

**DESSINS FOR MODULAR OPERAD
AND GROTHENDIECK–TEICHMÜLLER GROUP**
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*... j’avais retenu la flèche, une cible qui ne sera jamais
atteinte, une division à l’infini, le mystère du diable.
Bernard Cunéo. “Le Chat du Typographe”.*

ABSTRACT. A part of Grothendieck’s program for studying the Galois group $G_{\overline{\mathbf{Q}}}$ of the field of all algebraic numbers $\overline{\mathbf{Q}}$ emerged from his insight that one should lift its action upon $\overline{\mathbf{Q}}$ to the action of $G_{\mathbf{Q}}$ upon the (appropriately defined) profinite completion of $\pi_1(\mathbf{P}^1 \setminus \{0, 1, \infty\})$. The latter admits a good combinatorial encoding via finite graphs “dessins d’enfant”.

This part was actively developing during the last decades, starting with foundational works of A. Belyi, V. Drinfeld and Y. Ihara.

Our brief note concerns another part of Grothendieck program, in which its geometric environment is extended to moduli spaces of algebraic curves, more specifically, stable curves of genus zero with marked/labelled points. Our main goal is to show that dual graphs of such curves may play the role of “modular dessins” in an appropriate operadic context.

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0. Introduction and summary

An approach to the description of the (profinite completion of) “absolute Galois group” $G_{\mathbf{Q}}$ of the field of all algebraic numbers starts with observation that for any algebraic manifold (or more generally, integral scheme X) $G_{\mathbf{Q}}$ acts by outer automorphisms upon étale fundamental group of $X \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$ via exact sequence

$$1 \rightarrow \pi_1(X \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}) \rightarrow \pi_1(X) \rightarrow G_{\mathbf{Q}} \rightarrow 1$$

(see [Gr63], [SchGr64], and [Fr17], Ch. 12, for further details and references).

In the most studied case, that of $X = \mathbf{P}^1 \setminus \{0, 1, \infty\}$, the action of $G_{\mathbf{Q}}$ is further reduced to its action upon the so called *dessins d’enfant*, that are finite graphs of very special origin and structure. Each dessin is the inverse image of $[0, 1]$ upon a Riemannian surface Y which is finite covering $Y \rightarrow \mathbf{P}^1$ ramified only over $\{0, 1, \infty\}$.

In this article our aim consists in demonstrating that if we replace above $X = \mathbf{P}^1 \setminus \{0, 1, \infty\}$ by the family of moduli spaces of stable genus zero algebraic curves with labelled points, the role of dessins d’enfant can be played by dual combinatorial graphs of such curves.

This demonstration is the theme of the central Sections 3 and 4 of the paper, whereas Sections 1 and 2 are preparatory. These sections introduce a not quite standard combinatorial approach to the operadic formalism, sufficient for our goals, but avoiding appeal to the machinery of Quillen’s model categories, that was used in the article [BriHoRo19] and the monograph [Fr17] for similar purposes.

In Section 1 we describe, following [BoMa07], theory of graphs in the categorical language, and provide details of operad theory based on it.

In Section 2, we describe in more details the genus zero modular operad in this environment and show why related combinatorial graphs deserve their name of “modular dessins” as geometric objects.

In Section 3 we introduce a modification/enrichment of modular operad necessary for defining the action of the absolute Galois group upon modular dessins.

Finally, Section 4 transplants the powerful machinery of quantum statistics from [BosCo95], [CoKr00] and [CoMar08] to the operadic context, and more specifically, the context of enriched genus zero modular operad.

It shows, in particular, how one can extract a rich arithmetic information about the absolute Galois group, using this machinery.

1. Graphs and operads

All the following constructions are made in a fixed small universe. The basic objects we will be considering in this preparatory section are *trees* and their categories, *operads* as functors on a category of trees, and *symmetries* of these categories and functors.

This environment can be considered as a version of “dendroidal” constructions described in [MoeWe07], [CiMoe13], but for the restricted purposes of this paper we prefer to use basic definitions and constructions from [BoMa07].

Below we offer a very condensed survey of them. An interested reader will find much more details in [BoMa07].

1.1. Finite graphs, trees, and stable trees. In [BoMa07], Definition 1.1.1, a combinatorial finite graph τ is defined as a family of finite sets (F_τ, V_τ) (*flags* and *vertices*) and maps $(\partial_\tau : F_\tau \rightarrow V_\tau, j_\tau : F_\tau \rightarrow F_\tau)$ (*boundary maps* and *structure involution* satisfying $j_\tau^2 = id.$)

Two–element orbits of j_τ form the set E_τ of *edges* of τ . Elements of one such orbit are sometimes called “halves” of the respective edge, and two points, – boundaries of a member of this orbit, – the boundary of the respective edge itself.

One–element orbits of j_τ are called *tails*, or *leaves* (we will use both words as synonymous). A graph τ with one vertex and no edges is called *corolla*.

The *multiplicity* of a vertex v is the number of flags whose boundary is v .

Let us call a sequence of vertices v_1, v_2, \dots, v_n , $n \geq 2$ of a graph τ a *path* connecting v_1 and v_n , if for each i , $v_i \neq v_{i+1}$ constitute the boundary of an edge. We say that the graph τ is *connected*, if any two its vertices can be connected by a path. Corollas are also connected.

Graphs τ_1 and τ_2 are called *disjoint*, if $V_{\tau_1} \cap V_{\tau_2} = \emptyset$ and $E_{\tau_1} \cap E_{\tau_2} = \emptyset$. We can define the disjoint union \sqcup of any finite family of pairwise disjoint graphs in an obvious way. Clearly, each graph is a disjoint union of its maximal connected subgraphs, that can be called its *connected components*.

A (connected) graph is called a (connected) *tree*, if no vertex is connected by two different edges, and more generally, there is no “cyclic” path in it of length ≥ 2 . A tree is called *stable*, if each vertex is a boundary of at least three different flags. If we say simply *stable tree*, without mentioning connectedness, this means that each connected component of this graph is a connected stable tree. Later in Sec. 4, we will call non–necessarily connected trees also *forests*.

In many papers involving graphs, authors prefer to bypass a set–theoretic step, to start directly with categorical definitions, and illustrate their constructions by pictures.

To the contrary, here we are inclined to stay as long as possible with set–theoretic (“combinatorial”) notions. A passage to “pictures” is also described in this way in [BoMa07], 1.1.2, under the name *geometric realisation of a graph*.

Our next goal consists in defining morphisms between graphs, in such a way that we get a category *Graph* whose objects are (some) graphs, and each morphism $h : \tau \rightarrow \sigma$ is a triple (h^F, h_V, j_h) of the following structure: $h^F : F_\sigma \rightarrow F_\tau$ is a contravariant map, $h_V : V_\sigma \rightarrow V_\tau$ is a covariant map, and j_τ is an involution on the set of flags of τ contained in $F_\tau \setminus h^F(F_\sigma)$.

These data must satisfy a pretty long list of conditions/restrictions, for which we refer the reader to [BoMa07], Definition 1.1.2. Quite important is the end of this Definition, saying that *composition of morphisms corresponds to the set–theoretic compositions of h^F and h_V* , and in addition explaining the behavior of j .

Example: isomorphisms of graphs. According to this definition, any isomorphism $\sigma \rightarrow \tau$ induces a bijection $p_V : V_\sigma \rightarrow V_\tau$ of vertices, and a bijection $p^F : F_\tau \rightarrow F_\sigma$. These two bijections must be compatible in the following sense: if $f \in F_\sigma$, then $\partial_\tau((p^F)^{-1}f) = p_V^{-1}(\partial_\sigma(f))$. Finally, the induced map upon edges also must be a bijection.

1.2. Operadic composition of graphs. Let (\mathcal{C}, \otimes) be a monoidal category. Generally, an operad in \mathcal{C} based upon a category of graphs *Graph* is a functor $(\text{Graph}, \sqcup) \rightarrow (\mathcal{C}, \otimes)$ endowed with additional structures. These structures would admit a natural description, *if* the operation \sqcup were a monoidal structure, or even in a more enriched environment, if we first define upon *Graph* a structure of (simplicial) model category: see [Ho17], [BriHoRo19] and references therein.

However, \sqcup is *not* a monoidal product even in the category of sets: see a brief discussion in [BoMa07], sec. 1.6 and 1.7.

