Max-Planck-Institut für Mathematik
Bonn

Symplectic 4-manifolds on the Noether line and between the Noether and half Noether lines

by

Sümeyra Sakalli

Max-Planck-Institut für Mathematik
Preprint Series 2019 (52)

Symplectic 4-manifolds on the Noether line and between the Noether and half Noether lines

Sümeyra Sakalli

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

MPIM 19-52

SYMPLECTIC 4-MANIFOLDS ON THE NOETHER LINE AND BETWEEN THE NOETHER AND HALF NOETHER LINES

SÜMEYRA SAKALLI

Abstract. We construct minimal, simply connected and symplectic 4-manifolds on the Noether line and between the Noether and half Noether lines by star surgeries introduced by Karakurt and Starkston, and by using complex singularities. We show that our manifolds have exotic smooth structures and each of them has one basic class. We also present a completely geometric way of constructing certain configurations of complex singularities in the rational elliptic surfaces, without using any monodromy arguments.

1. INTRODUCTION

For a closed, simply connected, symplectic 4-manifold X, a pair of invariants are defined as follows: $\chi_h(X) := (e(X) + \sigma(X))/4$ and $c_1^2(X) := 2e(X) + 3\sigma(X)$, where $e(X)$ and $\sigma(X)$ denote the Euler characteristic and the signature of X, respectively. The (χ_h, c_1^2)-plane is called the geography chart on which the following lines

$$c_1^2 = 2\chi_h - 6 \quad \text{and} \quad c_1^2 = \chi_h - 3 \tag{1}$$

are called the Noether and half Noether lines, respectively. Note that for minimal complex surfaces S of general type, the Noether inequality $c_1^2(S) \geq 2\chi_h(S) - 6$ holds (see e.g. [BHPVdV04]). Moreover, it is known that all minimal complex surfaces of general type have exactly one (Seiberg-Witten) basic class, up to sign [Wit94]. Thus, it is natural to ask if one can construct smooth 4-manifolds with one basic class. In [FS00], Fintushel and Stern built a family of simply connected, spin, smooth, nonsymplectic 4-manifolds with one basic class. Then, Fintushel, Park and Stern constructed a family of simply connected, noncomplex, *symplectic* 4-manifolds with one basic class which fill the region between the half-Noether and Noether lines in the (χ_h, c_1^2)-chart [FPS02]. Later Akhmedov constructed infinitely many simply connected, nonsymplectic and pairwise nondiffeomorphic 4-manifolds with nontrivial Seiberg-Witten invariants [Akh07]. Park and Yun also gave a construction of an infinite family of simply connected, nonspin, smooth, nonsymplectic 4-manifolds with one basic class [PY07]. All these manifolds were obtained via knot surgeries, blow-ups and rational blow-downs.

In [KS16], Karakurt and Starkston introduced *star surgeries* which are new 4dimensional symplectic operations. A star surgery is the operation of cutting out the neighborhood of a star shaped plumbing of symplectic 2-spheres inside a symplectic 4-manifold, and replacing it with a convex symplectic filling of strictly smaller Euler characteristic. Also in [Sta16], Starkston showed that infinitely many star surgeries are not equivalent to any sequences of generalized symplectic rational blow-downs.

In this paper we will give new constructions of minimal, simply connected and symplectic 4-manifolds on the Noether line and between the Noether and half Noether lines by using various types of star surgeries and complex singularities. We will also show that each of our manifolds has one basic class and

2010 *Mathematics Subject Classification.* Primary 57R55, 57R57, 57R17; Secondary 14J17, 14J27, 32Q15.
Key words and phrases. symplectic 4-manifolds, Seiberg-Witten basic class, complex singularities.

has exotic smooth structure. By the latter we mean that they are homeomorphic but not diffeomorphic to the manifolds with standard smooth structures. We would like to note that in [Ham08], symplectic 4-manifolds on the Noether line with bigger Euler characteristics were built. On the other hand, in literature there are different constructions of our symplectic 4-manifolds having the same topological invariants (e.g. [FPS02, ABB$^+$10]). However, here we will give completely different construction and we do not know if our manifolds are diffeomorphic to the previously constructed ones. In fact, giving different constructions of smooth or symplectic 4-manifolds with the same invariants is interesting and an active research area. For instance, see [Par05, SS05, FS06, Mic07, KS16, AN, AS19] for distinct constructions of symplectic 4-manifolds which are all exotic copies of $\mathbb{CP}^2\#6\overline{\mathbb{CP}}^2$ and $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$. However, as of today, it is not known how to distinguish the smooth structures of symplectic, exotic 4-manifolds that have the same topological invariants but are obtained in different ways, and it is an intriguing question. *Let us also point out that there are many other simply connected, exotic 4-manifolds, e.g.* [Don87, Par05, PSS05, SS05, FS06, Akh07, FPS07, Mic07, Akh08, BK08, AP08, ABB$^+$10, AP10, FS11] *are just a few papers, among many others. Today, the exotic copies of* $\mathbb{CP}^2\#2\,\overline{\mathbb{CP}}^2$ *constructed in* [AP10] *and* [FS11] *are the smallest known exotic 4-manifolds, i.e., they have the smallest Euler characteristic among the all known exotic 4-manifolds.*

Lastly, let us mention the complex singularities part of our paper. To be able to apply star surgeries in building our manifolds, first we need to construct star shaped plumbings of symplectic 2-spheres inside symplectic 4-manifolds. To build such plumbings, we will proceed as follows. At first, in Section 2, we will present a geometric way of acquiring configurations of I_n singular fibers inside the elliptic surface $E(1) := \mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$. Here the I_n singular fiber is a plumbing of n complex 2-spheres of self intersections -2 arranged in a cycle and was given by Kodaira in his famous work [Kod63]. Indeed, in [SSS07], Section 8.4 the authors constructed *a single* I_n fiber in $E(1)$ from a pencil. However they noted the following fact: "To understand the other fibers in such a(n elliptic) fibration (over S^2) is considerably harder, and when studying more singular fibers, we rather use the monodromy theoretic approach." In the following section, we will explain how to obtain more than one I_n singularity in $E(1)$ in a completely geometric way, without using any monodromies. Namely, starting from specific pencils of cubic curves, by symplectic blow-ups we will realize configurations of I_n fibers, by following the work in [Nar87].

Then, in sections 3, 4 and 5, by using the I_n singularities, we will construct star shaped plumbings in certain symplectic 4-manifolds and apply star surgeries. In this way, in Section 3 and 4 we will obtain minimal, simply connected and symplectic 4-manifolds with one basic class that are lying on the Noether line and between the half Noether and Noether lines, respectively. In the last section we will give additional constructions and show that our manifolds are on and above the Noether line.

Conventions and Notations. It is well-known that blow-ups and blow-downs can be done symplectically thanks to McDuff's result [McD90], and on this paper all blow-ups are performed in the symplectic category. (For an excellent exposition of these operations in the complex and symplectic categories, the reader may see [MS95]).

Let us end this section by recalling the fiber sum operation. First, an elliptic surface is a complex surface which admits a genus one fibration over a complex curve with finitely many singular fibers. We take two elliptic surfaces $S_1,\ S_2$, from each we take out regular neighborhoods of the generic fibers $T^2 \times D^2$. Then we glue the remaining pieces $S_i \setminus (T^2 \times D^2)$ along their boundaries by a fiber preserving, orientation reversing diffeomorphism. This operation is called the fiber sum and the resulting manifold $S_1\#_f S_2$ also admits an elliptic fibration. In the sequel, $E(n)$ will denote the elliptic surface which is the n-fold fiber sum of copies of $E(1) := \mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$, where $E(1)$ is equipped with an elliptic fibration. In particular $E(n) = E(n-1)\#_f E(1)$, with $e(E(n)) = 12n$, $\sigma(E(n)) = -8n$ and $\pi_1(E(n)) = 1$ (Example 5.2 in

[Gom95], Chapter 3 in [GS99]). Moreover $E(n)$ could be described as an n-cyclic branched cover of $E(1)$ and hence it admits a complex structure ([GS99], Remark 3.1.8 and 7.3.11). In addition, since $E(n)$ is Kähler, it is symplectic. On this paper, we consider $E(n)$ as a symplectic 4-manifold.

