

**GEOMETRIC AND TOPOLOGICAL RECURSION
AND
INVARIANTS OF THE MODULI SPACE OF CURVES**

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SUMMARY

A thread common to many problems of enumeration of surfaces is the idea that complicated cases can be recovered from simpler ones through a recursive procedure. Solving the problem for the simplest topologies and expressing how to glue them together provides an algorithm to solve the enumerative problem of interest. In this dissertation, we consider three distinct but interconnected topics: integration over the moduli space of curves and its combinatorial model, the enumeration of curves and quadratic differentials, and the enumeration of branched covers of the Riemann sphere. The leitmotif that will connect them all is a recursive procedure known as topological recursion.

The moduli space of curves is a key object of study in algebraic geometry. Its combinatorial model has provided powerful tools to compute various invariants of the moduli space, such as the Euler characteristic and Witten's intersection numbers. In this dissertation we further develop the (symplectic) geometry of this combinatorial model, providing a complete parallel with the Weil–Petersson geometry of the hyperbolic model. In particular, we show that certain length and twist coordinates are Darboux, and propose a new geometric approach to Witten's conjecture/Kontsevich's theorem. Namely, it is obtained by integration of a Mirzakhani-type identity on the combinatorial Teichmüller space, which recursively computes the constant function 1 by excision of embedded pairs of pants.

The second topic of interest is the enumeration of multicurves with respect to either the hyperbolic or the combinatorial notion of length. Following ideas of Mirzakhani and Andersen–Borot–Orantin, we show that such problems can again be recursively solved by excision of embedded pairs of pants. As a consequence, the average number of multicurves over the corresponding moduli space can be computed by topological recursion. On the other hand, since the work of Mirzakhani, the average number of multicurves is known to be related to the Masur–Veech volumes of the principal stratum of the moduli space of quadratic differentials. Combining these two results, we find a topological recursion formula to compute Masur–Veech volumes.

To conclude, we turn our attention to spin Hurwitz theory, that is the enumeration of branched covers of the Riemann sphere with respect to their ramification and parity. Thanks to the connection between the fermion formalism and Hurwitz theory, we are able to formulate a precise conjecture to recursively compute spin Hurwitz numbers from the simplest topologies. We also prove that this recursive formula is equivalent to a description of spin Hurwitz numbers as intersection numbers on the moduli space of curves, that is a spin version of the celebrated ELSV formula.

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PART I
INTRODUCTION

CHAPTER I — INTRODUCTIONS AND OUTLINE

The main topic of this dissertation is the enumerative geometry of surfaces and the computation of such invariants through topological and geometric recursion techniques, as well as their relation to the intersection theory on the moduli space of curves.

In this chapter we start by giving a short non-technical introduction to the concept of moduli spaces and topological recursion, aimed at those who have not yet been initiated into these topics. We then move to a more technical introduction, which gives a complete overview of the concepts and the motivations that are crucial for this thesis, as well as the main results obtained. For more in depth introductions to each topic, we refer to the specific chapters. We conclude with a detailed outline of the dissertation.

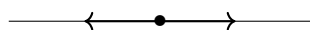
I.1 — NON-TECHNICAL INTRODUCTION

In this first introduction we will not be technical, but rather try to explain the most important concepts of this dissertation in an intuitive way. In particular, it is aimed at people with little to no mathematical background. Because of this, some concepts might be formally ill-defined or even wrong.

The research of this dissertation is a part of the mathematical field of geometry. Since ancient times, humans have tried to study shapes and geometrical objects like lines and triangles. Sometimes it is useful to study all geometric objects of interest at once, and pack them together in a moduli space.

A moduli space is a space whose points correspond to some objects of interest.

For instance, let us consider all possible directions along a straight line. As we all experienced by driving on a street, there are only two possible directions.



If we identify these directions with “left” and “right”, we can simply say that the moduli space of directions along a straight line is a collection of two points:

$$\{ \text{directions in a straight line} \} = \{ \leftarrow, \rightarrow \}.$$

Another similar example would be to consider all possible directions in a plane. In this case, there are infinitely many directions: if you find yourself in the middle of a room, you are allowed to go straight ahead, go back, left and right, and all possible directions in between. We can identify each direction in a plane with a point on a circle surrounding you. In other words, we can say that the moduli space of directions in a plane is a circle.

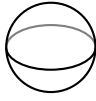
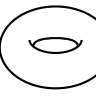

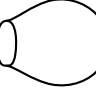





$n \setminus g$	0	1	2	...
0				...
1				
2			...	
3		...		

Figure 1.1: Surfaces classified by their genus g and number of boundary components n .

$$\{ \text{directions in a plane} \} = \bigcirc$$

Similarly, one can identify the moduli space of directions in $3d$ space with a sphere.

$$\{ \text{directions in } 3d \text{ space} \} = \bigcirc$$

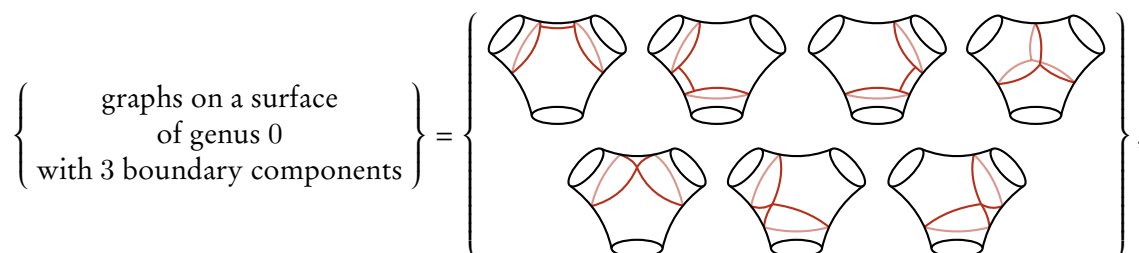
So, what is the point of identifying families of objects with a specific space? The main advantage here is that the “shape” of the moduli space can tell us properties of the objects of interest.

For example, the moduli space of directions along a straight line is formed of two separate points. This means that you cannot change direction while staying in line without “jumping” from one direction to the other. In contrast, the moduli space of directions in a plane is connected: you can easily rotate from one direction to another without having to jump.

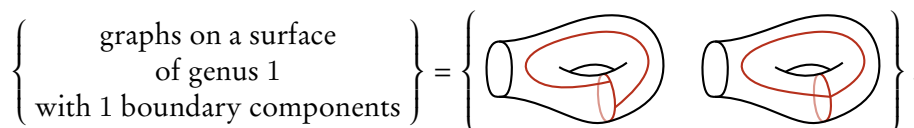
One can also study how many objects such moduli spaces contain. For example, the moduli space of directions along a straight line contains two objects. On the other hand, directions in a plane are infinite. However, we can still “count” them by saying that there are as many directions as the perimeter of the circle parametrising them, that is 2π . Similarly, the “number” of directions in $3d$ space is the same as the area of the sphere parametrising them, that is 4π .

In this dissertation, we will consider specific moduli spaces parametrising certain objects on surfaces. A surface is a gadget that locally looks like a plane: if you zoom way in, it is flat. We will always consider surfaces that are bounded, like a sphere or a torus, which is the surface of a doughnut shape. In fact, any surface looks like a torus with many holes, the number of which is called the genus, which we often denote by g . We will also allow surfaces to have a boundary, which can be obtained by cutting out a certain number n of small disks around the surface. If we allow surfaces to be elastic, as if they were made out of rubber, and we define two surfaces to be equivalent whenever you can stretch and bend one to the other, you can easily convince yourself that the genus and the number of boundary components uniquely determine the surface itself (see Figure 1.1).

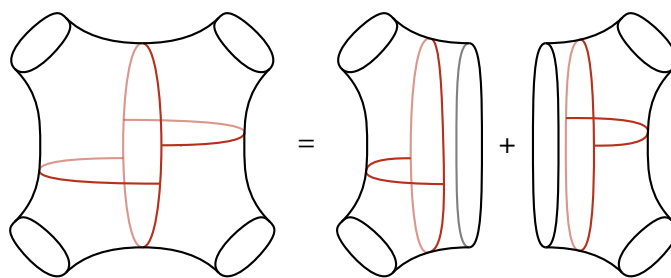
What we can consider now is the moduli space of *surfaces with some additional structure*, like surfaces with a graph drawn on them. When we impose certain rules for the allowed graphs, we obtain a finite number of possibilities for each type of surface. For example, one can consider graphs on surfaces such that, when you cut out the graph from the surface, you only obtain annuli around the boundary components. In this specific case, we have seven possible graphs on a sphere with three boundary components



and two graphs on a torus with one boundary component:



But the main question is: how can we count the number of graphs on a surface of arbitrary genus and number of boundary components? The basic idea, which is the leitmotif of this dissertation and goes under the name of *topological recursion*, is to reconstruct structures on surfaces by a cutting and gluing procedure. In this way, one can remove pieces of the surface and recursively compute the desired quantities. For example, one can start from a graph on a surface with four boundaries, and cut it in half to obtain two graphs on two surfaces with only three boundaries each.



To summarise, in this dissertation we will consider various moduli spaces of objects on surfaces, such as graphs, but also curves, quadratic differentials and covers. The aim is to count the number of such objects (or, more generally, compute the volumes of the associated moduli spaces) by means of a recursion procedure that simplifies the surfaces and the objects of interest, reducing the computation to gadgets of lower complexity.

1.2 — MATHEMATICAL INTRODUCTION

In the early '90s, Witten formulated a fundamental conjecture [Wit90], proved shortly after by Kontsevich [Kon92], establishing a new connection between two-dimensional quantum

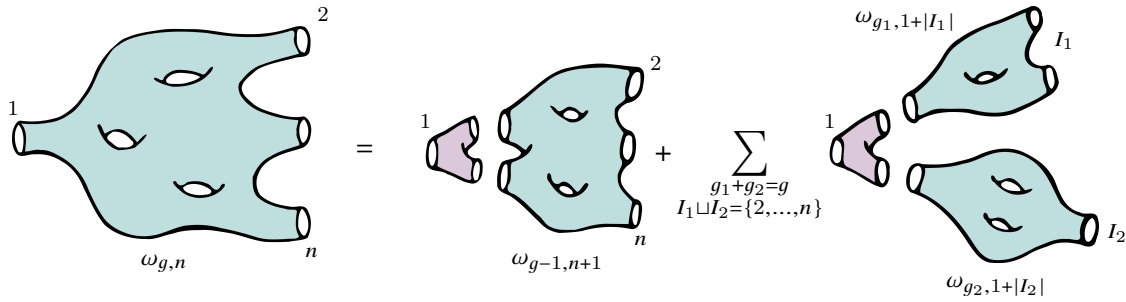


Figure 1.2: A schematic representation of topological recursion.

gravity and algebraic geometry. More precisely, he considered some intersection numbers over the *moduli space of curves*, called ψ -classes intersection numbers, and he conjectured that their generating series satisfies an *integrable hierarchy* of Korteweg–de Vries (KdV) type.

Many more generating series are solutions to more involved integrable hierarchies. Often the geometric meaning of these numbers still lies in the intersection theory of certain cohomology classes on the moduli space of curves. The appropriate mathematical tools capturing the behaviour of these numbers are called *cohomological field theories*, introduced by Kontsevich and Manin in order to formalise the properties of the virtual fundamental class in Gromov–Witten theory.

The Witten–Kontsevich result turned out to be just the tip of the iceberg of deep interactions between (1) theoretical physics, (2) algebraic geometry, and (3) mathematical physics. It corresponds to the simplest possible instance of this interplay, namely (1) the Gromov–Witten theory of a single point, (2) the trivial cohomological field theory, and (3) a topological solution of the KdV hierarchy.

Timely, Eynard and Orantin [EO07a] proposed a new innovative method motivated by random matrix theory: *topological recursion*, which recursively computes invariants through the topology of the moduli spaces of curves. When specialised, it recovers many known invariants such as ψ -classes intersection numbers, Mirzakhani’s volumes, knot invariants, asymptotics of expectation values in random matrix ensembles, Hurwitz numbers and Gromov–Witten invariants. It admits a dictionary with Givental theory, and has deep connections with Hitchin systems, WKB analysis, conformal field theories and many other topics. Concretely, topological recursion takes as input a spectral curve – a Riemann surface with some additional structure – and produces a family of differentials $\omega_{g,n}$ by a *universal recursion* on $2g-2+n$ based on surfaces’ gluing, as schematically represented in Figure 1.2. The enumerative invariants of interest are often the coefficients of these differentials.

MAIN RESULTS

THE COMBINATORIAL MODEL OF THE MODULI SPACE OF CURVES

In his proof of Witten’s conjecture, Kontsevich [Kon92] realised ψ -classes intersection numbers as symplectic volumes of a certain combinatorial model of the moduli space of curves, parametrising *metric ribbon graphs*. A similar situation occurs in Mirzakhani’s work [Miro7a], where different intersection numbers are realised as symplectic volumes of yet another model of the moduli space of curves, parametrising hyperbolic metrics. Albeit the striking parallelism

between the two problems and their solutions via topological recursion, a parallel proof of these two results was unknown.

In Part II we fill this gap and propose a proof of Kontsevich’s result for the combinatorial volumes of the moduli space of curves that is completely parallel to Mirzakhani’s work. More precisely, we study the geometry of the combinatorial Teichmüller space, that is the space of marked metric ribbon graphs on a fixed bordered surface. This space carries a natural symplectic structure, the Kontsevich symplectic form, for which we prove a *combinatorial Wolpert formula*: we show how some length and twist coordinates (analogous to the Fenchel–Nielsen coordinates) are Darboux for the Kontsevich symplectic form. We then set up geometric recursion in the sense of Andersen–Borot–Orantin [ABO17], proving a *combinatorial analogue of Mirzakhani’s identity*: we produce the constant function 1 by recursive excision of embedded pairs of pants.

As applications, we obtain geometric proofs of Witten’s conjecture/Kontsevich’s theorem and of Norbury’s recursion for the lattice point count in the combinatorial moduli space. These proofs arise now as part of a unified theory taking care of their natural geometric nature. We then conclude with the study of a rescaling flow, which allows to effectively carry natural constructions on Teichmüller spaces to their combinatorial analogues.

ENUMERATION OF MULTICURVES AND QUADRATIC DIFFERENTIALS

Another problem considered by Mirzakhani is the *enumeration of multicurves* with bounded length on a fixed hyperbolic surface. In [Miro8b] she proved that the average over the moduli space of this multicurve count is computed by a certain sum over graphs.

In Part III we complete her description, proving that the multicurve counting function is computed by a Mirzakhani-type identity, and its average over the moduli space satisfies topological recursion. Moreover, following the parallelism drawn in the first part of the dissertation, we prove analogous results in the combinatorial setting. This second count admits a sort of discretised version, that is the count of square-tiled surfaces with boundaries, for which we prove topological recursion too.

We then shift our attention to moduli spaces of quadratic differentials with simple poles and their associated *Masur–Veech volumes*. In [Miro8a], Mirzakhani showed how such volumes are related to the asymptotic number of multicurves on a hyperbolic surface. Combining her result with the topological recursion for the enumeration of multicurves, we obtain a new recursive formula for Masur–Veech volumes. By computing a large amount of them, we were able to propose conjectural formulas for these volumes as functions of the genus and the number of poles.

Shortly after its formulation, Chen, Möller and Sauvaget [CMS+19] were able to prove our conjecture by showing that Masur–Veech volumes arise from the *Segre class* of the *quadratic Hodge bundle*. To complete their description, we then prove how these intersection numbers can be computed by a second, different topological recursion.

SPIN HURWITZ THEORY

The enumeration of Hurwitz covers, i.e. branched covers of the Riemann sphere with specified ramification profiles, represents a classical enumerative problem studied since the end of the 19th century. In more recent years, Hurwitz numbers have again become an object of interest, due to strong ties with the integrable hierarchies of the Kyoto school [Okooo], the intersection

theory of the moduli spaces of curves via the Ekedahl–Lando–Shapiro–Vainshtein (ELSV) formula [ELSV01], and topological recursion [BM08].

Part IV focuses on a *spin* version of Hurwitz numbers, which enumerate branched covers of the Riemann sphere with respect to their ramification and *parity*. Thanks to their fermionic realisation related to simple Lie algebras of type B, the generating function of spin Hurwitz numbers is known to be a tau function of the BKP hierarchy. In the last part of the dissertation, we study the connections between spin Hurwitz numbers, topological recursion, and intersection theory on the moduli space of curve. Via Fock space techniques, we compute the $(0, 1)$ and $(0, 2)$ correlators, which suggest a conjectural spectral curve associated to spin Hurwitz numbers. In particular, we prove that topological recursion on this spectral curve is equivalent to an *ELSV-type formula* that involves Witten 2-spin class (and, in the easiest case, it reduces to double Hodge integrals).

Shortly after the formulation of our conjecture, Alexandrov and Shadrin [AS21] proved topological recursion for a wide class of BKP tau functions, including the ones associated to spin Hurwitz numbers. In particular, combining their result with the aforementioned equivalence, we obtain a complete proof of a spin ELSV formula.

I.3 — OUTLINE

This dissertation is based on the following preprints.

- [And+19] J. E. Andersen, G. Borot, S. Charbonnier, V. Delecroix, A. Giacchetto, D. Lewański, and C. Wheeler. “Topological recursion for Masur–Veech volumes” (2019). Submitted. arXiv: [1905.10352](#) [math.GT].
- [And+20] J. E. Andersen, G. Borot, S. Charbonnier, A. Giacchetto, D. Lewański, and C. Wheeler. “On the Kontsevich geometry of the combinatorial Teichmüller space” (2020). Submitted. arXiv: [2010.11806](#) [math.DG].
- [CMS+19] D. Chen, M. Möller, and A. Sauvaget. “Masur–Veech volumes and intersection theory: the principal strata of quadratic differentials” (2019). Appendix by G. Borot, A. Giacchetto and D. Lewański. Submitted. arXiv: [1912.02267](#) [math.AG].
- [GKL21] A. Giacchetto, R. Kramer, and D. Lewański. “A new spin on Hurwitz theory and ELSV via theta characteristics” (2021). Submitted. arXiv: [2104.05697](#) [math-ph].
- [GLN] A. Giacchetto, D. Lewański, and P. Norbury. In preparation.

It is organised in the following way.

- Part II deals with the combinatorial model of the moduli space of curves, and is all based on [And+20].
 - In Chapter 3 we introduce the combinatorial moduli spaces parametrising metric ribbon graphs and the associated Teichmüller spaces, and prove various topological and geometric properties. In particular, we discuss cutting and gluing of metric ribbon graphs, as well as Fenchel–Nielsen-type coordinates.

- In Chapter 4 we recall the symplectic properties of the combinatorial moduli and Teichmüller spaces, due to Kontsevich, and prove a Wolpert-type formula for this symplectic structure.
- In Chapter 5 we set-up geometric recursion in this combinatorial setting. In particular, we prove a combinatorial analogue of the Mirzakhani–McShane identity, which yields to a recursion formula for the symplectic volumes of the combinatorial moduli spaces. After integration, it gives a new geometric proof of Witten’s conjecture.
- In Chapter 6 we extend known results about the connection between hyperbolic structures and metric ribbon graphs, exploiting the geometric idea that metric ribbon graphs approximate hyperbolic surfaces with large boundaries.
- Part III deals with the enumeration of multicurves in both the hyperbolic and combinatorial settings, as well as its connection with Masur–Veech volumes and area Siegel–Veech constants of the principal stratum of the moduli space of quadratic differentials. It contains some results from [And+19; And+20], and new unpublished material originated from these works.
 - Chapter 7 is based on [And+19] and on the last section of [And+20]. We discuss the enumeration of multicurves in the hyperbolic and combinatorial setting, proving a Mirzakhani-type identity and a recursion for the average number of multicurves. We also discuss the enumeration of square-tiled surfaces with boundaries, weighted by their core area.
 - Chapter 8 builds again on ideas from [And+19; And+20]. We show that the asymptotic number of multicurves in the hyperbolic and combinatorial settings are equal, and that they coincide with the Masur–Veech volumes of the principal stratum of the moduli space of quadratic differentials. This gives a way to compute such volumes, and we were able to conjecture the behaviour of Masur–Veech volumes and Siegel–Veech constants as a function of the genus and the number of simple poles. We also discuss the connection between the asymptotic enumeration of square-tiled surfaces with boundaries and Masur–Veech volumes.
 - Chapter 9 is based on [CMS+19]. We briefly summarise how the authors proved the above conjecture from [And+19] through intersection theory of the Segre class of the quadratic Hodge bundle, and present our contribution which is a recursion formula for the Segre class intersection numbers. Based on [GLN], we also show how the Chern class of the quadratic Hodge bundle computes the Euler characteristic of the moduli space of curves, providing a new intersection-theoretic proof of the Harer–Zagier formula.
- Part IV deals with spin Hurwitz numbers, and is all based on [GKL21].
 - In Chapter 10 we review the representation theory of the spin algebra and the theory of neutral fermions. This allows us to represent spin Hurwitz numbers in terms of characters of the Sergeev group and vacuum expectation values on the neutral Fock space.
 - In Chapter 11 we derive the spin analogue of the Okounkov–Pandharipande operators on neutral fermions, which is then employed for the analysis of the

polynomiality structure of spin double Hurwitz numbers and their wall-crossing formulae. We also provide an explicit expression for the generating series of spin cut-and-join-operators, which can then be computed directly and algorithmically.

- Chapter 12 contains the main conjecture concerning spin Hurwitz theory: single spin Hurwitz numbers are generated by topological recursion on a specific spectral curve. We also give evidence for this conjecture by proving it in genus zero. Since the conjectural spectral curve differs from the usual definition, we define and analyse G -quotients of spectral curves, and reduce them to the usual setting of topological recursion. We then employ the correspondence with cohomological field theories to derive a representation of spin Hurwitz numbers as intersection numbers on $\overline{\mathcal{M}}_{g,n}$. To conclude, we express the cohomological field theory as the Chiodo class twisted by the Witten 2-spin class.

CHAPTER 2 — PREREQUISITES

The main purpose of this chapter is to make the thesis as self-contained as possible, giving an overview of different (but quite related) topics that are relevant for this dissertation: the moduli space of curves and cohomological field theories, topological recursion, geometric recursion and Teichmüller theory, fermion formalism and its connections with integrable hierarchies, and Hurwitz theory. This of course very often implies providing references for more details whenever we consider it necessary or interesting.

2.1 — MODULI SPACE OF CURVES

In this section we recall some facts about smooth connected compact complex curves of genus g , also called *Riemann surfaces*, with n marked points. Their moduli space $\mathcal{M}_{g,n}$ has been a central object in mathematics since Riemann's work in the middle of the 19th century, and its compactification $\overline{\mathcal{M}}_{g,n}$ was defined more than 50 years ago by Deligne and Mumford [DM69] by including stable curves. Either such moduli spaces can be seen as smooth Deligne–Mumford stacks (in the algebraic-geometric setting) or as smooth complex orbifolds (in the analytic setting). The latter notion is simpler and will be discussed here.

DEFINITION 2.1.1. Let $g, n \geq 0$ such that $2g - 2 + n > 0$. A *stable curve* of type (g, n) is a complex algebraic curve C of arithmetic genus g with n labeled marked points x_1, \dots, x_n such that

- the only singularities of C are simple nodes,
- the marked points are distinct and do not coincide with the nodes, and
- the curve (C, x_1, \dots, x_n) has a finite number of automorphisms.

We will not formally construct the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$, but we will list some of its properties and refer to [ACG11, Chapter XII] for further readings.

PROPOSITION 2.1.2. *The moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$ is a smooth complex compact orbifold of dimension $3g - 3 + n$. Moreover, it contains the moduli space of smooth curves $\mathcal{M}_{g,n}$ as a smooth open dense suborbifold.*

We will call $\partial\overline{\mathcal{M}}_{g,n} = \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ the *boundary* of the moduli space of stable curves. Beware that, as $\overline{\mathcal{M}}_{g,n}$ is a closed smooth orbifold, the boundary here is not meant in the sense of orbifold with boundary.

EXAMPLE 2.1.3. We describe here the moduli spaces of curves for the simplest topologies, namely genus zero (or *rational*) curves with three or four marked points, and genus one curves with one marked point (also called *elliptic curves*).

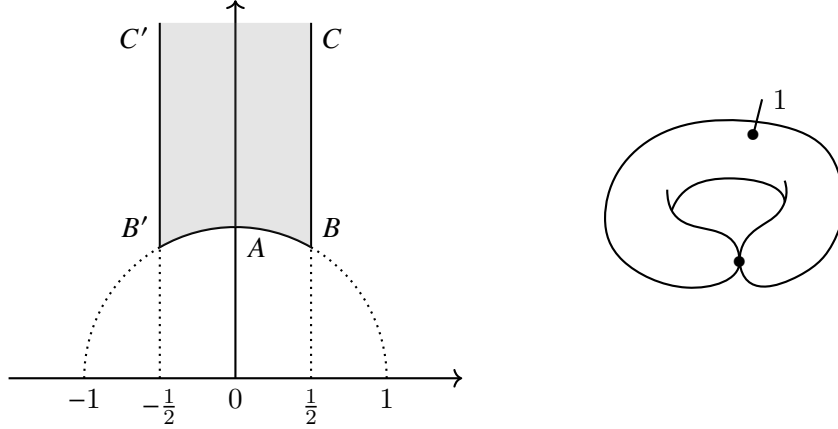


Figure 2.1: On the left, the moduli space $\mathcal{M}_{1,1}$. The arcs AB and AB' and the half-lines BC and $B'C'$ are identified. On the right, the point at infinity, corresponding to a pinched torus with one marked point.

- $\overline{\mathcal{M}}_{0,3} = \{*\}$. Every stable rational curve (C, x_1, x_2, x_3) with three marked points can be identified with $(\mathbb{P}^1, 0, 1, \infty)$ in a unique way.
- $\overline{\mathcal{M}}_{0,4} = \mathbb{P}^1$. Every smooth rational curve (C, x_1, x_2, x_3, x_4) can be uniquely identified with $(\mathbb{P}^1, 0, 1, \infty, \lambda)$, for some $\lambda \neq 0, 1, \infty$. The value λ is determined by the positions of the marked points on C via the cross-ratio:

$$\lambda = \frac{(x_4 - x_1)(x_2 - x_3)}{(x_4 - x_3)(x_2 - x_1)}.$$

The moduli space $\mathcal{M}_{0,4}$ is the set of values of λ , that is $\mathcal{M}_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. The nodal curves in $\overline{\mathcal{M}}_{0,4}$ correspond to two rational curves intersecting in one point and with two marked points each. These three nodal curves, corresponding to the three way of splitting four points in two sets of two, can be identified with λ tending to 0, 1 and ∞ .

- $\overline{\mathcal{M}}_{1,1}$. Every smooth elliptic curve is given by a quotient \mathbb{C}/Λ of the complex plane by a lattice, and the image of Λ is a natural marked point on the quotient. Further, two elliptic curves \mathbb{C}/Λ_1 and \mathbb{C}/Λ_2 are isomorphic if and only if $\Lambda_2 = a\Lambda_1$ for some $a \in \mathbb{C}^\times$. Consider now $\Lambda = e_1\mathbb{Z} \oplus e_2\mathbb{Z}$; multiplying by $1/e_1$, we obtain $\mathbb{Z} \oplus \tau\mathbb{Z}$ with τ lying in the upper-half plane \mathfrak{h} . Moreover, the elliptic curve defined by the lattice $\mathbb{Z} \oplus \tau\mathbb{Z}$ is isomorphic to the curve defined by $\mathbb{Z} \oplus \tau'\mathbb{Z}$, for τ' given by the modular action

$$\tau' = \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}).$$

Thus, $\mathcal{M}_{1,1} = [\mathfrak{h}/\mathrm{SL}(2, \mathbb{C})]$ as an orbifold, with generic point of stabiliser $\mathbb{Z}/2\mathbb{Z}$. Notice that the moduli space has two non-smooth points as a variety, corresponding to $\tau = i$ and $\tau = e^{\pi i/3}$, with stabiliser given by $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$ respectively. In this case there is only one nodal curve, that is a rational curve with two points identified (a pinched torus), and it corresponds to the point at infinity in Figure 2.1.

Some of the main features of the moduli spaces of stable curves come from the existence of natural maps between them: the forgetful and gluing maps.

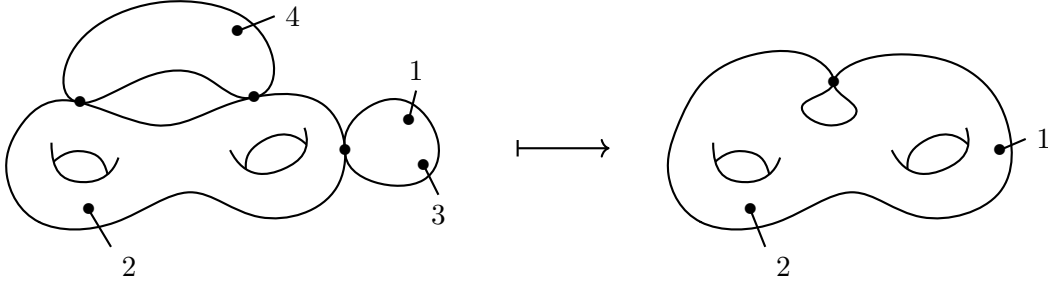


Figure 2.2: An example of forgetful morphism $p_2: \overline{\mathcal{M}}_{3,4} \rightarrow \overline{\mathcal{M}}_{3,2}$. Forgetting the last two marked points, both rational components become unstable and have to be contracted.

The idea of a forgetful map is to assign to a genus g stable curve (C, x_1, \dots, x_{n+m}) the curve (C, x_1, \dots, x_n) , where we have forgotten the last m marked points. However, the resulting curve is not necessarily stable. Assuming that $2g - 2 + n > 0$, then either the curve (C, x_1, \dots, x_n) is stable, or it has at least one rational component with one or two special points (that is, a marked point or a node). In the latter case, this component can be contracted into a point (see Figure 2.2). If the curve thus obtained is not stable, we can find another component to contract. Since the number of irreducible components decreases with each operation, in the end we will obtain a stable curve $(C, x_1, \dots, x_n)^{\text{st}}$.

DEFINITION 2.1.4. Define the *forgetful map*

$$p_m: \overline{\mathcal{M}}_{g,n+m} \longrightarrow \overline{\mathcal{M}}_{g,n}, \quad (C, x_1, \dots, x_{n+m}) \longmapsto (C, x_1, \dots, x_n)^{\text{st}}. \quad (2.1.1)$$

In the following, we will denote p_1 simply by p .

The forgetful map is very important from the deformation theory point of view: it coincides with the universal curve.

PROPOSITION 2.1.5. *The forgetful map $p: \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$ is the universal curve*

$$\pi: \overline{\mathcal{C}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}. \quad (2.1.2)$$

In other words, the following universal property holds: for any family $X \rightarrow B$ of genus g stable curves with n marked points, there exists a unique morphism $\varphi: B \rightarrow \overline{\mathcal{M}}_{g,n}$ such that the family X is a pullback by φ .

For the gluing maps, the idea is simply to identify marked points of stable curve(s), creating a new curve of simpler type.

DEFINITION 2.1.6. Define the *gluing map of non-separating kind* by identifying the last two marked points of a single stable curve:

$$q: \overline{\mathcal{M}}_{g-1,n+2} \longrightarrow \overline{\mathcal{M}}_{g,n}. \quad (2.1.3)$$

Define the *gluing map of separating kind* by identifying the last marked points of different stable curves:

$$r: \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \longrightarrow \overline{\mathcal{M}}_{g,n}, \quad (2.1.4)$$

where $g_1 + g_2 = g$ and $n_1 + n_2 = n$.

Together, we will call the maps p, q and r the *tautological maps*. Notice that tautological maps simplify the topology of a curve by making its Euler characteristic higher. This simple feature will be the fil rouge of the whole dissertation, appearing in the context of cohomological field theories, topological and geometric recursion, Hurwitz theory, etc.

Notice that the image of both q and r lie in the boundary $\partial\overline{\mathcal{M}}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$. In particular, this defines a stratification of the moduli space of curves via combinatorial data called stable graphs.

DEFINITION 2.1.7. A *stable graph* is the data $\Gamma = (V, H, \Lambda, E, g, \nu, \iota)$ of

- a set V of vertices,
- a set H of half-edges, together with a map $\nu: H \rightarrow V$ that sends a half-edge to the vertex it is attached to, and an involution $\iota: H \rightarrow H$ pairing half-edges together,
- a subset $\Lambda \subseteq H$ of leaves, that is the set of fixed points of ι ,
- the set E of edges is the set of orbits of ι of cardinality 2,
- a genus map $g: V \rightarrow \mathbb{Z}_{\geq 0}$,

such that the following conditions hold:

- the graph (V, E) is connected,
- for each vertex $v \in V$, the stability condition holds: $2g(v) - 2 + n(v) > 0$, where $n(v) = |\nu^{-1}(v)|$ is the valence of v (i.e. the number of edges and leaves attached to v).

For a given stable graph Γ , we define its genus as $g(\Gamma) = \sum_{v \in V} g(v) + h^1(\Gamma)$, where $h^1(\Gamma)$ is the first Betti number of the graph Γ . An automorphism of a stable graph is an automorphism of the underlying graph that preserves individually the leaves and the genus of each vertex. Define the type of Γ as $(g(\Gamma), |\Lambda|)$, and denote the set of stable graphs of type (g, n) by $\mathcal{G}_{g,n}$.

If necessary, we will denote the sets of vertices, edges, half-edges and leaves of a stable graph Γ with a subscript Γ .

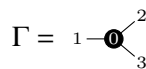
For a given stable graph Γ , we define

$$\overline{\mathcal{M}}_{\Gamma} = \prod_{v \in V_{\Gamma}} \overline{\mathcal{M}}_{g(v), n(v)}, \quad \xi_{\Gamma}: \overline{\mathcal{M}}_{\Gamma} \longrightarrow \overline{\mathcal{M}}_{g,n}, \quad (2.1.5)$$

where the map ξ_{Γ} is defined by gluing all the marked points on the components as indicated by the edges of Γ . The images of $\mathcal{M}_{\Gamma} = \prod_{v \in V_{\Gamma}} \mathcal{M}_{g(v), n(v)}$ (resp. $\overline{\mathcal{M}}_{\Gamma}$) via ξ_{Γ} under all stable graphs of type (g, n) gives the *open* (resp. *closed*) *boundary stratification* of $\overline{\mathcal{M}}_{g,n}$. Clearly, $\mathcal{M}_{g,n} \subseteq \overline{\mathcal{M}}_{g,n}$ corresponds to the open boundary stratum given by the unique stable graph with one vertex of genus g , n leaves and no edges.

EXAMPLE 2.1.8. In the following, the genus of a vertex is represented by a number inside the vertex itself, while the leaves are labelled by natural numbers $1, 2, \dots$

- $\mathcal{G}_{0,3}$. There is a single stable graph of type $(0, 3)$, namely



This corresponds to the identification $\overline{\mathcal{M}}_{0,3} = \mathcal{M}_{0,3} = \{*\}$.

- $\mathcal{G}_{0,4}$. There are four stable graphs of type $(0, 4)$, namely

$$\Gamma = \begin{array}{c} 1 \quad 3 \\ \diagdown \quad / \\ \bullet \\ / \quad \diagdown \\ 2 \quad 4 \end{array}, \quad \Gamma_{12|34} = \begin{array}{c} 1 \quad 3 \\ \diagdown \quad / \\ \bullet \text{---} \bullet \\ / \quad \diagdown \\ 2 \quad 4 \end{array},$$

and similarly $\Gamma_{13|24}$ and $\Gamma_{14|23}$, all with trivial automorphism group. This corresponds to the open boundary stratification $\overline{\mathcal{M}}_{0,4} = \mathcal{M}_{0,4} \sqcup (\mathcal{M}_{0,3} \times \mathcal{M}_{0,3})^{\sqcup 3}$.

- $\mathcal{G}_{1,1}$. There are two stable graphs of type $(1, 1)$, namely

$$\Gamma = 1 \text{---} \bullet \quad \text{and} \quad \Gamma' = 1 \text{---} \bullet \circlearrowright$$

with automorphism groups of order 1 and 2 respectively. This corresponds to the open boundary stratification $\overline{\mathcal{M}}_{1,1} = \mathcal{M}_{1,1} \sqcup (\mathcal{M}_{0,3}/(\mathbb{Z}/2\mathbb{Z}))$, with the single point corresponding to the pinched torus in Figure 2.1.

One of the main application of the boundary structure of the moduli space is discussed in the next section: intersection theory on $\overline{\mathcal{M}}_{g,n}$.

2.1.1 — TAUTOLOGICAL RING

The generalisation of various topological invariants, such as homology and cohomology, from the manifold setting to that of orbifold is relatively easy, albeit it brings some technical differences. In particular, for an orbifold X it is natural to consider its cohomology ring with rational coefficients, rather than integers, which coincides with the cohomology ring of its underlying topological space (also over \mathbb{Q}). In particular, we can safely consider the rational cohomology ring of the moduli space of stable curves $H^*(\overline{\mathcal{M}}_{g,n})$, where Poincaré duality holds. However, as we will see, it is more natural to consider subrings which behave well under pushforwards by tautological maps.

DEFINITION 2.1.9. Define the *tautological ring* of the moduli spaces of stable curves as the minimal family of unital subrings $R^*(\overline{\mathcal{M}}_{g,n}) \subseteq H^{2*}(\overline{\mathcal{M}}_{g,n})$ that is stable under pushforwards by tautological maps[†]. Elements of the tautological ring are called tautological classes.

Clearly, $1 \in R^0(\overline{\mathcal{M}}_{g,n})$, since a subring contains the unity element by definition. Moreover, all boundary strata are tautological, since they are pushforward of the unit element by gluing maps. The same holds for intersections and self-intersections of boundary strata.

Let us introduce now some more tautological classes. The definition of λ - and κ -classes on the moduli spaces without marked points was firstly given by Mumford in [Mum83], along with the term “tautological classes”. The ψ -classes were first defined by Miller in [Mil86], and became truly important after Witten formulated his fundamental conjecture [Wit90] on their intersection numbers in connection with the Korteweg–de Vries (KdV) hierarchy.

DEFINITION 2.1.10. Let ω_π be the relative dualising sheaf for $\pi: \overline{\mathcal{C}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$, that is ω_π restricted to each fiber is the canonical bundle of the corresponding curve.

[†]From the algebraic-geometric point of view, a more natural choice is to consider Chow groups $A^d(\overline{\mathcal{M}}_{g,n})$ instead of even cohomology group $H^{2d}(\overline{\mathcal{M}}_{g,n})$. Except for the Givental–Teleman classification of CohFTs, Theorem 2.2.12, every construction and result from this chapter holds in the Chow setting.

- Define the Hodge bundle as $\mathcal{E} = \pi_*\omega_\pi$. The fiber over a point (C, x_1, \dots, x_n) is the vector space of abelian differentials $H^0(C, \omega_C)$, i.e. the space of forms over C that are meromorphic, with poles at the nodes only and with residues on the two branches meeting at a node of opposite sign. Hence, \mathcal{E} is a rank g vector bundle, and define the *Hodge classes*

$$\lambda_k = c_k(\mathcal{E}) \in H^{2k}(\overline{\mathcal{M}}_{g,n}), \quad k = 0, \dots, g. \quad (2.1.6)$$

Define the full Hodge class as the Chern polynomial $\Lambda(t) = c(\mathcal{E}; t) = \sum_{k=0}^g t^k \lambda_k$.

- Let σ_i be the section of π corresponding to the i -th marked point. Define the line bundles $\mathcal{L}_i = \sigma_i^*\omega_\pi$. The fiber over a point (C, x_1, \dots, x_n) is the cotangent space $T_{x_i}^*C$ of C at x_i . Define the *ψ -classes*

$$\psi_i = c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n}). \quad (2.1.7)$$

- Define the *κ -classes*

$$\kappa_d = p_*(\psi_{n+1}^{d+1}) \in H^{2d}(\overline{\mathcal{M}}_{g,n}), \quad (2.1.8)$$

where $p: \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$ is the forgetful map. We also define the multi-index κ -classes: for $\mu = (\mu_1, \dots, \mu_m)$, define $\kappa_\mu = p_{m,*}(\psi_{n+1}^{\mu_1+1} \cdots \psi_{n+m}^{\mu_m+1})$, where $p_m: \overline{\mathcal{M}}_{g,n+m} \rightarrow \overline{\mathcal{M}}_{g,n}$ is the m -th forgetful map.

PROPOSITION 2.1.11 (See for instance [Zvo12, Theorem 2.27]). *All λ -, ψ - and κ -classes are tautological classes.*

TAUTOLOGICAL RELATIONS

One of the most natural questions to ask is an explicit presentation of the tautological ring in terms of generators and relations. This can be achieved via strata algebra classes and tautological relations. The following definition is due to Pixton [Pix13].

DEFINITION 2.1.12. Fix a stable graph Γ . A *basic class* on $\overline{\mathcal{M}}_\Gamma$ is a product of monomials in κ -classes at each vertex of the graph and powers of ψ -classes at each half-edge:

$$\gamma = \prod_{v \in V_\Gamma} \prod_{d \geq 0} \kappa_d(v)^{m_d(v)} \cdot \prod_{h \in H_\Gamma} \psi_h^{k(h)} \in R^\bullet(\overline{\mathcal{M}}_\Gamma), \quad (2.1.9)$$

where $\kappa_d(v)$ is the d -th κ -class on $\overline{\mathcal{M}}_{g(v),n(v)}$. We suppose that the weights satisfy

$$\sum_{d \geq 0} d \cdot m_d(v) + \sum_{h \in H_\Gamma(v)} k(h) \leq 3g(v) - 3 + 2n(v) \quad (2.1.10)$$

at each vertex to avoid trivial vanishing, where $H_\Gamma(v) \subseteq H_\Gamma$ denotes the set of half-edges (including the leaves) incident to v . Define the degree of $[\Gamma, \gamma]$ by setting

$$\deg[\Gamma, \gamma] = \deg_{\mathbb{C}} \gamma + |E|. \quad (2.1.11)$$

Consider the graded \mathbb{Q} -vector space $\mathcal{S}_{g,n}^\bullet$ whose basis is given by isomorphism classes of pairs $[\Gamma, \gamma]$, where Γ is a stable graph of type (g, n) and γ is a basic class on $\overline{\mathcal{M}}_\Gamma$. Since there are

only finitely many pairs $[\Gamma, \gamma]$ up to isomorphism, $\mathcal{S}_{g,n}^\bullet$ is finite dimensional. Define a graded algebra structure on $\mathcal{S}_{g,n}^\bullet$ as follows:

$$[\Gamma_1, \gamma_1] \cdot [\Gamma_2, \gamma_2] = \sum_{\Gamma} [\Gamma, \gamma_1 \gamma_2 \epsilon_{\Gamma}], \quad \epsilon_{\Gamma} = - \prod_{\substack{e \in E_1 \cap E_2 \\ e=(h,h')}} (\psi_h + \psi_{h'}) \quad (2.1.12)$$

where the sum is over all stable graphs Γ whose set of edges E_{Γ} is a union (not necessarily disjoint) of two subsets $E = E_1 \cup E_2$ such that contracting all edges outside E_i results in Γ_i . Here ψ_h and $\psi_{h'}$ are the ψ -classes corresponding to the two half-edges of e , and ϵ_{Γ} is the excess class given by Fulton's excess theory.

DEFINITION 2.1.13. Via the above intersection product, $\mathcal{S}_{g,n}^\bullet$ is a finite dimensional graded \mathbb{Q} -algebra called the *strata algebra*. Pushforward along ξ_{Γ} defines a canonical surjective ring homomorphism

$$\sigma: \mathcal{S}_{g,n}^\bullet \longrightarrow R^\bullet(\overline{\mathcal{M}}_{g,n}), \quad \sigma[\Gamma, \gamma] = \xi_{\Gamma,*}\gamma. \quad (2.1.13)$$

An element of the kernel of σ is called a *tautological relation*.

A natural question, then, is how to explicitly construct tautological relations. In his PhD thesis [Pix13], Pixton constructed a set of relations based on the known Faber–Zagier relations for the moduli space of smooth curves $\mathcal{M}_{g,0}$ and conjectured that they constitute all tautological relations. Such relations were later proved to hold in cohomology by Pandharipande–Pixton–Zvonkine [PPZ15] using cohomological field theories techniques (a more precise definition of such relations will be given in Section 2.2 via the Witten 3-spin class). The above (conjectural) presentation has many practical applications. For instance, Delecroix–Schmitt–van Zelm [DSZ20] implemented Pixton's presentation of the tautological ring as a SageMath package, called `admcycles`, which allows various checks on cohomology or intersection theory on the moduli space of curves. In particular, `admcycles` has been used to numerically check Theorem 12.3.6, expressing spin Hurwitz numbers in terms of intersection theory on the moduli space of curve.

INTERSECTION THEORY

Being a smooth compact complex orbifold of dimension $3g - 3 + n$, it make sense to consider intersection numbers of the form

$$\int_{\overline{\mathcal{M}}_{g,n}} \alpha \in \mathbb{Q}, \quad \alpha \in H^{6g-6+2n}(\overline{\mathcal{M}}_{g,n}). \quad (2.1.14)$$

It is not difficult to show that the computation of intersection numbers involving λ - and κ -classes can be reduced to the computation of ψ -classes only (see [Zvo12]). A simple example of such relations is the following result.

LEMMA 2.1.14. *Let P be a polynomial in the κ - and ψ -classes. Denote by \hat{P} the polynomial obtained from P by substituting $\kappa_d \mapsto \kappa_d - \psi_{n+1}^d$ for every d . Then we have*

$$\int_{\overline{\mathcal{M}}_{g,n}} P \cdot \kappa_m = \int_{\overline{\mathcal{M}}_{g,n+1}} \hat{P} \cdot \psi_{n+1}^{m+1}. \quad (2.1.15)$$

I. Introduction

Computing ψ -classes intersection numbers is much more difficult. A formula was first conjectured by Witten [Wit90], and proved shortly after by Kontsevich in [Kon92]. To state Witten–Kontsevich result, let us introduce Witten’s notation for intersection numbers:

$$\langle \tau_{k_1} \cdots \tau_{k_n} \rangle_g = \int_{\mathcal{M}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n}. \quad (2.1.16)$$

THEOREM 2.1.15 (Witten conjecture/Kontsevich theorem [Wit90; Kon92]). *For $2g - 2 + n > 0$, the following relations hold.*

STRING EQUATION.

$$\langle \tau_{k_1} \cdots \tau_{k_n} \tau_0 \rangle_g = \sum_{i=1}^n \langle \tau_{k_{i-1}} \tau_{k_1} \cdots \widehat{\tau_{k_i}} \cdots \tau_{k_n} \rangle_g \quad (2.1.17)$$

DILATON EQUATION.

$$\langle \tau_{k_1} \cdots \tau_{k_n} \tau_1 \rangle_g = (2g - 2 + n) \langle \tau_{k_1} \cdots \tau_{k_n} \rangle_g \quad (2.1.18)$$

TOPOLOGICAL RECURSION EQUATION. *The following relations uniquely determine the intersection numbers $\langle \tau_{k_1} \cdots \tau_{k_n} \rangle_g$ from the initial data $\langle \tau_0^3 \rangle_0 = 1$ and $\langle \tau_1 \rangle_1 = \frac{1}{24}$.*

$$\begin{aligned} \langle \tau_{k_1} \cdots \tau_{k_n} \rangle_g = & \sum_{m=2}^n \frac{(2k_1 + 2k_m - 1)!!}{(2k_1 + 1)!! (2k_m - 1)!!} \langle \tau_{k_1+k_m-1} \tau_{k_2} \cdots \widehat{\tau_{k_m}} \cdots \tau_{k_n} \rangle_g \\ & + \frac{1}{2} \sum_{a+b=k_1-2} \frac{(2a+1)!! (2b+1)!!}{(2k_1+1)!!} \left(\langle \tau_a \tau_b \tau_{k_2} \cdots \tau_{k_n} \rangle_{g-1} \right. \\ & \left. + \sum_{\substack{g_1+g_2=g \\ T_1 \sqcup T_2 = \{\tau_{k_2}, \dots, \tau_{k_n}\}}} \langle \tau_a T_1 \rangle_{g_1} \langle \tau_b T_2 \rangle_{g_2} \right). \end{aligned} \quad (2.1.19)$$

We can package the above numbers in a generating series:

$$F^{\text{WK}}(t_0, t_1, t_2, \dots) = \sum_{\substack{g \geq 0, n \geq 1 \\ 2g-2+n > 0}} \frac{1}{n!} \sum_{k_1, \dots, k_n \geq 0} \langle \tau_{k_1} \cdots \tau_{k_n} \rangle_g \prod_{i=1}^n t_i^{k_i}. \quad (2.1.20)$$

Witten’s original formulation of his conjecture states that F^{WK} is the unique τ -function of the KdV hierarchy, satisfying the string equation and a certain initial condition:

$$\partial_{t_0} F^{\text{WK}} = \frac{t_0^2}{2} + \sum_{k \geq 1} t_k \partial_{t_{k-1}} F^{\text{WK}}, \quad F^{\text{WK}}(t_0, 0, 0, \dots) = \frac{t_0^3}{6}. \quad (2.1.21)$$

Such conjecture was proved to be equivalent to Theorem 2.1.15 by Dijkgraaf–Verlinde–Verlinde [DVV91] using Virasoro constraints: the partition function $Z^{\text{WK}} = \exp(F^{\text{WK}})$ satisfies the linear, homogeneous differential equations

$$L_m Z^{\text{WK}} = 0, \quad m \geq -1, \quad (2.1.22)$$

(g, n)	$\langle \tau_{k_1} \cdots \tau_{k_n} \rangle_g$	*	(g, n)	$\langle \tau_{k_1} \cdots \tau_{k_n} \rangle_g$	*	(g, n)	$\langle \tau_{k_1} \cdots \tau_{k_n} \rangle_g$	*
(0, 3)	$\langle \tau_0^3 \rangle_0$	1	(1, 1)	$\langle \tau_1 \rangle_1$	$\frac{1}{24}$	(2, 1)	$\langle \tau_4 \rangle_2$	$\frac{1}{1152}$
(0, 4)	$\langle \tau_0^3 \tau_1 \rangle_0$	1	(1, 2)	$\langle \tau_0 \tau_2 \rangle_1$	$\frac{1}{24}$	(2, 2)	$\langle \tau_0 \tau_5 \rangle_2$	$\frac{1}{1152}$
(0, 5)	$\langle \tau_0^4 \tau_2 \rangle_0$	1	(1, 3)	$\langle \tau_1^2 \rangle_1$	$\frac{1}{24}$		$\langle \tau_1 \tau_4 \rangle_2$	$\frac{1}{384}$
	$\langle \tau_0^3 \tau_1^2 \rangle_0$	2		$\langle \tau_0^2 \tau_3 \rangle_1$	$\frac{1}{24}$	$\langle \tau_2 \tau_3 \rangle_2$	$\frac{29}{5760}$	
(0, 6)	$\langle \tau_0^5 \tau_3 \rangle_0$	1	(1, 4)	$\langle \tau_0 \tau_1 \tau_2 \rangle_1$	$\frac{1}{12}$	(3, 1)	$\langle \tau_7 \rangle_3$	$\frac{1}{82944}$
	$\langle \tau_0^4 \tau_1 \tau_2 \rangle_0$	3		$\langle \tau_1^3 \rangle_1$	$\frac{1}{12}$	(3, 2)	$\langle \tau_0 \tau_8 \rangle_3$	$\frac{1}{82944}$
(0, 7)	$\langle \tau_0^3 \tau_1^3 \rangle_0$	6	$\langle \tau_0^3 \tau_4 \rangle_1$	$\frac{1}{24}$	$\langle \tau_1 \tau_7 \rangle_3$		$\frac{5}{82944}$	
	$\langle \tau_0^6 \tau_4 \rangle_0$	1	$\langle \tau_0^2 \tau_1 \tau_3 \rangle_1$	$\frac{1}{8}$	$\langle \tau_2 \tau_6 \rangle_3$	$\frac{77}{414720}$		
	$\langle \tau_0^5 \tau_1 \tau_3 \rangle_0$	4	$\langle \tau_0^2 \tau_2^2 \rangle_1$	$\frac{1}{6}$	$\langle \tau_3 \tau_5 \rangle_3$	$\frac{503}{1451520}$		
	$\langle \tau_0^5 \tau_2^2 \rangle_0$	6	$\langle \tau_0 \tau_1^2 \tau_2 \rangle_1$	$\frac{1}{4}$	$\langle \tau_4^2 \rangle_3$	$\frac{607}{1451520}$		
	$\langle \tau_0^4 \tau_1^2 \tau_2 \rangle_0$	12	$\langle \tau_1^4 \rangle_1$	$\frac{1}{4}$	(4, 1)	$\langle \tau_{10} \rangle_4$	$\frac{1}{7962624}$	
$\langle \tau_0^3 \tau_1^4 \rangle_0$	24							

Table 2.1: Some ψ -classes intersection numbers, computed using the topological recursion relation (2.1.19).

where L_m denote the differential operator

$$\begin{aligned}
L_{-1} &= \partial_{t_0} - \frac{t_0^2}{2} - \sum_{k \geq 1} t_k \partial_{t_{k-1}}, \\
L_0 &= \partial_{t_1} - \sum_{k \geq 0} \frac{2k+1}{3} t_k \partial_{t_k} - \frac{1}{24}, \\
L_m &= \partial_{t_{m+1}} - \sum_{k \geq 0} \frac{(2m+2k+1)!!}{(2m+3)!!(2k-1)!!} t_k \partial_{t_{k+m}} + \frac{1}{2} \sum_{a+b=m-1} \frac{(2a+1)!!(2b+1)!!}{(2m+3)!!} \partial_{t_a, t_b}.
\end{aligned} \tag{2.1.23}$$

Let us briefly explain Witten's motivation of his conjecture, which originates from two-dimensional quantum gravity. As a toy model for the more complicated gravity theory in four-dimensional space-time, in $2d$ gravity the space-time is a surface, while the gravitational field is a Riemannian metric on the surface itself. In the attempt to quantise such theory, i.e. to compute the partition function of $2d$ quantum gravity, one should compute a certain integral over the space of all possible Riemannian metrics on all possible surfaces. The space of Riemannian metrics over a fixed topological surface is infinite-dimensional, and physicists found two possible ways to give a meaning to such ill-defined quantity.

- The first way is to approximate the Riemann surface by small equilateral triangles. Thus, the integral over all metrics is replaced by a sum over triangulations. Such combinatorial problem can be solved, and the KdV hierarchy appeared in the works devoted to enumeration of triangulations on surfaces, which can be related to matrix models.
- Alternatively, one can compute the partition function by integrating first over all conformally equivalent metrics. After that, the remaining integral is performed over the moduli space of Riemann surfaces, and more precisely one has to compute integrals of the form $\langle \tau_{k_1} \cdots \tau_{k_n} \rangle_g$.

Witten’s conjecture states that the partition functions resulting from the two approaches coincide, based on the physics expectation that there is a unique theory of gravity.

