbolic graphs.

Theorem 1.3 (Survival of FPP_λ). *Let G be a vertex transitive, hyperbolic graph. For any $\mu \in (0, 1)$ and $\lambda > 0$, FPP_λ survives almost surely.*

As an immediate corollary, we establish coexistence for FPPHE.

Corollary 1.4 (Coexistence). *Let G be a vertex transitive, hyperbolic, non-amenable graph. For all $\lambda > 0$ there is a value $\mu_0 = \mu_0(G, \lambda) > 0$ such that for all $\mu \in (0, \mu_0)$, FPP_1 and FPP_λ coexist with positive probability.*

Proof. The proof follows directly by putting together the statements of Theorem 1.1 and Theorem 1.3. □

1.2 Related processes

Even though FPPHE was invented to analyze MDLA, growth processes with competition is a classical area of research. One example is the two-type Richardson model, which consists of FPP_1 starting from o and FPP_λ starting from one active seed (for example, located at a neighbor of o); thus both FPP_1 and FPP_λ start spreading from time 0. It is conjectured that coexistence occurs if and only if $\lambda = 1$. The fact that coexistence occurs when $\lambda = 1$ was established in [23] on the two-dimensional lattice, and extended to other dimensions by [26] and [17] (see also [27, 1]). The converse has not been fully resolved, but [24] established that the set of values of λ for which there is coexistence is at most a countable subset of $(0, \infty)$. Simulation suggests that, unlike the two-type Richardson model, FPPHE on \mathbb{Z}^d has a regime of coexistence even when $\lambda \neq 1$; this seems to be a consequence of the richer set of behaviors coming from the interplay between λ and μ in FPPHE.

For a simplified, deterministic version of the two-type Richardson model, [3] shows that, on a hyperbolic graph, coexistence is possible for *all* values of $\lambda > 0$. Our proof of Theorem 1.3 can also be used to establish this result for the actual two-type Richardson model. This is the content of the next corollary, which does not need non-amenability.

Corollary 1.5. *Consider the two-type Richardson model defined above on a vertex transitive, hyperbolic graph. Then, for any $\lambda > 0$, with positive probability both FPP_1 and FPP_λ survive.*

FPPHE can also be regarded as a natural model for the spread of two conflicting rumors throughout a network. In this setting, FPP_1 represents a false rumor that starts spreading from time 0 from the origin. Vertices hosting seeds of FPP_λ represent the nodes of the network that can verify that the rumor is actually false. Thus when they receive the rumor (thereby becoming an activated seed), they start spreading the correct information. In this setting, G being hyperbolic and non-amenable is of particular relevance, since several authors observed that some important real-world networks can be modeled by graphs equipped with a hyperbolic structure [33, 29, 13, 38]. The last one, in particular, investigates a randomized algorithm to stop the spread of a false rumor within a large network, by using its hyperbolicity properties. This line of work substantially deviates from ours. Nonetheless, our theorems show that, in a hyperbolic graph, in order to stop the spread of the false rumor via FPPHE, it is not enough to have $\lambda \geq 1$ (as simulations suggest for \mathbb{Z}^d), but one needs to have a sufficiently large density of seeds of FPP_λ .

1.3 Ideas of the proofs

Proof idea of Theorem 1.1. In order to show survival of FPP_1 , we cannot employ the same strategy as in [36] for \mathbb{Z}^d . The reason is that [36] showed *strong survival*, and their proof technique resorts to showing that every activated seed of FPP_λ gets eventually encapsulated by FPP_1 . In our case, we know from Theorem 1.3 that strong survival does not occur for any λ, μ . So we had to develop a new strategy, which uses the hyperbolicity and non-amenableity.

First non-amenableity gives that one can embed a binary tree inside G (cf. Theorem 2.1 below). Using this, we will construct a tree \mathbb{T} where the “edges” of \mathbb{T} will represent *almost* geodesic paths of G of some fixed length. Exploring the exponential growth of \mathbb{T} , one could suspect that it is possible to find an infinite path v_1, v_2, \dots in \mathbb{T} such that the distance between v_i and the set of seeds increases with i . This would be convenient, as it would imply existence of an infinite path in G that gets further and further away from seeds, which would give a positive probability for this path to get entirely occupied by FPP_1 . However, one can show that an infinite path satisfying the above requirement simply does not exist.

Our approach is to resort to a multi-scale analysis: we will look for an infinite path in \mathbb{T} such that the passage times near this path are “good” (in the sense that they are close to their expected value) at all scales. To do this, we define certain *cylinders* of different sizes, and each size is identified with a *scale*. The first scale consists of cylinders around each edge of \mathbb{T} , and each subsequent scale $\mathbf{j} \geq 2$ is formed by cylinders whose axis is a path of length \mathbf{j} in \mathbb{T} , and whose width will grow linearly with \mathbf{j} .

We say that a cylinder \mathcal{C} is good if FPP (of a single type, without competition) started anywhere inside \mathcal{C} advances linearly in time (during a time interval of order \mathbf{j}). For scale 1, we will also require that \mathcal{C} is completely free of seeds to be good. As a consequence, if \mathcal{C} is good then it will follow that the paths traversed by FPPHE from one end of \mathcal{C} to the other end will remain close to the axis of \mathcal{C} . This is a consequence of the fact that hyperbolic graphs have long detours away from the geodesics (cf. Proposition 2.3 below), and such detours will have typical (or linear) passage times since \mathcal{C} is a good cylinder.

We will then show that there is an infinite path Ξ in the tree \mathbb{T} that only intersects good cylinders. The existence of such a path implies that FPP_1 advances quickly enough near a d_G -geodesic ray close enough to Ξ . This does not allow FPP_λ to get anywhere near Ξ because for that to happen, a seed located outside of scale-1 cylinders has to be activated by FPP_1 coming from Ξ and then propagate FPP_λ back to Ξ . However, such a detour out of scale-1 cylinders is too long, and since Ξ is covered by good cylinders at all scales, the passage time around such a detour will not succeed in bringing FPP_λ back to Ξ before FPP_1 . We emphasize here that this

proof can be carried out even when G is not transitive.

Proof idea of Theorem 1.3. To establish survival of FPP_λ we proceed as follows. Let FPP_1 run from the origin until it hits a large ball (of fixed radius) completely occupied by seeds. This will eventually occur. While FPP_1 tries to go around this ball (for which, with high probability, it will require an amount of time that is exponential in the radius of the ball), FPP_λ is free to proceed, activate all seeds in the ball and then continue to occupy further regions. In particular, FPP_λ will have enough time to occupy a larger ball, which will then force FPP_1 to do an even longer detour. We will show that, with positive probability, FPP_λ succeeds in occupying an infinite sequence of balls of increasing radii, before they can be reached by FPP_1 . When this happens, we say that this iteration succeeded.

An iteration succeeding implies that FPP_λ survives. However the probability that an iteration succeeds is only positive, and there are difficulties in showing that FPP_λ actually survives almost surely. For example, if an iteration is unsuccessful, one needs to take special care of the measurability of the events that have been observed. In particular, we need to define each iteration in such a way that we only observe events that are measurable locally (that is, with respect to a finite subset of edges). So, once an iteration fails, we can carry out another iteration inductively. Another technical difficulty is given by the fact that whenever there is a failure, both FPP_1 and FPP_λ have already occupied large regions of G . Thus, we need to carry out each iteration such that its success probability does not depend on the size of the previously explored areas of G . This is crucial since the set occupied by FPP_1 grows over time, and each iteration starts from a ball of seeds of FPP_λ of a large but fixed radius. Using hyperbolicity, we show that this can be done in such a way that the probability of success of each iteration remains bounded away from zero as time goes, implying that eventually an iteration will succeed. Finally, we also need to ensure that at each iteration FPP_λ has room to continue expanding “towards infinity”, without getting trapped inside a dead-end of G , or curving back towards o . It is in this part that we use transitivity of G .

Outline of the paper

In Section 2 we present fundamental properties of first passage percolation and hyperbolic graphs, and give a construction of FPPHE in terms of passage times. In Section 3 we construct the tree \mathbb{T} described in the proof overview above. In Sections 4 and 5 we set up an inductive argument which is the core of the proof of Theorem 1.1, which is described in Section 6. The proof of Theorem 1.3 can be found in Section 7. We present conclusive remarks and open questions in Section 8, and prove a technical results regarding detours away from geodesics on hyperbolic graphs in Appendix A.

2 Preliminaries

All graphs considered in this paper will be infinite and of bounded degree. For a graph G , we denote by $V(G)$ its set of vertices and by $E(G)$ its set of edges.

2.1 Hyperbolic and non-amenable graphs

For any subset of vertices $S \subset V(G)$ the *internal boundary* of S is

$$\partial S := \{v \in S : \exists x \in V(G) \setminus S \text{ such that } \{x, v\} \in E(G)\}. \quad (2.1)$$

Moreover, we recall that the *Cheeger constant* of G is defined as

$$h(G) := \inf_{S \subset V(G)} \frac{|\partial S|}{|S|}, \quad (2.2)$$

where $|S|$ denotes the cardinality of S , and the infimum is taken over all *finite* sets $S \subset V(G)$. A graph is non-amenable if and only if it has positive Cheeger constant.

Roughly speaking, G is non-amenable if each finite subset has a large boundary, otherwise G is called *amenable*. Non-amenableity is often responsible for the appearance of intriguing phenomena in random processes. Notable examples are *percolation*, in which non-amenableity allows the appearance of infinitely many infinite components [31, Chapter 7], and *branching random walks*, in which non-amenableity brought about a fascinating regime where the process survives indefinitely but is nonetheless transient [4].

Let d_G denote the *metric* induced by the shortest-path distance in G . The graph G is called δ -*hyperbolic*, where $\delta \in [0, \infty)$, if for any three vertices $x, y, z \in V(G)$ and any three connecting geodesic segments $\gamma_{x,y}, \gamma_{y,z}, \gamma_{z,x}$ between these vertices we have

$$\forall u \in \gamma_{x,y}, \text{ there is } v \in \gamma_{y,z} \cup \gamma_{z,x} \text{ such that } u \in B_G(v, \delta), \quad (2.3)$$

where $B_G(v, \delta)$ is the ball (with respect to the d_G metric) centered at v with radius δ , and we regard a path in G (such as $\gamma_{x,y}$) as a sequence of vertices. It is known (cf. e.g. [25] and references therein) that $\delta = 0$ if and only if G is a tree, since our results are trivial for trees, we will always assume that $\delta > 0$. In words, (2.3) means that the entire geodesic $\gamma_{x,y}$ is contained in the union of balls of radius δ centered at vertices of $\gamma_{y,z} \cup \gamma_{z,x}$. In this case we shall say that the triangle with vertices x, y, z is δ -thin. Thus a graph is δ -hyperbolic if and only if each of its triangles is δ -thin. If G is δ -hyperbolic for some $0 \leq \delta < \infty$ we will simply say that G is *hyperbolic*. From now on, whenever we consider triangles on G we always mean *geodesic* triangles, that is, their sides lie on geodesic segments.

A class of hyperbolic graphs of large interest is that of Cayley graphs of *hyperbolic groups*. These were introduced in [22] in order to study properties of a randomly chosen group. For example, it is shown in [22] that if we choose a group with a finite (symmetric) set of generators and a random set of finite representations, then the resulting group is hyperbolic with high probability; see also [37, 35]. Recently there have been several works about the study of random processes on hyperbolic groups, usually establishing very different behavior with respect to such processes on lattices. For instance, [19, 18, 32, 30] studied the asymptotic behavior of random walks on hyperbolic groups, [6] investigated *first-passage percolation* on hyperbolic groups and showed the presence of doubly-infinite geodesics, and [28] showed that Bernoulli bond percolation on a non-amenable, hyperbolic, quasi-transitive graph has an intermediate phase in which there are infinitely many infinite clusters.

Embedding of a tree into G . A *bilipschitz embedding* of a metric space (X, d_X) into another metric space (Y, d_Y) is a map $f : X \rightarrow Y$ such that for some constant $\hat{\alpha} \geq 1$ and all vertices $u, v \in X$ we have

$$\hat{\alpha}^{-1} d_X(u, v) \leq d_Y(f(u), f(v)) \leq \hat{\alpha} d_X(u, v).$$

The following result is going to be crucial in our proofs. It gives that one can embed a binary tree in any non-amenable graph (without requiring hyperbolicity).

Theorem 2.1 ([5, Theorem 1.5]). *Let H be a bounded degree, non-amenable graph, considered together with the graph metric. Then there is a tree T contained in H , with positive Cheeger constant $h(T) > 0$, such that the inclusion map $T \rightarrow H$ is a bilipschitz embedding, and there is a bilipschitz embedding of the binary tree into H .*

Let \mathbb{T}_3 denote the infinite regular tree of degree 3. All graphs G considered in this paper will have bounded degrees. Whenever G is also non-amenable, we will denote by $\hat{\alpha}$ the constant obtained from the above bilipschitz embedding; that is, we will take a map $f : \mathbb{T}_3 \rightarrow G$ and denote by $\hat{\alpha} \geq 1$ the best constant such that, for all $x, y \in V(\mathbb{T}_3)$,

$$\hat{\alpha}^{-1}d_{\mathbb{T}_3}(x, y) \leq d_G(f(x), f(y)) \leq \hat{\alpha}d_{\mathbb{T}_3}(x, y). \quad (2.4)$$

To simplify the notation, after fixing the map f and the value of $\hat{\alpha}$ so that (2.4) is satisfied, we identify the vertices of \mathbb{T}_3 with their images in G through f , by saying simply “take $x \in V(\mathbb{T}_3)$ ” instead of “take $x \in V(G)$ such that $f^{-1}(x) \in V(\mathbb{T}_3)$ ”.

One of the characteristics of hyperbolic graphs is that whenever there is a bilipschitz embedding such as in (2.4), then for any pair of vertices $x, y \in V(\mathbb{T}_3)$ the image of any geodesic segment (on \mathbb{T}_3) connecting x with y via f is “close” to some geodesic segment (in G) connecting $f(x)$ with $f(y)$. This is the content of the next result, whose proof can be found in [34, 14]. The statement that we write here is adapted to our context from [34, Theorem 2.31] and [14, Corollaire 2.6].

Proposition 2.2. *Let X_1 and X_2 be two δ -hyperbolic, bounded-degree graphs, and let $f : X_1 \rightarrow X_2$ be a bilipschitz embedding with constant $\hat{\alpha} \geq 1$. Then there is a constant $\kappa = \kappa(\hat{\alpha}, \delta)$ so that the following holds. For any pair of vertices $x, y \in V(X_1)$, let $\gamma_{f(x), f(y)}$ be any geodesic in X_2 from $f(x)$ to $f(y)$, and $\gamma_{x, y}$ be any geodesic in X_1 from x to y . Let $\tilde{\gamma}_{x, y}$ be a quasi-geodesic in X_2 obtained from the concatenation of the geodesic segments $\gamma_{f(z_1), f(z_2)}$ of X_2 for each edge $\{z_1, z_2\}$ in $\gamma_{x, y}$. Then,*

$$\forall z \in \tilde{\gamma}_{x, y}, d_{X_2}(z, \gamma_{f(x), f(y)}) \leq \kappa \quad \text{and} \quad \forall z \in \gamma_{f(x), f(y)}, d_{X_2}(z, \tilde{\gamma}_{x, y}) \leq \kappa.$$

We point out that since we are dealing with discrete graphs, the constant δ will always be a non-negative integer.

Any graph G with $h(G) > 0$ has exponential growth, and its growth rate is bounded from below by $1 + h(G) > 1$. Furthermore, let $1 < \Delta < \infty$ denote the maximum degree of G . Then, for all $n > 0$ and all vertices $x \in V(G)$, we have

$$(1 + h(G))^n \leq |B_G(x, n)| \leq \Delta^n. \quad (2.5)$$

A crucial property of hyperbolic graphs that we are going to use is that detours from geodesic segments are very long.

Proposition 2.3. *Let G be a δ -hyperbolic graph. Given a vertex $x_0 \in V(G)$ and a value $r > 0$, consider the ball $B_G(x_0, r)$ with center x_0 and radius r . Now take a geodesic segment γ that goes through x_0 and has its endpoints y, z outside of $B_G(x_0, r)$. Then, any path started at y and ended at z which does not intersect $B_G(x_0, r)$ has length bounded from below by $\delta 2^{r/\delta}$.*

The proof of this fact can be found in [21, Sections 6 and 7], in the more general context of δ -thin, geodesic metric spaces.

2.2 First-passage percolation and coexistence

We start by defining the *first passage time*, following the notation of [2, Section 1]. For any edge $\{x_0, x_1\} \in E(G)$ take a non-negative random variable $t_{\{x_0, x_1\}}$, which we will refer to as *the passage time* of the edge $\{x_0, x_1\}$. We assume that the collection $\{t_e\}_{e \in E(G)}$ is i.i.d. with a common distribution fixed beforehand. In this work we will always assume that the common distribution is exponential of mean 1. We will later show how FPPHE can be constructed from

$\{t_e\}_{e \in E(G)}$. The random variable $t_{\{x_0, x_1\}}$ can be interpreted as the time needed to cross the edge $\{x_0, x_1\}$. For any finite path $\gamma = (x_0, x_1, \dots, x_n)$ we define the FPP-time of γ to be

$$T(\gamma) := \sum_{i=0}^{n-1} t_{\{x_i, x_{i+1}\}}. \quad (2.6)$$

Moreover, for any pair of vertices $x, y \in V(G)$ we set

$$T(x \rightarrow y) := \inf_{\gamma: x \rightarrow y} T(\gamma),$$

where the infimum is taken over all paths from x to y . Roughly speaking, the quantity $T(x \rightarrow y)$ can be seen as the “shortest time” needed to go from vertex x to vertex y .

A standard interpretation of FPP is that of a *random metric* on G , in fact one can define the new distance d_{FPP} as $d_{\text{FPP}}(x, y) := T(x \rightarrow y)$. In this way one can think of FPP as a process “spreading” in time, in the sense that at time $T = 0$ the set of vertices “occupied” by the process consists only of the starting vertex. Inductively, this set will grow and for all times $T > 0$ the set of vertices occupied by the process consists of all vertices at d_{FPP} -distance at most T from the starting vertex. We point out, however, that by the definition of the model, the (random) metric structure induced by FPP is lost whenever $\lambda \neq 1$, which reminds the behavior of other FPP-based competition models, such as the Richardson model with unequal rates and chase-escape.

First-passage percolation spreads linearly in time

We start with a lemma showing that on a graph with bounded degree, FPP is likely to move linearly in time. To fix the notation, for $x \in V(G)$ and $T \geq 0$ we let $A_T^x := \{y \in V(G) : T(x \rightarrow y) \leq T\}$; and for simplicity we set $A_T := A_T^x$. In words, for all $T \geq 0$ and each $x \in V(G)$ we have

$$A_T^x = \{\text{vertices reached by FPP started at } x \text{ and run for time } T\}.$$

Recall that $B_G(x, L)$ is the ball centered at x of radius $L > 0$ with respect to the metric d_G .

Lemma 2.4. *Suppose that G has maximum degree Δ , then for any constant $c_1 > 0$ there exists a constant $\mathbf{c}_{\text{out}} := \mathbf{c}_{\text{out}}(\Delta, c_1) > 1$, such that for every $x \in V(G)$ and all $T > 0$,*

$$\mathbb{P}[A_T^x \subseteq B_G(x, \mathbf{c}_{\text{out}}T)] \geq 1 - e^{-c_1 T}. \quad (2.7)$$

Moreover, for any constant $0 < c_0 < 1$ there exists a positive constant $\mathbf{c}_{\text{in}} := \mathbf{c}_{\text{in}}(\Delta, c_0) < 1$, such that for every $x \in V(G)$ and all $T > 0$

$$\mathbb{P}[B_G(x, \mathbf{c}_{\text{in}}T) \subseteq A_T^x] \geq 1 - e^{-c_0 T}. \quad (2.8)$$

The lemma above follows from the estimate below, which we collect in another lemma since we will need to refer to it later.

Lemma 2.5. *For any positive integer ℓ , any $S \leq \ell/2$ and any path $P = (x_0, x_1, \dots, x_\ell)$ of length ℓ in G , we have $\mathbb{P}[T(P) \leq S] \leq 2 \frac{e^{-S} S^\ell}{\ell!}$. Similarly, for any $T \geq 1$, any integer $\ell \leq T$ and any path $P = (x_0, x_1, \dots, x_\ell)$ of length ℓ in G , we have $\mathbb{P}[T(P) \geq T] \leq \ell \left(\frac{T e}{\ell}\right)^\ell e^{-T}$.*

Proof. Note that $\mathbb{P}[T(P) \leq S] = \mathbb{P}[\text{Poi}(S) \geq \ell]$, where $\text{Poi}(S)$ denotes a Poisson random variable with parameter S . This is due to the following. The first-passage time $T(P)$ of a path P is a sum of ℓ i.i.d. exponential random variables with mean 1. Thus, to demand that a sum of ℓ

i.i.d. random variables distributed as $\text{Exp}(1)$ is at most S , is equivalent to demand that within a time-interval of length S , one has witnessed at least ℓ observations of a Poisson process of rate 1. This is equivalent to ask that a Poisson random variable with parameter S is at least ℓ . Hence,

$$\mathbb{P}[T(P) \leq S] \leq \sum_{k \geq \ell} \frac{e^{-S} S^k}{k!} \leq 2 \frac{e^{-S} S^\ell}{\ell!},$$

where in the last step we use the fact that $\ell \geq 2S$. The second part of the lemma is analogous:

$$\mathbb{P}[T(P) \geq T] \leq \mathbb{P}[\text{Poi}(T) \leq \ell] \leq \sum_{k \leq \ell} \frac{e^{-T} T^k}{k!}.$$

For all $k \leq \ell \leq T$ the quantity $\frac{T^k}{k!}$ is increasing in k , then for all values $\ell \leq T$ we have

$$\mathbb{P}[T(P) \geq T] \leq e^{-T} \sum_{k \leq \ell} \frac{T^k}{k!} \leq e^{-T} \ell \frac{T^\ell}{\ell!} \leq e^{-T} \ell \left(\frac{Te}{\ell} \right)^\ell,$$

thus finishing the proof. \square

Proof of Lemma 2.4. We start with the first part of the statement. Observe that

$$\mathbb{P}[A_T^x \subseteq B_G(x, \mathbf{c}_{\text{out}}T)] = \mathbb{P} \left[\sup_{w \in A_T^x} d_G(x, w) \leq \mathbf{c}_{\text{out}}T \right] \geq 1 - \mathbb{P} \left[\sup_{w \in A_T^x} d_G(x, w) > \mathbf{c}_{\text{out}}T \right].$$

Recalling that, for any path $P_\ell = (x_0, x_1, \dots, x_\ell)$ of length $\ell \geq 1$, $T(P_\ell)$ denotes the FPP time of P_ℓ as defined in (2.6). Using that $\mathbb{P} \left[\sup_{w \in A_T^x} d_G(x, w) > \mathbf{c}_{\text{out}}T \right]$ is the probability that there exists a path started at x of length larger than $\mathbf{c}_{\text{out}}T$ with FPP-time at most T , and applying Lemma 2.5, we get that for all c_1 large enough we can choose \mathbf{c}_{out} such that

$$\mathbb{P} \left[\sup_{w \in A_T^x} d_G(x, w) > \mathbf{c}_{\text{out}}T \right] \leq \sum_{\ell > \mathbf{c}_{\text{out}}T} \sum_{P_\ell \text{ started at } x} \mathbb{P}[T(P_\ell) \leq T] \leq \sum_{\ell > \mathbf{c}_{\text{out}}T} \sum_{P_\ell \text{ started at } x} e^{-2c_1 \ell}.$$

Then we observe that the number of paths of length $\ell \geq 1$ started at a fixed vertex x is at most Δ^ℓ , by the bounded degree assumption. Thus, we take c_1 large enough so that, for all $\ell \geq \mathbf{c}_{\text{out}}T$, we have $\mathbb{P} \left[\sup_{w \in A_T^x} d_G(x, w) > \mathbf{c}_{\text{out}}T \right] \leq \sum_{\ell > \mathbf{c}_{\text{out}}T} \Delta^\ell e^{-2c_1 \ell} \leq e^{-c_1 \ell}$.