Acknowledgements. I would like to thank Anar Akhmedov for his useful comments and discussions on an earlier draft of this paper. I also thank Tian-Jun Li for his comments and pointing out a typo. I acknowledge the financial support and hospitality of the Max Planck Institute for Mathematics, Bonn.

2. Configurations of I_n singularities in the rational elliptic surfaces

Let us recall the following facts from [MP86]. A (Jacobian) rational elliptic surface is the complex projective plane blown up at nine points, $\mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$, which admits an elliptic fibration over $\mathbb{CP}^1$ (with a section). A cubic pencil in $\mathbb{CP}^2$ is a one-dimensional linear system of cubics, which has nine basepoints by Bezout's theorem. By blowing up the basepoints of a cubic pencil we obtain a rational elliptic surface. The exceptional divisors of square -1 correspond exactly to the sections. Moreover,

Proposition 2.1. *(*[MP86]*) Every Jacobian rational elliptic surface is the blow up of the basepoints of a cubic pencil.*

When we drop the assumption of being Jacobian, a rational elliptic surface is still a blow up of $\mathbb{CP}^2$ at nine points, though blow-ups are not necessarily at the basepoints of a cubic pencil [MP86]. In [Nar87], Naruki explicitly constructed configurations of singular fibers in rational elliptic surfaces by blowing up cubic pencils. See also [Per90], pp. 7-14, for the complete list of singular fibers in the global elliptic fibrations on $\mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$, where Persson also noted that the configurations other than type I_1 and II are obtained by blowing up cubic pencils. Here, we will not consider the global fibrations. We will work with the configurations without I_1 and II fibers. In addition, in [Kur14], Naruki's work was generalized; starting with cubic pencils more configurations were shown to exist. All in all, from [Nar87, Per90, Kur14] we have the following configurations in $\mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$ that will be used in the sequel:

$$\begin{aligned} I_6, I_3, I_2 \\ I_5, I_4 \\ I_4, I_3, 2I_2 \\ 2I_5 \end{aligned} \tag{2}$$

We would like to note that all of these configurations were obtained from blowing up cubic pencils. Let us take the I_6, I_3, I_2 configuration which was constructed in $\mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$ (Section 2.9 in [Nar87]). Note that in the notation of [Nar87], $\tilde{A}_k$ corresponds to I_{k+1} fiber (p.318). Let us also note that the Euler numbers of the I_k fibers are $e(I_k) = k, \;\; k \geq 2$. From the simple Euler number computation we see that inside $\mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$ there has to be an additional fiber I_1. After constructing I_6, I_3, I_2 fibers by blow-ups, Naruki showed the existence of the I_1 fiber via Cremona transformations which correspond to holomorphic automorphisms of the corresponding variety.

We would like to give a more detailed construction of I_6, I_3, I_2, I_1 configuration, find the homology classes of the sphere components of each fiber and verify that their self intersections are -2. Finding homology classes also enables us to do computations (for instance in finding the symplectic Kodaira dimension of the resulting manifolds). In this way, we will make this configuration more accessible to work with. First, Naruki showed the existence of a pencil P generated by a nodal cubic C and the union of a conic Q and a line L in $\mathbb{CP}^2$, by giving their defining polynomials. C and Q intersect only at the node p of C with multiplicity 6, and C and L intersect at a point q. Here q is the inflexion point and L is the corresponding inflexion line as we show in Figure 1. Hence the base points of the pencil are p and q. Then he takes a member C_1 of the pencil P, which has a node at q (and of course passes through the other basepoint p). The intersection multiplicity of C_1 and L is 3. See Figure 1 for the sketch of this pencil.

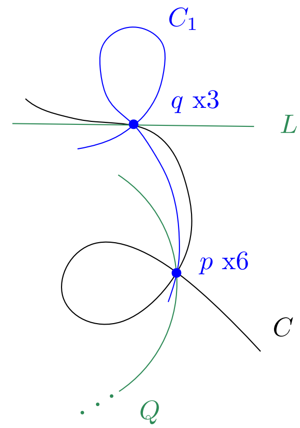

FIGURE 1. Pencil of cubic curves

Then he blew up $\mathbb{CP}^2$ three times over q and six times over p. These blow-ups give us I_3 and I_6 fibers, respectively, in $\mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$. We see this as follows. Take the cubic C_1 having a node at q, and the union $L \cup Q$. Both C_1 and $L \cup Q$ has equal homology classes $3h$. C_1 and L intersect at q with multiplicity three. See the first part of Figure 2, where C_1 is shown in blue and L and Q are in green.

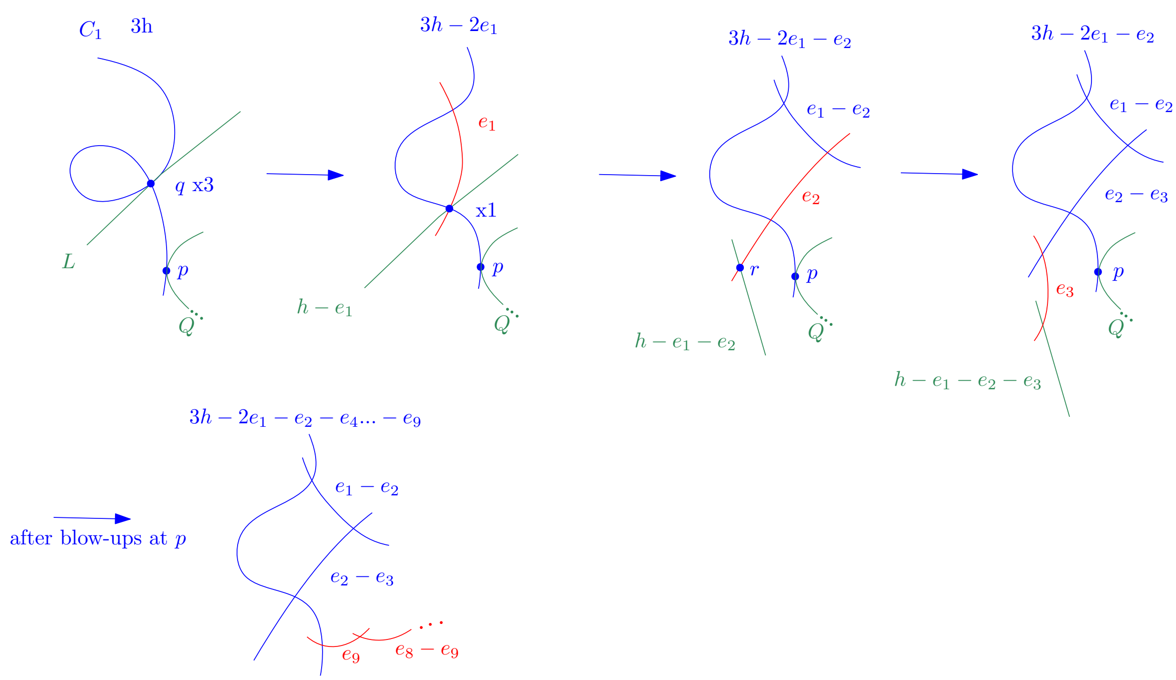

FIGURE 2. Obtaining I_3 fiber

We start with blowing up at q and denote the resulting exceptional sphere by e_1. We note that after the first blow up, the intersection multiplicity drops down by two and the node of the cubic is resolved. The homology class of the strict transform $\widetilde{C_1}$ of C_1 is $3h - 2e_1$. On the other hand, the total homology class of Q and the strict transform $\widetilde{L}$ of L is $2h + (h - e_1)$ (step 2 of Figure 2). To equate the homology classes of $\widetilde{C_1}$ and $Q \cup \widetilde{L}$, we add e_1 to $\widetilde{C_1}$ (to the blue part), and blow up at the point q again. The homology class of the strict transform of C_1 becomes $3h - 2e_1 - e_2$ where e_2 is the second exceptional sphere (step 3 of Figure 2). To equate the total homology classes of the blue part and green part again, we add e_2 to the blue part. Then we blow up the intersection point r of e_2 with the strict transform of L. We denote the exceptional divisor by e_3 that separates the green line from the blue part (step 4 of Figure 2). Next, we blow up over the point p six times. The homology class of the strict transform of C_1 becomes $3h - 2e_1 - e_2 - e_4 - ... - e_9$ so that its self intersection is -2. This -2 sphere with the -2 spheres $e_1 - e_2$ and $e_2 - e_3$ give the I_3 fiber in $\mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$ (last part of Figure 2).