The Witten–Kontsevich theorem is a fundamental result in mathematical physics, as it relates two areas that were not thought before to have anything in common: intersection theory on the moduli space of curves and integrable system. However, such result turned out to be just the tip of the iceberg of a deep interaction between algebraic geometry and mathematical physics. Since then, these connections have been explored intensively, for example in Hurwitz theory (see Section 2.6.1) and more generally in connection with many cohomological field theories and enumerative problems solved by topological recursion.

There are many known proofs of the Witten–Kontsevich result: Kontsevich’s original proof uses ribbon graphs and matrix model techniques, Mirzakhani’s proof [Miro7a] through hyperbolic geometry, Okounkov–Pandharipande [OP09], Kazarian–Lando [KL07] and Kazarian [Kaz09] via the ELSV formula, Bennett–Cochran–Safnuk–Woskoff using symplectic reduction [BCSW12], and many more using different ideas. In Chapter 5 we will give a new geometric proof of the Witten–Kontsevich result based on geometric recursion and Mirzakhani’s ideas.

2.2 — COHOMOLOGICAL FIELD THEORIES

A fundamental tool in the construction of classes on the moduli space of curves, which formalises the idea of cohomology classes compatible with tautological maps, is that of cohomological field theory (CohFT). CohFTs were defined in the mid 1990s by Kontsevich and Manin [KM94] in order to capture the formal properties of the virtual fundamental class in Gromov–Witten theory, and have deep connections with Frobenius manifolds and topological recursion (cfr. Section 2.3.1). A powerful computational tool is Teleman’s classification result [Tel12] of semisimple CohFTs via the action of the Givental group. The aim of the section is to present such ideas, following the exposition of [Pan19].

DEFINITION 2.2.1. Let V be a finite dimensional \mathbb{Q} -vector space equipped with a non-degenerate symmetric 2-form η . A *cohomological field theory* on (V, η) consists of a collection $\Omega = (\Omega_{g,n})_{2g-2+n>0}$ of elements

$$\Omega_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n}) \otimes (V^*)^{\otimes n} \quad (2.2.1)$$

satisfying the following axioms.

- i) SYMMETRY. Each $\Omega_{g,n}$ is \mathfrak{S}_n -invariant, where the action of the symmetric group \mathfrak{S}_n permutes simultaneously the marked points of $\overline{\mathcal{M}}_{g,n}$ and the copies of $(V^*)^{\otimes n}$.
- ii) NATURALITY. Considering the gluing maps

$$\begin{aligned} q: \overline{\mathcal{M}}_{g-1,n+2} &\longrightarrow \overline{\mathcal{M}}_{g,n}, \\ r: \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} &\longrightarrow \overline{\mathcal{M}}_{g,n}, \quad g_1 + g_2 = g, \quad n_1 + n_2 = n, \end{aligned} \quad (2.2.2)$$

we have

$$\begin{aligned} q^* \Omega_{g,n}(v_1 \otimes \cdots \otimes v_n) &= \Omega_{g-1,n+2}(v_1 \otimes \cdots \otimes v_n \otimes \eta^\dagger), \\ r^* \Omega_{g,n}(v_1 \otimes \cdots \otimes v_n) &= (\Omega_{g_1,n_1+1} \otimes \Omega_{g_2,n_2}) \left(\bigotimes_{i=1}^{n_1} v_i \otimes \eta^\dagger \otimes \bigotimes_{j=1}^{n_2} v_{n_1+j} \right), \end{aligned} \quad (2.2.3)$$

where $\eta^\dagger \in V^{\otimes 2}$ is the bivector dual to η .

If the vector space comes with a distinguished element $\mathbb{1} \in V$, we can also ask for a third axiom:

iii) **FLAT UNIT.** Consider the forgetful map

$$p: \overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}. \quad (2.2.4)$$

Then

$$\begin{aligned} p^* \Omega_{g,n}(v_1 \otimes \cdots \otimes v_n) &= \Omega_{g,n+1}(v_1 \otimes \cdots \otimes v_n \otimes \mathbb{1}), \\ \Omega_{0,3}(v_1 \otimes v_2 \otimes \mathbb{1}) &= \eta(v_1, v_2). \end{aligned} \quad (2.2.5)$$

In this case, Ω is called a cohomological field theory with *flat unit*.

A CohFT determines a product \star on V , called the *quantum product*: $v_1 \star v_2$ is defined as the unique vector such that for all $v_3 \in V$ the following holds:

$$\eta(v_1 \star v_2, v_3) = \Omega_{0,3}(v_1 \otimes v_2 \otimes v_3). \quad (2.2.6)$$

Commutativity and associativity of \star follow from (i) and (ii) respectively. If the CohFT has flat unit, the quantum product is unital, with $\mathbb{1} \in V$ being the identity by (iii).

The degree 0 part of a CohFT

$$\varpi_{g,n} = \deg_0 \Omega_{g,n} \in H^0(\overline{\mathcal{M}}_{g,n}) \otimes (V^*)^{\otimes n} \cong (V^*)^{\otimes n} \quad (2.2.7)$$

is a $2d$ topological field theory (TFT), and is uniquely determined by the values of $\varpi_{0,3}$ and by the bilinear form η (or equivalently, by the associated quantum product and the bilinear form η).

Associated to any CohFT Ω , we also have a collection of rational numbers called *CohFT correlators* (or ancestor invariants), defined as

$$\langle \tau_{k_1}(v_1) \cdots \tau_{k_n}(v_n) \rangle_g^\Omega = \int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n}(v_1 \otimes \cdots \otimes v_n) \prod_{i=1}^n \psi_i^{k_i}. \quad (2.2.8)$$

Notice that, for degree reasons, $\sum_{i=1}^n k_i \leq 3g - 3 + n$.

EXAMPLE 2.2.2. Let us give some examples of CohFTs in dimension 1. Let us take $V = \mathbb{Q}$ and $\eta(1, 1) = 1$. Then a CohFT on (V, η) is uniquely determined by $\Omega_{g,n} = \Omega_{g,n}(1^{\otimes n})$.

- Setting $\Omega_{g,n} = 1$, we get a CohFT (with flat unit $\mathbb{1} = 1$) concentrated in degree zero, called the *trivial CohFT*.
- The class $\exp(2\pi^2 \kappa_1)$ defines a CohFT, appearing in hyperbolic geometry in relation to Weil–Petersson volumes (see Section 2.4.1). It is not a CohFT with flat unit.
- The Hodge class $\Lambda(t)$ defines a 1-parameter family of CohFTs with flat unit $\mathbb{1} = 1$.
- In [Nor17], Norbury defines a CohFT, denoted by $\Theta_{g,n}$, that satisfies a different version of the flat unit axiom, namely

$$\psi_{n+1} \cdot p^* \Theta_{g,n+1} = \Theta_{g,n}.$$

It coincides with Chiodo’s class (see Section 2.2.2) in degree $2g - 2 + n$ for the specific values $r = 2$ and $s = -1$, and appears in the supersymmetric version of Mirzakhani’s recursion [SW19; Nor20].

Here are some higher dimensional CohFTs appearing in the literature.

- For $r \geq 2$, let $V = \mathbb{Q}\langle v_0, \dots, v_{r-2} \rangle$ with pairing $\eta(v_i, v_j) = \delta_{i+j, r-2}$ and unit $\mathbb{1} = v_0$. Witten r -spin class is a CohFT with flat unit

$$W_{g,n}^r: V^{\otimes n} \longrightarrow H^\bullet(\overline{\mathcal{M}}_{g,n})$$

of pure complex degree

$$\begin{aligned} \deg W_{g,n}^r(v_{a_1} \otimes \dots \otimes v_{a_n}) &= D_{g,n}^r(a_1, \dots, a_n) \\ &= \frac{(r-2)(g-1) + \sum_{i=1}^n a_i}{r}. \end{aligned}$$

If $D_{g,n}^r(a_1, \dots, a_n)$ is not an integer, the corresponding Witten's class vanishes. For instance, the case $r = 2$ (cf. [JKV01, Corollary 4.7]) is given by

$$W_{g,n}^2(v_{a_1} \otimes \dots \otimes v_{a_n}) = \begin{cases} 1 & \text{if } a_1 = \dots = a_n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

In genus 0, the construction was first carried out by Witten [Wit93] using r -spin structures. The construction of Witten's class in higher genera was first obtained by Polishchuk and Vaintrob [PV00] and later simplified by Chiodo [Chio06]. In [PPZ15], it was shown that the class

$$\hat{W}_{g,n}^r(v_{a_1} \otimes \dots \otimes v_{a_n}) = \sum_{m \geq 0} \frac{1}{m!} p_{m,*} W_{g,n+m}^r(v_{a_1} \otimes \dots \otimes v_{a_n} \otimes (rv_{r-2})^{\otimes m})$$

extends Witten r -spin class to lower degrees, and that it still constitutes a CohFT with flat unit on $(V, \eta, \mathbb{1})$.

- Let X be a smooth projective variety and fix $g \geq 0$, $n \geq 1$, $\beta \in H_2(X, \mathbb{Z})$. Consider the moduli space $\overline{\mathcal{M}}_{g,n}(X, \beta)$ of maps $\phi: C \rightarrow X$ from a stable genus g curve (C, x_1, \dots, x_n) , such that $[\phi(C)] = \beta$. This space is in general singular, but Behrend and Fantechi [BF97] and many others could construct a *virtual* fundamental cycle $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\text{vir}}$ over which cohomology classes can be integrated, as if they were integrated on a cycle of complex dimension

$$d_{g,n}(X, \beta) = \dim X + (3 - \dim X)g + n - 3 + \int_{\beta} c_1(TX).$$

Note that we have a proper fibration $\pi: \overline{\mathcal{M}}_{g,n}(X, \beta) \rightarrow \overline{\mathcal{M}}_{g,n}$, which forgets about the map ϕ , and n evaluation morphisms $\text{ev}_i: \overline{\mathcal{M}}_{g,n}(X, \beta) \rightarrow X$, which remember the image of π via ϕ . For any $v_1, \dots, v_n \in H^\bullet(X, \mathbb{Z})$, we can form the cohomology class on $\overline{\mathcal{M}}_{g,n}$

$$\Omega_{g,n}^{X,\beta}(v_1, \dots, v_n) = \pi_* \left([\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\text{vir}} \cap \prod_{i=1}^n \text{ev}_i^* v_i \right).$$

Define a graded CohFT (over the space $\text{Eff}(X)$ of effective 2-cycles, i.e. the subspace of cycles in $H_2(X, \mathbb{Z})$ that can be realised as the image of a curve) as

$$V = \bigoplus_{\beta \in \text{Eff}(X)} H^\bullet(X, \mathbb{Z}), \quad \eta(v, w) = \int_X v \cap w, \quad \Omega_{g,n}^X = \sum_{\beta \in \text{Eff}(X)} \Omega_{g,n}^{X,\beta},$$

which has a flat unit $\mathbb{1} = 1 \in H^0(X, \mathbb{Z})$. See [KM94] for the original discussion on CohFTs motivated by Gromov–Witten theory, and [BCM20] for a modern account on virtual classes.

- Other examples of higher dimensional CohFTs are Chiodo classes [Chio8b; JPPZ17; LPSZ17], described in details in Section 2.2.2, and the Chern character of the Verlinde bundle [Mar+17].

2.2.1 — GIVENTAL ACTION AND CLASSIFICATION OF SEMISIMPLE COHFTS

In [Giv01] Givental defined a certain action on Gromov–Witten potentials, and this action was lifted to cohomological field theories in the works of Teleman [Tel12] and Shadrin [Shao9]. A careful proof that the resulting collection of cohomology classes satisfies the cohomological field theory axioms can be found in [PPZ15]. Here we recall the basic definitions.

R-MATRIX ACTION

Fix a vector space V with a symmetric bilinear form η . An R -matrix is an $\text{End}(V)$ -valued power series that is the identity in degree 0

$$R(u) = \text{Id} + \sum_{k \geq 1} R_k u^k, \quad R_k \in \text{End}(V), \quad (2.2.9)$$

and satisfying the symplectic condition:

$$R(u)R^\dagger(-u) = \text{Id}. \quad (2.2.10)$$

Here R^\dagger is the adjoint with respect to η . The inverse matrix $R^{-1}(u)$ also satisfies the symplectic condition. In particular, we can consider the $V^{\otimes 2}$ -valued power series²

$$E(u, v) = \frac{\text{Id} \otimes \text{Id} - R^{-1}(u) \otimes R^{-1}(v)}{u + v} \eta^\dagger \in V^{\otimes 2}[[u, v]]. \quad (2.2.11)$$

We will write $E(u, v) = \sum_{k, \ell \geq 0} E_{k, \ell} u^k v^\ell$, with $E_{k, \ell} \in V^{\otimes 2}$.

Remark 2.2.3. We can write the above conditions in components. Fix a basis (e_i) of V , denote by (e^i) the dual basis of V^* and by $\langle \cdot, \cdot \rangle : V^* \times V \rightarrow \mathbb{Q}$ the canonical pairing. The symplectic condition and the bivector E can be written as

$$\begin{aligned} \sum_{a, b} R_a^i(u) \eta^{a, b} R_b^j(-u) &= \eta^{i, j}, \\ E(u, v) &= \sum_{i, j} \frac{\eta^{i, j} - \sum_{a, b} (R^{-1})_a^i(u) \eta^{a, b} (R^{-1})_b^j(v)}{u + v} e_i \otimes e_j, \end{aligned} \quad (2.2.12)$$

where $R_\ell^k(u) = \langle e^k, R(u)e_\ell \rangle$, $\eta_{i, j} = \eta(e_i, e_j)$ and $(\eta^{i, j})$ is the inverse matrix. We will denote by $E^{i, j}(u, v) = \sum_{k, \ell \geq 0} E_{k, \ell}^{i, j} u^k v^\ell$ the formal power series with coefficients given by

$$E^{i, j}(u, v) = \frac{\eta^{i, j} - \sum_{a, b} (R^{-1})_a^i(u) \eta^{a, b} (R^{-1})_b^j(v)}{u + v} \in \mathbb{Q}[[u, v]]. \quad (2.2.13)$$

²The reason why we use R^{-1} instead of R is so that it defines a left action on the set of CohFTs. Beware that some authors use a different notation.

DEFINITION 2.2.4. Consider a CohFT Ω on (V, η) , together with an R -matrix. We define a collection of cohomology classes

$$R\Omega_{g,n} \in H^\bullet(\overline{\mathcal{M}}_{g,n}) \otimes (V^*)^{\otimes n} \quad (2.2.14)$$

as follows. Let $\mathcal{G}_{g,n}$ be the finite set of stable graphs of genus g with n leaves. For each $\Gamma \in \mathcal{G}_{g,n}$, define a contribution $\text{Cont}_\Gamma \in H^\bullet(\overline{\mathcal{M}}_\Gamma) \otimes (V^*)^{\otimes n}$ by the following construction:

- place $\Omega_{g(v),n(v)}$ at each vertex v of Γ ,
- place $R^{-1}(\psi_\lambda)$ at each leaf λ of Γ ,
- at every edge $e = (h, h')$ of Γ , place $E(\psi_h, \psi_{h'})$.

Define $R\Omega_{g,n}$ to be the sum of contributions of all stable graphs, after pushforward to the moduli space weighted by automorphism factors:

$$R\Omega_{g,n} = \sum_{\Gamma \in \mathcal{G}_{g,n}} \frac{1}{|\text{Aut}(\Gamma)|} \xi_{\Gamma,*} \text{Cont}_\Gamma. \quad (2.2.15)$$

PROPOSITION 2.2.5. *The data $R\Omega = (R\Omega_{g,n})_{2g-2+n>0}$ form a CohFT on (V, η) . Moreover, the R -matrix action on CohFTs is a left group action.*

TRANSLATIONS

There is also another action on the space of CohFTs: a *translation* is a V -valued power series vanishing in degree 0 and 1:

$$T(u) = \sum_{d \geq 1} T_d u^{d+1}, \quad T_d \in V. \quad (2.2.16)$$

DEFINITION 2.2.6. Consider a CohFT Ω on (V, η) , together with a translation T . We define a collection of cohomology classes

$$T\Omega_{g,n} \in H^\bullet(\overline{\mathcal{M}}_{g,n}) \otimes (V^*)^{\otimes n} \quad (2.2.17)$$

by setting

$$T\Omega_{g,n}(v_1 \otimes \cdots \otimes v_n) = \sum_{m \geq 0} \frac{1}{m!} p_{m,*} \Omega_{g,n+m}(v_1 \otimes \cdots \otimes v_n \otimes T(\psi_{n+1}) \otimes \cdots \otimes T(\psi_{n+m})). \quad (2.2.18)$$

Here $p_m: \overline{\mathcal{M}}_{g,n+m} \rightarrow \overline{\mathcal{M}}_{g,n}$ is the map forgetting about the last m marked points. Notice that the vanishing of T in degree 0 and 1 ensures that the above sum is actually finite.

PROPOSITION 2.2.7. *The data $T\Omega = (T\Omega_{g,n})_{2g-2+n>0}$ form a CohFT on (V, η) . Moreover, translations form an abelian group action on CohFTs.*

For a translation proportional to the unity acting on a TFT with unit, one can express the resulting CohFT as multiplication of the original TFT by exponential of κ -classes.

LEMMA 2.2.8. Consider $T(u) = \sum_{d \geq 1} T_d u^{d+1} \in u^2 V[[u]]$. Define $\hat{T}(u) = \sum_{m \geq 1} \hat{T}_m u^m \in uV[[u]]$ by setting

$$T(u) = u \left(\mathbb{1} - \exp(-\hat{T}(u)) \right). \quad (2.2.19)$$

Here $\exp(-\hat{T}(u)) = \sum_{k \geq 0} \frac{(-1)^k}{k!} \hat{T}(u)^{\star k}$. Then following relation holds on $H^\bullet(\overline{\mathcal{M}}_{g,n}) \otimes V$:

$$\exp(\hat{T}(\kappa)) = \sum_{m \geq 0} \frac{1}{m!} p_{m,*} (T(\psi_n) \star \cdots \star T(\psi_{n+m})). \quad (2.2.20)$$

In general, if we start from a CohFT with flat unit on $(V, \eta, \mathbb{1})$, acting by an R -matrix or by translation does not preserve flatness. However, there is a specific circumstance for which this happens.

PROPOSITION 2.2.9. Let Ω be a CohFT with flat unit on $(V, \eta, \mathbb{1})$. Let R be an R -matrix, and consider the V -valued power series

$$T_L(u) = u(R^{-1}(u)\mathbb{1} - \mathbb{1}), \quad T_R(u) = u(\mathbb{1} - R^{-1}(u)\mathbb{1}). \quad (2.2.21)$$

Then $T_L R \Omega$ and $R T_R \Omega$ coincide, and form a CohFT with flat unit.

DEFINITION 2.2.10. Let Ω be a CohFT on $(V, \eta, \mathbb{1})$ with flat unit, and R be an R -matrix. We define the *unit-preserving R -matrix action* as

$$R \cdot \Omega = T_L R \Omega = R T_R \Omega. \quad (2.2.22)$$

GIVENTAL–TELEMAN CLASSIFICATION

Having an action on the set of CohFTs, a natural question is the description of the orbit structure. The answer was given by Teleman in the specific case of semisimple CohFTs.

DEFINITION 2.2.11. A CohFT Ω on $(V, \eta, \mathbb{1})$ with flat unit is *semisimple* if $(V, \star, \mathbb{1})$ is a semisimple algebra, i.e. if there exists a basis (e_i) of idempotents

$$e_i \star e_j = \delta_{i,j} e_i \quad (2.2.23)$$

after an extension of scalars to \mathbb{C} .

THEOREM 2.2.12 (Givental–Teleman classification [Tel12]). Let Ω be a semisimple CohFT on $(V, \eta, \mathbb{1})$ with flat unit. Denote by ϖ the associated TFT. There exists a unique R -matrix such that

$$\Omega = R \cdot \varpi. \quad (2.2.24)$$

The Givental–Teleman classification is an important and useful tool to study cohomological field theory, as it allows for explicit constructions of classes and relations. Before giving some examples, let us show how in the 1-dimensional case a semisimple CohFT can be rewritten as the exponential of a combination of κ -, ψ -, and boundary divisor classes.

PROPOSITION 2.2.13. Let $\varpi_{g,n}$ be a TFT on \mathbb{Q} with $\eta(1, 1) = 1$, which is uniquely determined by a scalar as $\varpi_{g,n}(1^{\otimes n}) = a^{2g-2+n}$, and let R be an R -matrix. By definition, the latter can be written as $R(u) = \exp(\sum_{m \geq 1} r_m u^m)$ for some coefficients $(r_m)_{m \geq 1}$. Then the CohFT $R \cdot \varpi$ is given by

$$R \cdot \varpi_{g,n}(1^{\otimes n}) = a^{2g-2+n} \exp \left(\sum_{m \geq 1} r_m \left(\kappa_m - \sum_{i=1}^n \psi_i^m + \delta_m \right) \right) \quad (2.2.25)$$

where $\delta_m = \frac{1}{2} j_* (\sum_{k+\ell=m-1} \psi^k (\psi')^\ell)$, and $j: \partial \overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$ is the boundary map.

EXAMPLE 2.2.14. Let us go back to the semisimple cases of Example 2.2.2.

- The Hodge CohFT $\Lambda(t)$ is the first non-trivial example of CohFT with unit (semisimplicity is trivial in dimension 1). The associated TFT is trivial, and the R -matrix is given by

$$R^{-1}(u) = \exp\left(-\sum_{m \geq 1} \frac{B_{m+1}}{m(m+1)} (tu)^m\right),$$

where B_m denotes the m -th Bernoulli number. In particular, Proposition 2.2.13 specialises to *Mumford's formula* [Mum83] for the Hodge class:

$$\Lambda(t) = \exp\left(\sum_{m \geq 1} \frac{B_{m+1}}{m(m+1)} t^m \left(\kappa_m - \sum_{i=1}^n \psi_i^m + \delta_m\right)\right). \quad (2.2.26)$$

- The shifted Witten r -spin class $\hat{W}_{g,n}^r$ is semisimple. The associated TFT and R -matrix was computed in [PPZ15], with the former being

$$\hat{w}_{g,n}^r(v_{a_1} \otimes \cdots \otimes v_{a_n}) = \left(\frac{r}{2}\right)^{g-1} \sum_{k=1}^{r-1} (-1)^{(k-1)(g-1)} \frac{\prod_{i=1}^n \sin\left(\frac{(a_i+1)k\pi}{r}\right)}{\sin^{2g-2+n}\left(\frac{k\pi}{r}\right)}.$$

For the R -matrix, consider the hypergeometric series ($a = 0, \dots, r-2$)

$$B_{r,a}(u) = \sum_{m \geq 0} \left(\prod_{k=1}^m \frac{((2k-1)r - 2(a+1))((2k-1)r + 2(a+1))}{k} \right) \left(-\frac{u}{16r^2}\right)^m,$$

and denote by $B_{r,a}^{\text{even}}$ (resp. $B_{r,a}^{\text{odd}}$) the even (resp. odd) summands of $B_{r,a}$. Then the R -matrix is given by

$$(R^{-1})_a^a = B_{r,a}^{\text{even}}, \quad (R^{-1})_a^{r-2-a} = B_{r,a}^{\text{odd}},$$

and 0 elsewhere. If r is even, the coefficient at the diagonals' intersection is set to be 1.

Among other things, the Givental–Teleman classification of CohFTs can be used to obtain relations in the cohomology ring. For instance, one knows from geometric reasons that the Hodge class $\Lambda(t)$ vanishes in degree $d > g$. On the other hand, Mumford's formula for $\Lambda(t)$ gives a certain class in any degree. Denoting by $\mathcal{H}_{g,n}^d$ the degree d component of Mumford's formula (i.e. the coefficient of t^d in the right-hand side of Equation (2.2.26)), we obtain the following tautological relations: for every $d > g$, $\mathcal{H}_{g,n}^d = 0$ in $R^d(\overline{\mathcal{M}}_{g,n})$. The first non-trivial example of such tautological relations is the degree 1 relation in genus 0:

$$\mathcal{H}_{0,n}^1 = \kappa_1 - \sum_{i=1}^n \psi_i + \delta_1 = 0 \quad \text{in } R^1(\overline{\mathcal{M}}_{0,n}). \quad (2.2.27)$$

Pixton–Pandharipande–Zvonkine [PPZ15] exploited such argument in the case of Witten 3-spin class. From the R -matrix action on \hat{w}^3 , we can write the shifted Witten 3-spin class as the map σ of Definition 2.1.13 applied to a certain combination of strata algebra classes:

$$\hat{W}_{g,n}^3(v_{a_1} \otimes \cdots \otimes v_{a_n}) = \sigma\left(\sum_{\Gamma \in \mathcal{G}_{g,n}} \frac{1}{|\text{Aut}(\Gamma)|} [\Gamma, \gamma(a_1, \dots, a_n)]\right). \quad (2.2.28)$$

Denoting by $\mathcal{R}_{g,n}^d(a_1, \dots, a_n)$ the degree d component the strata algebra class on the right-hand side, one has

$$\sigma(\mathcal{R}_{g,n}^d(a_1, \dots, a_n)) = 0 \quad \text{for } d > D_{g,n}^3(a_1, \dots, a_n) = \frac{g-1 + \sum_{i=1}^n a_i}{3}. \quad (2.2.29)$$

These relations are *Pixton's relations* on $R^\bullet(\overline{\mathcal{M}}_{g,n})$. A natural question to ask is whether such relations generate all possible tautological relations.

CONJECTURE 2.2.15. *Pixton's relations coincide with the kernel of the natural map $\sigma: \mathcal{S}_{g,n}^\bullet \rightarrow R^\bullet(\overline{\mathcal{M}}_{g,n})$.*

Another natural question involves tautological relations in the Chow setting. Since the Givental–Teleman classification has not been proved in such context, the validity of Pixton's relations is still an open problem in the Chow setting.

CONJECTURE 2.2.16. *Pixton's relations hold in the Chow setting, and generate all tautological relations in the Chow tautological ring.*

2.2.2 — CHIODO CLASSES

Another important class of CohFTs that generalise the Hodge class are Chiodo classes [Chio8b]. We refer to [Jaroo; AJo3; CCCo7; Chio8b] for further details on the moduli space of twisted spin curves.

DEFINITION 2.2.17. For a fixed positive integer r , and integers k, a_1, \dots, a_n satisfying the modular constraint

$$\sum_{i=1}^n a_i \equiv k(2g - 2 + n) \pmod{r}, \quad (2.2.30)$$

consider the moduli space $\overline{\mathcal{M}}_{g,a}^{r,k}$ of objects (C, x_1, \dots, x_n, L) , where (C, x_1, \dots, x_n) is a stable curve of genus g with n marked points, and $L \rightarrow C$ is a line bundle such that

$$L^{\otimes r} \cong \omega_{\log}^{\otimes k} \left(- \sum_{i=1}^n a_i x_i \right). \quad (2.2.31)$$

Here $\omega_{\log} = \omega(\sum_{i=1}^n x_i)$ is the log canonical bundle. Such moduli space, called the *moduli space of twisted spin curves*, has a universal curve and a universal line bundle:

$$\pi: \overline{\mathcal{C}}_{g,a}^{r,k} \longrightarrow \overline{\mathcal{M}}_{g,a}^{r,k}, \quad \mathcal{L} \longrightarrow \overline{\mathcal{C}}_{g,a}^{r,k}. \quad (2.2.32)$$

Moreover, it comes with a forgetful map $\epsilon: \overline{\mathcal{M}}_{g,a}^{r,k} \rightarrow \overline{\mathcal{M}}_{g,n}$.

Define the *Chiodo class* as

$$C_{g,n}^{r,k}(a_1, \dots, a_n) = \epsilon_* c(-R^\bullet \pi_* \mathcal{L}) \in H^\bullet(\overline{\mathcal{M}}_{g,n}). \quad (2.2.33)$$

Here $R^\bullet \pi_* \mathcal{L}$ is the derived pushforward of \mathcal{L} , and c is its total Chern class.

From Chiodo's formula [Chio8b] for the Chern character of $R^\bullet \pi_* \mathcal{L}$, together with a careful analysis of the morphism ϵ_* , Janda–Pixton–Pandharipande–Zvonkine [JPPZ17] obtained an explicit expression for Chiodo classes in terms of strata algebra classes.

DEFINITION 2.2.18. Let $\Gamma \in \mathcal{G}_{g,n}$ be a stable graph, and fix an n -tuple $a = (a_1, \dots, a_n)$ of integers satisfying the modular constraint $\sum_{i=1}^n a_i \equiv k(2g - 2 + n) \pmod{r}$. A k -weighting modulo r of Γ with boundary data a is a map $w: H_\Gamma \rightarrow \{0, \dots, r-1\}$ satisfying the following axioms.

- VERTEX CONDITION. For every vertex $v \in V_\Gamma$,

$$\sum_{h \in H_\Gamma(v)} w(h) \equiv k(2g(v) - 2 + n(v)) \pmod{r}. \quad (2.2.34)$$

- EDGE CONDITION. For every edge $e = (h, h') \in E_\Gamma$,

$$w(h) + w(h') \equiv 0 \pmod{r}. \quad (2.2.35)$$

- LEAF CONDITION. For every leaf $\lambda_i \in \Lambda_\Gamma$ corresponding to the marking $i \in \{1, \dots, n\}$,

$$w(\lambda_i) \equiv a_i \pmod{r}. \quad (2.2.36)$$

Denote by $W_\Gamma^{r,k}(a)$ the set of k -weighting modulo r of Γ with boundary data a .

Define the m -th Bernoulli polynomial $B_m(x)$ through the generating series

$$\frac{te^{tx}}{e^t - 1} = \sum_{m \geq 0} B_m(x) \frac{t^m}{m!}. \quad (2.2.37)$$

PROPOSITION 2.2.19 ([JPPZ17]). Chiodo's class $C_{g,n}^{r,k}(a_1, \dots, a_n)$ is given by

$$\begin{aligned} & \sum_{\Gamma \in \mathcal{G}_{g,n}} \sum_{w \in W_\Gamma^{r,k}(a)} \frac{r^{2g-1-h^1(\Gamma)}}{|\text{Aut}(\Gamma)|} \xi_{\Gamma,*} \prod_{v \in V_\Gamma} \exp \left(\sum_{m \geq 1} \frac{(-1)^m B_{m+1}(\frac{k}{r})}{m(m+1)} \kappa_m(v) \right) \\ & \times \prod_{\substack{e \in E_\Gamma \\ e=(h,h')}} \frac{1 - \exp \left(- \sum_{m \geq 1} \frac{(-1)^m B_{m+1}(\frac{w(h)}{r})}{m(m+1)} ((\psi_h)^m - (-\psi_{h'})^m) \right)}{\psi_h + \psi_{h'}} \\ & \times \prod_{\lambda_i \in \Lambda_\Gamma} \exp \left(- \sum_{m \geq 1} \frac{(-1)^m B_{m+1}(\frac{a_i}{r})}{m(m+1)} \psi_{\lambda_i}^m \right). \end{aligned} \quad (2.2.38)$$

In particular, it is tautological.

COROLLARY 2.2.20 ([JPPZ17; LPSZ17]). Chiodo classes form a CohFT $(C_{g,n}^{r,k})_{2g-2+n>0}$ on $V = \mathbb{Q} \langle v_1, \dots, v_r \rangle$, $\eta(v_a, v_b) = \frac{1}{r} \delta_{a+b,r}$ by setting

$$C_{g,n}^{r,k}: v_{a_1} \otimes \dots \otimes v_{a_n} \mapsto C_{g,n}^{r,k}(a_1, \dots, a_n). \quad (2.2.39)$$

Moreover, the following holds.

- $C_{g,n}^{r,k}$ is obtained from the TFT with unit on (V, η, v_r) defined by

$$c_{g,n}^{r,k}(v_{a_1} \otimes \cdots \otimes v_{a_n}) = r^{2g-1} \delta_{a_1+\cdots+a_n-k(2g-2+n),r} \quad (2.2.40)$$

through a translation and an R -matrix actions: $C^{r,k} = RTc^{r,k}$, with

$$\begin{aligned} T(u) &= u \left(1 - \exp \left(- \sum_{m \geq 1} \frac{(-1)^m B_{m+1}(\frac{k}{r})}{m(m+1)} u^m \right) \right) v_r, \\ R^{-1}(u) &= \exp \left(- \sum_{m \geq 1} \frac{(-1)^m \text{diag}_{a=1}^r(B_{m+1}(\frac{a}{r}))}{m(m+1)} u^m \right). \end{aligned} \quad (2.2.41)$$

- The action is unit-preserving (i.e. $T = T_R$ and $C^{r,k} = R.c^{r,k}$ in accordance with Definition 2.2.10) if and only if $0 \leq k \leq r$.

Notice that, for $r = k = a_i = 1$, Chiodo's class coincides with the Hodge class $\Lambda(-1)$.

EXAMPLE 2.2.21. Proposition 2.2.19 gives an explicit formula to compute Chiodo classes. For instance, one can compute $C_{1,1}^{r,k}(a) = \text{Cont}_\Gamma + \frac{\xi_{\Gamma'}^{r,*}}{2} \text{Cont}_{\Gamma'}$, where the stable graphs are

$$\Gamma = \text{---} \bullet \quad \text{and} \quad \Gamma' = \text{---} \circ$$

and their contributions are given by

$$\begin{aligned} \text{Cont}_\Gamma &= \delta_{k-a} r \left(1 - \frac{B_2(\frac{k}{r})}{2} \kappa_1 \right) \left(1 + \frac{B_2(\frac{a}{r})}{2} \psi_1 \right) \in H^\bullet(\overline{\mathcal{M}}_{1,1}), \\ \text{Cont}_{\Gamma'} &= \delta_{k-a} \left(- \sum_{w=0}^{r-1} \frac{B_2(\frac{w}{r})}{2} \right) \in H^\bullet(\overline{\mathcal{M}}_{0,3}). \end{aligned}$$

The Kronecker delta indices are considered modulo r . Using the relations $\kappa_1 = \psi_1$ and $\frac{\xi_{\Gamma'}^{r,*}}{2}[\text{pt}] = 12\psi_1$, together with the Bernoulli polynomial identities $B_2(x) = x^2 - x + \frac{1}{6}$, $6r \sum_{w=0}^{r-1} B_2(\frac{w}{r}) = 1$, we find

$$C_{1,1}^{r,k}(a) = \delta_{k-a} r \left(1 + \frac{(a-k)(a+k-r)-2}{2r^2} \psi_1 \right) \in H^\bullet(\overline{\mathcal{M}}_{1,1}).$$

Integrating over the moduli space, we find the CohFT correlators

$$\int_{\overline{\mathcal{M}}_{1,1}} C_{1,1}^{r,k}(a) = \delta_{k-a} \frac{(a-k)(a+k-r)-2}{48r}, \quad \int_{\overline{\mathcal{M}}_{1,1}} C_{1,1}^{r,k}(a) \psi_1 = \delta_{k-a} \frac{r}{24}.$$

In this dissertation, we will see many applications of Chiodo classes, namely orbifold Hurwitz numbers with completed cycles in Section 2.6, Masur–Veech volumes in Section 9.1, the Euler characteristic of the moduli space of curves in Section 9.3, and spin Hurwitz numbers in Section 12.4. Other applications of Chiodo classes include the double ramification cycle [JPPZ17], Norbury's Θ -class [Nor17], and double Hurwitz numbers [DL20; Bor+20].

2.3 — TOPOLOGICAL RECURSION

Topological recursion (TR), as originally defined by Eynard and Orantin [EO07a], is a general formalism to recursively define a set of symmetric multidifferentials $\omega_{g,n}$ on a spectral curve, i.e. a Riemann surface with some additional structure.

$$\text{SPECTRAL CURVE} \xrightarrow{\text{TR}} (\omega_{g,n})_{g,n \geq 0} \quad (2.3.1)$$

The original motivation of topological recursion can be found in matrix model theory [CE06a; CE06b; CEO06], where such differentials encode the topological expansion when the matrix size N tends to infinity. However, topological recursion quickly found several applications in many different areas of enumerative geometry. When specialised, it recovers several known invariants such as Witten’s intersection numbers, Weil–Peterson volumes, knots invariants, Hurwitz numbers and Gromov–Witten invariants. It has a correspondent construction in Givental theory, and has deep connections with integrable hierarchies, Hitchin system, JWKB method³, conformal field theories and many others.

In this dissertation, we will present the original formulation of Eynard–Orantin, followed by the Kontsevich–Soibelman reformulation [KS18; ABCO17] in Section 2.3.2 and its “geometrisation” by Andersen–Borot–Orantin [ABO17] in Section 2.4. We refer to [Eyn14] for a short overview, and [Bor20] for both an algebraic and a geometric approach.

DEFINITION 2.3.1. A *spectral curve* is the data $\mathcal{S} = (C, x, y, B)$ of

- a Riemann surface C , not necessarily connected nor compact,
- a function $x: C \rightarrow \mathbb{C}$, such that its differential dx is meromorphic and has finitely many zeros that are simple (called *ramification points*),
- a meromorphic function $y: C \rightarrow \mathbb{C}$ that is holomorphic at the ramification points and such that dy is non-zero at the ramification points,
- a symmetric bidifferential B on $C \times C$, having a double pole on the diagonal with biresidue 1 and no other poles. In other words, for every choice of local coordinates ζ ,

$$B(z_1, z_2) = \left(\frac{1}{(\zeta(z_1) - \zeta(z_2))^2} + O(1) \right) d\zeta(z_1) d\zeta(z_2), \quad (2.3.2)$$

that is, $B \in H^0(C^2, \omega_C^{\otimes 2}(2\Delta))^{\otimes 2}$, where ω is the canonical bundle and Δ is the diagonal in $C \times C$.

Remark 2.3.2. In some applications, the function x is not required to be meromorphic, but only its differential dx is. This is the case, for instance, of x containing a logarithmic term: then x will be meromorphic on \mathbb{P}^1 minus a cut from 0 to ∞ , and its differential will be a meromorphic function on the whole \mathbb{P}^1 .

³This method is commonly known as “WKB method”, named after G. Wentzel, H. A. Kramers, and L. Brillouin, who all developed it in 1926. However, H. Jeffreys had developed in 1923 a general method of approximating solutions to linear, second-order ODEs.

Remark 2.3.3. In some applications, the bidifferential B can be rescaled to have biresidue $\beta \in \mathbb{C}^\times$ along the diagonal (see also the homogeneity property of Theorem 2.3.6). Moreover, the bidifferential B is allowed to have other poles, as long as there are no poles when the two arguments approach different ramification points. In Chapter 12, we will see an example where this property does not hold and a workaround is needed.

Remark 2.3.4. If C is a connected compact surface of genus g equipped with a Torelli marking, i.e. a symplectic basis $(A_i, B_i)_{i=1, \dots, g}$ of $H_1(C, \mathbb{Z})$, then there exists a unique element $B \in H^0(C^2, \omega_C^{\otimes 2}(2\Delta))^{\mathfrak{S}_2}$ that is normalised along the A -cycles:

$$\oint_{A_i} B(\cdot, z) = 0 \quad \forall i = 1, \dots, g. \quad (2.3.3)$$

It is called the *canonical bidifferential of the second kind*. For instance, on \mathbb{P}^1 we have

$$B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}, \quad (2.3.4)$$

while on an elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ with A -cycle $[0, 1)$ and B -cycle $[0, \tau)$, we have

$$B(z_1, z_2) = \left(\wp(z_1 - z_2, \tau) + \frac{\pi}{\mathfrak{J}(\tau)} \right) dz_1 dz_2, \quad (2.3.5)$$

where \wp is the Weierstraß function on E_τ .

Denote by \mathfrak{a} the set of ramification points of a spectral curve \mathcal{S} . Since all ramification points are simple, one can find local coordinates ζ_a around each $a \in \mathfrak{a}$ such that

$$x(z) = \frac{\zeta_a^2(z)}{2} + x(a). \quad (2.3.6)$$

Denote by $U_a \subseteq C$ the (small enough) neighbourhood of the ramification point a in which the local coordinates ζ_a is defined. On U_a , there is a well-defined holomorphic involution $\iota_a: \zeta_a \mapsto -\zeta_a$. It is uniquely determined by the conditions $x \circ \iota_a = x$, $\iota_a(a) = a$ and $\iota \neq \text{id}$. Define the topological recursion kernel

$$K_a(z_1, z) = \frac{1}{2} \frac{\int_{\iota_a(z)}^z B(z_1, \cdot)}{(y(z) - y(\iota_a(z))) dx(z)}, \quad (2.3.7)$$

which is a well-defined 1-form in z_0 and inverse of a 1-form in z on $C \times U_a$.

DEFINITION 2.3.5. For a given spectral curve (C, x, y, B) , define the *topological recursion correlators* $(\omega_{g,n})_{g \geq 0, n \geq 1}$ as follows. Define the unstable cases as

$$\omega_{0,1} = y dx, \quad \omega_{0,2} = B. \quad (2.3.8)$$

For $2g - 2 + n > 0$, define the multidifferential $\omega_{g,n}$ on C^n recursively on $2g - 2 + n$ by setting

$$\begin{aligned} \omega_{g,n}(z_1, \dots, z_n) = \sum_{a \in \mathfrak{a}} \text{Res}_{z \rightarrow a} K_a(z_1, z) & \left(\omega_{g-1, n+1}(z, \iota_a(z), z_2, \dots, z_n) \right. \\ & \left. + \sum_{\substack{\text{no } (0,1) \\ g_1+g_2=g \\ I_1 \sqcup I_2 = \{2, \dots, n\}}} \omega_{g_1, 1+|I_1|}(z, z_{I_1}) \omega_{g_2, 1+|I_2|}(\iota_a(z), z_{I_2}) \right). \end{aligned} \quad (2.3.9)$$

The second sum excludes all terms containing $\omega_{0,1}$. Here we also used the shorthand notation $z_I = (z_i)_{i \in I}$ for any finite set I .

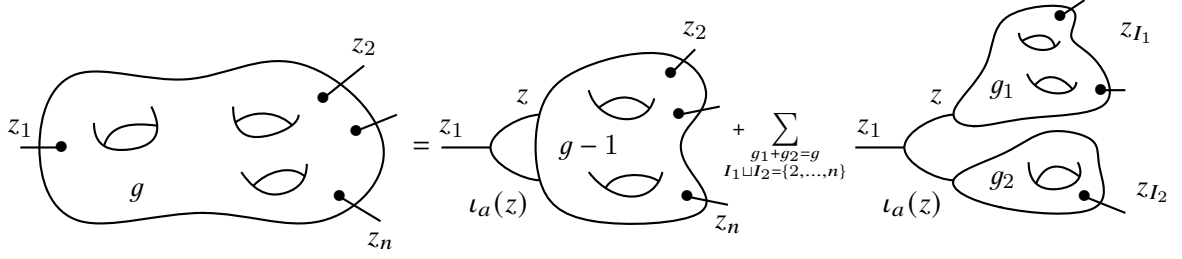


Figure 2.3: A schematic representation of topological recursion.

At first sight, the topological recursion correlators $\omega_{g,n}$ are not necessarily symmetric in the variables z_1, \dots, z_n , as z_1 is playing a different role in the definition. However, it turns out that $\omega_{g,n}$ is actually symmetric.

Notice also that the terms appearing in the recursion formula are in bijection with the tautological maps of Definitions 2.1.4 and 2.1.6. Indeed:

- the first term $\omega_{g-1,n+1}$ corresponds to the gluing map of non-separating kind,
- the terms of the form $\omega_{g_1,1+|I_1|} \omega_{g_2,1+|I_2|}$ with $2g_i - 2 + (|I_i| + 1) > 0$ correspond to all possible gluing maps of separating kind, and
- the terms of the form $\omega_{0,2} \omega_{g,n-1}$ correspond to the forgetful maps.

The meaning of such correspondence will become more explicit in Section 2.3.1, when the relation between topological recursion and CohFTs will be explained. Furthermore, a geometric explanation in terms of excision of pairs of pants from topological surfaces will be given in Section 2.4. See also Figure 2.3 for a pictorial representation of the topological recursion formula.

The following properties can be found in the original work of Eynard and Orantin.

THEOREM 2.3.6 ([EO07a, Section 4.4]). *The topological recursion correlators $(\omega_{g,n})_{g \geq 0, n \geq 1}$ satisfy the following properties.*

- **SYMMETRY AND POLE STRUCTURE.** For $2g - 2 + n > 0$, $\omega_{g,n}$ is a well-defined meromorphic multidifferential on C^n , with poles only at ramification points of order at most $6g - 4 + 2n$ and vanishing residue, and symmetric with respect to all variables:

$$\omega_{g,n} \in H^0(C^n, \omega_C^{\otimes n}((6g - 4 + 2n)\mathbf{a}))^{\mathfrak{S}_n}. \quad (2.3.10)$$

- **HOMOGENEITY.** Under rescaling $\omega_{0,1} \mapsto \lambda \omega_{0,1}$, we have $\omega_{g,n} \mapsto \lambda^{-(2g-2+n)} \omega_{g,n}$. Moreover, under rescaling $\omega_{0,2} \mapsto \beta \omega_{0,2}$, we have $\omega_{g,n} \mapsto \beta^{3g-3+2n} \omega_{g,n}$.
- **DILATON EQUATION.** The following equation holds:

$$\sum_{a \in \mathbf{a}} \operatorname{Res}_{z \rightarrow a} \Phi(z) \omega_{g,n+1}(z, z_1, \dots, z_n) = (2g - 2 + n) \omega_{g,n}(z_1, \dots, z_n), \quad (2.3.11)$$

where $d\Phi(z) = \omega_{0,1}(z)$.

- *LOOP EQUATION.* For every ramification point $a \in \mathfrak{a}$, the expression

$$\omega_{g-1,n+1}(z, t_a(z), z_2, \dots, z_n) + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2, \dots, n\}}} \omega_{g_1, 1+|I_1|}(z, z_{I_1}) \omega_{g_2, 1+|I_2|}(t_a(z), z_{I_2}) \quad (2.3.12)$$

is a meromorphic quadratic differential in z , with at least a double zero for $z \rightarrow a$. Notice that the sum includes $(0, 1)$ terms.

We remark that the dilaton equation allows to extend the definition of the correlator differentials to $n = 0$ and $g > 1$, that are scalars, usually denoted by F_g .

DEFINITION 2.3.7. For a given spectral curve (C, x, y, B) , define the *free energies* by

$$F_g = \frac{1}{2g-2} \sum_{a \in \mathfrak{a}} \operatorname{Res}_{z \rightarrow a} \Phi(z) \omega_{g,1}(z), \quad (2.3.13)$$

where $d\Phi(z) = \omega_{0,1}(z)$. A definition of F_0 and F_1 can be found in [EO07a, Subsection 4.2.2].

Remark 2.3.8. Notice that the topological recursion depends only on the local behaviour of y and B around the ramification points. More formally, given a spectral curve as in Definition 2.3.1, we get neighbourhoods U_a of the ramification points $a \in \mathfrak{a}$ in which

$$x(z) = \frac{\zeta_a^2(z)}{2} + x(a). \quad (2.3.14)$$

In such neighbourhoods, we can expand y and B as

$$\begin{aligned} y &= \sum_{k \geq 0} t_{k,a} \zeta_a^k \\ B &= \left(\frac{\delta_{a_1, a_2}}{(\zeta_{a_1} - \zeta_{a_2})^2} + \sum_{k_1, k_2 \geq 0} \phi_{(k_1, a_1), (k_2, a_2)} \zeta_{a_1}^{k_1} \zeta_{a_2}^{k_2} \right) d\zeta_{a_1} d\zeta_{a_2}, \end{aligned} \quad (2.3.15)$$

for some coefficients $t_{k,a}$ and $\phi_{(k_1, a_1), (k_2, a_2)}$ in \mathbb{C} . Moreover, we define the auxiliary meromorphic functions ξ^a and the meromorphic differentials $d\xi^{k,a}$ as

$$\xi^a(z) = \int^z \frac{B(w, \cdot)}{d\zeta_a(w)} \Big|_{w=a}, \quad d\xi^{k,a}(z) = d \left(\left(-\frac{1}{\zeta_a} \frac{d}{d\zeta_a} \right)^k \xi^a(z) \right). \quad (2.3.16)$$

Then the topological recursion correlators can be expressed as linear combinations of the differentials $(d\xi^\alpha(z))_{\alpha=(k,a) \in \mathbb{N} \times \mathfrak{a}}$, with coefficients $F_{g; \alpha_1, \dots, \alpha_n}$ being polynomials⁴ in t_α and $\phi_{\alpha, \beta}$:

$$\omega_{g,n}(z_1, \dots, z_n) = \sum_{\alpha_1, \dots, \alpha_n \in \mathbb{N} \times \mathfrak{a}} F_{g; \alpha_1, \dots, \alpha_n} \prod_{i=1}^n d\xi^{\alpha_i}(z_i). \quad (2.3.17)$$

One can easily show that $d\xi^{a,k}$ has poles of order at most $2k+2$ at $z=a$ and no other poles. From the pole structure of $\omega_{g,n}$, we deduce that the above sum is finite, and more precisely that $F_{g; \alpha_1, \dots, \alpha_n}$ with $\alpha_i = (k_i, a_i)$ vanishes for $k_1 + \dots + k_n > 3g - 3 + n$.

⁴To be more precise, $F_{g; \alpha_1, \dots, \alpha_n}$ is a polynomial in the Taylor coefficients of the expansion of the inverse of $y(z) - y(t_a(z))$, namely $t_{1,a}^{-1}$ and $(t_{2k+1,a})_{k \geq 0, a \in \mathfrak{a}}$, and in the even coefficients of B , namely $(\phi_{(2k,a), (2\ell, b)})_{k, \ell \geq 0, a, b \in \mathfrak{a}}$.

Remark 2.3.9. Since the original work of Eynard and Orantin, topological recursion has been generalised in various directions, notably by Bouchard–Hutchinson–Loliencar–Meiers–Rupert and Bouchard–Eynard [BE13; Bou+14] to allow for non-simple ramifications of x (see also [Bor+18]), by Borot–Shadrin [BS17] as blobbed topological recursion, by Chekhov–Norbury [CN19] for irregular spectral curves, by Osuga and Bouchard [Osu19; BK20] in the context of super spectral curves, and by Borot–Kramer–Schüler [BKS20] for singular spectral curves. Different generalisations that do not involve spectral curves are discussed in Section 2.3.2 and Section 2.4.2.

EXAMPLE 2.3.10. We give here some examples of spectral curves that are relevant in enumerative geometry and mathematical physics.

- The Airy curve [EO07a]. Let $C = \mathbb{P}^1$ and

$$x(z) = \frac{z^2}{2}, \quad y(z) = -z, \quad B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}.$$

The associated multidifferentials (see Table 2.2) are generating functions of ψ -intersection numbers, or in other words correlators of the trivial CohFT:

$$\omega_{g,n}(z_1, \dots, z_n) = \sum_{k_1, \dots, k_n \geq 0} \int_{\mathcal{M}_{g,n}} \prod_{i=1}^n \psi_i^{k_i} \frac{(2k_i + 1)!!}{z_i^{2k_i+2}} dz_i.$$

As we will see in Part II, such correlators are also the (Laplace transform of the) Kontsevich volumes of the combinatorial moduli space of curves, and the topological recursion formula is a consequence of a Mirzakhani-type identity on the associated combinatorial Teichmüller space. The name “Airy curve” comes from the quantisation of the associated curve $y^2 - 2x = 0$, that when quantised becomes the Airy differential equation.

- The Mirzakhani curve [EO07b]. Let $C = \mathbb{P}^1$ and

$$x(z) = \frac{z^2}{2}, \quad y(z) = -\frac{\sin(2\pi z)}{2\pi}, \quad B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}.$$

The associated multidifferentials are the generating functions of correlators for the CohFT $\exp(2\pi^2 \kappa_1)$:

$$\omega_{g,n}(z_1, \dots, z_n) = \sum_{k_1, \dots, k_n \geq 0} \int_{\mathcal{M}_{g,n}} e^{2\pi^2 \kappa_1} \prod_{i=1}^n \psi_i^{k_i} \frac{(2k_i + 1)!!}{z_i^{2k_i+2}} dz_i.$$

Such correlators are also the (Laplace transform of the) Weil–Peterson volumes of the moduli space of bordered hyperbolic surfaces, and the topological recursion formula is a consequence of the Mirzakhani identity on the associated Teichmüller space (see Section 2.4.2).

- The Lambert curve [BM08; BEMS11; EMS11]. Let $C = \mathbb{P}^1$ and

$$x(z) = \log(z) - z, \quad y(z) = z, \quad B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}.$$

(g, n)	$\omega_{g,n}(z) \frac{m_{((3g-2+n)^n)}}{dz_1 \cdots dz_n}$
(0, 3)	1
(0, 4)	$3m_{(1^3)}$
(0, 5)	$15m_{(2^4)} + 18m_{(2^3, 1^2)}$
(0, 6)	$105m_{(3^5)} + 135m_{(3^4, 2, 1)} + 162m_{(3^3, 2^3)}$
(0, 7)	$945m_{(4^6)} + 1260m_{(4^5, 3, 1)} + 1350m_{(4^5, 2^2)} + 1620m_{(4^4, 3^2, 2)} + 1944m_{(4^3, 3^4)}$
(1, 1)	8
(1, 2)	$\frac{5}{8}m_{(2)} + \frac{3}{8}m_{(1^2)}$
(1, 3)	$\frac{35}{8}m_{(3^2)} + \frac{15}{4}m_{(3, 2, 1)} + \frac{9}{4}m_{(2^3)}$
(1, 4)	$\frac{315}{8}m_{(4^3)} + \frac{315}{8}m_{(4^2, 3, 1)} + \frac{75}{2}m_{(4^2, 2^2)} + \frac{135}{4}m_{(4, 3^2, 2)} + \frac{81}{4}m_{(3^3, 1)}$
(2, 1)	$\frac{105}{128}$
(2, 2)	$\frac{1155}{128}m_{(5)} + \frac{945}{128}m_{(4, 1)} + \frac{1015}{128}m_{(3, 2)}$
(3, 1)	$\frac{25025}{1024}$

Table 2.2: A list of topological recursion correlators associated to the Airy spectral curve for low values of $2g - 2 + n$, normalised as $\omega_{g,n}(z) \frac{m_{((3g-2+n)^n)}}{dz_1 \cdots dz_n}$. Here m_λ is the monomial symmetric polynomial associated to the partition λ , evaluated at z_1^2, \dots, z_n^2 .

Notice that x contains a logarithmic term, for which we can still consider $C = \mathbb{P}^1$ (see Remark 2.3.2). The associated multidifferentials are the generating functions of simple Hurwitz numbers (see Section 2.6):

$$\omega_{g,n}(z_1, \dots, z_n) = \sum_{\mu \vdash n} h_{g;\mu} \prod_{i=1}^n \mu_i e^{\mu_i x(z_i)} dx(z_i).$$

Here the notation $\mu \vdash n$ means that μ is a partition of n . The name ‘‘Lambert curve’’ comes from the equation satisfied by x and y , namely $y = -W(-e^x)$, where W is the Lambert function.