The working version of operads that we will adopt here, starts with definition of what we mean by *operadic compositions* in *Graph* and its subcategory of stable trees, and proceeds by extending it to operadic composition in some algebraic manifolds.

1.3. Definition. Let (τ_i, t_i) , $i = 1, 2$, $t_i \in F_{\tau_i}$, $t_1 \neq t_2$, be two pairs (graph, tail). Its composition (“grafting”) is the graph, denoted

$$(\tau_1, t_1) * (\tau_2, t_2),$$

or else $\tau_1 *_{(t_1, t_2)} \tau_2$. It is the set theoretic union of τ_1 and τ_2 , in which t_1 and t_2 are now halves of a new edge.

Notice that if τ_1, τ_2 are disjoint (stable) trees, then their composition is a (stable) tree as well.

Below we will use only the following very restricted definition of operad of stable trees $Tree$:

1.4. Definition. *An operad $Tree$ consists of a family of stable trees, together with a family of binary grafting operations, and their iterations in all possible ways.*

The reader will easily see that these iterations satisfy an appropriate reformulation of usual operadic axioms.

1.5. Example: magma operad Mag . We will give two equivalent formalisms describing this operad (see [BriHoMo19], Definitions 6.9 and 6.10).

Description 1. For a finite set S of cardinality $n \geq 2$, define a finite set of stable trees $Tree^S$, in the following way: $\tau \in Tree^S$ if and only if it is connected, each vertex is the boundary of precisely three flags, and moreover, there exists a linear ordering $s := (s_1, s_2, \dots, s_n)$ of S such that:

(a) $F_\tau = \{(s_i, i, +) \mid i = 1, \dots, n\} \cup \{(s_i, i, -) \mid i = 1, \dots, n\} \cup (*_S, -)$. Intuitively, $+$, resp. $-$ describes orientation of of flag towards, resp. outwards, the vertex of the flag.

(b) For each vertex $v \in V_\tau$, of three flags whose boundary is v , two flags are oriented towards v , and one outwards.

(c) Halves of each edge have opposite orientation.

(d) For each vertex v , there exists exactly one oriented path of edges connecting v to the root vertex $*_S$.

Finally, binary operadic compositions between $Tree^S$ and $Tree^T$ are allowed only if they produce another three with the same properties, i.e. root of $\tau_1 \in Tree^S$ must be grafted to a non-root of $\tau_2 \in Tree^T$.

We may add to this operad a degenerate tree, corresponding to $n = 1$. We omit its description.

Description 2. For a non-empty finite set S of cardinality $\geq n$ define inductively the set of nonassociative (and noncommutative) monomials Mag_n^S :

(a) $Mag_1^S = S$.

(b) $Mag_m^S = \cup_{p+q=m; p, q \geq 1} M_p^S \times M_q^S$ for all $2 \leq m \leq n$.

Elements of Mag_n^S are written in [BriHoMo19], Definition 6.10, as linear words in the alphabet $S \cup \{(\,)\}$, and their identification with binary trees becomes intuitively clear.

Finally, we will define general operads in this context as follows.

1.6. Definition. *Let (\mathcal{C}, \otimes) be a monoidal category. Then a stable tree operad A in (\mathcal{C}, \otimes) encoded by a tree operad $Tree$ consists of a family of objects of \mathcal{C} labelled by stable trees from $Tree$, and family of binary operators between them, labelled by graftings $*_{(t_1, t_2)}$.*

1.7. Comments. Let us explain, how the most standard definition of the operad fits into our one.

Usually by an operad in (\mathcal{C}, \otimes) one means a collection of objects $A(n)$ of \mathcal{C} and a family of morphisms (operadic compositions)

$$A(n) \otimes A(k_1) \otimes \dots \otimes A(k_n) \rightarrow A(k_1 + \dots + k_n - n)$$

satisfying standard associativity conditions and some non-universal additional data and restrictions such as structural actions of S_n upon $A(n)$, inequalities upon n, k_i etc. Usually one distinguishes them, calling the respective modified operads cyclic ones, PROP's etc.

Our approach, developed in [BoMa07], insists on encoding all initial data and axioms for them in the definition of an appropriate category of graphs (in our context trees) and and their morphisms. The magma operad is a good illustration of this principle.

So from our viewpoint, an ordinary description of operad above means that $A(n)$ is its value at the corolla with n tails oriented towards the vertex v and one tail oriented outside (“root”). The operadic composition mentioned above comes from grafting of n such corollas to all non-tails of the corolla with n incoming tails.

2. Dessins for modular operad: geometry and combinatorics

2.1. Combinatorics. In this section we shall introduce the central example of stable tree operad in the category of algebraic manifolds with direct product: modular genus zero operad, described here in the context of Definition 1.6.

Modular dessins (“*dessins d’un vieillard*”) that we will be considering here are *combinatorial graphs* encoding stable curves of genus zero with a finite subset of marked/labelled nonsingular points. We will start working over the field of algebraic numbers $\overline{\mathbf{Q}}$.

One such stable curve C has only double points as singularities, and all its irreducible components are isomorphic to \mathbf{P}^1 . Each its irreducible component must have ≥ 3 points each of which is either labelled, or singular.

2.2. Encoding stable curves. It is known (see below) that up to deformation every such stable curve is encoded by the following combinatorial tree $\tau = \tau_C$:

$E_\tau :=$ the set of double points of C .

$F_\tau :=$ union of the set of labelled points (together with their labels) of C and halves of the edges from E_τ . It is convenient to identify these halves with preimages of double points on the normalization of C .

j_τ sends each labelled point to itself, and each half of the edge to another half.

$V_\tau :=$ the set of irreducible components of C .

$\partial_\tau(f) = v$, if either f encodes a labelled point, and v encodes the irreducible component, to which this point belongs; or else f encodes a half of the edge, and v encodes the the respective irreducible component of the normalisation.

2.3. Moduli spaces and their strata. We will start with some basic facts about moduli spaces of genus zero stable curves with marked points. Our principal sources here are [Ke92], [Ka93], and their extension and generalisation in [BrMe13].

The main facts can be concisely stated as follows.

(a) Let S be a finite set of cardinality $n + 1$, $n \geq 3$. Then stable genus zero curves with $n + 1$ points labelled by S are parametrised by points of the smooth projective irreducible manifold $\overline{M}_{0,S}$ of dimension $n - 2$.

The subspace of points corresponding to only *irreducible* curves is an open Zariski dense submanifold $M_{0,S} \subset \overline{M}_{0,S}$. The graph of any such curve is corolla with S tails.

(b) More generally, given a stable connected tree τ with the set of tails (labelled by) S , all stable genus zero modular curves with graph τ and their further specialisations/degenerations are parametrised by the Zariski closed smooth projective manifold $\overline{M}_{0,\tau} \subset \overline{M}_{0,S}$.

Those curves whose graph is exactly τ are parametrised by the Zariski open dense submanifold $M_{0,\tau} \subset \overline{M}_{0,\tau}$.

We will call the submanifolds $\overline{M}_{0,\tau}$, resp. $M_{0,\tau}$, *closed*, resp. *open* strata of structural stratification of $\overline{M}_{0,S}$.

2.4. Operadic compositions. We can now sketch the definition of the stable tree operad \overline{M} in the monoidal category of algebraic manifolds with direct product, in the framework of Definition 1.6 above.

Its objects labelled by stable trees are $\overline{M}_{0,\tau}$, and the (binary) operadic composition is defined by the simple-minded formula

$$\overline{M}_{0,\tau_1} * \overline{M}_{0,\tau_2} := \overline{M}_{0,\tau_1 * \tau_2},$$

where for brevity we omitted notation for tails.

2.5. Examples. (a) We start with strata of codimension one.

Closed strata correspond to labelled trees having one edge. Up to isomorphism, they are classified by unordered 2-partitions $S = S' \sqcup S''$, both parts of each have cardinalities ≥ 2 . Each part labels tails at one vertex of the edge.

(b) More generally, closed stratum $\overline{M}_{0,\sigma}$ is a substratum of another one $\overline{M}_{0,\tau}$ of relative codimension one, iff σ can be obtained from τ by inserting one extra edge in place of a vertex v of τ and distributing half edges (or tails) at v according to a two-partition as above.

By induction, we see that embeddings $\overline{M}_{0,\sigma} \subset \overline{M}_{0,\tau}$ of relative codimension $d \geq 1$ are classified by subsets of edges of σ of cardinality d such that their “blowing down” produces τ . In particular, they can be obtained by iterating embeddings of codimension one.