Similarly, by blowing up six times at p, the strict transform of the cubic C (shown in black in the previous figure (Figure 1)) and five more -2 spheres coming from the blow ups give us the I_6 fiber. On the other hand, we note that the conic Q and the line L intersect twice and thus the I_2 fiber comes from their strict transforms. This gives the I_6, I_3, I_2 configuration in $\mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$. Lastly the I_1 fiber comes from another cubic passing through the points p, q ([Nar87]), but we will not use it in our constructions.

Other configurations in Equation 2 could be constructed similarly.

3. Constructions of a minimal, simply connected and symplectic 4-manifold on the Noether line

We will begin with constructing a minimal, simply connected and symplectic 4-manifold on the Noether line by using a star surgery introduced in [KS16]. We consider the elliptic surface $E(5)$ which is the 5-fold fiber sum of copies of $E(1) := \mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$ equipped with an elliptic fibration. We will apply the so called $(\mathcal{Q}, \mathcal{R})$-*star surgery* to $E(5)$. Let us review this surgery first.

$\mathcal{Q}$ is the configuration of symplectic spheres which intersect according to a star shaped graph with 4 arms. The central vertex u_0 is a -5 sphere, and the arms respectively contain one -3 sphere u_1; one -2 sphere u_2; -2 and -3 spheres u_3 and u_4; and lastly two -2 spheres u_5 and u_6 (see Figure 3 and also figure 4 in [KS16]).

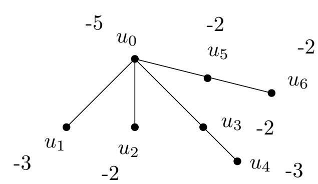

FIGURE 3. The configuration $\mathcal{Q}$

The intersection form $[\mathcal{Q}]$ for $H_2(\mathcal{Q}, \mathbb{Z})$ is given by a 7×7 matrix

$$\begin{bmatrix} -5 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & -3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

and its inverse is

$$-1/261\begin{bmatrix} 90 & 30 & 45 & 54 & 18 & 60 & 30\\ 30 & 97 & 15 & 18 & 6 & 20 & 10\\ 45 & 15 & 153 & 27 & 9 & 30 & 15\\ 54 & 18 & 27 & 189 & 63 & 36 & 18\\ 18 & 6 & 9 & 63 & 108 & 12 & 6\\ 60 & 20 & 30 & 36 & 12 & 214 & 107\\ 30 & 10 & 15 & 18 & 6 & 107 & 184 \end{bmatrix}$$

We diagonalize $[\mathcal{Q}]$ over $\mathbb{R}$ and find that the signature $\sigma(\mathcal{Q})$ of $\mathcal{Q}$ is -7. On the other hand, $\mathcal{R}$ is the simply connected, symplectic 4-manifold of Euler characteristic 3, with convex boundary. In addition its intersection form for $H_2(\mathcal{R}, \mathbb{Z})$ is given by the 2×2 negative definite matrix (Lemma 3.8 in [KS16]):

$$\begin{bmatrix} -10 & -23\\ -23 & -79 \end{bmatrix}$$

Thus we find that its signature is -2. Let ξ_{can} denote the canonical contact structure on the boundary $\partial\mathcal{Q}$ of $\mathcal{Q}$. The boundary of $\mathcal{R}$ with the induced contact structure is contactomorphic to $(\partial\mathcal{Q}, \xi_{can})$ (Proposition 2.6 in [KS16]).

Definition 3.1. *Replacing the neighborhood of $\mathcal{Q}$ in a symplectic 4-manifold by the filling $\mathcal{R}$ is called the $(\mathcal{Q}, \mathcal{R})$ surgery.*

We will construct $\mathcal{Q}$ in $E(5)\#\overline{\mathbb{CP}}^2$ and symplectically replace its neighborhood by the filling $\mathcal{R}$. First, as we discussed in Section 2, $E(1)$ contains one I_4, one I_3 and two I_2 singular fibers and the -1 section hits each I_n fiber at one of its -2 sphere components. (Here we note that the homomorphism from the Mordell-Weil group (group of sections) to the group of every singular fiber is injective [Per90].) In the construction of $E(5)$ we take 5-fold fiber sum of $E(1)$'s and sew the -1 sections together. This gives us the -5 section of $E(5)$ and the I_n singularities descend from the 5 copies of $E(1)$. Hence, there are 5 copies of (I_4, I_3, $2I_2$) singular fibers in $E(5)$ obtained in this way and the -5 section hits each I_n fiber at one of its -2 sphere components.

Let us fix one copies of I_4, I_3, and I_2 in $E(5)$ (see Figure 4) and denote the -2 spheres of I_4 by A, B, C, D, the -2 spheres of I_3 by E, F, G and the -2 spheres of I_2 by H, I such that D, E, H have intersections with the section.

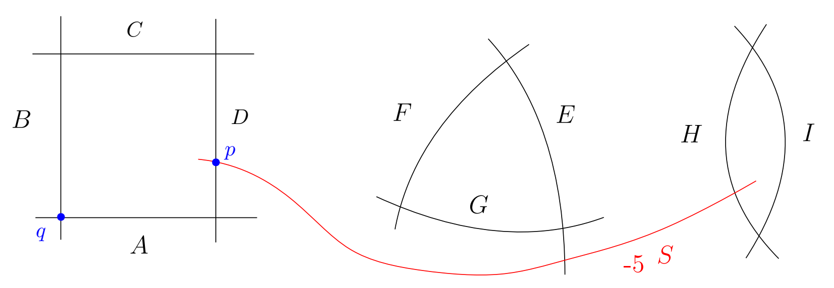

FIGURE 4. Obtaining $\mathcal{Q}$ from three of the singular fibers in $E(5)$

We take the -5 section S and the -2 sphere D of I_4 intersecting S, and symplectically resolve their intersection. This gives us a -5 symplectic sphere which we take as the central vertex u_0 of $\mathcal{Q}$. Next, we blow-up the intersection point q of the -2 spheres A and B in I_4. The proper transform of A is the -3 sphere and we take it as u_1 of $\mathcal{Q}$. For the other arm of $\mathcal{Q}$ containing the spheres u_3, u_4, we take -2 sphere C as u_3

and the proper transform of B as the -3 sphere u_4. Note that A and B are separated after the blow-up. For u_5, u_6 we take the -2 spheres E, F of the I_3 fiber. Lastly, the -2 sphere H is taken as u_2. Hence we obtain $\mathcal{Q}$ in $E(5)\#\overline{\mathbb{CP}}^2$ (see Figure 4). Let

$$X = ((E(5)\#\overline{\mathbb{CP}}^2) \setminus \mathcal{Q}) \cup \mathcal{R} := (W \setminus \mathcal{Q}) \cup \mathcal{R}.$$

Then $\sigma(X) = \sigma(W) - \sigma(\mathcal{Q}) + \sigma(\mathcal{R}) = -41 + 7 - 2 = -36$ and $e(X) = e(W) - e(\mathcal{Q}) + e(\mathcal{R}) = 61 - 8 + 3 = 56$. Thus,

$$\chi_h(X) = 5 \quad \text{and} \quad c_1^2(X) = 4 = 2\chi_h - 6 \tag{3}$$

which shows that X is on the Noether line. From Van Kampen's theorem, we easily see that X is simply connected as $\mathcal{R}$ is simply connected.