- The (qr, q) -Lambert curve [SSZ15; LPSZ17; Bor+21; KLPS19; DKPS19]. More generally, let $C = \mathbb{P}^1$ and, for non-negative integers q and r , consider

$$x(z) = \log(z) - z^{qr}, \quad y(z) = z^q, \quad B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}.$$

The associated multidifferentials are the generating functions of q -orbifold Hurwitz numbers with $(r + 1)$ -completed cycles (see Section 2.6):

$$\omega_{g,n}(z_1, \dots, z_n) = \sum_{\mu \vdash n} h_{g;\mu}^{q,r} \prod_{i=1}^n \mu_i e^{\mu_i x(z_i)} dx(z_i).$$

The result for $r = 1$ and general q was proved in [BHLMI4; DLN16]. A general relation between topological recursion and hypergeometric KP tau function (including many Hurwitz problems) has been solved uniformly in [BDKS20].

- The Bessel curve [DN18; Nor17]. Let $C = \mathbb{P}^1$ and

$$x(z) = \frac{z^2}{2}, \quad y(z) = -\frac{1}{z}, \quad B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}.$$

Although y has a pole at the (unique) ramification point of x , the topological recursion can still be defined. The associated multidifferentials are conjecturally computing the correlators of Norbury’s class $\Theta_{g,n}$:

$$\omega_{g,n}(z_1, \dots, z_n) = \sum_{d_1, \dots, d_n \geq 0} \int_{\mathcal{M}_{g,n}} \Theta_{g,n} \prod_{i=1}^n \psi_i^{d_i} \frac{(2d_i + 1)!!}{z_i^{2d_i+2}} dz_i.$$

The name “Bessel curve” comes from the quantisation of the associated curve $y^2 - \frac{1}{2x} = 0$, which when quantised becomes the (degenerate) Bessel differential equation.

There are many more examples, that we briefly list here: monotone and strictly monotone Hurwitz numbers (and their orbifold generalisations) [DOPS18; KPS19], double Hurwitz numbers [Bor+20], singularity theory [Mil15; Dun+19], stationary Gromov–Witten invariants of \mathbb{P}^1 [NS14; DOSS14], Gromov–Witten invariants of toric Calabi–Yau 3-folds [BKMP09; EO15; FLZ20], Hitchin system [DM18; BH19], integrable systems and JWKB analysis [BE12; BE17; IKT18; IKT19; Iwa20], BPS states arising from hypergeometric-type spectral curves [IK20], perturbative knots invariants [BME12; BEW17; BB18; Dun+20], formal asymptotics of knot invariants [DFM11; BE15] (see also [GS12] for quantisation of the A -polynomial curves and a topological recursion perspective), and $\mathcal{N} = 2$ four-dimensional supersymmetric gauge theories [BBCC21].

In Part II, we will also discuss the application of topological recursion in the context of the combinatorial moduli space, in Part III for the enumeration of multicurves, Masur–Veech volumes, and the Euler characteristic of the moduli space, and in Part IV for computing spin Hurwitz numbers.

A useful property of topological recursion is the variation formula with respect to the bidifferential B . As we will see later, it is the analogue of the R -matrix action on CohFTs.

THEOREM 2.3.11. *Let (C, x, y, B) and (C, x, y, \tilde{B}) be two spectral curves, $\omega_{g,n}$ and $\tilde{\omega}_{g,n}$ the respective topological recursion multidifferentials. Define two projectors \mathcal{P} and $\tilde{\mathcal{P}}$ acting on the space of meromorphic 1-forms on C as*

$$\mathcal{P}[\phi](z_0) = \sum_{\alpha \in \mathfrak{a}} \operatorname{Res}_{z=\alpha} \left(\int_{\alpha}^z B(\cdot, z_0) \right) \phi(z), \quad (2.3.18)$$

and likewise $\tilde{\mathcal{P}}$ with \tilde{B} . Denote by \mathcal{V} the image of \mathcal{P} . Assume there exists a 2-cycle $\gamma \subset (C \setminus \mathfrak{a})^2$ and a germ Υ of an holomorphic function at γ such that \tilde{B} is obtained by shifting B as

$$\tilde{B}(z_1, z_2) = B(z_1, z_2) + \int_{(w_1, w_2) \in \gamma} \Upsilon(w_1, w_2) B(z_1, w_1) B(z_2, w_2), \quad (2.3.19)$$

and define a linear form Θ on $\operatorname{Sym}^2 \mathcal{V}$ by the formula

$$\Theta[\varpi] = \int_{(w_1, w_2) \in \gamma} \Upsilon(w_1, w_2) \varpi(w_1, w_2). \quad (2.3.20)$$

Then, we have

$$\begin{aligned} \tilde{\omega}_{g,n}(z_1, \dots, z_n) &= \\ &= \sum_{\Gamma \in \mathcal{G}_{g,n}} \frac{1}{|\text{Aut}(\Gamma)|} \left(\bigotimes_{\lambda \in \Lambda_\Gamma} \tilde{\mathcal{P}}_{z_\lambda} \otimes \bigotimes_{\substack{e \in E_\Gamma \\ e=(h_1, h_2)}} \mathcal{O}_{z_{h_1}, z_{h_2}} \right) \left[\bigotimes_{v \in V_\Gamma} \omega_{g(v), n(v)}((z_h)_{h \in H(v)}) \right]. \end{aligned} \quad (2.3.21)$$

In this formula, $H(v)$ is the set of half-edges attached to the vertex v , and we indicate in subscripts the variables on which the operators act.

Sketch of the proof. Eynard–Orantin [EO07a, Theorem 6.1] establishes a formula for the first derivative of $\tilde{\omega}_{g,n}$ with respect to the matrix elements of κ when:

- C is a compact Riemann surface,
- B is the canonical bidifferential of the second kind (see Remark 2.3.4),
- $\tilde{B} = B + 2\pi i \, du^t \kappa du$ for a fixed symmetric matrix κ , where u is the Abel map (considered as a column vector).

Integrating their relation with respect to κ yields the result – in that case γ is an element of $\text{Sym}^2 H^1(C, \mathbb{Z})$ and $Y = (2\pi i)^{-2}$. The same proof in fact works under the assumptions of the theorem. Notice that the order of integration of the variables $(z_h, z_{h'})$ is irrelevant, by the assumptions on (γ, Y) and the fact that $\omega_{g,n}$ only has poles on the ramification divisor. \square

2.3.1 — EYNARD–DOSS CORRESPONDENCE

Consider a spectral curve $\mathcal{S} = (C, x, y, B)$. Under some conditions, Eynard [Eyn11] and Dunin-Barkowski–Orantin–Shadrin–Spitz [DOSS14] showed that the expansion coefficients $F_{g; \alpha_1, \dots, \alpha_n}$ from Equation (2.3.17) are correlators of a certain CohFT (over \mathbb{C}) associated to \mathcal{S} that we now describe.

In most applications, it is convenient to fix some normalisation constants $(c[a])_{a \in \mathfrak{a}}$ in \mathbb{C}^\times and an additional constant $c \in \mathbb{C}^\times$. Choose local coordinates ζ_a around a ramification point a such that $x = (c[a]\zeta_a)^2 + x(a)$. Consider the auxiliary functions ξ^a and the associated meromorphic differentials $d\xi^{k,a}$, defined as

$$\xi^a(z) = \int^z \frac{B(w, \cdot)}{d\zeta_a(w)} \Big|_{w=a}, \quad d\xi^{k,a}(z) = d \left(\left(-\frac{1}{\zeta_a} \frac{d}{d\zeta_a} \right)^k \xi^a(z) \right). \quad (2.3.22)$$

Notice that, for $c[a] = \frac{1}{\sqrt{2}}$, we get the differentials of Equation (2.3.16). Moreover, they are globally defined, as

$$-\frac{1}{\zeta_a} \frac{d}{d\zeta_a} = -2c[a]^2 \frac{d}{dx}. \quad (2.3.23)$$

We also set $\Delta^a = \frac{dy(z)}{d\zeta_a(z)} \Big|_{z=a}$ and $t^a = -2c[a]^2 c \Delta^a$. Define a unital, semisimple TFT on $V = \mathbb{C} \langle e_1, \dots, e_r \rangle$ by setting $\eta(e_a, e_b) = \delta_{a,b}$ and

$$\mathbb{1} = \sum_{a \in \mathfrak{a}} t^a e_a, \quad \varpi_{g,n}(e_{a_1} \otimes \dots \otimes e_{a_n}) = \frac{\delta_{a_1, \dots, a_n}}{(t^{a_i})^{2g-2+n}}. \quad (2.3.24)$$

Define the R -matrix $R \in \text{End}(V)[[u]]$ and the translation $T \in u^2V[[u]]$ by setting

$$(R^{-1})_a^b(u) = -\sqrt{\frac{u}{2\pi}} \int_{\gamma_b} d\xi^a e^{-\frac{x-x(b)}{2c[b]^2u}}, \quad (2.3.25)$$

$$T^a(u) = \left(t^a u - (-2c[a]^2c) \sqrt{\frac{u}{2\pi}} \int_{\gamma_a} dy e^{-\frac{x-x(a)}{2c[a]^2u}} \right). \quad (2.3.26)$$

Here $\gamma_a = \{z \in C \mid x(z) - x(a) > 0\}$ is the *steepest descent* from a , oriented from the negative to the positive values of the local coordinate ζ_a . Moreover, the equations are intended as equalities between formal power series in u , where on the right-hand side we take the asymptotic expansion as $u \rightarrow 0$. Through the Givental action, we can then define a CohFT

$$\Omega_{g,n} = RT\varpi_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n}) \otimes (V^*)^{\otimes n} \quad (2.3.27)$$

from the data (ϖ, R, T) , through a sum over stable graphs as explained in Section 2.2. The link with the topological recursion correlators is given by the following theorem.

THEOREM 2.3.12 (Eynard–DOSS correspondence [Eyn11; DOSS14]). *Suppose we have a compact spectral curve $\mathcal{S} = (C, x, y, B)$. Then its topological recursion correlators are given by*

$$\omega_{g,n}(z_1, \dots, z_n) = c^{2g-2+n} \sum_{\alpha_1, \dots, \alpha_n \in \mathbb{N} \times \mathfrak{a}} \langle \tau_{\alpha_1} \cdots \tau_{\alpha_n} \rangle_g^\Omega \prod_{i=1}^n d\xi^{\alpha_i}(z_i), \quad (2.3.28)$$

where we have used the following shorthand notation for the CohFT correlators:

$$\langle \tau_{\alpha_1} \cdots \tau_{\alpha_n} \rangle_g^\Omega = \int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n}(e_{a_1} \otimes \cdots \otimes e_{a_n}) \prod_{i=1}^n \psi_i^{k_i}, \quad \alpha_i = (k_i, a_i). \quad (2.3.29)$$

Moreover, all the ingredients on the right-hand side depend on the choice of constants $(c[a])_{a \in \mathfrak{a}}$ and c , while the left-hand side is independent of it.

Remark 2.3.13. A weaker version of the above theorem holds for local curves (cf. [DOSS14]). More precisely, the topological recursion correlators are still expressed in terms of the basis $d\xi^{k,a}$ and the coefficients are intersection numbers of a certain class on the moduli space of curves. However, such class is not necessarily a CohFT. See also [Dun+18] for further readings.

Remark 2.3.14. As the translation of Equation (2.3.26) is acting on a unital TFT, one can rewrite the translation action on ϖ as a multiplication by κ -classes (see Lemma 2.2.8) via the V -valued power series

$$\Delta^a \exp(-\hat{T}^a(u)) = \frac{1}{\sqrt{2\pi u}} \int_{\gamma_a} dy e^{-\frac{x-x(a)}{2c[a]^2u}}. \quad (2.3.30)$$

Moreover, the compatibility between the translation and the R -matrix given by Proposition 2.2.9 is equivalent to the following condition (sometimes called the DOSS test):

$$t^b \exp(-\hat{T}^b(u)) = \sum_{a \in \mathfrak{a}} t^a (R^{-1})_a^b(u). \quad (2.3.31)$$

If such equation (which can be seen as a compatibility condition between y and B) holds, the resulting cohomological field theory coincides with $R.\varpi$, which is flat.

Remark 2.3.15. It is now clear that a change in the bidifferential B reflects in a Givental-type action on the topological recursion multidifferentials, as explained in Theorem 2.3.11. Indeed, the bidifferential B is the ingredient determining the R -matrix for the associated CohFT, and the R -matrix action is defined as a sum over stable graphs.

EXAMPLE 2.3.16. Here are some examples of CohFTs associated to the spectral curves of Example 2.3.10 (for specific choices of constants).

Spectral curve	CohFT
Airy	1
Mirzakhani	$e^{2\pi^2\kappa_1}$
Lambert	$\Lambda(-1)$
(qr, q) -Lambert	$C_{g,n}^{qr,q}$
Bessel	$\Theta_{g,n}$

Chiodo class with parameter (qr, q) , associated to the (qr, q) -Lambert curve, requires a change of basis (see [LPSZ17]). Moreover, the spectral curve on \mathbb{P}^1 given by

$$x(z) = \log(z) - z^r, \quad y(z) = z^k, \quad B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2} \quad (2.3.32)$$

gives the Chiodo class with general parameters (r, k) .

2.3.2 — QUANTUM AIRY STRUCTURES

Recently, Kontsevich and Soibelman [KS18] proposed a new point of view for topological recursion, which actually generalise the one of Eynard and Orantin. The Kontsevich–Soibelman approach starts from a collection of quadratic differential operators $(L_\alpha)_{\alpha \in I}$ that form a Lie subalgebra of the Weyl algebra, and constructs a formal power series

$$Z(\hbar; \mathbf{x}) = \exp \left(\sum_{g \geq 0, n > 0} \frac{\hbar^{g-1}}{n!} \sum_{\alpha_1, \dots, \alpha_n \in I} F_{g; \alpha_1, \dots, \alpha_n} x^{\alpha_1} \cdots x^{\alpha_n} \right) \quad (2.3.33)$$

that is annihilated by the differential operators L_α .

DEFINITION 2.3.17. Let V be a (possibly infinite-dimensional) vector space over \mathbb{C} . Fix a basis $(e_\alpha)_{\alpha \in I}$ and let $(x^\alpha)_{\alpha \in I}$ be the dual basis. Define the *Weyl algebra* as

$$\mathcal{W}_\hbar(V) = \mathbb{C}[\hbar] \langle (x^\alpha, \partial_\alpha)_{\alpha \in I} \rangle / \langle [\partial_\alpha, x^\alpha] = \hbar \rangle. \quad (2.3.34)$$

A *quantum Airy structure* on V is a collection $(L_\alpha)_{\alpha \in I}$ of elements of $\mathcal{W}_\hbar(V)$ of the form

$$L_\alpha = \hbar \partial_\alpha - \sum_{i, j \in I} \left(\frac{1}{2} A_{\alpha, i, j} x^i x^j + \hbar B_{\alpha, i}^j x^i \partial_j + \frac{\hbar^2}{2} C_{\alpha}^{i, j} \partial_i \partial_j \right) - \hbar D_\alpha \quad (2.3.35)$$

that form a Lie subalgebra of $\mathcal{W}_\hbar(V)$:

$$[L_\alpha, L_\beta] = \hbar \sum_{\gamma \in I} f_{\alpha, \beta}^\gamma L_\gamma. \quad (2.3.36)$$

In this definition, we can always assume that $A_{\alpha, \beta, \gamma} = A_{\alpha, \gamma, \beta}$ and $C_{\alpha}^{\beta, \gamma} = C_{\alpha}^{\gamma, \beta}$. Moreover, if V is infinite-dimensional, we require that only finitely many coefficients in Equation (2.3.35) are non-zero. For a basis-free definition of quantum Airy structures, see [Bor20].

The subalgebra condition can be recast into a set of relations between the coefficients (A, B, C, D) .

LEMMA 2.3.18. *Let $(L_\alpha)_{\alpha \in I}$ be a collection of differential operators of the form (2.3.35). They form a Lie subalgebra if and only if the following relations hold $\forall \alpha, \beta, \gamma, \delta \in I$.*

- *A-symmetry.*

$$A_{\alpha, \beta, \gamma} = A_{\beta, \alpha, \gamma} \quad (2.3.37)$$

- *IHX-type relations.*

$$\begin{aligned} \sum_{i \in I} \left(B_{\alpha, \beta}^i A_{i, \gamma, \delta} + B_{\alpha, \gamma}^i A_{i, \beta, \delta} + B_{\alpha, \delta}^i A_{i, \beta, \gamma} \right) &= (\alpha \leftrightarrow \beta) \\ \sum_{i \in I} \left(B_{\alpha, \beta}^i C_i^{\gamma, \delta} + C_\alpha^{\gamma, i} B_{\beta, i}^\delta + C_\alpha^{\delta, i} B_{\beta, i}^\gamma \right) &= (\alpha \leftrightarrow \beta) \\ \sum_{i \in I} \left(B_{\alpha, \beta}^i B_{i, \gamma}^\delta + B_{\alpha, \gamma}^i B_{\beta, i}^\delta + C_\alpha^{\delta, i} A_{i, \beta, \gamma} \right) &= (\alpha \leftrightarrow \beta) \end{aligned} \quad (2.3.38)$$

- *D-relation.*

$$\sum_{i \in I} B_{\alpha, \beta}^i D_i + \frac{1}{2} \sum_{j, k \in I} C_\alpha^{j, k} A_{\beta, j, k} = (\alpha \leftrightarrow \beta) \quad (2.3.39)$$

Moreover, the tensors (A, B, C, D) fix the Lie subalgebra structure constants as $f_{\alpha, \beta}^\gamma = B_{\alpha, \beta}^\gamma - B_{\beta, \alpha}^\gamma$.

The main feature of quantum Airy structures is the existence of a unique partition function annihilated by the operators L_α .

THEOREM 2.3.19 ([KS18]). *There exists a unique formal series*

$$Z(\hbar; \mathbf{x}) = \exp \left(\sum_{g \geq 0, n > 0} \frac{\hbar^{g-1}}{n!} \sum_{\alpha_1, \dots, \alpha_n \in I} F_{g; \alpha_1, \dots, \alpha_n} x^{\alpha_1} \dots x^{\alpha_n} \right) \quad (2.3.40)$$

such that the scalars $F_{g; \alpha_1, \dots, \alpha_n}$, called quantum Airy structure correlators, are symmetric under the permutation of the indices $\alpha_1, \dots, \alpha_n$, $F_{0; \alpha} = F_{0; \alpha, \beta} = 0$ and

$$L_\alpha \cdot Z(\hbar; \mathbf{x}) = 0 \quad \forall \alpha \in I. \quad (2.3.41)$$

Moreover, the scalars $F_{g; \alpha_1, \dots, \alpha_n}$ are uniquely determined by the following recursion on $2g - 2 + n > 0$

$$\begin{aligned} F_{g; \alpha_1, \dots, \alpha_n} &= \sum_{m=2}^n \sum_{i \in I} B_{\alpha_1, \alpha_m}^i F_{g; i, \alpha_2, \dots, \widehat{\alpha_m}, \dots, \alpha_n} \\ &+ \frac{1}{2} \sum_{j, k \in I} C_{\alpha_1}^{j, k} \left(F_{g-1; j, k, \alpha_2, \dots, \alpha_n} + \sum_{\substack{g_1 + g_2 = g \\ I_1 \sqcup I_2 = \{\alpha_2, \dots, \alpha_n\}}} F_{g_1; j, I_1} F_{g_2; k, I_2} \right), \end{aligned} \quad (2.3.42)$$

together with the initial conditions $F_{0; \alpha, \beta, \gamma} = A_{\alpha, \beta, \gamma}$ and $F_{1; \alpha} = D_\alpha$. In the following, we will refer to the above recursion formula as the Kontsevich–Soibelman topological recursion.

Remark 2.3.20. Let us briefly explain the motivation of Kontsevich and Soibelman. Their starting point is the notion of classical Airy structure, i.e. a Lagrangian defined by quadratic equations in the symplectic vector space T^*V . The initial datum is then a lift of the former to a Lie subalgebra of the Weyl algebra of V , which they call a quantum Airy structure. Thus, the corresponding partition function Z may be viewed as JWKB wave function of a quantum system whose symmetry is generated by Hamiltonians L_α .

EXAMPLE 2.3.21. The easiest example of a quantum Airy structure is that of $V = \mathbb{C}$, and $A = B = C = D = 1$. We have a unique differential operator

$$L = \hbar \partial_x - \frac{x^2}{2} - \hbar x \partial_x - \frac{\hbar^2}{2} \partial_x^2 - \hbar. \quad (2.3.43)$$

In particular, there is no relation to be checked. It is not difficult to show that the differential equation $L \cdot Z(\hbar; x) = 0$ is mapped to the Airy differential equation, after a suitable change of variable. Thus, the corresponding formal series Z coincides with the (properly normalised asymptotic expansion of the) Bairy function:

$$Z(\hbar; x) = \frac{1}{3^{1/6} \Gamma(2/3)} e^{\frac{2x-x^2}{2\hbar}} \text{Bi} \left(\frac{1-\hbar-2x}{(2\hbar)^{2/3}} \right). \quad (2.3.44)$$

This example motivates the name ‘‘Airy structures’’.

More generally, for a vector space V of dimension d , one can count the number of possible choices for the tensors (A, B, C, D) and the number of relations given by Lemma 2.3.18:

$$\begin{aligned} \#\text{choices} &= \frac{d(5d^2 + 3d + 4)}{3} = O(d^3), \\ \#\text{relations} &= \frac{d(d-1)(2d^2 + d + 1)}{2} = O(d^4). \end{aligned} \quad (2.3.45)$$

Notice that the number of relations grows faster than the one of possible choices, and already for $d \geq 3$ the system of relations is overdetermined. Thus, it is not clear whether quantum Airy structures can exist at all. However, quantum Airy structures do exist, many of which arise from vertex operator algebras and geometry [ABCO17; Bor+18; HR21; BKS20]. There are also many examples of infinite-dimensional quantum Airy structure, in particular there is one associated to any spectral curve (cf. Section 2.3.2). A similar statement holds for more general spectral curves, although there needs to be some assumptions on the behaviour at ramification points [BKS20].

Remark 2.3.22. Since the work of Kontsevich and Soibelman, quantum Airy structures has been generalised in various directions, notably by Borot–Bouchard–Chidambaram–Creutzig–Noshchenko [Bor+18] to allow for higher order differential operators, and by Bouchard–Ciosmak–Hadasz–Osuga–Ruba–Sułkowski [Bou+20] in the context of super vertex operator algebras.

A useful property of quantum Airy structures, also called *twisting*, is the analogue of the Givental action on CohFTs. It was studied for the first time in [ABCO17]. The version we present here can be found in [And+19].

PROPOSITION 2.3.23. *Let $(L_\alpha)_{\alpha \in I}$ be a quantum Airy structure on V , determined by the data (A, B, C, D) , and let $(u^{\alpha,\beta})_{\alpha,\beta \in I}$ be a collection of scalars satisfying $u^{\alpha,\beta} = u^{\beta,\alpha}$. Define the operator $U = \exp(\frac{\hbar}{2} \sum_{\alpha,\beta \in I} u^{\alpha,\beta} \partial_\alpha \partial_\beta)$ and the differential operators*

$$\tilde{L}_\alpha = UL_\alpha U^{-1}. \quad (2.3.46)$$

Then $(\tilde{L}_\alpha)_{\alpha \in I}$ form a quantum Airy structure on V , with twisted data $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ given by

$$\begin{aligned} \tilde{A}_{\alpha,\beta,\gamma} &= A_{\alpha,\beta,\gamma} \\ \tilde{B}_{\alpha,\beta}^\gamma &= B_{\alpha,\beta}^\gamma + \sum_{i \in I} A_{\alpha,\beta,i} u^{i,\gamma} \\ \tilde{C}_\alpha^{\beta,\gamma} &= C_\alpha^{\beta,\gamma} + \sum_{i \in I} \left(B_{\alpha,i}^\beta u^{i,\gamma} + B_{\alpha,i}^\gamma u^{i,\beta} \right) + \sum_{j,k \in I} A_{\alpha,j,k} u^{j,\beta} u^{k,\gamma} \\ \tilde{D}_\alpha &= D_\alpha + \frac{1}{2} \sum_{i,j \in I} A_{\alpha,i,j} u^{i,j}. \end{aligned} \quad (2.3.47)$$

Moreover, the partition function is given by $\tilde{Z} = U \cdot Z$ or, more explicitly, as the following sum over stable graphs:

$$\tilde{F}_{g;\alpha_1,\dots,\alpha_n} = \sum_{\Gamma \in \mathcal{G}_{g,n}} \frac{1}{|\text{Aut}(\Gamma)|} \sum_{\alpha: H_\Gamma \rightarrow I} \prod_{v \in V_\Gamma} F_{g(v);(\alpha_h)_{h \in H(v)}} \prod_{\substack{e \in E_\Gamma \\ e=(h,h')}} u^{\alpha_h, \alpha_{h'}}. \quad (2.3.48)$$

The sum over half-edges decorations is restricted to those which respect the boundary condition, i.e. if $\lambda \in \Lambda_\Gamma$ corresponds to the label $i \in \{1, \dots, n\}$, then $\alpha_\lambda = \alpha_i$. Again, $H(v)$ is the set of half-edges attached to the vertex v .

AIRY STRUCTURES FROM TOPOLOGICAL RECURSION

Consider a (local) spectral curve $\mathcal{S} = (C, x, y, \omega_{0,2})$. Here we use $\omega_{0,2}$ instead of B to avoid confusion with the quantum Airy structure initial data. Andersen–Borot–Chekhov–Orantin showed that the expansion coefficients $F_{g;\alpha_1,\dots,\alpha_n}$ of the topological recursion correlators on a certain basis of differentials coincide with the correlators of a quantum Airy structure associated to \mathcal{S} that we now describe.

Choose local coordinates ζ_a near a ramification point a such that $x = \frac{\zeta_a^2}{2} + x(a)$, and consider the meromorphic differential $\eta^{k,a}$, defined in a neighbourhood of a as

$$\eta^{k,a}(z_0) = \text{Res}_{z=a} \left(\int_a^z \omega_{0,2}(z_0, \cdot) \right) \frac{(2k+1)!!}{\zeta_a^{2k+2}(z)} d\zeta_a(z). \quad (2.3.49)$$

Define also the meromorphic functions and the inverse of a 1-form (both well-defined in a neighbourhood of a)

$$\xi_{k,a}^*(z) = \frac{\zeta_a^{2k+1}(z)}{(2k+1)!!}, \quad \theta_a(z) = \frac{-2}{(y(z) - y(\iota_a(z))) dx(z)}. \quad (2.3.50)$$

We can then define the tensors

$$\begin{aligned}
 A_{\alpha,\beta,\gamma} &= \operatorname{Res}_{z=a} \theta_a(z) \xi_\alpha^*(z) d\xi_\beta^*(z) d\xi_\gamma^*(z) \\
 B_{\alpha,\beta}^\gamma &= \operatorname{Res}_{z=a} \theta_a(z) \xi_\alpha^*(z) d\xi_\beta^*(z) \eta^\gamma(z) \\
 C_{\alpha}^{\beta,\gamma} &= \operatorname{Res}_{z=a} \theta_a(z) \xi_\alpha^*(z) \eta^\beta(z) \eta^\gamma(z) \\
 D_\alpha &= \delta_{k,0} \left(\frac{\theta_{-1,a}}{2} \phi_{(0,a),(0,a)} + \frac{\theta_{0,a}}{8} \right) + \delta_{k,1} \frac{\theta_{-1,a}}{24}
 \end{aligned} \quad \alpha = (k, a). \quad (2.3.51)$$

In the definition of D_α , the coefficients are given by the expansion of θ_a and $\omega_{0,2}$ as

$$\theta_a = \sum_{k \geq -1} \theta_{k,a} \zeta_a^{2k} (d\zeta_a)^{-1}, \quad (2.3.52)$$

$$\omega_{0,2} = \left(\frac{\delta_{a_1,a_2}}{(\zeta_{a_1} - \zeta_{a_2})^2} + \sum_{k_1, k_2 \geq 0} \phi_{(k_1,a_1),(k_2,a_2)} \zeta_{a_1}^{k_1} \zeta_{a_2}^{k_2} \right) d\zeta_{a_1} d\zeta_{a_2}. \quad (2.3.53)$$

PROPOSITION 2.3.24 ([[ABCO17](#), Proposition 9.2]). *The tensors (A, B, C, D) defined by Equation (2.3.51) form a quantum Airy structure on the free vector space spanned by $I = \mathbb{N} \times \mathfrak{a}$. Moreover, the topological recursion multidifferentials satisfy*

$$\omega_{g,n}(z_1, \dots, z_n) = \sum_{\alpha_1, \dots, \alpha_n \in \mathbb{N} \times \mathfrak{a}} F_{g;\alpha_1, \dots, \alpha_n} \prod_{i=1}^n \eta^{\alpha_i}(z_i), \quad (2.3.54)$$

and the sum is finite since $F_{g;\alpha_1, \dots, \alpha_n}$ vanishes for $k_1 + \dots + k_n > 3g - 3 + n$, with $\alpha_i = (k_i, a_i)$.

Remark 2.3.25. Notice here that the (local) basis of differentials $(\eta^\alpha)_{\alpha \in I}$ is different from the (global) basis $(d\xi^\alpha)_{\alpha \in I}$ defined in (2.3.16) and considered in the statement of the Eynard–DOSS correspondence Theorem 2.3.12. This is because here we are considering the more general case of *local* spectral curves, while in Theorem 2.3.12 only *compact* curves are considered. If we start from a compact curve, then the change of basis from (η^α) to $(d\xi^\alpha)$ is well-defined, and one can translate the quantum Airy structure accordingly.

For a fixed spectral curve, we can also explicitly find a correspondence between the shifting of the bidifferential $\omega_{0,2}$ in the Eynard–Orantin formalism (Theorem 2.3.11) and the twisting procedure of the associated quantum Airy structure (Proposition 2.3.23). Namely, consider two spectral curve \mathcal{S} and $\tilde{\mathcal{S}}$ differing in the choice of the symmetric bidifferential as

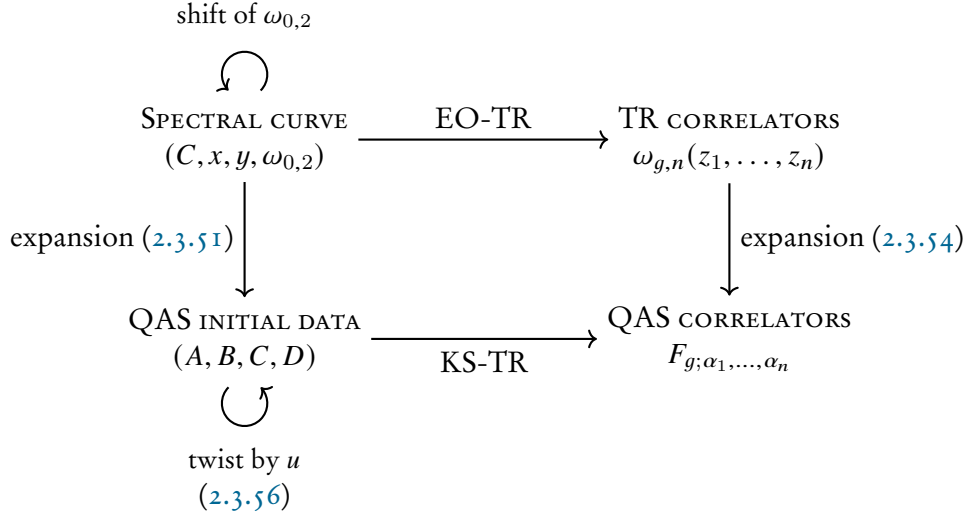
$$\begin{aligned}
 \tilde{\omega}_{0,2}(z_1, z_2) &= \omega_{0,2}(z_1, z_2) + U(z_1, z_2) \\
 &= \omega_{0,2}(z_1, z_2) + \sum_{k_1, k_2 \geq 0} U_{(k_1,a_1),(k_2,a_2)} \zeta_{a_1}^{k_1} \zeta_{a_2}^{k_2} d\zeta_{a_1} d\zeta_{a_2}
 \end{aligned} \quad (2.3.55)$$

for a symmetric bidifferential $U(z_1, z_2)$ that is holomorphic near the ramification points. Denote by (A, B, C, D) and $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ the quantum Airy structures associated to \mathcal{S} and $\tilde{\mathcal{S}}$ respectively. Define the symmetric matrix (u^{α_1, α_2}) as

$$u^{\alpha_1, \alpha_2} = (2k_1 - 1)!!(2k_2 - 1)!! U_{(2k_1, a_1), (2k_2, a_2)} \quad \alpha_i = (k_i, a_i). \quad (2.3.56)$$

Then the quantum Airy structure $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is obtained from (A, B, C, D) by the twisting procedure of Proposition 2.3.23.

Schematically, the relation between the Eynard–Orantin and Kontsevich–Soibelman formalisms can be pictured as follows:



Notice that the vertical arrows are not correspondences, as not all Airy structures come from spectral curves.

EXAMPLE 2.3.26. Applying the above theorem to the Airy curve of Example 2.3.10, that is

$$x(z) = \frac{z^2}{2}, \quad y(z) = -z, \quad \omega_{0,2}(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2},$$

we find the quantum Airy structure given by $A_{i,j,k} = \delta_{i,j,k}$, $D_i = \frac{\delta_{i,1}}{24}$, and

$$B_{i,j}^k = \delta_{i+j,k+1} \frac{(2k+1)!!}{(2i+1)!!(2j-1)!!}, \quad C_i^{j,k} = \delta_{i,j+k+2} \frac{(2j+1)!!(2k+1)!!}{(2i+1)!!}.$$

In particular, the recursive equation for the scalars $F_{g;k_1, \dots, k_n}$ coincide with the Witten–Kontsevich recursion for ψ -classes intersection numbers of Theorem 2.1.15: $F_{g;k_1, \dots, k_n} = \langle \tau_{k_1} \cdots \tau_{k_n} \rangle_g$. More generally, if the curve is compact and we perform the aforementioned change of bases, we obtain the following equality between quantum Airy structure and CohFT correlators:

$$F_{g;\alpha_1, \dots, \alpha_n} = \langle \tau_{\alpha_1} \cdots \tau_{\alpha_n} \rangle_g^\Omega.$$

Thus, the CohFT Ω satisfies the topological recursion formula of Equation (2.3.42). Moreover, the operators $(L_\alpha)_{\alpha \in \mathbb{N} \times \mathfrak{a}}$ annihilating the partition function are a generalisation of the celebrated *Virasoro constraints* for ψ -classes intersection numbers given by Equation (2.1.23).

2.4 — GEOMETRIC RECURSION AND TEICHMÜLLER THEORY

Geometric recursion (GR), introduced by Andersen–Borot–Orantin [ABO17], is a “geometrisation” or “categorification” of topological recursion. Concretely, it is a general formalism that constructs functorial assignments

$$\Sigma \mapsto \Omega_\Sigma \in E(\Sigma) \tag{2.4.1}$$

for some functors E from the category of bordered surfaces to a suitable target category of topological vector spaces. Geometric recursion proceeds by successive excisions of homotopy

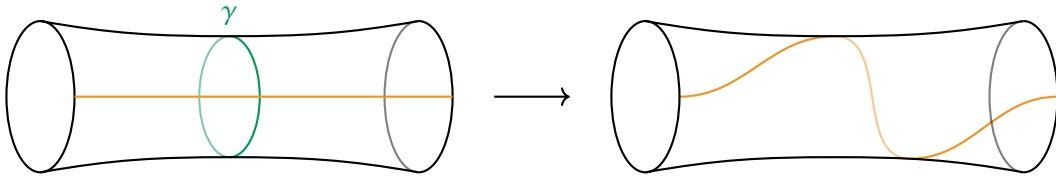


Figure 2.4: A Dehn twist along γ . The yellow curve is modified as shown.

classes of embedded pairs of pants on the surface Σ , and it produces mapping class group invariant vectors Ω_Σ starting from some initial data (that correspond to Σ being a pair of pants or a torus with one boundary) and some gluing data.

In the following, we will explain geometric recursion in the context of Teichmüller theory and Mirzakhani's recursion [Miro7a], which is the main source of inspiration for the theory itself, and refer to [ABO17] for the general formalism. In Parts II and III we will adapt such formalism to a more combinatorial setting.

2.4.1 — HYPERBOLIC GEOMETRY AND TEICHMÜLLER SPACES

The following exposition is based on [FM11; FLP12].

DEFINITION 2.4.1. A *bordered surface* Σ is a smooth, compact, connected, oriented surface with non-empty boundary and labelled boundary components $\partial_1\Sigma, \dots, \partial_n\Sigma$. We assume Σ to be stable, i.e. its Euler characteristic is negative. If Σ is of genus g , we call (g, n) the type of Σ . We use P (resp. T) to refer to bordered surfaces with the topology of a pair of pants (resp. of a torus with one boundary component).

Notice that we assume bordered surface to be connected. In case we consider disconnected bordered surfaces, we will assume each component to be stable, with non-empty boundary and with labelled boundary components.

DEFINITION 2.4.2. Denote by Mod_Σ the *mapping class group* of Σ :

$$\text{Mod}_\Sigma = \text{Diff}^+(\Sigma) / \text{Diff}_0^+(\Sigma), \quad (2.4.2)$$

where $\text{Diff}^+(\Sigma)$ is the group of orientation-preserving diffeomorphisms of the surface Σ and $\text{Diff}_0^+(\Sigma)$ denotes its subgroup consisting of those diffeomorphisms isotopic to the identity. The *pure mapping class group* Mod_Σ^0 is the subgroup of Mod_Σ consisting of mapping classes which preserve the labellings of boundary components.

A particular example of mapping class is the *Dehn twist* associated to a homotopy class of simple closed curve γ on Σ . To define it, fix a tubular neighbourhood of a representative of γ , together with a homeomorphism to the annulus $S^1 \times [0, 1]$. A twist on the annulus may be defined by sending $(z, t) \mapsto (\exp(2\pi it)z, t)$, which restricts to the identity on the boundary circles where $t = 0$ and $t = 1$. This defines a self-homeomorphism (mod boundary) on the annulus. The corresponding Dehn twist on Σ is obtained by extending the self-homeomorphism on the annulus to the identity outside the tubular neighbourhood, and taking its class modulo $\text{Diff}_0^+(\Sigma)$.

We can define now the Teichmüller space associated to a bordered surface Σ , which parametrises hyperbolic metrics on Σ up to isotopy.

DEFINITION 2.4.3. Let Σ be a bordered surface. An hyperbolic marking on Σ is a pair (X, f) where X is a hyperbolic surface with labelled geodesic boundaries and $f: \Sigma \rightarrow X$ is an orientation-preserving diffeomorphism respecting the labelling. Define the (hyperbolic) *Teichmüller space* as

$$\mathcal{T}_\Sigma = \left\{ (X, f) \mid (X, f) \text{ is a hyperbolic marking on } \Sigma \right\} / \sim \quad (2.4.3)$$

where $(X, f) \sim (X', f')$ if and only if there exists an isometry $\varphi: X \rightarrow X'$ respecting the labelling of the boundaries and such that $\varphi \circ f$ is isotopic to f' . We denote points in \mathcal{T}_Σ by $\sigma = [X, f]$, and we call them hyperbolic structures on Σ . By considering hyperbolic lengths of the labeled boundary components, we have a perimeter map $p: \mathcal{T}_\Sigma \rightarrow \mathbb{R}_+^n$ and we set $\mathcal{T}_\Sigma(L) = p^{-1}(L)$, i.e.

$$\mathcal{T}_\Sigma(L) = \left\{ (X, f) \mid \begin{array}{l} (X, f) \text{ is a hyperbolic marking on } \Sigma \\ \text{with labelled geodesics boundaries of lengths } L \end{array} \right\} / \sim \quad (2.4.4)$$

for $L \in \mathbb{R}_+^n$.

PROPOSITION 2.4.4. *Let Σ be a bordered surface of type (g, n) . The Teichmüller space $\mathcal{T}_\Sigma(L)$ is a real manifold of dimension $6g - 6 + 2n$.*

The pure mapping class group of Σ acts on the Teichmüller spaces $\mathcal{T}_\Sigma(L)$ properly discontinuously, and the quotient $\mathcal{M}_\Sigma(L)$ is called the *moduli space of hyperbolic structures* on Σ . It only depends on the type (g, n) of the surface and on the boundary constraint $L \in \mathbb{R}_+^n$, and it is naturally homeomorphic to

$$\mathcal{M}_{g,n}(L) = \left\{ X \mid \begin{array}{l} X \text{ is a hyperbolic surface of type } (g, n) \\ \text{with labelled geodesics boundaries of lengths } L \end{array} \right\} / \sim \quad (2.4.5)$$

where $X \sim X'$ if and only if there exists an isometry from X to X' preserving the labelling of the boundary components. A non-trivial result, which is a consequence of the Riemann uniformisation theorem, is that $\mathcal{M}_{g,n}(L)$ is homeomorphic to the moduli space of curves introduced in Section 2.1.

THEOREM 2.4.5. *The space $\mathcal{M}_{g,n}(L)$ is a smooth real orbifold of dimension $6g - 6 + 2n$. Moreover, for all $L \in \mathbb{R}_+^n$, it is homeomorphic (as a smooth real orbifold) to the moduli space of curves:*

$$\mathcal{M}_{g,n}(L) \cong \mathcal{M}_{g,n}. \quad (2.4.6)$$

CURVES, LENGTH AND TWIST

For a fixed bordered surface Σ , we denote by

- \mathcal{S}_Σ the set of homotopy classes of simple closed curves on Σ consisting of non boundary-parallel (also called *essential*) curves,
- \mathcal{M}_Σ the set of multicurves, i.e. homotopy classes of finite unions of pairwise disjoint essential simple closed curves on Σ ,
- \mathcal{M}'_Σ the set of primitive multicurves, i.e. those multicurves whose components are pairwise non-homotopic.

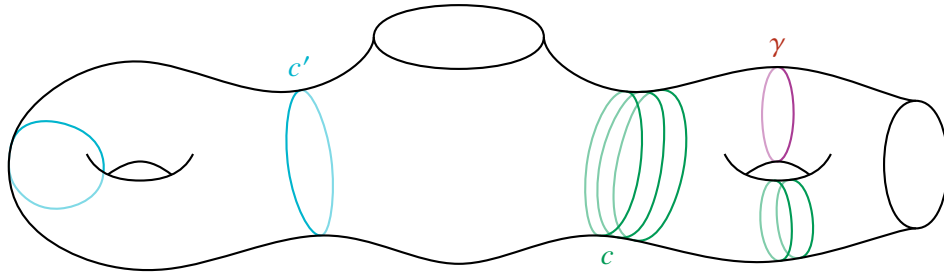


Figure 2.5: Examples of a simple closed curve γ (in purple), a multicurve c (in green), and a primitive multicurve c' (in blue).

From now on, curves are always considered up to homotopy. By convention \mathcal{M}_Σ and \mathcal{M}'_Σ contain the empty multicurve, whereas \mathcal{S}_Σ does not. See Figure 2.5 for some examples.

If Σ is a bordered surface and $\gamma \in \mathcal{S}_\Sigma$ (or more generally in \mathcal{M}'_Σ), we can consider the closed surface Σ_γ defined as the result of cutting Σ along a chosen representative of γ . The assumptions on γ imply that every connected component of Σ_γ is stable. Among such connected components, there is one containing $\partial_1 \Sigma$. We label the boundaries of such surface by putting the components of $\partial \Sigma$ first (in the order they appear in Σ), followed by those of γ (in some order). For the connected components of Σ_γ that do not contain $\partial_1 \Sigma$, we label the boundaries by putting the components of γ first (in some order), followed by those of $\partial \Sigma$ (in the order they appear in Σ). In the following we specify the choice of order only in case it has an actual relevance in the argument.

DEFINITION 2.4.6. Let Σ be a bordered surface. If $\sigma \in \mathcal{T}_\Sigma(L)$ is a hyperbolic structure on Σ and γ is a simple closed curve, there is a unique shortest geodesic in the homotopy class of γ for each hyperbolic metric representing σ , and we denote by $\ell_\sigma(\gamma)$ its *hyperbolic length* (which does not depend on the choice of representative). The length of a multicurve is by definition the sum of lengths of its components.

One of the main feature of simple closed curves is that they give a parametrisation of the entire Teichmüller space.

THEOREM 2.4.7 (See for instance [FLP12, Theorem 7.9]). *Let us equip $\mathbb{R}_+^{\mathcal{S}_\Sigma}$ with the product topology. The hyperbolic length of simple closed curves gives a map*

$$\ell_* : \mathcal{T}_\Sigma(L) \longrightarrow \mathbb{R}_+^{\mathcal{S}_\Sigma} \tag{2.4.7}$$

which is a homeomorphism onto its image.

In other words, knowing the length of all simple closed curves is enough to reconstruct the hyperbolic metric. However, something much stronger holds: if Σ is a bordered surface of type (g, n) , it suffices to know the hyperbolic length of only $9g - 9 + 3n$ curves to determine the corresponding point of $\mathcal{T}_\Sigma(L)$. In order to choose such curves, we need to fix a seamed pants decomposition of Σ .

DEFINITION 2.4.8. Given a bordered surface Σ of type (g, n) , a *seamed pants decomposition* consists of

- a pants decomposition $\mathcal{P} = (\gamma_1, \dots, \gamma_{3g-3+n})$, that is a maximal collection of pairwise non-homotopic, essential, simple closed curves, labelled by $1, \dots, 3g - 3 + n$, and that cut the surface into pairs of pants,

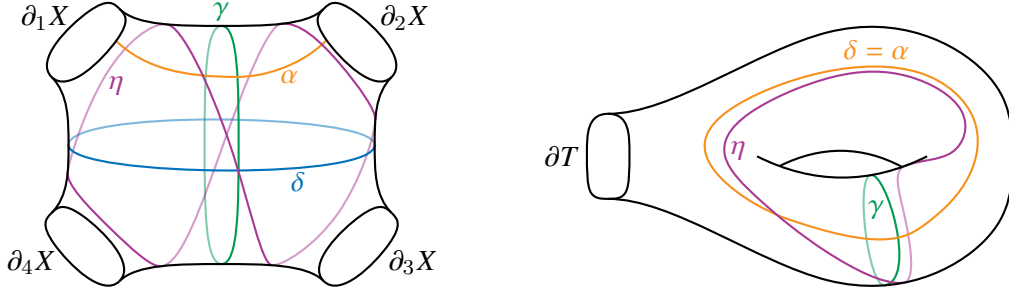


Figure 2.6: The curves δ and η for Σ_i of type $(0, 4)$ (on the left) and of type $(1, 1)$ (on the right). We omit the subscripts.

- a collection \mathcal{S} of non-homotopic, essential simple closed curves or simple arcs connecting boundary components of Σ , pairwise non-homotopic relative to the boundary, such that the intersection of \mathcal{S} with any of the pair of pants P in the decomposition specified by \mathcal{P} is a union of three disjoint arcs connecting the boundary components of P pairwise.

Given \mathcal{P} , we can construct an \mathcal{S} by first choosing three disjoint arcs on each pair of pants and then matching up endpoints in any fashion.

Let Σ be of type (g, n) and fix a seamed pants decomposition $(\mathcal{P}, \mathcal{S})$, with $\mathcal{P} = (\gamma_1, \dots, \gamma_{3g-3+n})$. The union of the pair of pants in the decomposition that are adjacent to γ_i is a surface Σ_i of type $(0, 4)$ or $(1, 1)$. We choose $\alpha_i \in \mathcal{S}$ crossing γ_i in Σ_i , with the condition that, if Σ_i is of type $(1, 1)$, then α_i does not intersect $\partial\Sigma_i$. We now define two other homotopy classes of curves on Σ_i (see Figure 2.6).

- If Σ_i has type $(0, 4)$, we let δ_i be the curve determined by a tubular neighbourhood of α_i union the boundary component it connects. If Σ_i has type $(1, 1)$, we let δ_i be the curve α_i .
- Let η_i be the image of δ_i after a Dehn twist along γ_i .

In the $(0, 4)$ case there are two possible choices of α_i as above, but both choices give the same (δ_i, η_i) .

THEOREM 2.4.9 ($(9g - 9 + 3n)$ -theorem, see for instance [FM11, Theorem 10.7]). *Let Σ be a bordered surface of type (g, n) and $(\mathcal{P}, \mathcal{S})$ a seamed pants decomposition. The map*

$$\mathcal{T}_\Sigma(L) \longrightarrow \mathbb{R}_+^{9g-9+3n}, \quad \sigma \longmapsto (\ell_\sigma(\gamma), \ell_\sigma(\delta), \ell_\sigma(\eta)) \quad (2.4.8)$$

is continuous and injective.

Although $\mathcal{T}_\Sigma(L)$ injects into $\mathbb{R}_+^{9g-9+3n}$, the above map is not surjective. In order to construct a homeomorphism, we use the idea that pairs of pants can be used as building blocks to create surfaces with negative Euler characteristic.

Fix a seamed pants decomposition $(\mathcal{P}, \mathcal{S})$ on Σ . We define the *length parameters* of a point $\sigma \in \mathcal{T}_\Sigma(L)$ to be the tuple of positive real numbers

$$\ell(\sigma) = (\ell_1(\sigma), \dots, \ell_{3g-3+n}(\sigma)), \quad (2.4.9)$$

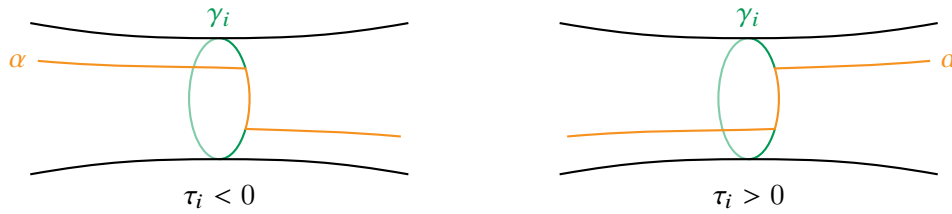


Figure 2.7: The sign convention for the twist parameter.

where $\ell_i(\sigma) = \ell_\sigma(\gamma_i)$. Notice that $\ell(\sigma)$ is sufficient to determine the hyperbolic structure on each pair of pants, as $\mathcal{T}_P \cong \mathbb{R}_+^{\partial P}$, where the isomorphism is given by the perimeter map.

To completely determine the hyperbolic metric, one needs to take into account how pairs of pants are glued together⁵. We call the *twist parameters* of a point $\sigma \in \mathcal{T}_\Sigma(L)$ the tuple of real numbers

$$\tau(\sigma) = (\tau_1(\sigma), \dots, \tau_{3g-3+n}(\sigma)), \quad (2.4.10)$$

defined as follows. To construct the twist parameter τ_i , take a curve $\alpha \in \mathcal{S}$ that meets γ_i . Homotopic to α , relative to the boundary of Σ , is a unique length-minimising piecewise geodesic curve which is entirely contained in the seams of the pairs of pants and the curves $\gamma_1, \dots, \gamma_{3g-3+n}$ (see [FLP12, Exposé 7] for a definition of pants' seams). The twist parameter τ_i is the signed distance that this curve travels along γ_i , according to the sign convention of Figure 2.7. It can be shown that the twist parameter is independent of the choice of curve $\alpha \in \mathcal{S}$.

Despite the fact that length and twist parameters, collectively known as *Fenchel–Nielsen coordinates*, depend on the choice of a seamed pants decomposition, we have the following result.

THEOREM 2.4.10 (Fenchel–Nielsen coordinates). *For seamed pants decomposition of Σ and any $L \in \mathbb{R}_+^n$, the map*

$$\mathcal{T}_\Sigma(L) \rightarrow (\mathbb{R}_+ \times \mathbb{R})^{3g-3+n}, \quad \sigma \mapsto (\ell(\sigma), \tau(\sigma)) \quad (2.4.11)$$

is a homeomorphism.

WEIL–PETERSSON GEOMETRY

By work of Goldman [Gol84], the space $\mathcal{T}_\Sigma(L)$ carries a natural symplectic form that is invariant under the action of the pure mapping class group, called the *Weil–Petersson symplectic form* and denoted by ω_{WP} . In particular, it descends to a symplectic form on the moduli space $\mathcal{M}_{g,n}(L)$, that will be denoted with the same symbol. In [Miro7b], expanding on the fundamental work of Wolpert [Wol85] for $n = 0$, Mirzakhani proves that the Weil–Petersson form extends as a closed form to the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$ and defines a cohomology class on $H^2(\overline{\mathcal{M}}_{g,n})$.

⁵Citing Thurston's book [Thu97], “to determine a point in Teichmüller space we need to consider how many times the leg of the pajama suit is twisted before it fits onto the baby's foot”.

THEOREM 2.4.11 ([Wol85; Miro7b]). *Under the homeomorphism $\mathcal{M}_{g,n}(L) \cong \mathcal{M}_{g,n}$, the Weil–Petersson form extends as a closed form to $\overline{\mathcal{M}}_{g,n}$ and defines the cohomology class*

$$2\pi^2\kappa_1 + \frac{1}{2} \sum_{i=1}^n L_i^2 \psi_i \in H^2(\overline{\mathcal{M}}_{g,n}). \quad (2.4.12)$$

In particular, the Weil–Petersson measure $d\mu_{\text{WP}} = \frac{\omega_{\text{WP}}^{3g-3+n}}{(3g-3+n)!}$ makes $\mathcal{M}_{g,n}(L)$ into a finite measure space. Its volume, called the *Weil–Petersson volume*, is then a homogeneous polynomial in $2\pi^2, L_1^2, \dots, L_n^2$ of degree $3g-3+n$ with rational coefficients storing intersection numbers on $\overline{\mathcal{M}}_{g,n}$:

$$V_{g,n}^{\text{WP}}(L) = \int_{\mathcal{M}_{g,n}(L)} d\mu_{\text{WP}} = \sum_{k_0, k_1, \dots, k_n \geq 0} \int_{\overline{\mathcal{M}}_{g,n}} \frac{(2\pi^2\kappa_1)^{k_0}}{k_0!} \prod_{i=1}^n \psi_i^{k_i} \frac{L_i^{2k_i}}{2^{k_i} k_i!}. \quad (2.4.13)$$

In particular, any statement about the volume $V_{g,n}^{\text{WP}}(L)$ yields a statement about the intersection theory of $\exp(2\pi^2\kappa_1)$ on $\overline{\mathcal{M}}_{g,n}$, and vice versa.

Another important feature of the Weil–Petersson form, proved by Wolpert [Wol85], is its expression in Fenchel–Nielsen coordinates: lengths and twists are Darboux coordinates for ω_{WP} . Geometrically, this implies that the Weil–Petersson symplectic structure is compatible with cutting and gluing along simple closed curves.