(c) Consider now dessins of strata having maximal codimension $n-2 = \dim \overline{M}_{0,S}$. From the description in (b) one sees that if one forgets the labelling of tails (half-edges) of such a graph, all such dessins have the same structure: $n+1$ vertices are linearly ordered, say as $\{v_1, \dots, v_{n+1}\}$; consecutive pairs $(v_1, v_2), (v_2, v_3), \dots, (v_q, v_{n+1})$ are connected by one edge each; finally, v_2, \dots, v_n carry one additional leaf (or tail), whereas v_1 and v_{n+1} carry two additional leaves each.

Then the labelling is simply a bijection between n and the set of half-edges of τ .

Surprisingly, our modular dessins of *maximal codimension* with forgotten labelling form a subclass of dessins d’enfant that occur also in the classical Grothendieck–Teichmüller context: they are what Grothendieck called “*clean dessins*”.

In fact, in order to pass from our description to Grothendieck’s one should do some re–encoding: we must introduce extra vertices (and edges) and label the set of all resulting vertices as “black” ones and “white” ones. This operation is (almost) uniquely determined by the geometry of our trees:

(c1) Add one vertex at the free end of each leaf and one vertex in the middle of all edges (v_i, v_{i+1}) .

(c2) Call all old vertices v_i and $n + 3$ new ones *black vertices*.

(c3) Call n new vertices in the middle of old edges *white ones*.

In the initial Grothendieck’s approach, the resulting “bipartite” graphs (black/white vertices) encode a subfamily of Belyi maps $\Sigma \rightarrow \mathbf{P}^1$ with a very special ramification profile.

In the last subsections 4.13–4.23 of Section 4, we will show how modular dessins encode both geometric and Galois symmetries of the genus zero modular operad.

(d) Now we can clarify somewhat the geometry of locally closed strata $M_{0,\tau}$.

From the definition, it follows that

$$M_{0,\tau} = \overline{M}_{0,\tau} \setminus \left(\bigcup_{\sigma} \overline{M}_{0,\sigma} \right)$$

where the union is taken over all substrata of relative codimension one, that in turn bijectively correspond to edges of τ .

2.6. Combinatorics of admissible projections. Let now $S \subset S'$ be a finite set and its subset. We call the respective *admissible projection* the morphism $\overline{M}_{0,S'} \rightarrow \overline{M}_{0,S}$ forgetting points with labels in $S' \setminus S$. We consider relationships between dessins for $\overline{M}_{0,S'}$ and for $\overline{M}_{0,S}$.

(a) *Divisorial strata.* Divisorial strata of $\overline{M}_{0,S}$ correspond to (stable) 2–partitions $S = S_1 \sqcup S_2$, and divisorial strata of $\overline{M}_{0,S'}$ correspond to (stable) 2–partitions $S = S'_1 \sqcup S'_2$.

Under the admissible projection $p : \overline{M}_{0,S'} \rightarrow \overline{M}_{0,S}$, one such stratum projects to another one precisely when p induces admissible projections of components

$$S'_i \sqcup \{pt'_i\} \rightarrow S_i \sqcup \{pt_i\}, \quad i = 1, 2.$$

Here pt_i, pt'_i correspond to halves of edges mentioned in 2.3 (a) above.

(b) *Strata of maximal codimension.* As in 2.3 (c) above, strata of maximal codimension in $\overline{M}_{0,S'}$ are encoded by "linear graphs" (sequence of neighbouring vertices connected pairwise by edges) that are stabilised by adding two labelled tails at each end of the graph and one labelled tail at each middle vertex. The total set of labels is S' .

Under an admissible projection $p : \overline{M}_{0,S'} \rightarrow \overline{M}_{0,S}$, one should first delete all tails labelled by elements of $S' \setminus S$. After that one should contract all edges that did not occur in the respective tree for a stratum of $\overline{M}_{0,S}$.

2.7. The Grothendieck–Teichmüller monoid and group. In the remaining part of this section, we will describe very sketchily, following [BriHoRo19] and [Fr17], how the Grothendieck–Teichmüller symmetries, first made explicit in [Dr90] and [Ih94], reappear in the context of combinatorial skeleton of genus zero modular operad.

Let $\widehat{\mathbf{F}}_2$ be the profinite completion of the free group with generators x, y , and $\widehat{\mathbf{Z}}$ the profinite completion of \mathbf{Z} .

The Grothendieck–Teichmüller monoid \widehat{GT} is the monoid of endomorphisms of $\widehat{\mathbf{F}}_2$ of the form

$$x \mapsto x^\lambda, \quad y \mapsto f^{-1}y^\lambda f$$

where $(\lambda, f) \in \widehat{\mathbf{Z}} \times \widehat{\mathbf{F}}_2$ satisfy the following equations:

- (a) $f(x, y)f(y, x) = 1$,
- (b) $f((xy)^{-1}, x)(xy)^{-m}f(y, (xy)^{-1})y^m f(x, y)x^m = 1$, $m = (\lambda - 1)/2$,
- (c) "pentagon relation", whose precise form we omit here; see our basic references.

The Grothendieck–Teichmüller group \widehat{GT} , by definition, is the subgroup of invertible elements of \widehat{GT} .

2.8. Embedding $G_{\mathbf{Q}} \rightarrow \widehat{GT}$. The absolute Galois group $G_{\mathbf{Q}}$ acts upon dessins d'enfants, used to encode coverings $Y \rightarrow \mathbf{P}^1$ unramified outside $\{0, 1, \infty\}$ (cf. Section 0 above and basic references). This action can be translated into the embedding $G_{\mathbf{Q}} \rightarrow \widehat{GT}$.

The problem of characterisation of the image of this embedding still remains unsolved.

2.9. Example: braids and their encoding by graphs. An action of the Grothendieck–Teichmüller monoid and group upon combinatorial modular operad,

which plays the central role in [BriHoRo19] and [Fr17], proceeds via replacement of whole families of objects and morphisms by their homotopical versions.

As a part of this replacement, several operads governing “braiding relations” of the type 2.7 (a), (b), (c) are defined, in particular operads of parenthesized braids PaB and parenthesized ribbon braids $PaRB$: see [BriHoRo19], Definitions 6.11 and 6.12.

In our on-going project, striving to avoid the introduction of homotopical algebra, we have to engineer encoding such braiding operations by graphs.

We finish this section by saying a few words about it.

In Example 1.5 above, we have shown how to encode orientation of flags in a graph in order to define “inputs” and “outputs” in the combinatorial presentation of ordinary operads.

Here we can use the similar, albeit slightly more sophisticated trick. In order to explain it, focus on the description of braids as morphisms in the Remark 6.3 of [BriHoRo19]. Cut such a braid in two halves and represent it as the composition of two other braids: *input braid* and *output braid*. Encode these halves by some words in a fixed finite alphabet, and include in the notation information about input/output, such as $+$ and $-$ in Example 1.5.

In the last Section 4, we will see that similar tricks are needed in order to introduce quantum statistical counting of modular dessins: see 4.4.

3. Dessins for modular operad: Galois symmetries

Although the geometric origins of Grothendieck’s dessins d’enfant and of our modular dessins are very different, moduli spaces of curves appeared at a very early stage of Grothendieck’s research dedicated to Galois groups of algebraic numbers (at least 1984, or earlier): cf. [Gr97].

In our present context, the Galois group $Gal(\overline{\mathbf{Q}}/\mathbf{Q})$ enters the scene via its action upon the family of sets $\overline{M}_{0,S}(\overline{\mathbf{Q}})$ compatibly with its tower structure for appropriate “admissible” categories of labelling sets S .

We shall start with some preparatory considerations.

Our approach here is based upon the fact that manifolds $\overline{M}_{0,S}$ and some of the structural morphisms between them have canonical models defined over \mathbf{Q} , and thus also over algebraic extensions $K \supset \mathbf{Q}$ obtained by the scalar extensions: see [Ka93] and [BrMe13].

Therefore, representations of the profinite Galois group G of $\overline{\mathbf{Q}}/\mathbf{Q}$ in the automorphism groups of Kapranov models define forms of $\overline{M}_{0,S}$ over field of algebraic numbers: see [Se13].

We will show that these forms can be united in an enriched genus zero modular operad, upon which G acts compatibly with operadic structure.