Next we will prove that X is minimal. The Seiberg-Witten basic classes of $E(5)\#\overline{\mathbb{CP}}^2 = W$ are $\pm f \pm E_1$ and $\pm 3f \pm E_1$ where $f, E_1 \in H^2(W, \mathbb{Z})$ are the Poincaré duals of the homology classes of the regular fiber and the exceptional sphere coming from the blow-up, respectively ([GS99]). We need to determine which classes extend to X. Let $P = f + E_1$ and $\gamma_0, ..., \gamma_6$ be the basis of $H^2(\mathcal{Q}, \mathbb{Q})$ which is dual to $u_0, ..., u_6$. Then

$$P|_{\mathcal{Q}} \quad = (P \cdot u_0)\gamma_0 + (P \cdot u_1)\gamma_1 + (P \cdot u_4)\gamma_4 \quad = \gamma_0 + \gamma_1 + \gamma_4.$$

From inverse of the intersection matrix $[\mathcal{Q}]$ we find that

$$(P|_{\mathcal{Q}})^2 = -1.54 \tag{4}$$

Now let us prove the following

Lemma 3.2. *Let us assume that there is a basic class $\widetilde{P}$ on X such that $\widetilde{P}|_{X-\mathcal{R}} = P|_{W-\mathcal{Q}}$. Then $(\widetilde{P}|_{\mathcal{R}})^2 < 0$.*

Proof. The intersection form of $\mathcal{R}$ is given by ([KS16]):

$$\begin{bmatrix} -10 & -23 \\ -23 & -79 \end{bmatrix}$$

and its inverse is

$$1/261 \begin{bmatrix} -79 & 23 \\ 23 & -10 \end{bmatrix}$$

Let r_1, r_2 be the generators of the second homology of $\mathcal{R}$ and s_1, s_2 be their duals. Then for some a, b

$$\begin{aligned} (\widetilde{P}|_{\mathcal{R}})^2 &= ((\widetilde{P} \cdot r_1)s_1 + (\widetilde{P} \cdot r_2)s_2)^2 \\ &:= (a \cdot s_1 + b \cdot s_2)^2 \\ &= a^2(-79/261) + 2ab(23/261) + b^2(-10/261) \\ &= (-79a^2 + 46ab - 10b^2)/261 \\ &< 0 \end{aligned}$$

where the last inequality follows from the fact that $(a\sqrt{79} - b\sqrt{10})^2 \geq 0$ so $79a^2 + 10b^2 > 56ab$ if a and b are of the same sign. (If they are of the opposite sign, we are already done.) $\square$

Therefore the dimension of the SW moduli space satisfies the following:

$$\begin{aligned} d_X(\widetilde{P}) &= \frac{\widetilde{P}^2 - 3\sigma(X) - 2\chi(X)}{4} \\ &= \frac{P^2 - (P|_{\mathcal{Q}})^2 + (\widetilde{P}|_{\mathcal{R}})^2 - 3\sigma(X) - 2\chi(X)}{4} \\ &= \frac{-1 + 1.54 + (\widetilde{P}|_{\mathcal{R}})^2 - 4}{4} \\ &= \frac{-5 + 1.54 + (\widetilde{P}|_{\mathcal{R}})^2}{4} \\ &< 0 \end{aligned}$$

where the last inequality follows from the Lemma 3.2. This contradicts our assumption that $\widetilde{P}$ is a basic class of X.

Next, let $L = f - E_1$ and assume that there is a basic class $\widetilde{L}$ on X such that $\widetilde{L}|_{X-\mathcal{R}} = L|_{W-\mathcal{Q}}$. Similarly as above,

$$\begin{aligned} L|_{\mathcal{Q}} &= \gamma_0 - \gamma_1 - \gamma_4, \\ (L|_{\mathcal{Q}})^2 &= -0.8 \end{aligned}$$

which implies

$$\begin{aligned} d_X(\widetilde{L}) &= \frac{-5 + 0.8 + (\widetilde{L}|_{\mathcal{R}})^2}{4} \\ &< 0 \end{aligned}$$

since $(\widetilde{L}|_{\mathcal{R}})^2 < 0$, too by Lemma 3.2. (Note that the computations in Lemma 3.2 only depends on the filling $\mathcal{R}$). Hence, again we reach a contradiction.

Let us look at the class $N := 3f - E_1$. Under the assumption that there is a basic class $\widetilde{N}$ on X such that $\widetilde{N}|_{X-\mathcal{R}} = N|_{W-\mathcal{Q}}$, we have

$$\begin{aligned} N|_{\mathcal{Q}} &= 3\gamma_0 - \gamma_1 - \gamma_4, \\ (N|_{\mathcal{Q}})^2 &= 9(\gamma_0)^2 + (\gamma_1)^2 + (\gamma_4)^2 - 6\gamma_0\gamma_1 - 6\gamma_0\gamma_4 + 2\gamma_1\gamma_4 \\ &= -1/261(810 + 97 + 108 - 180 - 108 + 12) \\ &= -2.83 \end{aligned}$$

Therefore,

$$\begin{aligned} d_X(\widetilde{N}) &= \frac{-5 + 2.83 + (\widetilde{N}|_{\mathcal{R}})^2}{4} \\ &< 0. \end{aligned}$$

This contradiction shows that N does not extend to X as a basic class, either.

However, up to sign, the last class $M := 3f + E_1$ extends to the symplectic manifold X as a basic class $\widetilde{M}$. In fact, if $\widetilde{M}$ was not a basic class of X, this would contradict the fact that X has at least one pair of basic classes by Taubes' theorem ([Tau94]). Hence we conclude that $\widetilde{M}$ is, up to sign, the only basic class of X. By the blow-up formula ([GS99, FS95]), we conclude that X is minimal.

Hence we proved the following theorem:

Theorem 3.3. *There exists a minimal, simply connected, symplectic 4-manifold X lying on the Noether line. The manifold X has one basic class up to sign and is obtained by the $(\mathcal{Q}, \mathcal{R})$ star surgery.*

□

Remark 3.4. **1.** *We can repeat the same construction with $E(1)$ instead of $E(5)$. Namely, by blowing up the -1 section of $E(1)$ four times, away from the singular fibers, we obtain the configuration as in Figure 4. Then, as above we do the symplectic resolution at the point p and one blow-up at q. Next, we apply the $(\mathcal{Q}, \mathcal{R})$-surgery to $E(1)\#5\overline{\mathbb{CP}}^2$. The resulting symplectic manifold has $e = 12$, $\sigma = -8$, thus $\chi_h = 1$, and $c_1^2 = 0$. It can be shown that it is an exotic copy of $\mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$. We will return to this construction in a follow-up paper.*

2. *We note that in this construction we have used one -1 section of $E(5)$ and three of the I_n singular fibers. There are 20 singular fibers of types I_n in the manifold $E(5)$.*

a. *By using another -5 section and four of the unused singularities I_4, I_3, I_2, I_2 in the manifold X, we can construct the plumbing $\mathcal{S}_2$ and apply $(\mathcal{S}_2, \mathcal{T}_2)$-star surgery (see Section 4.2 for the description of this plumbing and the $(\mathcal{S}_2, \mathcal{T}_2)$-star surgery). Then, by using the third -5 section, and an unused -2 sphere from one of the unused singular fibers we obtain the plumbing -5 -2, and apply rational blow down. It gives us a symplectic 4-manifold with invariants $\chi_h = 5$, and $c_1^2 = 9$.*

b. *Another possible construction is as follows. In the manifold X, by doing 4 more blow-ups and using two of the -5 sections, we can construct 2 more copies of the plumbing $\mathcal{Q}$ and apply $(\mathcal{Q}, \mathcal{R})$-star surgery twice. The resulting manifold has the invariants $\chi_h = 5$, and $c_1^2 = 10$.*

We will turn back to these constructions in a subsequent project.