THEOREM 2.4.12 (Wolpert formula [Wol85]). *For a fixed seamed pants decomposition with associated Fenchel–Nielsen coordinates $(\ell_i, \tau_i)_{i=1, \dots, 3g-3+n}$, the Weil–Petersson symplectic form is given by*

$$\omega_{\text{WP}} = \sum_{i=1}^{3g-3+n} d\ell_i \wedge d\tau_i. \quad (2.4.14)$$

2.4.2 — MIRZAKHANI-TYPE FORMULAE

MIRZAKHANI’S RECURSION

One of the main obstacles in the computation of Weil–Petersson volumes is the fact that Fenchel–Nielsen coordinates do not behave well under the action of the mapping class group. In [Miro7a], Mirzakhani was able to work around this issue by unfolding the volume integral. Let us explain the argument for the simple case of $\mathcal{M}_{1,1}(L)$. Consider the space

$$\mathcal{M}_{1,1}^*(L) = \{ (X, \gamma) \mid X \in \mathcal{M}_{1,1}(L), \gamma \text{ a simple closed geodesic on } X \}, \quad (2.4.15)$$

together with the projection $\pi: \mathcal{M}_{1,1}^* \rightarrow \mathcal{M}_{1,1}$. Consider the map $\ell: \mathcal{M}_{1,1}^* \rightarrow \mathbb{R}_+$ defined by $\ell(X, \gamma) = \ell_X(\gamma)$. A simple argument shows that⁶

$$\mathcal{M}_{1,1}^*(L) = \left\{ (\ell, \tau) \in \mathbb{R}_+ \times \mathbb{R} \mid 0 \leq \ell \leq \frac{\tau}{2} \right\} / \sim, \quad (2.4.16)$$

where we have set $(\ell, 0) \sim (\ell, \frac{\tau}{2})$. In particular, thanks to Wolpert’s formula, the pullback of the Weil–Peterson measure reads $\pi^* d\mu_{\text{WP}} = d\ell d\tau$.

⁶Beware that Mirzakhani [Miro7a] considers a different orbifold structure on $\mathcal{M}_{1,1}(L)$, ignoring the elliptic involution. Hence, our result for the Weil–Petersson volume differs from hers by a factor of 2.

We can write now the constant function 1 (the integrand in the volume integral) using an identity by McShane⁷ [McS98]: for any $X \in \mathcal{M}_{1,1}(L)$

$$1 = \sum_{\pi(Y)=X} f(\ell(Y)), \quad f(\ell) = \frac{2}{L} \log \left(\frac{e^{\frac{\ell}{2}} + e^\ell}{e^{-\frac{\ell}{2}} + e^\ell} \right). \quad (2.4.17)$$

As a consequence, one can unfold the volume integral as

$$\begin{aligned} \int_{\mathcal{M}_{1,1}(L)} d\mu_{\text{WP}} &= \int_{\mathcal{M}_{1,1}(L)} \sum_{\pi(Y)=X} f(\ell(Y)) d\mu_{\text{WP}} \\ &= \int_{\mathcal{M}_{1,1}^*(L)} f(\ell(Y)) d\ell d\tau \\ &= \int_0^\infty \int_0^{\frac{\ell}{2}} f(\ell) d\ell d\tau \\ &= \frac{1}{2} \int_0^\infty f(\ell) \ell d\ell, \end{aligned} \quad (2.4.18)$$

from which one can easily get $V_{1,1}^{\text{WP}}(L) = \frac{L^2}{48} + \frac{\pi^2}{12}$.

In order to unfold the volume integral required to calculate $V_{g,n}^{\text{WP}}(L)$, Mirzakhani proved a more general version of McShane's identity.

THEOREM 2.4.13 (Mirzakhani identity [Miro7a]). *Fix a bordered surface Σ with $\chi_\Sigma < -1$. For every $L = (L_1, \dots, L_n) \in \mathbb{R}_+^n$ and $\sigma \in \mathcal{T}_\Sigma(L)$,*

$$1 = \sum_{m=2}^n \sum_{\gamma} B^M(L_1, L_m, \ell_\sigma(\gamma)) + \frac{1}{2} \sum_{\gamma, \gamma'} C^M(L_1, \ell_\sigma(\gamma), \ell_\sigma(\gamma')), \quad (2.4.19)$$

where the first summation is over simple closed geodesics γ which bound a pair of pants with $\partial_1 \Sigma$ and $\partial_m \Sigma$, while the second summation is over ordered pairs (γ, γ') of simple closed geodesics which bound a pair of pants with $\partial_1 \Sigma$. Moreover, the functions $B^M : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ and $C^M : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ are given by

$$\begin{aligned} B^M(L, L', \ell) &= 1 - \frac{1}{L} \log \left(\frac{\cosh(\frac{L'}{2}) + \cosh(\frac{L+\ell}{2})}{\cosh(\frac{L'}{2}) + \cosh(\frac{L-\ell}{2})} \right), \\ C^M(L, \ell, \ell') &= \frac{2}{L} \log \left(\frac{e^{\frac{\ell}{2}} + e^{\frac{\ell+\ell'}{2}}}{e^{-\frac{\ell}{2}} + e^{\frac{\ell+\ell'}{2}}} \right). \end{aligned} \quad (2.4.20)$$

The main idea behind the proof is to consider, for each point $p \in \partial_1 \Sigma$, the orthogeodesic α_p starting at p orthogonally to $\partial_1 \Sigma$. If we start at p and walk along α_p , then one of the following mutually excluding situations must arise (cf. Figure 2.8).

- A) The geodesic α_p never intersects itself or a boundary component.
- B_m) The geodesic α_p intersects $\partial_m \Sigma$ for some $m \in \{2, \dots, n\}$, without intersecting itself.
- c) The geodesic α_p intersects $\partial_1 \Sigma$ or it intersects itself.

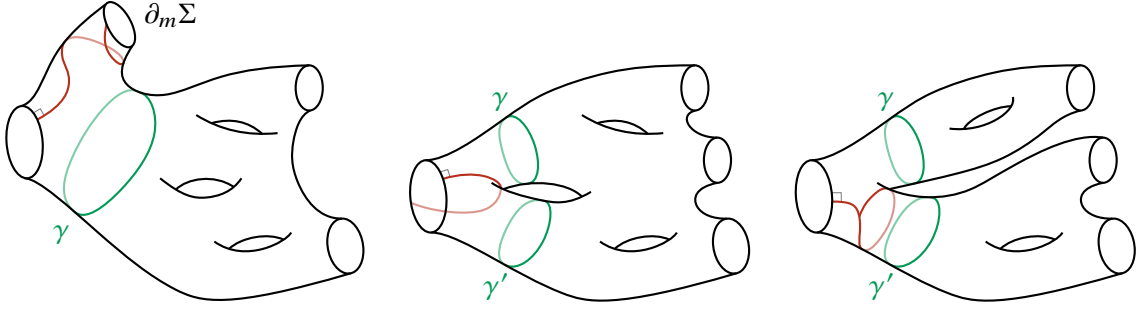


Figure 2.8: The orthogeodesic α_p (in red) and some of its possible behaviour, together with the simple closed curve(s) it determines (in green). On the left, the arc α_p intersect the boundary component $\partial_m \Sigma$ (B_m -type), and it determines a single simple closed curve γ . In the two other cases, α_p intersect $\partial_1 \Sigma$ and itself respectively (c-type), determining two simple closed curves (γ, γ') .

Denote by $\partial_{1,\bullet} \Sigma$ the subset of $\partial_1 \Sigma$ for which the condition $\bullet \in \{A, B_2, \dots, B_n, C\}$ occurs. By a result of Birman–Series [BS85], $\partial_{1,A} \Sigma$ is a measure zero subset with respect to the curvilinear measure μ_σ induced by σ on $\partial_1 \Sigma$. On the other hand, we can construct maps

$$\begin{aligned}
 b_m : \partial_{1,B_m} \Sigma &\longrightarrow \mathcal{B}_{\Sigma,m} = \left\{ \begin{array}{l} \text{homotopy classes of embedded pair of pants} \\ \text{with labelled boundary components } (\partial_1 \Sigma, \partial_m \Sigma, \gamma) \\ \text{for a simple closed geodesic } \gamma \end{array} \right\} \\
 c : \partial_{1,C} \Sigma &\longrightarrow \mathcal{C}_\Sigma = \left\{ \begin{array}{l} \text{homotopy classes of embedded pair of pants} \\ \text{with labelled boundary components } (\partial_1 \Sigma, \gamma, \gamma') \\ \text{for two simple closed geodesics } \gamma \text{ and } \gamma' \end{array} \right\}
 \end{aligned} \tag{2.4.21}$$

as follows (see Figure 2.8 again). Consider the union of $\partial_1 \Sigma$, the orthogeodesic α_p from p to the intersection point, and (in the B_n case) $\partial_m \Sigma$. A sufficiently small neighbourhood of this embedded graph is topologically a pair of pants. By taking geodesic representatives in the homotopy classes of the boundary components, we obtain an embedded hyperbolic pair of pants P , whose geodesic boundary is $(\partial_1 \Sigma, \partial_m \Sigma, \gamma)$ in the B_m case, and $(\partial_1 \Sigma, \gamma, \gamma')$ in the c case. In the latter situation, we can label the boundary components of P by saying that γ is the one on the right-hand side of α_p , while γ' is one on the left-hand side of α_p . As a consequence of the above discussion, we find that

$$L_1 = \mu_\sigma(\partial_1 \Sigma) = \sum_{m=2}^n \sum_{P \in \mathcal{B}_{\Sigma,m}} \mu_\sigma(b_m^{-1}(P)) + \frac{1}{2} \sum_{P \in \mathcal{C}_\Sigma} \mu_\sigma(c^{-1}(P)). \tag{2.4.22}$$

The factor $1/2$ comes from the symmetric role of the curves γ and γ' .

Mirzakhani’s identity follows now from the following lemma, together with the identification of $\mathcal{B}_{\Sigma,m}$ with the set of simple geodesics γ bounding a pair of pants with pair of pants with $\partial_1 \Sigma$ and $\partial_m \Sigma$, and the identification of \mathcal{C}_Σ with the set of ordered pairs (γ, γ') of simple closed geodesics bounding a pair of pants with $\partial_1 \Sigma$.

LEMMA 2.4.14. *The functions defined in Equation (2.4.20) have the following geometric meaning.*

⁷More precisely, McShane discovered such identity in the case $L = 0$, i.e. on the moduli space of complete hyperbolic structures on the once-punctured torus. The identity for general $L \in \mathbb{R}_+$ is due to Mirzakhani.

- If $P \in \mathcal{B}_{\Sigma, m}$ bounds a simple closed curve γ with $\partial_1 \Sigma$ and $\partial_m \Sigma$, then

$$\mu_\sigma(b_m^{-1}(P)) = L_1 \cdot B^M(L_1, L_m, \ell_\sigma(\gamma)). \quad (2.4.23)$$

- If $P \in \mathcal{C}_\Sigma$ bounds two simple closed curves γ and γ' with $\partial_1 \Sigma$, then

$$\mu_\sigma(c^{-1}(P)) = L_1 \cdot C^M(L_1, \ell_\sigma(\gamma), \ell_\sigma(\gamma')). \quad (2.4.24)$$

Remark 2.4.15. Notice that that above computation is “local” in the sense that it depends only on the geometry of P and not on the geometry of the entire surface. Hence, it can be done via hyperbolic trigonometry. Moreover, the function B^M can be alternatively interpreted as follows. Fix an embedded pair of pants $P \in \mathcal{B}_{\Sigma, m}$, and take a point $p \in \partial_1 \Sigma$ at random uniformly with respect to the hyperbolic line element. The probability that the orthogeodesic α_p defines an embedded pair of pants given by P is equal to $B^M(L_1, L_m, \ell_\sigma(\gamma))$. Similarly for $P \in \mathcal{C}_\Sigma$. Mirzakhani’s identity simply states that such events are disjoint and exhaustive.

One can now apply the unfolding argument presented before to Mirzakhani’s identity, obtaining a recursion formula for the Weil–Petersson volumes.

THEOREM 2.4.16 (Mirzakhani’s recursion [Miro7a]). *The Weil–Petersson volumes are uniquely determined by the following recursion on $2g - 2 + n > 1$*

$$\begin{aligned} V_{g,n}^{\text{WP}}(L_1, \dots, L_n) &= \\ &= \sum_{m=2}^n \int_{\mathbb{R}_+} B^M(L_1, L_m, \ell) V_{g,n-1}^{\text{WP}}(\ell, L_2, \dots, \widehat{L_m}, \dots, L_n) \ell d\ell \\ &\quad + \frac{1}{2} \int_{\mathbb{R}_+^2} C^M(L_1, \ell, \ell') \left(V_{g-1, n+2}^{\text{WP}}(\ell, \ell', L_2, \dots, L_n) \right. \\ &\quad \left. + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2, \dots, n\}}} V_{g_1, 1+|I_1|}^{\text{WP}}(\ell, L_{I_1}) V_{g_2, 1+|I_2|}^{\text{WP}}(\ell', L_{I_2}) \right) \ell \ell' d\ell d\ell' \end{aligned} \quad (2.4.25)$$

with the conventions $V_{0,1}^{\text{WP}} = V_{0,2}^{\text{WP}} = 0$, and the base cases

$$V_{0,3}^{\text{WP}}(L_1, L_2, L_3) = 1 \quad \text{and} \quad V_{1,1}^{\text{WP}}(L) = \frac{L^2}{48} + \frac{\pi^2}{12}. \quad (2.4.26)$$

One can immediately recognise in Mirzakhani’s recursion the same structure of the topological recursion formula, introduced in Section 2.3. This is not a coincidence, as shown by Eynard–Orantin and briefly explained in Example 2.3.10.

PROPOSITION 2.4.17 ([EO07b]). *Consider the spectral curve given by $C = \mathbb{P}^1$ and*

$$x(z) = \frac{z^2}{2}, \quad y(z) = -\frac{\sin(2\pi z)}{2\pi}, \quad B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}. \quad (2.4.27)$$

The associated multidifferentials are the Laplace transform of the Weil–Petersson volumes:

$$\omega_{g,n}(z_1, \dots, z_n) = d_1 \cdots d_n \int_{\mathbb{R}_+^n} V_{g,n}^{\text{WP}}(L_1, \dots, L_n) \prod_{i=1}^n e^{-z_i L_i} L_i dL_i. \quad (2.4.28)$$

(g, n)	$V_{g,n}^{\text{WP}}(L)$
(0, 3)	1
(0, 4)	$\frac{1}{2}m_{(1)} + 2\pi^2$
(0, 5)	$\frac{1}{8}m_{(2)} + \frac{1}{2}m_{(1^2)} + 3\pi^2m_{(1)} + 10\pi^4$
(0, 6)	$\frac{1}{48}m_{(3)} + \frac{3}{16}m_{(2,1)} + \frac{3}{4}m_{(1^3)} + \frac{3\pi^2}{2}m_{(2)} + 6\pi^2m_{(1^2)} + 26\pi^4m_{(1)} + \frac{244\pi^6}{3}$
(0, 7)	$\frac{1}{384}m_{(4)} + \frac{1}{24}m_{(3,1)} + \frac{3}{32}m_{(2^2)} + \frac{3}{8}m_{(2,1^2)} + \frac{3}{2}m_{(1^4)} + \frac{5\pi^2}{12}m_{(3)} + \frac{15\pi^2}{12}m_{(2,1)}$ $+ 15\pi^2m_{(1^3)} + 20\pi^4m_{(2)} + 80\pi^4m_{(1^2)} + \frac{910\pi^6}{3}m_{(1)} + \frac{2758\pi^8}{3}$
(1, 1)	$\frac{1}{48}m_{(1)} + \frac{\pi^2}{12}$
(1, 2)	$\frac{1}{192}m_{(2)} + \frac{1}{96}m_{(1^2)} + \frac{\pi^2}{12}m_{(1)} + \frac{\pi^4}{4}$
(1, 3)	$\frac{1}{1152}m_{(3)} + \frac{1}{192}m_{(2,1)} + \frac{1}{96}m_{(1^3)} + \frac{\pi^2}{24}m_{(2)} + \frac{\pi^2}{8}m_{(1^2)} + \frac{13\pi^4}{24}m_{(1)} + \frac{14\pi^6}{9}$
(1, 4)	$\frac{1}{9216}m_{(4)} + \frac{1}{768}m_{(3,1)} + \frac{1}{384}m_{(2^2)} + \frac{1}{128}m_{(2,1^2)} + \frac{1}{64}m_{(1^4)} + \frac{7\pi^2}{576}m_{(3)}$ $+ \frac{\pi^2}{12}m_{(2,1)} + \frac{\pi^2}{4}m_{(1^3)} + \frac{41\pi^4}{96}m_{(2)} + \frac{17\pi^4}{12}m_{(1^2)} + \frac{187\pi^6}{36}m_{(1)} + \frac{529\pi^8}{36}$
(2, 1)	$\frac{1}{442368}m_{(4)} + \frac{29\pi^2}{138240}m_{(3)} + \frac{139\pi^4}{23040}m_{(2)} + \frac{169\pi^6}{2880}m_{(1)} + \frac{29\pi^8}{192}$
(2, 2)	$\frac{1}{4423680}m_{(5)} + \frac{1}{294912}m_{(4,1)} + \frac{29}{2211840}m_{(3,2)} + \frac{11\pi^2}{276480}m_{(4)} + \frac{29\pi^2}{69120}m_{(3,1)} + \frac{7\pi^2}{7680}m_{(2^2)}$ $+ \frac{19\pi^4}{7680}m_{(3)} + \frac{181\pi^4}{11520}m_{(2,1)} + \frac{551\pi^6}{8640}m_{(2)} + \frac{7\pi^6}{36}m_{(1^2)} + \frac{1085\pi^8}{1728}m_{(1)} + \frac{787\pi^{10}}{480}$
(3, 1)	$\frac{1}{53508833280}m_{(7)} + \frac{77\pi^2}{9555148800}m_{(6)} + \frac{3781\pi^4}{2786918400}m_{(5)} + \frac{47209\pi^6}{418037760}m_{(4)} + \frac{127189\pi^8}{26127360}m_{(3)}$ $+ \frac{8983379\pi^{10}}{87091200}m_{(2)} + \frac{8497697\pi^{12}}{9331200}m_{(1)} + \frac{9292841\pi^{14}}{4082400}$

Table 2.3: A list of Weil–Petersson polynomials $V_{g,n}^{\text{WP}}(L)$ for low values of $2g - 2 + n$ computed via the topological recursion formula Equation (2.4.25). Here m_λ is the monomial symmetric polynomial associated to the partition λ , evaluated at L_1^2, \dots, L_n^2 .

An immediate consequence of Mirzakhani’s result is a recursion for the following intersection numbers

$$[\tau_{k_1} \cdots \tau_{k_n}]_g = \int_{\mathcal{M}_{g,n}} \kappa_1^{3g-3+n-|k|} \prod_{i=1}^n \psi_i^{k_i}, \quad |k| = \sum_{i=1}^n k_i. \quad (2.4.29)$$

More precisely, one can compute the quantum Airy structure associated to the above spectral curve, and the recursion for the quantum Airy structure correlators $F_{g;k_1, \dots, k_n}$ can be easily recast into a recursion for $[\tau_{k_1} \cdots \tau_{k_n}]_g$. Moreover, taking the top coefficients of the Weil–Petersson polynomials in L_1^2, \dots, L_n^2 leads to a recursion for the ψ -classes intersection numbers $\langle \tau_{k_1} \cdots \tau_{k_n} \rangle_g$, namely the Witten–Kontsevich recursion equation (2.1.19). In Chapter 6 we will present a geometric interpretation of such fact.

GEOMETRIC RECURSION AND MULTICURVE COUNT

In [ABO17], Andersen–Borot–Orantin generalised Mirzakhani’s argument, defining a general framework to recursively construct mapping class group invariant functions by excision of embedded pairs of pants. We begin by reviewing such framework, called *geometric recursion*, in a simplified form.

Let us introduce the category \mathbf{B}_1 as follows.

- Objects: bordered surfaces, possibly disconnected. By convention, we include the empty surface in \mathbf{B}_1 .

- Morphisms: isotopy classes of diffeomorphisms relatively to the boundary which preserve the first boundary component but are allowed to permute the labellings of the other boundary components. If the surface is disconnected, the morphism preserves the first boundary component of each connected component of the surface.

Geometric recursion starts with a functor E from \mathbf{B}_1 to the category $\mathbf{TVect}_{\mathbb{R}}$ of topological vector spaces and aims at constructing E -valued functorial assignments

$$\Sigma \longmapsto \Omega_{\Sigma} \in E(\Sigma). \quad (2.4.30)$$

For this purpose, E must come with extra functorial data that satisfy a number of axioms, subsumed in the notion of *target theory*. Instead of repeating the fully general definition of target theories and associated geometric recursion [ABO17], we shall describe in concrete terms the geometric recursion for the functor E used in the present section, namely the spaces $\text{Mes}(\mathcal{T}_{\Sigma})$ of \mathbb{C} -valued measurable functions on the Teichmüller space of Σ (cf. [ABO17, Sections 7–10]). A similar example will be developed in Part II. Most results of this section still hold true after replacing “measurable” by “continuous”.

DEFINITION 2.4.18. Geometric recursion *initial data* consist of a quadruple (A, B, C, D) where

- A, B, C are measurable functions on $\mathcal{T}_P \cong \mathbb{R}_+^3$,
- D is an assignment $T \mapsto D_T \in \text{Mes}(\mathcal{T}_T)$, for each T torus with one boundary component,

satisfying the following axioms.

- $A(L_1, L_2, L_3) = A(L_1, L_3, L_2)$ and $C(L_1, L_2, L_3) = C(L_1, L_3, L_2)$.
- The assignment $T \mapsto D_T$ is functorial, and in particular D_T is a mapping class group invariant function. We also denote by D the induced function on $\mathcal{M}_{1,1}(L)$.

DEFINITION 2.4.19. We recursively construct an assignment $\Sigma \mapsto \Omega_{\Sigma} \in \text{Mes}(\mathcal{T}_{\Sigma})$ as follows. We let

$$\Omega_{\emptyset} = 1, \quad \Omega_P(\sigma) = A(\vec{\ell}_{\sigma}(\partial P)), \quad \Omega_T = D_T, \quad (2.4.31)$$

where $\vec{\ell}_{\sigma}(\partial P)$ is the ordered triple of hyperbolic lengths of the boundary components of P . For disconnected surfaces we set

$$\Omega_{\Sigma_1 \sqcup \dots \sqcup \Sigma_k}(\sigma_1, \dots, \sigma_k) = \prod_{i=1}^k \Omega_{\Sigma_i}(\sigma_i). \quad (2.4.32)$$

For connected surfaces with Euler characteristic $\chi_{\Sigma} < -1$, we define Ω_{Σ} inductively on χ_{Σ} by *geometric recursion*:

$$\Omega_{\Sigma}(\sigma) = \sum_{m=2}^n \sum_{P \in \mathcal{B}_{\Sigma, m}} B(\vec{\ell}_{\sigma}(\partial P)) \Omega_{\Sigma-P}(\sigma|_{\Sigma-P}) + \frac{1}{2} \sum_{P \in \mathcal{C}_{\Sigma}} C(\vec{\ell}_{\sigma}(\partial P)) \Omega_{\Sigma-P}(\sigma|_{\Sigma-P}). \quad (2.4.33)$$

Here $\mathcal{B}_{\Sigma, m}$ and \mathcal{C}_{Σ} are the sets of homotopy classes of embedded pairs of pants bounding $\partial_1 \Sigma$ introduced in Equation (2.4.21) and appearing in Mirzakhani’s recursion. We choose as representative of P the embedding as a pair of pants with geodesic boundaries, so that the restriction of σ to $\Sigma - P$ makes it a hyperbolic surface with geodesic boundaries. Moreover, to define the labelling of the boundary components of $\Sigma - P$, we say that the (labelled) boundary components of P that appear in $\Sigma - P$ are put first, followed by the (labelled) boundary components of Σ that appear in $\Sigma - P$.

The sum (2.4.33) has countably many terms and therefore its convergence should be discussed. Denote by $\mathcal{T}_\Sigma^{(\epsilon)} \subset \mathcal{T}_\Sigma$ the ϵ -thick part of the Teichmüller space, i.e. the set of $\sigma \in \mathcal{T}_\Sigma$ such that $\ell_\sigma(\gamma) \geq \epsilon$ for any simple closed curve γ (including boundary components).

DEFINITION 2.4.20. We say that the initial data (A, B, C, D) are *admissible* if for any $\epsilon > 0$ there exists $t \geq 0$ such that, for all $s \geq 0$, there exists $M_{\epsilon,s} > 0$ for which

$$\begin{aligned}
 \sup_{L_1, L_2, L_3 \geq \epsilon} \frac{|A(L_1, L_2, L_3)|}{((1+L_1)(1+L_2)(1+L_3))^t} &\leq M_{\epsilon,0}, \\
 \sup_{L, L', \ell \geq \epsilon} \frac{|B(L, L', \ell)| (1 + [\ell - L - L']_+)^s}{((1+L)(1+L'))^t} &\leq M_{\epsilon,s}, \\
 \sup_{L, \ell, \ell' \geq \epsilon} \frac{|C(L, \ell, \ell')| (1 + [\ell + \ell' - L]_+)^s}{(1+L)^t} &\leq M_{\epsilon,s}, \\
 \sup_{\sigma \in \mathcal{T}_\Sigma^{(\epsilon)}} \frac{|D_T(\sigma)|}{(1 + \ell_\sigma(\partial T))^t} &\leq M_{\epsilon,0}.
 \end{aligned} \tag{2.4.34}$$

Here $[x]_+ = \max\{x, 0\}$.

At first sight, the above bounds might seem unreasonable, but they come from the proof's strategy of [ABO17] to estimate geometric recursion amplitudes. They implement a decay faster than inverse power laws in ℓ and ℓ' (playing the role of the cutting curves) which only take place for ℓ much bigger than the boundary lengths L . Moreover, they implement at most polynomial growth in the boundary lengths L . Then the basic idea is to split the geometric recursion sum into contributions from “small pairs of pants”, which are in a finite number, and big pairs of pants, which grow polynomially. The above bounds fit together with the geometric estimates for the number of pairs of pants, and are proved to be sufficient for the absolute convergence of $\Omega_\Sigma(\sigma)$. We will implement the same idea in Theorem 5.1.4 to prove absolute convergence of geometric recursion amplitudes in the combinatorial setting.

THEOREM 2.4.21 ([ABO17, Corollary 8.3]). *If (A, B, C, D) are admissible initial data, then for any bordered surface Σ :*

- *the series (2.4.33) converges absolutely and uniformly on any compact of \mathcal{T}_Σ ,*
- *$\Sigma \mapsto \Omega_\Sigma \in \text{Mes}(\mathcal{T}_\Sigma)$ is a well-defined functorial assignment (in particular, Ω_Σ is a pure mapping class group invariant measurable function),*
- *there exists $u \geq 0$, depending only on the topological type of Σ , such that for any $\epsilon > 0$ we have*

$$\sup_{\sigma \in \mathcal{T}_\Sigma^{(\epsilon)}} |\Omega_\Sigma(\sigma)| \leq K_\epsilon \prod_{b \in \pi_0(\partial\Sigma)} (1 + \ell_\sigma(b))^u \tag{2.4.35}$$

for some constant K_ϵ depending only on ϵ and the topological type of Σ .

The above theorem assures that admissible initial data produce pure mapping class group invariant measurable functions that descends to the moduli space $\mathcal{M}_{g,n}(L)$, denoted $\Omega_{g,n}$. Such moduli space is naturally endowed with the Weil–Peterson measure μ_{WP} , and it is natural to ask whether functions produced by geometric recursion are integrable. In fact, it is important to note that μ_{WP} is compatible with cutting along simple closed curves, as expressed by Wolpert’s formula. As a consequence of this fact, integration of functions obtained by

geometric recursion against μ_{WP} over the moduli space with fixed boundary lengths produces functions on \mathbb{R}_+^n that also satisfy a recursion on the Euler characteristic of the same nature. In order to guarantee integrability we are going to introduce stronger assumptions on the initial data. We are also going to denote by

$$\langle f \rangle(L) = \int_{\mathcal{M}_{g,n}(L)} f d\mu_{\text{WP}} \quad (2.4.36)$$

the average over the moduli space of any integrable function f on $(\mathcal{M}_{g,n}(L), \mu_{\text{WP}})$.

DEFINITION 2.4.22. The initial data (A, B, C, D) are called *strongly admissible* if there exists $\eta \in [0, 2)$ and $t \geq 0$ such that, for any $s \geq 0$, there exists $M_s > 0$ such that for any $L_1, L_2, L_3, \ell, \ell' > 0$

$$\begin{aligned} |A(L_1, L_2, L_3)| &\leq M_0 \left((1+L_1)(1+L_2)(1+L_3) \right)^t, \\ |B(L, L', \ell)| &\leq \frac{M_s \left((1+L)(1+L') \right)^t}{\ell^\eta (1 + [\ell - (L+L')]_+)^s}, \\ |C(L, \ell, \ell')| &\leq \frac{M_s (1+L)^t}{(\ell\ell')^\eta (1 + [\ell + \ell' - L]_+)^s}, \end{aligned} \quad (2.4.37)$$

and D is integrable on $\mathcal{M}_{1,1}(L_1)$ and satisfies

$$|\langle D \rangle(L)| = \left| \int_{\mathcal{M}_{1,1}(L)} D d\mu_{\text{WP}} \right| \leq M_0 (1+L)^t. \quad (2.4.38)$$

THEOREM 2.4.23 (TR from GR [ABO17, Theorem 8.8]). *Let (A, B, C, D) be strongly admissible initial data and Ω_Σ be the resulting functions. Then, $\Omega_{g,n}$ is integrable against μ_{WP} on $\mathcal{M}_{g,n}(L)$ for any $L \in \mathbb{R}_+^n$, and the integrals satisfy the following recursion on $2g - 2 + n > 1$*

$$\begin{aligned} \langle \Omega_{g,n} \rangle(L_1, \dots, L_n) &= \\ &= \sum_{m=2}^n \int_{\mathbb{R}_+} B(L_1, L_m, \ell) \langle \Omega_{g,n-1} \rangle(\ell, L_2, \dots, \widehat{L}_m, \dots, L_n) \ell d\ell \\ &\quad + \frac{1}{2} \int_{\mathbb{R}_+^2} C(L_1, \ell, \ell') \left(\langle \Omega_{g-1,n+2} \rangle(\ell, \ell', L_2, \dots, L_n) \right. \\ &\quad \left. + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2, \dots, n\}}} \langle \Omega_{g_1, 1+|I_1|} \rangle(\ell, L_{I_1}) \langle \Omega_{g_2, 1+|I_2|} \rangle(\ell', L_{I_2}) \right) \ell \ell' d\ell d\ell' \end{aligned} \quad (2.4.39)$$

with the conventions $\langle \Omega_{0,1} \rangle = \langle \Omega_{0,2} \rangle = 0$, and the base cases

$$\langle \Omega_{0,3} \rangle(L_1, L_2, L_3) = A(L_1, L_2, L_3) \quad \text{and} \quad \langle \Omega_{1,1} \rangle(L) = \langle D \rangle(L). \quad (2.4.40)$$

In many applications, it is natural to induce the initial datum D from the function C . Geometrically, it means that the one-holed torus amplitude can be inferred by self-gluing a pair of pants.

LEMMA 2.4.24. *If we are only given (A, B, C) satisfying the conditions in Definition 2.4.20, the series*

$$D_T(\sigma) = \sum_{\gamma \in \delta_T} C(\ell_\sigma(\partial T), \ell_\sigma(\gamma), \ell_\sigma(\gamma)) \quad (2.4.41)$$

converges absolutely on any compact of \mathcal{T}_T to a Mod_T -invariant function, and (A, B, C, D) are admissible initial data. Furthermore, if (A, B, C) satisfy the conditions in Definition 2.4.22, where in the bound for C one assumes $0 \leq \eta < 1$, then (A, B, C, D) are strongly admissible and

$$\langle D \rangle(L) = \frac{1}{2} \int_{\mathbb{R}_+} C(L, \ell, \ell) \ell d\ell. \quad (2.4.42)$$

In the last statement, the stronger condition $\eta < 1$ for C (instead of $\eta < 2$) guarantees that $\ell \mapsto \ell C(L, \ell, \ell)$ is integrable near 0.

We can now reformulate Mirzakhani's identity by saying that the constant function 1 can be obtained from geometric recursion.

THEOREM 2.4.25 (Mirzakhani identity revisited). *The following initial data*

$$\begin{aligned} A^M(L_1, L_2, L_3) &= 1 \\ B^M(L, L', \ell) &= 1 - \frac{1}{L} \log \left(\frac{\cosh(\frac{L'}{2}) + \cosh(\frac{L+\ell}{2})}{\cosh(\frac{L'}{2}) + \cosh(\frac{L-\ell}{2})} \right), \\ C^M(L, \ell, \ell') &= \frac{2}{L} \log \left(\frac{e^{\frac{L}{2}} + e^{\frac{\ell+\ell'}{2}}}{e^{-\frac{L}{2}} + e^{\frac{\ell+\ell'}{2}}} \right), \end{aligned} \quad (2.4.43)$$

are strongly admissible, and lead by geometric recursion to $\Omega_\Sigma(\sigma) = 1$ for any Σ and $\sigma \in \mathcal{T}_\Sigma$. In other words, for any bordered surface Σ with $\chi_\Sigma < -1$ and any $\sigma \in \mathcal{T}_\Sigma$, we have

$$1 = \sum_{m=2}^n \sum_{P \in \mathcal{B}_{\Sigma, m}} B^M(\vec{\ell}_\sigma(\partial P)) + \frac{1}{2} \sum_{P \in \mathcal{C}_\Sigma} C^M(\vec{\ell}_\sigma(\partial P)). \quad (2.4.44)$$

Moreover, for a torus T with one boundary component and any $\sigma \in \mathcal{T}_T$, we have

$$1 = \sum_{\gamma \in \delta_T} C^M(\ell_\sigma(\partial T), \ell_\sigma(\gamma), \ell_\sigma(\gamma)). \quad (2.4.45)$$

A generalised Mirzakhani identity was obtained by Andersen–Borot–Orantin, which allows the computation of statistics of lengths of primitive multicurves via geometric recursion.

THEOREM 2.4.26 (Hyperbolic length statistics of multicurves [ABO17, Theorem 10.1]). *Let (A, B, C, D) be admissible initial data and denote by Ω_Σ the associated geometric recursion amplitudes. Let $f: \mathbb{R}_+ \rightarrow \mathbb{C}$ be a measurable function such that for any $\epsilon > 0$ and $s \geq 0$, there exists $M_{s, \epsilon}$ such that*

$$\sup_{\ell \geq \epsilon} |f(\ell)| \ell^s \leq M_{s, \epsilon}. \quad (2.4.46)$$

Then, the following initial data twisted by f are admissible:

$$\begin{aligned} A[f](L_1, L_2, L_3) &= A(L_1, L_2, L_3), \\ B[f](L, L', \ell) &= B(L, L', \ell) + A(L, L', \ell) f(\ell), \\ C[f](L, \ell, \ell') &= C(L, \ell, \ell') + B(L, \ell, \ell') f(\ell) + B(L, \ell', \ell) f(\ell') + A(L, \ell, \ell') f(\ell) f(\ell'), \\ D_T[f](\sigma) &= D_T(\sigma) + \sum_{\gamma \in \delta_T} A(\ell_\sigma(\partial T), \ell_\sigma(\gamma), \ell_\sigma(\gamma)) f(\ell_\sigma(\gamma)). \end{aligned} \quad (2.4.47)$$

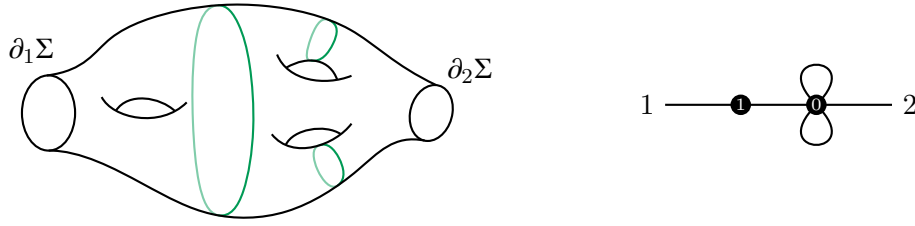


Figure 2.9: A primitive multicurve and the associated stable graph.

Denote by $\Omega_\Sigma[f]$ the corresponding geometric recursion amplitudes. If for all Σ , Ω_Σ is invariant under all braidings of boundary components of Σ , then $\Omega_\Sigma[f]$ is given by the length statistics of primitive multicurves \mathcal{M}'_Σ weighted by f :

$$\Omega_\Sigma[f](\sigma) = \sum_{c \in \mathcal{M}'_\Sigma} \Omega_{\Sigma_c}(\sigma|_{\Sigma_c}) \prod_{\gamma \in \pi_0(c)} f(\ell_\sigma(\gamma)). \quad (2.4.48)$$

Here Σ_c is the bordered surface obtained by cutting Σ along c (the choice of the first boundary component is irrelevant due to the assumed invariance).

Remark 2.4.27. A useful version of the above result can be stated for length statistics of multicurves (not only primitive ones). Consider $F: \mathbb{R}_+ \rightarrow \mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ a measurable function with values in the unit disk, such that

$$f(x) = \sum_{k \geq 1} F(x)^k = \frac{F(x)}{1 - F(x)} \quad (2.4.49)$$

satisfies the conditions of Theorem 2.4.26. Then the geometric recursion amplitudes $\Omega_\Sigma[f]$ associated to the initial data (2.4.47) are given by the length statistics of multicurves \mathcal{M}_Σ weighted by F :

$$\Omega_\Sigma[f](\sigma) = \sum_{c \in \mathcal{M}_\Sigma} \Omega_{\Sigma_c}(\sigma|_{\Sigma_c}) \prod_{\gamma \in \pi_0(c)} F(\ell_\sigma(\gamma)). \quad (2.4.50)$$

We can now integrate the amplitudes $\Omega_\Sigma[f]$ over the moduli space with respect to μ_{WP} , and the result can be calculated in two ways: by the topological recursion, and by direct integration. To express the latter, we notice that the quotient of the set of primitive multicurve by the pure mapping class group is in natural bijection with the set of stable graphs (cf. Definition 2.1.7):

$$\mathcal{G}_{g,n} = \mathcal{M}'_\Sigma / \text{Mod}_\Sigma^g, \quad (2.4.51)$$

where Σ is a bordered surface of type (g, n) . More precisely, a stable graph Γ encodes the class of a primitive multicurve c in the following way: vertices $v \in V_\Gamma$ correspond to connected components of Σ_c and the genera of the components is recorded in a function $g: V_\Gamma \rightarrow \mathbb{Z}_{\geq 0}$ which is part of the data of Γ ; edges $e \in E_\Gamma$ correspond to the components of c ; leaves $\lambda \in \Lambda_\Gamma$ correspond to boundary components of the surface. See Figure 2.9 for an example. Finally, the automorphism group of Γ is identified with that of the multicurve c .

COROLLARY 2.4.28. *Assume that (A, B, C, D) are strongly admissible initial data with Ω_Σ invariant under all braidings of boundary components of Σ and f is a measurable function for which there exists $\eta \in [0, 2)$ such that $\sup_{\ell > 0} |f(\ell)| \ell^\eta < +\infty$. Then $(A[f], B[f], C[f], D[f])$*

are strongly admissible and $\langle \Omega_{g,n}[f] \rangle$ satisfy the topological recursion (2.4.39) with these initial data. Besides, they are given by the following sum over stable graphs:

$$\begin{aligned} \langle \Omega_{g,n}[f] \rangle(L_1, \dots, L_n) &= \\ &= \sum_{\Gamma \in \mathcal{G}_{g,n}} \frac{1}{|\text{Aut}(\Gamma)|} \int_{\mathbb{R}_+^{E_\Gamma}} \prod_{v \in V_\Gamma} \langle \Omega_{g(v),n(v)} \rangle((\ell_e)_{e \in E(v)}, (L_\lambda)_{\lambda \in \Lambda(v)}) \prod_{e \in E_\Gamma} f(\ell_e) \ell_e d\ell_e. \end{aligned} \quad (2.4.52)$$

In Part III we will exploit the above results to compute the multicurve counting function, as well as its asymptotics, studied by Mirzakhani in [Miro8b].

RELATION TO QUANTUM AIRY STRUCTURE AND EYNARD–ORANTIN FORMALISM

Let $\phi: \mathbb{R}_+ \rightarrow \mathbb{R}$ and $\psi: \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$ be measurable functions. The operators

$$\hat{B}[\phi](L, L') = \int_{\mathbb{R}_+} B(L, L', \ell) \phi(\ell) \ell d\ell, \quad \hat{C}[\psi](L) = \int_{\mathbb{R}_+^2} C(L, \ell, \ell') \psi(\ell, \ell') \ell \ell' d\ell d\ell' \quad (2.4.53)$$

play an essential role in the topological recursion of Theorem 2.4.23 obtained from the integration of geometric recursion amplitudes. It turns out that for most of the natural initial data, the associated operators \hat{B} and \hat{C} preserve, respectively, the space of polynomials in one and two variables that are even (we call them even polynomials). Since in most examples the base cases $(g, n) = (0, 3)$ and $(1, 1)$ are even polynomials in the length variables, it implies that all topological recursion amplitudes are even polynomials.

DEFINITION 2.4.29. We say that initial data (A, B, C, D) are *polynomial* if (B, C) are such that the operators \hat{B} and \hat{C} preserve the spaces of even polynomials and A and $\langle D \rangle$ are themselves even polynomials.

For polynomial initial data, it is sometimes more efficient for computations to decompose $\langle \Omega_{g,n} \rangle$ on a basis of monomials and write the action of \hat{B} and \hat{C} on these monomials. For instance, let $e^k(L) = \frac{L^{2k}}{(2k+1)!}$ and decompose the initial data $A, \langle D \rangle$ and operators \hat{B}, \hat{C} as

$$\begin{aligned} A(L_1, L_2, L_3) &= \sum_{i,j,k \geq 0} A_{i,j,k} e^i(L_1) e^j(L_2) e^k(L_3), \\ \hat{B}[e^k](L, L') &= \sum_{i,j \geq 0} B_{i,j}^k e^i(L) e^j(L'), \\ \hat{C}[e^j \otimes e^k](L) &= \sum_{i \geq 0} C_i^{j,k} e^i(L), \\ \langle D \rangle(L) &= \sum_{k \geq 0} D_k e^k(L). \end{aligned} \quad (2.4.54)$$

Then, writing the topological recursion formula for $\langle \Omega_{g,n} \rangle$ gives the Kontsevich–Soibelman recursion for its expansion coefficients.

PROPOSITION 2.4.30. *The tensors (A, B, C, D) defined by Equation (2.4.54) form a quantum Airy structure on the free vector space spanned by $I = \mathbb{N}$. Moreover, the expansion coefficients of*

the topological recursion amplitudes $\langle \Omega_{g,n} \rangle$ in the basis $(e^k)_{k \geq 0}$ are the quantum Airy structure correlators associated to (A, B, C, D) :

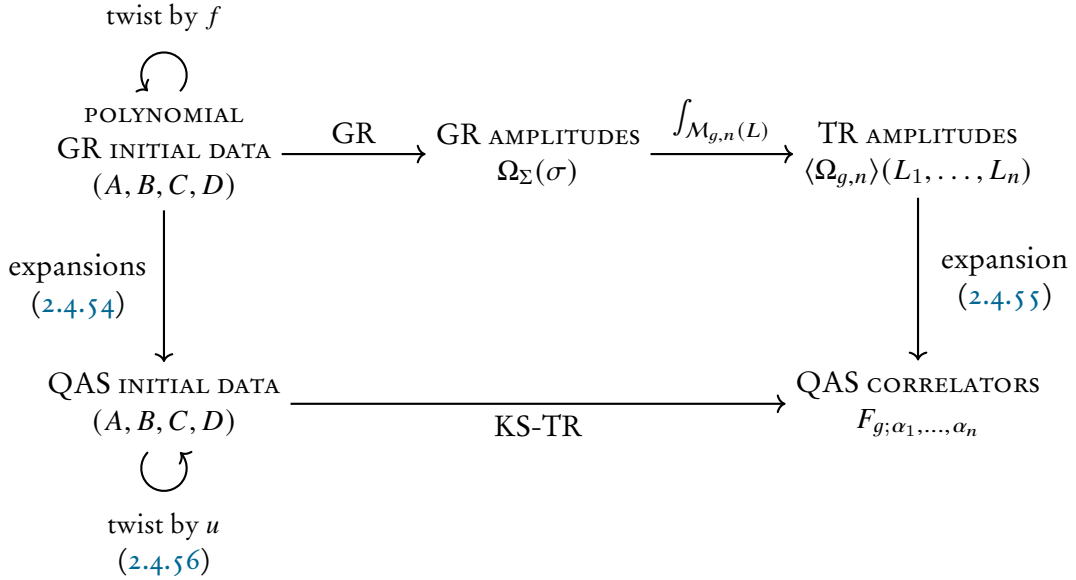
$$\langle \Omega_{g,n} \rangle(L_1, \dots, L_n) = \sum_{k_1, \dots, k_n \geq 0} F_{g;k_1, \dots, k_n} \prod_{i=1}^n e^{k_i}(L_i), \quad (2.4.55)$$

We can also give a correspondence between the twisting operations geometric recursion initial data (Theorem 2.4.26 and Corollary 2.4.28) and quantum Airy strictures (Proposition 2.3.23). Indeed, the twisting of geometric recursion initial data preserves polynomiality, as the decaying condition $\sup_{\ell > 0} |f(\ell)| \ell^\eta < +\infty$ guarantees that all moments of the test function f exist. Setting

$$u^{a,b} = \int_{\mathbb{R}_+} \frac{\ell^{2a+2b+1}}{(2a+1)!(2b+1)!} f(\ell) d\ell, \quad (2.4.56)$$

we obtain that the quantum Airy structure corresponding to the twisted geometric recursion initial data $(A[f], B[f], C[f], D[f])$ coincide with the quantum Airy structure (A, B, C, D) twisted by $(u^{a,b})$.

Schematically, the relation between geometric and topological recursion and quantum Airy structures can be pictured as follows.



Remark 2.4.31. Sometimes it is useful to expand the polynomials $\langle \Omega_{g,n} \rangle$ on a different basis. For instance, expanding the Weil–Petersson polynomials in the basis $e^k(L) = \frac{L^{2k}}{(2k+1)!}$ we obtain the coefficients

$$F_{g;k_1, \dots, k_n} = \int_{\mathcal{M}_{g,n}} e^{2\pi^2 \kappa_1} \prod_{i=1}^n (2k_i + 1)!! \psi_i^{k_i}, \quad (2.4.57)$$

while expanding in the basis $\hat{e}^k(L) = \frac{L^{2k}}{(2k)!!}$ gives

$$\hat{F}_{g;k_1, \dots, k_n} = \int_{\mathcal{M}_{g,n}} e^{2\pi^2 \kappa_1} \prod_{i=1}^n \psi_i^{k_i}. \quad (2.4.58)$$

2.5 — FERMION FORMALISM AND INTEGRABLE HIERARCHIES

In this section we introduce the fermionic formalism, also known as semi-infinite wedge formalism, as well as its connection with integrable hierarchies of Kadomtsev–Petviashvili (KP) type. We refer the reader to [MJD00] for a complete exposition. Nowadays, it is a standard tool in Hurwitz theory (see for instance [OP06]) and it can be generalised in different directions, as we will see in Part IV. The notation here is taken from [MJD00], and the reader should be aware that different sources in the literature have different sign conventions.

2.5.1 — FERMIONS AND THEIR FOCK REPRESENTATION

DEFINITION 2.5.1. Let $(V, (\cdot, \cdot))$ be a (possibly infinite-dimensional) inner-product space. The Clifford algebra $\mathcal{Cl}(V, (\cdot, \cdot))$ is the free algebra on V (including the empty word, which is the unit of the free algebra) modulo the relations

$$\{v, w\} = 2(v, w) \quad \text{for all } v, w \in V. \quad (2.5.1)$$

There is a canonical embedding $V \hookrightarrow \mathcal{Cl}(V, (\cdot, \cdot))$, and we identify V with its image.

EXAMPLE 2.5.2. For V^d a d -dimensional \mathbb{C} -vector space with basis (ξ_1, \dots, ξ_d) and Euclidean inner product $(\xi_i, \xi_j) = \delta_{i,j}$, the associated Clifford algebra is denoted by $\mathcal{Cl}_d = \mathcal{Cl}(V^d, (\cdot, \cdot))$.

DEFINITION 2.5.3. Let \mathcal{V} be the infinite-dimensional complex vector space with basis ψ_s and ψ_s^\dagger , for $s \in \mathbb{Z}' = \mathbb{Z} + \frac{1}{2}$ half-integers. Consider the bilinear form

$$(\psi_r, \psi_s^\dagger) = \frac{\delta_{r+s}}{2}, \quad (\psi_r, \psi_s) = (\psi_r^\dagger, \psi_s^\dagger) = 0, \quad (2.5.2)$$

and define the space of (charged) fermions as $\mathcal{A} = \mathcal{Cl}(\mathcal{V}, (\cdot, \cdot))$. It has canonical anticommutation relations (CAR) given by

$$\{\psi_r, \psi_s^\dagger\} = \delta_{r+s}, \quad \{\psi_r, \psi_s\} = \{\psi_r^\dagger, \psi_s^\dagger\} = 0. \quad (2.5.3)$$

We define the charge C and energy E of a fermion through the following table

Fermions	ψ_s	ψ_s^\dagger
Charge (C)	1	-1
Energy (E)	-s	-s

and extend it to monomials in ψ_s and ψ_s^\dagger as the sum of the charges and energies of their factors.

DEFINITION 2.5.4. Consider the subspace \mathcal{L} of \mathcal{V} generated by ψ_s and ψ_{-s}^\dagger for $s < 0$ (the positive energy fermions), which is maximal isotropic for $(\mathcal{V}, (\cdot, \cdot))$. Define the (fermionic) Fock space as the unique highest-weight left module of \mathcal{A} :

$$\mathfrak{F} = \mathcal{A}/(\mathcal{A} \cdot \mathcal{L}) \quad (2.5.4)$$

We write $|0\rangle$ for the class of 1, also called the vacuum.

With the dual construction, i.e. considering the unique highest-weight right module, we define the dual Fock space \mathfrak{F}^* and the covacuum $\langle 0|$. In particular, we have a pairing $\mathfrak{F}^* \times \mathfrak{F} \rightarrow \mathbb{C}$ denoted by

$$\langle \omega | \eta \rangle = (\langle \omega |, | \eta \rangle). \quad (2.5.5)$$

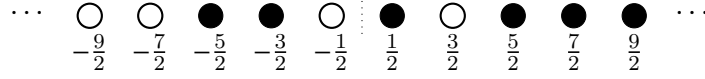


Figure 2.10: The Maya diagram corresponding to $|\mathbf{t}\rangle = |-\frac{5}{2}, -\frac{3}{2}, \frac{1}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \dots\rangle$.

Moreover, for any $O \in \mathcal{A}$ we can define its *vacuum expectation value* $\langle O \rangle$ as $\langle 0|O|0\rangle$. Since the (right) action of \mathcal{A} on the dual Fock space is the adjoint of the (left) action on the Fock space, there is no ambiguity in the notation.

The vacuum expectation value of a quadratic expression in the fermions is given by

$$\langle \psi_r \psi_s \rangle = \langle \psi_r^\dagger \psi_s^\dagger \rangle = 0, \quad \langle \psi_r \psi_s^\dagger \rangle = \delta_{r+s} \delta_{r>0}. \quad (2.5.6)$$

Remark 2.5.5. A basis for \mathfrak{F} is given by the vectors

$$\psi_{r_1} \cdots \psi_{r_m} \psi_{s_1}^\dagger \cdots \psi_{s_n}^\dagger |0\rangle \quad (2.5.7)$$

for $r_1 < \cdots < r_m < 0$ and $s_1 < \cdots < s_n < 0$. In other words, negatively charged fermions with decreasing positive energy acting on the vacuum, followed by positively charged fermions with decreasing positive energy.

DEFINITION 2.5.6. Define the *charge* and *energy* on the Fock space by setting

$$\begin{aligned} C(|0\rangle) &= 0, & E(|0\rangle) &= 0, \\ C(A|0\rangle) &= C(A), & E(A|0\rangle) &= E(A), \end{aligned} \quad (2.5.8)$$

for any $A|0\rangle \neq 0$ in the form of Equation (2.5.7). Define \mathfrak{F}_ℓ^d to be the subspace spanned by vectors of charge ℓ and energy d :

$$\mathfrak{F}_\ell^d = \text{span} \left\{ \psi_{r_1} \cdots \psi_{r_m} \psi_{s_1}^\dagger \cdots \psi_{s_n}^\dagger |0\rangle \left| \begin{array}{l} r_1 < \cdots < r_m < 0 \text{ and } s_1 < \cdots < s_n < 0, \\ m - n = \ell, \sum_i r_i + \sum_j s_j = d \end{array} \right. \right\}, \quad (2.5.9)$$

and define the charge ℓ subspace as $\mathfrak{F}_\ell = \bigoplus_{d \in \mathbb{Z}} \mathfrak{F}_\ell^d$.

The Fock space can alternatively realised as the space spanned by Maya diagrams, or as the space spanned by semi-infinite wedges.

MAYA DIAGRAMS AND SEMI-INFINITE WEDGES

Consider a *Maya diagram*, that is a diagram made up of white and black stones, lined up along the real line and with positions indexed by half-integers $s \in \mathbb{Z}'$. We require that far away on the right (i.e. $s \gg 0$) all the stones are black, and far away on the left (i.e. $s \ll 0$) all the stones are white (see Figure 2.10). A Maya diagram is determined by a sequence of half-integers $\mathbf{t} = (t_i)_{i>0}$ in increasing order $t_1 < t_2 < \cdots$, corresponding to the position of the black stones. In particular, we have $t_{i+1} = t_i + 1$ for $i \gg 0$. For any sequence satisfying such property, we denote by $|\mathbf{t}\rangle$ the associated Maya diagram.

DEFINITION 2.5.7. Define the *space of Maya diagrams* as the vector space spanned by Maya diagrams. We have a left action of the space of fermions, defined by

$$\psi_s |\mathbf{t}\rangle = \begin{cases} (-1)^{i-1} |\dots, t_{i-1}, t_{i+1}, \dots\rangle & \text{if } t_i = -s \text{ for some } i > 0, \\ 0 & \text{otherwise,} \end{cases} \quad (2.5.10)$$

$$\psi_s^\dagger |\mathbf{t}\rangle = \begin{cases} (-1)^i |\dots, t_i, s, t_{i+1}, \dots\rangle & \text{if } t_i < s < t_i + 1 \text{ for some } i > 0, \\ 0 & \text{otherwise.} \end{cases} \quad (2.5.11)$$

We can then interpret ψ_s as creating a white stone at $-s$ (or equivalently, annihilating a black stone there), while ψ_s^\dagger creates a black stone at s .

With this picture, it is clear that the space of Maya diagrams corresponds to the Fock space of Definition 2.5.4. Moreover, the vacuum state corresponds to the Maya diagram $|\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\rangle$, that is the Maya diagrams with only black stones on the right ($s > 0$) and white stones on the left ($s < 0$).

Remark 2.5.8. The above description can also be interpreted in terms of elementary particle physics, thinking of black stones as (charged) fermionic particles and white stones as their antiparticles. Thus, the vacuum corresponds to the Dirac sea and ψ_s, ψ_s^\dagger are fermionic creation operators for $s < 0$ and annihilation operators for $s > 0$. Moreover, the charge of a Maya diagram is equal to the number of particles minus the number of antiparticles.