3.1. Kapranov model of $\overline{M}_{0,n+1}$. Consider a family of n points p_1, \dots, p_n in \mathbf{P}^{n-2} . Assume that they are in general position, in the sense that any subfamily of $k \leq n - 1$ of these points spans a projective subspace $\mathbf{P}^{k-1} \subset \mathbf{P}^{n-2}$.

Now construct the following tower of successive blow ups of \mathbf{P}^{n-2} : first, blow up all points p_i ; second, in the resulting manifold, blow up all inverse images of lines, spanned by pairs of points (p_i, p_j) (notice that these inverse images have empty intersections); third, blow up inverse images of planes spanned by triples of points (p_i, p_j, p_k) , and so on.

The upper floor of this tower will be our standard model of $\overline{M}_{0,n+1}$. Clearly, it is defined over \mathbf{Q} , as well as the action of \mathbf{S}_n upon it, corresponding to all possible renumberings of (p_1, \dots, p_n) . As was proved in [BrMe13], after an arbitrary extension K of ground field, the full automorphism group of $\overline{M}_{0,n+1} \otimes_{\mathbf{Q}} K$ remains the same.

Generally, for an arbitrary finite set S , the automorphisms of $\overline{M}_{0,S}$ act upon Kapranov models by permutation of S .

3.2. K -forms of $\overline{M}_{0,S}$. It follows that if K is a normal algebraic extension of \mathbf{Q} with Galois group G^K , then K -forms of $\overline{M}_{0,S}$ are in a natural bijection with actions of G^K upon S : see [Se13], Chapter III.

This will allow us to define a tree operad upon which there is a natural action of the (profinite completion) $G := Gal(\overline{\mathbf{Q}}/\mathbf{Q})$ and connect it with a similar extension of the genus zero modular operad.

3.3. Stable tree operad with Galois action. We will now enrich our definition of stable tree operad by declaring that

(a) *flags of any corolla, and hence of any tree, are not just finite sets, but finite G -sets.*

(b) *Binary compositions (and their iterations) are allowed only if they are compatible with actions of G .*

3.4. Genus zero modular operad with Galois action. Similarly, we will now consider enriched modular operad, whose objects are all possible forms of $\overline{M}_{0,S}$, and compositions are compatible with respective actions of G upon labelling sets S .

In order to better visualise this definition, remark that S can be easily reconstructed from the geometry of universal family of curves $C_{0,S} \rightarrow \overline{M}_{0,S}$. Denote by $\sigma_i : \overline{M}_{0,S} \rightarrow C_{0,S}$, $i \in S$, the family of its structural sections. Clearly,

$$\bigcup_{i \in S} \sigma_i(\overline{M}_{0,S}(K)) \subset C_{0,S}(K).$$

Moreover, two sections with different labels $i \neq j \in S$ have empty intersection.

3.5. Proposition. *S is the set of equivalence classes of those points of $C_{0,S}(K)$ that lie on any of structure sections, modulo the equivalence relation “lying on the same section”.*

3.6. Galois action upon modular dessins. It is important to understand in more details the Galois action that we have summarily described above.

Two examples are worth special consideration.

(a) Dessins of strata of maximal codimension: see 2.5 (c) above.

Is their Galois behaviour the same as that Grothendieck’s *clean dessins* after re-encoding?

(b) M. Kapranov suggested to consider strata corresponding to tri-valent trees. The magma operad (see 1.5 above) furnishes an obvious motivation for this: one might conjecture that this will reproduce the embedding of G into the Grothendieck–Teichmüller group invoked in 2.9 above.

Our last Section is dedicated to this task.

4. From operads to quantum statistical mechanical systems

In this Section, we outline a strategy that translates the operadic setting into a quantum statistical mechanical system. We outline the main steps of this strategy in general abstract form, but we also comment on subtleties and difficulties that arise when one implements this general strategy in specific cases, such as the genus zero modular operad with Galois action discussed in the previous sections.

Our starting point is a stable tree operad \mathcal{O} in a monoidal category (\mathcal{C}, \otimes) , which we can think of, as discussed in Section 1 above, as a family of objects of \mathcal{C} labelled by stable trees, together with a family of binary operators between them labelled by the grafting operations on trees. We also assume that there is an action of a group G on the operad \mathcal{O} in the sense that G acts on the family of objects labelled by trees (through an action on the trees) compatibly with the grafting operations.

What we want to construct out of this operad is a *quantum statistical mechanical system*. Such a system consists of a complex *algebra of observables* \mathcal{A} , represented by bounded operators on a Hilbert space \mathcal{H} of states, together with a time evolution.

Representation by bounded operators is denoted $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$.

A *time evolution* is 1-parameter family of automorphisms $\sigma : \mathbf{R} \rightarrow \text{Aut}(\mathcal{A})$, which is generated in the representation on \mathcal{H} by a Hamiltonian operator H in the following sense: $\pi(\sigma_t(A)) = e^{itH} A e^{-itH}$, for all $A \in \mathcal{A}$.

The group G of our initial data should act as *symmetries* of the quantum statistical mechanical system, namely as automorphisms of the algebra \mathcal{A} that commute with the time evolution.

Further data that one derives from such a system include the *partition function* $Z(\beta) := \text{Tr}(e^{-\beta H})$, with β is the inverse temperature, and *equilibrium states*: certain linear functionals on \mathcal{A} .

Among equilibrium states there are *Gibbs states*

$$\varphi_\beta(A) = \frac{1}{Z(\beta)} \text{Tr}(A e^{-\beta H})$$

whenever these are defined, and more generally the *KMS states* at inverse temperature β , whenever that set is non-empty.

We refer the reader to [BraRob97] for the general operator algebraic setting for quantum statistical mechanics, and to Chapter 3 of [CoMa08] for a discussion of several quantum statistical mechanical systems of arithmetic origin.

In the case where the group G of symmetries is the Galois group of an extension \mathbf{K} of \mathbf{Q} , one would also like, as part of the quantum statistical mechanical system construction, to obtain an *arithmetic subalgebra* $\mathcal{A}_{\mathbf{Q}}$ of the algebra of observables \mathcal{A} . It should have the following properties:

(a) KMS states φ_∞ at zero temperature evaluated on observables in $\mathcal{A}_{\mathbf{Q}}$ take values in an embedding $j(\mathbf{K})$ of \mathbf{K} in \mathbf{C} :

$$\varphi_\infty(\mathcal{A}_{\mathbf{Q}}) \subset j(\mathbf{K}).$$

(b) KMS states φ_∞ intertwine the action of G as symmetries of the algebra with the action on the values in \mathbf{K} :

$$\varphi_\infty \circ \gamma(A) = \gamma \circ \varphi_\infty(A), \quad \forall \gamma \in G, \quad \forall A \in \mathcal{A}_{\mathbf{Q}}.$$

We outline in the following subsections a strategy in several steps aimed at this general construction. We also highlight the typical technical difficulties that one expects to encounter at each step.

4.1. Operads and commutative Hopf algebras. Let \mathcal{O} be an operad in the category *Sets*.

There is a general construction of an associated commutative Hopf algebra, which we will denote here by $\mathcal{A}_{\mathcal{O}}$, see [ChaLiv07]. The grafting operation of trees that gives the operad structure gives rise to a coproduct on this Hopf algebra that is closely related to the admissible cuts coproduct in the Connes–Kreimer Hopf algebra of rooted trees, see [CoKr00], [ChaLiv07], [ChaLiv01], [LaMoer06], [Moe01].

Here are some details.

The construction of [ChaLiv07] starts with associating some posets to an operad \mathcal{O} in *Sets*.

Let S be a finite set, and let $\mathcal{T}(S)$ be the set of rooted trees with the vertex set S endowed with the operadic grafting compositions

$$\mathcal{T}(L) \times \prod_{\ell \in L} \mathcal{T}(L_\ell) \rightarrow \mathcal{T}(S)$$

where in $(t, (t_\ell))$ the trees t_ℓ are grafted at their root to the leaves of the tree t . This operation is extended to the set $\mathcal{F}(S)$ of forests with the vertex set S . A partial order structure is determined by the following construction: $f \leq f'$ in $\mathcal{F}(S)$ if f' can be obtained from a subforest of f by a composition map. Each poset obtained in this way in $\mathcal{F}(S)$ has a unique minimal element and a maximal element, that is a rooted tree. One calls such posets “intervals”. In particular, in the collection of posets constructed in this way from forests in $\mathcal{F}(S)$, every interval is isomorphic to a product of maximal intervals. We denote intervals in this partial order by $[f, f']$.