To show the second part of the statement we proceed analogously:

$$\mathbb{P}[B_G(x, \mathbf{c}_{\text{in}}T) \subseteq A_T^x] = \mathbb{P} \left[\inf_{w \notin A_T^x} d_G(x, w) > \mathbf{c}_{\text{in}}T \right] \geq 1 - \mathbb{P} \left[\inf_{w \notin A_T^x} d_G(x, w) \leq \mathbf{c}_{\text{in}}T \right].$$

Then, $\mathbb{P} \left[\inf_{w \notin A_T^x} d_G(x, w) \leq \mathbf{c}_{\text{in}}T \right] \leq \mathbb{P}[\exists \text{ a path of length } \leq \mathbf{c}_{\text{in}}T \text{ with FPP-time } \geq T]$, and

$$\mathbb{P} \left[\inf_{w \notin A_T^x} d_G(x, w) \leq \mathbf{c}_{\text{in}}T \right] \leq \sum_{\ell \leq \mathbf{c}_{\text{in}}T} \sum_{P_\ell \text{ started at } x} \mathbb{P}[T(P_\ell) \geq T] \leq \sum_{\ell \leq \mathbf{c}_{\text{in}}T} \sum_{P_\ell \text{ started at } x} e^{-T \ell} \left(\frac{Te}{\ell} \right)^\ell.$$

Therefore, we obtain

$$\mathbb{P} \left[\inf_{w \notin A_T^x} d_G(x, w) \leq \mathbf{c}_{\text{in}}T \right] \leq \sum_{\ell \leq \mathbf{c}_{\text{in}}T} e^{-T \ell} \left(\frac{\Delta T e}{\ell} \right)^\ell \leq (\mathbf{c}_{\text{in}}T)^2 e^{-T} \left(\frac{\Delta T e}{\mathbf{c}_{\text{in}}T} \right)^{\mathbf{c}_{\text{in}}T},$$

where in the last step we used that $\ell \leq T$. The above can be expressed as

$$(\mathbf{c}_{\text{in}}T)^2 e^{-T} \frac{(\Delta T e)^{\mathbf{c}_{\text{in}}T}}{(\mathbf{c}_{\text{in}}T)^{\mathbf{c}_{\text{in}}T}} = \exp \left\{ 2 \ln(\mathbf{c}_{\text{in}}T) + \mathbf{c}_{\text{in}}T \ln \left(\frac{\Delta}{\mathbf{c}_{\text{in}}} \right) + (-1 + \mathbf{c}_{\text{in}})T \right\},$$

which implies that by taking $\mathbf{c}_{\text{in}} = \mathbf{c}_{\text{in}}(\Delta, c_0) < 1$ small enough, we have $2 \ln(\mathbf{c}_{\text{in}}T) + \mathbf{c}_{\text{in}}T \ln \left(\frac{\Delta}{\mathbf{c}_{\text{in}}} \right) + (-1 + \mathbf{c}_{\text{in}})T < -c_0T$, concluding the proof of the lemma. \square

Remark 2.6. In the proof of Lemma 2.4 we used that the distribution of each t_e is exponential with mean 1. However, the very same proof can be repeated if the $\{t_e\}_{e \in E(G)}$ are i.i.d. exponential random variables with rate λ . In fact, in this case, note that $\{\lambda t_e\}_{e \in E(G)}$ are exponential random variables of rate 1, and hence, (2.7) and (2.8) could be simply replaced by

$$\mathbb{P} \left[A_{T/\lambda}^x \subseteq B_G(x, \mathbf{c}_{\text{out}}T) \right] \geq 1 - e^{-c_1T} \quad \text{and} \quad \mathbb{P} \left[B_G(x, \mathbf{c}_{\text{in}}T) \subseteq A_{T/\lambda}^x \right] \geq 1 - e^{-c_0T},$$

respectively. Another way would be to write

$$\mathbb{P} \left[A_T^x \subseteq B_G(x, \mathbf{c}_{\text{out}}\lambda T) \right] \geq 1 - e^{-c_1\lambda T} \quad \text{and} \quad \mathbb{P} \left[B_G(x, \mathbf{c}_{\text{in}}\lambda T) \subseteq A_T^x \right] \geq 1 - e^{-c_0\lambda T}.$$

Remark 2.7. Note that (2.7) is stronger than (2.8) in the sense that c_1 can be arbitrarily large, while c_0 is restricted to be smaller than 1. The reason for this is that it is much easier to violate the event $\{B_G(x, \mathbf{c}_{\text{in}}T) \subseteq A_T^x\}$ in (2.8), since for this it is enough that all edges adjacent to x have a large passage time. On the other hand, to violate the event $\{A_T^x \subseteq B_G(x, \mathbf{c}_{\text{out}}T)\}$ from (2.7) one needs to have several edges with a small passage time.

2.3 Construction of FPPHE from $\{t_e\}_{e \in E(G)}$.

Let $\{\text{seeds}\}$ denote the set of seeds of FPP_λ ; so $\{\text{seeds}\}$ is the subset of $V(G) \setminus \{o\}$ obtained by adding each vertex independently with probability μ . We start with a collection of independent random variables $\{t_e\}_{e \in E(G)}$, with common distribution $\text{Exp}(1)$, and a collection of *seeds*. Then FPP_1 uses the passage times given by $\{t_e\}_{e \in E(G)}$ to spread from o , activating seeds whenever it tries to occupy vertices already occupied by a seed. Activated seeds of FPP_λ spread using the passage times $\{t_e/\lambda\}_{e \in E(G)}$. In other words, if $x \in V(G)$ gets occupied by FPP_λ at time t , and a neighbor y of x is not occupied, then y gets occupied by FPP_λ at time $t + t_{\{x,y\}}/\lambda$ unless it gets occupied through another edge first.

3 Construction of the embedded tree \mathbb{T}

Recall that \mathbb{T}_3 denotes the embedded tree found in Theorem 2.1. When G is a vertex-transitive graph we can safely assume that $o \in V(\mathbb{T}_3)$, but for general graphs this might not be the case. In order to solve this problem, we consider an ‘‘augmented’’ version of \mathbb{T}_3 in the following sense. We take any d_G -geodesic path between the origin o and the set $V(\mathbb{T}_3)$, and look at it as if it was a (unique) finite extra branch of \mathbb{T}_3 . More precisely, let $o' \in V(\mathbb{T}_3)$ be such that the following relation is satisfied

$$d_G(o, o') = \min_{v \in V(\mathbb{T}_3)} \{d_G(o, v)\}.$$

If the above holds for more than one such o' we just fix one arbitrarily.

Now we proceed to the construction of another embedded tree \mathbb{T} starting from \mathbb{T}_3 . Recall that $\hat{\alpha} \geq 1$ is the bilipschitz constant appearing in (2.4), and recall Theorem 2.1. Let $\gamma_{o,o'}$

denote a d_G -geodesic between o and o' . We now define a tree $\mathbb{T}_3 \cup \gamma_{o,o'}$, whose vertex set is $V(\mathbb{T}_3) \cup \gamma_{o,o'}$, and for every $u \in \gamma_{o,o'}$ and $x \in V(\mathbb{T}_3) \cup \gamma_{o,o'}$ we have

$$d_{\mathbb{T}_3 \cup \gamma_{o,o'}}(u, x) := \begin{cases} d_G(u, o') + d_{\mathbb{T}_3}(o', x) & \text{if } x \in V(\mathbb{T}_3) \\ d_G(u, x) & \text{if } x \in \gamma_{o,o'}. \end{cases} \quad (3.1)$$

It is straightforward to verify that this notation is well defined and there is a bilipschitz embedding of $\mathbb{T}_3 \cup \gamma_{o,o'}$ into G with the same constant $\hat{\alpha}$ that we had before.

We now start the construction of the embedded tree $\mathbb{T} \subset \mathbb{T}_3 \cup \gamma_{o,o'}$. First set the root of \mathbb{T} to be o . Now let

$$\mathbf{r}_1 := \mathbf{r}_1(G, \hat{\alpha}) := \lceil 2\hat{\alpha}d_G(o, o') \rceil,$$

and fix a large integer $\mathbf{r} > 0$ (which will be specified later on) such that $\mathbf{r} \geq \mathbf{r}_1$. Note that, from this choice, we have that $d_G(o, o') \leq \mathbf{r}/2$.

Remark 3.1. In the following we will deal with a sequence of values $\mathbf{r}_1, \dots, \mathbf{r}_9$ all of them are “large enough” so that several constraints are satisfied. For our proof to work it suffices to choose a value $\mathbf{r} \geq \max\{\mathbf{r}_1, \dots, \mathbf{r}_9\}$, and leave it fixed throughout.

Note that if G is transitive then we should go through the same construction with the only simplification that $o' \equiv o$.

Consider two vertices $w, z \in V(\mathbb{T}_3)$ such that the following occur:

$$d_{\mathbb{T}_3}(o', w) = d_{\mathbb{T}_3}(o', z) = \mathbf{r}, \quad \text{and} \quad d_{\mathbb{T}_3}(w, z) = 2\mathbf{r}.$$

(Since o' is now the *root* of \mathbb{T}_3 , we are picking w and z to be two arbitrary vertices in generation \mathbf{r} of \mathbb{T}_3 .) Note that their images via f will be vertices at d_G -distance $r \in [\hat{\alpha}^{-1}\mathbf{r}, \hat{\alpha}\mathbf{r}]$ from the o' . We add w and z to \mathbb{T} as children of o so that the two paths joining o with w and o with z on $\mathbb{T}_3 \cup \gamma_{o,o'}$ correspond to *edges* of \mathbb{T} . We say that w and z together form the “first generation” of \mathbb{T} . Inductively, suppose that we have defined \mathbb{T} up to generation k , for some $k \geq 1$ and denote the vertices at generation k by u_1, \dots, u_{2^k} . Then, for each such u_ℓ , $\ell \in \{1, \dots, 2^k\}$, consider two vertices w_ℓ, z_ℓ of \mathbb{T}_3 such that all the following occur:

- (i) $d_{\mathbb{T}_3}(o', w_\ell) = d_{\mathbb{T}_3}(o', z_\ell) = (k+1)\mathbf{r}$,
- (ii) $d_{\mathbb{T}_3}(w_\ell, z_\ell) = 2\mathbf{r}$,
- (iii) $d_{\mathbb{T}_3}(u_\ell, w_\ell) = d_{\mathbb{T}_3}(u_\ell, z_\ell) = \mathbf{r}$.

The set of vertices $\{w_1, z_1, \dots, w_{2^k}, z_{2^k}\}$ form the $(k+1)$ -th generation of \mathbb{T} . By proceeding inductively in this way we obtain a (rooted) binary tree \mathbb{T} embedded in G whose edges are *almost* geodesic segments, in the sense that their length corresponds to the distance of the two endpoints in G , up to a multiplicative constant. More precisely, for any two vertices $u, v \in V(\mathbb{T})$ we say that u, v are \mathbb{T} -neighbors whenever u and v are neighboring vertices in \mathbb{T} , that is, $d_{\mathbb{T}}(u, v) = 1$. From now on, by “*the embedded tree*” we will always mean \mathbb{T} , unless otherwise stated.

Lemma 3.2. *For \mathbb{T} defined as above there is a bilipschitz embedding of \mathbb{T} into G .*

Proof. From the definition of \mathbf{r} it follows that for any pair of distinct vertices $x, y \in V(\mathbb{T})$ we have the following two possibilities:

- (i) $x = \{o\}$, and $y \in V(\mathbb{T}_3) \setminus \{o'\}$;
- (ii) $x, y \in V(\mathbb{T}_3)$.

We start by showing that there is a constant $\alpha := \alpha(G) \geq \widehat{\alpha} \geq 1$ such that $d_G(x, y) \leq \alpha \mathbf{r} d_{\mathbb{T}}(x, y)$, for all pairs $x, y \in V(\mathbb{T})$. In case (i) we have

$$d_G(x, y) = d_G(o, y) \leq \frac{\mathbf{r}}{2} + \widehat{\alpha} d_{\mathbb{T}_3}(o', y) = \frac{\mathbf{r}}{2} + \widehat{\alpha} \mathbf{r} d_{\mathbb{T}}(o, y) \leq \left(\widehat{\alpha} + \frac{1}{2} \right) \mathbf{r} d_{\mathbb{T}}(o, y).$$

Analogously, in case (ii) we have

$$d_G(x, y) \leq \widehat{\alpha} d_{\mathbb{T}_3}(x, y) \leq \mathbf{r} \widehat{\alpha} d_{\mathbb{T}}(x, y).$$

Now we proceed to show the reversed inequality, that is, there is $\alpha > \widehat{\alpha}$ such that $d_G(x, y) \geq \alpha^{-1} \mathbf{r} d_{\mathbb{T}}(x, y)$, for all pairs $x, y \in V(\mathbb{T})$. We start with case (i) as above.

$$d_G(x, y) = d_G(o, y) \geq |d_G(o, o') - d_G(o', y)|.$$

From the definition of \mathbf{r} it follows that

$$|d_G(o, o') - d_G(o', y)| = d_G(o', y) - d_G(o, o') = d_G(o', y) \left(1 - \frac{d_G(o, o')}{d_G(o', y)} \right).$$

Using the definitions of \mathbb{T} and \mathbf{r} we deduce that

$$d_G(o, o') \frac{1}{d_G(o', y)} \leq \left(\frac{\mathbf{r}}{2\widehat{\alpha}} \right) \frac{1}{\widehat{\alpha}^{-1} d_{\mathbb{T}_3}(o', y)} \leq \frac{\mathbf{r}}{2} \frac{1}{\mathbf{r} d_{\mathbb{T}}(o', y)} \leq \frac{1}{2}.$$

Therefore we have

$$d_G(o, y) \geq d_G(o', y) - d_G(o, o') \geq d_G(o', y) \left(1 - \frac{1}{2} \right) \geq \widehat{\alpha}^{-1} d_{\mathbb{T}_3}(o', y) \frac{1}{2} \geq \frac{\mathbf{r}}{2\widehat{\alpha}} d_{\mathbb{T}}(o, y).$$

Case (ii) is a straightforward consequence of the fact that \mathbb{T}_3 is a bilipschitz embedding, in fact $d_G(x, y) \geq \widehat{\alpha}^{-1} d_{\mathbb{T}_3}(x, y) \geq \widehat{\alpha}^{-1} \mathbf{r} d_{\mathbb{T}}(x, y)$. \square

4 Construction of multiscale and good cylinders

In this section we set up the definitions and methods for an inductive argument that will be carried on to prove Theorem 1.1. In order to proceed, we need to introduce some more notation, including the concept of *cylinder*, which will be used to prove Theorem 1.1.

As before, for any two vertices $x, y \in V(G)$, let $\Gamma_{x,y}$ denote the *set* of geodesics (with respect to the graph distance d_G) that connect x to y on G . Note that this set might have more than one element because we are not assuming that geodesics are unique.

Definition 4.1 (Cylinder). For any two vertices $x, y \in V(G)$ and constant $L \geq 1$ we define the *cylinder* $\mathcal{C}_{x,y}^{(L)}$ as the union of all balls (with respect to the metric d_G) of radius L centered at vertices on any geodesic connecting x with y . More precisely, we set

$$\mathcal{C}_{x,y}^{(L)} := \bigcup_{\gamma \in \Gamma_{x,y}} \bigcup_{w \in \gamma} B_G(w, L).$$

A straightforward application of δ -hyperbolicity implies that for any $x, y \in V(G)$, for any two geodesics γ_1 and γ_2 in $\Gamma_{x,y}$ we have

$$\sup_{u \in \gamma_1} \inf_{v \in \gamma_2} d_G(u, v) \leq \delta. \quad (4.1)$$

Proposition 4.2. *Consider two vertices $x, y \in V(G)$ such that $d_G(x, y) \geq 50\delta$, and take any geodesic $\gamma_{x,y}$ between x and y . Set $d := d_G(x, y)$, and fix an integer value L such that $9\delta \leq L \leq d/5$. Moreover, fix two distinct vertices u, v on $\gamma_{x,y}$ such that both $d_G(x, u)$ and $d_G(y, v)$ are at least $L + 1$ and $d_G(u, v) \geq 2.5L$. Then any path started at x and ended at y that avoids the cylinder $\mathcal{C}_{u,v}^{(L)}$ has length bounded from below by $\frac{1}{3L}\delta d_G(u, v) \cdot 2^{L/(2\delta)}$.*

The above result is just an extension of the fact that detours away from geodesics are exponentially large (cf. Proposition 2.3); we defer the proof to Appendix A.

Remark 4.3. We emphasize that if the path under consideration starts from somewhere in $B_G(x, L)$ and ends somewhere in $B_G(y, L)$ (instead of starting at vertex x and ending at vertex y), then it suffices to modify the result by a multiplicative constant. In fact, by excluding the two balls $B_G(x, L)$ and $B_G(y, L)$, we would obtain a new lower bound

$$\frac{1}{3L}\delta d_G(u, v) \cdot 2^{L/(2\delta)} - 2L \geq \frac{1}{3L}\delta[d_G(u, v) - 2L] \cdot 2^{L/(2\delta)} \geq \frac{1}{12L}\delta d_G(u, v) \cdot 2^{L/(2\delta)}, \quad (4.2)$$

where the last inequality follows because $d_G(u, v) \geq 2.5L$ and $L \leq d/5$.

4.1 First Scale (Scale 1)

Recall that we are dealing with two different FPP processes, FPP_1 and FPP_λ described in the Introduction, and that by “seeds” we mean the starting points of FPP_λ . Recall also that $T(P)$ is the passage time of a path P with respect to the passage times $\{t_e\}_{e \in E(G)}$, which are of rate 1 and are used in the construction of FPPHE, and that A_t^w is the ball of radius t centered at $w \in V(G)$ according to (the random metric induced by) $t_e, e \in E(G)$. We then define

$$A_t^{w, \max} := \text{ball of radius } t \text{ centered at } w \text{ according to } \max\{1, \lambda\}t_e, e \in E(G) \quad (4.3)$$

and

$$A_t^{w, \min} := \text{ball of radius } t \text{ centered at } w \text{ according to } \min\{1, \lambda\}t_e, e \in E(G). \quad (4.4)$$

Choose a constant $\varepsilon > 0$ arbitrarily small which will be kept fixed throughout, and recall that $\mathbf{r} \geq 1$ is a fixed integer used in the definition of \mathbb{T} . We will take \mathbf{r} to be large enough with respect to ε . Now, for $w \in V(G)$ let

$$\mathcal{P}_w := \{\text{all self-avoiding and finite } G\text{-paths starting from } w\},$$

and for $P \in \mathcal{P}_w$ let $|P|$ denote its d_G -length.

Consider any pair $x, y \in V(\mathbb{T})$ and denote by $\mathbf{j} := d_{\mathbb{T}}(x, y)$. By the bilipschitz embedding, this implies (cf. Lemma 3.2) that there is a value $\alpha \geq \hat{\alpha} \geq 1$ such that $d_G(x, y) \in [\alpha^{-1}\mathbf{j}\mathbf{r}, \alpha\mathbf{j}\mathbf{r}]$. Recall the constants \mathbf{c}_{in} and \mathbf{c}_{out} from Lemma 2.4, and define the event

$$\mathcal{G}_1(x, y; \mathbf{j}\mathbf{r}) := \left\{ \begin{array}{l} \forall t \in \left[\varepsilon^{1/2}\mathbf{j}\mathbf{r}, 4\frac{\mathbf{c}_{\text{out}}}{\mathbf{c}_{\text{in}}^2}\mathbf{j}\mathbf{r} \right] \cap \mathbb{Z}, \text{ and } \forall w \in \mathcal{C}_{x,y}^{(\varepsilon\mathbf{j}\mathbf{r})} \text{ we have} \\ a) B_G(w, \min\{1, \lambda\}\mathbf{c}_{\text{in}}t) \subseteq A_t^{w, \min} \subseteq A_t^{w, \max} \subseteq B_G(w, \max\{1, \lambda\}\mathbf{c}_{\text{out}}t) \\ b) \forall P \in \mathcal{P}_w \text{ with } \sqrt{\varepsilon}\mathbf{c}_{\text{out}}\mathbf{j}\mathbf{r} \leq |P| \leq 4\frac{\mathbf{c}_{\text{out}}^2}{\mathbf{c}_{\text{in}}}\mathbf{j}\mathbf{r}, \text{ we have } \frac{|P|}{\mathbf{c}_{\text{out}}} \leq T(P) \end{array} \right\}. \quad (4.5)$$

Note that the dependence on $\mathbf{j}\mathbf{r}$ is written in order to avoid confusion, but the notation is redundant as this quantity is fixed once we fix the pair x and y .

For any pair $x, y \in V(\mathbb{T})$ of \mathbb{T} -neighboring vertices, paths satisfying condition b) described in $\mathcal{G}_1(x, y; \mathbf{r})$ for some initial vertex $w \in \mathcal{C}_{x,y}^{(\varepsilon\mathbf{r})}$ will be called *typical* at scale 1.