With this notation, we can define a basis of the charge zero Fock space: for any partition $\mu = (\mu_1, \dots, \mu_n) \vdash d$, define the vector

$$|\mu\rangle = |\mu_1 + \frac{1}{2}, \mu_2 + \frac{3}{2}, \dots\rangle, \quad (2.5.12)$$

and we have the decomposition $\mathfrak{F}_0 = \bigoplus_{d \in \mathbb{N}} \bigoplus_{\mu \vdash d} \mathbb{C} |\mu\rangle$. Notice that the vacuum corresponds to the empty partition.

It will also be useful to define a distinguished vector for non-zero charges: the charge ℓ vacuum $|\ell\rangle$ is defined as

$$|\ell\rangle = |\frac{1}{2} - \ell, \frac{3}{2} - \ell, \dots\rangle \in \mathfrak{F}_\ell. \quad (2.5.13)$$

A second representation of the Fock space is through semi-infinite wedges.

DEFINITION 2.5.9. Let V be the vector space spanned by the elements $(e_s)_{s \in \mathbb{Z}'}$. Define the *semi-infinite wedge space* as the vector space spanned by semi-infinite wedges on V , that is wedge products of the form

$$e_{t_1} \wedge e_{t_2} \wedge \dots, \quad (2.5.14)$$

for any increasing sequence $\mathbf{t} = (t_i)_{i>0}$ of half-integers satisfying $t_{i+1} = t_i + 1$ for $i \gg 0$. We have a left action of the space of fermions, defined through exterior and interior products:

$$\psi_s = e_{-s} \wedge, \quad \psi_s^\dagger = e_s^* \lrcorner. \quad (2.5.15)$$

Here e_s^* is the vector dual to e_s .

With this picture, it is clear that the space of semi-infinite wedges corresponds to the Fock space of Definition 2.5.4. Moreover, the vacuum state corresponds to the semi-infinite wedge $e_{1/2} \wedge e_{3/2} \wedge e_{5/2} \wedge \dots$.

SATO GRASSMANNIAN AND PLÜCKER RELATIONS

DEFINITION 2.5.10. Define the bi-infinite *general linear algebra* $\mathfrak{gl}(\infty)$ as the vector space of band matrices $(a_{r,s})_{r,s \in \mathbb{Z}'}$, that is, $a_{r,s}$ is non-zero only for finitely many possible values of $r - s$, together with the commutator bracket.

Most of the computation in $\mathfrak{gl}(\infty)$ can be expressed in terms of the standard “basis” given by the elements $\{E_{r,s} \mid r, s \in \mathbb{Z}'\}$ such that $(E_{r,s})_{u,v} = \delta_{r,u}\delta_{s,v}$ and their commutation relation is given by

$$[E_{r,s}, E_{u,v}] = \delta_{s,u}E_{r,v} - \delta_{r,v}E_{u,s}. \quad (2.5.16)$$

More precisely, every element of $\mathfrak{gl}(\infty)$ can be expressed as $\sum_{r,s} a_{r,s}E_{r,s}$ with $a_{r,s}$ non-zero only for finitely many possible values of $r - s$.

PROPOSITION 2.5.11. *There is a representation of the central extension $\widehat{\mathfrak{gl}}(\infty) = \mathfrak{gl}(\infty) \oplus \mathbb{C}$ to the space of fermions \mathcal{A} , defined on the central factor by $1 \mapsto 1$ and on $\mathfrak{gl}(\infty)$ by*

$$E_{r,s} \mapsto \hat{E}_{r,s} = :\psi_{-r}\psi_s^\dagger:, \quad (2.5.17)$$

where $:\psi_u\psi_v^\dagger: = \psi_u\psi_v^\dagger - \langle \psi_u\psi_v^\dagger \rangle$ is the normal ordered product. Moreover, the commutation relation between basis elements given by

$$[\hat{E}_{r,s}, \hat{E}_{u,v}] = \delta_{s,u}\hat{E}_{r,v} - \delta_{r,v}\hat{E}_{u,s} + \delta_{s,u}\delta_{r,v}(\delta_{s>0} - \delta_{v>0}). \quad (2.5.18)$$

EXAMPLE 2.5.12. Examples of elements in $\widehat{\mathfrak{gl}}(\infty)$ are given as follows.

- For any $m \in \mathbb{Z}_+$, we have the diagonal operators

$$\mathcal{F}_m = \sum_{s \in \mathbb{Z}'} s^m \hat{E}_{s,s}. \quad (2.5.19)$$

Notice that \mathcal{F}_0 acts as multiplication by the charge, and thus we call it the charge operator. Likewise, \mathcal{F}_1 acts as multiplication by the energy, and thus we call it the energy operator. For $m > 1$, the operators \mathcal{F}_m are also called (*fermionic*) *completed cut-and-join operators*⁸.

- For any $n \in \mathbb{Z}$, we have the elements

$$J_n = \sum_{s \in \mathbb{Z}'} \hat{E}_{s-n,s}, \quad (2.5.20)$$

also called *currents*. They form a Lie subalgebra of $\widehat{\mathfrak{gl}}(\infty)$, called the *Heisenberg algebra*, and they satisfy the *canonical commutation relation* (CCR):

$$[J_m, J_n] = m\delta_{m+n}. \quad (2.5.21)$$

We also define the generating series $J(t) = \sum_{n>0} t_n J_n$, that will play an important role in the boson-fermion correspondence of Section 2.5.1. In the literature, they are also denoted with the symbols α_n or H_n , and called “bosons” or “Hamiltonians”.

⁸In the literature, there is a different normalisation which defines \mathcal{F}_m as $\frac{1}{m!} \sum_{s \in \mathbb{Z}'} s^m \hat{E}_{s,s}$.

- A better way to pack both completed cut-and-join and current operators is through the *Okounkov–Pandharipande operators* [OP06, Section 2]: for any $n \in \mathbb{Z}$, set

$$\mathcal{E}_n(z) = \sum_{s \in \mathbb{Z}'} e^{z(s-\frac{n}{2})} \hat{E}_{s-n,s} + \frac{\delta_n}{\varsigma(z)}, \quad \varsigma(z) = 2 \sinh\left(\frac{z}{2}\right). \quad (2.5.22)$$

They satisfy the commutation relations

$$[\mathcal{E}_m(z), \mathcal{E}_n(w)] = \varsigma(\det \begin{bmatrix} m & z \\ n & w \end{bmatrix}) \mathcal{E}_{m+n}(z+w). \quad (2.5.23)$$

Notice that $\mathcal{E}_n(0) = J_n$ for $n \neq 0$. In particular, taking the limit $z, w \rightarrow 0$, we recover the CCRs of Equation (2.5.21). Moreover, $\mathcal{E}_0(z)$ is the generating series of the completed cut-and-join operators, “corrected” in degree zero:

$$\mathcal{E}_0(z) = \sum_{m \geq 0} \frac{\mathcal{F}_m}{m!} z^m + \frac{1}{\varsigma(z)}. \quad (2.5.24)$$

We can upgrade the action of the Lie algebra $\widehat{\mathfrak{gl}}(\infty)$ to that of a group by setting

$$\widehat{\mathbb{GL}}(\infty) = \left\{ e^{X_1} \dots e^{X_N} \mid N \in \mathbb{N}, X_i \in \widehat{\mathfrak{gl}}(\infty) \right\}. \quad (2.5.25)$$

The action of an element of this group is defined by expanding the exponentials, which is well-defined since only finitely many monomials in the X_i ’s have a non-vanishing action.

DEFINITION 2.5.13. Define the (big cell of the) *Sato Grassmannian* as the $\widehat{\mathbb{GL}}(\infty)$ -orbit of the vacuum vector $|0\rangle$ in the projectivisation of the Fock space:

$$\widehat{\mathbb{GL}}(\infty) \cdot |0\rangle \subseteq \mathbb{P}\mathfrak{F}_0. \quad (2.5.26)$$

Remark 2.5.14. In the semi-infinite wedge formalism of Definition 2.5.9, we can alternatively realise the Sato Grassmannian as follows. Let V be the vector space spanned by the elements $(e_s)_{s \in \mathbb{Z}'}$, and V_+ the subspace spanned by $(e_s)_{s > 0}$. Then, we have that the Sato Grassmannian is given by

$$\text{Gr}(V) = \left\{ W \subseteq V \mid \pi_W : W \rightarrow V_+ \text{ is an isomorphism} \right\}, \quad (2.5.27)$$

where π_W is the projection of W to V_+ . Recall that the Fock space is the semi-infinite wedge product on V . We then have the *Plücker embedding*

$$\text{Gr}(V) \hookrightarrow \mathbb{P}\mathfrak{F}_0, \quad (2.5.28)$$

defined as $W \mapsto \pi_W^{-1}(e_{1/2}) \wedge \pi_W^{-1}(e_{3/2}) \wedge \dots$.

As in the final-dimensional case, elements of the Grassmannian can be characterised by quadratic relations, which can be compactly written in terms of fermions’ generating series

$$\psi(z) = \sum_{s \in \mathbb{Z}'} \psi_s z^{-s-\frac{1}{2}}, \quad \psi^\dagger(z) = \sum_{s \in \mathbb{Z}'} \psi_s^\dagger z^{-s-\frac{1}{2}}, \quad (2.5.29)$$

THEOREM 2.5.15 (Plücker relations). *An element $|\omega\rangle \in \mathbb{P}\mathfrak{F}_0$ belongs to the Sato Grassmannian if and only if it satisfies the following quadratic relations, also called the Plücker relations:*

$$\text{Res}_{z=\infty} \psi(z) |\omega\rangle \otimes \psi^\dagger(z) |\omega\rangle dz = 0. \quad (2.5.30)$$

The remaining part of this section is devoted to the description of the Sato Grassmannian in terms of bosons. To this end, we first review some basics facts about symmetric functions and refer to [McD98] for further readings.

SYMMETRIC FUNCTIONS

DEFINITION 2.5.16. Let Λ be the algebra of symmetric functions (or bosonic Fock space):

$$\Lambda = \varprojlim_n \mathbb{C}[x_1, \dots, x_n]^{\mathfrak{S}_n}, \quad (2.5.31)$$

where \mathfrak{S}_n acts by permuting the variables and the inverse limit is taken with respect to the restriction maps $P(x_1, \dots, x_n, x_{n+1}) \mapsto P(x_1, \dots, x_n, 0)$.

There are three important bases on Λ , that we now recall. Consider three families of symmetric polynomials first, called elementary symmetric polynomials e_k , complete homogeneous symmetric polynomials h_k , and power-sum symmetric polynomials p_k :

$$e_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 < \dots < i_k \leq n} x_{i_1} \cdots x_{i_k}, \quad (2.5.32)$$

$$h_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 \leq \dots \leq i_k \leq n} x_{i_1} \cdots x_{i_k}, \quad (2.5.33)$$

$$p_k(x_1, \dots, x_n) = \sum_{i=1}^n x_i^k. \quad (2.5.34)$$

Their images in Λ are called *elementary*, *complete homogeneous*, and *power-sum symmetric functions* respectively.

LEMMA 2.5.17. *The algebra of symmetric polynomials is a free algebra on the elementary, complete homogeneous, and power-sum symmetric functions:*

$$\Lambda = \mathbb{C}[e_1, e_2, \dots] = \mathbb{C}[h_1, h_2, \dots] = \mathbb{C}[p_1, p_2, \dots]. \quad (2.5.35)$$

Moreover, the change of basis is given by the following Newton's identities:

$$\sum_{k \geq 0} e_k u^k = \exp\left(-\sum_{m \geq 1} \frac{(-1)^m p_m}{m} u^m\right), \quad \sum_{k \geq 0} h_k u^k = \exp\left(\sum_{m \geq 1} \frac{p_m}{m} u^m\right). \quad (2.5.36)$$

Another important basis is the one consisting of Schur functions.

DEFINITION 2.5.18. Let λ be a partition of length $\leq n$, and define the polynomials $a_\lambda(x_1, \dots, x_n) = \det(x_i^{\lambda_j + n - j})_{1 \leq i, j \leq n}$. Define the Schur symmetric polynomials as

$$s_\lambda(x_1, \dots, x_n) = \frac{a_\lambda(x_1, \dots, x_n)}{a_\emptyset(x_1, \dots, x_n)}. \quad (2.5.37)$$

Their images in Λ are called the *Schur symmetric functions*.

The following lemma expresses the duality between Schur and power-sum symmetric functions. For a partition μ , define $p_\mu = \prod_{i=1}^{\ell(\mu)} p_{\mu_i}$.

LEMMA 2.5.19. *The change of basis from Schur functions to power-sum symmetric functions is given by the irreducible characters of the symmetric group:*

$$s_\lambda = \sum_{\mu \vdash |\lambda|} \frac{\chi_\lambda(\mu)}{\mathfrak{z}_\mu} p_\mu, \quad p_\mu = \sum_{\lambda \vdash |\mu|} \chi_\lambda(\mu) s_\lambda. \quad (2.5.38)$$

Here $\mathfrak{z}_\mu = \prod_{i=1}^{\ell(\mu)} \mu_i! \prod_{m>0} |\{i \mid \mu_i = m\}|!$ is the order of the centraliser of an element of cycle type μ in the symmetric group.

BOSON-FERMION CORRESPONDENCE

We can now relate the fermionic and bosonic Fock spaces as follows.

THEOREM 2.5.20 (Boson-fermion correspondence, see for instance [MJD00, Theorem 5.1]).
 Let $t_n = \frac{p_n}{n}$. The map

$$\sigma: \mathfrak{F} \longrightarrow \Lambda[\zeta, \zeta^{-1}], \quad |\omega\rangle \longmapsto \sum_{\ell \in \mathbb{Z}} \langle \ell | e^{J(t)} | \omega \rangle \zeta^\ell \quad (2.5.39)$$

is an isomorphism, called the boson-fermion correspondence. Moreover, the action of the currents is given by

$$\sigma(J_n |\omega\rangle) = \begin{cases} \partial_{t_n} \sigma(|\omega\rangle) & \text{if } n > 0, \\ -n t_{-n} \sigma(|\omega\rangle) & \text{if } n < 0, \end{cases} \quad (2.5.40)$$

and that of the fermions by

$$\sigma(\psi(z) |\omega\rangle) = \exp\left(\sum_{k>0} z^k t_k\right) \exp\left(-\sum_{k>0} \frac{z^{-k}}{k} \partial_{t_k}\right) \zeta \exp(z\zeta \partial_\zeta) \sigma(|\omega\rangle), \quad (2.5.41)$$

$$\sigma(\psi^\dagger(z) |\omega\rangle) = \exp\left(-\sum_{k>0} z^k t_k\right) \exp\left(\sum_{k>0} \frac{z^{-k}}{k} \partial_{t_k}\right) \zeta^{-1} \exp(-z\zeta \partial_\zeta) \sigma(|\omega\rangle). \quad (2.5.42)$$

Via the boson-fermion correspondence, one can compute the action of the completed cut-and-join operators and that of currents on the (charge-zero sector of the) fermionic Fock space as follows:

PROPOSITION 2.5.21.

- For any partition λ , the action of the completed cut-and-join operator \mathcal{F}_m is given by

$$\mathcal{F}_m |\lambda\rangle = \mathbf{p}_m(\lambda) |\lambda\rangle, \quad (2.5.43)$$

where \mathbf{p}_m are the shifted power-sum symmetric functions⁹

$$\mathbf{p}_m(\lambda) = \sum_{i>0} [(\lambda_i - i + \frac{1}{2})^m - (-i + \frac{1}{2})^m]. \quad (2.5.44)$$

- For a partition μ , set $J_{\pm\mu} = J_{\pm\mu_1} \cdots J_{\pm\mu_n}$. Then

$$J_{-\mu} |0\rangle = \sum_{\lambda \vdash |\mu|} \chi_\lambda(\mu) |\lambda\rangle, \quad J_\mu |\lambda\rangle = \chi_\mu(\lambda) |0\rangle. \quad (2.5.45)$$

⁹The standard definition [OP06] involves certain additive constants that we have dropped to simplify the expression, since these constants play no role in this dissertation. Notice also that, as customary, we denote shifted power-sums in bold to differentiate them from the non-shifted version.

2.5.2 — INTEGRABLE HIERARCHIES

Consider a classical mechanical system with N degrees of freedom, described by a Hamiltonian function H :

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, \dots, N. \quad (2.5.46)$$

We say that the mechanical system is *completely integrable* if it has N independent first integrals $F_1(q, p), \dots, F_N(q, p)$. In this case, the general solution can then be obtained by solving $F_i(q, p) = C_i$ for some arbitrary constants C_1, \dots, C_N . The complete integrability of a mechanical system reflects the presence of a high degree of symmetry, or equivalently the presence of the action of a huge transformation group.

Following this guiding principle, the Kyoto school gave a complete description of the KP hierarchy (an infinite-dimensional integrable system) in terms of the infinite-dimensional Lie group $\widehat{\text{GL}}(\infty)$. In this exposition we will skip the (pseudo-)differential operators approach, and directly connect integrability to the Fock spaces. A complete exposition can be found in [MJDoo].

DEFINITION 2.5.22. A *tau function of the KP hierarchy* is an element $\tau \in \Lambda$ in the image via the boson-fermion correspondence Φ of an element of the big cell of the Sato Grassmannian. It is a tau function of the *KdV hierarchy* if moreover τ does not depend on even times.

As such, a tau function is defined up to a multiplicative constant. Notice that, from the above discussion, a tau function of the KP hierarchy can always be expressed as

$$\tau(\mathbf{t}) = \left\langle e^{J(\mathbf{t})} g \right\rangle \quad (2.5.47)$$

for some $g \in \widehat{\text{GL}}(\infty)$. The Plücker relations translate into the Hirota bilinear relations satisfied by tau functions.

THEOREM 2.5.23 (Hirota bilinear relations). *An element $\tau \in \Lambda$ is a tau function of the KP hierarchy if and only if the following relations, known as Hirota bilinear relations, hold:*

$$\text{Res}_{z=\infty} \exp\left(2 \sum_{k>0} z^k u_k\right) \exp\left(-2 \sum_{k>0} \frac{z^{-k}}{k} \partial_{u_k}\right) \tau(\mathbf{t} + \mathbf{u}) \tau(\mathbf{t} - \mathbf{u}) dz = 0. \quad (2.5.48)$$

By expanding the above relation order by order in Taylor expansion when $\mathbf{u} \rightarrow 0$, we obtain a collection of non-linear PDEs satisfied by τ . For instance, collecting the coefficient of u_1^k we get the first few PDEs of the hierarchy:

$$\begin{aligned} k = 1 & \quad -D_{t_2}[\tau, \tau] = 0, \\ k = 2 & \quad \frac{1}{6}(D_{t_1}^3 - 4D_{t_3})[\tau, \tau] = 0, \\ k = 3 & \quad \frac{1}{18}(D_{t_1}^4 + 3D_{t_2}^2 - 4D_{t_1}D_{t_3} + 3D_{t_1}^2D_{t_2} - 6D_{t_4})[\tau, \tau] = 0, \end{aligned} \quad (2.5.49)$$

expressed in term of the *Hirota derivative*:

$$f(\mathbf{t} + \mathbf{u})g(\mathbf{t} - \mathbf{u}) = \sum_{\alpha \text{ multi-index}} D^\alpha[f, g](\mathbf{t}) \frac{\mathbf{u}^\alpha}{\alpha!}. \quad (2.5.50)$$

Since odd polynomials of D_{t_i} applied to $[\tau, \tau]$ vanish, the first two equations are trivially satisfied, and the first non-trivial equation is the KP equation:

$$(D_{t_1}^4 + 3D_{t_2}^2 - 4D_{t_1}D_{t_3})[\tau, \tau] = 0. \quad (2.5.51)$$

More generally, the coefficients of monomials in u generate a sequence of non-linear PDEs, called the KP hierarchy, which are all compatible by construction.

To conclude this section, let us briefly describe a larger hierarchy, called the $2d$ Toda lattice hierarchy, introduced by Ueno–Takasaki [UT84] (see also [Tak91]). It depends on two infinite sets of times $\mathbf{t}_+ = (t_1, t_2, \dots)$ and $\mathbf{t}_- = (t_{-1}, t_{-2}, \dots)$ and an extra discrete parameter ℓ . Here we take the fermionic formalism perspective to tau functions of $2d$ Toda lattice hierarchy.

DEFINITION 2.5.24. Consider the decomposition of $\mathfrak{gl}(\infty)$ into strictly upper triangular, diagonal and strictly lower triangular matrices:

$$\mathfrak{gl}(\infty) = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-. \quad (2.5.52)$$

We can exponentiate the action of the Lie subalgebras on the Fock space to the action of the associated Lie groups.

DEFINITION 2.5.25. A tau function of the $2d$ Toda lattice hierarchy is an element $\tau_\ell \in \mathbb{C}[\mathbf{t}_+, \mathbf{t}_-]$ of the form

$$\tau_\ell(\mathbf{t}_+, \mathbf{t}_-) = \left\langle \ell \left| e^{J_+(\mathbf{t}_+)} g_+ e^\Delta g_- e^{-J_-(\mathbf{t}_-)} \right| \ell \right\rangle, \quad (2.5.53)$$

for some $\ell \in \mathbb{Z}$ and some $(g_+, e^\Delta, g_-) \in \mathfrak{n}_+ \times \widehat{\mathfrak{h}} \times \mathfrak{n}_-$. Here we considered $J_\pm(\mathbf{t}_\pm) = \sum_{\pm k > 0} t_k J_k$.

As for the KP hierarchy, $2d$ Toda tau functions satisfy a bilinear identity equivalent to an infinite set of compatible non-linear PDEs. See [Tak91] for further readings. In particular, notice that evaluating the second set of times in a $2d$ Toda tau function to specific values, finitely many non-zero, and setting $\ell = 0$, recovers a KP tau function.

2.6 — HURWITZ THEORY

Hurwitz theory, originated from the work of Hurwitz [Hur91], is the enumerative study of branched covers of compact Riemann surfaces with specified ramifications. Ramified covers naturally give rise to monodromy representations, i.e. homomorphisms from the fundamental group of the punctured target surface to a symmetric group. The count of all such representations can be identified with a coefficient of a specific product of elements in the class algebra of the symmetric group, leading to closed formulae for Hurwitz numbers in terms of characters of the symmetric group.

To summarise, Hurwitz numbers admit two different representations: a geometric count of topological covers and an algebraic count of group homomorphisms. In this dissertation, we will only focus on ramified covers of \mathbb{P}^1 , presenting the two different representations of Hurwitz numbers. We refer to [CM16] for a modern account of the theory.

DEFINITION 2.6.1. Let p_1, \dots, p_k be distinct points on \mathbb{P}^1 , and $\mu^1, \dots, \mu^k \vdash d$ partitions of $d > 0$. A Hurwitz cover of degree d and ramification data μ^1, \dots, μ^k is a degree d holomorphic

map $f: C \rightarrow \mathbb{P}^1$, where C is a connected compact Riemann surface, with ramification profile μ^i over p_i , and unramified everywhere else.

An isomorphism of Hurwitz covers from $f: C \rightarrow \mathbb{P}^1$ to $f': C' \rightarrow \mathbb{P}^1$ is an isomorphism $\varphi: C \rightarrow C'$ such that $f' \circ \varphi = f$. Denote by $\text{Aut}(f)$ the group of automorphisms of $f: C \rightarrow \mathbb{P}^1$.

We remark that, for a given Hurwitz cover, its genus is determined by the ramification data via the *Riemann–Hurwitz formula*:

$$2 - 2g = 2d - \sum_{i=1}^k (d - \ell(\mu^k)). \quad (2.6.1)$$

As the number of isomorphism classes of Hurwitz covers with specified ramification data is finite, it is natural to count them, weighted by automorphism.

DEFINITION 2.6.2. Let $\mu^1, \dots, \mu^k \vdash d$. Define the *Hurwitz number* as

$$H_d(\mu^1, \dots, \mu^k) = \sum_{[f]} \frac{1}{|\text{Aut}(f)|}, \quad (2.6.2)$$

where the sum runs over all isomorphism classes of Hurwitz covers $f: C \rightarrow \mathbb{P}^1$ of degree d , and ramification data μ^1, \dots, μ^k .

Sometimes it is useful to allow the source curve of the maps we count to be disconnected. We call the corresponding count a disconnected Hurwitz number and denote it by $H_d^\bullet(\mu^1, \dots, \mu^k)$. See Equation (2.6.19) for the connection between connected and disconnected counts.

A Hurwitz cover $f: C \rightarrow \mathbb{P}^1$ is determined, up to isomorphism, by its *monodromy representation* $\phi: \pi_1(\mathbb{P}^1 \setminus \{p_1, \dots, p_k\}, p_0) \rightarrow \mathfrak{S}_d$. Here \mathfrak{S}_d denotes the symmetric group of degree d . This motivates the following

DEFINITION 2.6.3. Let $\mu^1, \dots, \mu^k \vdash d$. We say that $(\sigma_1, \dots, \sigma_k)$ is a *factorisation* of type (μ^1, \dots, μ^k) if

- $\sigma_i \in \mathfrak{S}_d$, and $\sigma_k \cdot \sigma_{k-1} \cdots \sigma_1 = \text{id}$,
- the cycle type of σ_i is given by μ^i .

A factorisation is called *transitive* if

- the group generated by $\sigma_1, \dots, \sigma_k$ acts transitively on $\{1, \dots, d\}$.

Denote the set of transitive factorisations of type (μ^1, \dots, μ^k) by $F_d(\mu^1, \dots, \mu^k)$, and the set of factorisations of type (μ^1, \dots, μ^k) by $F_d^\bullet(\mu^1, \dots, \mu^k)$.

The relation between Hurwitz numbers and monodromy representations states the following equivalence (see [CM16, Chapter 7] for further readings).

PROPOSITION 2.6.4. *The counts of Hurwitz covers and factorisations are equivalent:*

$$H_d(\mu^1, \dots, \mu^k) = \frac{1}{d!} |F_d(\mu^1, \dots, \mu^k)|. \quad (2.6.3)$$

The same holds for the disconnected count: $H_d^\bullet(\mu^1, \dots, \mu^k) = \frac{1}{d!} |F_d^\bullet(\mu^1, \dots, \mu^k)|$.

The count of permutations in the symmetric group can be recast into a counting problem in the symmetric group algebra, allowing for the use of representation-theoretic tools in Hurwitz theory. Indeed, consider the *symmetric group algebra* $\mathbb{C}[\mathfrak{S}_d]$, and for any partition $\mu \vdash d$, define the elements

$$C_\mu = \sum_{\sigma \in \mathcal{O}_\mu} \sigma, \quad (2.6.4)$$

where \mathcal{O}_μ is the conjugacy class in \mathfrak{S}_d of a permutation of cycle type μ . They are called the *conjugacy class elements of type μ* . It follows from the above proposition that disconnected Hurwitz numbers correspond to the coefficient of the identity in a particular product of conjugacy class elements:

$$H_d^\bullet(\mu^1, \dots, \mu^k) = \frac{1}{d!} [\text{id}] C_{\mu^1} \cdots C_{\mu^k}. \quad (2.6.5)$$

Here $[\sigma]$ selects the coefficient of σ in the expression that follows.

The above expression can be further simplified using classical results regarding the centre of the symmetric group algebra $\mathcal{Z}_d = Z\mathbb{C}[\mathfrak{S}_d]$, also called the *class algebra*. Indeed, \mathcal{Z}_d is a semisimple algebra with basis given by conjugacy class elements $(C_\mu)_{\mu \vdash d}$, and the change of basis to that of semisimple elements

$$e_\lambda \cdot e_\nu = \delta_{\lambda, \nu} e_\lambda \quad (2.6.6)$$

is given in terms of characters of the symmetric group:

$$e_\lambda = \frac{\dim(\lambda)}{d!} \sum_{\mu \vdash d} \chi_\lambda(\mu) C_\mu, \quad C_\mu = |\mathcal{O}_\mu| \sum_{\lambda \vdash d} \frac{\chi_\lambda(\mu)}{\dim(\lambda)} e_\lambda. \quad (2.6.7)$$

Here $\dim(\lambda) = \chi_\lambda(1^d)$ is the dimension of the irreducible representation of \mathfrak{S}_d labelled by λ . Implementing the change of basis in the expression for Hurwitz numbers, we obtain the following result, known as Burnside character formula.

THEOREM 2.6.5 (Burnside character formula). *Hurwitz numbers are given by*

$$H_d^\bullet(\mu^1, \dots, \mu^k) = \sum_{\lambda \vdash d} \left(\frac{\dim(\lambda)}{d!} \right)^2 \prod_{i=1}^k f_{\mu^i}(\lambda). \quad (2.6.8)$$

Here $f_\mu(\lambda) = |\mathcal{O}_\mu| \frac{\chi_\lambda(\mu)}{\dim(\lambda)}$.

The elements f_μ , considered as functions from the set \mathcal{P}_d of partitions of d to \mathbb{C} , play an important role in Hurwitz theory. It can be shown that the map

$$\phi_d: \mathcal{Z}_d \longrightarrow \mathbb{C}^{\mathcal{P}_d}, \quad C_\mu \longmapsto f_\mu. \quad (2.6.9)$$

is an algebra isomorphism. If we upgrade the definition of f_μ to a function on the set of all partitions $\mathcal{P} = \bigcup_{d \geq 0} \mathcal{P}_d$ (including the empty partition \emptyset) by setting

$$f_\mu(\lambda) = \begin{cases} \binom{|\mu|}{|\lambda|} |\mathcal{O}_\mu| \frac{\chi_\lambda(\mu)}{\dim(\lambda)} & \text{if } |\mu| \leq |\lambda|, \\ 0 & \text{otherwise,} \end{cases} \quad (2.6.10)$$

where the character is define through the inclusion of symmetric groups $\mathfrak{S}_{|\mu|} \subseteq \mathfrak{S}_{|\lambda|}$ for $|\mu| \leq |\lambda|$, then the above map extends to an algebra morphism

$$\phi: \bigoplus_{d \geq 0} \mathcal{Z}_d \longrightarrow \mathbb{C}^{\mathcal{P}}, \quad C_\mu \mapsto \mathbf{f}_\mu. \quad (2.6.11)$$

By a result of Vershik–Kerov [VK81], the highest degree term of \mathbf{f}_μ is given by the shifted power-sum symmetric functions of Proposition 2.5.21:

$$\mathbf{f}_\mu = \frac{1}{\prod_i \mu_i} \mathbf{p}_\mu + \text{lower order terms}. \quad (2.6.12)$$

In particular, it is natural to consider the following elements.

DEFINITION 2.6.6. For $\mu \vdash d$, define the *completed conjugacy class* elements as

$$\overline{C}_\mu = \frac{1}{\prod_i \mu_i} \phi^{-1}(\mathbf{p}_\mu) \in \bigoplus_{m=0}^d \mathcal{Z}_m. \quad (2.6.13)$$

For $\mu = (d)$, we call the associated element $\overline{(d)} = \overline{C}_{(d)}$ *completed cycle*.

The formulae for the first few completed cycles are given below.

$$\begin{aligned} \overline{(1)} &= (1) \\ \overline{(2)} &= (2) \\ \overline{(3)} &= (3) + (1, 1) + \frac{1}{12}(1) \\ \overline{(4)} &= (4) + 2(2, 1) + \frac{5}{4}(2) \end{aligned} \quad (2.6.14)$$

The term “completed cycle” was first suggested by Eskin–Okounkov–Zorich in an unpublished work, motivated by the enumeration of degenerations of Hurwitz covers. The Gromov–Witten/Hurwitz (GW/H) correspondence of Okounkov–Pandharipande [OP06] explains the geometric meaning of the completed cycles: stationary Gromov–Witten invariants of \mathbb{P}^1 correspond to Hurwitz numbers with completed cycles, and in particular the lower order terms in the completed cycles reflect the contributions from the boundary of the moduli space of stable maps.

In order to connect Hurwitz theory and the fermionic formalism of Section 2.5.1, we will restrict our attention to a particular type of Hurwitz numbers.

DEFINITION 2.6.7. Let $\mu, \nu \vdash d$. Define the disconnected *double Hurwitz numbers with $(r + 1)$ -completed cycles* as

$$h_{g; \mu, \nu}^{\bullet, r} = \frac{|\text{Aut}(\mu)| |\text{Aut}(\nu)|}{b!} H_d^\bullet \left(\mu, \overline{(r+1)}^b, \nu \right), \quad (2.6.15)$$

where the right-hand side is defined through the Burnside character formula¹⁰, i.e.

$$h_{g; \mu, \nu}^{\bullet, r} = \frac{|\text{Aut}(\mu)| |\text{Aut}(\nu)|}{b!} \sum_{\lambda \vdash d} \left(\frac{\dim(\lambda)}{d!} \right)^2 \mathbf{f}_\mu(\lambda) \binom{\mathbf{p}_{r+1}(\lambda)}{r+1}^b \mathbf{f}_\nu(\lambda). \quad (2.6.16)$$

¹⁰Beware that some authors (for instance [SSZ12]) use a different normalisation, namely in Burnside character formula they consider $\frac{\mathbf{p}_{r+1}}{(r+1)!}$ instead of $\frac{\mathbf{p}_{r+1}}{r+1}$. Moreover, other authors (for instance [ELSV01]) define Hurwitz numbers without the division by $b!$.

The value b is determined by the Riemann–Hurwitz formula: $rb = 2g - 2 + \ell(\mu) + \ell(\nu)$. In particular, Hurwitz numbers are set to be zero if $r \nmid 2g - 2 + \ell(\mu) + \ell(\nu)$. The automorphism group $|\text{Aut}(\lambda)| = \prod_{\ell > 0} |\{i \mid \lambda_i = \ell\}|!$ is added for convenience, and should be interpreted as labelling the inverse images of the branch points.

Remark 2.6.8. Many authors call such counting “spin Hurwitz numbers” rather than “Hurwitz numbers with completed cycles”. This is because such Hurwitz numbers are related to the moduli space of r -spin curves through Chiodo classes (see Section 2.6.1). However, since one can define Hurwitz numbers counting covers with respect to the parity of the underlying spin structure, we will avoid this terminology and reserve it to the counting of Part IV.

The connected counting $h_{g;\mu,\nu}^r$ is defined as usual via exponentiation: define the generating functions Z^r and H^r of the disconnected and connected counting respectively.

$$Z^r(\beta; \mathbf{p}, \mathbf{q}) = \sum_{g,d} \sum_{\mu, \nu \vdash d} h_{g;\mu,\nu}^{\bullet,r} \beta^b \left(\frac{1}{\ell(\mu)!} \prod_{i=1}^{\ell(\mu)} p_{\mu_i} \right) \left(\frac{1}{\ell(\nu)!} \prod_{i=1}^{\ell(\nu)} q_{\nu_i} \right), \quad (2.6.17)$$

$$H^r(\beta; \mathbf{p}, \mathbf{q}) = \sum_{g,d} \sum_{\mu, \nu \vdash d} h_{g;\mu,\nu}^r \beta^b \left(\frac{1}{\ell(\mu)!} \prod_{i=1}^{\ell(\mu)} p_{\mu_i} \right) \left(\frac{1}{\ell(\nu)!} \prod_{i=1}^{\ell(\nu)} q_{\nu_i} \right). \quad (2.6.18)$$

Then we have relation

$$Z^r(\beta; \mathbf{p}, \mathbf{q}) = \exp(H^r(\beta; \mathbf{p}, \mathbf{q})). \quad (2.6.19)$$

Remark 2.6.9. Geometrically, double Hurwitz numbers with $(r + 1)$ -completed cycles count Hurwitz covers of \mathbb{P}^1 with ramification profiles μ and ν over 0 and ∞ respectively, and ramification profile given by $(r + 1, 1^{d-r-1})$, up to a combinatorial factor and up to lower order terms, as expressed by Equation (2.6.12). See [SSZ12, Subsection 2.5] for further details about their geometric meaning. We remark again that, although such terms seems unnatural, they take into account degeneration of Hurwitz covers and appear naturally in the context of GW/H correspondence.

Combining the definition of Hurwitz numbers with completed cycles and the action of the completed cut-and-join operators and the current operators (cf. Proposition 2.5.21), we obtain the following expression for Hurwitz numbers in terms of vacuum expectation values.

THEOREM 2.6.10 ([OP06; SSZ12]). *The disconnected double Hurwitz numbers with $(r + 1)$ -completed cycles can be extracted from the following vacuum expectation value on the Fock space:*

$$h_{g;\mu,\nu}^{\bullet,r} = \frac{1}{b!} \left\langle \frac{J_\mu}{\prod_{i=1}^{\ell(\mu)} \mu_i} \left(\frac{\mathcal{F}_{r+1}}{r+1} \right)^b \frac{J_{-\nu}}{\prod_{j=1}^{\ell(\nu)} \nu_j} \right\rangle, \quad (2.6.20)$$

with $rb = 2g - 2 + \ell(\mu) + \ell(\nu)$.

As a simple corollary, we find that the generating function of Hurwitz numbers with completed cycles is a tau function of the $2d$ Toda lattice hierarchy of Definition 2.5.25. The result is due to Okounkov [Ok00], who proved it for the case $r = 1$ (also known as *simple double Hurwitz numbers*), and later generalised by Shadrin–Spitz–Zvonkine [SSZ12].

COROLLARY 2.6.11. *The generating function of disconnected double Hurwitz numbers with $(r + 1)$ -completed cycles can be represented as the following vacuum expectation value*

$$Z^r(\beta; \mathbf{p}, \mathbf{q}) = \left\langle e^{\sum_{m>0} \frac{\beta^m}{m} J_m} e^{\beta \frac{\mathcal{F}_{r+1}}{r+1}} e^{\sum_{n>0} \frac{\beta^n}{n} J_{-n}} \right\rangle \quad (2.6.21)$$

and, in particular, it is a tau function for the 2d Toda lattice hierarchy (identically in β), after the identification $p_m = mt_m$ and $q_n = -nt_{-n}$.

Another simple corollary of Equation (2.6.20) is an evolution equation for $Z^r(\beta; \cdot)$, also known as the cut-and-join equation. The idea is to define the (bosonic) completed cut-and-join operators \mathcal{W}_m , acting on the space of symmetric functions Λ , as the equivalent of the action of \mathcal{F}_m under the boson-fermion correspondence:

$$\sigma\left(\frac{\mathcal{F}_m}{m}|\omega\rangle\right) = \mathcal{W}_m\sigma(|\omega\rangle). \quad (2.6.22)$$

We refer to [SSZ12] for a geometric interpretation of such operators, and a generating series for them. The first two completed cut-and-join operators (in the p variables) are given as follows:

$$\begin{aligned} \mathcal{W}_1 &= \sum_{k>0} kp_k\partial_{p_k}, \\ \mathcal{W}_2 &= \frac{1}{2} \sum_{k,\ell>0} (k\ell p_{k+\ell}\partial_{p_k}\partial_{p_\ell} + (k+\ell)p_k p_\ell\partial_{p_{k+\ell}}). \end{aligned} \quad (2.6.23)$$

THEOREM 2.6.12 (Cut-and-join equation). *The following evolution equation holds:*

$$\partial_\beta Z^r = \mathcal{W}_{r+1} Z^r. \quad (2.6.24)$$

We also call it cut-and-join equation for double Hurwitz numbers with $(r+1)$ -completed cycles.

Let us briefly discuss some interesting specialisations of double Hurwitz numbers.

DEFINITION 2.6.13. Fix a positive integer q . Define the disconnected q -orbifold Hurwitz numbers with $(r+1)$ -completed cycles, denoted $h_{g;\mu}^{\bullet,q,r}$, by specialising the generating function of double Hurwitz numbers with $(r+1)$ -completed cycles to $q_n = \delta_{n,q}$. In other words,

$$h_{g;\mu}^{\bullet,q,r} = \frac{1}{b!} \left\langle \frac{J_\mu}{\prod_{i=1}^{\ell(\mu)} \mu_i} \left(\frac{\mathcal{F}_{r+1}}{r+1}\right)^b \frac{1}{\left(\frac{d}{q}\right)!} \left(\frac{J_{-q}}{q}\right)^{\frac{d}{q}} \right\rangle \quad (2.6.25)$$

for $d = |\mu|$ that is divisible by q and $rb = 2g - 2 + \ell(\mu) + \frac{d}{q}$. The case $q = 1$ and general r is also called Hurwitz numbers with $(r+1)$ -completed cycles, denoted $h_{g;\mu}^{\bullet,r}$, and the case $q = r = 1$ is called simple Hurwitz numbers, denoted $h_{g;\mu}^{\bullet}$. For the connected counting, we omit the bullet.

Okounkov's result specialises to the KP hierarchy for the generating function of q -orbifold Hurwitz numbers with $(r+1)$ -completed cycles:

$$Z^{q,r}(\beta; \mathbf{p}) = \sum_{g,d} \sum_{\mu \vdash d} h_{g;\mu}^{\bullet,q,r} \beta^b \frac{1}{\ell(\mu)!} \prod_{i=1}^{\ell(\mu)} p_{\mu_i}. \quad (2.6.26)$$

Moreover, the cut-and-join equation is still satisfied (but with different initial conditions).

COROLLARY 2.6.14. *The generating function of q -orbifold Hurwitz numbers with $(r+1)$ -completed cycles can be represented as the following vacuum expectation value*

$$Z^{q,r}(\beta; \mathbf{p}) = \left\langle e^{\sum_{m>0} \frac{p_m}{m} J_m} e^{\beta \frac{\mathcal{F}_{r+1}}{r+1}} e^{\frac{J_{-q}}{q}} \right\rangle \quad (2.6.27)$$

and, in particular, it is a tau function for the KP hierarchy (identically in β), after the identification $p_m = mt_m$. Moreover, it satisfies the cut-and-join equation:

$$\partial_\beta Z^{q,r} = \mathcal{W}_{r+1} Z^{q,r}. \quad (2.6.28)$$

2.6.1 — ELSV-TYPE FORMULAE

In 2001, Ekedahl–Lando–Shapiro–Vainshtein [ELSV01] discovered an interesting connection between Hurwitz theory and intersection theory on the moduli spaces of curves. Since then, it has been generalised in several directions to different ELSV-type formulae. See [Lew17] for an overview.

THEOREM 2.6.15 (ELSV formula). *Connected simple Hurwitz numbers are expressed in terms of Hodge integrals: for $\mu = (\mu_1, \dots, \mu_n)$,*

$$h_{g;\mu} = \prod_{i=1}^n \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\mathcal{M}_{g,n}} \frac{\Lambda(-1)}{\prod_{i=1}^n (1 - \mu_i \psi_i)}. \quad (2.6.29)$$

Important applications of the ELSV formula include Witten’s conjecture and the λ_g -conjecture. Apart from the original proof, obtained through localisation techniques on the moduli space of stable maps from a genus g curve to \mathbb{P}^1 with prescribed pole structure, one can prove it using the Eynard–DOSS correspondence of Section 2.3.1, knowing that Hurwitz numbers can be computed via topological recursion. Using this argument, a generalisation of the original ELSV formula involving Chiodo classes was proved in a series of papers [SSZ15; LPSZ17; KLPS19; Bor+21; DKPS19]. The formula was originally conjectured in unpublished notes by Zvonkine [Zvo06], motivated by Gromov–Witten theory.

THEOREM 2.6.16 (Zvonkine’s ELSV formula). *Connected q -orbifold Hurwitz numbers with $(r + 1)$ -completed cycles are expressed in terms of Chiodo integrals: for $\mu = (\mu_1, \dots, \mu_n)$,*

$$h_{g;\mu}^{q,r} = r^{2g-2+n} (qr)^{\frac{(2g-2+n)q+d}{qr}} \prod_{i=1}^n \frac{\left(\frac{\mu_i}{qr}\right)^{[\mu_i]}}{[\mu_i]!} \int_{\mathcal{M}_{g,n}} \frac{C_{g,n}^{qr,q}(qr - \langle \mu_1 \rangle, \dots, qr - \langle \mu_n \rangle)}{\prod_{i=1}^n (1 - \frac{\mu_i}{qr} \psi_i)}, \quad (2.6.30)$$

where $\lambda = qr[\lambda] + \langle \lambda \rangle$ is the integral division of λ by q .

A geometric proof using localisation techniques for the $q = 1$ case was recently found by Leigh [Lei20].

Generalisations of the ELSV formula include:

- weakly monotone Hurwitz numbers [ALS16; DK17],
- strictly monotone Hurwitz numbers [BG20; GGR20], and
- spin Hurwitz numbers with completed cycles (see Part IV).

We will encounter other ELSV-type formulae throughout this dissertation, connecting different enumerative geometric problems (not necessarily Hurwitz numbers) solved by topological recursion and the intersection theory on the moduli space of curves. These include:

- Weil–Peterson volumes of the moduli spaces of hyperbolic surfaces and κ_1 intersected with ψ -classes (as discussed in Section 2.4),
- Kontsevich volumes of the combinatorial moduli spaces and ψ -classes intersection numbers (see Section 5.3),
- Masur–Veech volumes and the Segre class of the quadratic Hodge bundle (see Section 9.1),
- The (orbifolds) Euler characteristic of $\mathcal{M}_{g,n}$ and the Chern class of the quadratic Hodge bundle (see Section 9.3).

PART II

THE COMBINATORIAL MODEL OF THE MODULI SPACE OF CURVES

CHAPTER 3 — TOPOLOGY OF COMBINATORIAL SPACES

In his proof of Witten’s conjecture and following the pioneering works of Strebel, Harer, Mumford, Penner and Thurston [Str67; Har86; HZ86; Pen88], Kontsevich [Kon92] studied a combinatorial model $\mathcal{M}_{g,n}^{\text{comb}}$ for the moduli space of curves, namely the space of metric ribbon graphs on a stable surface Σ of genus g with $n > 0$ boundaries. He equipped it with a 2-form $\omega_{\mathbb{K}}$ whose restriction to the slice with fixed boundary lengths is almost everywhere symplectic, and so that the symplectic volume gives access to the intersection of ψ -classes on $\overline{\mathcal{M}}_{g,n}$. The corresponding *combinatorial Teichmüller space*

$$\mathcal{T}_{\Sigma}^{\text{comb}}(L) = \left\{ \begin{array}{l} \text{embedded metric ribbon graphs of genus } g \text{ with } n \text{ faces} \\ \text{of lengths } L = (L_1, \dots, L_n) \text{ that are retract of } \Sigma \end{array} \right\} / \text{isotopy}, \quad (3.0.1)$$

parametrising marked metric ribbon graphs, was first considered in [Mono4] and its topology studied via the arc complex. Its quotient by the pure mapping class group is $\mathcal{M}_{g,n}^{\text{comb}}$, and we can equip $\mathcal{T}_{\Sigma}^{\text{comb}}$ with the pullback of Kontsevich 2-form, which we still denote $\omega_{\mathbb{K}}$.

Many aspects of the geometry of hyperbolic structures have an analogue for combinatorial structures, and the aim of this chapter is to develop such structures following this guiding principle. First of all, one can define the length $\ell_{\mathbb{G}}(\gamma)$ of a simple closed curve γ with respect to an embedded metric ribbon graph $\mathbb{G} \in \mathcal{T}_{\Sigma}^{\text{comb}}(L)$: realising the latter as a measured foliation $\mathcal{F}_{\mathbb{G}}$ on Σ , this is Thurston’s intersection pairing between $\mathcal{F}_{\mathbb{G}}$ and γ .

Thanks to the notion of length, one can parametrise the combinatorial Teichmüller space using a maximal set of simple closed curves, i.e. a pants decomposition. On a surface Σ of genus g with n boundary components, there are $3g - 3 + n$ length parameters that determine the combinatorial structure on each pair of pants, and there are $3g - 3 + n$ twist parameters that determine how the pairs of pants are glued together. All together, they constitute a combinatorial analogue of Fenchel–Nielsen coordinates (cf. Theorem 2.4.10), and they are known in the measured foliation and train track settings as Dehn–Thurston coordinates (cf. [FLP12, Exposé 6] and [PH92, Theorem 3.1.1]). The main difference with the hyperbolic world lies in the twist τ : for some values of τ , it is not possible to glue combinatorial structures; in the measured foliation description this corresponds to the creation of saddle connections that cannot be removed by Whitehead equivalences, and are not allowed in the combinatorial Teichmüller space. Metrically, they would in fact correspond to nodal surfaces. However, we show that the set of twists for which we cannot perform the gluing is a countable subset of \mathbb{R} with open dense complement. We give here a concise form of the stronger Theorem 3.4.5.

THEOREM 3.A (Combinatorial Fenchel–Nielsen coordinates).

1. (Adaptation of Dehn–Thurston–Penner arguments [Deh22; FLP12; PH92]). *Given a seamed pants decomposition for a bordered surface Σ of genus g with n boundary components, we have an open map*

$$\mathcal{T}_{\Sigma}^{\text{comb}}(L) \longrightarrow \mathbb{R}_+^{3g-3+n} \times \mathbb{R}^{3g-3+n}, \quad \mathbb{G} \longmapsto (\ell(\mathbb{G}), \tau(\mathbb{G})) \quad (3.0.2)$$

II. The combinatorial model of the moduli space of curves

that is a homeomorphism onto its image.

2. The image of $\mathcal{T}_\Sigma^{\text{comb}}(L)$ inside $\mathbb{R}_+^{3g-3+n} \times \mathbb{R}^{3g-3+n}$ has a complement of zero measure.

In the subsequent chapters we will analyse the symplectic geometry of $\mathcal{T}_\Sigma^{\text{comb}}(L)$, proving an analogue of Wolpert’s formula showing that the combinatorial Fenchel–Nielsen coordinates are Darboux for ω_K . Moreover, we will prove an analogue of Mirzakhani’s identity, and set up the geometric recursion to produce mapping class group invariant functions on the combinatorial Teichmüller space by a cut-and-paste approach (cf. Section 2.4). In particular the second point of the above theorem, which to the best of our knowledge is new, will be crucial in the integration of the combinatorial Mirzakhani identity, giving a geometric proof of Witten’s conjecture.

3.0.1 — RELATION WITH PREVIOUS WORKS AND OPEN QUESTIONS

As we already mentioned, the combinatorial model of the moduli space of curves was intensively studied in the last decades, and properties of the associated universal cover (the combinatorial Teichmüller space) was analysed in various directions via the arc complex [Mono4; Luo07; Mon09], measured foliations [FLP12] and train tracks [PH92].

The novelty of our work lies in the new perspective of considering a complete parallelism between the Teichmüller space of combinatorial and hyperbolic structures, which we tried to convey in this chapter. From this point of view, it is natural to ask for a combinatorial analogue of various result in Teichmüller theory, such as the length spectrum map $\mathcal{T}_\Sigma^{\text{comb}}(L) \hookrightarrow \mathbb{R}_+^{\delta_\Sigma}$, Fenchel–Nielsen coordinates, and the $(9g - 9 + 3n)$ -theorem.

The above perspective will persist in the following chapter, where we are going to prove the combinatorial analogue of results of Wolpert [Wol85], Mirzakhani–McShane [McS98; Miro7a; Miro7b] and Andersen–Borot–Orantin [ABO17].

We conclude with a natural question regarding the second point of Theorem 3.A.

QUESTION 3.B. *For any seamed pants decomposition of Σ , describe the image of $\mathcal{T}_\Sigma^{\text{comb}}(L)$ in $\mathbb{R}_+^{3g-3+n} \times \mathbb{R}^{3g-3+n}$.*

3.0.2 — ORGANISATION OF THE CHAPTER

The chapter is organised as follows.

- Section 3.1 contains no new results, but it introduces the basic concepts and relations necessary for the subsequent constructions. In particular, we recall the definition of the combinatorial moduli space and combinatorial Teichmüller space, and explain their relation with the ordinary spaces of hyperbolic structures, the arc complex, and the space of measured foliations.
- Section 3.2 introduces the notion of combinatorial length of simple closed curves with respect to a point on the combinatorial Teichmüller space. We discuss the geometry of embedded pairs of pants, and define the embedding $\mathcal{T}_\Sigma^{\text{comb}}(L) \hookrightarrow \mathbb{R}_+^{\delta_\Sigma}$ through the combinatorial length functions adapting ideas of Thurston [FLP12] (cf. Theorem 2.4.7 for the hyperbolic analogue).

- Section 3.3 discusses the notion of cutting an glued embedded metric ribbon graphs along simple closed curves. It turns out that the gluing process is possible for all but countably many configurations.
- Building on the previous discussions, in Section 3.4 we construct a combinatorial analogue of the Fenchel–Nielsen coordinates, as explained in Theorem 3.A.
- To conclude, in Section 3.5 we prove a combinatorial version of the $(9g - 9 + 3n)$ -theorem (cf. Theorem 2.4.9), following the spirit of a result of Thurston for measured foliations on closed surfaces.

3.1 — COMBINATORIAL SPACES

3.1.1 — COMBINATORIAL MODULI SPACES

A ribbon graph (sometimes called fat graph) is a graph equipped with a cyclic ordering on the edges incident to each vertex. To each ribbon graph, one can associate an oriented surface with boundary by replacing edges by thin ribbons and vertices by disks, and pasting rectangles to disks according to the chosen cyclic orders at the vertices.

DEFINITION 3.1.1. A *ribbon graph* is a triple $G = (\vec{E}, i, s)$ consisting of the following data.

- A set \vec{E} of oriented edges.
- A fixed-point-free involution $i: \vec{E} \rightarrow \vec{E}$, sending an oriented edge to its opposite. The set of i -orbits is denoted by E and describes the unoriented edges; the equivalence class of an oriented edge \vec{e} is simply denoted by e .
- A permutation $s: \vec{E} \rightarrow \vec{E}$. The set of s -orbits is denoted by V and describes the vertices.

Let $\phi = i \circ s^{-1}$; the set of ϕ -orbits is denoted by F and describes the faces (or boundaries) of G . A ribbon graph is connected if the underlying graph is.

The genus of a connected ribbon graph G , denoted by g , is defined by

$$2 - 2g = |V| - |E| + |F|. \quad (3.1.1)$$

If furthermore it has $n = |F|$ boundaries, it is said to be of type (g, n) . We call G reduced if all its vertices have valency ≥ 3 , and labelled if its boundaries are labelled $\partial_1 G, \dots, \partial_n G$. We define $\mathcal{R}_{g,n}$ to be the set of connected, reduced, labelled ribbon graphs of type (g, n) .

A *metric ribbon graph* \mathbf{G} consists of a ribbon graph $G = (\vec{E}, i, s)$, together with the assignment of $\ell_{\mathbf{G}}: E_G \rightarrow \mathbb{R}_+$. The *perimeter* of a boundary component $\partial_m G$ consisting of the cyclic sequence of edges (e_1, \dots, e_k) is defined by

$$\ell_{\mathbf{G}}(\partial_m G) = \sum_{i=1}^k \ell_{\mathbf{G}}(e_i). \quad (3.1.2)$$

If necessary, we will denote the sets of vertices, edges, and faces of a ribbon graph G with a subscript G .