Given such a collection of posets, one can construct an associated commutative Hopf algebra over \mathbf{Q} , the “incidence Hopf algebra”, which we denote here by $\mathcal{A}_{\mathcal{F}(S)}$. It is spanned by the isomorphism classes of products of maximal intervals. The

commutative multiplication of the Hopf algebra is the product of intervals and the coproduct is given by

$$\Delta[f, f'] = \sum_{f \leq f'' \leq f'} [f, f''] \otimes [f'', f'].$$

Note that in general $\mathcal{A}_{\mathcal{F}(S)}$ is a free commutative algebra, but not necessarily generated by the maximal intervals, since there can be isomorphisms of products of maximal intervals with non-pairwise isomorphic factors and with different numbers of factors.

It is shown in sec 6.3 of [ChaLiv07] that the incidence Hopf algebra $\mathcal{A}_{\mathcal{F}(S)}$ is isomorphic to the Connes–Kreimer Hopf algebra of rooted trees. As an algebra, this is the polynomial algebra on the rooted trees τ in which a product of rooted trees τ_i is identified with a forest $f = \tau_1 \sqcup \cdots \sqcup \tau_n$. The coproduct is defined on a tree using admissible cuts C (and extended multiplicatively to forests):

$$\Delta(\tau) = \sum_{C \in \text{Cuts}(\tau)} \rho_C(\tau) \otimes \pi_C(\tau), \quad (4.1)$$

where $\text{Cuts}(\tau)$ is the set of admissible cuts of τ . One admissible cut C is defined as a set of edges of τ that contains at most one edge in any path from the root to a leaf (including the case of the empty set). The term $\pi_C(\tau)$ is the forest consisting of the branches removed by the cut and the term $\rho_C(\tau)$ is the remaining pruned rooted tree after the cut is performed. The antipode is defined recursively by $S(1) = 1$,

$$S(\tau) = -m(S \otimes id - \iota\epsilon)\Delta(\tau)$$

where m is the multiplication, ι the unit and ϵ the counit.

In our setting, we are considering operads \mathcal{O} in a monoidal category (\mathcal{C}, \otimes) . However, for our construction of a quantum statistical mechanical system we still want to associate to them Hopf algebras $\mathcal{A}_{\mathcal{O}}$ over \mathbf{Q} . Thus, it is convenient to still work here with the same kind of Connes–Kreimer Hopf algebra. Namely we let $\mathcal{A}_{\mathcal{O}}$ be the commutative algebra over \mathbf{Q} generated by the isomorphism classes $X_{\tau} := [C_{\tau}]$ of the objects C_{τ} in \mathcal{C} parametrised by the trees τ , with $X_f := X_{\tau_1} \cdots X_{\tau_n}$ for a forest $f = \tau_1 \sqcup \cdots \sqcup \tau_n$. The coproduct is modelled on (4.1):

$$\Delta(X_{\tau}) = \sum_{C \in \text{Cuts}(\tau)} X_{\rho_C(\tau)} \otimes X_{\pi_C(\tau)}. \quad (4.2)$$

4.2. Group of symmetries. Now we want to include in the list of initial data the action of a group G , and describe a modification of the construction of a commutative Hopf algebra $\mathcal{A}_{\mathcal{O}}$ such that the action of G on the trees would induce an action on the Hopf algebra.

This a priori need not be the case in general: depending on the action, admissible cuts for a tree τ may not map under the G -action to admissible cuts for other trees $\gamma\tau$ in the same orbit, and this can potentially create a problem with the compatibility of the G -action with the coproduct (4.2), which we must somehow avoid. Here are some details.

4.3. Definition. (i) An admissible cut of τ is called G -balanced cut, if for each $\gamma \in G$, the pair $(\gamma\rho_C(\tau), \gamma\pi_C(\tau))$ is an admissible cut of the tree $\gamma\tau$. Denote the set of such cuts $Cuts_{G(\tau)}$.

(ii) The action $\gamma \in G$ is the action of γ on the tree components of the forest $\pi_C(\tau)$. The Hopf algebra $\mathcal{A}_{\mathcal{O},G}$ is defined as above as a commutative algebra over \mathbf{Q} , with coproduct

$$\Delta(X_\tau) = \sum_{C \in Cuts_{G(\tau)}} X_{\rho_C(\tau)} \otimes X_{\pi_C(\tau)}. \quad (4.3)$$

The condition that the action of the group G on the set of trees is compatible with the grafting operations of the operad, however, gives a stronger constraint on how G transforms the trees.

4.4. Lemma. The compatibility of the grafting operations of trees with the G -action ensures that $Cuts_{G(\tau)} = Cuts(\tau)$, hence that $\mathcal{A}_{\mathcal{O},G} = \mathcal{A}_{\mathcal{O}}$.

Proof. Compatibility of the grafting operations of trees in the operad \mathcal{O} with the G -action means that for all $\gamma \in G$

$$\gamma\tau_1 *_{(\gamma t_1, \gamma t_2)} \gamma\tau_2 = \gamma \cdot \tau_1 *_{(t_1, t_2)} \tau_2. \quad (4.4)$$

Given an admissible cut $C \in Cuts(\tau)$, the tree τ is given by a grafting

$$\tau = \rho_C(\tau) *_{(\ell_i, r_i)} \pi_C(\tau),$$

where $*_{(\ell_i, r_i)}$ means the successive grafting of the root r_i of the i -th component of the forest $\pi_C(\tau)$ to the i -th leaf ℓ_i of the tree $\rho_C(\tau)$. Under the action of $\gamma \in G$, the compatibility (4.4) of the grafting operations implies that

$$\gamma\tau = \gamma\rho_C(\tau) *_{(\gamma\ell_i, \gamma r_i)} \gamma\pi_C(\tau).$$

This shows that $(\gamma\rho_C(\tau), \gamma\pi_C(\tau))$ is indeed an admissible cut of $\gamma\tau$. The admissibility is guaranteed by the fact that each component in the forest $\gamma\pi_C(\tau)$ has its root γr_i grafted to a leaf $\gamma\ell_i$ of the tree $\gamma\rho_C(\tau)$ rather than to a leaf of one of the previously attached component of $\gamma\pi_C(\tau)$.

Thus, the compatibility requirement (4.4) generally is very strong. But one can cope with it by encoding action of G by appropriate labellings of flags. Then this action will not change combinatorics of trees themselves.

4.5. A semigroup action. The next part of the construction of a quantum statistical mechanical system associated to the operad \mathcal{O} is a semigroup \mathcal{S} acting by endomorphisms of the commutative algebra $\mathcal{A}_{\mathcal{O}}$. Here we will require only that \mathcal{S} acts by commutative algebra homomorphisms, $\mathcal{S} \subset \text{Hom}_{\text{Alg}_{\mathbb{Q}}}(\mathcal{A}_{\mathcal{O}}, \mathcal{A}_{\mathcal{O}})$.

We will not require a compatibility of this semigroup action with the coproduct of $\mathcal{A}_{\mathcal{O}}$.

There is a natural candidate for such a semigroup given an operad \mathcal{O} . Indeed $\mathcal{O}(1)$ of an operad is always a semigroup with multiplication given by the operadic composition $\mathcal{O}(1) \otimes \mathcal{O}(1) \rightarrow \mathcal{O}(1)$. Moreover, the semigroup $\mathcal{O}(1)$ also acts on the components $\mathcal{O}(n)$ of the operad, for $n \geq 2$, again by the operadic composition $\mathcal{O}(1) \otimes \mathcal{O}(n) \rightarrow \mathcal{O}(n)$. In terms of trees, the semigroup $\mathcal{O}(1)$ corresponds to linear trees with one root and one leaf and the grafting of the leaf of one tree to the root of the next. The action of $\mathcal{O}(1)$ upon $\mathcal{O}(n)$ is similarly determined by grafting the leaf of a linear tree to the root of a tree with n leaves. Thus, we obtain the following.

4.6. Lemma. *The action of the semigroup $\mathcal{S} = \mathcal{O}(1)$ on the algebra $\mathcal{A}_{\mathcal{O}}$ is given on the generators by $X_{\tau} \mapsto X_{\ell*\tau}$, where ℓ is a linear tree and $\ell*\tau$ is the grafting of a leaf of ℓ to the root of the tree τ . The action is extended multiplicatively to forests.*

Notice that an admissible cut of a linear tree cuts just one edge. Therefore $\pi_C(\ell)$ is also a linear tree and $\pi_C(\ell)*\tau$ then denotes the grafting of its leaf to the root of τ .