Definition 4.4 (Good cylinders, scale 1). For \mathbf{c}_{in} and \mathbf{c}_{out} as in Lemma 2.4 and for any pair of \mathbb{T} -neighboring vertices x, y we define the cylinder $\mathcal{C}_{x,y}^{(\varepsilon\mathbf{r})}$ to be *good at scale 1* if the following two conditions are satisfied:

- (i) The first condition is that the event $\mathcal{G}_1(x, y; \mathbf{r})$ holds cf. (4.5) with $\mathbf{j} = 1$.
- (ii) Choose a constant β so that

$$\beta := (6 + \varepsilon)(1 + \alpha)\alpha^2 \frac{\mathbf{c}_{\text{out}}}{\mathbf{c}_{\text{in}}} \max\{\lambda, \lambda^{-1}\}. \quad (4.6)$$

Then the second requirement is that the event $\mathcal{G}_2(x, y)$ below holds:

$$\mathcal{G}_2(x, y) := \left\{ \mathcal{C}_{x,y}^{(\varepsilon\mathbf{r} + \beta\mathbf{r})} \cap \{\text{seeds}\} = \emptyset \right\}.$$

In words, condition (i) requires that for all integer times t in the interval $\left[\varepsilon^{1/2}\mathbf{r}, 4\frac{\mathbf{c}_{\text{out}}}{\mathbf{c}_{\text{in}}}\mathbf{r} \right]$, for each vertex w of the cylinder the set of vertices reached by a FPP process of rate λ or 1 started at w contains a ball of radius $\min\{1, \lambda\}\mathbf{c}_{\text{in}}t$ and is contained inside a ball of radius $\max\{1, \lambda\}\mathbf{c}_{\text{out}}t$. Furthermore, the passage time $T(P)$ of any path P is not too short. Note that a path P has an expected passage time of $|P|$, and we use the factor $1/\mathbf{c}_{\text{out}}$ to get the event to hold with high probability for all such paths. Part b) of event $\mathcal{G}_1(x, y; \mathbf{r})$ will be used to show that long detours away from geodesics cannot happen, in particular, if FPP_1 tries to deviate from $\gamma_{x,y}$ inside a good cylinder $\mathcal{C}_{x,y}^{(\varepsilon\mathbf{r})}$, the time to traverse such detour cannot be too small.

The choice of the radius of the cylinder in $\mathcal{G}_2(x, y)$ is technical and will become clear later on (cf. the proof of Theorem 1.1 in Section 6).

We first aim to show that for a careful choice of the parameters the probability that a given cylinder is *good at scale 1* is high. Recall that $x, y \in V(\mathbb{T})$. Our first result shows that the event $\mathcal{G}_2(x, y)$ occurs with high probability, uniformly for every choice of \mathbb{T} -neighboring vertices x and y .

Lemma 4.5. *For any $\mathbf{r} \geq 1$ and constant $c_2 > 0$, there is a $\mu_0 := \mu_0(c_2, \Delta, \varepsilon, \beta, \delta, \alpha, \mathbf{r}) > 0$ small enough such that for all $\mu < \mu_0$ and all \mathbb{T} -neighboring vertices x, y , we have*

$$\mathbb{P}[\mathcal{G}_2(x, y)] \geq 1 - e^{-c_2\mathbf{r}}.$$

Before proving this result, we state a fundamental fact. Recall the bounds determined in (2.5), then we have that each cylinder $\mathcal{C}_{x,y}^{(\varepsilon\mathbf{r})}$ is such that

$$\left| \mathcal{C}_{x,y}^{(\varepsilon\mathbf{r})} \right| \leq \Delta^{(\varepsilon\mathbf{r} + \delta)} \alpha \mathbf{r}. \quad (4.7)$$

This bound holds due to the following. Each vertex inside the cylinder has maximum degree Δ . By the bilipschitz embedding (Lemma 3.2), if $d_{\mathbb{T}}(x, y) = 1$, then $\alpha^{-1}\mathbf{r} \leq d_G(x, y) \leq \alpha\mathbf{r}$. Each pair of vertices is joined by geodesics that are not necessarily unique, but by δ -hyperbolicity two geodesics joining x and y are at distance at most δ from each other (cf. Equation (4.1)). Thus, it follows that by fixing a geodesic $\gamma_{x,y}$, then $\mathcal{C}_{x,y}^{(\varepsilon\mathbf{r})} \subset \cup_{w \in \gamma_{x,y}} B_G(w, \varepsilon\mathbf{r} + \delta)$.

Proof of Lemma 4.5. Recall the definition of β from (4.6) and that the value of \mathbf{r} is fixed. Then, we can choose $\mu_0 := \mu_0(c_2, \Delta, \varepsilon, \beta, \delta, \alpha, \mathbf{r})$ such that for all $\mu < \mu_0$ the cylinder $\mathcal{C}_{x,y}^{(\varepsilon\mathbf{r} + \beta\mathbf{r})}$ does not contain seeds with high probability. More precisely, given $c_2 > 0$ we can choose μ_0 such that for all $\mu < \mu_0$ we have $\Delta^{\varepsilon\mathbf{r} + \beta\mathbf{r} + \delta} \alpha \mathbf{r} \mu < e^{-c_2\mathbf{r}}$. In this way, by using the union bound we obtain $\mathbb{P}[\mathcal{G}_2^c(x, y)] \leq \left| \mathcal{C}_{x,y}^{(\varepsilon\mathbf{r} + \beta\mathbf{r})} \right| \mu \stackrel{(2.5), (4.7)}{\leq} \Delta^{\varepsilon\mathbf{r} + \beta\mathbf{r} + \delta} \alpha \mathbf{r} \mu. \quad \square$

The next result states that paths that are typical at scale 1 are very likely.

Lemma 4.6. *Recall the constants c_0 and c_1 from Lemma 2.4. For any $c_0 \in (0, 1)$ and $c_1 > 0$, there exists $c_3 > 0$, a small enough $\varepsilon = \varepsilon(c_0, c_1, \Delta) > 0$ and a large enough $\mathbf{r}_2 = \mathbf{r}_2(c_3, \varepsilon, \alpha)$ so that for all $\mathbf{r} > \mathbf{r}_2$ and all $x, y \in V(\mathbb{T})$, $\mathbf{j} := d_{\mathbb{T}}(x, y)$, we have*

$$\mathbb{P}[\mathcal{G}_1^c(x, y; \mathbf{j}\mathbf{r})] \leq e^{-c_3\sqrt{\varepsilon}\mathbf{j}\mathbf{r}}.$$

Moreover, $\mathcal{G}_1(x, y; \mathbf{j}\mathbf{r})$ is measurable with respect to the passage times in $\mathcal{C}_{x,y}^{\lceil \varepsilon\mathbf{j}\mathbf{r} + 4\max\{1, \lambda\} \frac{c_{\text{out}}^2}{c_{\text{in}}^2} \mathbf{j}\mathbf{r} \rceil + 1}$.

Proof. From Lemma 2.4 it follows that for any $w \in \mathcal{C}_{x,y}^{(\varepsilon\mathbf{j}\mathbf{r})}$ and $t \in \left[\lceil \varepsilon^{1/2} \mathbf{j}\mathbf{r} \rceil, 4 \frac{c_{\text{out}}^2}{c_{\text{in}}^2} \mathbf{j}\mathbf{r} \right]$

$$\mathbb{P}\left(\{B_G(w, \min\{1, \lambda\}c_{\text{in}}t) \not\subseteq A_t^{w, \min}\} \cup \{A_t^{w, \max} \not\subseteq B_G(w, \max\{1, \lambda\}c_{\text{out}}t)\}\right) \leq e^{-c_1t} + e^{-c_0t}.$$

Furthermore, the first part of Lemma 2.5 shows that for any path of length ℓ such that $\varepsilon^{1/2}c_{\text{out}}\mathbf{j}\mathbf{r} \leq \ell \leq 4 \frac{c_{\text{out}}^2}{c_{\text{in}}^2} \mathbf{j}\mathbf{r}$, we have that for any fixed $w \in \mathcal{C}_{x,y}^{(\varepsilon\mathbf{j}\mathbf{r})}$

$$\begin{aligned} \mathbb{P}\left[\exists P \in \mathcal{P}_w \text{ with } |P| = \ell \text{ such that } T(P) < \frac{\ell}{c_{\text{out}}}\right] &\leq \Delta \ell 2 \frac{e^{-\ell/c_{\text{out}}} (\ell/c_{\text{out}})^\ell}{\ell!} \\ &\leq 2e^{-\ell/c_{\text{out}}} \left(\frac{\Delta \ell}{c_{\text{out}}}\right)^\ell. \end{aligned}$$

Now, for any c_1 , one can choose c_{out} large enough so that the above is at most $e^{-c_1\ell}$. Thus, by the union bound over all possible starting points $w \in \mathcal{C}_{x,y}^{(\varepsilon\mathbf{j}\mathbf{r})}$, as well as all possible values of ℓ and t :

$$\mathbb{P}[\mathcal{G}_1^c(x, y; \mathbf{j}\mathbf{r})] \leq \left| \mathcal{C}_{x,y}^{(\varepsilon\mathbf{j}\mathbf{r})} \right| \left[\sum_{\ell=\sqrt{\varepsilon}\mathbf{j}\mathbf{r}}^{4 \frac{c_{\text{out}}^2}{c_{\text{in}}^2} \mathbf{j}\mathbf{r}} e^{-c_1\ell} + \sum_{t=\sqrt{\varepsilon}\mathbf{j}\mathbf{r}}^{4 \frac{c_{\text{out}}^2}{c_{\text{in}}^2} \mathbf{j}\mathbf{r}} (e^{-c_1t} + e^{-c_0t}) \right] \leq \left(\alpha \mathbf{j}\mathbf{r} \Delta^{\varepsilon\mathbf{j}\mathbf{r} + \delta}\right) e^{-2c_3\sqrt{\varepsilon}\mathbf{j}\mathbf{r}},$$

for some constant $c_3 > 0$ depending on c_0 and c_1 . Now taking $\varepsilon > 0$ small enough, and then \mathbf{r} large enough establishes the first part of the lemma. The second part follows directly from the definition, since we have that $A_t^{w, \max} \subseteq B_G(w, \max\{1, \lambda\}c_{\text{out}}t)$, which means that we only need to check all passage times inside the (finite) ball $B_G(w, \max\{1, \lambda\}c_{\text{out}}t)$. \square

The next lemma shows that cylinders at scale 1 are very likely to be good.

Lemma 4.7. *Recall the constants c_0 and c_1 from Lemma 2.4. There is a constant $c_4 = c_4(c_0, c_1, \varepsilon) > 0$ so that for all $\mathbf{r} \geq \mathbf{r}_2$, where \mathbf{r}_2 is from Lemma 4.6, there exists a $\mu_0 > 0$ for which whenever $\mu < \mu_0$ we have that for all \mathbb{T} -neighboring vertices x, y ,*

$$\mathbb{P}\left[\mathcal{C}_{x,y}^{(\varepsilon\mathbf{r})} \text{ is good at scale } 1\right] \geq 1 - e^{-c_4\mathbf{r}}.$$

Proof. The definition of good cylinder at scale 1 implies that for any fixed pair of \mathbb{T} -neighbors $x, y \in V(\mathbb{T})$ we have $\mathbb{P}\left[\mathcal{C}_{x,y}^{(\varepsilon\mathbf{r})} \text{ is not good}\right] \leq \mathbb{P}[\mathcal{G}_1^c(x, y; \mathbf{r})] + \mathbb{P}[\mathcal{G}_2^c(x, y)]$. The lemma follows from Lemmas 4.5 and 4.6. \square

4.2 Higher scales

In this section we define what *higher scales* are, and what a *good cylinder at a higher scale* is. These concepts will be used in Section 6 to show survival of FPP_1 .

Definition 4.8 (Good cylinders, higher scales). Recall the definition of a cylinder from Definition 4.1. Consider two distinct and not \mathbb{T} -neighboring vertices $x, y \in V(\mathbb{T})$ and let $\mathbf{j} := d_{\mathbb{T}}(x, y)$. Note that $\mathbf{j} \geq 2$. Then the cylinder $\mathcal{C}_{x,y}^{(\varepsilon\mathbf{j}\mathbf{r})}$ is *good at scale $\mathbf{j} \geq 2$* if the event $\mathcal{G}_1(x, y; \mathbf{j}\mathbf{r})$ defined in (4.5) is realized.

We emphasize that Lemma 4.6 implies that for all $x, y \in V(\mathbb{T})$ with $\mathbf{j} = d_{\mathbb{T}}(x, y) \geq 2$

$$\mathbb{P} \left[\mathcal{C}_{x,y}^{(\varepsilon\mathbf{j}\mathbf{r})} \text{ is good at scale } \mathbf{j} \right] = \mathbb{P}[\mathcal{G}_1(x, y; \mathbf{j}\mathbf{r})] \geq 1 - e^{-c_3\sqrt{\varepsilon}\mathbf{j}\mathbf{r}} \geq 1 - e^{-c_4\mathbf{j}\mathbf{r}}.$$

5 Good \mathbb{T} -paths

In what follows, we will need the definition of a *good \mathbb{T} -path* (such path is to be found on the tree \mathbb{T}). Roughly speaking, a sequence of \mathbb{T} -neighboring vertices $\{v_i\}_{i \geq 0} \in V(\mathbb{T})$ (where we set $v_0 := o$) is a good \mathbb{T} -path if it is covered by good cylinders at all scales.

5.1 Definition and properties of Good \mathbb{T} -paths

For any infinite \mathbb{T} -path Ξ we define

$$\{\Xi \text{ is good}\} := \bigcap_{k=1}^{\infty} \bigcap_{\substack{u, v \in \Xi \\ d_{\mathbb{T}}(u, v) = k}} \left\{ \mathcal{C}_{u,v}^{(\varepsilon k\mathbf{r})} \text{ is good} \right\}. \quad (5.1)$$

Now suppose that we have found an infinite good \mathbb{T} -path Ξ (we will show in Section 5.2 that indeed it exists), and fix it throughout this section. In the following, we set

$$V(\Xi) := \left\{ v \in V(G) : v \in (\Xi \cup V(E(\Xi))) \right\}, \quad (5.2)$$

where $V(E(\Xi))$ is the set of vertices in $V(G)$ that lie on the “edges” of Ξ ; more precisely, we define the set $V(E(\Xi))$ as follows. For any pair of \mathbb{T} -neighboring vertices $w, z \in V(\mathbb{T})$, there is a geodesic (with respect to $d_{\mathbb{T}_3}$) path on the binary tree \mathbb{T}_3 that connects w and z , denote it by $\gamma_{\mathbb{T}_3}(w, z)$. (Note that $d_{\mathbb{T}_3}(w, z) = \mathbf{r}$.) For all $w, z \in V(\mathbb{T}_3)$ consider the image of the geodesic $\gamma_{\mathbb{T}}(w, z)$ via the bilipschitz embedding. This gives a sequence of $d_{\mathbb{T}_3}$ -adjacent vertices, and each pair can be connected by a d_G -geodesic path. The concatenation of such d_G -geodesic paths is itself a path in G , denote it by $\pi_G(w, z)$. Therefore for all pairs w, z such that $d_{\mathbb{T}}(w, z) = 1$ we define

$$V(w, z) := \{v \in G : v \in \pi_G(w, z)\},$$

and finally we set

$$V(E(\Xi)) := \bigcup_{w, z \in \Xi, d_{\mathbb{T}}(w, z) = 1} V(w, z).$$

For a graphical representation see Figure 2.

Once we have fixed the path Ξ , we define

$$\mathcal{C}_{o, \infty}^{(\varepsilon\mathbf{r}/2)} := \bigcup_{v \in V(\Xi)} B_G \left(v, \frac{\varepsilon\mathbf{r}}{2} \right).$$

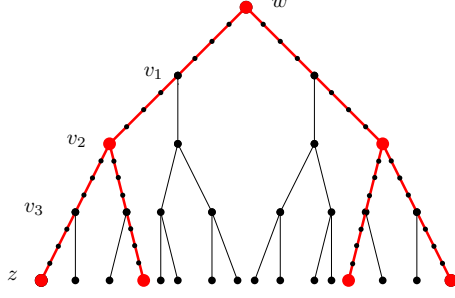


Figure 2: Red vertices such as w, v_2 and z are those on \mathbb{T} , large black vertices on a red path such as v_1, v_3 are those on \mathbb{T}_3 , whereas the smallest vertices are those of G . Here $\mathbf{r} = 2$.

Note that this quantity (and those defined throughout this section) depends on Ξ , but we omit this dependence in order to make the notation less heavy. For any vertex $z \in \Xi$ define

$$N(z) := \left\{ w \in \mathcal{C}_{0,\infty}^{(\varepsilon\mathbf{r}/2)} : \forall z' \in \Xi \text{ we have } d_G(w, z) \leq d_G(w, z') \right\},$$

with ties broken according to an arbitrary rule so that $N(z), z \in \Xi$, forms a partition of $\mathcal{C}_{0,\infty}^{(\varepsilon\mathbf{r}/2)}$. In words, for any $z \in \Xi$, $N(z)$ is the set of all vertices inside $\mathcal{C}_{0,\infty}^{(\varepsilon\mathbf{r}/2)}$ that are closer to z than to any other vertex of the tree \mathbb{T} .

At this point we use a fundamental fact, namely that for large enough \mathbf{r} , for any vertex $x \in \Xi$ there are vertices $u', v' \in \Xi$ such that

$$N(x) \subset \mathcal{C}_{u',v'}^{(2\varepsilon\mathbf{r}/3)}; \quad (5.3)$$

in particular, u' and v' can be the vertices on Ξ before and after x , which from now on we denote simply by u and v . Therefore for all vertices $x \in \Xi$ we can define

$$\mathcal{C}(N(x)) := \mathcal{C}_{u,v}^{(2\varepsilon\mathbf{r})}. \quad (5.4)$$

Furthermore, note that we have $d_G(N(x), \partial\mathcal{C}(N(x))) \geq \frac{4}{3}\varepsilon\mathbf{r}$. However, later on we will simply use the fact that $d_G(N(x), \partial\mathcal{C}(N(x))) \geq \varepsilon\mathbf{r}/2$.

Remark 5.1. Note that by construction it follows that the vertices $u, v \in \Xi$ defined above are such that $d_{\mathbb{T}}(u, v) = 2$, thus, the event $\{\mathcal{C}(N(x)) \text{ is good}\}$ is contained in the event $\{\text{all cylinders } \mathcal{C}_{w,z}^{(2\varepsilon\mathbf{r})} \text{ with } w, z \in \Xi \text{ and } d_{\mathbb{T}}(w, z) = 2 \text{ are good}\}$.

Now we proceed with the following technical result. Recall the definition of the internal boundary of a set, given in (2.1), and the result from Proposition 2.2.

Lemma 5.2. *Let $\kappa = \kappa(\alpha, \delta)$ be the constant appearing in Proposition 2.2, and let $\mathbf{r}_3 = \mathbf{r}_3(\varepsilon, \delta, \kappa)$ be defined so that $\mathbf{r}_3 \geq \frac{2}{\varepsilon}(3\delta + \kappa)$. Then for all $\mathbf{r} \geq \mathbf{r}_3$, for any pair of vertices $x, y \in \Xi$, any two points $\mathbf{a} \in \partial N(x)$ and $\mathbf{b} \in \partial N(y)$, all d_G -geodesics $\gamma_{\mathbf{a},\mathbf{b}} \in \Gamma_{\mathbf{a},\mathbf{b}}$ are completely contained inside the set $\mathcal{C}_{\mathbf{a},x}^{(\varepsilon\mathbf{r}/2)} \cup \mathcal{C}_{x,y}^{(\varepsilon\mathbf{r}/2)} \cup \mathcal{C}_{y,\mathbf{b}}^{(\varepsilon\mathbf{r}/2)}$. Furthermore, the set $\mathcal{C}_{\mathbf{a},x}^{(\varepsilon\mathbf{r}/2)} \cup \mathcal{C}_{x,y}^{(\varepsilon\mathbf{r}/2)} \cup \mathcal{C}_{y,\mathbf{b}}^{(\varepsilon\mathbf{r}/2)}$ also contains the (unique) $d_{\mathbb{T}}$ -geodesic $\gamma_{\mathbb{T}}(x, y)$.*

Proof. In order to proceed, we need to make use of the fact that triangles are δ -thin. More precisely, let \mathbf{a}' and \mathbf{b}' be any two projections (that can be chosen arbitrarily if they are not unique) on the geodesic segment $\gamma_{x,y}$ of the vertices \mathbf{a} and \mathbf{b} respectively. In formulas:

$$\begin{aligned} \mathbf{a}' &:= \text{any vertex } v \in \gamma_{x,y} \text{ such that } d_G(\mathbf{a}, v) = d_G(\mathbf{a}, \gamma_{x,y}); \\ \mathbf{b}' &:= \text{any vertex } v \in \gamma_{x,y} \text{ such that } d_G(\mathbf{b}, v) = d_G(\mathbf{b}, \gamma_{x,y}). \end{aligned}$$

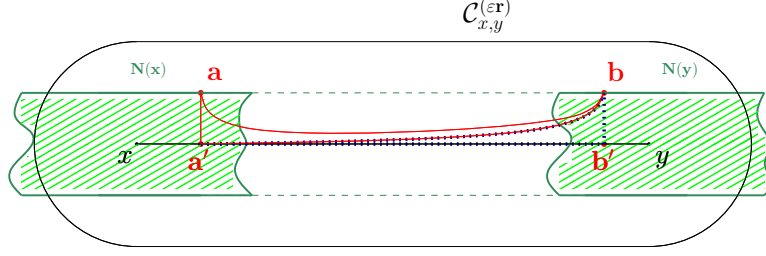


Figure 3: The sets $N(x)$ and $N(y)$ are drawn in green (and they exit $\mathcal{C}_{x,y}^{(\epsilon r)}$). The triangles described above are drawn with a thin red line and a blue thick dotted line respectively.

Now consider the two triangles $\{\mathbf{a}, \mathbf{a}', \mathbf{b}\}$ and $\{\mathbf{a}', \mathbf{b}, \mathbf{b}'\}$ (cf. Figure 3). These are δ -thin. A direct consequence of this fact is that any (arbitrarily chosen) geodesic $\gamma_{\mathbf{a}, \mathbf{b}}$ is contained inside the union of balls of radius δ centered at $\Gamma_{\mathbf{a}, \mathbf{a}'} \cup \Gamma_{\mathbf{a}', \mathbf{b}}$. Using again δ -thinness, we have that any geodesic $\gamma_{\mathbf{a}', \mathbf{b}} \in \Gamma_{\mathbf{a}', \mathbf{b}}$ has to be contained inside the union of the balls of radius δ centered at $\Gamma_{\mathbf{a}', \mathbf{b}'} \cup \Gamma_{\mathbf{b}', \mathbf{b}}$. Thus, it immediately follows that $\gamma_{\mathbf{a}, \mathbf{b}}$ is contained inside the union of balls of radius 2δ centered at $\Gamma_{\mathbf{a}, \mathbf{a}'} \cup \Gamma_{\mathbf{a}', \mathbf{b}'} \cup \Gamma_{\mathbf{b}', \mathbf{b}}$. A straightforward consequence of this is that any geodesic $\gamma_{\mathbf{a}, \mathbf{b}}$ is such that

$$\gamma_{\mathbf{a}, \mathbf{b}} \subset \mathcal{C}_{\mathbf{a}, \mathbf{a}'}^{(2\delta)} \cup \mathcal{C}_{\mathbf{a}', \mathbf{b}'}^{(2\delta)} \cup \mathcal{C}_{\mathbf{b}', \mathbf{b}}^{(2\delta)}.$$

By construction, this union is contained inside $\mathcal{C}_{\mathbf{a}, x}^{(3\delta)} \cup \mathcal{C}_{x, y}^{(3\delta)} \cup \mathcal{C}_{y, \mathbf{b}}^{(3\delta)}$ which, together with $\mathbf{r} \geq \mathbf{r}_3$, gives the first part of the statement. The final sentence of the statement follows from Proposition 2.2. \square

5.2 Existence of a good \mathbb{T} -path

The next step consists in showing that with positive probability there is an infinite good \mathbb{T} -path Ξ containing the origin o .