4. Constructions of minimal, simply connected and symplectic 4-manifolds between the Noether and half Noether lines

In this section, we will give new constructions of minimal, simply connected and symplectic 4-manifolds between the Noether and half Noether lines by using star surgeries introduced in [KS16].

4.1. First construction.

We consider the elliptic surface $E(6)$ which is the 6-fold fiber sum of copies of $E(1) = \mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$ equipped with an elliptic fibration. We will apply the so called $(\mathcal{K}, \mathcal{L})$-star surgery to $E(6)$. First we will review this surgery from [KS16]. $\mathcal{K}$ is the configuration of symplectic spheres which intersect according to a star shaped graph with 4 arms. Each arm contains one -2 sphere $u_i, i = 1, \cdots 4$ and the central vertex u_0 is a -6 sphere (see Figure 5, also figure 6 in [KS16]).

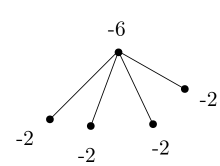

FIGURE 5. The configuration $\mathcal{K}$

The intersection form $[\mathcal{K}]$ for $H_2(\mathcal{K},\mathbb{Z})$ is given by a 5×5 matrix

$$\begin{bmatrix} -6 & 1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & -2 \end{bmatrix}$$

and its inverse is

$$-1/16 \begin{bmatrix} 4 & 2 & 2 & 2 & 2 \\ 2 & 9 & 1 & 1 & 1 \\ 2 & 1 & 9 & 1 & 1 \\ 2 & 1 & 1 & 9 & 1 \\ 2 & 1 & 1 & 1 & 9 \end{bmatrix}$$

We diagonalize $[\mathcal{K}]$ over $\mathbb{R}$ and find that the signature $\sigma(\mathcal{K})$ of $\mathcal{K}$ is -5. On the other hand, $\mathcal{L}$ is the symplectic 4-manifold of Euler characteristic 2, $c_1(\mathcal{L}) = 0$, $\pi_1(\mathcal{L}) = \mathbb{Z}/4$, $H_2(\mathcal{L}) = \mathbb{Z}$ and intersection form is the matrix $[-4]$ (Section 3D in [KS16]). Hence $\sigma(\mathcal{L}) = -1$. It is shown that $\mathcal{K}$ can be replaced by the symplectic filling $\mathcal{L}$ and we have

Definition 4.1. *Replacing the neighborhood of $\mathcal{K}$ in a symplectic 4-manifold by the filling $\mathcal{L}$ is called the $(\mathcal{K},\mathcal{L})$ surgery.*

In [Sta16] it was shown that the $(\mathcal{K},\mathcal{L})$-surgery is not equivalent to any sequences of generalized symplectic rational blow-downs.

We can easily construct the configuration $\mathcal{K}$ in $E(6)$. For instance, as above, we can consider $E(1)$ with one I_4, one I_3 and two I_2 singular fibers with the -1 section hitting each I_n fiber once, along one of its -2 sphere components. In the construction of $E(6)$ we take the 6-fold fiber sum and sew the -1 sections together, so these singular fibers descend to $E(6)$ and the -6 section intersects one -2 sphere component of each I_n fiber. Thus, for the central vertex of $\mathcal{K}$ we take the -6 section S. The -2 spheres of the I_4, I_3, and 2 I_2's intersecting S will be the symplectic spheres u_i of $\mathcal{K}$, for $i = 1,\cdots 4$.

Now we symplectically replace the neighborhood of $\mathcal{K}$ in $E(6)$ by the filling $\mathcal{L}$. Let

$$Y = ((E(6)\setminus\mathcal{K})\cup\mathcal{L}.$$

Then $\sigma(Y) = \sigma(E(6)) - \sigma(\mathcal{K}) + \sigma(\mathcal{L}) = -48 + 5 - 1 = -44$ and $e(Y) = e(E(6)) - e(\mathcal{K}) + e(\mathcal{L}) = 72 - 6 + 2 = 68$. Thus,

$$\chi_h(Y) = 6 \quad \text{and} \quad c_1^2(Y) = 4 \tag{5}$$

So, we have

$$2\chi_h(Y) - 6 > c_1^2(Y) > \chi_h(Y) - 3 \tag{6}$$

which shows that Y is in between the Noether and the half Noether lines.

The manifold Y is simply connected. In fact, the generator of $\pi_1(\mathcal{L})$ can be isotoped into the boundary of $\mathcal{L}$ and it restricts to the boundary Seifert fibered space as a meridian of any of the -2 surgery curves in the plumbing diagram (Proposition 3.11 in [KS16]). On the other hand, by our construction, one of the spheres u_j of $\mathcal{K}$ is a part of the I_4 fiber and the other transversally intersecting spheres are not cut out in the star surgery. Hence the meridian of u_j bounds a disk in the complement of $\mathcal{K}$ which is contained in a sphere component of the I_4 fiber transversely intersecting u_j. That is to say, the generator of $\pi_1(\mathcal{L})$ is isotopic to the meridian of u_j in the embedding which is homotopically trivial, hence Y is simply connected (see also the proof of Theorem 5.22 in [KS16]).

Next we will prove that Y is minimal. The Seiberg-Witten basic classes of $E(6)$ are $\pm 2f$ and $\pm 4f$ where $f \in H^2(Y, \mathbb{Z})$ is the Poincaré dual of the homology class of the fiber. We need to determine which classes extend to Y. Let $P = 2f$ and $\gamma_0, ..., \gamma_4$ be the basis of $H^2(\mathcal{K}, \mathbb{Q})$ which is dual to $u_0, ..., u_4$. Then

$$P|_{\mathcal{K}} \quad = (P \cdot u_0)\gamma_0 \quad = 2\gamma_0.$$

From inverse of the intersection matrix $[\mathcal{K}]$ above we find that

$$(P|_{\mathcal{K}})^2 = 4\gamma_0^2 = 4(-4/16) = -1.$$

Now we assume that there is a basic class $\widetilde{P}$ on Y such that $\widetilde{P}|_{Y-\mathcal{L}} = P|_{E(6)-\mathcal{K}}$. We have $(\widetilde{P}|_{\mathcal{L}})^2 < 0$ since the intersection form of $\mathcal{L}$ is negative definite.

Therefore the dimension of the SW moduli space:

$$\begin{aligned} d_Y(\widetilde{P}) &= \frac{\widetilde{P}^2 - 3\sigma(Y) - 2\chi(Y)}{4} \\ &= \frac{P^2 - (P|_{\mathcal{K}})^2 + (\widetilde{P}|_{\mathcal{L}})^2 - 3\sigma(Y) - 2\chi(Y)}{4} \\ &= \frac{0 + 1 + (\widetilde{P}|_{\mathcal{L}})^2 - 4}{4} \\ &= \frac{(\widetilde{P}|_{\mathcal{L}})^2 - 3}{4} \\ &< 0 \end{aligned}$$

This contradicts our assumption that $\widetilde{P}$ is a basic class of Y. Therefore, the class P does not descend to a basic class of Y.

On the other hand, as in the previous section, by Taubes' theorem ([Tau94]) the top class $R = 4f$ descends to a basic class of Y. By the blow-up formula ([GS99, FS95]), we conclude that Y is minimal.