4.7. Remark. If there is an action of a group G on the set of trees, compatible with the grafting operations of the operad \mathcal{O} , for a reason that will become more transparent below, instead of considering the semigroup $\mathcal{S} = \mathcal{O}(1)$ we will only consider the subsemigroup \mathcal{S}^G of the invariants with respect to the G -action in $\mathcal{O}(1)$.

4.8. Remark. In the case of the modular operad with stable components $\{\overline{M}_{0,S}\}$, an implementation of a semigroup playing the role of $\mathcal{O}(1)$ in our environ-

ment may be defined with help of the moduli spaces \overline{L}_n of pointed curves studied in [LoMa00].

4.9. A semigroup crossed product algebra. Starting with a semigroup action $\alpha : \mathcal{S} \rightarrow \text{End}(\mathcal{A})$ on an algebra \mathcal{A} , we will define a semigroup crossed product algebra in the following way.

Assume that all α_ℓ are invertible on their range, and denote by β_ℓ the partial inverses given by $\beta_\ell(\alpha_\ell(A)) = A$ and $\beta_\ell(A) = 0$ if $A \neq \alpha_\ell(A')$ for some $A' \in \mathcal{A}$. Note that this condition of invertibility on the range is satisfied for the action of linear trees ℓ by grafting $\ell * \tau$ on trees τ .

4.10. Definition. Let $\alpha : \mathcal{S} \rightarrow \text{End}(\mathcal{A})$ be an action of a semigroup \mathcal{S} by endomorphisms of an algebra \mathcal{A} with partial inverses β_ℓ . The semigroup crossed product algebra $\mathcal{A} \rtimes \mathcal{S}$ is generated by \mathcal{A} and by elements S_ℓ, S_ℓ^* satisfying the following relations:

- (i) $S_\ell S_{\ell'} = S_{\ell'\ell}$, for all $\ell, \ell' \in \mathcal{S}$.
- (ii) $S_\ell^* S_\ell = 1$, for all $\ell \in \mathcal{S}$.
- (iii) $S_\ell^* A S_\ell = \alpha_\ell(A)$, for all $\ell \in \mathcal{S}$ and all $A \in \mathcal{A}$.
- (iv) $S_\ell A S_\ell^* = \beta_\ell(A)$, for all $\ell \in \mathcal{S}$ and all $A \in \mathcal{A}$.

The first two conditions mean that the S_ℓ define a representation by isometries of the opposite semigroup \mathcal{S}^{op} .

In the case of $\mathcal{A}_{\mathcal{O}}$ with the action of the semigroup of linear trees by grafting on trees, we have $\alpha_\ell X_\tau = X_{\ell * \tau}$. The partial inverse β_ℓ can be applied to an element X_τ when the tree τ starts at the root with a linear tree ℓ , that is, when $\tau = \ell * \tau'$ for some other tree τ' , in which case we have $\beta_\ell(X_\tau) = X_{\tau'}$. One can use the notation $\beta_\ell(X_\tau) = X_{\ell^{-1} * \tau}$, where $\ell^{-1} * \tau = \tau'$ if $\tau = \ell * \tau'$, and $\beta_\ell(X_\tau) = 0$ otherwise.

4.11. Remark. The action of the group G on the algebra $\mathcal{A}_{\mathcal{O}}$ extends to an action on the crossed product $\mathcal{A}_{\mathcal{O}} \rtimes \mathcal{S}$, where the action on the \mathcal{S} part of the crossed product algebra is trivial since we are assuming that the semigroup consists of the elements of $\mathcal{O}(1)$ that are fixed by G .

4.12. Constructing a Hilbert space representation. Let $\mathcal{H} = \ell^2(\mathcal{S})$ be the Hilbert space of square integrable functions on the semigroup \mathcal{S} endowed with the discrete topology. We can represent the elements S_ℓ of the crossed product algebra as bounded operators on \mathcal{H} by

$$S_\ell \epsilon_{\ell'} = \epsilon_{\ell' * \ell}, \tag{4.5}$$

where $\{\epsilon_\ell\}_{\ell \in \mathcal{S}}$ is the standard orthonormal basis of \mathcal{H} , and $\ell' * \ell$ is the multiplication in \mathcal{S} given by the grafting of the leaf of ℓ' to the root of ℓ . Similarly, we let the elements S_ℓ^* act as

$$S_\ell^* \epsilon_{\ell'} = \epsilon_{\ell''} \quad (4.6)$$

if $\ell'' * \ell = \ell'$, and 0 otherwise.

For grafting of linear trees condition that $\ell'' * \ell = \ell'$ is satisfied whenever ℓ' has more nodes than ℓ and it's equivalent to ℓ'' and ℓ being the two parts of an admissible cut of ℓ' .

4.13. Lemma. *The operators S_ℓ and S_ℓ^* acting as in (4.5) and (4.6) are isometries, and they define a representation on \mathcal{H} of the opposite semigroup \mathcal{S}^{op} .*

Proof. The operators S_ℓ and S_ℓ^* acting as in (4.5) and (4.6) are clearly bounded operators on \mathcal{H} satisfying the isometry condition $S_\ell^* S_\ell = 1$, for all $\ell \in \mathcal{S}$. The composition satisfies the identity $S_{\ell_1} S_{\ell_2} = S_{\ell_2 \ell_1}$, hence the operators define a representation of the opposite semigroup \mathcal{S}^{op} .

We then need to construct a representation of the algebra $\mathcal{A}_\mathcal{O}$ on $\mathcal{H} = \ell^2(\mathcal{S})$ that is compatible with (4.5) and (4.6) through the crossed product relation. To this purpose, and so that the construction we make here would be suitable for the goals that we will discuss later, we now focus more specifically on the case where the group G acting on trees and grafting operations is the Galois group $G = Gal(\overline{\mathbf{Q}}/\mathbf{Q})$.

4.14. Definition. *Let \mathcal{O} be an operad with an action of $G = Gal(\overline{\mathbf{Q}}/\mathbf{Q})$ on the set of trees compatibly with the grafting operations. Let $\mathcal{A}_\mathcal{O}$ be the commutative Hopf algebra constructed as above. Let $Hom_{Alg_{\mathbf{Q}}}(\mathcal{A}_\mathcal{O}, \overline{\mathbf{Q}})$ be the set of commutative algebra homomorphisms from $\mathcal{A}_\mathbf{Q}$ to $\overline{\mathbf{Q}}$. Since $\mathcal{A}_\mathcal{O}$ is a Hopf algebra, this set is a group $\mathcal{G}(\overline{\mathbf{Q}}) = Hom_{Alg_{\mathbf{Q}}}(\mathcal{A}_\mathcal{O}, \overline{\mathbf{Q}})$, where \mathcal{G} is the affine group scheme dual to the Hopf algebra. An element $\varphi \in \mathcal{G}(\overline{\mathbf{Q}})$ is called a balanced character if it intertwines the G -action on $\mathcal{A}_\mathcal{O}$ and the G -action on $\overline{\mathbf{Q}}$ in the following sense:*

$$\varphi \circ \gamma = \gamma \circ \varphi, \quad \forall \gamma \in G.$$

We say that a character $\varphi \in \mathcal{G}(\overline{\mathbf{Q}})$ is bounded if (under a fixed choice of an embedding $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$) it satisfies the condition $|\varphi(A)| < C$ for some $C > 0$ and for all $A \in \mathcal{A}_\mathcal{O}$.

The following example should convince the reader that the set of bounded balanced characters of $\mathcal{A}_\mathcal{O}$ is non-empty.

4.15. Example. Let G act on the set of trees of \mathcal{O} so that all the orbits are finite sets. Let $\{\tau\}$ be a set of representatives of the G -orbits on trees of \mathcal{O} , such that the corresponding orbit has size $d_\tau := \text{card Orb}(\tau)$. Choose a set $\{\lambda_\tau\}$ of algebraic numbers such that $|\gamma\lambda_\tau| \leq 1$ for all $\gamma \in G$ (in a fixed embedding $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$), with $\deg(\lambda_\tau) = d_\tau$. Then setting $\varphi(X_{\gamma\tau}) = \gamma\lambda_\tau$ defines a bounded balanced character.