Proposition 5.3. *Recall the definition of good path from (5.1), and assume that \mathbf{r} is large enough. Then, $\mathbb{P}[\text{There is an infinite } \mathbb{T}\text{-path } \Xi \text{ containing } o \text{ and such that } \Xi \text{ is good}] > 0$.*

Before proceeding to the proof, we need to introduce some terminology.

Cutsets. We refer to the term *cutset* whenever we mean a subset of vertices of \mathbb{T} that separates the root o from infinity. In particular, we will need the so-called *minimal* cutsets, which are cutsets that do not have any subsets which are cutsets themselves. More precisely, a cutset Π_k of cardinality k is such that

$$\Pi_k \subseteq V(\mathbb{T}) : |\Pi_k| = k, \text{ and any infinite path containing } o \text{ passes through } \Pi_k.$$

Furthermore, Π_k is a *minimal* cutset of cardinality k if

$$\Pi_k \text{ is a cutset, } |\Pi_k| = k, \text{ and for all subsets } S \subsetneq \Pi_k, S \text{ is not a cutset.}$$

Consider the tree \mathbb{T} rooted at o . Since \mathbb{T} has no cycles, any vertex of the tree is a cutpoint, i.e., its removal separates the tree into two disjoint connected components. Note that if we remove a minimal cutset from \mathbb{T} , the external boundary of the finite connected component containing o is the minimal cutset. Therefore, we can identify any minimal cutset with the internal boundary of a finite induced subtree of \mathbb{T} containing o . Since the boundary of this subtree is the set of its

leaves, all minimal cutsets of cardinality k (for any $k \geq 1$) correspond to the boundary of some *rooted* induced subtree of \mathbb{T} that has exactly k leaves. The next lemma finds an upper bound on the number of minimal cutsets of cardinality k .

Lemma 5.4. *For all $k \geq 1$ we have*

$$\left| \{ \Pi_k : \Pi_k \text{ is a minimal cutset of } \mathbb{T} \text{ of cardinality } k \} \right| < 4^{k-1}.$$

Proof. Start by recalling the definition of *Catalan number*: set $C_0 := 1$ and for all $n \geq 1$ set $C_n := \frac{1}{n+1} \binom{2n}{n}$. From the definition, for all $n \geq 2$, C_n satisfies the recursive relation $\frac{C_n}{C_{n-1}} = \frac{4n-2}{n+1} < 4$. Hence,

$$C_n \leq 4^n, \text{ for all } n \geq 1. \quad (5.5)$$

In the case of a binary tree, a minimal cutset Π_k is the (internal) boundary of a rooted (binary) tree which has exactly k leaves. Thus, by our previous discussion, the number of such Π_k 's is known to be the Catalan number C_{k-1} . Hence, for all $k \geq 1$ we have

$$\left| \{ \Pi_k : \Pi_k \text{ is a minimal cutset of } \mathbb{T} \text{ of cardinality } k \} \right| = C_{k-1} \stackrel{(5.5)}{<} 4^{k-1},$$

finishing the proof. □

We proceed now with a proof of the proposition.

Proof of Proposition 5.3. Consider two vertices $x, y \in V(\mathbb{T})$ and let $\mathbf{j} := d_{\mathbb{T}}(x, y)$, and suppose that $\mathcal{C}_{x,y}^{(\varepsilon \mathbf{j}^{\mathbf{r}})}$ is bad. To simplify the notation, we assume that x is an ancestor of y , and denote

$$\mathbf{i} := d_{\mathbb{T}}(o, x);$$

thus $\mathbf{i} + \mathbf{j} = d_{\mathbb{T}}(o, y)$. Moreover, for any vertex $v \in \mathbb{T}$ and integer $n \geq 0$ we denote by $\mathbf{u}(v, n)$ the ancestor of v at generation n , and we set this to be o , if such a vertex does not exist. More precisely, $\mathbf{u}(v, n)$ is the vertex in $V(\mathbb{T})$ satisfying the following properties:

- $d_{\mathbb{T}}(o, \mathbf{u}(v, n)) = n$, with $\mathbf{u}(v, n) := o$ whenever $d_{\mathbb{T}}(o, v) \leq n$;
- $\mathbf{u}(v, n)$ belongs to the shortest path from o to v .

To determine whether a cylinder $\mathcal{C}_{x,y}^{(\varepsilon \mathbf{j}^{\mathbf{r}})}$ is good or not, we need to observe all passage times contained in the (larger) set

$$\mathcal{C}_{x,y}^{(\lceil \varepsilon + 4 \max\{1, \lambda\} \frac{c_{\text{out}}^2}{c_{\text{in}}^2} \rceil \mathbf{j}^{\mathbf{r}} + 1)} \subseteq \mathcal{C}_{x,y}^{(\lceil \varepsilon + 4 \max\{1, \lambda\} \frac{c_{\text{out}}^2}{c_{\text{in}}^2} + 1 \rceil \mathbf{j}^{\mathbf{r}})}.$$

This is a consequence of the second part of Lemma 4.6. In order to make the notation less heavy, we set

$$\eta := \left\lceil \varepsilon + 4 \max\{1, \lambda\} \frac{c_{\text{out}}^2}{c_{\text{in}}^2} + 1 \right\rceil. \quad (5.6)$$

For every pair of vertices x, y as described above, for which $\mathcal{C}_{x,y}^{(\varepsilon \mathbf{j}^{\mathbf{r}})}$ is bad, we declare that the entire (infinite) subtree rooted at $\mathbf{u}(x, \mathbf{i} - 3\alpha^2 \eta \mathbf{j})$ is also bad, that is, all descendants of $\mathbf{u}(x, \mathbf{i} - 3\alpha^2 \eta \mathbf{j})$ (including itself) are declared to be bad. Throughout this proof we will say that we will *remove* (or *discard*) the root $\mathbf{u}(x, \mathbf{i} - 3\alpha^2 \eta \mathbf{j})$ and when this happens we also remove/discard the entire subtree. Recall that for all $x \in V(\mathbb{T})$ such that $\mathbf{i} - 3\alpha^2 \eta \mathbf{j} \leq 0$ we set $\mathbf{u}(x, \mathbf{i} - 3\alpha^2 \eta \mathbf{j}) := o$.

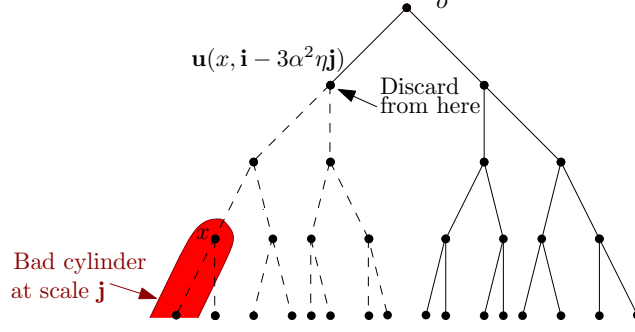


Figure 4: A graphical representation of the definition of $\mathbf{u}(x, \mathbf{i} - 3\alpha^2\eta\mathbf{j})$. The dashed subtree is the one that will be discarded due to the bad cylinder $\mathcal{C}_{x,y}^{(\varepsilon\mathbf{j}\mathbf{r})}$.

Any vertex of \mathbb{T} has at most $2^{\lceil 3\alpha^2\eta \rceil \mathbf{j}}$ descendants at $d_{\mathbb{T}}$ -distance $\lceil 3\alpha^2\eta \rceil \mathbf{j}$. This implies that for any given $v \in V(\mathbb{T}) \setminus \{o\}$ there are at most $2^{\lceil 3\alpha^2\eta \rceil \mathbf{j}}$ vertices x such that $d_{\mathbb{T}}(o, x) = \mathbf{i} \geq \lceil 3\alpha^2\eta \rceil \mathbf{j}$ and $v = \mathbf{u}(x, \mathbf{i} - 3\alpha^2\eta\mathbf{j})$. For a graphical representation see Figure 5.2.

By Lemmas 4.7 and 4.6, there is a constant c_5 such that for all $\mathbf{j} \geq 1$ and all large enough \mathbf{r} we have

$$\mathbb{P}[\text{a given cylinder at scale } \mathbf{j} \text{ is not good}] \leq e^{-c_5\mathbf{j}\mathbf{r}}.$$

Thus, by the union bound over all possible bad “scale- \mathbf{j} cylinders” that could cause any vertex $v \in V(\mathbb{T}) \setminus \{o\}$ to be discarded (as the root of an infinite induced subtree as described above) we obtain

$$\mathbb{P}[\text{vertex } v \in V(\mathbb{T}) \setminus \{o\} \text{ is discarded due to a bad cylinder at scale } \mathbf{j}] \leq 2^{3\alpha^2\eta\mathbf{j}} e^{-c_5\mathbf{j}\mathbf{r}}. \quad (5.7)$$

In particular, this quantity can be made arbitrarily small by choosing \mathbf{r} very large, as we argue below. The above holds for any vertex that is not o , since there are more vertices x for which $o = \mathbf{u}(x, \mathbf{i} - 3\alpha^2\eta\mathbf{j})$. The number of such vertices x is at most $2^{3\alpha^2\eta\mathbf{j}+1}$, the size of a complete binary tree of depth $3\alpha^2\eta\mathbf{j}$. Thus, by taking also the union bound over all values of $\mathbf{j} \geq 1$, relation (5.7) implies that

$$\mathbb{P}[\text{a given vertex } v \in V(\mathbb{T}) \text{ is removed}] \leq \sum_{\mathbf{j} \geq 1} 2^{3\alpha^2\eta\mathbf{j}+1} e^{-c_5\mathbf{j}\mathbf{r}}.$$

At this point for any given constant $c_6 > 0$ arbitrarily large, we can pick a value $\mathbf{r}_4 = \mathbf{r}_4(\alpha, \eta, c_5, c_6) > 0$ so large that for all $\mathbf{r} \geq \mathbf{r}_4$

$$\mathbb{P}[\text{a given vertex } v \in V(\mathbb{T}) \text{ is removed}] \leq \sum_{\mathbf{j} \geq 1} 2^{3\alpha^2\eta\mathbf{j}+1} e^{-c_5\mathbf{j}\mathbf{r}} < e^{-c_6}.$$

Now the next claim shows on each branch of \mathbb{T} vertices are removed independently of one another.

Claim 5.5. *Consider any two vertices $u, v \in V(\mathbb{T})$, such that*

- *v is not an ancestor of u (that is, $v \neq \mathbf{u}(u, n)$, for all $n \geq 0$) and*
- *u is not an ancestor of v (that is, $u \neq \mathbf{u}(v, n)$, for all $n \geq 0$).*

Then $\mathbb{P}[\{u \text{ is removed}\} \cap \{v \text{ is removed}\}] = \mathbb{P}[u \text{ is removed}] \mathbb{P}[v \text{ is removed}]$.

Proof. Suppose that there is a scale \mathbf{j} such that vertex u is removed because of the existence of a bad cylinder $\mathcal{C}_{w,z}^{(\varepsilon\mathbf{j}\mathbf{r})}$ at scale \mathbf{j} . Note that in this case $u = \mathbf{u}(w, \mathbf{i} - 3\alpha^2\eta\mathbf{j})$ where $\mathbf{i} = d_{\mathbb{T}}(o, w)$. Similarly, suppose that there is a scale \mathbf{j}' such that vertex v is removed because of the existence of a bad cylinder $\mathcal{C}_{w',z'}^{(\varepsilon\mathbf{j}'\mathbf{r})}$ at scale \mathbf{j}' . In this case $v = \mathbf{u}(w', \mathbf{i}' - 3\alpha^2\eta\mathbf{j}')$ where $\mathbf{i}' = d_{\mathbb{T}}(o, w')$.

Now take two vertices $a, a' \in V(G)$ such that $a \in \gamma_{w,z} \subset \Gamma_{w,z}$ and $a' \in \gamma_{w',z'} \subset \Gamma_{w',z'}$. We want to show that $d_G(a, a') \geq \eta\mathbf{j}\mathbf{r} + \eta\mathbf{j}'\mathbf{r}$. In the following, let

$$\begin{aligned} a_{\mathbb{T}} &\text{ be a vertex in } V(\mathbb{T}) \text{ such that } d_G(a, a_{\mathbb{T}}) = d_G(a, \gamma_{\mathbb{T}}(w, z)), \text{ and} \\ a'_{\mathbb{T}} &\text{ be a vertex in } V(\mathbb{T}) \text{ such that } d_G(a', a'_{\mathbb{T}}) = d_G(a', \gamma_{\mathbb{T}}(w', z')). \end{aligned}$$

Then, because of the bilipschitz embedding we deduce that

$$\begin{aligned} d_G(a, a') &\geq -d_G(a, a_{\mathbb{T}}) + d_G(a_{\mathbb{T}}, a'_{\mathbb{T}}) - d_G(a', a'_{\mathbb{T}}) \\ &\geq -\kappa - \alpha\mathbf{r}/2 + \alpha^{-1}\mathbf{r}d_{\mathbb{T}}(a_{\mathbb{T}}, a'_{\mathbb{T}}) - \kappa - \alpha\mathbf{r}/2. \end{aligned} \tag{5.8}$$

The last inequality follows from Proposition 2.2 together with the bilipschitz embedding. In fact, the factor κ is an upper bound on the distance between a (resp. a') and the image of $\gamma_{w,z}$ (resp. $\gamma_{w',z'}$) on $V(\mathbb{T}_3)$, and the largest possible d_G -distance between any vertex of $V(\mathbb{T}_3)$ and $V(\mathbb{T})$ is $\alpha\mathbf{r}/2$. At this point we can define a value $\mathbf{r}_5 = \mathbf{r}_5(\kappa, \alpha) > 0$ such that

$$\mathbf{r}_5 \geq \frac{2\kappa}{2\alpha - 1}. \tag{5.9}$$

Subsequently, by taking $\mathbf{r} \geq \mathbf{r}_5$ we observe that relation (5.8) is bounded from below by

$$\begin{aligned} -2\kappa - \alpha\mathbf{r} + \alpha^{-1}\mathbf{r}d_{\mathbb{T}}(w, w') &\geq -2\kappa - \alpha\mathbf{r} + \alpha^{-1}\mathbf{r}(d_{\mathbb{T}}(w, u) + d_{\mathbb{T}}(v, w')) \\ &\geq -2\kappa - \alpha\mathbf{r} + 3\alpha\eta(\mathbf{j} + \mathbf{j}')\mathbf{r} \\ &\stackrel{\mathbf{r} \geq \mathbf{r}_5}{\geq} \eta(\mathbf{j} + \mathbf{j}')\mathbf{r}. \end{aligned}$$

This finishes the proof of the Claim. \square

We now complete the proof of Proposition 5.3. Now we bound the probability of finding any cutset of vertices that have been removed, separating the root from infinity. For each $k \geq 2$ fixed, given a (fixed) minimal cutset Π_k , the probability that Π_k consists only of removed vertices is bounded from above by

$$\mathbb{P}[\Pi_k \text{ consists only of removed vertices}] \stackrel{\text{Claim 5.5}}{=} \prod_{\mathbf{u} \in \Pi_k} \mathbb{P}[\mathbf{u} \text{ is removed}] \leq (e^{-c_6})^{|\Pi_k|} = e^{-c_6k}.$$

By the union bound, together with Lemma 5.4 one has that

$$\begin{aligned} &\mathbb{P}[\exists \text{ a minimal cutset consisting of discarded vertices}] \\ &\leq \sum_{k=1}^{\infty} e^{-c_6k} \cdot |\{\text{minimal cutsets of cardinality } k\}| \stackrel{\text{Lemma 5.4}}{\leq} \sum_{k=1}^{\infty} e^{-c_6k} \cdot 4^{k-1}. \end{aligned}$$

Now fix a large constant c_7 so that, whenever c_6 is large enough, we have $\sum_{k=1}^{\infty} e^{-c_6k} \cdot 4^{k-1} < e^{-c_7} < 1/1000$. This shows that with probability at least $1 - e^{-c_7} > 999/1000$, there is at least one good path covered by good cylinders at all scales. This concludes the proof of Proposition 5.3. \square

6 Survival of FPP_1 (Proof of Theorem 1.1)

Recall the definition of a good \mathbb{T} -path from (5.1). In this section we show that if we have a good \mathbb{T} -path Ξ containing o on \mathbb{T} (by Proposition 5.3 this event occurs with probability bounded away from 0), then FPP_1 survives indefinitely with positive probability. Recall the notation introduced in (5.2), namely $V(\Xi) = \{v \in V(G) : v \in (\Xi \cup V(E(\Xi)))\}$. To achieve our goal, we make the following assumptions and, by *contradiction*, we show that the presence of a good path makes it impossible for FPP_λ to surround FPP_1 .

Assumptions. Recall the definition of the constant β from (4.6). We assume the following conditions:

(A.1) There is an infinite good \mathbb{T} -path Ξ on \mathbb{T} containing the origin o . Note that since Ξ is good we have (cf. Section 4.1)

$$\left\{ \bigcup_{v \in V(\Xi)} B_G(v, \beta \mathbf{r} + \varepsilon \mathbf{r}) \right\} \cap \{\text{seeds}\} = \emptyset.$$

(A.2) There is a vertex y on Ξ such that FPP_1 started from o activates a seed $\mathbf{s} \in V(G)$ and the FPP_λ originated at \mathbf{s} occupies a vertex in $N(y)$. From now on, y will denote the first (in time) vertex of Ξ satisfying this assumption.

Definition 6.1. From now on, with the terminology “a geodesic of FPPHE” we mean the following. For each vertex draw an oriented edge toward the neighbor from which it got occupied (by FPP_1 or FPP_λ), then each vertex will have an oriented path to the origin and the collection of oriented edges forms a tree. A geodesic of FPPHE is defined as an oriented path in this tree. Note that a vertex x will be occupied by FPP_1 if there is no seed on the oriented path from x to o , whereas it will be occupied by FPP_λ otherwise.

Idea of the proof. Fix the good path Ξ . If (A.2) occurs for some $y \in \Xi$, then there must be a vertex $x \in \Xi$, such that

$$\begin{aligned} d_{\mathbb{T}}(o, x) &< d_{\mathbb{T}}(o, y), \\ \text{there is a geodesic of FPPHE from } N(x) &\text{ to } N(y), \quad \text{and} \\ \text{a geodesic of FPPHE between } N(x) &\text{ and } N(y) \text{ passes through } \mathbf{s}. \end{aligned} \tag{6.1}$$

If there is more than one such x , we consider the one which is closest to y .

Remark 6.2. Note that y cannot coincide with the root o . In fact, assumption (A.1) rules out the possibility that FPP_1 finds a seed before having completely occupied $N(o)$. More precisely, since ε is much smaller than β , by the time FPP_1 reaches a seed, the set $N(o)$ will be completely occupied by FPP_1 , giving no chance to FPP_λ to ever reach it. The same reasoning shows that in general $x \neq y$.

We will show that if (A.2) occurs, then by definition the cylinder $\mathcal{C}_{x,y}^{(\varepsilon d_{\mathbb{T}}(x,y)\mathbf{r})}$ has to be bad, which contradicts (A.1), as $x, y \in \Xi$.

Now we are ready to give a proof of Theorem 1.1.

Proof of Theorem 1.1. First, since Lemma 2.4 holds for some values $\mathbf{c}_{\text{in}} \in (0, 1)$ small enough, and $\mathbf{c}_{\text{out}} > 0$ large enough, we can safely assume that $\mathbf{c}_{\text{out}} > \alpha \max\{\lambda^{-1}, \lambda\} = \alpha \frac{\max\{1, \lambda\}}{\min\{1, \lambda\}}$. Moreover, we choose $\varepsilon' = \varepsilon'(\alpha, \mathbf{c}_{\text{out}}, \lambda) > 0$ so small, that for all $\varepsilon < \varepsilon'$ we have

$$\varepsilon < \min \left\{ \frac{1}{\alpha}, \left(\frac{1}{2\alpha \mathbf{c}_{\text{out}} \max\{1, \lambda\}} \right)^2 \right\}. \tag{6.2}$$

Recall the definition of $\mathcal{C}(N(x))$ from (5.4). An immediate consequence of Lemma 5.2 is that

$$\forall \mathbf{a} \in \partial N(x), \mathbf{b} \in \partial N(y), \text{ all } \gamma_{\mathbf{a},\mathbf{b}} \in \Gamma_{\mathbf{a},\mathbf{b}} \text{ are contained inside } \mathcal{C}_{x,y}^{(\varepsilon\mathbf{r})} \cup \mathcal{C}(N(x)) \cup \mathcal{C}(N(y)).$$

Now suppose that assumptions (A.1) and (A.2) both hold. From now on, we will denote by y the vertex in Ξ that satisfies Assumption (A.2) and by x the one satisfying (6.1), and let $\mathbf{j} = d_{\mathbb{T}}(x, y)$. Observe that (6.2) together with the fact that $d_G(x, y) \in [\alpha^{-1}\mathbf{j}\mathbf{r}, \alpha\mathbf{j}\mathbf{r}]$ imply

$$\frac{d_G(x, y)}{\mathbf{c}_{\text{out}} \max\{1, \lambda\}} \geq \frac{\alpha^{-1}}{\mathbf{c}_{\text{out}} \max\{1, \lambda\}} \mathbf{j}\mathbf{r} \geq 2\varepsilon^{1/2}\mathbf{j}\mathbf{r} > \varepsilon^{1/2}\mathbf{j}\mathbf{r}.$$

Using a similar reasoning we obtain

$$\frac{d_G(x, y)}{\mathbf{c}_{\text{in}} \min\{1, \lambda\}} \leq \frac{\alpha\mathbf{j}\mathbf{r}}{\mathbf{c}_{\text{in}} \min\{1, \lambda\}} \leq \frac{\alpha \max\{1, \lambda\}\mathbf{j}\mathbf{r}}{\mathbf{c}_{\text{in}} \min\{1, \lambda\}} \leq \frac{\mathbf{c}_{\text{out}}}{\mathbf{c}_{\text{in}}} \mathbf{j}\mathbf{r} \leq 4 \frac{\mathbf{c}_{\text{out}}}{\mathbf{c}_{\text{in}}^2} \mathbf{j}\mathbf{r}.$$

We will split the remaining part of the proof of Theorem 1.1 into two cases, according to whether \mathbf{j} is small or large. The value that separates the two cases is an arbitrary value which simplifies the computations, but has no interpretation in the description of the model.