Hence we have the following theorem:

Theorem 4.2. *There exists a minimal, simply connected, symplectic 4-manifold Y lying in between the Noether and the half Noether lines. The manifold Y has one basic class up to sign and is obtained by the $(\mathcal{K}, \mathcal{L})$-star surgery.*

□

Remark 4.3. *We note that we can do this construction with $E(1)\#5\overline{\mathbb{CP}}^2$ instead of $E(6)$. This gives us an exotic 4-manifold with $b_2^+ = 1$. We will come back to this construction in a follow-up paper.*

4.2. **Second construction.** In this construction we consider the elliptic surface $E(5)$ and we will apply the star surgery $(\mathcal{S}_2, \mathcal{T}_2)$ to it. Here $\mathcal{S}_2$ is the configuration of symplectic spheres which intersect according to a star shaped graph with 4 arms. Each arm contains one -2 sphere $u_i, i = 1, \cdots 4$ and the central vertex u_0 is a -5 sphere (see Figure 6, also figure 2 in [KS16]).

The intersection form $[\mathcal{S}_2]$ for $H_2(\mathcal{S}_2, \mathbb{Z})$ is given by a 5×5 matrix

$$\begin{bmatrix} -5 & 1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & -2 \end{bmatrix}$$

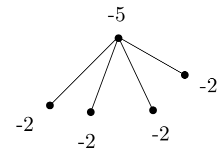

FIGURE 6. The configuration $\mathcal{S}_2$

and its inverse is

$$-1/12\begin{bmatrix}4&2&2&2&2\\2&7&1&1&1\\2&1&7&1&1\\2&1&1&7&1\\2&1&1&1&7\end{bmatrix}$$

Thus, the signature $\sigma(\mathcal{S}_2)$ of $\mathcal{S}_2$ is -5. On the other hand, $\mathcal{T}_2$ is the symplectic 4-manifold of Euler characteristic 3, $\pi_1(\mathcal{L}) = \mathbb{Z}/2$, $\sigma(\mathcal{T}_2) = -2$. It is shown that $\mathcal{S}_2$ can be replaced by the symplectic filling $\mathcal{T}_2$ ([KS16]). Hence,

Definition 4.4. *Replacing the neighborhood of $\mathcal{S}_2$ in a symplectic 4-manifold by the filling $\mathcal{T}_2$ is called the $(\mathcal{S}_2, \mathcal{T}_2)$ surgery.*

Analogous to the previous section, by using I_4, I_3 and 2 I_2 fibers and the -5 section, we construct the configuration $\mathcal{S}_2$ in $E(5)$: For the central vertex of $\mathcal{S}_2$, we take the -5 section S. S intersects one -2 sphere component of each I_n fiber. For the four -2 spheres of the plumbing $\mathcal{S}_2$, we take the -2 sphere components of I_4, I_3 and 2 I_2 fibers intersecting the section S. Hence, we have the configuration $\mathcal{S}_2$ symplectically embedded in $E(5)$. Then we symplectically replace the neighborhood of $\mathcal{S}_2$ in $E(5)$ by the filling $\mathcal{T}_2$. Let

$$Z = ((E(5) \setminus \mathcal{S}_2) \cup \mathcal{T}_2.$$

Then $\sigma(Z) = \sigma(E(5)) - \sigma(\mathcal{S}_2) + \sigma(\mathcal{T}_2) = -40 + 5 - 2 = -37$ and $e(Z) = e(E(5)) - e(\mathcal{S}_2) + e(\mathcal{T}_2) = 60 - 6 + 3 = 57$. Thus,

$$\chi_h(Z) = 5 \quad \text{and} \quad c_1^2(Z) = 3. \tag{7}$$

So, we have

$$2\chi_h(Y) - 6 > c_1^2(Y) > \chi_h(Y) - 3 \tag{8}$$

which shows that Z is in between the Noether and the half Noether lines.

The manifold Z is simply connected as in the previous example (see the proof of Lemma 5.2 in [KS16]). We also note that $b_2^+(Z) > 1$.

Next we will prove that Z is minimal. The Seiberg-Witten basic classes of $E(5)$ are $\pm f$ and $\pm 3f$. We will determine which classes extend to Z. Let $\gamma_0, ..., \gamma_4$ be the basis of $H^2(\mathcal{S}_2, \mathbb{Q})$ which is dual to $u_0, ..., u_4$. Then

$$f|_{\mathcal{S}_2} \quad = (f \cdot u_0)\gamma_0 \quad = \gamma_0.$$

From inverse of the intersection matrix $[\mathcal{S}_2]$ we find that

$$(f|_{\mathcal{S}_2})^2 = \gamma_0^2 = -4/12.$$

Now we assume that there is a basic class $\widetilde{P}$ on Z such that $\widetilde{P}|_{Z-\mathcal{T}_2} = f|_{E(5)-\mathcal{S}_2}$. We have $(\widetilde{P}|_{\mathcal{T}_2})^2 < 0$ since the intersection form of $\mathcal{T}_2$ is negative definite (see Section 6 in [KS16]). Therefore the dimension of the SW moduli space is

$$\begin{aligned} d_Z(\widetilde{P}) &= \frac{\widetilde{P}^2 - 3\sigma(Z) - 2\chi(Z)}{4} \\ &= \frac{f^2 - (f|_{\mathcal{S}_2})^2 + (\widetilde{P}|_{\mathcal{T}_2})^2 - 3\sigma(Z) - 2\chi(Z)}{4} \\ &= \frac{0 + 4/12 + (\widetilde{P}|_{\mathcal{T}_2})^2 - 3}{4} \\ &= \frac{4/12 + (\widetilde{P}|_{\mathcal{T}_2})^2 - 3}{4} \\ &< 0. \end{aligned}$$

This contradicts our assumption that $\widetilde{P}$ is a basic class of Z. Therefore, the class f does not descend to a basic class of Z.

As in the previous sections, we conclude that only the top class $3f$ descends to a basic class of Z and by the blow-up formula Z is minimal.

Hence we have proved the following:

Theorem 4.5. *There exists a minimal, simply connected, symplectic 4-manifold Z lying in between the Noether and the half Noether lines. The manifold Z has one basic class up to sign and is obtained by the $(\mathcal{S}_2, \mathcal{T}_2)$-star surgery.*

□

Remark 4.6. *We note that we can do this construction with $E(1)\#4\overline{\mathbb{CP}}^2$ instead of $E(5)$. This gives us an exotic 4-manifold with $b_2^+ = 1$. We will return to this construction in a follow-up paper.*

Remark 4.7. *In sections 3 and 4 we have used $I_4, I_3, 2I_2$ fibers. Instead we can use $2I_4, 2I_2$ fibers as well, whose existence in $E(1)$ were shown in* [Nar87]*, Section 2.11. Indeed, for our constructions in Section 4, we can also use $4I_3$ configuration which was constructed in* [Nar87]*, Section 2.12.*

5. Additional constructions

In this section we will give alternative constructions by using different singularities, and $(\mathcal{U}, \mathcal{V})$-star surgery which is again introduced in [KS16]. Here $\mathcal{U}$ is the configuration of symplectic spheres as in Figure 7

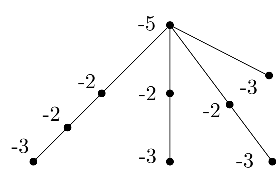

FIGURE 7. The configuration $\mathcal{U}$

and $e(\mathcal{U}) = 10$, $\sigma(\mathcal{U}) = -9$. On the other hand, the symplectic filling $\mathcal{V}$ is the symplectic 4-manifold with $e(\mathcal{V}) = 3$, $\sigma(\mathcal{V}) = -2$.

Definition 5.1. *The $(\mathcal{U}, \mathcal{V})$ surgery is symplectically replacing the neighborhood of $\mathcal{U}$ in a symplectic 4-manifold by the filling $\mathcal{V}$.*

By using this surgery let us give one more construction of a symplectic 4-manifold on the Noether line.