In order to see it, for each representative τ with size of the corresponding G -orbit $d_\tau = \text{card Orb}(\tau)$, choose first an algebraic number λ_τ such that $\text{card Orb}(\lambda_\tau) = [\mathbf{Q}(\lambda_\tau) : \mathbf{Q}] = \deg(\lambda_\tau) = d_\tau$. It is always possible to divide such λ_τ by a sufficiently large integer so that all the G -orbits are contained inside the unit disk, so we can assume that this property is satisfied for λ_τ . Then we obtain a character $\varphi \in \mathcal{G}(\overline{\mathbf{Q}})$ which is both bounded and balanced.

Now we construct a representation of $\mathcal{A}_{\mathcal{O}}$ on $\mathcal{H} = \ell^2(\mathcal{S})$ in the following way.

4.16. Lemma. Let $\varphi \in \mathcal{G}(\overline{\mathbf{Q}})$ be a bounded balanced character as in Definition 4.14. Then setting

$$\pi_\varphi(X_\tau)\epsilon_\ell = \varphi(X_{\ell*\tau})\epsilon_\ell \quad (4.7)$$

defines a representation of the algebra $\mathcal{A}_{\mathcal{O}}$ by bounded operators on the Hilbert space $\mathcal{H} = \ell^2(\mathcal{S})$. Together with (4.5) and (4.6), this determines a representation of the crossed product algebra $\mathcal{A}_{\mathcal{O}} \rtimes \mathcal{S}$ on \mathcal{H} .

Proof. Since φ is a homomorphism in $\text{Hom}_{\text{Alg}\overline{\mathbf{Q}}}(\mathcal{A}_{\mathcal{O}}, \overline{\mathbf{Q}})$ and \mathcal{S} acts by algebra endomorphisms, we have

$$\pi_\varphi(X_\tau X_{\tau'})\epsilon_\ell = \varphi(X_{\ell*\tau})\varphi(X_{\ell*\tau'})\epsilon_\ell = \pi_\varphi(X_\tau)\pi_\varphi(X_{\tau'})\epsilon_\ell.$$

The property that $\pi_\varphi(X_\tau)$ is a bounded operator follows from the boundedness property of the character. We have

$$S_\ell \pi_\varphi(X_\tau) S_\ell^* \epsilon_{\ell'} = \varphi(X_{\ell''*\tau})\epsilon_{\ell'}$$

if $\ell*\ell'' = \ell'$, and 0 otherwise. Furthermore

$$S_\ell^* \pi_\varphi(X_\tau) S_\ell \epsilon_{\ell'} = \pi_\varphi(X_{\ell*\tau}).$$

This gives $S_\ell \pi_\varphi(X_\tau) S_\ell^* = \pi_\varphi(\beta_\ell(X_\tau))$ and $S_\ell^* \pi_\varphi(X_\tau) S_\ell = \pi_\varphi(\alpha_\ell(X_\tau))$. Thus, the relations of the semigrossed product algebra $\mathcal{A}_{\mathcal{O}} \rtimes \mathcal{S}$ are satisfied.

4.17. **Remark.** Let $\mathcal{A}_{\mathcal{O},\mathbf{C}} = \mathcal{A}_{\mathcal{O}} \otimes_{\mathbf{Q}} \mathbf{C}$. The representation $\pi_{\varphi} : \mathcal{A}_{\mathcal{O}} \rightarrow \mathcal{B}(\ell^2(\mathcal{S}))$ of Lemma 4.16 extends to a representation of $\mathcal{A}_{\mathcal{O},\mathbf{C}}$ and of the crossed product $\mathcal{A}_{\mathcal{O},\mathbf{C}} \rtimes \mathcal{S} = (\mathcal{A}_{\mathcal{O}} \rtimes \mathcal{S}) \otimes_{\mathbf{Q}} \mathbf{C}$. Let $\mathcal{B}_{\mathcal{O}}$ denote the C^* -algebra obtained from the crossed product algebra $\mathcal{A}_{\mathcal{O},\mathbf{C}} \rtimes \mathcal{S}$ by including adjoints of the elements of $\pi_{\varphi}(\mathcal{A}_{\mathcal{O},\mathbf{C}})$ and completing it in the operator norm of the algebra of bounded operators $\mathcal{B}(\ell^2(\mathcal{S}))$.

The C^* -algebra $\mathcal{B}_{\mathcal{O}}$ is the semigroup crossed product C^* -algebra $\mathcal{B}_{\mathcal{O}} = \mathcal{A}_{\mathcal{O},\pi_{\varphi}} \rtimes \mathcal{S}$, where $\mathcal{A}_{\mathcal{O},\pi_{\varphi}}$ is the C^* -completion of $\pi_{\varphi}(\mathcal{A}_{\mathcal{O},\mathbf{C}})$ in $\mathcal{B}(\ell^2(\mathcal{S}))$. The \mathbf{Q} -algebra $\mathcal{B}_{\mathcal{O}}^{ar} := \mathcal{A}_{\mathcal{O}} \rtimes \mathcal{S}$ is then referred to as the *arithmetic subalgebra* of $\mathcal{B}_{\mathcal{O}}$.

4.18. **Semigroup homomorphisms and time evolution.** Now we pass to the construction of a time evolution operator on the C^* -algebra $\mathcal{B}_{\mathcal{O}} = \mathcal{A}_{\mathcal{O},\pi_{\varphi}} \rtimes \mathcal{S}$ of Remark 4.17. The central requirement is that the time evolution commutes with the symmetries given by the action of the group G . We use the following strategy.

4.19. **Proposition.** *Suppose that there exists a semigroup homomorphism $\lambda : \mathcal{S} \rightarrow \mathbf{N}$ to the multiplicative semigroup of natural numbers \mathbf{N} , with the property that the growth of the multiplicities $a_n = \{\ell \in \mathcal{S} \mid \lambda(\ell) = n\}$ is such that the Dirichlet series*

$$\sum_{n \geq 1} a_n n^{-\beta} \quad (4.8)$$

converges for sufficiently large β . Then setting

$$\sigma_t(X_{\tau}) = X_{\tau} \quad \& \quad \sigma_t(S_{\ell}) = \lambda(\ell)^{it} S_{\ell}, \quad \sigma_t(S_{\ell}^*) = \lambda(\ell)^{-it} S_{\ell}^*, \quad (4.9)$$

defines a time evolution on the C^ -algebra $\mathcal{B}_{\mathcal{O}} = \mathcal{A}_{\mathcal{O},\pi_{\varphi}} \rtimes \mathcal{S}$ that commutes with the action of G by automorphisms. In the representation of Lemma 4.16 this time evolution is generated by the Hamiltonian*

$$H \epsilon_{\ell} = \log \lambda(\ell) \epsilon_{\ell}, \quad (4.10)$$

with partition function

$$Z(\beta) = \text{Tr}(e^{-\beta H}) = \sum_{\ell \in \mathcal{S}} \lambda(\ell)^{-\beta} \quad (4.11)$$

that converges for sufficiently large inverse temperature β .

Proof. Since $\lambda : \mathcal{S} \rightarrow \mathbf{N}$ is a semigroup homomorphisms with values in a commutative semigroup, we have $\sigma_t(S_{\ell} S_{\ell'}) = \sigma_t(S_{\ell' \ell}) = \lambda(\ell)^{it} \lambda(\ell')^{it} S_{\ell' \ell}$. The action

(4.9) is moreover compatible with the relations of the semigroup crossed product algebra.

Since the time evolution is trivial on the subalgebra $\mathcal{A}_{\mathcal{O},\mathbf{C}}$ and on $\mathcal{A}_{\mathcal{O},\pi_\varphi}$, and nontrivial on the \mathcal{S} part of the crossed product, while the action of G by automorphisms is nontrivial on the $\mathcal{A}_{\mathcal{O},\pi_\varphi}$ part and trivial on the semigroup \mathcal{S} part, the two actions on the crossed product algebra $\mathcal{B}_{\mathcal{O}} = \mathcal{A}_{\mathcal{O},\pi_\varphi} \rtimes \mathcal{S}$ commute.

To see that the time evolution is generated by the Hamiltonian (4.10) we need to check that, for all $A \in \mathcal{A}_{\mathcal{O},\mathbf{C}} \rtimes \mathcal{S}$,

$$\pi_\varphi(\sigma_t(A)) = e^{itH} \pi_\varphi(A) e^{-itH}.$$

This is the case for $A \in \mathcal{A}_{\mathcal{O},\mathbf{C}}$ where the time evolution acts trivially.