First case: $1 \leq \mathbf{j} \leq \left\lceil \min\{\lambda, \lambda^{-1}\} \frac{\mathbf{c}_{\text{in}}}{\mathbf{c}_{\text{out}}} \frac{\beta}{\alpha} - 3 - \varepsilon \right\rceil$. In this case we will show that before FPP_1 can move from $N(x)$ to activate a seed \mathbf{s} , it already goes from $N(x)$ to $N(y)$. From Assumption (A.1) it follows that the cylinder $\mathcal{C}_{x,y}^{(\varepsilon\mathbf{j}\mathbf{r})}$ is good, and thus any path (in particular a geodesic of FPP_1) from any fixed vertex $\mathbf{a} \in N(x)$ to a seed \mathbf{s} takes time bounded from below by

$$\frac{\beta}{\max\{1, \lambda\} \mathbf{c}_{\text{out}}} \mathbf{r}. \quad (6.3)$$

Now we consider an upper bound for any FPP_1 geodesic that passes close to the graph geodesics. Note that by definition $\min\{\lambda, \lambda^{-1}\} = \frac{\min\{1, \lambda\}}{\max\{1, \lambda\}}$. Once again we exploit Proposition 2.2 and Lemma 5.2, which guarantee that the $d_{\mathbb{T}}$ -geodesic $\gamma_{\mathbb{T}}(x, y)$ is contained in $\mathcal{C}_{x,y}^{(\varepsilon\mathbf{r})}$. The maximum distance between any vertex of $N(x)$ and any vertex of $N(y)$ is at most

$$\alpha\mathbf{j}\mathbf{r} + 2 \left\lceil \alpha \frac{\mathbf{r}}{2} \right\rceil + \varepsilon\mathbf{r} \leq \alpha\mathbf{j}\mathbf{r} + \alpha\mathbf{r} + 2 + \varepsilon\mathbf{r} \leq \alpha\mathbf{j}\mathbf{r} + 2\alpha\mathbf{r} + \varepsilon\mathbf{r},$$

and using the range limitation of \mathbf{j} we obtain

$$\alpha\mathbf{j}\mathbf{r} + 2\alpha\mathbf{r} + \varepsilon\mathbf{r} \stackrel{\text{(1st case)}}{\leq} \alpha \left(\min\{\lambda, \lambda^{-1}\} \frac{\mathbf{c}_{\text{in}}}{\mathbf{c}_{\text{out}}} \frac{\beta}{\alpha} - 2 - \varepsilon \right) \mathbf{r} + 2\alpha\mathbf{r} + \varepsilon\mathbf{r} \leq \alpha \frac{\min\{1, \lambda\}}{\max\{1, \lambda\}} \frac{\mathbf{c}_{\text{in}}}{\mathbf{c}_{\text{out}}} \frac{\beta}{\alpha} \mathbf{r}.$$

Otherwise stated, we have

$$\sup_{\mathbf{a} \in N(x), \mathbf{b} \in N(y)} d_G(\mathbf{a}, \mathbf{b}) \leq \frac{\min\{1, \lambda\}}{\max\{1, \lambda\}} \frac{\mathbf{c}_{\text{in}}}{\mathbf{c}_{\text{out}}} \beta\mathbf{r}.$$

Now we consider a path from \mathbf{a} to \mathbf{b} inside $\mathcal{C}_{x,y}^{(\varepsilon\mathbf{r})} \cup N(x) \cup N(y)$ as a concatenation of shorter paths of length between $\sqrt{\varepsilon}\mathbf{c}_{\text{out}}\mathbf{r}$ and $4 \frac{\mathbf{c}_{\text{out}}^2}{\mathbf{c}_{\text{in}}^2} \mathbf{r}$ which we refer to as *sub-paths*. Note that these sub-paths are *good* in the sense of part (b) in the definition of $\mathcal{G}_1(x, y; \mathbf{r})$, cf. (4.5). Hence, their passage time is at most their length multiplied by $(\mathbf{c}_{\text{in}} \min\{1, \lambda\})^{-1}$. Thus their concatenation will give that disregarding the interaction with FPP_λ , FPP_1 goes from \mathbf{a} to \mathbf{b} in time at most

$$\frac{1}{\mathbf{c}_{\text{in}} \min\{1, \lambda\}} \frac{\min\{1, \lambda\}}{\max\{1, \lambda\}} \frac{\mathbf{c}_{\text{in}}}{\mathbf{c}_{\text{out}}} \beta\mathbf{r} = \frac{1}{\max\{1, \lambda\} \mathbf{c}_{\text{out}}} \beta\mathbf{r}.$$

Thus, by comparing this result with what we found in (6.3), we see that this implies that FPP_1 manages to occupy $N(y)$ before it manages to activate a seed. Thus in this case the proof is concluded.

Second case: $\mathbf{j} > \left\lceil \min\{\lambda, \lambda^{-1}\} \frac{\mathbf{c}_{\text{in}}}{\mathbf{c}_{\text{out}}} \frac{\beta}{\alpha} - 3 - \varepsilon \right\rceil$. It is easy to verify from (4.6) that this necessarily implies $\mathbf{j} \geq 3$. This case corresponds to the situation where the path leaves from $N(x)$, stays completely outside of at least $\mathbf{j} - 2$ sets $N(z)$ for some $z \in \Xi$ such that $d_G(o, x) < d_G(o, z) < d_G(o, y)$, and subsequently enters $\mathcal{C}_{o, \infty}^{(\varepsilon \mathbf{r}/2)}$ at $N(y)$.

We start by computing an upper bound on the time of all quickest geodesic paths (with respect to FPPHE, recall Definition 6.1). A geodesic path starting at some vertex $w \in N(x)$ and ending at some vertex $w' \in N(y)$ is completely contained inside the set $\mathcal{C}_{x, y}^{(\varepsilon \mathbf{r}/2)} \cup \mathcal{C}(N(x)) \cup \mathcal{C}(N(y))$ by Lemma 5.2, and thus it has length bounded from above by

$$d_G(w, x) + d_G(x, y) + d_G(y, w') \leq \left(\frac{\varepsilon \mathbf{r}}{2} + \left\lceil \frac{\alpha \mathbf{r}}{2} \right\rceil \right) + \alpha \mathbf{j} \mathbf{r} + \left(\frac{\varepsilon \mathbf{r}}{2} + \left\lceil \frac{\alpha \mathbf{r}}{2} \right\rceil \right).$$

For $\varepsilon > 0$ satisfying (6.2), we can take $\mathbf{r}_6 = \mathbf{r}_6(\varepsilon) > 0$ so large that for all $\mathbf{r} \geq \mathbf{r}_6$ we have $\varepsilon \mathbf{r} + 2 \leq \mathbf{r}$. Thus for all $\mathbf{j} \geq 1$ (and therefore for all $\mathbf{j} \geq \lceil 2\alpha(1 + \alpha) \rceil$) we obtain

$$\varepsilon \mathbf{r} + \alpha \mathbf{j} \mathbf{r} + 2 \left\lceil \frac{\alpha \mathbf{r}}{2} \right\rceil \leq \varepsilon \mathbf{r} + \alpha \mathbf{j} \mathbf{r} + (\alpha \mathbf{r} + 2) \leq 4 \frac{\alpha}{\mathbf{c}_{\text{in}}} \mathbf{j} \mathbf{r}.$$

Stated in a more clear way, this is saying that

$$\sup_{\mathbf{a} \in N(x), \mathbf{b} \in N(y)} d_G(\mathbf{a}, \mathbf{b}) \leq 4 \frac{\alpha}{\mathbf{c}_{\text{in}}} \mathbf{j} \mathbf{r}. \quad (6.4)$$

Recall the definition of the event $\mathcal{G}_1(x, y; \mathbf{j} \mathbf{r})$, then we proceed as in the first case, namely we consider sub-paths inside $\mathcal{C}_{x, y}^{\varepsilon \mathbf{r}/2} \cup N(x) \cup N(y)$. Then, it follows that disregarding the interactions with FPP_λ , FPP_1 goes from $N(x)$ to $N(y)$ in time at most

$$\frac{1}{\min\{1, \lambda\} \mathbf{c}_{\text{in}}} (\varepsilon \mathbf{r} + 2 + \alpha \mathbf{r} + \alpha \mathbf{j} \mathbf{r}) \leq 4 \frac{\alpha}{\min\{1, \lambda\} \mathbf{c}_{\text{in}}^2} \mathbf{j} \mathbf{r}. \quad (6.5)$$

By construction, this is an upper bound on the time needed by FPP_1 started in $N(x)$ to completely occupy the set $\mathcal{C}_{x, y}^{(\varepsilon \mathbf{r}/2)} \cup N(x) \cup N(y)$.

Now we proceed with a lower bound on the time needed to any geodesic path of FPPHE that avoids $\mathbf{j} - 2$ regions $N(\cdot)$ to go from $N(x)$ to $N(y)$. By Lemma 5.2, for any $\mathbf{a} \in N(x)$ and $\mathbf{b} \in N(y)$ we have that \mathbf{a} and \mathbf{b} are connected by a geodesic that is completely contained inside $\mathcal{C}_{x, y}^{(\varepsilon \mathbf{r}/2)} \cup \mathcal{C}_{\mathbf{a}, x}^{(\varepsilon \mathbf{r}/2)} \cup \mathcal{C}_{y, \mathbf{b}}^{(\varepsilon \mathbf{r}/2)}$. Now, relation (2.4) implies that $\alpha^{-1} \mathbf{j} \mathbf{r}$ is the smallest possible d_G -distance between x and y . Thus, by construction, the d_G -distance between $N(x)$ and $N(y)$ is bounded from below by

$$\alpha^{-1} \mathbf{j} \mathbf{r} - 2 \left\lceil \alpha \frac{\mathbf{r}}{2} \right\rceil - \varepsilon \mathbf{r} \geq \alpha^{-1} \mathbf{j} \mathbf{r} - 2\alpha \mathbf{r} - \varepsilon \mathbf{r} - 2.$$

Define $\mathbf{r}_7 := \mathbf{r}_7(\alpha, \varepsilon) := 2/(\alpha \varepsilon)$ and let $\mathbf{r} \geq \mathbf{r}_7$. By using the lower bound on \mathbf{j} , we deduce

$$\begin{aligned} \alpha^{-1} \mathbf{j} \mathbf{r} - 2\alpha \mathbf{r} - \varepsilon \mathbf{r} - 2 &\geq \alpha^{-1} \left(\min\{\lambda, \lambda^{-1}\} \frac{\mathbf{c}_{\text{in}}}{\mathbf{c}_{\text{out}}} \frac{\beta}{\alpha} - 3 - \varepsilon \right) \mathbf{r} - 2\alpha \mathbf{r} - \varepsilon \mathbf{r} - 2 \\ &\stackrel{(4.6)}{\geq} \alpha^{-1} \left[\min\{\lambda, \lambda^{-1}\} \frac{\mathbf{c}_{\text{in}}}{\mathbf{c}_{\text{out}}} \left((6 + \varepsilon)(1 + \alpha) \alpha \frac{\mathbf{c}_{\text{out}}}{\mathbf{c}_{\text{in}}} \max\{\lambda, \lambda^{-1}\} \right) - 3 - \varepsilon \right] \mathbf{r} - 2\alpha \mathbf{r} - \varepsilon \mathbf{r} - 2 \\ &= \alpha^{-1} ((6 + \varepsilon)(1 + \alpha) \alpha - 3 - \varepsilon) \mathbf{r} - 2\alpha \mathbf{r} - \varepsilon \mathbf{r} - 2 \\ &\geq [(6 + 6\alpha + \varepsilon + \varepsilon \alpha) - 3\alpha^{-1} - \varepsilon \alpha^{-1} - 2\alpha - \varepsilon] \mathbf{r} - 2 \\ &\stackrel{\varepsilon < 1}{\geq} [6 + 4\alpha + \varepsilon \alpha - 4\alpha^{-1}] \mathbf{r} - 2 \stackrel{\alpha \geq 1}{\geq} (6 + \varepsilon \alpha) \mathbf{r} - 2 \stackrel{\mathbf{r} \geq \mathbf{r}_7}{\geq} 6\mathbf{r}. \end{aligned}$$

Now, by simply choosing $\mathbf{r}_8 = \mathbf{r}_8(\delta, \varepsilon)$ so large that for all $\mathbf{r} \geq \mathbf{r}_8$ we have $6\mathbf{r} \geq \max\{50, 50\delta\}$ and $\varepsilon\mathbf{r} \geq 9\delta$. It follows that $d_G(\mathbf{a}, \mathbf{b}) \geq \max\{50, 50\delta\}$, and therefore we are in the conditions to apply Proposition 4.2 with $L = \varepsilon\mathbf{r}$. As a consequence, the length of the detour that the geodesic of FPPHE makes is bounded from below by $(\alpha^{-1}\mathbf{j}\mathbf{r} - 2\alpha\mathbf{r} - \varepsilon\mathbf{r} - 2)\delta 2^{\varepsilon\mathbf{r}/(4\delta)}$ which, for \mathbf{r} large enough, is much larger than $\frac{4\mathbf{c}_{\text{out}}^2\mathbf{j}\mathbf{r}}{\mathbf{c}_{\text{in}}^2}$. Then, since $\mathcal{C}_{x,y}^{(\varepsilon\mathbf{j}\mathbf{r})}$ is good, any path from $N(x)$ of length $\frac{4\mathbf{c}_{\text{out}}^2\mathbf{j}\mathbf{r}}{\mathbf{c}_{\text{in}}^2}$ incurs a passage time of at least

$$\frac{1}{\max\{1, \lambda\}} \left(\frac{4\mathbf{c}_{\text{out}}^2\mathbf{j}\mathbf{r}}{\mathbf{c}_{\text{in}}^2} \right) \frac{1}{\mathbf{c}_{\text{out}}} > \frac{4\alpha\mathbf{j}\mathbf{r}}{\min\{1, \lambda\}\mathbf{c}_{\text{in}}^2}.$$

So, by comparing with (6.5), it becomes clear that the passage time along this path takes longer than the time FPP₁ takes to go from anywhere in $N(x)$ to anywhere in $N(y)$, establishing the desired contradiction and completing the proof of Theorem 1.1. \square

7 Survival of FPP_λ (Proof of Theorem 1.3)

Here we show that with positive probability FPP_λ will be able to “escape” towards infinity, before FPP₁ can reach it and surround it. We will show that the event “FPP_λ survives indefinitely” occurs with probability one.

7.1 Step 1: Construction of a quasi-geodesic infinite ray

Let $t > 0$ be a fixed integer, and start FPP₁ from the origin o . Since we are dealing with the two processes FPP₁ and FPP_λ, we need to define the set of vertices of the graph that have been reached by either process by time t . We set

$$\bar{A}_t := \{v \in V(G) : v \text{ has been occupied by FPP}_1 \text{ or FPP}_\lambda \text{ by time } t\},$$

where if v hosts a seed then it is included in \bar{A}_t only if the seed has already been activated by time t . Recall that $\partial\bar{A}_t$ denotes the internal boundary of \bar{A}_t . At first sight, it is tempting to believe that, in some cases (for example when G is vertex-transitive), one could have that for every $t \geq 0$ the vertex $x \in \partial\bar{A}_t$ that is the furthest from o lies on an infinite geodesic ray (w.r.t. the graph metric) which intersects \bar{A}_t only at x . But this statement is not necessarily true even when G is vertex-transitive and \bar{A}_t is close to a ball; the interested reader is referred to [8] and references therein. We now prove a technical result, based on [7, Lemma 2], that allows us to overcome this problem.

Lemma 7.1. *Let G be a hyperbolic graph. Fix two integers $\sigma \geq 8\delta + 1$ and $s \geq \sigma$, and take any vertex $x \in V(G)$ of distance at least σ from o . Let $x_1 \in \gamma_{x,o}$ be the vertex at distance σ from x . Then, any vertex $y \in B_G(x, s)$ such that*

$$d_G(o, y) < d_G(o, x) + d_G(x, y) - 2\sigma \tag{7.1}$$

belongs to $B_G(x_1, s - \sigma + 4\delta)$.

Before proceeding to the proof of the above lemma, we will need another technical result.

Lemma 7.2. *Let G be a hyperbolic graph. Take any geodesic triangle $\{a, b, c\}$ in G . Then, for all vertices $B, C \in V(G)$ with $B \in \gamma_{a,b}$ and $C \in \gamma_{a,c}$ such that*

$$d_G(a, B) = d_G(a, C) \leq \frac{1}{2} (d_G(a, b) + d_G(a, c) - d_G(b, c)), \tag{7.2}$$

we have that $d_G(B, C) \leq 4\delta$.

Proof. We have to split the proof of this fact into two cases, which are drawn in Figure 7.1:

- (i) The vertices B and C satisfy (7.2) and $d_G(B, \gamma_{b,c}) > \delta$ or $d_G(C, \gamma_{b,c}) > \delta$.
- (ii) The vertices B and C satisfy (7.2) and $d_G(B, \gamma_{b,c}) \leq \delta$ and $d_G(C, \gamma_{b,c}) \leq \delta$.

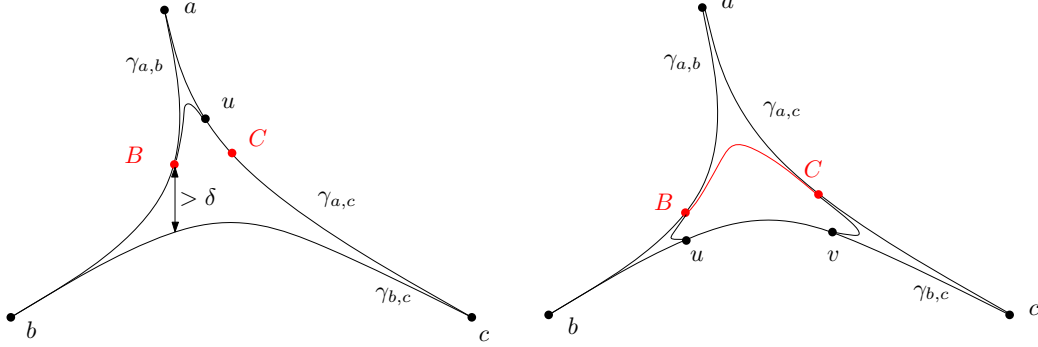


Figure 5: A representation of case (i) (on the left) and case (ii) (on the right) of Lemma 7.2.

Case (i). Assume that $d_G(B, \gamma_{b,c}) > \delta$ (the proof if $d_G(C, \gamma_{b,c}) > \delta$ is symmetric). Since G is δ -hyperbolic, then there must be a vertex $u \in \gamma_{a,c}$ such that $d_G(u, B) \leq \delta$. The triangle inequality implies that

$$d_G(a, C) \stackrel{(7.2)}{=} d_G(a, B) \leq d_G(a, u) + d_G(u, B) \leq d_G(a, u) + \delta. \quad (7.3)$$

If $d_G(a, u) \leq d_G(a, C)$, then

$$d_G(B, C) \leq d_G(B, u) + d_G(u, C) \leq \delta + (d_G(a, C) - d_G(a, u)) \stackrel{(7.3)}{\leq} 2\delta.$$

If $d_G(a, C) \leq d_G(a, u)$, then

$$\begin{aligned} d_G(B, C) &\leq d_G(B, u) + d_G(u, C) \leq \delta + d_G(u, C) \\ &= \delta + (d_G(a, u) - d_G(a, C)) \stackrel{(7.2)}{\leq} \delta + d_G(a, u) - d_G(a, B) \\ &\leq \delta + (d_G(a, B) + d_G(B, u)) - d_G(a, B) \leq 2\delta. \end{aligned}$$

Thus the lemma holds in case (i).

Case (ii). By assumption, there are vertices $u, v \in \gamma_{b,c}$ such that $d_G(B, u) \leq \delta$ and $d_G(C, v) \leq \delta$. First we consider the case $d_G(b, u) \leq d_G(b, v)$, for which $d_G(b, c) = d_G(b, u) + d_G(u, v) + d_G(v, c)$. Relation (7.2) clearly implies that

$$d_G(a, B) + d_G(a, C) = 2d_G(a, B) \leq d_G(a, b) + d_G(a, c) - d_G(b, c),$$

that is

$$d_G(b, c) \leq d_G(a, b) + d_G(a, c) - (d_G(a, B) + d_G(a, C)). \quad (7.4)$$

Now, since $u, v \in \gamma_{b,c}$ and $B \in \gamma_{a,b}$ and $C \in \gamma_{a,c}$ we have

$$\begin{aligned} d_G(b, c) &= d_G(b, u) + d_G(u, v) + d_G(v, c) \stackrel{(7.4)}{\leq} d_G(a, b) + d_G(a, c) - (d_G(a, B) + d_G(a, C)) \\ &\leq d_G(a, b) + d_G(a, c) - (d_G(a, b) - d_G(b, B)) - (d_G(a, c) - d_G(c, C)) \\ &= d_G(b, B) + d_G(c, C). \end{aligned}$$

Again using the definition of u and v we find

$$d_G(b, B) \leq d_G(b, u) + d_G(u, B) \quad \text{and} \quad d_G(c, C) \leq d_G(c, v) + d_G(v, C).$$

Therefore we obtain:

$$d_G(b, c) = d_G(b, u) + d_G(u, v) + d_G(v, c) \leq d_G(b, u) + d_G(u, B) + d_G(c, v) + d_G(v, C),$$

which in turn implies $d_G(u, v) \leq d_G(u, B) + d_G(v, C) \leq 2\delta$. At this point we obtain

$$d_G(B, C) \leq d_G(u, B) + d_G(v, C) + d_G(u, v) \leq 4\delta,$$

showing that the lemma holds in case (ii) when $d_G(b, u) \leq d_G(b, v)$.