5.1. **First construction.** First, as we showed in Section 2, the elliptic surface $E(1)$ admits 3 singular fibers of types I_6, I_3, I_2, and the -1 section intersects one -2 sphere component of each I_n fiber. As above we take the 5-fold fiber sums of $E(1)$'s in which we sew the -1 sections and obtain a -5 section. This gives us the complex surface $E(5)$ with 5 copies of (I_6, I_3, I_2)-singularities. Let us consider one copy of (I_6, I_3, I_2) fibers inside $E(5)$. We have the following picture (Figure 8), where every labeled irreducible components of the fibers are -2 spheres and S is the -5 section:

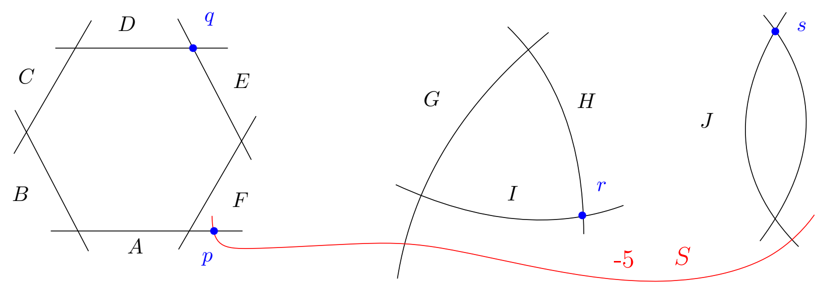

FIGURE 8. Obtaining $\mathcal{U}$ from three of the singular fibers in $E(5)$

To construct the configuration $\mathcal{U}$, we symplectically resolve the intersection point p of the -5 section with the -2 sphere A. The resulting symplectic -5 sphere will be the central vertex of $\mathcal{U}$. Next, we blow up the points q, r, s shown on the figure above (Figure 8). After blow-ups, we obtain -3 spheres $\widetilde{D}, \widetilde{E}, \widetilde{H}, \widetilde{I}, \widetilde{J}, \widetilde{K}$ as the proper transforms of the -2 spheres D, E, H, I, J, K respectively. The -2 spheres B, C and the -3 sphere $\widetilde{D}$ give the first arm of $\mathcal{U}$. The -2 sphere F and -3 sphere $\widetilde{E}$ constitute the second arm, the -2 sphere G and -3 sphere $\widetilde{H}$ give the third arm. Lastly, the -3 sphere $\widetilde{J}$ is the fourth arm of $\mathcal{U}$. Hence we have acquired the plumbing $\mathcal{U}$ symplectically embedded in $E(5)\#3\overline{\mathbb{CP}}^2$. Let

$$T = ((E(5)\#3\overline{\mathbb{CP}}^2) \setminus \mathcal{U}) \cup \mathcal{V}$$

Then $\sigma(T) = \sigma(E(5)\#3\overline{\mathbb{CP}}^2) - \sigma(\mathcal{U}) + \sigma(\mathcal{V}) = -43 + 9 - 2 = -36$ and $e(T) = e(E(5)\#3\overline{\mathbb{CP}}^2) - e(\mathcal{U}) + e(\mathcal{V}) = 63 - 10 + 3 = 56$. Thus, as in Section 3 we again have

$$\chi_h(X) = 5 \quad \text{and} \quad c_1^2(X) = 4 = 2\chi_h - 6 \tag{9}$$

This shows that T is on the Noether line. From Van Kampen's theorem, we easily see that T is simply connected as $\mathcal{V}$ is simply connected [KS16]. By Freedman's theorem we conclude that T is homeomorphic to X which is constructed in Section 3. Minimality of T can be shown as in the previous cases. Since the proof is similar, we skip the details. As a result, we have

Theorem 5.2. *There exists a minimal, simply connected, symplectic 4-manifold T lying on the Noether line. T is obtained by the $(\mathcal{U}, \mathcal{V})$-star surgery and homeomorphic to the manifold X constructed in Section 3.*

□

Now we remark that starting with the configuration $\mathcal{U}$ in $E(5)\#3\overline{\mathbb{CP}}^2$, by two blow-downs and two symplectic resolutions we obtain the configuration $\mathcal{Q}$ in $E(5)\#\overline{\mathbb{CP}}^2$. Indeed, let us blow-down the exceptional spheres coming from blowing up the points r and s (see Figure 8). Now recall that we obtained the -5 sphere of $\mathcal{U}$ from resolving the intersection point p of S and A. Next let us symplectically resolve the intersection of the -5 sphere $S + A$ with the sphere B and then C. (Also recall that we did blow up the point q). The resulting plumbing is $\mathcal{Q}$ lying in $E(5)\#\overline{\mathbb{CP}}^2$ as in the construction of the manifold X. However, we do not know if X and T are diffeomorphic to each other and it is an alluring question.

5.2. **Remark.** Altenatively, we can construct the plumbing $\mathcal{U}$ inside $E(5)\#3\overline{\mathbb{CP}}^2$ as follows. Let us consider another singularity coming from $E(1)$. As we showed in Section 2, $E(1)$ admits the singular fibers; I_5 I_4, $3I_1$ and the -1 section intersects one -2 sphere component of each I_n fiber. After taking 5-fold fiber sum, as above, we obtain 5 copies of ($I_5, I_4, 3I_1$) singular fibers in $E(5)$. Let us take 2 copies of (I_5, I_4) fibers in $E(5)$. We have the following picture where each irreducible component is a -2 sphere and the red curve is the -5 section S (Figure 9):

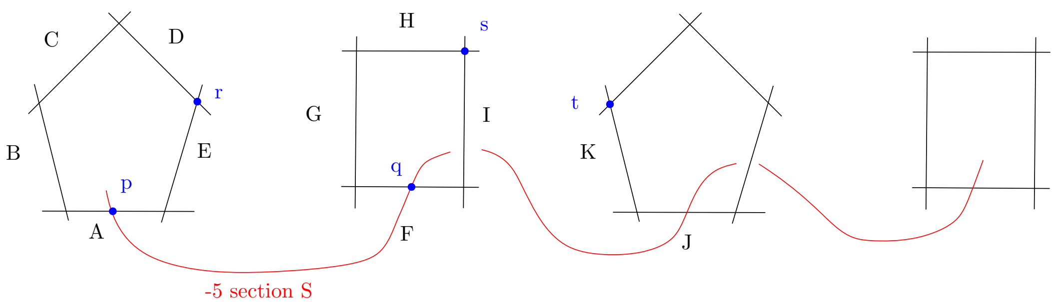

FIGURE 9. Obtaining $\mathcal{U}$ from 2 copies of (I_5, I_4) singular fibers in $E(5)$

Once again we will construct $\mathcal{U}$ in $E(5)\#3\overline{\mathbb{CP}}^2$. We first symplectically resolve the intersection points p, q of the section S with the -2 spheres A and F, respectively. The resulting symplectic -5 sphere is the central vertex of $\mathcal{U}$. Then we blow up the points r, s, t which are the intersection points of the -2 spheres as shown in Figure 9. As above, for a sphere P, let $\widetilde{P}$ denote the proper transform of P. The -2 spheres B, C and the -3 sphere $\widetilde{D}$ give one arm of the plumbing $\mathcal{U}$. The -3 sphere $\widetilde{E}$ is another arm of $\mathcal{U}$. We note that $\widetilde{D}$ and $\widetilde{E}$ are separated by the exceptional sphere. The -2 sphere G and the -3 sphere $\widetilde{H}$ constitute another arm of $\mathcal{U}$. Lastly, the -2 sphere J and the -3 sphere $\widetilde{K}$ form the last arm of $\mathcal{U}$. Thereby we again get $\mathcal{U}$ inside $E(5)\#3\overline{\mathbb{CP}}^2$ and we can apply ($\mathcal{U}, \mathcal{V}$)-surgery as above.