II. The combinatorial model of the moduli space of curves

DEFINITION 3.1.2. An automorphism of a ribbon graph $G = (\vec{E}, i, s)$ is a permutation $\varphi: \vec{E} \rightarrow \vec{E}$ that commutes with i and s and acts trivially on the set of faces F . We denote by $\text{Aut}(G)$ the automorphism group of G . An automorphism of a metric ribbon graph \mathbf{G} is an automorphism of the underlying ribbon graph preserving $\ell_{\mathbf{G}}$. We denote by $\text{Aut}(\mathbf{G})$ the automorphism group of \mathbf{G} ; it is a subgroup of $\text{Aut}(G)$.

For any ribbon graph G , its automorphism group $\text{Aut}(G)$ acts on the space of metrics on G , by edge permutation. Given a point in $\mathbb{R}_+^{E_G}$, namely a metric ribbon graph \mathbf{G} , its stabiliser under the action of $\text{Aut}(G)$ is precisely $\text{Aut}(\mathbf{G})$. It is then natural to make the following definition.

DEFINITION 3.1.3. For $2g - 2 + n > 0$, the *combinatorial moduli space* $\mathcal{M}_{g,n}^{\text{comb}}$ is the orbicell-complex parametrising metric ribbon graphs of type (g, n) , i.e.

$$\mathcal{M}_{g,n}^{\text{comb}} = \bigcup_{G \in \mathcal{R}_{g,n}} \frac{\mathbb{R}_+^{E_G}}{\text{Aut}(G)}, \quad (3.1.3)$$

where the cells naturally glue together via edge degeneration: when an edge length goes to zero, the edge contracts to give a metric ribbon graph with fewer edges as discussed in [Kon92]. Define the perimeter map $p: \mathcal{M}_{g,n}^{\text{comb}} \rightarrow \mathbb{R}_+^n$ by setting

$$p(\mathbf{G}) = (\ell_{\mathbf{G}}(\partial_1 G), \dots, \ell_{\mathbf{G}}(\partial_n G)). \quad (3.1.4)$$

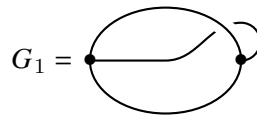
We also denote $\mathcal{M}_{g,n}^{\text{comb}}(L) = p^{-1}(L)$ for $L \in \mathbb{R}_+^n$.

From the above definition, it follows that the combinatorial moduli space $\mathcal{M}_{g,n}^{\text{comb}}$ has a natural real orbifold structure of dimension $6g - 6 + 3n$. As we remarked before, for a point \mathbf{G} in the moduli space its orbifold stabiliser is $\text{Aut}(\mathbf{G})$. Further, we have an orbicell decomposition given by the sets $\mathbb{R}_+^{E_G}/\text{Aut}(G)$ consisting of those metric ribbon graphs whose underlying ribbon graph is G . Notice that the dimension of such a cell is $|E_G|$, and in particular the top-dimensional cells are the ones associated to trivalent metric ribbon graphs, for which $|E_G| = 6g - 6 + 3n$.

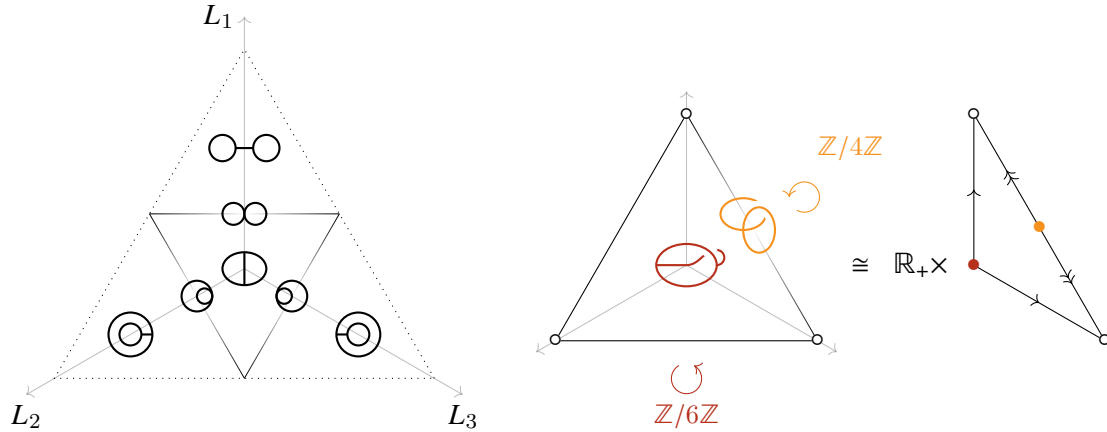
EXAMPLE 3.1.4. The moduli space $\mathcal{M}_{0,3}^{\text{comb}}$ is homeomorphic through the perimeter map to the open cone \mathbb{R}_+^3 , obtained as the union of seven cells corresponding to the seven ribbon graphs of type $(0, 3)$ (see Figure 3.1a). In this case there are no orbifold points and $\mathcal{M}_{0,3}^{\text{comb}}(L_1, L_2, L_3)$ is just a point.

A more complicated example is that of $\mathcal{M}_{1,1}^{\text{comb}}$. The space is obtained by gluing two orbicells, as depicted in Figure 3.1b.

- The first orbicell is \mathbb{R}_+^3 quotiented by $\mathbb{Z}/6\mathbb{Z}$, that is the automorphism group of the unique ribbon graph of type $(1, 1)$ with three edges:



All metric ribbon graphs \mathbf{G}_1 whose underlying graph is G_1 have $\text{Aut}(\mathbf{G}_1) = \mathbb{Z}/2\mathbb{Z}$ except for the metric ribbon graph where all three edges have the same length which has $\text{Aut}(\mathbf{G}_1) = \mathbb{Z}/6\mathbb{Z}$ instead.

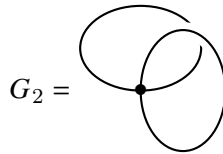


(a) The 7 cells composing $\mathcal{M}_{0,3}^{\text{comb}}$. The picture represents the cone \mathbb{R}_+^3 together with a slice $\{L_1 + L_2 + L_3 = \text{const}\}$. The dotted lines are not part of the space.

(b) The two orbicells composing $\mathcal{M}_{1,1}^{\text{comb}}$. On the right, the red point has stabiliser $\mathbb{Z}/6\mathbb{Z}$, the orange point has stabiliser $\mathbb{Z}/4\mathbb{Z}$, and all other points have stabiliser $\mathbb{Z}/2\mathbb{Z}$. The white points are not part of the space.

Figure 3.1: The combinatorial moduli spaces of type (0, 3) and (1, 1).

- The second orbicell is \mathbb{R}_+^2 quotiented by $\mathbb{Z}/4\mathbb{Z}$, the automorphism group of the unique ribbon graph of type (1, 1) with two edges:



All metric ribbon graphs G_2 whose underlying graph is G_2 have $\text{Aut}(G_2) = \mathbb{Z}/2\mathbb{Z}$, except for the metric ribbon graph where both edges have the same length which rather has $\text{Aut}(G_1) = \mathbb{Z}/4\mathbb{Z}$.

The main reason for considering $\mathcal{M}_{g,n}^{\text{comb}}(L)$ is the following result.

THEOREM 3.1.5 (Mumford, Thurston, Penner, Bowditch–Epstein). *The moduli spaces $\mathcal{M}_{g,n}$ and $\mathcal{M}_{g,n}^{\text{comb}}(L)$ are orbifold-homeomorphic.*

Historically, Mumford was the first to notice such stratification of the moduli space, building on ideas of Jenkins and Strebel [Jen57; Str67] and unpublished works of Thurston on the geometry of meromorphic quadratic differentials. A second approach uses hyperbolic geometry and is due to Penner and Bowditch–Epstein [Pen87; BE88]. In the next subsection we discuss the latter and its generalisation to bordered surfaces due to Luo and Mondello [Luo07; Mon09].

3.1.2 — COMBINATORIAL TEICHMÜLLER SPACES

One can think of points in the combinatorial Teichmüller space as points in the combinatorial moduli space together with a marking, the advantage being that there is now a well-defined

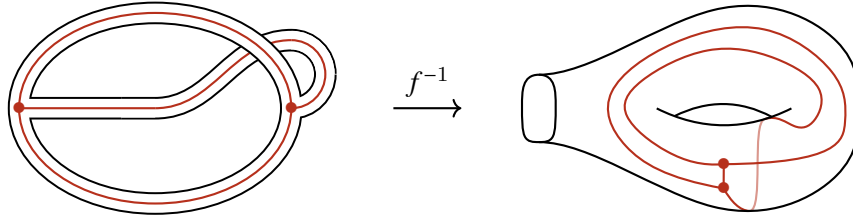


Figure 3.2: The geometric realisation of a ribbon graph and a marking on a one-holed torus via f .

notion of length of curves. We review its definition and its known relation with hyperbolic surfaces via the spine construction. Although the topological properties of such spaces have already been studied in the literature, the main focus has been on the proper arc complex description. Here we put forward an equivalent description via measured foliations which later facilitates our constructions.

PRIMARY DEFINITIONS

DEFINITION 3.1.6. The *geometric realisation of a metric ribbon graph* \mathbf{G} of type (g, n) is the oriented, compact, genus g surface with n boundary components

$$|\mathbf{G}| = \left\{ (u, t, \vec{e}) \in [-1, 1] \times \mathbb{R} \times \vec{E} \mid t \in [0, \ell_{\mathbf{G}}(e)] \right\} / \sim \quad (3.1.5)$$

where the equivalence relation \sim is given by

$$\begin{cases} (u, t, \vec{e}) \sim (u, \ell_{\mathbf{G}}(e) - t, i(\vec{e})) & \text{for } u \in [-1, 1], \vec{e} \in \vec{E}, t \in [0, \ell_{\mathbf{G}}(e)], \\ (u, \ell_{\mathbf{G}}(e), \vec{e}) \sim (-u, 0, s(\vec{e})) & \text{for } u \in [0, 1], \vec{e} \in \vec{E}. \end{cases} \quad (3.1.6)$$

We can picture the graph underlying G as a subset of $|\mathbf{G}|$, and the inclusion is a deformation retract; see the left-hand side of Figure 3.2 for an example.

Recall from Definition 2.4.1 the notion of bordered surface.

DEFINITION 3.1.7. A *combinatorial marking* of a bordered surface Σ is an ordered pair (\mathbf{G}, f) where \mathbf{G} is a metric ribbon graph and $f: \Sigma \rightarrow |\mathbf{G}|$ is a homeomorphism respecting the labellings of boundaries of \mathbf{G} and Σ . The *combinatorial Teichmüller space*¹ is defined as

$$\mathcal{T}_{\Sigma}^{\text{comb}} = \left\{ (\mathbf{G}, f) \mid (\mathbf{G}, f) \text{ is a combinatorial marking on } \Sigma \right\} / \sim. \quad (3.1.7)$$

Here we set $(\mathbf{G}, f) \sim (\mathbf{G}', f')$ if and only if there exists a metric ribbon graph isomorphism $\varphi: \mathbf{G} \rightarrow \mathbf{G}'$ such that $\varphi \circ f$ is isotopic to f' . We call such equivalence classes *combinatorial structures* and denote by $\mathbb{G} = [\mathbf{G}, f]$ the points in $\mathcal{T}_{\Sigma}^{\text{comb}}$.

¹Although the combinatorial moduli space was well-studied in the '80s by Mumford, Thurston, Harer, Penner and many others, the idea of consider *embedded* metric ribbon graphs up to isotopy appeared for the first in the work of Mondello [Mon04, Definition 3.2], to the best of our knowledge. In his work, he uses the identification of the set of embedded metric ribbon graphs with the proper arc complex, and in particular he does not use the term “combinatorial Teichmüller space”. The only instance we found of the name “combinatorial Teichmüller space” is [ACG11, Chapter XVIII]. Since our approach is completely parallel to the hyperbolic setting, we decided to use this terminology.

Notice that, for a fixed $\mathbb{G} \in \mathcal{T}_\Sigma^{\text{comb}}$ each boundary component $\partial_m \Sigma$ corresponds to a unique face of the embedded graph. Thus, we can talk about the length (with respect to \mathbb{G}) of the boundary $\partial_m \Sigma$, denoted by $\ell_{\mathbb{G}}(\partial_m \Sigma)$. In particular, we can define the perimeter map $p: \mathcal{T}_\Sigma^{\text{comb}} \rightarrow \mathbb{R}_+^n$ by setting

$$p(\mathbb{G}) = (\ell_{\mathbb{G}}(\partial_1 \Sigma), \dots, \ell_{\mathbb{G}}(\partial_n \Sigma)), \quad (3.1.8)$$

and denote $\mathcal{T}_\Sigma^{\text{comb}}(L) = p^{-1}(L)$.

We remark that, for a fixed representative (G, f) of $\mathbb{G} \in \mathcal{T}_\Sigma^{\text{comb}}$, we have a retraction from the surface to the geometric realisation of the graph underlying G . Thus, we can picture elements of $\mathcal{T}_\Sigma^{\text{comb}}$ as metric ribbon graphs embedded into Σ up to isotopy, such that the embedded graph is a retract of the surface (see Figure 3.2).

As recalled below, the combinatorial Teichmüller space can be endowed with a natural topology via the so-called proper arc complex that makes it into a cell complex. The cells, denoted $\mathfrak{Z}_{\Sigma, G}$, are indexed by isotopy classes of embedding of G into Σ , onto which Σ retracts, and they parametrise the metrics of G . The practical consequence of this topological discussion is that the edge lengths form a coordinate system in each cell. In particular, this makes it easier to check whether a function defined on $\mathcal{T}_\Sigma^{\text{comb}}$ is piecewise continuous, once it is expressed in terms of edge lengths.

The mapping class group of Σ (see Definition 2.4.2) naturally acts on $\mathcal{T}_\Sigma^{\text{comb}}$ by setting

$$[\phi] \cdot [G, f] = [G, f \circ \phi], \quad (3.1.9)$$

for $[\phi] \in \text{Mod}_\Sigma$ and $[G, f] \in \mathcal{T}_\Sigma^{\text{comb}}$. When Σ has type (g, n) , by forgetting the marking we have the natural isomorphism

$$\mathcal{M}_{g,n}^{\text{comb}}(L) \cong \mathcal{T}_\Sigma^{\text{comb}}(L) / \text{Mod}_\Sigma^\partial. \quad (3.1.10)$$

Here $\text{Mod}_\Sigma^\partial$ is the pure mapping class group of Σ (recall that the boundary components are labelled). We denote the quotient $\mathcal{T}_\Sigma^{\text{comb}}(L) / \text{Mod}_\Sigma^\partial$ by $\mathcal{M}_\Sigma^{\text{comb}}(L)$ when we want to refer to the actual surface Σ .

EXAMPLE 3.1.8. For a pair of pants P , the perimeter map gives the isomorphism $\mathcal{T}_P^{\text{comb}} \cong \mathbb{R}_+^3$. The pure mapping class group is trivial, so that $\mathcal{M}_{0,3}^{\text{comb}} \cong \mathbb{R}_+^3$, see Figure 3.3a.

For a torus T with one boundary component, $\mathcal{T}_T^{\text{comb}}$ is the union of infinitely many cells homeomorphic to \mathbb{R}_+^3 , glued together through infinitely many cells homeomorphic to \mathbb{R}_+^2 . In Figure 3.3b we presented some adjacent cells of $\mathcal{T}_T^{\text{comb}}$. In the quotient by Mod_T^∂ , all the 3- and 2-cells are identified, and we are left with a further action of $\mathbb{Z}/6\mathbb{Z}$ for the top-dimensional cell and an action of $\mathbb{Z}/4\mathbb{Z}$ for the codimension 1 cell. The result is the combinatorial moduli space $\mathcal{M}_{1,1}^{\text{comb}}$ described in Example 3.1.4.

RELATION WITH THE ARC COMPLEX

The combinatorial Teichmüller spaces have been endowed with a natural topology associated to a dual construction: the proper arc complex (see [Luo07; Mon09]).

DEFINITION 3.1.9. Fix a bordered surface Σ . Define the *arc complex* \mathcal{A}_Σ to be the simplicial complex whose vertices are non-trivial homotopy classes of proper embeddings² $\alpha: [0, 1] \hookrightarrow \Sigma$

²If X and Y are topological manifolds with boundaries, a continuous map $f: X \rightarrow Y$ is called a proper embedding if $f^{-1}(\partial Y) = \partial X$, and we use here the natural notion of homotopies among such.

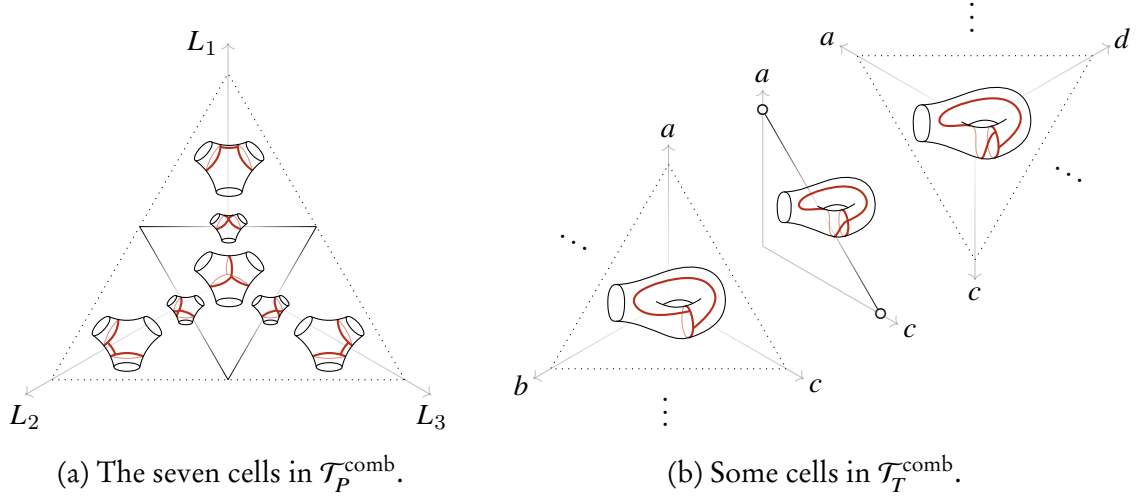


Figure 3.3: The combinatorial Teichmüller spaces of type (0, 3) and (1, 1).

with endpoints $\alpha(0), \alpha(1) \in \partial\Sigma$. A simplex in \mathcal{A}_Σ is a collection $\alpha = (\alpha_1, \dots, \alpha_k)$ of distinct vertices such that the arcs α_i admit representatives which do not intersect. The non-proper subcomplex $\mathcal{A}_\Sigma^\infty$ of \mathcal{A}_Σ consists of those simplices α such that one connected component of $\Sigma \setminus \bigcup_{i=1}^k \alpha_i$ is not simply connected. The simplices in $\mathcal{A}_\Sigma \setminus \mathcal{A}_\Sigma^\infty$ are called *proper*.

Consider the geometric realisation spaces $|\mathcal{A}_\Sigma|$ and $|\mathcal{A}_\Sigma^\infty|$. The geometric realisation of a simplicial complex comes with two natural topologies: the coherent topology, namely the finest topology that makes the realisation of all simplicial maps continuous, and the metric topology, for which every k -simplex is isometric to the standard simplex $\Delta^k \subset \mathbb{R}^{k+1}$ and every attachment map is a local isometry. The metric topology on $|\mathcal{A}_\Sigma|$ is coarser than the coherent one, but they agree where the complex is locally finite, and this is the case for $|\mathcal{A}_\Sigma| \setminus |\mathcal{A}_\Sigma^\infty|$.

Consider then the topological space $(|\mathcal{A}_\Sigma| \setminus |\mathcal{A}_\Sigma^\infty|) \times \mathbb{R}_+$, called *metrised arc complex*, whose points are of the form $x = \sum_{i=1}^k \ell_i \alpha_i$, where $\ell_i > 0$ and $(\alpha_1, \dots, \alpha_k)$ is a proper simplex. There is a bijection between the spaces $\mathcal{T}_\Sigma^{\text{comb}}$ and $(|\mathcal{A}_\Sigma| \setminus |\mathcal{A}_\Sigma^\infty|) \times \mathbb{R}_+$ defined as follows. Given a combinatorial structure \mathbb{G} , we define for each edge e the dual arc α_e connecting the two (possibly equal) boundary components on the two sides of e (see Figure 3.5). Thus, we define the map $\mathcal{T}_\Sigma^{\text{comb}} \rightarrow (|\mathcal{A}_\Sigma| \setminus |\mathcal{A}_\Sigma^\infty|) \times \mathbb{R}_+$ by setting $\mathbb{G} \mapsto \sum_{e \in E_{\mathbb{G}}} \ell_{\mathbb{G}}(e) \alpha_e$. The map is clearly invertible and we topologise the combinatorial Teichmüller space $\mathcal{T}_\Sigma^{\text{comb}}$ by pulling back the topology of the proper arc complex.

RELATION WITH ORDINARY TEICHMÜLLER SPACES

Recall from Section 2.4.1 that we assign to each bordered surface Σ of type (g, n) its Teichmüller space of hyperbolic structures

$$\mathcal{T}_\Sigma(L) = \left\{ (X, f) \left| \begin{array}{l} (X, f) \text{ is a hyperbolic marking on } \Sigma \\ \text{with labelled geodesic boundaries of lengths } L \end{array} \right. \right\} / \sim. \quad (3.1.11)$$

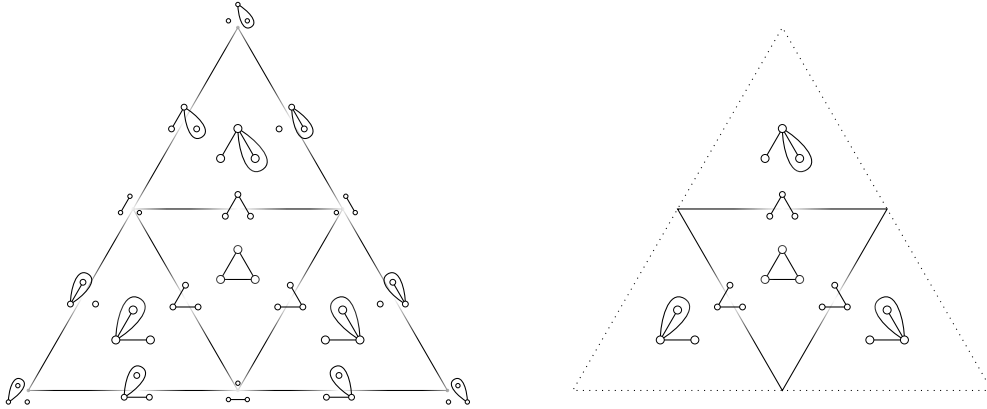


Figure 3.4: The simplicial structure of $|\mathcal{A}_P|$ and the set $|\mathcal{A}_P| \setminus |\mathcal{A}_P^\infty|$ associated to a pair of pants P . The latter is in natural bijection with the slice $\{\mathbb{G} \in \mathcal{T}_P^{\text{comb}} \mid \ell_{\mathbb{G}}(\partial_1 P) + \ell_{\mathbb{G}}(\partial_2 P) + \ell_{\mathbb{G}}(\partial_3 P) = 1\}$.

The quotient of $\mathcal{T}_\Sigma(L)$ by the pure mapping class group $\text{Mod}_\Sigma^\partial$ is orbifold-homeomorphic to the moduli space

$$\mathcal{M}_{g,n}(L) = \left\{ X \mid \begin{array}{l} X \text{ is a hyperbolic surface of type } (g, n) \\ \text{with labelled geodesic boundaries of lengths } L \end{array} \right\} / \sim. \quad (3.1.12)$$

Further, for each $L \in \mathbb{R}_+^n$, $\mathcal{M}_{g,n}(L)$ is orbifold-homeomorphic to the moduli space of curves $\mathcal{M}_{g,n}$. For later use, we review the description of a mapping class group equivariant homeomorphism between the combinatorial Teichmüller space and the ordinary one, due to Luo and Mondello [Luo07; Mon09]. It adapts to the bordered case the construction of Penner and Bowditch–Epstein [Pen87; BE88].

Let $\sigma \in \mathcal{T}_\Sigma(L)$ and fix a representative (X, f) . Define the valency $v_\sigma(q)$ of a point q in the interior of Σ as the number of shortest geodesics joining q to $\partial\Sigma$ that realise the distance $\text{dist}_\sigma(q, \partial\Sigma)$. Clearly $v_\sigma(q) \geq 1$. Define the loci $A_\sigma = \{q \in \Sigma \mid v_\sigma(q) = 2\}$ and $V_\sigma = \{q \in \Sigma \mid v_\sigma(q) \geq 3\}$. We have that A_σ is the disjoint union of simple, open geodesic arcs α_e indexed by $E = \pi_0(A_\sigma)$, the set of edges, and V_σ is a finite collection of points, the vertices. We define the *spine* $\text{sp}(\sigma)$ of σ as the 1-dimensional CW-complex embedded in Σ given by $V_\sigma \cup A_\sigma$.

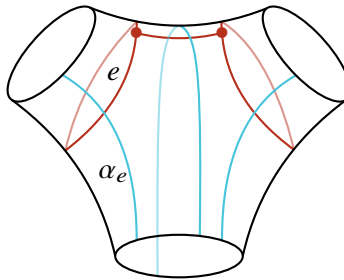


Figure 3.5: Example of duality between a combinatorial structure (red) and an arc system (light blue).

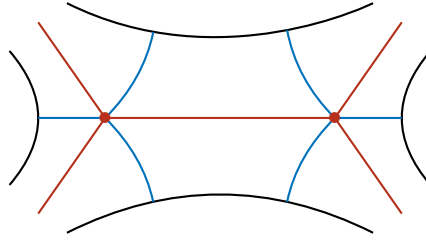


Figure 3.6: Example of the spine construction. In red, the spine $\text{sp}(\sigma)$. In blue, the ribs emanating from two vertices.

We can naturally assign to $\text{sp}(\sigma)$ a metric $\ell_{\text{sp}(\sigma)} : E \rightarrow \mathbb{R}_+$ in the following way. For each vertex q of $\text{sp}(\sigma)$, consider the $\nu_\sigma(q)$ shortest geodesics from q to the boundary – which are called *ribs*. Cutting Σ along its ribs yields a union of hexagons; the diagonal of each hexagon whose endpoints are the vertices of the spine corresponds to the edges of the spine (see Figure 3.6). We assign to it the length of the side of the hexagon which lies along the boundary of Σ ; there are two such sides, but they have the same length since the reflection with respect to the edge is a hyperbolic isometry of the hexagon. In this way, $\text{sp}(\sigma)$ induces a combinatorial marking on Σ and the perimeters of $\text{sp}(\sigma)$ correspond precisely to the hyperbolic lengths of the boundaries of Σ . Further, isotopy classes of hyperbolic markings correspond to isotopy classes of combinatorial markings. Thus, we are led to the following definition.

DEFINITION 3.1.10. There exists a well-defined map

$$\text{sp} : \mathcal{T}_\Sigma(L) \longrightarrow \mathcal{T}_\Sigma^{\text{comb}}(L), \quad (3.1.13)$$

called the *spine map*.

It is possible (although more difficult) to construct the inverse map and actually show that it is a homeomorphism, equivariant with respect to the action of $\text{Mod}_\Sigma^\partial$.

THEOREM 3.1.11 ([Luo07; Mon09]). *The spine map $\text{sp} : \mathcal{T}_\Sigma(L) \rightarrow \mathcal{T}_\Sigma^{\text{comb}}(L)$ is a homeomorphism, equivariant under the action of the pure mapping class group.*

We remark that in [Luo07; Mon09] the theorem is stated in terms of the proper arc complex, rather than the combinatorial Teichmüller space. As a direct consequence, we find that for each fixed $L \in \mathbb{R}_+^n$, there is an orbifold homeomorphism $\mathcal{M}_{g,n}(L) \cong \mathcal{M}_{g,n}^{\text{comb}}(L)$ and thus with $\mathcal{M}_{g,n}$ (Theorem 3.1.5).

RELATION WITH MEASURED FOLIATIONS

For a given metric ribbon graph, its geometric realisation is naturally endowed with a measured foliation, as we now explain. We refer to [FLP12, Section 5.1] for a complete discussion about measured foliations, but to be self-contained we recall here the basic definitions.

DEFINITION 3.1.12. Let Σ be a bordered surface, and \mathcal{F} a *foliation* on Σ with isolated singularities. A *transverse invariant measure* on \mathcal{F} is a measure μ defined on each arc transverse to the foliation, invariant under isotopy of arcs through transverse arcs whose endpoints remain in the same leaf. If the arc passes through a singularity, the transversality pertains to all points of the arc belonging to a regular leaf.

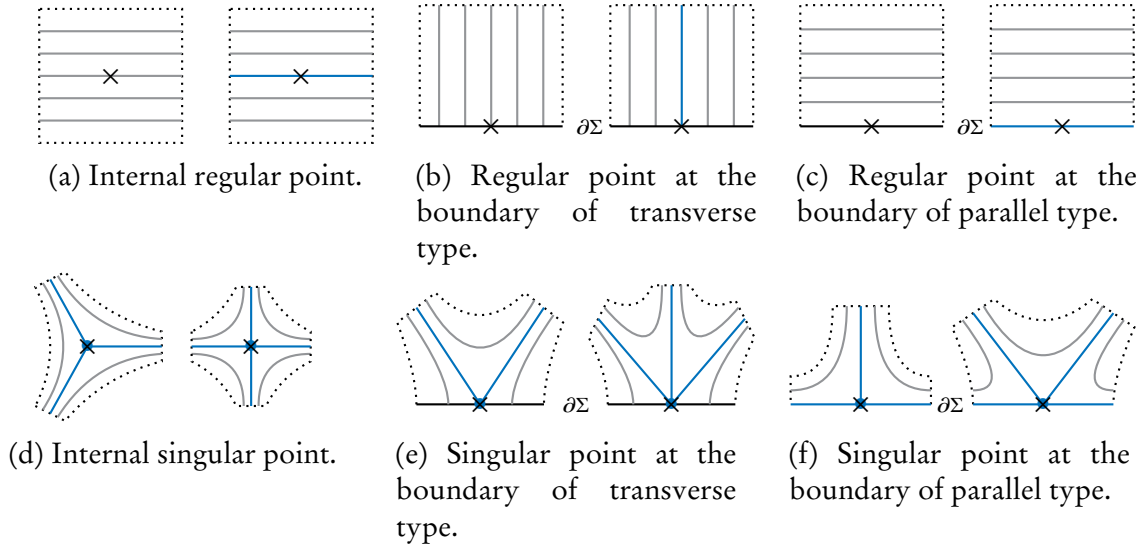


Figure 3.7: Possible models for points in a foliation. The singular leaves are depicted in blue, while smooth leaves in grey. Only singular points of low valency are depicted, but any higher valency is allowed.

In what follows, we also suppose that:

- the measure is regular with respect to the Lebesgue one: every regular point of Σ admits a smooth chart $U \ni (x, y)$ where the foliation is defined by foliation by constant y and the measure on each transverse arc is induced by $|dy|$;
- each point of Σ has a neighbourhood that is the domain of a chart isomorphically foliated as one of the models of Figure 3.7.

DEFINITION 3.1.13. We say that two measured foliations on Σ are *Whitehead equivalent* if they differ by isotopy or a Whitehead move, (see Figure 3.8). We denote by \mathcal{MF}_Σ^* the set of Whitehead equivalence classes of measured foliations on Σ .

We can now discuss the relation between foliations and metric ribbon graphs.

DEFINITION 3.1.14. Given a metric ribbon graph G of type (g, n) , the geometric realisation $|G|$ has a unique measured foliation (\mathcal{F}_G, μ_G) such that:

- the singularities of (\mathcal{F}_G, μ_G) are the vertices of the embedded graph,
- the measured foliation is transverse to the embedded graph,
- (\mathcal{F}_G, μ_G) on the hexagon around each edge $e \in E$ agrees with $|dt|$, where t is the natural coordinate on $[0, \ell_G(e)]$ as in Definition 3.1.6.

The singular leaves of the measured foliation cut $|G|$ into hexagons, each with two opposite edges consisting of arcs along the boundary of $|G|$, and the remaining four edges consisting of singular leaves (see Figure 3.9). The diagonals parallel to the boundary arcs are nothing but the edges of the embedded graph. In the description of Section 3.1.2, the singular leaves correspond to the ribs, and the singular points to the vertices of the spine. Notice that such

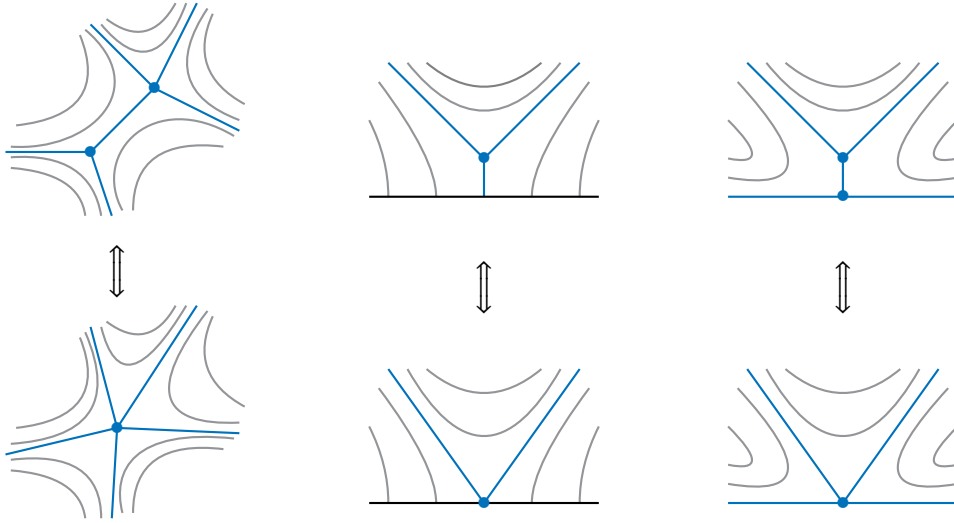


Figure 3.8: Whitehead moves.

a hexagon decomposition of $|\mathcal{G}|$, together with the assignment of a positive length to each diagonal, is sufficient to reconstruct the metric ribbon graph \mathcal{G} . For an element $\mathbb{G} = [\mathcal{G}, f]$ in $\mathcal{T}_\Sigma^{\text{comb}}$, we get a natural isotopy class of measured foliations $(\mathcal{F}_\mathbb{G}, \mu_\mathbb{G})$ on Σ by pushing $(\mathcal{F}_\mathbb{G}, \mu_\mathbb{G})$ forward along f . In the following, we omit the transverse measure $\mu_\mathbb{G}$ when there is no ambiguity.

The above construction defines a map

$$\mathcal{F}_* : \mathcal{T}_\Sigma^{\text{comb}} \longrightarrow \mathcal{MF}_\Sigma^*, \quad (3.1.14)$$

whose image is the set of classes of measured foliations admitting a representative, with respect to Whitehead equivalence, whose leaves are all transverse to the boundary of Σ (i.e. the local models Figures 3.7a, 3.7b, 3.7d and 3.7e are allowed, while Figures 3.7c and 3.7f are excluded), and there is no singular leaf connecting two singular points – such singular leaf is called a *saddle connection*. Moreover, the measured foliation can be used to give a hexagonal decomposition of the surface, and reconstruct the embedded metric ribbon graph. To summarise, we have the following lemma.

LEMMA 3.1.15. *The map \mathcal{F}_* is injective. Its image consists of measured foliations admitting a representative with respect to Whitehead equivalence whose leaves are all transverse to the boundary of the surface and with no saddle connections.*

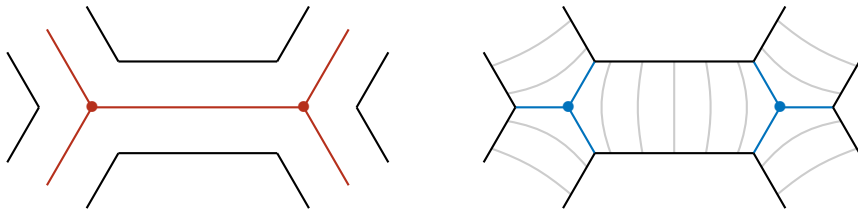


Figure 3.9: The geometric realisation of a metric ribbon graph (left) and the associated measured foliation. The edges of the embedded graph are depicted in red, the singular leaves of the associated foliation in blue.

In the remaining part of this chapter, we establish various topological properties of the combinatorial Teichmüller space. Many of these properties are established in [FLP12], but for a different space of measured foliations (namely, the completion of the set of multicurves), and in [PH92] in the language of train tracks. We include proofs and explicitly highlight results that are not simple adaptations from [FLP12; PH92]. The details of these proofs will in fact be used in later parts of the dissertation, specifically in the proof the combinatorial Mirzakhani–McShane identity (cf. Section 5.3).

3.2 — LENGTH FUNCTIONS

We introduce now combinatorial length functions for simple closed curves $\gamma \in \mathcal{S}_\Sigma$ (see Section 2.4.1 for notation and conventions), and show that combinatorial structures are (locally in $\mathcal{T}_\Sigma^{\text{comb}}$) completely determined by the knowledge of finitely many of these lengths. This gives an alternative description of the topology on $\mathcal{T}_\Sigma^{\text{comb}}$.

3.2.1 — COMBINATORIAL LENGTH FUNCTIONS

Consider a combinatorial structure $\mathbb{G} \in \mathcal{T}_\Sigma^{\text{comb}}$ and a simple closed curve γ in Σ (not necessarily essential). As discussed in the previous section, \mathbb{G} induces an isotopy class of measured foliations $(\mathcal{F}_\mathbb{G}, \mu_\mathbb{G})$ on Σ , so we have a well-defined length function $\ell_\mathbb{G}$ defined on \mathcal{S}_Σ (cf. [FLP12, Section 5.3], where the same quantity is denoted by $I(\mathcal{F}, \mu; \gamma)$)

$$\ell_\mathbb{G}(\gamma) = \inf_{\gamma_0} \sup_{\alpha_i} \left(\sum_{i=1}^k \mu_\mathbb{G}(\alpha_i) \right), \quad (3.2.1)$$

where the infimum is taken over all representatives γ_0 in the homotopy class γ , and the supremum is taken over all the possible ways of writing γ_0 as a sum of arcs $\alpha_1, \dots, \alpha_k$, mutually disjoint and transverse to $\mathcal{F}_\mathbb{G}$.

It can be shown [FLP12, Proposition 5.7] that the infimum is reached by quasitransverse representatives of γ .

DEFINITION 3.2.1. We say that a curve γ_0 is *quasitransverse* to a foliation \mathcal{F} if each connected component of γ_0 minus the singularities of \mathcal{F} is either a leaf or is transverse to \mathcal{F} . Further, in a neighbourhood of a singularity, we require that no transverse arc lies in a sector adjacent to an arc contained in a leaf, and that consecutive transverse arcs lie in distinct sectors.

In terms of the embedded metric ribbon graph, we have the following effective way to compute the length $\ell_\mathbb{G}(\gamma)$: consider the (unique) representative γ_0 of γ that has been homotoped to the embedded graph and is non-backtracking (cf. [Hato2, Section 1.A] for uniqueness of non-backtracking representatives). The length of the curve can now be computed as the sum of the lengths of the edges visited by the curve, since such γ_0 is a quasitransverse representative, perhaps after performing a sequence of Whitehead moves in a small disc neighbourhood of the vertices of \mathbb{G} , making a measured foliation $\mathcal{F}'_\mathbb{G}$ Whitehead equivalent to $\mathcal{F}_\mathbb{G}$ and thus representing the same point in $\mathcal{MF}_\Sigma^\star$. One can also conclude from [FLP12, Proposition 5.9] that the length is always positive.

Notice that, if the curve is homotopic to one of the boundary components, the notion of boundary length described before (Equation (3.1.8)) agrees with this more general definition.

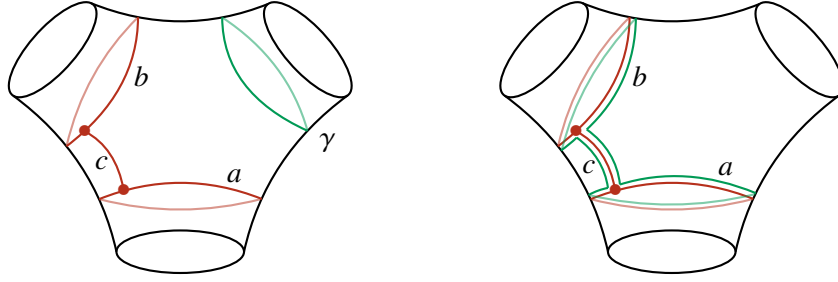


Figure 3.10: The length of a simple closed curve γ with respect to a combinatorial structure \mathbb{G} on a pair of pants. We have $\ell_{\mathbb{G}}(\gamma) = a + c + b + c = a + b + 2c$.

Moreover, the assignment $\mathbb{G} \mapsto \ell_{\mathbb{G}}(\gamma)$ is continuous on $\mathcal{T}_{\Sigma}^{\text{comb}}$, as it is a sum of edge lengths on the closure of each open cell.

Remark 3.2.2. The notion of length naturally extends to multicurves $c \in \mathcal{M}_{\Sigma}$ by adding the length of components. Using Equation (3.2.1), we can also define the length of homotopy classes (relative to $\partial\Sigma$) of arcs between boundaries. This is again a continuous assignment, but it can take zero values.

In Sections 3.2.2 and 3.2.3 we prove the following result regarding the combinatorial length spectrum, which is the analogue of [FLP12, Theorem 1.3] for measured foliations on a closed surface, and the analogue of Theorem 2.4.7 for the usual Teichmüller space of hyperbolic structures.

THEOREM 3.2.3. *Let us equip $\mathbb{R}_{+}^{\delta\Sigma}$ with the product topology. The combinatorial length of simple closed curves gives a map*

$$\ell_*: \mathcal{T}_{\Sigma}^{\text{comb}}(L) \longrightarrow \mathbb{R}_{+}^{\delta\Sigma} \quad (3.2.2)$$

which is a homeomorphism onto its image.

COROLLARY 3.2.4. *The topology on $\mathcal{T}_{\Sigma}^{\text{comb}}(L)$ defined via the arc-complex and the initial topology induced by the combinatorial length map $\ell_*: \mathcal{T}_{\Sigma}^{\text{comb}}(L) \rightarrow \mathbb{R}_{+}^{\delta\Sigma}$ coincide.*

Here we prove the theorem by a direct construction of the inverse map from the image of ℓ_* , via the study of lengths of curves and their relation with embedded pairs of pants. These computations are used again in Chapter 5.

3.2.2 — EMBEDDED PAIRS OF PANTS

We are interested in embedded pairs of pants and the way the lengths of their boundary components (with respect to a combinatorial structure) determine the combinatorial structure itself.

DEFINITION 3.2.5. When Σ is a bordered surface of type (g, n) such that $2g - 2 + n > 1$, and $m_0 \in \{1, \dots, n\}$, we let $\mathcal{P}_{\Sigma, m_0}$ be the set of homotopy classes of embeddings $\varphi: P \hookrightarrow \Sigma$ of pairs of pants P such that

- $\varphi(\partial_1 P) = \partial_{m_0} \Sigma$,
- $\overline{\Sigma \setminus \varphi(P)}$ (where the closure is taken in Σ) is stable,

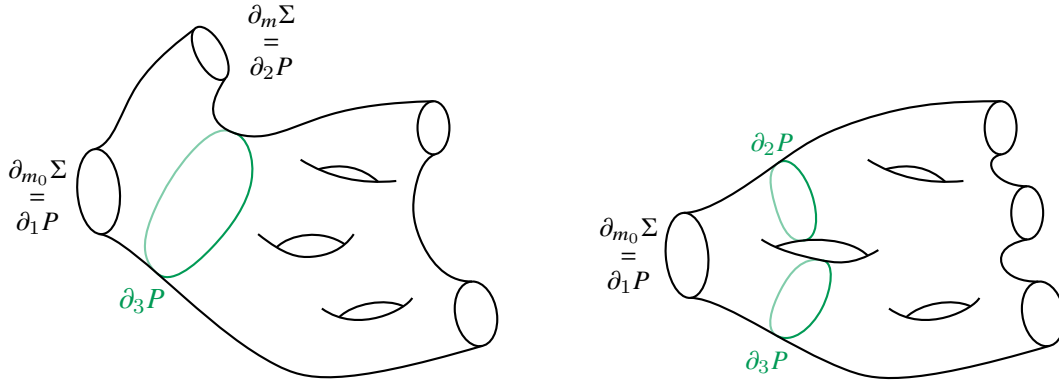


Figure 3.11: Two embedded pairs of pants in $\mathcal{B}_{\Sigma, m_0, m}$ (left) and $\mathcal{C}_{\Sigma, m_0}$.

- if $\varphi(\partial_i P) = \partial_m \Sigma$ for some $m \neq m_0$, then $i = 2$.

By abuse of notation, we denote by P the homotopy class of $\varphi: P \hookrightarrow \Sigma$, and we often identify P with its embedding in Σ . Moreover, we define the following sets of homotopy classes of embedded pairs of pants (see Figure 3.11)

$$\begin{aligned} \mathcal{B}_{\Sigma, m_0, m} &= \{ P \in \mathcal{P}_{\Sigma, m_0} \mid \partial_2 P = \partial_m \Sigma \text{ for some } m \neq m_0 \}, \\ \mathcal{C}_{\Sigma, m_0} &= \{ P \in \mathcal{P}_{\Sigma, m_0} \mid \partial_i P \subset \Sigma^\circ \text{ for } i = 2, 3 \}. \end{aligned} \quad (3.2.3)$$

We have a partition

$$\mathcal{P}_{\Sigma, m_0} = \left(\bigsqcup_{m \neq m_0} \mathcal{B}_{\Sigma, m_0, m} \right) \sqcup \mathcal{C}_{\Sigma, m_0}. \quad (3.2.4)$$

Notice that, for $m_0 = 1$, these are the sets of homotopy classes of pairs of pants appearing in Mirzakhani's recursion, or more generally in the definition of geometric recursion amplitudes in the hyperbolic setting (cf. Section 2.4).

As briefly mentioned in the explanations of Mirzakhani's argument, these pairs of pants can be described alternatively in terms of arcs between boundaries, which we recall here for the convenience of the reader.

DEFINITION 3.2.6. Denote by $\mathfrak{A}_{\Sigma, m_0}$ the set of non-trivial homotopy classes of proper embeddings $\alpha: [0, 1] \hookrightarrow \Sigma$ with $\alpha(0) \in \partial_{m_0} \Sigma$ and $\alpha(1) \in \partial \Sigma$. By abuse of notation, we denote by α the homotopy class of $\alpha: [0, 1] \hookrightarrow \Sigma$. There is a partition

$$\mathfrak{A}_{\Sigma, m_0} = \left(\bigsqcup_{m \neq m_0} \mathfrak{B}_{\Sigma, m_0, m} \right) \sqcup \mathfrak{C}_{\Sigma, m_0}, \quad (3.2.5)$$

where the elements $\alpha \in \mathfrak{B}_{\Sigma, m_0, m}$ are those with $\alpha(1) \in \partial_m \Sigma$, and the elements $\alpha \in \mathfrak{C}_{\Sigma, m_0}$ are those with $\alpha(1) \in \partial_{m_0} \Sigma$.

We define a surjective map

$$Q_{m_0}: \mathfrak{A}_{\Sigma, m_0} \longrightarrow \mathcal{P}_{\Sigma, m_0} \quad (3.2.6)$$

by assigning to an arc α the pairs of pants P whose boundaries are formed by the boundary of a closed tubular neighbourhood of α and the boundary components it connects (Figure 3.12).

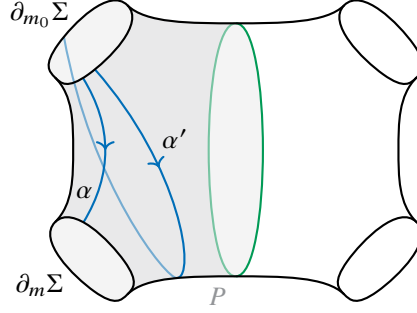


Figure 3.12: Arcs and pairs of pants: $Q_{m_0}(\alpha) = Q_{m_0}(\alpha') = Q_{m_0}(-\alpha') = P$.

The boundaries of P are labelled as follows. We always set $\partial_1 P = \partial_{m_0} \Sigma$. Then if $\varphi(\partial_i P) = \partial_m \Sigma$ for some $m \neq m_0$, then $i = 2$ and $\partial_3 P$ is determined; otherwise, we define $\partial_2 P$ (resp. $\partial_3 P$) to be the boundary component on the left-hand side (resp. right-hand side) of the curve α oriented from 0 to 1.

Remark 3.2.7. The map Q_{m_0} to $\mathfrak{A}_{\Sigma, m_0}$ is not injective. More precisely, $Q_{m_0}^{-1}(P)$ contains a single element when $P \in \mathcal{C}_{\Sigma, m_0}$, while it contains three elements when $P \in \mathcal{B}_{\Sigma, m_0, m}$ for any $m \neq m_0$. Indeed any given $P \in \mathcal{B}_{\Sigma, m_0, m}$ can be obtained by an arc α from $\partial_{m_0} \Sigma$ to $\partial_m \Sigma$, but also by an arc α' from $\partial_{m_0} \Sigma$ to itself and its inverse $-\alpha'$ (Figure 3.12). Notice that in this case α is the only arc in $\mathfrak{B}_{\Sigma, m_0, m}$, while $\alpha', -\alpha' \in \mathfrak{C}_{\Sigma, m_0}$.

We introduce the notion of small pairs of pants, which play a role in the rest of the paper.

DEFINITION 3.2.8. Let $\mathbb{G} \in \mathcal{T}_{\Sigma}^{\text{comb}}$. We say that $P \in \mathcal{P}_{\Sigma, m_0}$ is \mathbb{G} -small if

$$\ell_{\mathbb{G}}(\partial P \cap \partial \Sigma) \geq \ell_{\mathbb{G}}(\partial P \cap \Sigma^{\circ}). \quad (3.2.7)$$

The definition does not depend on the representative chosen for P . If the inequalities are strict, then we say the pair of pants is *strictly* \mathbb{G} -small.

We can rewrite the above definition by differentiating two possible cases: P is \mathbb{G} -small if

- $P \in \mathcal{B}_{\Sigma, m_0, m}$ implies $\ell_{\mathbb{G}}(\partial_3 P) \leq \ell_{\mathbb{G}}(\partial_1 P) + \ell_{\mathbb{G}}(\partial_2 P)$,
- $P \in \mathcal{C}_{\Sigma, m_0}$ implies $\ell_{\mathbb{G}}(\partial_2 P) + \ell_{\mathbb{G}}(\partial_3 P) \leq \ell_{\mathbb{G}}(\partial_1 P)$.

In other words, the internal boundary component(s) are small compared to the external one(s). We characterise small pairs of pants in terms of the corresponding arcs as follows.

LEMMA 3.2.9. Let $P \in \mathcal{P}_{\Sigma, m_0}$.

- $P \in \mathcal{B}_{\Sigma, m_0, m}$ is \mathbb{G} -small if and only if for the unique $\alpha \in Q_{m_0}^{-1}(P) \cap \mathfrak{B}_{\Sigma, m_0, m}$ we have $\ell_{\mathbb{G}}(\alpha) = 0$,
- $P \in \mathcal{C}_{\Sigma, m_0}$ is \mathbb{G} -small if and only if for the unique $\alpha \in Q_{m_0}^{-1}(P)$ we have $\ell_{\mathbb{G}}(\alpha) = 0$.

Proof. Let $\alpha \in \mathfrak{B}_{\Sigma, m_0, m}$ for $m \neq m_0$ and assume that $\ell_{\mathbb{G}}(\alpha) > 0$. It is then uniquely represented by some non-backtracking edgepath α in \mathbb{G} with initial and final vertices adjacent to $\partial_{m_0} \Sigma$ and $\partial_m \Sigma$. Then, $\partial_3 Q_{m_0}(\alpha)$ can be homotoped to a non-backtracking edgepath consisting in

travelling along α , then going around $\partial_m \Sigma$, then travelling backwards along α , and going around $\partial_{m_0} \Sigma$. This implies that

$$\ell_{\mathbb{G}}(\partial_{m_0} \Sigma) + \ell_{\mathbb{G}}(\partial_m \Sigma) < \ell_{\mathbb{G}}(\partial_3 Q_{m_0}(\alpha)),$$

hence $Q_{m_0}(\alpha)$ is not \mathbb{G} -small, and we conclude by the contrapositive. A similar argument works when $\alpha \in \mathfrak{C}_{\Sigma, m_0}$. \square

Remark 3.2.10. Notice that α can only have length zero with respect to \mathbb{G} if it has a quasitransverse representative given by an oriented non-singular leaf in the foliation associated to \mathbb{G} . There are finitely many homotopy classes of oriented leaves relative to the boundary, and they are in bijection with the oriented edges of the metric ribbon graph. Since $\mathfrak{A}_{\Sigma, m_0}$ surjects onto $\mathcal{P}_{\Sigma, m_0}$ for Σ of type (g, n) and $\mathbb{G} \in \mathcal{T}_{\Sigma}^{\text{comb}}$, there are at most $2(6g - 6 + 3n)$ \mathbb{G} -small pairs of pants in $\mathcal{P}_{\Sigma, m_0}$.

Small pairs of pants can be characterised in terms of the support of two functions that play an important role in the combinatorial Mirzakhani–McShane identity. Let $[x]_+ = \max\{x, 0\}$ and consider the functions on $\mathcal{T}_P^{\text{comb}} \cong \mathbb{R}_+^3$ defined by

$$\begin{aligned} B^K(L, L', \ell) &= \frac{1}{2L} \left([L - L' - \ell]_+ - [-L + L' - \ell]_+ + [L + L' - \ell]_+ \right), \\ C^K(L, \ell, \ell') &= \frac{1}{L} [L - \ell - \ell']_+. \end{aligned} \quad (3.2.8)$$

It is easy to check that these functions only take non-negative values. They encode aspects of the geometry of combinatorial pairs of pants, as described in the following lemma, and are the combinatorial analogs of the functions (2.4.20) introduced by Mirzakhani in [Miro7a] in the hyperbolic context. Moreover, they can be obtained as a limit of Mirzakhani’s functions in the large boundary limit, cf. Section 6.3.

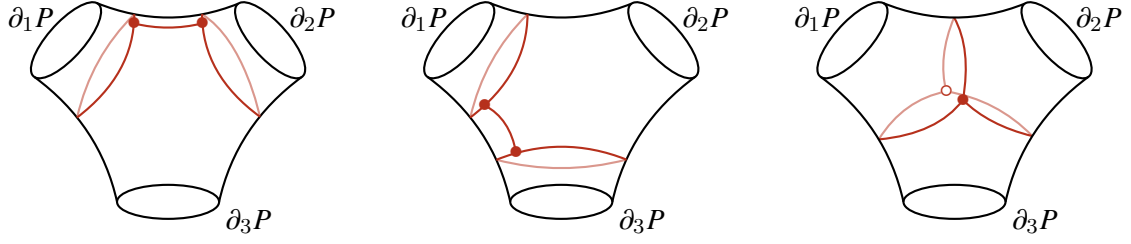
LEMMA 3.2.11.