If $d_G(b, v) < d_G(b, u)$ then we have two possible situations. Either $d_G(B, v) \leq \delta$, in which case $d_G(B, C) \leq d_G(B, v) + d_G(v, C) \leq 2\delta$, or $d_G(B, v) > \delta$. In the latter case we observe that if $d_G(v, u) \leq 2\delta$ we are fine since $d_G(B, C) \leq 4\delta$ follows as above. So, the only problematic case left to analyze is

$$d_G(B, v) > \delta \quad \text{and} \quad d_G(v, u) > 2\delta. \quad (7.5)$$

In this situation we see that $d_G(b, u) = d_G(b, v) + d_G(v, u) \stackrel{(7.5)}{>} d_G(b, v) + 2\delta$. On the other hand, by the triangle inequality

$$d_G(b, u) \leq d_G(b, B) + d_G(B, u) \leq d_G(b, B) + \delta,$$

and these two inequalities together imply

$$d_G(b, v) < d_G(b, B) - \delta. \quad (7.6)$$

Now, considering the triangle $\{a, b, v\}$ we have

$$d_G(a, v) \geq d_G(a, b) - d_G(b, v) \stackrel{(7.6)}{>} d_G(a, b) - d_G(b, B) + \delta = d_G(a, B) + \delta. \quad (7.7)$$

Using the triangle inequality we see that

$$d_G(a, v) \leq d_G(a, C) + d_G(C, v) \leq d_G(a, C) + \delta = d_G(a, B) + \delta. \quad (7.8)$$

Comparing (7.7) and (7.8) we obtain a contradiction, showing that the two conditions in (7.5) are not compatible in the current setting, meaning that at least one between $d_G(B, v) \leq \delta$ and $d_G(v, u) \leq 2\delta$ has to hold. This completes the proof of the lemma in Case (ii). \square

Proof of Lemma 7.1. Given Lemma 7.2, the proof is a simple generalization of that of [7, Lemma 2], but we include it for the sake of completeness. Let x be any fixed vertex and s as in the statement. Take an element $y \in B_G(x, s)$ and suppose that y satisfies (7.1), namely that

$$d_G(o, y) < d_G(o, x) + d_G(x, y) - 2\sigma. \quad (7.9)$$

Recall $x_1 \in \gamma_{o,x}$ in the statement and let $z \in \gamma_{x,y}$ be such that $d_G(x, z) = d_G(x, x_1) = \sigma$. By (7.9) it follows that

$$\frac{1}{2}(d_G(o, x) + d_G(x, y) - d_G(o, y)) > \sigma = d_G(x, x_1) = d_G(x, z).$$

Using Lemma 7.2 with $a = x, b = o, c = y, B = x_1$ and $C = z$, we obtain that $d_G(x_1, z) \leq 4\delta$. At this point we have

$$d_G(x_1, y) \leq d_G(x_1, z) + d_G(z, y) = 4\delta + (d_G(x, y) - d_G(x, z)) = 4\delta + s - \sigma.$$

Thus $y \in B_G(x_1, s - \sigma + 4\delta)$. \square

Remark 7.3. Note that from the proof of Lemma 7.1 and the fact that $\sigma \geq 8\delta + 1$ we can deduce that there is at least one element $y \in \partial B_G(x, s)$ such that $d_G(o, y) \geq d_G(o, x) + d_G(x, y) - (16\delta + 2) = d_G(o, x) + s - (16\delta + 2)$. In fact, all elements that satisfy (7.1) are contained inside $B_G(x_1, s - \sigma + 4\delta)$, but $\partial B_G(x, s) \not\subset B_G(x_1, s - \sigma + 4\delta)$. As we mentioned in the introduction, our results do hold for trees in a trivial way, hence we assume directly that $\delta \geq 1$. Thus, the above implies that there is at least one element $y \in \partial B_G(x, s)$ such that

$$d_G(o, y) \geq d_G(o, x) + s - 18\delta.$$

Now we will use Lemma 7.1 and Remark 7.3 inductively in order to find an infinite ray that will progressively go far away from the origin. We will henceforth assume that G is a vertex-transitive graph. This is needed to obtain an upper bound that is independent of x on the number of vertices y satisfying the requirement of Lemma 7.1. Such bound is at most $|B_G(x_1, s - 4\delta)|$.

Fix a finite value $R_1 > 0$ that is large enough with respect to δ , λ , \mathbf{c}_{in} and \mathbf{c}_{out} ; in particular, R_1 will be large enough with respect to all parameters except for μ . Let $\tau_0 \geq 0$ be an arbitrary time in the evolution of FPPHE.

Lemma 7.4. *For any $t > 0$, choose a vertex $x_t \in \partial \bar{A}_t$ so that*

$$d_G(o, x_t) = \max_{z \in \partial \bar{A}_t} d_G(o, z).$$

For any given shape of the aggregate \bar{A}_{τ_0} we can find an infinite ray $\bar{\gamma}$ started at o for which there is a sequence of vertices $w_{\tau_0}^{(0)}, w_{\tau_0}^{(1)}, w_{\tau_0}^{(2)}, \dots \in \bar{\gamma}$ such that $w_{\tau_0}^{(0)} = x_{\tau_0}$ and, for all $i \geq 1$, we have

$$\sum_{k=1}^{2i} R_1^k \leq d_G(\bar{A}_{\tau_0}, w_{\tau_0}^{(i)}) \leq 18\delta i + \sum_{k=1}^{2i} R_1^k.$$

Moreover, for all $i \geq 1$ it also holds that

$$d_G(w_{\tau_0}^{(i-1)}, w_{\tau_0}^{(i)}) = R_1^{2i} + R_1^{2i-1} + 18\delta$$

and

$$d_G(o, w_{\tau_0}^{(i)}) \geq d_G(o, w_{\tau_0}^{(i-1)}) + R_1^{2i} + R_1^{2i-1}.$$

Remark 7.5. Note that Lemma 7.4 would hold for any deterministic connected set that contains the origin. However, for sake of clarity, we wrote it only referred to the aggregate because we need to apply it to the set \bar{A}_t defined as above.

Proof. We will find the sequence of vertices inductively, and we start by setting a value $S_1 := R_1^2 + R_1 + 18\delta$. From Lemma 7.1 (together with Remark 7.3) we find that there is an element $w \in \partial B_G(x_{\tau_0}, S_1)$ such that

$$\begin{aligned} d_G(o, x_{\tau_0}) + S_1 &\geq d_G(o, w) \geq d_G(o, x_{\tau_0}) + d_G(x_{\tau_0}, w) - 18\delta \\ &= d_G(o, x_{\tau_0}) + S_1 - 18\delta, \end{aligned}$$

where the last equality follows from the fact that w is on the boundary of the ball centered at x_{τ_0} . By the definition of S_1 we have

$$d_G(o, x_{\tau_0}) + R_1^2 + R_1 + 18\delta \geq d_G(o, w) \geq d_G(o, x_{\tau_0}) + R_1^2 + R_1, \quad (7.10)$$

and by our definition of x_{τ_0} we have that

$$S_1 = d_G(w, x_{\tau_0}) \geq d_G(w, \bar{A}_{\tau_0}) \geq d_G(o, w) - d_G(o, x_{\tau_0}) \stackrel{(7.10)}{\geq} R_1^2 + R_1.$$

Such vertex w is the first element of the sought sequence, thus we call it $w_{\tau_0}^{(1)}$. The first segment of the ray $\bar{\gamma}$ consists of the concatenation of an arbitrary geodesic from 0 to x_{τ_0} and an arbitrary geodesic $\gamma_{x_{\tau_0}, w_{\tau_0}^{(1)}}$.

Now we proceed inductively. Suppose that we have found all the first $k \geq 1$ elements of the sought sequence, that is, we have defined $\{w_{\tau_0}^{(i)}\}_{i=1}^k$, and now we want to define the element $w_{\tau_0}^{(k+1)}$. Set

$$S_{k+1} := R_1^{2(k+1)} + R_1^{2(k+1)-1} + 18\delta,$$

and observe that by Lemma 7.1 and Remark 7.3 we have that there exists an element $w \in \partial B_G(w_{\tau_0}^{(k)}, S_{k+1})$ such that

$$d_G(o, w_{\tau_0}^{(k)}) + S_{k+1} \geq d_G(o, w) \geq d_G(o, w_{\tau_0}^{(k)}) + S_{k+1} - 18\delta.$$

We will set $w_{\tau_0}^{(k+1)} = w$. Thus, from the definition of $w_{\tau_0}^{(k)}$ and setting $w_{\tau_0}^{(0)} = x_{\tau_0}$, it follows that

$$d_G(w, \bar{A}_{\tau_0}) \geq d_G(o, w) - d_G(o, x_{\tau_0}) = \sum_{i=1}^{k+1} \left(d_G(o, w_{\tau_0}^{(i)}) - d_G(o, w_{\tau_0}^{(i-1)}) \right).$$

Hence, we obtain the bound

$$d_G(w, \bar{A}_{\tau_0}) \geq \sum_{i=1}^{k+1} (S_i - 18\delta) = \sum_{i=1}^{2(k+1)} R_1^i.$$

For the upper bound, we obtain

$$d_G(w, \bar{A}_{\tau_0}) \leq d_G(w, x_{\tau_0}) \leq \sum_{i=1}^{k+1} d_G(w_{\tau_0}^{(i)}, w_{\tau_0}^{(i-1)}) = 18\delta(k+1) + \sum_{i=1}^{2(k+1)} R_1^i.$$

Such vertex w is the next element of the sought sequence, thus we denote it by $w_{\tau_0}^{(k+1)}$, and the next segment of the ray $\bar{\gamma}$ consists of an arbitrarily chosen geodesic $\gamma_{w_{\tau_0}^{(k)}, w_{\tau_0}^{(k+1)}}$.

Simply using Remark 7.3 leads to $d_G(w_{\tau_0}^{(k)}, w_{\tau_0}^{(k+1)}) = S_{k+1}$, which concludes the proof. \square

The main idea of the proof of Theorem 1.3 is the following. We take a sequence of large balls $B_{\tau_0}^{(1)}, B_{\tau_0}^{(2)}, \dots$ (of increasing radii) centered at the vertices $\{w_{\tau_0}^{(k)}\}_{k \geq 1}$, and condition on the event that the first one $B_{\tau_0}^{(1)}$ is completely full of seeds (the ball has finite volume, hence this event has positive probability). This will force FPP_1 to make a long detour around it to occupy any vertex that is further away in $\bar{\gamma}$, which will require a very long time. Meanwhile, FPP_λ started from the seeds will spread along $\bar{\gamma}$ and start occupying the subsequent balls. The rest of the proof will consist in proving that for any such attempt, with positive probability, FPP_λ will succeed in occupying the infinite sequence of balls. We proceed now with the next step, which consists in defining the balls that should be occupied by FPP_λ and avoided by FPP_1 . For a graphic representation refer to Figure 6.

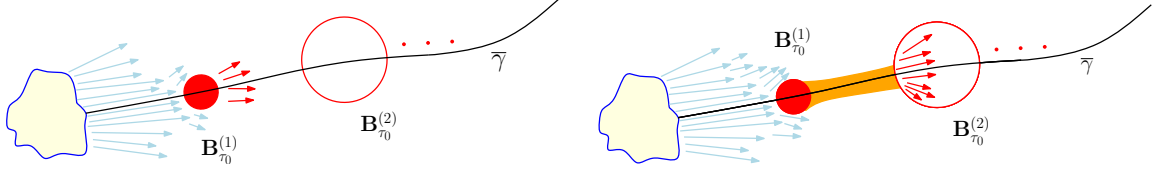


Figure 6: *Left:* The light blue arrows represent paths of FPP started from \bar{A}_{τ_0} which eventually activate the seeds in $\mathbf{B}_{\tau_0}^{(1)}$. While the geodesic of FPP₁ started in \bar{A}_{τ_0} (light blue arrows) try to go around the ball full of seeds, paths of FPP _{λ} (red arrows) have already occupied and exited $\mathbf{B}_{\tau_0}^{(1)}$. *Right:* While the paths of FPP started in \bar{A}_{τ_0} (blue arrows) are still trying to surround the first ball, paths of FPP _{λ} have already filled up a long region (colored in orange) and started to reach the following ball. This will be useful to show that (with positive probability) FPP₁ is too slow to ever surround FPP _{λ} .

7.2 Step 2: Construction of the balls

7.2.1 First ball

Recall the value of R_1 and the sequence of vertices $\{w_{\tau_0}^{(k)}\}_{k \geq 1} \in \bar{\gamma}$ defined above. From Lemma 7.4 it follows that

$$R_1 + R_1^2 + 18\delta \geq d_G(\bar{A}_{\tau_0}, w_{\tau_0}^{(1)}) \geq R_1 + R_1^2. \quad (7.11)$$

Now we place a ball of radius R_1 centered at $w_{\tau_0}^{(1)}$, and denote it by $\mathbf{B}_{\tau_0}^{(1)}$: this will be the “first ball” which we need in order to start the whole procedure. Refer to Figure 7. Note that with positive probability (bounded from below by $\mu^{\Delta R_1} > 0$), the ball $\mathbf{B}_{\tau_0}^{(1)}$ is completely filled with seeds, so we assume this event holds in the subsequent steps. To formalize this, we define the event

$$\mathcal{E}_{\tau_0}^{(1)} = \left\{ B_{\tau_0}^{(1)} \setminus \{\text{seeds}\} = \emptyset \right\}.$$

Remark 7.6. The upper bound on the distance in relation (7.11) will be useful later on in order to ensure that the ball $\mathbf{B}_{\tau_0}^{(1)}$ is activated quickly enough.

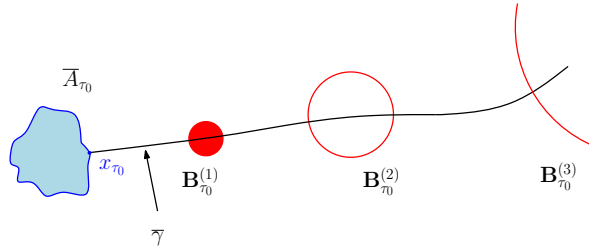


Figure 7: The blue cluster is \bar{A}_{τ_0} , and the solid red ball $B_{\tau_0}^{(1)}$ is the one full of seeds.

Now we proceed with the construction of the other balls using Lemma 7.4 inductively.

7.2.2 Second ball

Start by taking $w_{\tau_0}^{(2)}$ found in Lemma 7.4. As we know, we have $w_{\tau_0}^{(2)} \in \bar{\gamma}$, and $d_G(w_{\tau_0}^{(1)}, w_{\tau_0}^{(2)}) = R_1^4 + R_1^3 + 18\delta$. From now on we will denote

$$R_2 := R_1^2, \quad \text{and set} \quad \mathbf{B}_{\tau_0}^{(2)} := B_G(w_{\tau_0}^{(2)}, R_2).$$

We refer to Figure 7, and remark that from our definitions it follows that

$$d_G(\mathbf{B}_{\tau_0}^{(1)}, \mathbf{B}_{\tau_0}^{(2)}) = R_1^4 + R_1^3 - R_1^2 - R_1 + 18\delta \leq 2R_2R_1^2.$$

At this point, using δ -thinness, we obtain the next result, which is illustrated in Figure 8.

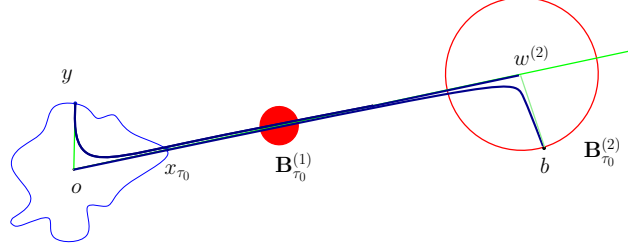


Figure 8: A representation of the situation described in Lemma 7.7, where the white shape on the left is \bar{A}_{τ_0} , and x_{τ_0}, y are as described above. The light green segments represent geodesics, as well as the dark blue ones. We did not draw the geodesic(s) between y and b , in order to avoid confusion.

Lemma 7.7. *For all $y \in \partial\bar{A}_{\tau_0}$ and each vertex $b \in \mathbf{B}_{\tau_0}^{(2)}$, all geodesics $\gamma_{y,b}$ connecting y with b are such that $\gamma_{y,b} \cap B(w_{\tau_0}^{(1)}, 23\delta) \neq \emptyset$.*

Proof. Fix any $y \in \partial\bar{A}_{\tau_0}$ and any vertex $b \in \mathbf{B}_{\tau_0}^{(2)}$. For this proof it suffices to choose $b \in \partial\mathbf{B}_{\tau_0}^{(2)}$, as every geodesic that enters $\mathbf{B}_{\tau_0}^{(2)}$ has to contain at least a vertex of $\partial\mathbf{B}_{\tau_0}^{(2)}$. The core of this proof consists in showing that

$$d_G(w_{\tau_0}^{(1)}, \gamma_{o, w_{\tau_0}^{(2)}}) \leq 21\delta. \quad (7.12)$$

In fact, suppose that (7.12) holds. By our choice of R_1 it follows that for the fixed $y \in \partial\bar{A}_{\tau_0}$ and $b \in \partial\mathbf{B}_{\tau_0}^{(2)}$

$$d_G(\gamma_{o,y}, w_{\tau_0}^{(1)}) > 25\delta, \quad d_G(w_{\tau_0}^{(1)}, \gamma_{w_{\tau_0}^{(2)}, b}) > 25\delta, \quad d_G(w_{\tau_0}^{(2)}, y) \geq d_G(w_{\tau_0}^{(2)}, \bar{A}_{\tau_0}) > 25\delta.$$

Then, by δ -thinness we obtain that

$$d_G(w_{\tau_0}^{(1)}, \gamma_{y,b}) \leq \delta + d_G(w_{\tau_0}^{(1)}, \gamma_{o,b}) \leq 2\delta + d_G(w_{\tau_0}^{(1)}, \gamma_{o, w_{\tau_0}^{(2)}}) \stackrel{(7.12)}{\leq} 23\delta,$$

as claimed. Thus we proceed to show the validity of (7.12). Consider the triangle $\{o, w_{\tau_0}^{(1)}, w_{\tau_0}^{(2)}\}$ (for a graphical representation see Figure 8), and recall that by Lemma 7.4 we know that

$$d_G(o, w_{\tau_0}^{(2)}) \geq d_G(o, w_{\tau_0}^{(1)}) + d_G(w_{\tau_0}^{(1)}, w_{\tau_0}^{(2)}) - 18\delta. \quad (7.13)$$

Let $v \in \gamma_{o, w_{\tau_0}^{(1)}}$ denote a vertex such that $19\delta < d_G(w_{\tau_0}^{(1)}, v) \leq 20\delta$. If we show that $d_G(v, \gamma_{o, w_{\tau_0}^{(2)}}) \leq \delta$, then we would immediately deduce that

$$d_G(w_{\tau_0}^{(1)}, \gamma_{o, w_{\tau_0}^{(2)}}) \leq d_G(w_{\tau_0}^{(1)}, v) + d_G(v, \gamma_{o, w_{\tau_0}^{(2)}}) \leq 21\delta,$$

that is (7.12). We will show this by contradiction; so from now on assume that $d_G(v, \gamma_{o, w_{\tau_0}^{(2)}}) > \delta$.

This and δ -thinness of the triangle $\{o, w_{\tau_0}^{(1)}, w_{\tau_0}^{(2)}\}$ implies that there exists a vertex $v' \in \gamma_{w_{\tau_0}^{(1)}, w_{\tau_0}^{(2)}}$ such that

$$d_G(v, v') \leq \delta. \quad (7.14)$$

Then we have that

$$d_G(o, w_{\tau_0}^{(1)}) = d_G(o, v) + d_G(v, w_{\tau_0}^{(1)}) \quad \text{and} \quad d_G(w_{\tau_0}^{(1)}, w_{\tau_0}^{(2)}) = d_G(w_{\tau_0}^{(1)}, v') + d_G(v', w_{\tau_0}^{(2)}).$$

In particular, by using these relations in (7.13) we deduce that

$$d_G(o, w_{\tau_0}^{(2)}) \geq d_G(o, v) + d_G(v, w_{\tau_0}^{(1)}) + d_G(w_{\tau_0}^{(1)}, v') + d_G(v', w_{\tau_0}^{(2)}) - 18\delta.$$

On the other hand, by the triangle inequality we always have

$$d_G(o, w_{\tau_0}^{(2)}) \leq d_G(o, v) + d_G(v, v') + d_G(v', w_{\tau_0}^{(2)}) \stackrel{(7.14)}{\leq} d_G(o, v) + \delta + d_G(v', w_{\tau_0}^{(2)}).$$

The above two inequalities imply that

$$d_G(v, w_{\tau_0}^{(1)}) + d_G(w_{\tau_0}^{(1)}, v') - 18\delta \leq \delta,$$

but this is a contradiction since $d_G(v, w_{\tau_0}^{(1)}) > 19\delta$. Therefore (7.12) holds and the lemma is proven. \square

We will now proceed to show that the probability that FPP₁ starting from anywhere in \bar{A}_{τ_0} gets close enough to $\mathbf{B}_{\tau_0}^{(2)}$ is very low. Consider the portion of the geodesic $\gamma_{w_{\tau_0}^{(1)}, w_{\tau_0}^{(2)}}$ that does not intersect $\mathbf{B}_{\tau_0}^{(1)}$, that is $\gamma_{w_{\tau_0}^{(1)}, w_{\tau_0}^{(2)}} \setminus \mathbf{B}_{\tau_0}^{(1)}$. Now define

$$\mathbf{P}_{\tau_0}^{(1)} := \mathbf{B}_{\tau_0}^{(2)} \cup \left(\gamma_{w_{\tau_0}^{(1)}, w_{\tau_0}^{(2)}} \setminus \mathbf{B}_{\tau_0}^{(1)} \right), \quad (7.15)$$

and let $G^{(1)}$ be the sub-graph of G induced by the ‘‘removal’’ of $\mathbf{B}_{\tau_0}^{(1)}$. In other words, all paths in the graph $G^{(1)}$ must avoid the ball $\mathbf{B}_{\tau_0}^{(1)}$, inducing possibly exponentially long detours.