Let us finish this section by giving two more constructions of symplectic 4-manifolds lying above the Noether line by using the ($\mathcal{U}, \mathcal{V}$)-star surgery.

5.3. **Second construction.** By taking $E(2)$ instead of $E(5)$ in the previous remark, we can build a symplectic 4-manifold lying above the Noether line in the following way. As above, from the I_5, I_4 singularities in $E(1)$ we get 2 copies of (I_5, I_4) fibers in $E(2)$ and the -2 section intersects the singular fibers as in Figure 10.

We first blow up the -2 section at the 3 unlabeled points away from the singularities as shown in Figure 10. The proper transform of the section is a -5 sphere and we take it as the central vertex of the plumbing $\mathcal{U}$. Then we repeat the exact same steps as in the previous case, i.e., we do two symplectic resolutions at the points p, q and three blow-ups at the points r, s, t. Hence we get the plumbing $\mathcal{U}$ in $E(2)\#6\overline{\mathbb{CP}}^2$. Let

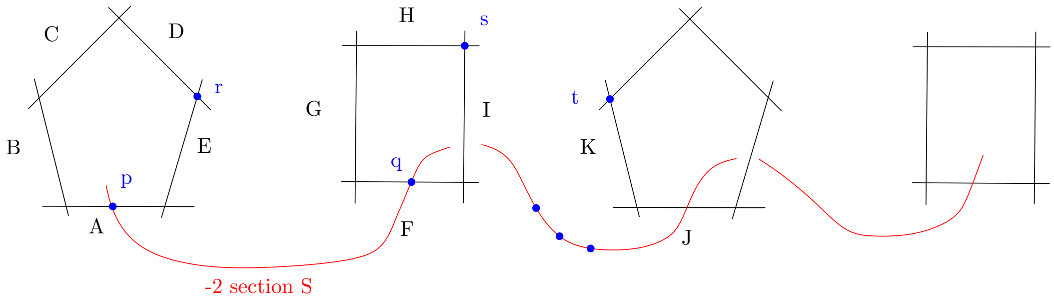

FIGURE 10. Obtaining $\mathcal{U}$ from 2 copies of (I_5, I_4) singular fibers in $E(2)$

$$M = ((E(2)\#6\overline{\mathbb{CP}}^2) \setminus \mathcal{U}) \cup \mathcal{V}$$

Then $\sigma(M) = \sigma(E(2)\#6\overline{\mathbb{CP}}^2) - \sigma(\mathcal{U}) + \sigma(\mathcal{V}) = -22 + 9 - 2 = -15$ and $e(M) = e(E(2)\#6\overline{\mathbb{CP}}^2) - e(\mathcal{U}) + e(\mathcal{V}) = 30 - 10 + 3 = 23$.

Hence
$$\chi_h(M) = (e(M) + \sigma(M))/4 = 2 \quad \text{and} \quad c_1^2(M) = 2e(M) + 3\sigma(M) = 1.$$

Therefore the symplectic manifold M lies above the Noether line. By Van Kampen's theorem we show that M is simply connected since V is simply connected. Minimality can be shown similarly as in the previous sections, thus we skip the computations. Thereby, we have the following result.

Theorem 5.3. *There exists a minimal, simply connected, symplectic 4-manifold M lying above the Noether line and is obtained by the $(\mathcal{U}, \mathcal{V})$-star surgery.*

□

5.4. **Last construction.** Lastly, let us consider $2I_5$ configuration which lies in $E(1)$ (see Section 2) and descends to $E(5)$. See Figure 11 where we denote the irreducible components of the I_5 fibers by $A, ..., E$ and $F, ..., J$, and S is the -5 section of $E(5)$.

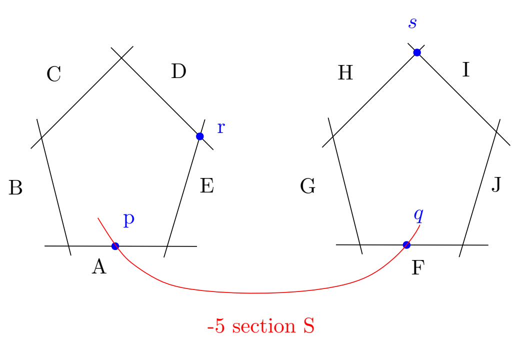

FIGURE 11. Obtaining $\mathcal{U}$ from $2I_5$ configuration in $E(5)$

We first symplectically resolve the intersection points p, q of the -5 section S with the -2 spheres A and F. The resulting symplectic -5 sphere will be the central vertex of the configuration $\mathcal{U}$. Next, we blow-up at the points r and s as we show in Figure 11. The spheres B, C and the proper transform $\widetilde{D}$ of D give the

$(-2,-2,-3)$ arm of $\mathcal{U}$, and the -3 sphere $\widetilde{E}$ is the -3 arm. The spheres $G, \widetilde{H}$ and $J, \widetilde{I}$ are the two $(-2,-3)$ arms of $\mathcal{U}$. Hence we obtain the plumbing $\mathcal{U}$ in $E(5)\#2\overline{\mathbb{CP}}^2$ and apply $(\mathcal{U},\mathcal{V})$-star surgery. Let

$$R = ((E(5)\#2\overline{\mathbb{CP}}^2) \setminus \mathcal{U}) \cup \mathcal{V}$$

Then $\sigma(R) = \sigma(E(5)\#2\overline{\mathbb{CP}}^2) - \sigma(\mathcal{U}) + \sigma(\mathcal{V}) = -42 + 9 - 2 = -35$ and $e(R) = e(E(5)\#2\overline{\mathbb{CP}}^2) - e(\mathcal{U}) + e(\mathcal{V}) = 62 - 10 + 3 = 55$. Therefore, we have

$$\chi_h(R) = 5 \quad \text{and} \quad c_1^2(R) = 5 \tag{10}$$

This shows that R is above the Noether line. From Van Kampen's theorem, we see that R is simply connected as $\mathcal{V}$ is simply connected. Minimality of R can be shown as in the previous cases. As a result, we have

Theorem 5.4. *There exists a minimal, simply connected, symplectic 4-manifold R lying above the Noether line, obtained by the $(\mathcal{U},\mathcal{V})$-star surgery.*

□

Remark 5.5. *In the above construction, we can take the $2I_5$ configuration inside $E(1)$ instead of $E(5)$. We blow up the -1 section of $E(1)$ four times, away from the singular fibers $2I_5$. Then, we repeat the same construction. Namely, as above, we do two symplectic resolutions at the points p, q and two more blow-ups at the points r and s. This gives us the plumbing $\mathcal{U}$ in $E(1)\#6\overline{\mathbb{CP}}^2$ and we apply $(\mathcal{U},\mathcal{V})$-star surgery. The resulting symplectic manifold has $e = 11$ and $\sigma = -7$ and thus $\chi_h = 1$, and $c_1^2 = 1$. It can be shown that it is an exotic copy of $\mathbb{CP}^2\#8\overline{\mathbb{CP}}^2$. We will work on this construction in a follow-up project.*

Remark 5.6. *We can easily conclude that all our manifolds constructed in this paper have exotic smooth structures. Indeed, by Freedman's theorem we first determine their homeomorphism types. Since they are symplectic, by Taubes' theorem* [Tau94], *they have nontrivial Seiberg-Witten invariants. On the other hand, the manifolds that our manifolds are homeomorphic to, have trivial Seiberg-Witten invariants by the connected sum formula for the Seiberg-Witten invariants* [GS99]. *Hence exoticness of our manifolds follows.*

REFERENCES

MAX PLANCK INSTITUTE FOR MATHEMATICS, VIVATSGASSE 7, 53111 BONN GERMANY

E-mail address: sakal008@math.umn.edu