For S_ℓ we have $e^{itH} S_\ell e^{-itH} \epsilon_{\ell'} = \lambda(\ell' * \ell)^{it} \lambda(\ell')^{-it} S_{\ell \in \ell'}$, hence $e^{itH} S_\ell e^{-itH} = \lambda(\ell)^{it} S_\ell = \sigma_t(S_\ell)$, and similarly for S_ℓ^* . Under the assumption that the multiplicities grow at most polynomially, $m_n \leq P(n)$, the partition function is given by the Dirichlet series (4.8)

$$Z(\beta) = \sum_{n \in \mathbf{N}} a_n n^{-\beta},$$

and converges for sufficiently large inverse temperature β .

4.20. Lemma. *Let G be an action on the trees of the operad \mathcal{O} , compatible with grafting, such that the orbits of G on the set of trees are finite. Let \mathcal{S} be the semigroup of linear trees fixed by the G -action, with the composition by grafting tail to root. Then there is a choice of a semigroup homomorphism $\lambda : \mathcal{S} \rightarrow \mathbf{N}$ satisfying the properties of Proposition 4.19, such that the partition function $Z(\beta) = \sum_{\ell \in \mathcal{S}} \lambda(\ell)^{-\beta}$ is convergent for all $\beta > 0$.*

Proof. In the case of the semigroup of linear trees $\ell \in \mathcal{S}$ fixed by the G -action, an example of a homomorphism $\lambda : \mathcal{S} \rightarrow \mathbf{N}$ satisfying the growth condition of Proposition 4.19 can be obtained in the following way. Let $L(\ell) := \text{card } E(\ell)$ be the length of the linear graph ℓ counted as number of edges. We assign $L(\ell) = 0$ to a graph ℓ consisting of a single vertex, which we include as the unit of the semigroup. Let \mathcal{L} be the set of labels of vertices and edges on which the group G also acts. Since we are assuming that the semigroup \mathcal{S} consists of elements that are fixed by G , the labels of such linear graphs must be in the subset \mathcal{L}^G of G -fixed points in \mathcal{L} . Let $k = \text{card } \mathcal{L}^G$ be the cardinality of this set, which we assume finite, since we

work under the assumptions that orbits of G on the set of trees of the operad \mathcal{O} are finite.

Let $N \in \mathbf{N}$ be chosen so that $N > 2k^2$. Then setting $\lambda(\ell) = N^{L(\ell)}$ defines a semigroup homomorphism $\lambda : \mathcal{S} \rightarrow \mathbf{N}$ to the multiplicative semigroup of positive integers, since the length is additive under grafting: $L(\ell * \ell') = L(\ell) + L(\ell')$.

A linear tree ℓ has $L(\ell)$ edges and $L(\ell) + 1$ vertices. If edges and vertices are labelled by \mathcal{L}^G , this gives $k^{L(\ell)} \cdot k^{L(\ell)+1}$ possible choices of ℓ . Thus, the partition function $Z(\beta)$ is computed by the series

$$\sum_{L \in \mathbf{N}} k^L \cdot k^{L+1} N^{-L}.$$

In view of the choice of N , this is bounded by

$$\sum_{L \in \mathbf{N}} k^{2L+1} k^{-2L} 2^{-L} = k \sum_{L \geq 1} 2^{-L} \leq k.$$

Thus, in this case the partition function is convergent for all $\beta > 0$.

Moreover, we see that in this situation all KMS states are Gibbs states of the form

$$\phi_\beta(A) = \frac{1}{Z(\beta)} \text{Tr}(\pi_\varphi(A) e^{-\beta H}).$$

We discuss the zero-temperature ground states and their properties in the next subsection.

4.21. Gibbs states and ground states. The ground states at zero temperature are defined in Chapter 3 of [CoMa09] as the weak limits of the KMS-states at large inverse temperature β , when $\beta \rightarrow \infty$:

$$\phi_\infty(A) = \lim_{\beta \rightarrow \infty} \phi_\beta(A) = \lim_{\beta \rightarrow \infty} \frac{1}{Z(\beta)} \text{Tr}(\pi_\varphi(A) e^{-\beta H}),$$

for Gibbs states ϕ_β at inverse temperature β . The group G acts on the set of Gibbs states at a given β by pullback, $\gamma^* \phi_\beta(A) = \phi_\beta(\gamma(A))$.

4.22. Proposition. *Consider the time evolution of Lemma 4.20. The ground states, when restricted to the arithmetic subalgebra $\mathcal{B}_\mathcal{O}^{ar}$, take values in $\overline{\mathbf{Q}}$ and satisfy the intertwining property with respect to the G -action*

$$\phi_\infty \circ \gamma = \gamma \circ \phi_\infty.$$

Proof. We consider the time evolution defined by the semigroup $\lambda : \mathcal{S} \rightarrow \mathbf{N}$ with $\lambda(\ell) = N^{L(\ell)}$ discussed in Lemma 4.20.

The kernel of the corresponding Hamiltonian $H\epsilon_\ell = \log \lambda(\ell)\epsilon_\ell = L(\ell) \log(N) \epsilon_\ell$ is spanned by a single vector ϵ_ℓ , corresponding to the graph ℓ consisting of a single vertex, which we have included as unit of the semigroup. We write this vector as ϵ_1 .

We then see from the above that we have

$$\phi_\infty(A) = \langle \epsilon_1, A \epsilon_1 \rangle,$$

namely the limit is the ground state in the usual sense of projection onto the kernel of the Hamiltonian.

Consider the case where A is an element of the arithmetic subalgebra $\mathcal{B}_{\mathcal{O}}^{ar}$. It suffices to consider the elements $A \in \mathcal{A}_{\mathcal{O}}$ since any element in $\mathcal{B}_{\mathcal{O}}^{ar}$ that is not contained in $\mathcal{A}_{\mathcal{O}}$ will have projection $\langle \epsilon_1, A \epsilon_1 \rangle = 0$. In fact, there would be a number of S_ℓ or S_ℓ^* terms that map ϵ_1 to some other $\epsilon_{\ell'}$ that is orthogonal to ϵ_1 . Thus, we can restrict ourselves to the case $A = X_\tau$ in $\mathcal{A}_{\mathcal{O}}$. The case of a forest is similar.

Since in the construction we have chosen $\varphi \in \mathcal{G}(\overline{\mathbf{Q}})$ to be a balanced character, we obtain

$$\langle \epsilon_1, \pi_\varphi(X_\tau) \epsilon_1 \rangle = \varphi(X_\tau).$$

Hence ϕ_∞ evaluated on $\mathcal{B}_{\mathcal{O}}^{ar}$ takes values in $\overline{\mathbf{Q}}$. We then have

$$\phi_\infty(\gamma X_\tau) = \varphi(\gamma X_\tau) = \gamma \varphi(X_\tau) = \gamma \phi_\infty(X_\tau),$$

so we obtain the intertwining condition.

We can also write more explicitly the Gibbs states at finite values of the inverse temperature as follows.

4.23. Corollary. *The values of Gibbs states at inverse temperature $\beta > 0$ on elements X_τ are given by*

$$\phi_\beta(X_\tau) = \frac{1}{Z(\beta)} \sum_{\ell \in \mathcal{S}} \varphi(X_{\ell \star \tau}) \lambda(\ell)^{-\beta}.$$

Proof. We have

$$\begin{aligned}\phi_\beta(X_\tau) &= \frac{1}{Z(\beta)} \text{Tr}(\pi_\varphi(X_\tau) e^{-\beta H}) = \frac{1}{Z(\beta)} \sum_{\ell \in \mathcal{S}} \langle \epsilon_\ell, \pi_\varphi(X_\tau) \epsilon_\ell \rangle \lambda(\ell)^{-\beta} \\ &= \frac{1}{Z(\beta)} \sum_{\ell \in \mathcal{S}} \langle \epsilon_1, S_\ell^* \pi_\varphi(X_\tau) S_\ell \epsilon_1 \rangle \lambda(\ell)^{-\beta} = \frac{1}{Z(\beta)} \sum_{\ell \in \mathcal{S}} \varphi(X_{\ell * \tau}) \lambda(\ell)^{-\beta}.\end{aligned}$$

Thus, we can regard these as normalised generating series of the values $\varphi(X_{\ell * \tau})$ weighted by $\lambda(\ell)^{-\beta}$.

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