Subsequently we set

$$\mathcal{T}_1 := R_1^6; \quad (7.16)$$

the reason for this choice will become clear later. Recall the construction of FPPHE from the passage times $\{t_e\}_{e \in E(G)}$ described in Section 2.3. From now on, for every two vertices $u, v \in V(G)$, we fix a geodesic $\gamma_{u,v}$ in G such that if u, v are vertices belonging to the same geodesic segment of $\bar{\gamma}$, then $\gamma_{u,v}$ is chosen as the same geodesic appearing in $\bar{\gamma}$. Then, we define

$$\bar{\mathbf{T}}_1(u \rightarrow v) := \sum_{e \in \gamma_{u,v}} t_e,$$

and note that $\bar{\mathbf{T}}_1(u \rightarrow v)$ corresponds to the passage time of a rate-1 FPP over the path $\gamma_{u,v}$. (The subscript in $\bar{\mathbf{T}}_1$ is to emphasize that passage times are of rate 1.) If $\bar{w}_{\tau_0}^{(1)}$ denotes the vertex of $\gamma_{x_{\tau_0}, w_{\tau_0}^{(1)}}$ at distance R_1 from $w_{\tau_0}^{(1)}$ (that is, $\bar{w}_{\tau_0}^{(1)} = \gamma_{x_{\tau_0}, w_{\tau_0}^{(1)}} \cap \partial B_{\tau_0}^{(1)}$), then we have that

$$\begin{aligned} \mathcal{T}_1 &\geq R_1 \left(\max\{1, \lambda^{-1}\} d_G(x_{\tau_0}, \bar{w}_{\tau_0}^{(1)}) + \lambda^{-1} d_G(\bar{w}_{\tau_0}^{(1)}, w_{\tau_0}^{(2)}) + \lambda^{-1} (R_1 + R_2) \right) \\ &= R_1 \left(\max\{1, \lambda^{-1}\} (R_1^2 + 18\delta) + \lambda^{-1} (R_1^4 + R_1^3 + R_1^2 + 2R_1 + 18\delta) \right). \end{aligned} \quad (7.17)$$

If we disregard the factor of R_1 outside the parenthesis, the above bounds have the following meaning. The term $\max\{1, \lambda^{-1}\} d_G(x_{\tau_0}, \bar{w}_{\tau_0}^{(1)})$ is the expected time of the slowest between FPP₁ and FPP _{λ} to go from x_{τ_0} along $\gamma_{x_{\tau_0}, w_{\tau_0}^{(1)}}$ until activating the seed located in $\bar{w}_{\tau_0}^{(1)}$. Then, $\lambda^{-1} d_G(\bar{w}_{\tau_0}^{(1)}, w_{\tau_0}^{(2)})$ is the expected time that FPP _{λ} takes to go from $\bar{w}_{\tau_0}^{(1)}$ along $\bar{\gamma}$ until hitting

$w_{\tau_0}^{(2)}$, and $\lambda^{-1}R_1$ and $\lambda^{-1}R_2$ are the times that (with high probability) FPP_λ takes to go from $w_{\tau_0}^{(1)}$ to a vertex in the boundary of $\mathbf{B}_{\tau_0}^{(1)}$ and from $w_{\tau_0}^{(2)}$ to a vertex in the boundary of $\mathbf{B}_{\tau_0}^{(2)}$, respectively. The factor R_1 comes into play just to assure that, with high probability, we have

$$\begin{aligned} \mathcal{T}_1 \geq & \max\{1, \lambda^{-1}\} \bar{\mathcal{T}}_1 \left(x_{\tau_0} \rightarrow \bar{w}_{\tau_0}^{(1)} \right) + \lambda^{-1} \bar{\mathcal{T}}_1 \left(\bar{w}_{\tau_0}^{(1)} \rightarrow w_{\tau_0}^{(2)} \right) \\ & + \lambda^{-1} \max_{x \in \mathbf{B}_{\tau_0}^{(1)}} \bar{\mathcal{T}}_1(w_{\tau_0}^{(1)} \rightarrow x) + \lambda^{-1} \max_{x \in \mathbf{B}_{\tau_0}^{(2)}} \bar{\mathcal{T}}_1(w_{\tau_0}^{(2)} \rightarrow x). \end{aligned}$$

We are ready to define an *enlargement* of $\mathbf{P}_{\tau_0}^{(1)}$ as follows:

$$\text{EP}_{\tau_0}^{(1)} := \bigcup_{x \in \mathbf{P}_{\tau_0}^{(1)}} B_{G^{(1)}}(x, \mathbf{c}_{\text{out}} \mathcal{T}_1).$$

The reason why this is needed, is because of measurability. In fact, as we will show, in order to understand crucial information about the process we only need to look at the passage times inside the set $\text{EP}_{\tau_0}^{(1)}$. For the same reason, we need to define the *boundary* of $\text{EP}_{\tau_0}^{(1)}$ in the graph $G^{(1)}$ and not in G . More precisely, we set

$$\partial_1 \text{EP}_{\tau_0}^{(1)} := \left\{ x \in \text{EP}_{\tau_0}^{(1)} : \exists y \in G^{(1)} \setminus \text{EP}_{\tau_0}^{(1)} \text{ such that } \{x, y\} \in E(G^{(1)}) \right\}.$$

It is fundamental to emphasize that Lemma 7.7, together with our choice of \mathcal{T}_1 (cf. (7.16)) leads to

$$d_{G^{(1)}}(\bar{\mathbf{A}}_{\tau_0}, \text{EP}_{\tau_0}^{(1)}) \stackrel{\text{Lem. 7.7, Prop. 2.3}}{\geq} \delta 2^{R_1 - 22\delta - 2} \geq 10\mathcal{T}_1.$$

In particular, it follows that $\text{EP}_{\tau_0}^{(1)}$ and $\bar{\mathbf{A}}_{\tau_0}$ are disjoint, so to go from $\bar{\mathbf{A}}_{\tau_0}$ to $\mathbf{P}_{\tau_0}^{(1)}$, FPP_1 needs to pass through $\partial_1 \text{EP}_{\tau_0}^{(1)}$, and to go from $\partial_1 \text{EP}_{\tau_0}^{(1)}$ to $\mathbf{P}_{\tau_0}^{(1)}$ it takes much longer than \mathcal{T}_1 . For a graphical representation refer to Figure 9.

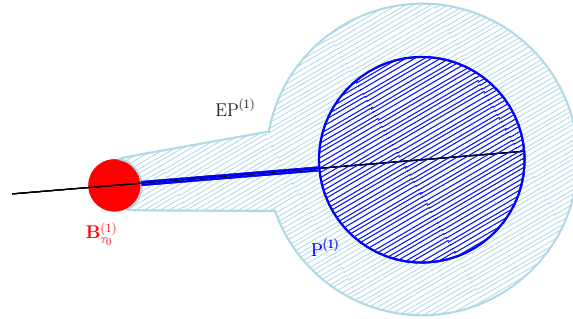


Figure 9: The set $\mathbf{P}_{\tau_0}^{(1)}$ is represented in dark blue (dashed ball $\mathbf{B}_{\tau_0}^{(2)}$ together with the thick segment), and its enlargement $\text{EP}_{\tau_0}^{(1)}$ is colored light blue (dashed). The red ball is $\mathbf{B}_{\tau_0}^{(1)}$.

We now let $T^{(1)}(\cdot)$ be the same function $T(\cdot)$ of the passage times of rate 1 as in (2.6) but using only edges from $G^{(1)}$. We formalize the discussion above by defining the following *good events*:

$$\mathcal{F}_1^{(2)} := \left\{ \begin{aligned} & \mathcal{T}_1 \geq \max\{1, \lambda^{-1}\} \bar{\mathcal{T}}_1 \left(x_{\tau_0} \rightarrow \bar{w}_{\tau_0}^{(1)} \right) + \lambda^{-1} \bar{\mathcal{T}}_1 \left(\bar{w}_{\tau_0}^{(1)} \rightarrow w_{\tau_0}^{(2)} \right) \\ & + \lambda^{-1} \max_{x \in \mathbf{B}_{\tau_0}^{(1)}} \bar{\mathcal{T}}_1(w_{\tau_0}^{(1)} \rightarrow x) + \lambda^{-1} \max_{x \in \mathbf{B}_{\tau_0}^{(2)}} \bar{\mathcal{T}}_1(w_{\tau_0}^{(2)} \rightarrow x) \end{aligned} \right\}, \quad (7.18)$$

and

$$\mathcal{F}_2^{(2)} := \left\{ T^{(1)} \left(\mathbf{P}_{\tau_0}^{(1)} \rightarrow \partial_1 \text{EP}_{\tau_0}^{(1)} \right) > \mathcal{T}_1 \right\}. \quad (7.19)$$

At this point we are able to define a crucial event:

$$\mathcal{E}_{\tau_0}^{(2)} := \mathcal{F}_1^{(2)} \cap \mathcal{F}_2^{(2)}. \quad (7.20)$$

In words, if the event $\mathcal{E}_{\tau_0}^{(2)}$ occurs, then it implies that FPP_λ completely occupies $\mathbf{P}_{\tau_0}^{(1)}$ before FPP_1 can move through $G^{(1)}$ from $\bar{\mathbf{A}}_{\tau_0}$ to $\mathbf{P}_{\tau_0}^{(1)}$.

Lemma 7.8. *We can set R_1 large enough with respect to G so that $\mathbb{P}[\mathcal{E}_{\tau_0}^{(2)}] \geq \frac{3}{4}$.*

Proof. For the purpose of this proof, denote by $\pi_1 := \pi_1(\bar{\gamma})$ and $\hat{\pi}_1 := \hat{\pi}_1(\bar{\gamma})$ the first and the last vertices along $\bar{\gamma}$ such that

$$\pi_1 \in \bar{\gamma} \cap \mathbf{B}_{\tau_0}^{(1)}, \quad \text{and} \quad \hat{\pi}_1 \in \bar{\gamma} \cap \mathbf{B}_{\tau_0}^{(1)}.$$

Note that $\pi_1 = \bar{w}_{\tau_0}^{(1)}$. Moreover, set $\Upsilon_{\tau_0}^{(1)} := C_1 + C_2 + C_3$, where we have defined the following quantities

$$\begin{aligned} C_1 &:= \max\{1, \lambda^{-1}\} \bar{\mathcal{T}}_1(x_{\tau_0} \rightarrow \pi_1), & C_2 &:= \lambda^{-1} \bar{\mathcal{T}}_1(\pi_1 \rightarrow w_{\tau_0}^{(2)}), \\ C_3 &:= \lambda^{-1} \max_{x \in \mathbf{B}_{\tau_0}^{(1)}} \bar{\mathcal{T}}_1(w_{\tau_0}^{(1)} \rightarrow x), & C_4 &:= \lambda^{-1} \max_{x \in \mathbf{B}_{\tau_0}^{(2)}} \bar{\mathcal{T}}_1(w_{\tau_0}^{(2)} \rightarrow x). \end{aligned}$$

The value of $\Upsilon_{\tau_0}^{(1)}$ is an over-estimate on the time needed for FPP started in $\bar{\mathbf{A}}$ to occupy the set $\gamma_{x_{\tau_0}, \pi_1} \cup \mathbf{B}_{\tau_0}^{(1)} \cup \mathbf{P}_{\tau_0}^{(1)}$. Note that the value of \mathcal{T}_1 is defined in such a way that

$$\begin{aligned} \mathbb{P}\left(\mathcal{F}_1^{(2)}\right) &= \mathbb{P}\left(\Upsilon_{\tau_0}^{(1)} \leq \mathcal{T}_1\right) \\ &\geq 1 - \mathbb{P}\left(\bar{\mathcal{T}}_1(x_{\tau_0} \rightarrow \pi_1) \geq R_1 d_G(x_{\tau_0}, \pi_1)\right) - \mathbb{P}\left(\bar{\mathcal{T}}_1(\pi_1 \rightarrow w_{\tau_0}^{(2)}) \geq R_1 d_G(\pi_1, w_{\tau_0}^{(2)})\right) \\ &\quad - \Delta^{R_1} \max_{x \in \mathbf{B}_{\tau_0}^{(1)}} \mathbb{P}\left(\bar{\mathcal{T}}_1(w_{\tau_0}^{(1)} \rightarrow x) \geq R_1^2\right) - \Delta^{R_2} \max_{x \in \mathbf{B}_{\tau_0}^{(2)}} \mathbb{P}\left(\bar{\mathcal{T}}_1(w_{\tau_0}^{(2)} \rightarrow x) \geq R_1 R_2\right). \end{aligned}$$

Using the second part of Lemma 2.5, as well as the fact that R_1 is large enough, we obtain

$$\mathbb{P}\left(\mathcal{F}_1^{(2)}\right) \geq 1 - \exp(-R_1^2/2).$$

Furthermore, since FPP_1 has to pass through $\partial_1 \text{EP}_{\tau_0}^{(1)}$ before reaching $\mathbf{P}_{\tau_0}^{(1)}$, we can exploit the symmetry of the process to obtain

$$\begin{aligned} &\mathbb{P}\left(\exists v \in \partial \text{EP}_{\tau_0}^{(1)} : T^{(1)}(v \rightarrow \mathbf{P}_{\tau_0}^{(1)}) \leq \mathcal{T}_1\right) = \mathbb{P}\left(\exists v \in \partial \mathbf{P}_{\tau_0}^{(1)} : T^{(1)}(v \rightarrow \text{EP}_{\tau_0}^{(1)}) \leq \mathcal{T}_1\right) \\ &\leq \sum_{v \in \partial \mathbf{P}_{\tau_0}^{(1)}} \mathbb{P}\left(T^{(1)}(v \rightarrow \partial_1 \text{EP}_{\tau_0}^{(1)}) \leq \mathcal{T}_1\right) \\ &\leq \left|\partial \mathbf{P}_{\tau_0}^{(1)}\right| e^{-c_1 \mathcal{T}_1} \\ &\leq (R_1^4 + R_1^3 + 18\delta + \Delta^{R_2}) e^{-c_1 \mathcal{T}_1}. \end{aligned}$$

The lemma then follows since $\mathbb{P}\left(\mathcal{E}_{\tau_0}^{(2)} \text{ fails}\right) \leq \mathbb{P}\left(\mathcal{F}_1^{(2)} \text{ fails}\right) + \mathbb{P}\left(\mathcal{F}_2^{(2)} \text{ fails}\right)$, $\mathcal{T}_1 = R_1^6$ and R_1 is large enough. \square

So far we have shown that it is likely that $\mathbf{B}_{\tau_0}^{(2)}$ gets completely occupied by FPP_λ before FPP_1 comes even close to it. In the next subsection we show that we can repeat this reasoning in an inductive way.

7.2.3 Inductive procedure

From now on, for all $k \geq 3$ set

$$R_k := R_1^{2(k-1)}.$$

Consider all the vertices $\{w_{\tau_0}^{(k)}\}_{k \geq 1}$ from Lemma 7.4. Subsequently we set

$$\mathbf{B}_{\tau_0}^{(k)} := B_G(w_{\tau_0}^{(k)}, R_k).$$

Note that

$$d_G(\mathbf{B}_{\tau_0}^{(k-1)}, \mathbf{B}_{\tau_0}^{(k)}) = R_1^{2k} + R_1^{2k-1} + 18\delta - R_1^{2k-2} - R_1^{2k-4} \leq 2R^{2k} = 2R_{k+1}.$$

Using δ -thinness we obtain the next result, which is a generalization of Lemma 7.7.

Lemma 7.9. *For all $k \geq 2$, for all $y \in \partial \bar{\mathbf{A}}_{\tau_0}$ and each vertex $b \in \mathbf{B}_{\tau_0}^{(k)}$, all geodesics $\gamma_{y,b}$ connecting y with b are such that $\gamma_{y,b} \cap B_G(w_{\tau_0}^{(k-1)}, 23\delta) \neq \emptyset$.*

Proof. The proof is analogous to that of Lemma 7.7 and thus we omit the details. \square

We will now proceed to show that the probability of FPP started in $\bar{\mathbf{A}}_{\tau_0}$ to get anywhere close to $\mathbf{B}_{\tau_0}^{(k)}$ is decaying exponentially fast in k .

We proceed inductively following the main ideas developed in the previous section. Take the portion of the geodesic segment $\gamma_{w_{\tau_0}^{(k-1)}, w_{\tau_0}^{(k)}} \subset \bar{\gamma}$ that does not intersect $\mathbf{B}_{\tau_0}^{(k-1)}$, that is $\gamma_{w_{\tau_0}^{(k-1)}, w_{\tau_0}^{(k)}} \setminus \mathbf{B}_{\tau_0}^{(k-1)}$. Now define

$$\mathbf{P}_{\tau_0}^{(k-1)} := \mathbf{B}_{\tau_0}^{(k)} \cup \left(\gamma_{w_{\tau_0}^{(k-1)}, w_{\tau_0}^{(k)}} \setminus \mathbf{B}_{\tau_0}^{(k-1)} \right), \quad (7.21)$$

and let $G^{(k-1)}$ be the sub-graph of G induced by the ‘‘removal’’ of $\mathbf{B}_{\tau_0}^{(k-1)}$. In other words, all paths in the graph $G^{(k-1)}$ must avoid the ball $\mathbf{B}_{\tau_0}^{(k-1)}$, inducing possibly exponentially long detours. We also define $T^{(k-1)}(\cdot)$ as the function that defines the passage times in the graph $G^{(k-1)}$. Subsequently we set

$$\mathcal{T}_{k-1} := R_1^{2k+2}. \quad (7.22)$$

We are now ready to define an *enlargement* of $\mathbf{P}_{\tau_0}^{(k-1)}$ as follows:

$$\mathbf{EP}_{\tau_0}^{(k-1)} := \bigcup_{x \in \mathbf{P}_{\tau_0}^{(k-1)}} B_{G^{(k-1)}} \left(x, \mathbf{c}_{\text{out}} \sum_{j=1}^{k-1} \mathcal{T}_j \right).$$

As before, we will define the boundary

$$\partial_{k-1} \mathbf{EP}_{\tau_0}^{(k-1)} := \left\{ x \in \mathbf{EP}_{\tau_0}^{(k-1)} : \exists y \in G^{(k-1)} \setminus \mathbf{EP}_{\tau_0}^{(k-1)} \text{ such that } \{x, y\} \in E(G^{(k-1)}) \right\}.$$

Lemma 7.9 together with our choice of R_1 guarantee that, for an appropriate constant c , we have $d_{G^{(k-1)}}(\bar{\mathbf{A}}_{\tau_0}, \mathbf{EP}_{\tau_0}^{(k-1)}) \geq c2^{\delta R_{k-1}} \geq 10 \sum_{i=1}^{k-1} T_i$, which in particular ensures that $\mathbf{EP}_{\tau_0}^{(k-1)}$ does not intersect $\bar{\mathbf{A}}_{\tau_0}$.

As we did in the case $k = 2$ we need to define some further good events and then we will show a generalization of Lemma 7.8. For all $k \geq 3$ we set $\bar{w}_{\tau_0}^{(k-1)}$ to be the last vertex of $\bar{\gamma}$ inside $\mathbf{B}_{\tau_0}^{(k-1)}$, and

$$\mathcal{F}_1^{(k)} := \left\{ \mathcal{T}_{k-1} \geq \lambda^{-1} \bar{T}_1 \left(\bar{w}_{\tau_0}^{(k-1)} \rightarrow \bar{w}_{\tau_0}^{(k)} \right) + \lambda^{-1} \max_{x \in \mathbf{B}_{\tau_0}^{(k)}} \bar{T}_1(w_{\tau_0}^{(k)} \rightarrow x) \right\}, \quad (7.23)$$

and

$$\mathcal{F}_2^{(k)} := \left\{ T^{(k-1)} \left(P_{\tau_0}^{(k-1)} \rightarrow \partial_1 \text{EP}_{\tau_0}^{(k-1)} \right) > \sum_{i=1}^{k-1} \mathcal{T}_i \right\}. \quad (7.24)$$

Thus we set

$$\mathcal{E}_{\tau_0}^{(k)} := \mathcal{F}_1^{(k)} \cap \mathcal{F}_2^{(k)}. \quad (7.25)$$

Note that the sum in (7.24) is needed since one cannot guarantee that FPP_1 has to enter some $\text{EP}_{\tau_0}^{(j)}$, $j < k-1$, before entering $\text{EP}_{\tau_0}^{(k-1)}$.

Lemma 7.10. *There is a constant $c > 0$ such that, for all large enough R_1 with respect to G , we have $\mathbb{P} \left[\mathcal{E}_{\tau_0}^{(k)} \right] \geq 1 - \exp \left(-cR_1^{2k+2} \right)$.*

Proof. The proof is very similar to that of Lemma 7.8; we will describe the main steps here. Again for the purpose of this proof, denote by $\widehat{\pi}_{k-1} := \widehat{\pi}_{k-1}(\bar{\gamma})$ the last vertices along $\bar{\gamma}$ such that $\widehat{\pi}_{k-1} \in \bar{\gamma} \cap \mathbf{B}_{\tau_0}^{(k-1)}$. Note that $\widehat{\pi}_{k-1} = \bar{w}_{\tau_0}^{(k-1)}$. Moreover, set

$$\Upsilon_{\tau_0}^{(k-1)} := \lambda^{-1} \bar{T}_1(\widehat{\pi}_{k-1} \rightarrow w_{\tau_0}^{(k)}) + \lambda^{-1} \sup_{x \in \partial \mathbf{B}_{\tau_0}^{(k)}} \bar{T}_1(w_{\tau_0}^{(k)} \rightarrow x).$$

As before, the value of $\Upsilon_{\tau_0}^{(k-1)}$ is an over-estimate on the time needed for FPP_λ to go from $\mathbf{B}_{\tau_0}^{(k-1)}$ to fully occupy the set $P_{\tau_0}^{(k-1)}$. As for the case $k=2$, we observe that T_{k-1} is such that

$$\mathbb{P} \left(\mathcal{F}_1^{(k)} \right) \geq \mathbb{P} \left(\Upsilon_{\tau_0}^{(k-1)} \leq \mathcal{T}_{k-1} \right) \geq 1 - e^{-R_1^{2k+2}/4}.$$

Reasoning as in Lemma 7.8 we obtain

$$\mathbb{P} \left(\mathcal{F}_2^{(k)} \right) \geq 1 - |P_{\tau_0}^{(k-1)}| \exp \left(-c_1 \sum_{i=1}^{k-1} \mathcal{T}_i \right).$$

Since $\sum_{i=1}^{k-1} \mathcal{T}_i = \sum_{i=1}^{k-1} R_1^{2i+4} \geq R_1^{2k+2}$ and $|P_{\tau_0}^{(k-1)}| \leq 2R_{k+1} + \Delta^{R_k} \leq 2R_1^{2k} + \Delta^{R_1^{2(k-1)}}$, we have

$$\mathbb{P} \left(\mathcal{F}_2^{(k)} \right) \geq 1 - \exp \left(-c_1 R_1^{2k+2}/2 \right).$$

The lemma then follows since $\mathbb{P} \left(\mathcal{E}_{\tau_0}^{(k)} \text{ fails} \right) \leq \mathbb{P} \left(\mathcal{F}_1^{(k)} \text{ fails} \right) + \mathbb{P} \left(\mathcal{F}_2^{(k)} \text{ fails} \right)$. \square

The end of this section is devoted to showing that the event that all balls $\left\{ \mathbf{B}_{\tau_0}^{(k)} \right\}_{k=2}^{\infty}$ are filled up by FPP_λ before FPP_1 can come any close to them is bounded away from zero.