- The function B^K associates to $(\ell_{\mathbb{G}}(\partial_1 P), \ell_{\mathbb{G}}(\partial_2 P), \ell_{\mathbb{G}}(\partial_3 P)) \in \mathbb{R}_+^3 \cong \mathcal{T}_P^{\text{comb}}$ the fraction of the $\partial_1 P$ that is not common with $\partial_3 P$ (once retracted to the graph).
- The function C^K associates to a point $\vec{\ell}_{\mathbb{G}}(\partial P) \in \mathbb{R}_+^3 \cong \mathcal{T}_P^{\text{comb}}$ the fraction of $\partial_1 P$ that is not common with $\partial_2 P \cup \partial_3 P$.

Proof. The result follows from direct computations in the closure of each of the four open cells of $\mathcal{T}_P^{\text{comb}}$. The various inequalities that define the cells are used to simplify B^K and C^K . Consider for example the leftmost pair of pants in Figure 3.13: we have $\ell_{\mathbb{G}}(\partial_3 P) \geq \ell_{\mathbb{G}}(\partial_1 P) + \ell_{\mathbb{G}}(\partial_2 P)$ so that $B^K(L_1, L_2, \ell) = 0$, and the portion of $\partial_1 P$ that is not common with $\partial_3 P$ is zero (all of $\partial_1 P$ intersects with $\partial_3 P$). Similarly, in the case of the central pair of pants of Figure 3.13 we find $\ell_{\mathbb{G}}(\partial_2 P) \geq \ell_{\mathbb{G}}(\partial_1 P) + \ell_{\mathbb{G}}(\partial_3 P)$, so that

$$B^K(L_1, L_2, \ell) = \frac{1}{2\ell_{\mathbb{G}}(\partial_1 P)} \left(\ell_{\mathbb{G}}(\partial_1 P) + \ell_{\mathbb{G}}(\partial_2 P) - \ell_{\mathbb{G}}(\partial_3 P) - (-\ell_{\mathbb{G}}(\partial_1 P) + \ell_{\mathbb{G}}(\partial_2 P) - \ell_{\mathbb{G}}(\partial_3 P)) \right) = 1,$$

and indeed all of $\partial_1 P$ intersects with $\partial_2 P$. Finally, in the rightmost pair of pants of Figure 3.13 we find $\ell_{\mathbb{G}}(\partial_i P) \leq \ell_{\mathbb{G}}(\partial_j P) + \ell_{\mathbb{G}}(\partial_k P)$ for all $i, j, k \in \{1, 2, 3\}$, which is the condition to be in this cell. We also see that the length of the edge adjacent to $\partial_1 P$ and $\partial_2 P$ is given by


 Figure 3.13: Three different cells in $\mathcal{T}_P^{\text{comb}}$.

$\frac{1}{2} (\ell_{\mathbb{G}}(\partial_1 P) + \ell_{\mathbb{G}}(\partial_2 P) - \ell_{\mathbb{G}}(\partial_3 P))$ and therefore its fraction of the first boundary is given by exactly

$$B^K(L_1, L_2, \ell) = \frac{1}{2\ell_{\mathbb{G}}(\partial_1 P)} (\ell_{\mathbb{G}}(\partial_1 P) + \ell_{\mathbb{G}}(\partial_2 P) - \ell_{\mathbb{G}}(\partial_3 P)).$$

The computation in other cells is similar, and analogously for C^K . \square

Notice that if B^K or C^K are non-zero, then there is at least an edge with a corresponding arc, that defines the pair of pants passing through the edge transversely, and therefore of zero length. As an immediate consequence this or simply by considering the support of B^K and C^K , we deduce the following characterisation of small pairs of pants.

COROLLARY 3.2.12. *Let $\mathbb{G} \in \mathcal{T}_{\Sigma}^{\text{comb}}$:*

- $P \in \mathcal{B}_{\Sigma, m_0, m}$ is \mathbb{G} -small if and only if $B^K(\vec{\ell}_{\mathbb{G}}(\partial P)) > 0$,
- $P \in \mathcal{C}_{\Sigma, m_0}$ is \mathbb{G} -small if and only if $C^K(\vec{\ell}_{\mathbb{G}}(\partial P)) > 0$.

Here $\vec{\ell}_{\mathbb{G}}(\partial P)$ is the ordered triple of combinatorial lengths of the boundary components of P .

3.2.3 — A PARTIAL INVERSE OF THE COMBINATORIAL LENGTH SPECTRUM MAP

In this paragraph we exhibit an inverse on the image of the combinatorial length map of Theorem 3.2.3. We first observe that, given $\mathbb{G} = [G, f] \in \mathcal{T}_{\Sigma}^{\text{comb}}$ and an oriented edge \vec{e} of the embedded graph, we can take the oriented dual arc $\alpha_{\vec{e}}$: its starting point (resp. ending point) lies on the component of $\partial\Sigma$ adjacent to $f(\vec{e})$ on the right (resp. left) and intersects the embedded graph exactly once through this edge. Note that the two boundary components adjacent to \vec{e} may coincide. We thus obtain a map

$$\vec{E}_G \rightarrow \mathfrak{A}_{\Sigma}^{\text{all}} = \bigsqcup_{m_0=1}^n \mathfrak{A}_{\Sigma, m_0}. \quad (3.2.9)$$

Composing with Q_{m_0} , we can associate to each oriented edge of \mathbb{G} a class of embedded pair of pants in $\mathcal{P}_{\Sigma}^{\text{all}} = \bigsqcup_{m_0=1}^n \mathcal{P}_{\Sigma, m_0}$.

LEMMA 3.2.13. *Let e be an edge in $\mathbb{G} \in \mathcal{T}_{\Sigma}^{\text{comb}}$ and fix an arbitrary orientation \vec{e} .*

- *If e is adjacent to $\partial_{m_1}\Sigma \neq \partial_{m_2}\Sigma$, let $P_1 \in \mathcal{B}_{\Sigma, m_1, m_2}$ and $P_2 \in \mathcal{B}_{\Sigma, m_2, m_1}$ be the pairs of pants corresponding to \vec{e} and $i(\vec{e})$ respectively. Then*

$$\begin{aligned} \ell_{\mathbb{G}}(e) &= \ell_{\mathbb{G}}(\partial_{m_1}\Sigma) \left(B^K(\vec{\ell}_{\mathbb{G}}(\partial P_1)) - C^K(\vec{\ell}_{\mathbb{G}}(\partial P_1)) \right) \\ &= \ell_{\mathbb{G}}(\partial_{m_2}\Sigma) \left(B^K(\vec{\ell}_{\mathbb{G}}(\partial P_2)) - C^K(\vec{\ell}_{\mathbb{G}}(\partial P_2)) \right). \end{aligned} \quad (3.2.10)$$

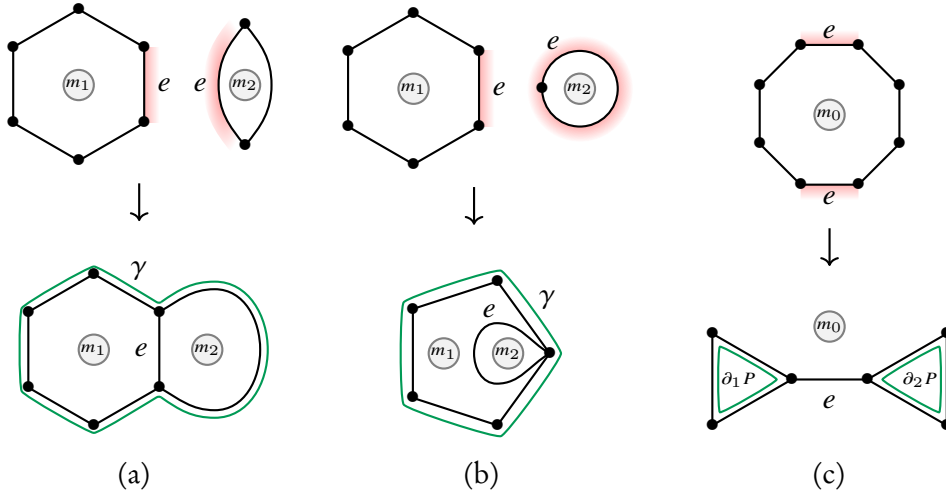


Figure 3.14: The three cases examined in Lemma 3.2.13.

- If e is adjacent to $\partial_{m_0}\Sigma$ on both sides, let $P \in \mathcal{C}_{\Sigma, m_0}$ be the pair of pants corresponding to \vec{e} or $i(\vec{e})$ (the two pairs of pants coincide). Then

$$\ell_{\mathbb{G}}(e) = \frac{1}{2}\ell_{\mathbb{G}}(\partial_{m_0}\Sigma) C^{\mathbb{K}}(\vec{\ell}_{\mathbb{G}}(\partial P)). \quad (3.2.11)$$

Proof. Given a boundary component, we can represent the edges around it by a polygon where some edges and vertices are identified. The sequence of edges around the boundary is non-backtracking.

- If e is adjacent to $\partial_{m_1}\Sigma \neq \partial_{m_2}\Sigma$, we have two polygons around $\partial_{m_1}\Sigma$ and $\partial_{m_2}\Sigma$. When neither of these are 1-gons, then we can represent $\gamma = \partial_3 P_1 = \partial_3 P_2$ as in Figure 3.14a. Due to the absence of bivalent vertices, this is again non-backtracking. We therefore have $\ell_{\mathbb{G}}(\gamma) = \ell_{\mathbb{G}}(\partial_{m_1}\Sigma) + \ell_{\mathbb{G}}(\partial_{m_2}\Sigma) - 2\ell_{\mathbb{G}}(e)$. Notice that we also have $|\ell_{\mathbb{G}}(\partial_{m_1}\Sigma) - \ell_{\mathbb{G}}(\partial_{m_2}\Sigma)| < \ell_{\mathbb{G}}(\gamma)$. Therefore, comparing the expression for $\ell_{\mathbb{G}}(e)$ with the expression in the statement, we see they agree. Now suppose without loss of generality that m_2 is a 1-gon. This implies that m_1 is not a 1-gon, as gluing two 1-gons together would produce a cylinder. It is also clear that $\ell_{\mathbb{G}}(e) = \ell_{\mathbb{G}}(\partial_{m_2}\Sigma)$. We can represent $\gamma = \partial_3 P_1 = \partial_3 P_2$ by Figure 3.14b. This implies that $\ell_{\mathbb{G}}(\partial_{m_1}\Sigma) \geq \ell_{\mathbb{G}}(\partial_{m_2}\Sigma) + \ell_{\mathbb{G}}(\gamma)$ and therefore, using this to calculate the expression in the statement, we see that they agree.
- If e is adjacent to $\partial_{m_0}\Sigma$ on both sides, we have one polygon around $\partial_{m_0}\Sigma$. We can then represent $\partial_2 P$ and $\partial_3 P$ as in Figure 3.14c. In absence of bivalent vertices, backtracking cannot occur around either side of e . Therefore we have $\ell_{\mathbb{G}}(\partial_{m_0}\Sigma) = \ell_{\mathbb{G}}(\partial_2 P) + \ell_{\mathbb{G}}(\partial_3 P) + 2\ell_{\mathbb{G}}(e)$. Comparing the expression for $\ell_{\mathbb{G}}(e)$ with the expression in the statement, we see they agree.

□

This lemma shows that we can reconstruct edge lengths from lengths of simple closed curves, which in turn proves that Theorem 3.2.3 holds when restricted to the closure of a cell. To fully reconstruct the ribbon graph (i.e. determine in which cell we are) from just the lengths

II. The combinatorial model of the moduli space of curves

of simple closed curves, we use the characterisations of small pairs of pants and their relation to edges in the embedded graph.

Proof of Theorem 3.2.3. We can now exhibit a global inverse on the image of the combinatorial length spectrum map $\ell_* : \mathcal{T}_\Sigma^{\text{comb}}(L) \rightarrow \mathbb{R}_+^{\delta_\Sigma}$. We first consider the following composition:

$$\begin{array}{ccc} \mathbb{R}_+^{\delta_\Sigma} & \rightarrow & (\mathbb{R}_+^3)^{\mathcal{P}_\Sigma^{\text{all}}} \rightarrow \mathbb{R}_{\geq 0}^{\mathfrak{A}_\Sigma^{\text{all}}} \\ \lambda & \longmapsto & l_\lambda. \end{array}$$

The first arrow associates to a length functional $\lambda \in \mathbb{R}_+^{\delta_\Sigma}$ the functional on $\mathcal{P}_\Sigma^{\text{all}}$ defined by $P \mapsto \vec{\lambda}(\partial P)$, where $\vec{\lambda}(\partial P)$ is the ordered triples of boundary lengths (here we define $\lambda(\partial_i \Sigma) = L_i$). The second arrow associates to a functional $\vec{\lambda}$ on $\mathcal{P}_\Sigma^{\text{all}}$ a functional on $\mathfrak{A}_\Sigma^{\text{all}}$ given by

$$\alpha \mapsto \begin{cases} \lambda_1(P_\alpha) \left(B^K(\vec{\lambda}(P_\alpha)) - C^K(\vec{\lambda}(P_\alpha)) \right) & \text{if } \alpha \in \mathfrak{B}_{\Sigma, m_0, m} \\ \frac{1}{2} \lambda_1(P_\alpha) C^K(\vec{\lambda}(P_\alpha)) & \text{if } \alpha \in \mathfrak{C}_{\Sigma, m_0} \end{cases}$$

where $P_\alpha = Q_{m_0}(\alpha)$ and λ_1 is the first component of the triple $\vec{\lambda}$. Note that these are exactly the expressions appearing in Lemma 3.2.13.

Consider now the restriction to $\text{Im}(\ell_*)$. If $\lambda = \ell_*(\mathbb{G})$ for some $\mathbb{G} \in \mathcal{T}_\Sigma^{\text{comb}}(L)$, we know from Lemma 3.2.9 and Remark 3.2.10 that l_λ , as a functional on $\mathfrak{A}_\Sigma^{\text{all}}$, is supported on the complement of those oriented arcs α homotopic to non-singular oriented leaves in the foliation associated to \mathbb{G} . These are in bijection with the oriented edges of \mathbb{G} . Forgetting about the orientation, consider the finite collection $(\alpha_1, \dots, \alpha_k)$ of all such arcs, choosing representatives that are pairwise non-intersecting in Σ . This defines a proper simplex in the arc complex \mathfrak{A}_Σ (see Section 3.1.2), and taking the dual we obtain a ribbon graph G with an embedding into Σ . We can equip it with a metric, assigning the length $l_\lambda(\alpha_i)$ to the edge dual to α_i . This represents a point in $\mathcal{T}_\Sigma^{\text{comb}}(L)$, so that we have a map

$$l_* : \text{Im}(\ell_*) \longrightarrow \mathcal{T}_\Sigma^{\text{comb}}(L).$$

By construction, $l_* \circ \ell_* = \text{id}$. Thus, ℓ_* is injective.

The map ℓ_* is clearly continuous, since the lengths of simple closed curves are linear combinations of lengths of edges. The inverse map l_* is also continuous on the ℓ_* -image of each cell, since we realised the edge lengths as piecewise linear (and thus continuous) functions of the length of locally finitely many simple closed curves. This completes the proof. \square

3.3 — CUTTING AND GLUING

Before describing the Dehn–Thurston coordinates, which in this setting we will call combinatorial Fenchel–Nielsen coordinates, we need the notion of cutting a combinatorial structure along an essential simple closed curve, and the reciprocal notion of gluing combinatorial structures along boundary components of the same length. In the context of measured foliations, this was already considered by Thurston. However, the main difference with [FLP12] is in the gluing: the combinatorial Teichmüller space does not contain measured foliations with

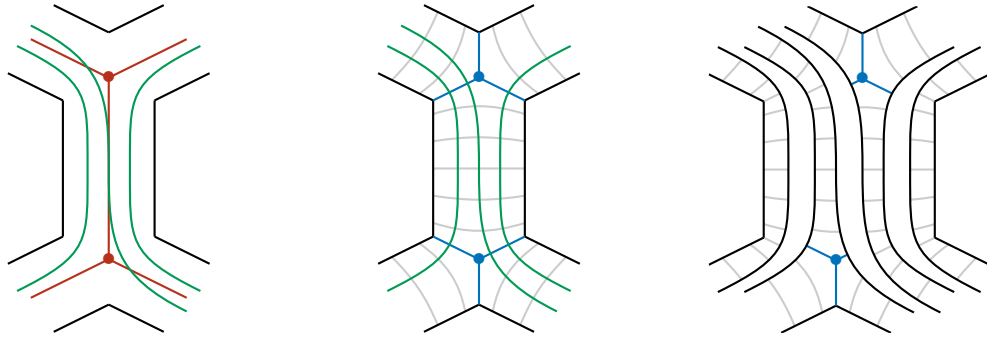


Figure 3.15: Cutting/gluing algorithm: the combinatorial structure \mathbb{G} (in red) pictured with the singular leaves (in blue), and the curve γ (in green).

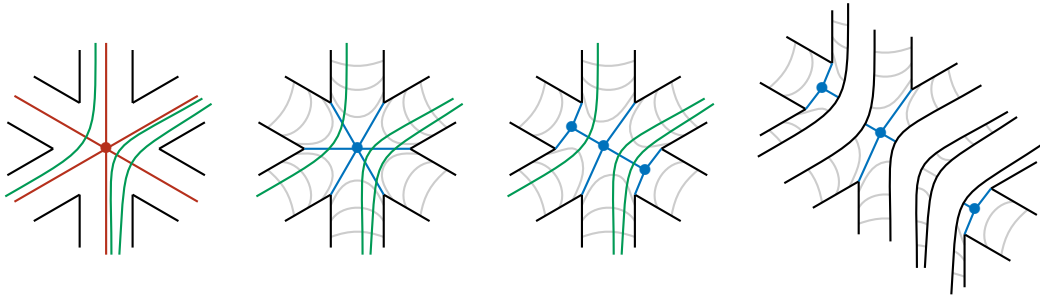


Figure 3.16: Cutting/gluing algorithm for vertices of higher valency. Two Whitehead moves are performed.

saddle connections, but saddle connections can be created from the gluing process. The main result of this section, Proposition 3.3.4, is the analysis of these “pathological gluings”, which turn out to occur only for a negligible set of twists.

CUTTING. Consider a bordered surface Σ , fix $\mathbb{G} \in \mathcal{T}_{\Sigma}^{\text{comb}}$ and γ an essential simple closed curve. We want to define a combinatorial structure on the surface Σ_{γ} obtained by cutting Σ along a chosen representative of γ . To this end, choose a representative (G, f) , so that we have an induced structure of measured foliation on Σ . If necessary, perform a minimal sequence of local Whitehead moves in small disc neighbourhoods of the vertices, in such a way that γ is transversal to the resulting foliation. We then restrict the measured foliation to Σ_{γ} , which is induced from a unique metric ribbon graph G_{γ} with an embedding which up to isotopy does not depend on the choices made. This defines a combinatorial structure $\mathbb{G}_{\gamma} \in \mathcal{T}_{\Sigma_{\gamma}}^{\text{comb}}$.

Cutting also makes sense when γ is a primitive multicurve, and it is equivalent to cutting along each component of γ in an arbitrary order. Note that the lengths of edges after cutting are again linear combinations of the edge lengths which agree on the closure of the open cells. This shows that the cutting, viewed as a map $\mathcal{T}_{\Sigma}^{\text{comb}} \rightarrow \mathcal{T}_{\Sigma_{\gamma}}^{\text{comb}}$, is continuous. See Figure 3.15 and Figure 3.16 for a local illustration of the cutting, and Appendix A for some global examples.

GLUING. Consider a bordered surface Σ , possibly disconnected, with a choice of two boundary components γ_{-} and γ_{+} . Let $\mathbb{G} \in \mathcal{T}_{\Sigma}^{\text{comb}}$ be such that $\ell_{\mathbb{G}}(\gamma_{-}) = \ell_{\mathbb{G}}(\gamma_{+})$. We want to define a combinatorial structure on the surface obtained by topologically gluing γ_{-} and γ_{+} . Fix a representative (G, f) , so that we have an induced structure of a measured foliation \mathcal{F} on Σ . First, we observe that once we pick a point p_{-} on γ_{-} , there is a unique action of \mathbb{R} on γ_{-} which

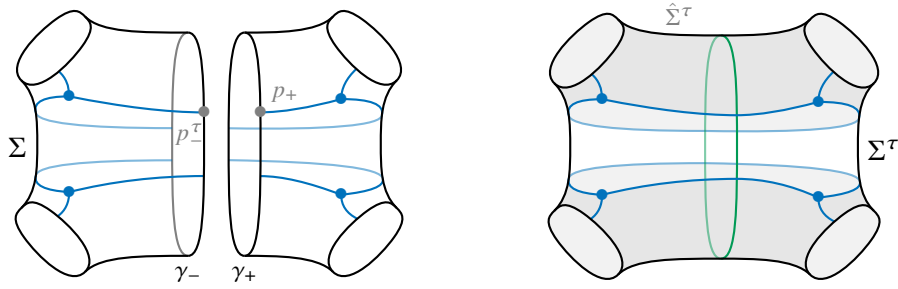


Figure 3.17: A glued measured foliation that is not dual to a combinatorial structure. Notice in grey $\hat{\Sigma}^\tau$, that is properly contained in Σ^τ , and the presence of saddle connections on the boundary of $\hat{\Sigma}^\tau$ that cannot be removed by Whitehead moves.

preserves the induced measure and orientation on γ_- . We let p_-^τ be the image of p_- under the action of $\tau \in \mathbb{R}$. Pick now a point p_+ on γ_+ , and identify γ_- with γ_+ in a measure preserving way, such that p_-^τ is identified with p_+ in an orientation reversing way. This means that we have a unique measured foliation \mathcal{F}^τ induced on the glued surface, which we denote Σ^τ .

3.3.1 — ADMISSIBLE GLUINGS

What is not clear from the above construction is whether the measured foliation \mathcal{F}^τ is associated to a combinatorial structure on Σ^τ . If this is true, we call such τ an *admissible twist*. We refer to Figure 3.15 and Figure 3.16 – read from right to left – for a local illustration of the gluing, Appendix A for some global examples, and Figure 3.17 for an example of \mathcal{F}^τ that is not associated to a combinatorial structure.

PROPOSITION 3.3.1. *There exists a unique metric ribbon graph \mathbf{G}^τ and a unique isotopy class of marking $f: \Sigma^\tau \rightarrow |\mathbf{G}^\tau|$ such that the measured foliation induced on Σ^τ agrees with \mathcal{F}^τ if and only if \mathcal{F}^τ has a representative without saddle connections, i.e. no leaf between two singularities.*

Proof. Perform a maximal sequence of Whitehead moves, i.e. that reduces the connected components of the compact singular leaves to a graph with one vertex. Let $\Lambda(\mathcal{F}^\tau)$ be the set of leaves of \mathcal{F}^τ , and define $\hat{\Sigma}^\tau = \{ \lambda \in \Lambda(\mathcal{F}^\tau) \mid \lambda \cap \partial \Sigma^\tau \neq \emptyset \}$ (cf. Figure 3.17). Then from Poincaré recurrence [FLP12, Theorem 5.2] this is nothing but the union of all leaves which go from boundary to boundary together with the finitely many leaves which connect the boundary to a singular point of \mathcal{F}^τ (i.e. no leaves starting from the boundary spiral in the surface). If $\hat{\Sigma}^\tau = \Sigma^\tau$, then \mathcal{F}^τ has no closed singular leaves and we see that the singular leaves of \mathcal{F}^τ split the surface into hexagons, which in turn determines \mathbf{G}^τ uniquely and its marking up to isotopy. If not, choose a good atlas for \mathcal{F}^τ as defined in [FLP12, Section 5.2], and observe that the complement of the singular leaves in $\hat{\Sigma}^\tau$ is a finite disjoint union of squares S_i , each with a non-singular foliation transverse to two open arcs of the boundary of Σ^τ , running between endpoints of the singular leaves in $\hat{\Sigma}^\tau$, and such that $\bar{S}_i - S_i \subset \Sigma^\tau$ are made up of a finite number of compact singular leaves of \mathcal{F}^τ . But since we are assuming $\hat{\Sigma}^\tau \neq \Sigma^\tau$, there must be at least one of these which connects two singular points in the interior of Σ^τ . As we took a representative of \mathcal{F}^τ with one singular point for each connected component of the compact singular leaves, we see that this implies there must be a cycle of singular leaves which cannot define a combinatorial structure. \square

We observe that, even though the foliation associated to \mathbb{G} has no saddle connections, they may occur for \mathcal{F}^τ as a result of the gluing process. However, Proposition 3.3.1 together with the next result imply that this is generically not the case, and that it is never the case for \mathbb{G} -strictly small pairs of pants.

LEMMA 3.3.2. *Let \mathbb{G} be such that every vertex around γ_- has exactly one singular leaf reaching γ_- and no singular leaf reaching γ_+ . Then all twists $\tau \in \mathbb{R}$ are admissible. The same statement holds if we exchange γ_- and γ_+ .*

Proof. Under such hypotheses, all smooth leaves starting from γ_- end at a boundary component of Σ that is neither γ_- nor γ_+ . Therefore for every twist τ , we glue the singular leaves to leaves of γ_+ which immediately reach a boundary component of $\partial\Sigma^\tau$. Gluing the singular leaves of γ_- can result in the creation of a leaf which returns back to γ_+ again. This however corresponds two gluing to leaves of γ_+ which immediately reach a boundary component of $\partial\Sigma^\tau$. \square

COROLLARY 3.3.3. *Let Σ be a bordered surface of Euler characteristic $\chi_\Sigma < -1$, take $P \in \mathcal{P}_{\Sigma, m_0}$ for some $m_0 \in \{1, \dots, n\}$, $\mathbb{G} \in \mathcal{T}_\Sigma^{\text{comb}}$, and consider the operation of cutting along $\partial P \cap \Sigma^\circ$, twisting and gluing back. If P is \mathbb{G} -strictly small, then any twist $\tau \in \mathbb{R}^{\pi_0(\partial P \cap \Sigma^\circ)}$ is admissible. The same is true if $\Sigma = T$ and we self-glue after twisting the pair of pants obtained from T by cutting along $\gamma \in \mathcal{S}_T$.*

PROPOSITION 3.3.4. *For $\mathbb{G} \in \mathcal{T}_\Sigma^{\text{comb}}$ with $l = \ell_{\mathbb{G}}(\gamma_-) = \ell_{\mathbb{G}}(\gamma_+)$ the set of admissible twists is an open dense subset of \mathbb{R} with countable complement.*

We need the result of the following lemma before proving the proposition. Let $S^- = \{\ell_1^-, \dots, \ell_M^-\}$ and $S^+ = \{\ell_1^+, \dots, \ell_N^+\}$ be the finite sets of lengths of edges in \mathbb{G} into which γ_- and γ_+ decompose into respectively. Without loss of generality, assume that p_- and p_+ are contained in singular leaves of \mathbb{G} .

LEMMA 3.3.5. *If we choose the points p_\pm such that $\tau = 0$ identifies two singular leaves, then for $\tau \notin \text{span}_{\mathbb{Q}}(S^- \cup S^+)$, \mathcal{F}^τ has a Whitehead representative with no singular leaf between two singularities.*

Proof. Denote by γ the curve in Σ^τ which is the image of γ_\pm , and identify $\gamma \sim \mathbb{R}/l\mathbb{Z}$ where $l = \ell_{\mathbb{G}}(\gamma_-) = \ell_{\mathbb{G}}(\gamma_+)$ and 0 corresponds to a singular leaf on the γ_+ side. Consider a point $p \in \gamma$. If we follow along the leaf passing through p (in either the γ_+ side or the γ_- side of the glued surface) until it gets back to γ at a point p' , we find that there exists some $R \in \text{span}_{\mathbb{Z}}(S^- \cup S^+)$ for each of the following cases, such that

- $p' = -p + R$, for a leaf going from γ_+ to γ_+ ;
- $p' = -p + 2\tau + R$, for a leaf going from γ_- to γ_- ;
- $p' = p + \tau + R$, for a leaf going from γ_+ to γ_- ;
- $p' = p - \tau + R$, for a leaf going from γ_- to γ_+ .

Indeed, we firstly notice that all singular leaves on the γ_+ side are identified as some points in $\text{span}_{\mathbb{Z}}(S^+)$, while on the γ_- side they are identified as some points in $\tau + \text{span}_{\mathbb{Z}}(S^-)$.

Suppose first that p' is obtained from p by following a leaf going from γ_+ to γ_+ (see left of Figure 3.18). We notice that p is given by $p = R_0 + a$, where $R_0 \in \text{span}_{\mathbb{Z}}(S^+)$ is the distance

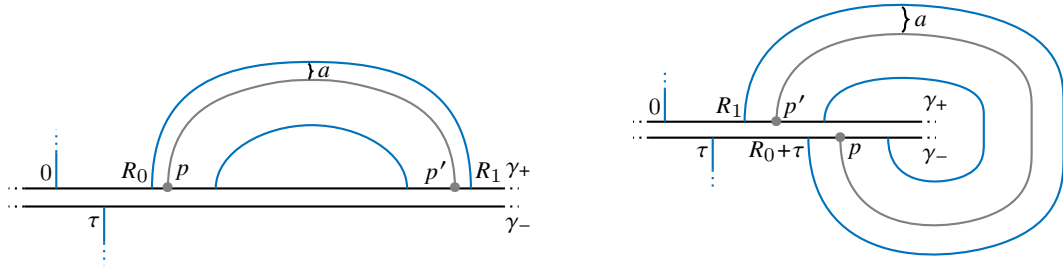


Figure 3.18: Examples of leaf dynamics, induced on γ by the foliation \mathcal{F} . The singular leaves before gluing are depicted in blue, the leaf connecting p and p' in grey (here it is depicted as a smooth leaf, i.e. $a > 0$). In between the two fully depicted singular leaves there is a strip of smooth leaves all homotopic to each other.

from the chosen singular leaf at 0 to the singular leaf just before p on the γ_+ side, and $a \geq 0$. Then, following the leaf, we find that the singular leaf just before p becomes the singular leaf just after p' , and $p' = R_1 - a = -p + (R_0 + R_1)$ where $R_1 \in \text{span}_{\mathbb{Z}}(S^+)$ is the distance from the chosen singular leaf at 0 to the leaf just after p' (following the orientation of γ_+). In particular, we obtain the claim with $R = R_0 + R_1 \in \text{span}_{\mathbb{Z}}(S^+)$.

Similarly, suppose now that p' is obtained from p by following a leaf going from γ_- to γ_+ (see right of Figure 3.18). Now we have $p = R_0 + \tau + a$, where $R_0 \in \text{span}_{\mathbb{Z}}(S^-)$ and $a > 0$ (here the singular leaf just before p on the γ_- side is at distance $R_0 + \tau$ from the chosen singular leaf at 0). Then, following the leaf, we find that the singular leaf at $R_0 + \tau$ is identified with a singular leaf at distance $R_1 \in \text{span}_{\mathbb{Z}}(S^+)$ from the chosen singular leaf at 0. Therefore $p' = R_1 + a = p - \tau + (R_0 - R_1)$. Thus, the claim with $R = R_0 - R_1 \in \text{span}_{\mathbb{Z}}(S^- \cup S^+)$.

The other cases follow similarly. Now, if p is a point at a singular leaf on the γ_{\pm} side, we see by induction that, after gluing, the singular leaf passes through γ at some other points of the form $\pm(p \mp n\tau) + R$ for some $n \in \mathbb{Z}_+$ and $R \in \text{span}_{\mathbb{Z}}(S^- \cup S^+)$. This implies that, if \mathbb{G}^{τ} has two singular points connected by a leaf, then a non-zero integral multiple of τ is contained in $\text{span}_{\mathbb{Z}}(S^- \cup S^+)$, or equivalently $\tau \in \text{span}_{\mathbb{Q}}(S^- \cup S^+)$. \square

Proof of Proposition 3.3.4. We first show that the set of admissible twists is an open subset of \mathbb{R} . Consider $\tau \in \mathbb{R}$ an admissible twist, and denote by \mathbb{G}^{τ} the associated combinatorial structure. For each edge e of \mathbb{G}^{τ} , let $n_{\gamma}(e)$ be the number of times which γ travels through the edge e . If we take τ' such that $|\tau' - \tau| < \epsilon$, then we see that the distance between the two singularities of the foliation changes by at most $\epsilon n_{\gamma}(e)$ (cf. Figure 3.19). This also holds at the boundary of the cells, when we have vertices of higher valency whose original distance would be zero. Therefore, if we choose $\epsilon > 0$ smaller than

$$\min_e \frac{\ell_{\mathbb{G}}(e)}{n_{\gamma}(e)}$$

where e runs over the edges of \mathbb{G}^{τ} visited by γ , then all singularities stay at the same or at a positive distance from each other. These lengths are realised by curves homotopic to the original length realising curve. As a consequence, $\mathcal{F}^{\tau'}$ cannot admit a cycle of singular leaves connecting singularities, and thus τ' is an admissible twist. The countable complement property follows from Lemma 3.3.5. \square

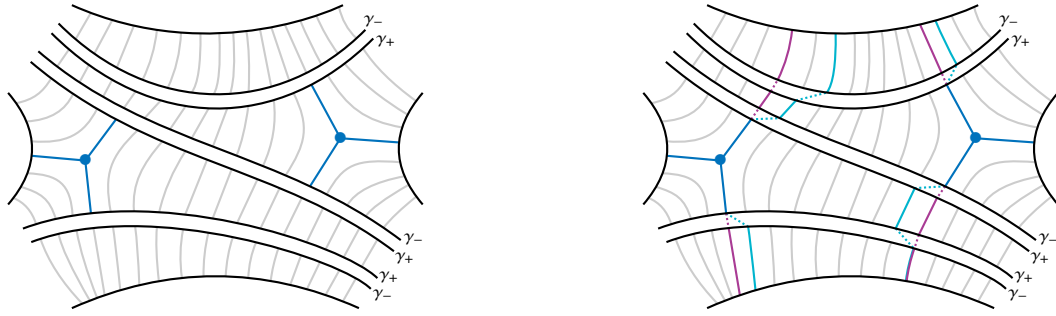


Figure 3.19: A combinatorial structure \mathbb{G} glued to \mathbb{G}^τ and $\mathbb{G}^{\tau'}$ for $|\tau - \tau'|$ small. The singular leaves of \mathbb{G} are shown in blue. After gluing, they are prolonged with the purple leaves in \mathbb{G}^τ , and with the light blue leaves in $\mathbb{G}^{\tau'}$. The dotted lines indicate the identification of γ_- and γ_+ for τ and τ' .

We remark that the set of non admissible twists can have accumulation points, and its set of accumulation points can be non isolated. However what is crucial for the next chapter is that the non-admissible twists form a measure-zero set in \mathbb{R} .

3.4 — COMBINATORIAL FENCHEL–NIELSEN COORDINATES

With the notions of cutting and gluing in the combinatorial spaces defined, we have the key tools to adapt the definition of Dehn–Thurston coordinates to our framework. The main difference with the hyperbolic setting is that the image of such a coordinate system does not cover the whole codomain, and this is due to the fact that certain (rare) values of the twists are forbidden.

3.4.1 — SEAMS AND PANTS DECOMPOSITIONS

Firstly, we need a technical ingredient, the pants seams, that allows us to define a canonical way of gluing pairs of pants. The following definition can be found in [FLP12, Section 6.3].

DEFINITION 3.4.1. Consider a combinatorial marking (\mathcal{G}, f) on a pair of pants P , with associated foliation \mathcal{F} . Define the *combinatorial seam* connecting two distinct boundary components γ and γ' of P to be the quasitransverse arc connecting γ and γ' , as indicated in Figure 3.20. In the cases 3.20c–3.20g, the seams are smooth leaves, located at exactly the same distance from the adjacent singular leaves.

We remark that the combinatorial seam realises the minimum among lengths of all essential arcs connecting one boundary component to another (the length can be zero). For a point $(L_1, L_2, L_3) \in \mathbb{R}_+^3 \cong \mathcal{T}_P^{\text{comb}}$, the length of the seam connecting $\partial_1 P$ and $\partial_2 P$ is given by the formula

$$\ell_{\text{comb}}(L_1, L_2, L_3) = \left\lfloor \frac{L_3 - L_1 - L_2}{2} \right\rfloor_+, \quad (3.4.1)$$

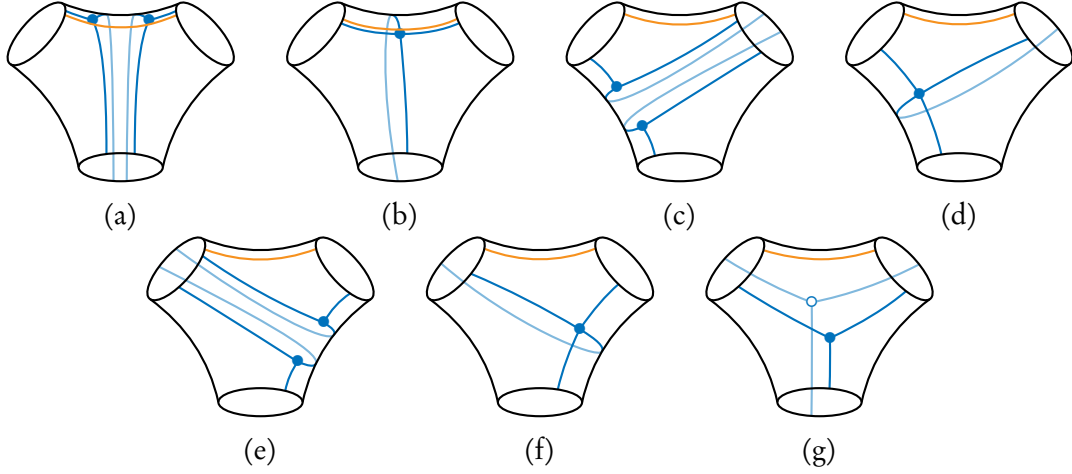


Figure 3.20: Combinatorial seams (in orange) on each cell of $\mathcal{T}_P^{\text{comb}}$. The singular leaves are depicted in blue.

while the length of a seam connecting $\partial_1 P$ to itself is given by

$$\begin{aligned} \ell_{\text{comb}}(L_1, L_2, L_3) &= \left\lfloor \frac{L_2 + L_3 - L_1}{2} \right\rfloor_+ + \left\lfloor \frac{L_2 - L_1 - L_3}{2} \right\rfloor_+ + \left\lfloor \frac{L_3 - L_1 - L_2}{2} \right\rfloor_+ \\ &= \max \left\{ \frac{L_2 + L_3 - L_1}{2}, L_2 - L_1, L_3 - L_1, 0 \right\} \end{aligned} \quad (3.4.2)$$

Notice that for a hyperbolic marking (X, φ) on P , there exists a notion of hyperbolic seam connecting γ and γ' , that is the shortest geodesic arc connecting the boundary components γ and γ' . On the other hand, we can consider the combinatorial marking (G, f) on P associated to (X, φ) defined through the spine map of Definition 3.1.10. The next elementary lemma, of which we omit the proof, shows that the hyperbolic and combinatorial seams are the same arcs.

LEMMA 3.4.2. *Consider a hyperbolic marking (X, φ) on P , and the associated combinatorial marking (G, f) . Through their aforementioned identification, the hyperbolic and combinatorial seams connecting two boundary components of P coincide.*

Remark 3.4.3. In the hyperbolic case, for a point $(L_1, L_2, L_3) \in \mathbb{R}_+^3 \cong \mathcal{T}_P$, the hyperbolic length of the seam connecting $\partial_1 P$ and $\partial_2 P$ is given by the formula

$$\cosh(\ell_{\text{hyp}}(L_1, L_2, L_3)) = \frac{\cosh(\frac{L_3}{2})}{\sinh(\frac{L_1}{2}) \sinh(\frac{L_2}{2})} + \coth(\frac{L_1}{2}) \coth(\frac{L_2}{2}), \quad (3.4.3)$$

while the hyperbolic length of a seam connecting $\partial_1 P$ to itself is given by

$$\begin{aligned} \cosh^2 \left(\frac{\ell_{\text{hyp}}(L_1, L_2, L_3)}{2} \right) \sinh^2 \left(\frac{L_1}{2} \right) &= \\ &= \cosh^2 \left(\frac{L_1}{2} \right) + \cosh^2 \left(\frac{L_2}{2} \right) + \cosh^2 \left(\frac{L_3}{2} \right) + 2 \cosh \left(\frac{L_1}{2} \right) \cosh \left(\frac{L_2}{2} \right) \cosh \left(\frac{L_3}{2} \right) - 1. \end{aligned} \quad (3.4.4)$$

Equations (3.4.1)–(3.4.2) and can be recovered from Equations (3.4.3)–(3.4.4) by taking the following limit:

$$\lim_{\beta \rightarrow +\infty} \frac{\ell_{\text{hyp}}(\beta L_1, \beta L_2, \beta L_3)}{\beta} = \ell_{\text{comb}}(L_1, L_2, L_3). \quad (3.4.5)$$

This is not a coincidence: in fact, in Chapter 6 we will systematically study a rescaling flow on Teichmüller space, and shown that many hyperbolic quantities equal the corresponding combinatorial ones in such a limit.

3.4.2 — COMBINATORIAL FENCHEL–NIELSEN COORDINATES AS GLOBAL COORDINATES

Recall from Definition 2.4.8 the notion of a pants decomposition $(\mathcal{P}, \mathcal{S})$ of a bordered surface Σ of type (g, n) :

- a maximal collection of simple closed curves $\mathcal{P} = (\gamma_1, \dots, \gamma_{3g-3+n})$ cutting Σ into pairs of pants,
- a collection of curves and arcs \mathcal{S} , that specifies three disjoint arcs connecting boundary components of each pair of pants of \mathcal{P} .

Fix once and for all a seams pants decomposition $(\mathcal{P}, \mathcal{S})$ on Σ . We define the *length parameters* of a point $\mathbb{G} \in \mathcal{T}_\Sigma^{\text{comb}}$ to be the tuple of positive real numbers

$$\ell(\mathbb{G}) = (\ell_1(\mathbb{G}), \dots, \ell_{3g-3+n}(\mathbb{G})), \quad (3.4.6)$$

where $\ell_i(\mathbb{G}) = \ell_{\mathbb{G}}(\gamma_i)$.

As a first step towards the definition of twist parameters, consider a combinatorial marking (\mathbf{G}, f) of a pair of pants P and an arc α connecting two distinct boundary components γ and γ' of P . Let δ be the combinatorial seam connecting γ and γ' , which depends on (\mathbf{G}, f) only as specified in Definition 3.4.1. Let \tilde{P} be a universal cover of P . It contains lifts $\tilde{\gamma}$ and $\tilde{\gamma}'$ of γ and γ' respectively, which acquire an orientation from \tilde{P} . Let $d = \delta \cap \gamma$ and $a = \alpha \cap \gamma$. We choose a lift \tilde{d} of d and call \tilde{a} the first lift of a met by travelling from \tilde{d} along $\tilde{\gamma}$ following its orientation. This determines lifts $\tilde{\delta}$ (resp. $\tilde{\alpha}$) of δ (resp. α) starting from \tilde{d} (resp. \tilde{a}). Now let $\tilde{d}' = \tilde{\delta} \cap \tilde{\gamma}'$ and $\tilde{a}' = \tilde{\alpha} \cap \tilde{\gamma}'$. Consider the path $c_{\tilde{d}'\tilde{a}'}$ along $\tilde{\gamma}'$ starting at \tilde{d}' and ending at \tilde{a}' . The measured foliation associated to (\mathbf{G}, f) lifts to a measured foliation $\tilde{\mathcal{F}}$ on the universal cover, and we can measure the length of $c_{\tilde{d}'\tilde{a}'}$. We then set $\text{sgn}(c_{\tilde{d}'\tilde{a}'}) = \pm 1$ depending on whether the orientation of $c_{\tilde{d}'\tilde{a}'}$ agrees with the one of $\tilde{\gamma}'$. We define the twisting number of α along γ' in P to be

$$t_{\alpha, \gamma'}^P(\mathbf{G}, f) = \text{sgn}(c_{\tilde{d}'\tilde{a}'}) \ell_{\tilde{\mathcal{F}}}(c_{\tilde{d}'\tilde{a}'}). \quad (3.4.7)$$

The definition does not depend on the choice of \tilde{d} , since all different choices are related by deck transformations which leave $t_{\alpha, \gamma'}^P(\mathbf{G}, f)$ fixed.

Given $\mathbb{G} \in \mathcal{T}_\Sigma^{\text{comb}}$, we define the i -th twist parameter $\tau_i(\mathbb{G})$ as follows. Fix a marking (\mathbf{G}, f) such that γ_i is quasitransverse to the measured foliation induced by the marking. Let α_i be one of the two arcs in \mathcal{S} crossing γ_i . There are two pairs of pants Q'_i and Q''_i (possibly the same) on each side of γ_i , and α_i determines two arcs $\alpha'_i = \alpha_i \cap Q'_i$ and $\alpha''_i = \alpha_i \cap Q''_i$. The i -th twist parameter of \mathbb{G} is defined to be

$$\tau_i(\mathbb{G}) = t_{\alpha'_i, \gamma_i}^{Q'_i}(\mathbf{G}|_{Q'_i}, f|_{Q'_i}) + t_{\alpha''_i, \gamma_i}^{Q''_i}(\mathbf{G}|_{Q''_i}, f|_{Q''_i}). \quad (3.4.8)$$

This twist parameter is invariant under isotopies, i.e. does not depend on the representative of \mathbb{G} . Besides, it does not depend on the choice of the arc in \mathcal{S} crossing γ_i . This can be seen by passing to the universal cover of a neighbourhood of γ_i – cf. [FM11, Section 10.6.1] for the

II. The combinatorial model of the moduli space of curves

analogue in the hyperbolic case. Finally, it only depends on the homotopy class of α_i , since a different choice of representative would modify both t' and t'' by the same quantity, but with different signs. Thus, we have well-defined *twist parameters*

$$\tau(\mathbb{G}) = (\tau_1(\mathbb{G}), \dots, \tau_{3g-3+n}(\mathbb{G})). \quad (3.4.9)$$

Notice that we may homotope the representative of α_i such that it is intersecting γ_i at a vertex of the combinatorial structure, showing that t' and t'' in (3.4.8) can be expressed as a sum of edge lengths in \mathbb{G} with half-integer coefficients. The half is coming from the definition of combinatorial seams, which were required to be equidistant from the adjacent singular leaves in the cells depicted in Figures 3.20c–3.20g.

DEFINITION 3.4.4. Let Σ be a bordered surface of type (g, n) equipped with a seamed pants decomposition $(\mathcal{P}, \mathcal{S})$. *Combinatorial Fenchel–Nielsen coordinates* relative to $(\mathcal{P}, \mathcal{S})$ is by definition the map $\Phi_L: \mathcal{T}_\Sigma^{\text{comb}}(L) \rightarrow \mathbb{R}_+^{3g-3+n} \times \mathbb{R}^{3g-3+n}$ defined by

$$\Phi_L(\mathbb{G}) = (\ell(\mathbb{G}), \tau(\mathbb{G})). \quad (3.4.10)$$

Using the gluing we can establish the following result. The first part is an adaptation of arguments by Dehn [Deh22], Thurston [FLP12] and Penner [PH92], who proved similar results for the set of multicurves, measured foliations and train tracks respectively. The second part, i.e. the zero-measure statement, will be crucial in Section 4.3 where we provide a general formula to integrate mapping class group invariant functions.

THEOREM 3.4.5.

1. For any $L \in \mathbb{R}_+^n$, the map

$$\Phi_L: \mathcal{T}_\Sigma^{\text{comb}}(L) \rightarrow (\mathbb{R}_+ \times \mathbb{R})^{3g-3+n} \quad (3.4.11)$$

is a homeomorphism onto its image.

2. The image of Φ_L is an open dense subset whose complement has zero measure. Moreover, if $\Phi_L(\mathbb{G}) = (\ell, \tau)$, then the image of the map τ_i restricted to

$$\left\{ \mathbb{G} \in \mathcal{T}_\Sigma^{\text{comb}}(L) \mid \ell_j(\mathbb{G}) = \ell_j \ \forall j, \ \tau_k(\mathbb{G}) = \tau_k \ \text{for } k \neq i \right\} \quad (3.4.12)$$

has a complement of zero measure in \mathbb{R} .

Proof. To prove the theorem, we use the gluing to construct a partial inverse map. More precisely, define the partial map $\Psi_L: (\mathbb{R}_+ \times \mathbb{R})^{3g-3+n} \rightarrow \mathcal{T}_\Sigma^{\text{comb}}(L)$ by setting

$$\Psi_L(\ell, \tau) = \mathbb{G},$$

where \mathbb{G} is defined as follows.

- For each pair of pants bounded by curves in \mathcal{P} , we assign boundary lengths defined by the perimeter lengths $L \in \mathbb{R}_+^n$ and the assigned lengths $\ell \in \mathbb{R}_+^{3g-3+n}$. This determines a unique combinatorial structure on each pair of pants.
- We glue the pairs of pants along each γ_i after twisting by τ_i . The twist zero corresponds to gluing the combinatorial seams of the pairs of pants together.

By partial map we mean that Ψ_L is not defined on the whole of $(\mathbb{R}_+ \times \mathbb{R})^{3g-3+n}$, as the gluing does not always define an embedded metric ribbon graph. Notice also that Ψ_L does not depend on the order on which we glue the pairs of pants together.

We proceed now with the proof. Firstly notice that the definition of the twist parameters implies that gluing with twist zero amounts to gluing all pairs of pants with matching seams. Also, since gluing back a cut combinatorial structure gives back the original one, we can see that Ψ_L is defined on the image of Φ_L and that $\Psi_L \circ \Phi_L$ is the identity on $\mathcal{T}_\Sigma^{\text{comb}}(L)$. Hence, Φ_L is a bijection onto its image.

On the closure of each open cell, the length and twists are linear functions of the edge lengths. Therefore, we have bijective linear functions that agree on boundaries of the open cells, and therefore, the inverse has the same properties and is therefore continuous which shows that Φ_L is a homeomorphism onto its image.

Now to prove openness we note that given any point $\mathbb{G} \in \mathcal{T}_\Sigma^{\text{comb}}(L)$ there exists a neighbourhood intersecting finitely many cells as there are finitely many ways to expand a singularity using Whitehead equivalence. We can therefore construct a finite simplicial complex containing \mathbb{G} as a vertex such that the intersection of a k -cell of $\mathcal{T}_\Sigma^{\text{comb}}(L)$ is a union of k -dimensional simplices. Then, as Φ_L is linear on each cell and a homeomorphism onto its image, Φ_L maps the simplicial complex to a simplicial complex in $(\mathbb{R}_+ \times \mathbb{R})^{3g-3+n}$.

A point on a finite simplicial complex in an Euclidean space of the same dimension is on the boundary if and only if it is contained in a codimension one simplex that is on the boundary of only one top-dimensional simplex. Every codimension one simplex containing \mathbb{G} is contained in two top-dimensional simplices and therefore $\Phi_L(\mathbb{G})$ is in the interior of the image of the simplex. Thus, Φ_L is open.

Finally, notice that by gluing one curve at a time and using Proposition 3.3.1 and Lemma 3.3.5 for each L and ℓ , we can see that the image is dense and its complement has zero measure or, more generally, that the set defined in Equation (3.4.12) has complement of zero measure in \mathbb{R} . \square

3.5 — A COMBINATORIAL $(9g - 9 + 3n)$ -THEOREM

In this paragraph we establish a combinatorial analogue of the hyperbolic $(9g - 9 + 3n)$ -theorem, that is, any combinatorial structure can be reconstructed from the data of the combinatorial lengths of $(9g - 9 + 3n)$ simple closed curves. Similar computations can be found in [FLP12, Exposé 6], where only measured foliations on closed surfaces are considered. These results are used in Section 6.1 to compare the hyperbolic and combinatorial twists, and in Section 6.2 to give a new proof of Penner's formulae [Pen82] for the action of the mapping class group on Dehn–Thurston coordinates.

Let Σ be of type (g, n) and fix a seamed pants decomposition $(\mathcal{P}, \mathcal{S})$, with $\mathcal{P} = (\gamma_1, \dots, \gamma_{3g-3+n})$. As in the hyperbolic case, each γ_i determine a surface of type $(0, 4)$ or $(1, 1)$ in Σ , which in turn determine two simple closed curve δ_i and η_i (see Theorem 2.4.9 for the definition, or Figure 3.21 for a quick reminder).

THEOREM 3.5.1. *Let Σ be a bordered surface of type (g, n) and $(\mathcal{P}, \mathcal{S})$ a seamed pants decomposition. The following map is continuous and injective:*

$$\mathcal{T}_\Sigma^{\text{comb}}(L) \longrightarrow \mathbb{R}_+^{9g-9+3n}, \quad \mathbb{G} \longmapsto (\ell_{\mathbb{G}}(\gamma), \ell_{\mathbb{G}}(\delta), \ell_{\mathbb{G}}(\eta)). \quad (3.5.1)$$

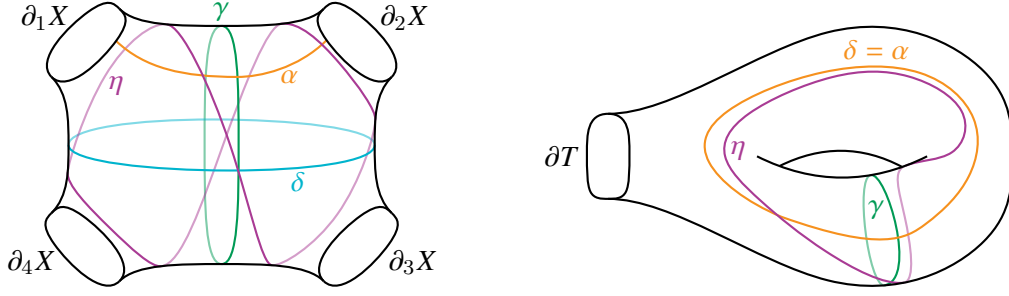


Figure 3.21: The curves δ and η obtained from a curve γ in the pants decomposition (we omit the subscript), and an arc/simple closed curve α in \mathcal{S} .

As a preparation to the proof, we present in Lemmas 3.5.2 and 3.5.4 closed formulae for $\ell_{\mathbb{G}}(\delta_i)$ and $\ell_{\mathbb{G}}(\eta_i)$ in the $(0, 4)$ and $(1, 1)$ cases respectively. For this purpose we can work locally on $\mathbb{G}|_{\Sigma_i}$ with a fixed seamed pants decomposition, which we denote by $\ell_i = \ell_{\mathbb{G}}(\gamma_i)$, $\ell'_i = \ell_{\mathbb{G}}(\delta_i)$ and $\ell''_i = \ell_{\mathbb{G}}(\eta_i)$.

FOUR-HOLED SPHERE

Let $\Sigma_i = X$ be a four-holed sphere. We remove the index i from the notation of γ_i , α_i , δ_i and η_i , as well as ℓ_i , ℓ'_i , ℓ''_i and τ_i . Label by $\partial_1 X, \dots, \partial_4 X$ the boundary components of X , so that γ is separating the components $\partial_1 X$ and $\partial_4 X$ from $\partial_2 X$ and $\partial_3 X$, and α is connecting the components $\partial_1 X$ and $\partial_2 X$. Finally, denote $L_i = \ell_{\mathbb{G}}(\partial_i X)$.