Theorem 7.11. *Let $\mathcal{E}_{\tau_0}^{(k)}$ be defined as (7.25) for all $k \geq 2$. Then, taking R_1 large enough with respect to G , we obtain $\mathbb{P} \left[\bigcap_{k=2}^{\infty} \mathcal{E}_{\tau_0}^{(k)} \right] \geq \frac{2}{3}$.*

Proof. This follows from Lemmas 7.8 and 7.10, since $\mathbb{P} \left[\bigcap_{k=2}^{\infty} \mathcal{E}_{\tau_0}^{(k)} \right] \geq 1 - \sum_{k \geq 2} \mathbb{P} \left(\mathcal{E}_{\tau_0}^{(k)} \text{ fails} \right)$. \square

7.3 Completing the proof of Theorem 1.3

Proof of Theorem 1.3. The above argument (summarized in Theorem 7.11) implies that the procedure succeeds with probability bounded from below by $2/3$, which is not enough for our purposes, as we want to show that it succeeds almost surely.

Suppose that there is a first value K_0 for which the event $\mathcal{E}_{\tau_0}^{(K_0)}$ fails to occur. If this happens, then one of the following two events has occurred:

- The passage times in $\gamma_{w_{\tau_0}^{(K_0-1)}, w_{\tau_0}^{(K_0)}}$ or in $\mathbf{B}_{\tau_0}^{(K_0)}$ are too large (delaying the progress of FPP_λ);
- The passage times from $\partial_{K_0-1}\text{EP}_{\tau_0}^{(K_0-1)}$ to $\text{P}_{\tau_0}^{(K_0-1)}$ are too small (speeding up the detour of FPP_1).

Now we observe that both these (bad) events are measurable with respect to the passage times inside the set $\cup_{k=2}^{K_0}\text{EP}_{\tau_0}^{(k-1)}$. Thus, the event

$$\{K_0 \text{ is the smallest value of } k \text{ for which } \mathcal{E}_{\tau_0}^{(k)} \text{ fails}\}$$

is measurable with respect to the passage times inside $\text{EP}_{\tau_0}^{(K_0-1)}$.

Subsequently, we inductively define a sequence of stopping times τ_1, τ_2, \dots such that $\tau_0 < \tau_1 < \dots$. At each attempt $j \geq 0$ we consider the procedure described above applied to the aggregate \bar{A}_{τ_j} . More precisely, for a given attempt j , if the j -th attempt is successful (for some $j \geq 0$) then FPP_λ will produce an infinite cluster, and we are done. Otherwise, let K_j be the first value for which $\mathcal{E}_{\tau_j}^{(K_j)}$ fails. Inductively we define

$$\tau_{j+1} = \tau_{j+1}(K_j) := \inf \left\{ t > \tau_j : \bigcup_{s=0}^j \left(\bigcup_{k=2}^{K_j} \text{EP}_{\tau_s}^{(k-1)} \cup \mathbf{B}_{\tau_s}^{(1)} \right) \text{ are fully occupied by either } \text{FPP}_1 \text{ or } \text{FPP}_\lambda \text{ by time } t \right\}.$$

Notice that for every $j \geq 0$ the value of τ_j is defined so that all edges whose passage times have been observed during the first $j - 1$ attempts have both endpoints occupied by either FPP_1 or FPP_λ . Therefore they will have no further influence on the future development of the process. Since every attempt will succeed with probability bounded from below by $2/3$, which is independent of A_{τ_s} , eventually the procedure will succeed almost surely. \square

8 Open questions and concluding remarks

Our Theorem 1.1, and consequently Corollary 1.4, holds when the underlying graph is both hyperbolic and non-amenable. However, it is easy to find graphs that are non-amenable but not hyperbolic for which our results hold. One example is a *free product of groups*, defined as follows. Consider $m \geq 2$ finitely generated groups $\Gamma_1, \Gamma_2, \dots, \Gamma_m$, with identity elements e_1, e_2, \dots, e_m respectively. Then the free product $\Gamma := \Gamma_1 * \Gamma_2 * \dots * \Gamma_m$ is the set of all words of the form $x_1 x_2 \dots x_n$, where $x_1, x_2, \dots, x_n \in \bigcup_{i=1}^m \Gamma_i \setminus \{e_i\}$.

A more intuitive way to visualize this product is as follows: consider a copy of Γ_1 and to each vertex $v \in \Gamma_1$ attach a copy of $\Gamma_2, \dots, \Gamma_m$ by “gluing” (i.e., identifying) e_2, \dots, e_m and v into a single vertex. Then, inductively, for every vertex on each copy of Γ_i attach a copy of Γ_j , for all $j \neq i$. This construction gives rise to a “*cactus-like*” structure, which whenever all factors $\{\Gamma_i\}_{i=1}^m$ are finite, turns into a *tree-like* structure. In this latter case, Γ is indeed hyperbolic

(for example, a d -regular tree is a free product of d copies of the trivial group of two elements). Free products have been intensively studied (cf. e.g., [9, 10] and references therein) in relation with different behaviors of random walks and branching random walks on graphs, and it turns out that many interesting results in these works appear when Γ is not hyperbolic.

Exploiting the cactus-like structure of Γ (both in the hyperbolic and the non-hyperbolic case), one can see that if FPP_1 survives with positive probability on at least one of the infinite factors, then the FPP_1 and FPP_λ will coexist forever. An example of this phenomenon occurring when Γ is not hyperbolic is when one of the free factors is \mathbb{Z}^d , with $d \geq 2$, and the initial density of seeds μ is small enough. The fact that FPP_1 can survive with positive probability on \mathbb{Z}^d is shown in [36], thus one considers the evolution of FPP_1 on the initial copy of \mathbb{Z}^d and that of FPP_λ on any copy whose origin (which, by definition coincides with the identity element of the group) is occupied by a seed. The cactus-like structure of Γ guarantees that the behavior of the process inside each factor does not interfere with what happens in other factors. This example shows that hyperbolicity is not a necessary condition for Theorem 1.1 and Corollary 1.4.

Question 8.1. *Is there an analogue of Theorem 1.1 and Corollary 1.4 for general non-amenable graphs?*

Our Theorem 1.3 shows that, on hyperbolic graphs, there is no regime of strong survival. It is natural to believe that the growth of the graph plays an important role in the survival of FPP_λ .

Question 8.2. *Is there an analogue of Theorem 1.3 (that is, survival of FPP_λ for all μ, λ) for general graphs of exponential growth?*

The above is not true on \mathbb{Z}^d , as shown in [36]. The proof in [36] uses the fact that FPP has a shape on \mathbb{Z}^d . It seems reasonable to believe that the same should hold for any graph G whenever FPP in G concentrates around a deterministic shape. This could be the case, for example, in graphs of polynomial growth (excluding trivial cases such as \mathbb{Z} , where the isoperimetric dimension is 1 — see [12]).

Question 8.3. *Is there a regime of strong survival in graphs of polynomial growth with isoperimetric dimension bigger than 1?*

Finally, a fascinating open question from [36] is whether there exists coexistence in \mathbb{Z}^d , where even the case $d = 2$ is open.

Question 8.4. *Is there coexistence on \mathbb{Z}^d , for $d \geq 2$?*

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Appendix

A Proof of Proposition 4.2

The aim of this section is to provide a proof of Proposition 4.2. We are given two vertices x, y at distance $d \geq 50\delta$ (cf. Proposition 4.2), and the constraint that the path under consideration must go from x to y avoiding the cylinder $\mathcal{C}_{u,v}^{(L)}$, where u and v are as in the statement.

Start by observing that the vertices u and v belong to a geodesic segment from x to y , hence the segment of $\gamma_{x,y}$ joining u and v is a geodesic segment from u to v , which we denote $\gamma_{u,v}$ (geodesic segments are piece-wise geodesics).

The classical result by Gromov (cf. Proposition 2.3, or [21, Section 7]) states that if on G all triangles are δ -thin, then the length of a path between two vertices x and y that avoids a ball of radius r centered at a point of the geodesic joining x and y has length at least $\delta 2^{r/\delta}$. We exploit this fact, using that $\mathcal{C}_{u,v}^{(L)}$ is defined as a union of balls centered at a d_G -geodesic. Each path that avoids the cylinder must avoid many balls (a number linear in the length d of the cylinder), hence by Gromov's result it will have length of order at least $\delta 2^{r/\delta} \cdot d$.

Now we proceed with a formal proof. Let P denote any path that goes from x to y that avoids the cylinder $\mathcal{C}_{u,v}^{(L)}$, and let n denote its d_G -length. In particular, it will be convenient to express P as a sequence of vertices such as

$$P = (P_0, P_1, \dots, P_n), \quad \text{with } P_0 = x \text{ and } P_n = y.$$

We will find a lower bound on n . A first consideration is that since P avoids $\mathcal{C}_{u,v}^{(L)}$, it must avoid the first ball of $\mathcal{C}_{u,v}^{(L)}$, meaning $P \cap B_G(u, L) = \emptyset$. For every vertex P_ℓ of the path P , let $\Gamma_{P_\ell, y}$ denote the set of geodesics starting at P_ℓ and ending at y .

Step 1. Consider the first (i.e. the smallest) index ℓ for which P_ℓ has a geodesic to y that does *not* intersect the ball $B_G(u, L)$, and set $\mathbf{w}_1 := P_\ell$. In formulas:

$$\mathbf{w}_1 := \min_{1 \leq \ell \leq n} \{P_\ell : \exists \gamma \in \Gamma_{P_\ell, y} \text{ s.t. } \gamma \cap B_G(u, L) = \emptyset\}.$$

Note that since P avoids the cylinder of radius L , we have that $d_G(\mathbf{w}_1, \gamma_{u,v}) \geq L$. Having found \mathbf{w}_1 , take the previous vertex in P (which in our previous notation corresponded to $P_{\ell-1}$), and denote it by $\bar{\mathbf{w}}_1$. For a graphic representation, see Figure 10.

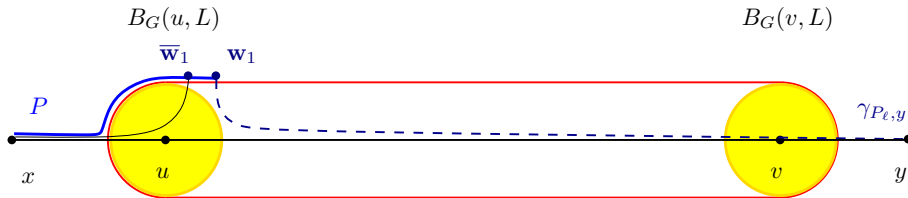


Figure 10: The construction described above: the yellow balls are $B_G(u, L)$ and $B_G(v, L)$. The path P is drawn in blue, and \mathbf{w}_1 is the first vertex that has a geodesic to y that does not touch $B_G(u, L)$ (the dashed line). The thin black line is a geodesic connecting x to $\bar{\mathbf{w}}_1$.

The next result will be important to show that the length of the path P between x and \mathbf{w}_1 is very large.

Claim A.1. *For all geodesics γ_{x, \mathbf{w}_1} between x and \mathbf{w}_1 we have $\gamma_{x, \mathbf{w}_1} \cap B_G(u, \delta) \neq \emptyset$.*

Proof. Let $\gamma_{\mathbf{w}_1, y}$ denote a geodesic such that $\gamma_{\mathbf{w}_1, y} \cap B_G(u, L) = \emptyset$. Then, clearly

$$d_G(u, \gamma_{\mathbf{w}_1, y}) \geq L \geq 9\delta > \delta. \quad (\text{A.1})$$

Moreover, since the triangle $\{x, \mathbf{w}_1, y\}$ is δ -thin, every vertex of $\gamma_{x,y}$ has to be contained in the set $\mathcal{C}_{x, \mathbf{w}_1}^{(\delta)} \cup \mathcal{C}_{\mathbf{w}_1, y}^{(\delta)}$. Since $u \in \gamma_{x,y}$, then (A.1) implies that $u \in \mathcal{C}_{x, \mathbf{w}_1}^{(\delta)}$, which is the claim. \square

Now choose any geodesic γ_{x, \mathbf{w}_1} , by Claim A.1 we know that there is at least a vertex on γ_{x, \mathbf{w}_1} at distance at most δ from u . Consider a ball of radius $L - \delta$ centered at any such vertex, and call it B_1 . Since $B_1 \subset B_G(u, L)$ we deduce that P when going from x to \mathbf{w}_1 avoids B_1 which is centered at a vertex of the geodesic γ_{x, \mathbf{w}_1} . By Proposition 2.3 we have that

$$d_P(x, \mathbf{w}_1) \geq \delta 2^{(L-\delta)/\delta}, \quad (\text{A.2})$$

where we have set

$$d_P(a, b) := \text{number of edges that } P \text{ crosses on its way from vertex } a \text{ to vertex } b. \quad (\text{A.3})$$

Step 2. Fix a vertex $\omega_1 \in \gamma_{u, v}$ such that $d_G(u, \omega_1) = L + 1 + 2\delta$ (i.e., at distance $d_G(x, u) + L + 1 + 2\delta$ from x). The next claim will be used to show that every geodesic $\gamma_{\mathbf{w}_1, y}$ passes at distance at most 2δ from ω_1 . For a graphical representation see Figure 11.

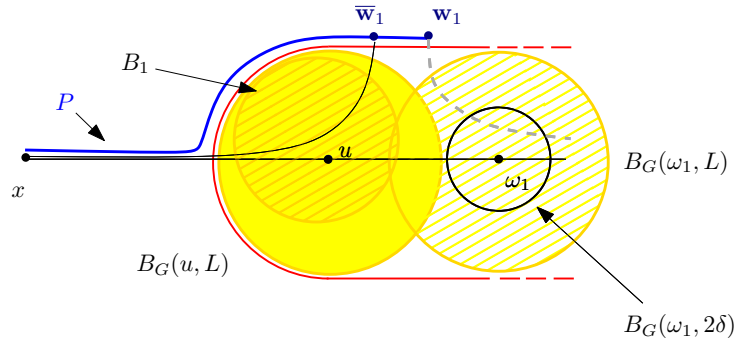


Figure 11: A closeup of Figure 10, with a representation of B_1 , of $B_G(\omega_1, 2\delta)$ and $B_G(\omega_1, L)$.

Claim A.2. For all vertices $b \in B_G(u, L)$ and all geodesics $\gamma_{b, y}$ we have $\gamma_{b, y} \cap B_G(\omega_1, \delta) \neq \emptyset$.

Proof. The proof of this fact follows again from δ -thinness. More precisely, if we consider the triangle $\{b, u, y\}$, then for any geodesic $\gamma_{b, y}$ we have $\gamma_{b, y} \subset \mathcal{C}_{b, u}^{(\delta)} \cup \mathcal{C}_{u, y}^{(\delta)}$. Since $b \in B_G(u, L)$, then $d_G(\gamma_{b, u}, \omega_1) \geq d_G(\omega_1, B_G(u, L)) = 2\delta + 1 > \delta$. Thus we must have that $d_G(\omega_1, \gamma_{b, y}) \leq \delta$, which is the claim. \square

By definition of $\bar{\mathbf{w}}_1$ we have that all geodesics $\gamma_{\bar{\mathbf{w}}_1, y} \cap B_G(u, L) \neq \emptyset$. Therefore, for every vertex in this intersection we can apply Claim A.2, obtaining that for all geodesics $\gamma_{\bar{\mathbf{w}}_1, y}$

$$\gamma_{\bar{\mathbf{w}}_1, y} \cap B_G(\omega_1, \delta) \neq \emptyset.$$

Now we observe that, as $d_G(\mathbf{w}_1, \bar{\mathbf{w}}_1) = 1$ and as we assume that $\delta > 0$, for every pair of geodesics $\gamma_{\mathbf{w}_1, y}$ and $\gamma_{\bar{\mathbf{w}}_1, y}$, we have $\sup_{z \in \gamma_{\mathbf{w}_1, y}} d_G(z, \gamma_{\bar{\mathbf{w}}_1, y}) \leq \max\{1, \delta\} = \delta$. Thus, for all geodesics $\gamma_{\mathbf{w}_1, y}$ we have

$$\gamma_{\mathbf{w}_1, y} \cap B_G(\omega_1, 2\delta) \neq \emptyset. \quad (\text{A.4})$$

Step 3. Now consider the ball $B_G(\omega_1, L)$, this will have a similar role to the previous $B_G(u, L)$. In order to continue, we define a vertex \mathbf{w}_2 (which will have a similar role to \mathbf{w}_1 in the previous step). On the path P find the first vertex \mathbf{w}_2 (after \mathbf{w}_1 in P) that has a geodesic to y that does not cross $B_G(\omega_1, L)$. In formulas:

$$\mathbf{w}_2 := \min_{1 \leq \ell \leq n} \{P_\ell > \mathbf{w}_1 : \exists \gamma \in \Gamma_{P_\ell, y} \text{ such that } \gamma \cap B_G(\omega_1, L) = \emptyset\},$$

where $P_\ell > \mathbf{w}_1$ means that P_ℓ comes after \mathbf{w}_1 in the path P . Moreover, let $\bar{\mathbf{w}}_2$ denote its predecessor on the path P . Note that from the respective definitions it follows that the vertices $\bar{\mathbf{w}}_1, \mathbf{w}_1, \mathbf{w}_2$ are all distinct.

At this point, a proof completely analogous to that of Claim A.1 (replacing u with ω_1 , x with \mathbf{w}_1 and \mathbf{w}_1 with \mathbf{w}_2) shows that for all geodesics $\gamma_{\mathbf{w}_1, \mathbf{w}_2} \in \Gamma_{\mathbf{w}_1, \mathbf{w}_2}$ we have $\gamma_{\mathbf{w}_1, \mathbf{w}_2} \cap B_G(\omega_1, 4\delta) \neq \emptyset$. The factor 4δ comes from the following facts. The distance between ω_1 and any geodesic $\gamma_{\mathbf{w}_1, y}$ is at most 2δ , as stated in (A.4). Now let ω'_1 denote any vertex that belongs to $\gamma_{\mathbf{w}_1, y} \cap B_G(\omega_1, 2\delta)$. The proof of Claim A.1 together with the subsequent reasoning shows that $d_G(\omega'_1, \gamma_{\mathbf{w}_1, \mathbf{w}_2}) \leq 2\delta$. Thus,

$$d_G(\omega_1, \gamma_{\mathbf{w}_1, \mathbf{w}_2}) \leq d_G(\omega_1, \omega'_1) + d_G(\omega'_1, \gamma_{\mathbf{w}_1, \mathbf{w}_2}) \leq 4\delta.$$

Subsequently, just as we did in Step 1 (cf. (A.2)) we can deduce that there is a ball B_2 of radius $L - 4\delta$ and centered at some point of any geodesic $\gamma_{\mathbf{w}_1, \mathbf{w}_2}$ such that the portion of P joining \mathbf{w}_1 to \mathbf{w}_2 avoids B_2 . Proposition 2.3 implies

$$d_P(\mathbf{w}_1, \mathbf{w}_2) \geq \delta 2^{(L-4\delta)/\delta}, \quad (\text{A.5})$$

which by construction yields to

$$d_P(x, \mathbf{w}_2) \stackrel{(\text{A.2}), (\text{A.5})}{\geq} d_P(x, \mathbf{w}_1) + d_P(\mathbf{w}_1, \mathbf{w}_2) \geq 2\delta 2^{(L-4\delta)/\delta}.$$

As we did at the beginning of Step 2, fix a vertex $\omega_2 \in \gamma_{u, v}$ such that $d_G(\omega_1, \omega_2) = L + 1 + 2\delta$. Then, with a similar proof to that of Claim A.2 we can show that for all vertices $b \in B_G(\omega_1, L)$ and all geodesics $\gamma_{b, y}$ we have $\gamma_{b, y} \cap B_G(\omega_2, \delta) \neq \emptyset$. As in Step 2, this yields that for all geodesics $\gamma_{\mathbf{w}_2, y}$ $\gamma_{\mathbf{w}_2, y} \cap B_G(\omega_2, 2\delta) \neq \emptyset$.

Step 4. At this point we can start an inductive procedure to find a lower bound on the length of P , in particular it suffices to repeat Step 3 until we get close to the end of the cylinder. More precisely, the k -th time that we start over with Step 3 we consider the ball $B_G(\omega_{k-2}, L)$ replaced with $B_G(\omega_{k-1}, L)$, define a vertex \mathbf{w}_k on the path P such that

$$\mathbf{w}_k := \min_{1 \leq \ell \leq n} \{P_\ell > \mathbf{w}_{k-1} : \exists \gamma \in \Gamma_{P_\ell, y} \text{ such that } \gamma \cap B_G(\omega_{k-1}, L) = \emptyset\},$$

and let $\bar{\mathbf{w}}_k$ denote its predecessor on the path P .

At this point, a proof completely analogous to that of Claim A.1 (inductively replacing u with ω_{k-2} , x with \mathbf{w}_{k-1} and \mathbf{w}_1 with \mathbf{w}_k) shows that for all geodesics $\gamma_{\mathbf{w}_{k-1}, \mathbf{w}_k}$ we have $\gamma_{\mathbf{w}_{k-1}, \mathbf{w}_k} \cap B_G(\omega_{k-1}, 4\delta) \neq \emptyset$. We note that the factor 4δ comes from the same reasoning as in Step 3. From this we deduce that $d_P(\mathbf{w}_{k-1}, \mathbf{w}_k) \geq \delta 2^{(L-4\delta)/\delta}$. In conclusion we obtain

$$d_P(x, \mathbf{w}_k) \geq d_P(x, \mathbf{w}_1) + \dots + d_P(\mathbf{w}_{k-1}, \mathbf{w}_k) \geq k\delta 2^{(L-4\delta)/\delta} \stackrel{L \geq 9\delta}{\geq} k\delta 2^{L/(2\delta)}. \quad (\text{A.6})$$

Then we define $\omega_k \in \gamma_{u, v}$ such that $d_G(\omega_{k-1}, \omega_k) = L + 1 + 2\delta$. Then we can show for all geodesics $\gamma_{\mathbf{w}_k, y}$ we have $\gamma_{\mathbf{w}_k, y} \cap B_G(\omega_k, 2\delta) \neq \emptyset$, and proceed inductively.

We have good control on the positions of the vertices $\{\mathbf{w}_j\}_j$, and therefore we can apply these iterations safely until we get to balls towards the end of the cylinder (i.e., close to v). To be on the safe side, we can perform this reasoning for almost the whole length of the cylinder, just ignoring the last few balls. We continue our iterations until the last ball reaches distance $2L$ from the end of the cylinder, which means distance $2L + d_G(v, y)$ from vertex y .

The total number K of such iterations is the number of balls that P avoids and are used to define the sequence of vertices $\{\mathbf{w}_j\}_{j=1}^K$. Thus, a lower bound on K is given by

$$K \geq \left\lfloor \frac{d_G(u, v) - 2L}{L + 1 + 2\delta} \right\rfloor \geq \frac{d_G(u, v)}{3L}. \quad (\text{A.7})$$

Finally, by putting together (A.6) and (A.7), we obtain the proposition.

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