LEMMA 3.5.2. *In the above setting, we have*

$$\ell'(\ell, \tau) = \max \left\{ L_1 + L_3 - \ell, L_2 + L_4 - \ell, 2|\tau| + M_{1,4}(\ell) + M_{2,3}(\ell) \right\}, \quad (3.5.2)$$

where $M_{i,j}(\ell) = \max \left\{ 0, L_i - \ell, L_j - \ell, \frac{L_i + L_j - \ell}{2} \right\}$. Further, $\ell''(\ell, \tau) = \ell'(\ell, \tau + \ell)$.

Proof. Let us assume that the ribbon graph underlying \mathbb{G} is trivalent, fix a marking of it, and cut \mathbb{G} along the curve γ . There are sixteen possibilities for the cut combinatorial structure: the marked ribbon graph on each pair of pants can belong to each of the four top-dimensional cells of the Teichmüller space of a pair of pants. Therefore, in order to check that Equation (3.5.2) holds for any \mathbb{G} , it is sufficient to check that it is satisfied in each of the sixteen cases. By symmetry considerations, the number of cases can actually be reduced to seven. We show the detailed argument for three particularly representative cases out of those seven, and argue that the other cases can be proven following the same strategy.

Firstly, suppose that $L_4 > L_1 + \ell$ and $L_3 > L_2 + \ell$ (see Figure 3.22a). Then

$$\max \left\{ 0, L_1 - \ell, L_4 - \ell, \frac{L_1 + L_4 - \ell}{2} \right\} = L_4 - \ell, \quad \max \left\{ 0, L_2 - \ell, L_3 - \ell, \frac{L_2 + L_3 - \ell}{2} \right\} = L_3 - \ell,$$

so the right-hand side of Equation (3.5.2) reduces to

$$\max \left\{ L_1 + L_3 - \ell, L_2 + L_4 - \ell, 2|\tau| + L_3 + L_4 - 2\ell \right\} = 2|\tau| + L_3 + L_4 - 2\ell.$$

In Figure 3.22b, a quasitransverse representative of δ is shown. Its orange part has length $L_4 - \ell$, its blue part has length $L_3 - \ell$, and its green part has length $2|\tau|$. In the end, we have $\ell' = 2|\tau| + L_3 + L_4 - 2\ell$, which is consistent with Equation (3.5.2).

Secondly, suppose that $L_4 > L_1 + \ell$ and $|L_2 - L_3| < \ell < L_2 + L_3$ (see Figure 3.22c). Then

$$\max \left\{ 0, L_1 - \ell, L_4 - \ell, \frac{L_1 + L_4 - \ell}{2} \right\} = L_4 - \ell, \quad \max \left\{ 0, L_2 - \ell, L_3 - \ell, \frac{L_2 + L_3 - \ell}{2} \right\} = \frac{L_2 + L_3 - \ell}{2}.$$

In this case, we also have $L_1 + L_3 < L_2 + L_4$, so the right-hand side of (3.5.2) reduces to

$$\max \left\{ L_1 + L_3 - \ell, L_2 + L_4 - \ell, 2|\tau| + L_4 - \ell + \frac{L_2 + L_3 - \ell}{2} \right\} = L_2 + L_4 - \ell + \left[2|\tau| - \frac{L_2 + \ell - L_3}{2} \right]_+.$$

Suppose first that $2|\tau| < \frac{L_2 + \ell - L_3}{2}$, which is depicted in Figure 3.22d together with a quasitransverse representative of δ . The orange part of δ has length $L_4 - \ell$, while the blue part of δ has length L_2 . Therefore

$$\ell' = L_2 + L_4 - \ell = L_2 + L_4 - \ell + \left[2|\tau| - \frac{L_2 + \ell - L_3}{2} \right]_+.$$

Suppose now that $2|\tau| > \frac{L_2 + \ell - L_3}{2}$, see Figure 3.22e. The orange part of δ has length $L_4 - \ell$, the blue part of δ has length $\frac{L_2 + L_3 - \ell}{2}$, and the green part of δ has length $2|\tau|$. Thus:

$$\ell' = 2|\tau| + L_4 - \ell + \frac{L_2 + L_3 - \ell}{2} = L_2 + L_4 - \ell + \left[2|\tau| - \frac{L_2 + \ell - L_3}{2} \right]_+.$$

Again, in both cases Equation (3.5.2) is satisfied.

Thirdly, suppose that $\ell > L_1 + L_4$ and $|L_2 - L_3| < \ell < L_2 + L_3$ (see Figure 3.22f). Then

$$\max \left\{ 0, L_1 - \ell, L_4 - \ell, \frac{L_1 + L_4 - \ell}{2} \right\} = 0, \quad \max \left\{ 0, L_2 - \ell, L_3 - \ell, \frac{L_2 + L_3 - \ell}{2} \right\} = \frac{L_2 + L_3 - \ell}{2}.$$

Without loss of generality, we can assume that $L_1 + L_3 > L_2 + L_4$. Then, the right-hand side of Equation (3.5.2) reduces to

$$\max \left\{ L_1 + L_3 - \ell, L_2 + L_4 - \ell, 2|\tau| + \frac{L_2 + L_3 - \ell}{2} \right\} = \max \left\{ L_1 + L_3 - \ell, 2|\tau| + \frac{L_2 + L_3 - \ell}{2} \right\}.$$

The case where $2|\tau| < L_1 + \frac{L_3 - L_2 - \ell}{2}$ is depicted in Figure 3.22g. The length of δ is then

$$\ell' = L_1 + L_3 - \ell = \max \left\{ L_1 + L_3 - \ell, 2|\tau| + \frac{L_2 + L_3 - \ell}{2} \right\}.$$

In the case where $2|\tau| > L_1 + \frac{L_3 - L_2 - \ell}{2}$, depicted in Figure 3.22h, the orange part of δ has length $\frac{L_2 + L_3 - \ell}{2}$, and the green part of δ has length $2|\tau|$, therefore

$$\ell' = 2|\tau| + \frac{L_2 + L_3 - \ell}{2} = \max \left\{ L_1 + L_3 - \ell, 2|\tau| + \frac{L_2 + L_3 - \ell}{2} \right\}.$$

In both cases, Equation (3.5.2) is satisfied.

The case of \mathbb{G} with higher valencies can be obtained from the trivalent case by continuity of the combinatorial lengths and twists. Lastly, since η is obtained from δ by performing a positive Dehn twist along γ , its length is given by $\ell'' = \ell'(\ell, \tau + \ell)$. \square

The above lemma expresses the lengths ℓ' and ℓ'' as functions of the Fenchel–Nielsen coordinates (ℓ, τ) . We can invert the perspective, expressing τ as a function of ℓ, ℓ' and ℓ'' .

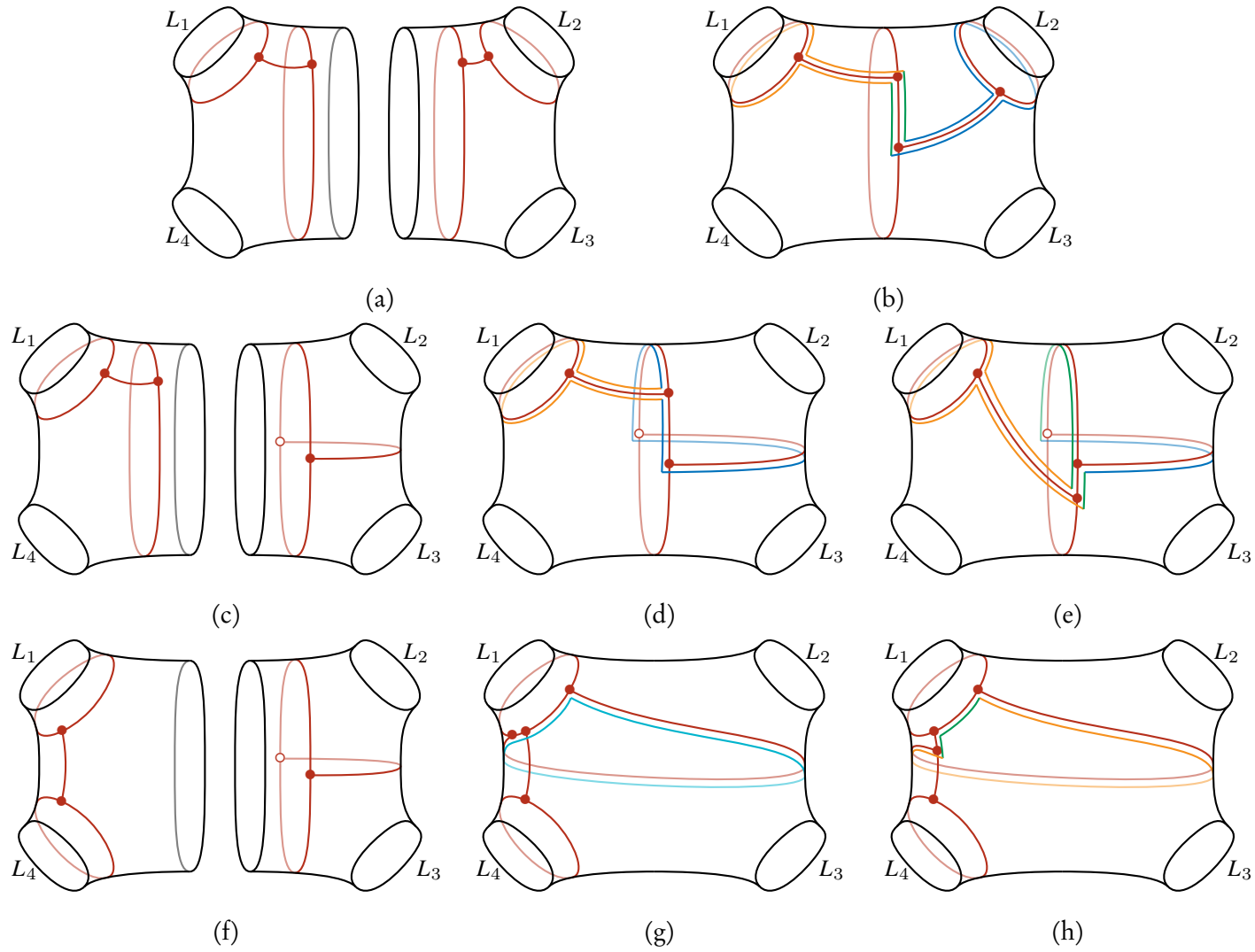


Figure 3.22: The three cases examined in the proof of Lemma 3.5.2.

COROLLARY 3.5.3. *In the previous situation, we have*

$$\tau = \begin{cases} \frac{1}{2}(\ell'' - M_{1,4} - M_{2,3}) - \ell & \text{if } \ell' = \ell^*, \\ -\frac{1}{2}(\ell' - M_{1,4} - M_{2,3}) & \text{if } \ell'' = \ell^*, \\ \frac{1}{2\ell} \left(\frac{\ell'' - M_{1,4} - M_{2,3}}{2} \right)^2 - \frac{1}{2\ell} \left(\frac{\ell' - M_{1,4} - M_{2,3}}{2} \right)^2 - \frac{\ell}{2} & \text{otherwise,} \end{cases} \quad (3.5.3)$$

where $\ell^* = \max \{ L_1 + L_3 - \ell, L_2 + L_4 - \ell \}$.

Proof. Let us denote $p = 2|\tau| + M_{1,4} + M_{2,3}$ and $q = \max \{ L_1 + L_3 - \ell, L_2 + L_4 - \ell \}$, so that $\ell' = \max \{ p, q \}$. We claim that $\ell' = q$ implies $2|\tau| \leq \ell$. If $L_2 + L_4 \geq L_1 + L_3$, this comes from the observation that $q = L_2 + L_4 - \ell$ and $\ell' = q + [\lambda]_+$ with

$$\begin{aligned} \lambda &= p - L_2 - L_4 + \ell \\ &= 2|\tau| - \ell + \max \left\{ \ell - L_4, L_1 - L_4, 0, \frac{L_1 - L_4 + \ell}{2} \right\} \\ &\quad + \max \left\{ \ell - L_2, 0, L_3 - L_2, \frac{L_3 - L_2 + \ell}{2} \right\} \geq 2|\tau| - \ell. \end{aligned}$$

If $L_1 + L_3 \geq L_2 + L_4$, we rather have $q = L_1 + L_3 - \ell$ and the claim follows by writing $\ell' = q + [\mu]_+$ with

$$\begin{aligned} \mu &= p - L_1 - L_3 + \ell \\ &= 2|\tau| - \ell + \max \left\{ \ell - L_1, 0, L_4 - L_1, \frac{L_4 - L_1 + \ell}{2} \right\} \\ &\quad + \max \left\{ \ell - L_3, L_2 - L_3, 0, \frac{L_2 - L_3 + \ell}{2} \right\} \geq 2|\tau| - \ell. \end{aligned}$$

Therefore, if $\ell' = \max \{ L_1 + L_3 - \ell, L_2 + L_4 - \ell \}$, we must have $|\tau| \leq \ell/2$, hence $|\tau + \ell| = \tau + \ell$. From Equation (3.5.2) we then find $\ell'' = 2|\tau + \ell| + M_{1,4} + M_{2,3}$, and solving for τ we get the first case of (3.5.3). The case $\ell'' = \max \{ L_1 + L_4 - \ell, L_2 + L_3 - \ell \}$ is similar. Finally, if none of those conditions are satisfied, then $\ell' = 2|\tau| + M_{1,4} + M_{2,3}$ and $\ell'' = 2|\tau + \ell| + M_{1,4} + M_{2,3}$. This covers the last case in Equation (3.5.3). \square

ONE-HOLED TORUS

Let $\Sigma_i = T$ be a one-holed torus. We remove the index i from the notation of $\gamma_i, \alpha_i, \delta_i$ and η_i , as well as $\ell_i, \ell'_i, \ell''_i$ and τ_i , and denote $L = \ell_{\mathbb{G}}(\partial T)$.

LEMMA 3.5.4. *In the above setting, we have*

$$\ell'(\ell, \tau) = |\tau| + \left[\frac{L - 2\ell}{2} \right]_+. \quad (3.5.4)$$

Further, $\ell''(\ell, \tau) = \ell'(\ell, \tau + \ell)$.

Proof. As before, we assume \mathbb{G} to be trivalent and we fix a marking. There are four cases, corresponding to the four open cells of the Teichmüller space of the pair of pants we obtain after cutting along γ .

We detail the case of Figure 3.23a, where $L > 2\ell$. In Figure 3.23b, a quasitransverse representative of δ is shown. Its orange part has length $L/2 - \ell$, its green part has length $|\tau|$. Thus, we find $\ell' = |\tau| + L/2 - \ell$, which is consistent with Equation (3.5.4) under the assumption $L > 2\ell$.

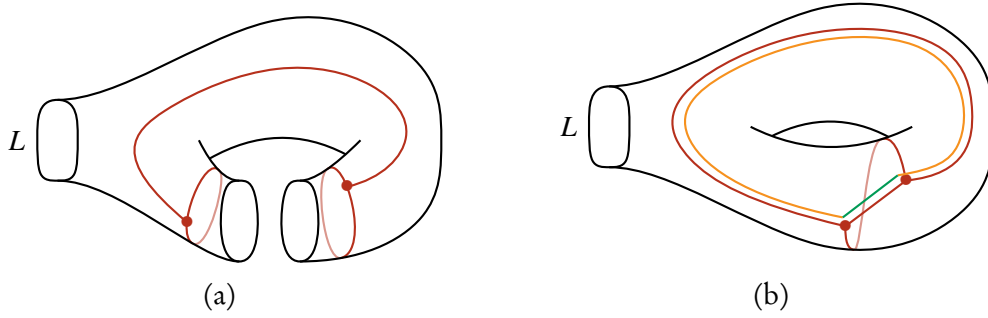


Figure 3.23: The case examined in the proof of Lemma 3.5.4.

The other cases are analogous. Again, the formula extends to higher valency by continuity, and since the curve η is obtained from the curve δ by performing a positive Dehn twist along γ , its length is given by $\ell'' = \ell'(\ell, \tau + \ell)$. \square

Again, we can recover from the above Lemma an expression for the twist parameter τ as a function of the lengths ℓ , ℓ' and ℓ'' . The proof is similar to the (0, 4) case.

COROLLARY 3.5.5. *In a one-holed torus T with the above setting, the twist parameter is given as a function of (ℓ, ℓ', ℓ'') by*

$$\tau = \frac{1}{2\ell} \left(\ell'' - \left[\frac{L - 2\ell}{2} \right]_+ \right)^2 - \frac{1}{2\ell} \left(\ell' - \left[\frac{L - 2\ell}{2} \right]_+ \right)^2 - \frac{\ell}{2}. \quad (3.5.5)$$

PROOF OF THE COMBINATORIAL $(9g - 9 + 3n)$ -THEOREM

Proof of Theorem 3.5.1. The map is clearly continuous. Further, if $\mathbb{G}, \mathbb{G}' \in \mathcal{T}_\Sigma^{\text{comb}}(L)$ are mapped to the same vector of lengths, then Corollaries 3.5.3 and 3.5.5 would give the same length and twist parameters. As the combinatorial Fenchel–Nielsen map is a homeomorphism into the image, we deduce that $\mathbb{G} = \mathbb{G}'$. This justifies the injectivity. \square

CHAPTER 4 — THE SYMPLECTIC STRUCTURE

The Kontsevich symplectic form ω_K on $\mathcal{M}_{g,n}^{\text{comb}}(L)$ was originally introduced by Kontsevich in [Kon92] as an ingredient in the proof of Witten’s conjecture. Its main feature is the connection with intersection theory of $\overline{\mathcal{M}}_{g,n}$: the symplectic volume of $\mathcal{M}_{g,n}^{\text{comb}}(L)$ compute ψ -classes intersections on the Deligne–Mumford compactification of the moduli space of curves:

$$V_{g,n}^K(L) = \int_{\mathcal{M}_{g,n}^{\text{comb}}(L)} \frac{\omega_K^{3g-3+n}}{(3g-3+n)!} = \sum_{\substack{k_1, \dots, k_n \geq 0 \\ k_1 + \dots + k_n = 3g-3+n}} \left(\int_{\mathcal{M}_{g,n}} \prod_{i=1}^n \psi_i^{k_i} \right) \prod_{i=1}^n \frac{L_i^{2k_i}}{2^{k_i} k_i!}. \quad (4.0.1)$$

After we lift ω_K to a mapping class group invariant 2-form on $\mathcal{T}_\Sigma^{\text{comb}}(L)$, the main result of this chapter is a combinatorial analogue of Wolpert’s formula [Wol85], expressing Kontsevich’s form in terms of combinatorial Fenchel–Nielsen coordinates (see Theorem 4.2.1 for a more precise statement).

THEOREM 4.A (Combinatorial Wolpert formula). *Let Σ be a bordered surface of type (g, n) , and fix any combinatorial Fenchel–Nielsen coordinates (ℓ_i, τ_i) for $\mathcal{T}_\Sigma^{\text{comb}}(L)$. Then*

$$\omega_K = \sum_{i=1}^{3g-3+n} d\ell_i \wedge d\tau_i. \quad (4.0.2)$$

In other words, we show that the Kontsevich form is compatible with cutting and gluing along simple closed curves. As a direct consequence, we achieve an explicit integration formula for a certain class of measurable functions on the combinatorial moduli space with respect to Kontsevich’s volume form (Proposition 4.3.1), constructed by summing over mapping class group orbits of simple closed curve. This integration is the key operation that connects topological and geometric recursion, and is the combinatorial analogue of Mirzakhani’s integration lemma on the moduli space of bordered Riemann surfaces [Miro7a].

4.0.1 — RELATION WITH PREVIOUS WORKS

The strategy proposed here to prove the combinatorial Wolpert formula is parallel, mutatis mutandi, to Wolpert’s original proof [Wol85, Theorem 1.3] for the Weil–Peterson form. The same holds for Mirzakhani’s integration lemma [Miro7a, Theorem 7.1].

In the context of measured foliations on closed surfaces, a similar result was proved by Papadopoulos [Pap86a; Pap86b], where the space is equipped with Thurston’s symplectic structure. Besides, we observe that the Kontsevich 2-form on $\mathcal{T}_\Sigma^{\text{comb}}$ is defined identically to Thurston’s symplectic form on \mathcal{MF}_Σ , compare e.g. with [Bon96, Section 3] in which one should consider the train track dual to the trivalent ribbon graph: switches correspond to corners of the ribbon graph and intersecting transverse arcs correspond to edges meeting at a vertex. Moreover, the definition of combinatorial Fenchel–Nielsen coordinates on $\mathcal{T}_\Sigma^{\text{comb}}$ is

identical to the definition of Dehn–Thurston on \mathcal{MF}_Σ . Hence, adapting the proof of Theorem 4.A to \mathcal{MF}_Σ therefore leads to the following result, which seems to be unnoticed in the literature (to the best of our knowledge and also to our surprise): for every punctured surface Σ , the Dehn–Thurston coordinates are almost everywhere Darboux coordinates for Thurston symplectic form on \mathcal{MF}_Σ .

We also remark that our result, specifically the Hamiltonian property of the vector field associated to the twisting along simple closed curves, generalises a result previously proved locally in [BCSW12, Lemma 3.2] for the special case of curves cutting out small pairs of pants.

To conclude, we note that another set of Darboux coordinates for ω_K have been constructed by Bertola–Korotkin [BK20] from periods of quadratic differentials. Such coordinates are a priori different, since they make sense at the level of moduli space (while our coordinates, constructed via length and twist along simple closed curves, do not). An advantage of our combinatorial Fenchel–Nielsen coordinates is their compatibility with the cutting and gluing operations, which allows for the integration of function constructed via sums over simple closed curves.

4.0.2 — ORGANISATION OF THE CHAPTER

The chapter is organised as follows.

- In Section 4.1 we recall the definition of Kontsevich’s form and its relation to the Deligne–Mumford compactification of the moduli space of curves, and we lift his construction to the combinatorial Teichmüller space.
- Section 4.2 proves the combinatorial Wolpert formula. The main technical step is to show that the vector field associated to the twist along a simple closed curve is the Hamiltonian vector field of the length function of the curve.
- We conclude in Section 4.3 with the aforementioned integration formula for certain functions constructed by summing over mapping class group orbits of simple closed curves.

4.1 — KONTSEVICH FORM

Consider a ribbon graph G of type (g, n) . For each index $i \in \{1, \dots, n\}$, we make the choice of a first edge $e_1^{[i]}$ on the i -th face of G . We label the edges around the i -th face by $e_1^{[i]}, \dots, e_{N_i}^{[i]}$ following the orientation of the face, which is opposite to the orientation of the boundary. Notice that every edge has a double label, as it bounds two faces or it appears twice in the cycle of a single face.

Let now Σ be a bordered surface of type (g, n) . On each cell of $\mathfrak{Z}_{\Sigma, G}(L) \subset \mathcal{T}_\Sigma^{\text{comb}}$ having G as underlying graph, we have length functions

$$\ell_j^{[i]} : \mathfrak{Z}_{\Sigma, G}(L) \longrightarrow \mathbb{R}_+, \quad \ell_j^{[i]}(\mathbb{G}) = \ell_{\mathbb{G}}(e_j^{[i]}). \quad (4.1.1)$$

DEFINITION 4.1.1. For each $i \in \{1, \dots, n\}$, consider the differential 2-form Ψ_i on the cell complex $\mathcal{T}_\Sigma^{\text{comb}}(L)$, defined on each cell $\mathfrak{Z}_{\Sigma, G}(L)$ by

$$\Psi_i = \sum_{1 \leq k < m \leq N_i} \frac{d\ell_k^{[i]}}{L_i} \wedge \frac{d\ell_m^{[i]}}{L_i}. \quad (4.1.2)$$

The form is $\text{Mod}_\Sigma^\partial$ -invariant (it depends only on the ribbon graph underlying the marking) and we denote the induced form on the quotient $\mathcal{M}_{g,n}^{\text{comb}}(L)$ with the same symbol¹.

It can be shown that the definition of Ψ_i does not depend on the choice of the first edge: the difference between two possible choices is of the form $L_i^{-2} dL_i \wedge \vartheta_i$ for some differential 1-forms ϑ_i , and thus is zero along the fibres $\mathcal{M}_{g,n}^{\text{comb}}(L)$ of the perimeter map.

In Kontsevich's original work [Kon92], completed by Zvonkine in [Zvo02], he related the above differential forms to the geometry of certain circle bundles. This proves that the associated cohomology class lies in the second cohomology of $\mathcal{M}_{g,n}^{\text{comb}}(L)$ with rational coefficients (rather than real coefficients.)

DEFINITION 4.1.2. For each $i \in \{1, \dots, n\}$, define $\mathcal{S}_i^{\text{comb}}$ as the space of ordered pairs (G, q) where $G \in \mathcal{M}_{g,n}^{\text{comb}}(L)$ and q is a point belonging to an edge that borders the i -th face of $|G|$. Its topology is the one induced by the natural cell structure. This defines a topological circle bundle $\mathcal{S}_i^{\text{comb}} \rightarrow \mathcal{M}_{g,n}^{\text{comb}}(L)$.

THEOREM 4.1.3 ([Kon92; Zvo02]).

- The class $[\Psi_i] \in H^2(\mathcal{M}_{g,n}^{\text{comb}}(L))$ equals $-c_1(\mathcal{S}_i^{\text{comb}})$.
- Under the identification $\mathcal{M}_{g,n} \cong \mathcal{M}_{g,n}^{\text{comb}}(L)$, the pullback of Ψ_i extends continuously to the Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$ and the associated cohomology class equals $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n})$.

The above theorem gives a differential-geometric interpretation of the ψ -classes of the moduli space of curves: they measure the variation of edge lengths around each face on the combinatorial moduli space.

DEFINITION 4.1.4. Define the *Kontsevich 2-form* on $\mathcal{T}_\Sigma^{\text{comb}}(L)$ as

$$\omega_K = \frac{1}{2} \sum_{i=1}^n L_i^2 \Psi_i. \quad (4.1.3)$$

THEOREM 4.1.5 ([Kon92]). *The differential form ω_K is non-degenerate when restricted to strata corresponding to graphs with vertices of odd valency only.*

A fortiori, ω_K descends to a symplectic form on the top-dimensional stratum of $\mathcal{M}_{g,n}^{\text{comb}}(L)$, that is denoted with the same symbol. In particular, we can define the *Kontsevich measure*

$$d\mu_K = \frac{\omega_K^{3g-3+n}}{(3g-3+n)!}. \quad (4.1.4)$$

¹See [Zvo02, Section 5.2] for a discussion on the differential geometry of cell complexes. What we need here is that the combinatorial Teichmüller spaces and the combinatorial moduli spaces have a well-defined notion of polytopal differential forms, and that the associated polytopal de Rham cohomology groups coincide with the usual cohomology groups over \mathbb{R} . In particular, we can consider the cohomology class $[\Psi_i] \in H^2(\mathcal{M}_{g,n}^{\text{comb}}(L))$.

II. The combinatorial model of the moduli space of curves

Strictly speaking, $d\mu_K$ is not a volume form on the whole $\mathcal{M}_{g,n}^{\text{comb}}(L)$, although it is a volume form on the top-dimensional stratum. In any case, we have a notion of volume

$$V_{g,n}^K(L) = \int_{\mathcal{M}_{g,n}^{\text{comb}}(L)} d\mu_K, \quad (4.1.5)$$

where the integral is taken over the top-dimensional stratum. More precisely, if G is a trivalent ribbon graph of type (g, n) and $P_G(L) \subseteq \mathbb{R}_+^{E_G}$ is the polytope corresponding to those metrics on G with fixed perimeter $L \in \mathbb{R}_+^n$, then we have

$$V_{g,n}^K(L) = \sum_{\substack{G \in \mathcal{R}_{g,n} \\ \text{trivalent}}} \frac{1}{|\text{Aut}(G)|} \int_{P_G(L)} d\mu_K. \quad (4.1.6)$$

We remark that, by abuse of notation, we are using the same symbol to denote the measure on $P_G(L)$ and its quotient $P_G(L)/\text{Aut}(G)$.

Notice that the volumes are finite, because the pullback of ω_K extends continuously to $\overline{\mathcal{M}}_{g,n}$. We moreover remark that, from Definition 4.1.4 and Theorem 4.1.3, the volumes are homogeneous polynomial in L_1^2, \dots, L_n^2 of degree $3g-3+n$ with coefficients given by ψ -classes intersections:

$$V_{g,n}^K(L) = \sum_{\substack{k_1, \dots, k_n \geq 0 \\ k_1 + \dots + k_n = 3g-3+n}} \left(\int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^n \psi_i^{k_i} \right) \prod_{i=1}^n \frac{L_i^{2k_i}}{2^{k_i} k_i!}. \quad (4.1.7)$$

It is useful to record the expression of the Kontsevich measure in terms of edge lengths.

LEMMA 4.1.6. *If G is a trivalent ribbon graph of type (g, n) , let $(l_e)_{e \in E_G}$ be the edge lengths. We have the equality of measures in $\mathbb{R}_+^{E_G}$*

$$d\mu_K \cdot \prod_{i=1}^n dL_i = 2^{2g-2+n} \prod_{e \in E_G} dl_e. \quad (4.1.8)$$

Proof. From [Kon92, Appendix C] or [CMS11], we get

$$\frac{1}{(3g-3+n)!} \left(\sum_{i=1}^n L_i^2 \Psi_i \right)^{3g-3+n} \prod_{i=1}^n dL_i = 2^{5g-5+2n} \prod_{e \in E_G} dl_e.$$

Dividing on both sides by 2^{3g-3+n} yields the result. \square

4.2 — A COMBINATORIAL WOLPERT'S FORMULA

The purpose of this section is to show that combinatorial Fenchel–Nielsen coordinates are Darboux for ω_K .

THEOREM 4.2.1. *Let Σ be a bordered surface of type (g, n) and fix any combinatorial Fenchel–Nielsen coordinates (ℓ_i, τ_i) for $\mathcal{T}_\Sigma^{\text{comb}}(L)$. Denote by $\iota_L: \mathfrak{Z}_{\Sigma, G}(L) \hookrightarrow \mathcal{T}_\Sigma^{\text{comb}}(L)$ the inclusion of a cell $\mathfrak{Z}_{\Sigma, G}(L)$. We have*

$$\omega_K = \iota_L^* \left(\sum_{i=1}^{3g-3+n} d\ell_i \wedge d\tau_i \right). \quad (4.2.1)$$

The advantage of this formula is that, while the left-hand side is clearly pure mapping class group invariant and it does not depend on the pants decomposition, the right-hand side has a simple expression in terms of global coordinates and does not rely on the cells decomposition.

4.2.1 — SYMPLECTIC PROPERTIES OF THE TWIST

As mentioned in the introduction to the chapter, the main technical ingredient for the proof is Proposition 4.2.3: the vector field associated to the twist along a simple closed curve is the Hamiltonian vector field of the length function of the curve. As mentioned in the introduction, this is analogous to the situation in the hyperbolic case explored by Wolpert [Wol83, Theorem 1.3], and has the same flavour of a result in the space of measured foliations on closed surfaces proved by Papadopoulos [Pap86a; Pap86b]. Moreover, Proposition 4.2.3 generalises a result previously proved locally in [BCSW12, Lemma 3.2] and only for very special curves cutting out small pairs of pants.

To prove such a result, we need to understand how small changes in the twist parameter affect the metric on the embedded ribbon graph. More precisely, fix $\mathbb{G} \in \mathcal{T}_\Sigma^{\text{comb}}$ in an open cell, and γ an essential, simple closed curve. Notice that, from Proposition 3.3.4, we can cut \mathbb{G} along γ and glue it back after twisting by a small amount $\tau \in \mathbb{R}$. In particular, since we are in an open cell, it makes sense to talk about the vector field ∂_τ generated by infinitesimal changes in the twist.

To get an expression for ∂_τ , we observe that each time γ passes along an edge, a twist by small τ has the effect of either adding τ to the edge length, subtracting τ to the edge length, or leaving the edge length invariant (see the proof of Proposition 3.3.4 and Figure 3.19). This depends on the direction taken by γ at two consecutive vertices. Then one simply sums the changes in the length of the edges visited by γ . In the notation of Figure 4.1 (edges may appear twice along γ), the vector field describing the twisting along γ is given by

$$\partial_\tau = \sum_{i=1}^F \left(\frac{\partial}{\partial \ell_{p_i}^{[b_i]}} - \frac{\partial}{\partial \ell_{q_i}^{[b_i]}} \right) = \sum_{i=1}^F \left(\frac{\partial}{\partial \ell_{r_i}^{[c_i]}} - \frac{\partial}{\partial \ell_{s_i}^{[c_i]}} \right), \quad (4.2.2)$$

or in a more symmetric expression,

$$\partial_\tau = \frac{1}{2} \sum_{i=1}^F \left(\frac{\partial}{\partial \ell_{p_i}^{[b_i]}} - \frac{\partial}{\partial \ell_{q_i}^{[b_i]}} + \frac{\partial}{\partial \ell_{r_i}^{[c_i]}} - \frac{\partial}{\partial \ell_{s_i}^{[c_i]}} \right). \quad (4.2.3)$$

To compute the contraction $\iota_{\partial_\tau} \omega_K$, we need the following technical lemma. Note that an edge $e_p^{[i]}$ is either adjacent to two different faces i.e. $e_p^{[i]} = e_r^{[j]}$ for $i \neq j$, or adjacent to the same face on both sides i.e. $e_p^{[i]} = e_r^{[i]}$ for $p \neq r$.

LEMMA 4.2.2. *In the interior of a top-dimensional cell, we have with the above notations*

$$\iota_{\partial_p^{[i]} + \partial_r^{[j]}} \omega_K = \sum_{k=p+1}^{N_i} d\ell_k^{[i]} - \sum_{k=1}^{p-1} d\ell_k^{[i]} + \sum_{u=r+1}^{N_j} d\ell_u^{[j]} - \sum_{u=1}^{r-1} d\ell_u^{[j]}. \quad (4.2.4)$$

Proof. We recall that $\omega_K = \frac{1}{2} \sum_{i=1}^n L_i^2 \Psi_i$ and Ψ_i defined in 4.1.1 only involves edges around the i -th face. Consider first the case $e_p^{[i]} = e_r^{[j]}$ for $i \neq j$. The interior product only receives

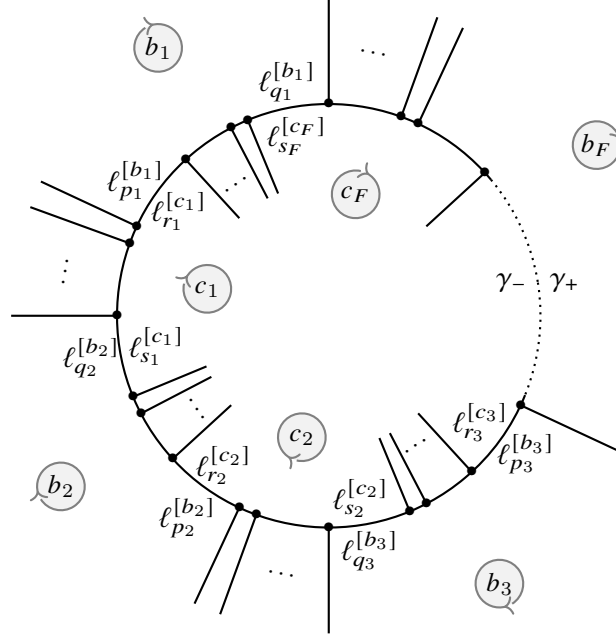


Figure 4.1: A schematic picture of γ used to calculate the vector field associated to the twist. The labels can be redundant if γ visits an edge multiple times.

contributions from Ψ_i and Ψ_j . The interior product with Ψ_i and Ψ_j gives

$$\frac{2}{L_i^2} \left(\sum_{k=p+1}^{N_i} d\ell_k^{[i]} - \sum_{k=1}^{p-1} d\ell_k^{[i]} \right) \quad \text{and} \quad \frac{2}{L_j^2} \left(\sum_{u=r+1}^{N_j} d\ell_u^{[i]} - \sum_{u=1}^{r-1} d\ell_u^{[j]} \right)$$

respectively. Therefore we obtain Equation (4.2.4). In the second case, we have $e_p^{[i]} = e_r^{[i]}$ for $p \neq r$, and the interior product only receives a contribution from Ψ_i . Assuming $p < r$, the interior product with Ψ_i is

$$\begin{aligned} & \frac{2}{L_i^2} \left(\sum_{k=p+1}^{r-1} d\ell_k^{[i]} + 2 \sum_{k=r+1}^{N_i} d\ell_k^{[i]} - 2 \sum_{k=1}^{p-1} d\ell_k^{[i]} - \sum_{k=p+1}^{r-1} d\ell_k^{[i]} \right) \\ &= \frac{2}{L_i^2} \left(\sum_{k=p+1}^{N_i} d\ell_k^{[i]} - \sum_{k=1}^{p-1} d\ell_k^{[i]} + \sum_{u=r+1}^{N_i} d\ell_u^{[i]} - \sum_{u=1}^{r-1} d\ell_u^{[i]} \right). \end{aligned}$$

Therefore, once again, we obtain Equation (4.2.4). The case $p > r$ is similar. \square

We are ready now to state the main property of the twist vector field: it is the Hamiltonian vector field associated to the combinatorial length function $\ell: \mathbb{G} \mapsto \ell_{\mathbb{G}}(\gamma)$.

PROPOSITION 4.2.3. *On the top-dimensional cells of $\mathcal{T}_{\Sigma}^{\text{comb}}(L)$, we have*

$$d\ell = \iota_{\partial_r} \omega_{\mathbb{K}}. \quad (4.2.5)$$

Proof. Fix a top-dimensional cell and suppose that γ is given by the schematic of Figure 4.1. Then we have

$$d\ell = \sum_{i=1}^F \left(\frac{1}{2} d\ell_{p_i}^{[b_i]} + \frac{1}{2} d\ell_{q_i}^{[b_i]} + \sum_{p_i < k < q_i} d\ell_k^{[b_i]} + \frac{1}{2} d\ell_{r_i}^{[c_i]} + \frac{1}{2} d\ell_{s_i}^{[c_i]} + \sum_{r_i < u < s_i} d\ell_u^{[c_i]} \right),$$

where the symbol $\sum_{\mu < \lambda < \nu}$ indicates the sum over all edges of a certain face indexed by λ , that are between the edges indexed by μ and ν , following the orientation of the face and excluding the extremes μ and ν . Notice that the orientation of the face is opposite to the orientation of the corresponding boundary. On the other hand, we can reduce the computation of $\iota_{\partial_\tau} \omega_K$ to the insertion of the coordinate vector fields appearing in Equation (4.2.3).

Let us explain why this calculation leads to a well-defined answer. Notice first that Definition 4.1.2 of Ψ_i makes perfect sense on the whole $\mathcal{T}_\Sigma^{\text{comb}}$, where the perimeter is not fixed, but the definition depends on the choice of a first edge in the i -th face. However, its pullback to $\mathcal{T}_\Sigma^{\text{comb}}(L)$ is independent of such a choice. Secondly, observe that the vector fields ∂_e are defined on $\mathcal{T}_\Sigma^{\text{comb}}$ but do not have a meaning on $\mathcal{T}_\Sigma^{\text{comb}}(L)$, as they do not preserve the boundary lengths L . However, particular linear combinations of them, such as ∂_τ , do. Therefore, it is legitimate to compute each contribution $\iota_{\partial_e} \Psi$ separately on $\mathcal{T}_\Sigma^{\text{comb}}$ (i.e. we can safely use Lemma 4.2.2), then sum them up to obtain $\iota_{\partial_\tau} \omega_K$, and eventually take the pullback to $\mathcal{T}_\Sigma^{\text{comb}}(L)$.

This said, a repeated use of Lemma 4.2.2 results in the following computation.

$$\begin{aligned} \iota_{\partial_\tau} \omega_K &= \frac{1}{2} \sum_{i=1}^F \left[\left(\iota_{\partial_{p_i}^{[b_i]} + \partial_{r_i}^{[c_i]}} \right) \omega_K - \left(\iota_{\partial_{q_i}^{[b_i]} + \partial_{s_i}^{[c_i]}} \right) \omega_K \right] \\ &= \frac{1}{2} \sum_{i=1}^F \left[\left(\sum_{k=p_i+1}^{N_{b_i}} d\ell_k^{[b_i]} - \sum_{k=1}^{p_i-1} d\ell_k^{[b_i]} + \sum_{u=r_i+1}^{N_{c_i}} d\ell_u^{[c_i]} - \sum_{u=1}^{r_i-1} d\ell_u^{[c_i]} \right) \right. \\ &\quad \left. - \left(\sum_{m=q_i+1}^{N_{b_i}} d\ell_m^{[b_i]} - \sum_{m=1}^{q_i-1} d\ell_m^{[b_i]} + \sum_{v=s_i+1}^{N_{c_i}} d\ell_v^{[c_i]} - \sum_{v=1}^{s_i-1} d\ell_v^{[c_i]} \right) \right] \\ &= \sum_{i=1}^F \left(\frac{1}{2} d\ell_{p_i}^{[b_i]} + \frac{1}{2} d\ell_{q_i}^{[b_i]} + \sum_{p_i < k < q_i} d\ell_k^{[b_i]} + \frac{1}{2} d\ell_{r_i}^{[c_i]} + \frac{1}{2} d\ell_{s_i}^{[c_i]} + \sum_{r_i < u < s_i} d\ell_u^{[c_i]} \right). \end{aligned}$$

This indeed coincides with $d\ell$. □

4.2.2 — PROOF OF THE COMBINATORIAL WOLPERT'S FORMULA

For a fixed oriented surface Σ , denote by $\bar{\Sigma}$ the surface with opposite orientation. The following lemma is based on the work of Wolpert [Wol85, Lemma 1.1].

LEMMA 4.2.4. *Let Σ be a bordered surface, $\rho: \Sigma \rightarrow \bar{\Sigma}$ be an isotopy class of orientation-reversing diffeomorphism that restricts to the identity on the boundary. Fix $\gamma \in \mathcal{S}_\Sigma$. Then ρ induces a homeomorphism $\mathcal{T}_\Sigma^{\text{comb}} \rightarrow \mathcal{T}_{\bar{\Sigma}}^{\text{comb}}$ and*

- $\rho^* d\ell(\gamma) = d\ell(\rho \circ \gamma)$,
- $\rho^* \omega_K = -\omega_K$,
- if ρ fixes γ , then $\rho^* d\tau(\gamma) = -d\tau(\gamma) + \frac{m}{2} d\ell(\gamma)$ for some $m \in \mathbb{Z}$.

Here $d\tau$ is the differential form dual with respect to ω_K to the vector field ∂_τ of Equation (4.2.3).

Proof. The map $\mathcal{T}_\Sigma^{\text{comb}} \rightarrow \mathcal{T}_{\bar{\Sigma}}^{\text{comb}}$ is simply the composition of ρ with the marking. It inverts the orientations of all curves, but it fixes the length functions, hence the first point. Further,

the Ψ -classes are going to be calculated using the opposite orientation, which yields the sign for the second point. The last point follows from the fact that the elements of $\text{Stab}(\gamma)$ are generated by (half-) Dehn twists along curves that do not intersect γ . Then, as ρ reverses the orientation of the surface, $d\tau(\gamma)$ acquires a sign from the orientation reversal and an ambiguity of $\frac{1}{2}\mathbb{Z}d\ell(\gamma)$ from potential (half-) Dehn twists along γ . \square

We are now ready to give a proof of the combinatorial Wolpert's formula.

Proof of Theorem 4.2.1. Fix a seamed pants decomposition. We know that (ℓ_i, τ_i) give global coordinates on $\mathcal{T}_\Sigma^{\text{comb}}(L)$. Therefore, on the top-dimensional cells

$$\omega_K = \sum_{i < j} a_{ij} d\ell_i \wedge d\ell_j + \sum_{i < j} b_{ij} d\tau_i \wedge d\tau_j + \sum_{i,j} c_{ij} d\ell_i \wedge d\tau_j$$

for some functions a_{ij} , b_{ij} and c_{ij} . Notice that from Proposition 4.2.3 we have

$$\iota_{\partial\tau_i} \omega_K = \sum_{i < j} b_{ij} d\tau_j - \sum_{j < i} b_{ji} d\tau_j + \sum_j c_{ji} d\ell_j = d\ell_i,$$

and hence $b_{ij} = 0$, $c_{ij} = \delta_{ij}$. Finally, if ρ is the isotopy class of an orientation-reversing diffeomorphism fixing γ_i , we have

$$\rho_* \left(\frac{\partial}{\partial \ell_i} \right) = \frac{\partial}{\partial \ell_i} + \frac{m_i}{2} \frac{\partial}{\partial \tau_i}.$$

Therefore from Lemma 4.2.4, for $i < j$, we find

$$\begin{aligned} a_{ij} &= \omega_K \left(\frac{\partial}{\partial \ell_i}, \frac{\partial}{\partial \ell_j} \right) = \omega_K \left(\frac{\partial}{\partial \ell_i} + \frac{m_i}{2} \frac{\partial}{\partial \tau_i}, \frac{\partial}{\partial \ell_j} + \frac{m_j}{2} \frac{\partial}{\partial \tau_j} \right) = \omega_K \left(\rho_* \frac{\partial}{\partial \ell_i}, \rho_* \frac{\partial}{\partial \ell_j} \right) \\ &= \rho^* \omega_K \left(\frac{\partial}{\partial \ell_i}, \frac{\partial}{\partial \ell_j} \right) = -\omega_K \left(\frac{\partial}{\partial \ell_i}, \frac{\partial}{\partial \ell_j} \right). \end{aligned}$$

and thus $a_{ij} = 0$. This proves the result on the top-dimensional cells. To extend it to cells $\iota_L: \mathcal{Z}_{\Sigma,G}(L) \hookrightarrow \mathcal{T}_\Sigma^{\text{comb}}(L)$ of positive codimension, we can consider it at the boundary of a top-dimensional cell. Then ι_L^* simply sets $d\ell_e = 0$ for each edge e of zero length on the boundary of the top-dimensional cell. This has exactly the same affect as excluding such edges from the sum in Definition 4.1.1 of Ψ_i , which coincides with the definition of Ψ_i on cells of positive codimension. \square

4.3 — INTEGRATION OVER THE COMBINATORIAL MODULI SPACES

In this section we establish an integration result, analogous to [Miro7a, Theorem 7.1] for $(\mathcal{M}_{g,n}(L), \omega_{\text{WP}})$, exploiting the symplectic structure of $(\mathcal{M}_{g,n}^{\text{comb}}(L), \omega_K)$ via the combinatorial Wolpert formula (4.2.1). This improves the results of [BCSW12, Theorem 1.1], which in fact can be extended from their original use to the integration of functions with support restricted² to “small pairs of pants”.

²This restriction on the support is related to the one appearing, e.g., in Equation (5.2.9).

Let us introduce some notation. Consider a bordered surface Σ of type (g, n) and let γ be a primitive multicurve with ordered components $(\gamma_j)_{j=1}^k$. We denote by Γ the orbit $\text{Mod}_\Sigma^\partial \cdot \gamma$ (although it is not important for what follows, such an object can be seen as a stable graph with ordered edges). Furthermore, consider an assignment

$$\Sigma' \mapsto \Xi_{\Sigma'} \in \text{Mes}(\mathcal{T}_{\Sigma'}^{\text{comb}}, \mu_{\mathbb{K}}) \quad (4.3.1)$$

of a measurable function on the combinatorial Teichmüller space to each bordered surface Σ' diffeomorphic to the cut surface Σ_γ . We assume that for any diffeomorphism $\phi: \Sigma' \rightarrow \Sigma''$ which preserves the labelling of the boundary components, we have $\phi_* \Xi_{\Sigma'} = \Xi_{\Sigma''}$ where ϕ_* is the map induced between the combinatorial Teichmüller spaces. In particular, $\Xi_{\Sigma'}$ is invariant under the action of $\text{Mod}_{\Sigma'}^\partial$, and descends to a function Ξ_Γ on the moduli space

$$\mathcal{M}_\Gamma^{\text{comb}} = \prod_{v \in \pi_0(\Sigma_\gamma)} \mathcal{M}_{g(v), n(v)}^{\text{comb}}, \quad (4.3.2)$$

which depends on Γ only. We can decompose $\pi_0(\partial\Sigma') = \pi_0(\gamma) \sqcup \pi_0(\gamma) \sqcup \pi_0(\partial\Sigma)$, so it makes sense to consider³ $\mathcal{M}_\Gamma^{\text{comb}}(\ell, \ell, L)$. We further assume that, for almost every $(\ell, L) \in \mathbb{R}_+^k \times \mathbb{R}_+^n$, Ξ_Γ is integrable with respect to the Kontsevich measure on $\mathcal{M}_\Gamma^{\text{comb}}(\ell, \ell, L)$, and we denote

$$\langle \Xi_\Gamma \rangle(\ell, \ell, L) = \int_{\mathcal{M}_\Gamma^{\text{comb}}(\ell, \ell, L)} \Xi_\Gamma d\mu_{\mathbb{K}}. \quad (4.3.3)$$

Finally, consider a measurable function $f: \mathbb{R}_+^n \times \mathbb{R}_+^k \rightarrow \mathbb{R}$ and define a new function $\Xi_\Sigma^{f, \Gamma}$ on $\mathcal{T}_\Sigma^{\text{comb}}$ by setting

$$\Xi_\Sigma^{f, \Gamma}(\mathbb{G}) = \sum_{\alpha \in \Gamma} f(\vec{\ell}_\mathbb{G}(\partial\Sigma), \vec{\ell}_\mathbb{G}(\alpha)) \Xi_{\Sigma_\alpha}(\mathbb{G}|_{\Sigma_\alpha}), \quad (4.3.4)$$

where $\vec{\ell}_\mathbb{G}(\partial\Sigma) = (\ell_\mathbb{G}(\partial_i\Sigma))_{i=1}^n$ and $\vec{\ell}_\mathbb{G}(\alpha) = (\ell_\mathbb{G}(\alpha_j))_{j=1}^k$. When the series (4.3.4) is absolutely convergent, it defines a $\text{Mod}_\Sigma^\partial$ -invariant function, which descends to a function $\Xi_{g,n}^{f, \Gamma}$ on the moduli space $\mathcal{M}_{g,n}^{\text{comb}}$.

PROPOSITION 4.3.1. *Assume that the series (4.3.4) is absolutely convergent, and that for almost every $L \in \mathbb{R}_+^n$ its limit is integrable with respect to $\mu_{\mathbb{K}}$ on $\mathcal{M}_\Sigma^{\text{comb}}(L)$. Assume as well that for almost every $(L, \ell) \in \mathbb{R}_+^n \times \mathbb{R}_+^k$ the function Ξ_Γ is integrable on $\mathcal{M}_\Gamma^{\text{comb}}(\ell, \ell, L)$ with respect to the Kontsevich measure. Then*

$$\int_{\mathcal{M}_{g,n}^{\text{comb}}(L)} \Xi_{g,n}^{f, \Gamma} d\mu_{\mathbb{K}} = \int_{\mathbb{R}_+^k} f(L, \ell) \langle \Xi_\Gamma \rangle(\ell, \ell, L) \prod_{j=1}^k \ell_j d\ell_j. \quad (4.3.5)$$

Proof. We adapt Mirzakhani's proof of [Miro7a, Theorem 7.1], which concerned the hyperbolic setting with $\Xi_{\Sigma'} = 1$, functions $f(L, \ell) = F(\ell_1 + \dots + \ell_k)$ and γ a primitive multicurve with unordered components. Because the components are ordered, our formula does not contain automorphism factors. The main difference is instead that, in the combinatorial setting, we have to remove the zero measure set of pathological twists.

³As explained in Section 2.4.1, the boundary components of the cut surface are labeled in a specific way. Thus, with the symbol $\mathcal{M}_\Gamma^{\text{comb}}(\ell, \ell, L)$, we mean the product of moduli spaces with fixed boundary lengths from $\ell_1, \dots, \ell_k, L_1, \dots, L_n$ ordered in such a way that they match the labeled boundary components of the cut surface.

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Consider the space

$$\mathcal{M}_{g,n}^{\text{comb},\Gamma}(L) = \mathcal{T}_{\Sigma}^{\text{comb}}(L) \Big/ \bigcap_{j=1}^k \text{Stab}(\gamma_j),$$

where $\text{Stab}(\gamma_j)$ is the stabiliser of γ_j in $\text{Mod}_{\Sigma}^{\partial}$. We denote by $\Pi^{\Gamma}: \mathcal{M}_{g,n}^{\text{comb},\Gamma}(L) \rightarrow \mathcal{M}_{g,n}^{\text{comb}}(L)$ the natural projection. Notice that

$$\mathcal{M}_{g,n}^{\text{comb},\Gamma}(L) \cong \left\{ (G, \alpha) \mid G \in \mathcal{M}_{g,n}^{\text{comb}}(L), \quad \alpha \in \Gamma \right\}.$$

Since the symplectic structure on $\mathcal{T}_{\Sigma}^{\text{comb}}$ is invariant under the action of the pure mapping class group, it induces a symplectic structure on $\mathcal{M}_{g,n}^{\text{comb},\Gamma}(L)$, which is the same as the pullback $(\Pi^{\Gamma})^* \omega_{\mathbb{K}}$. We denote the associated measure by $\mu_{\mathbb{K}}^{\Gamma}$.

Consider now the map $\mathcal{T}_{\Sigma}^{\text{comb}}(L) \rightarrow \mathbb{R}_+^k$ given by the tuple of combinatorial lengths of the components of γ . It descends to a map $\mathcal{L}^{\Gamma}: \mathcal{M}_{g,n}^{\text{comb},\Gamma}(L) \rightarrow \mathbb{R}_+^k$, and let $\mathcal{M}_{g,n}^{\text{comb},\Gamma}(L)[\ell] = (\mathcal{L}^{\Gamma})^{-1}(\ell)$ be the level sets for $\ell \in \mathbb{R}_+^k$. We have a map

$$\Pi: \mathcal{M}_{g,n}^{\text{comb},\Gamma}(L)[\ell] \longrightarrow \mathcal{M}_{\Gamma}^{\text{comb}}(\ell, \ell, L) \quad (4.3.6)$$

defined in the natural way: given an element $(G, \alpha) \in \mathcal{M}_{g,n}^{\text{comb},\Gamma}(L)[\ell]$, we take a lift $\mathbb{G} \in \mathcal{T}_{\Sigma}^{\text{comb}}(L)$ of G , we restrict \mathbb{G} to the cut surface Σ_{α} as explained in Section 3.3 and we project the restriction to the moduli space $\mathcal{M}_{\Sigma_{\alpha}}^{\text{comb}}(\ell, \ell, L) \cong \mathcal{M}_{\Gamma}^{\text{comb}}(\ell, \ell, L)$. The result does not depend on the choice of the lift \mathbb{G} since we are projecting to the combinatorial moduli space of Σ_{α} after restriction.

Notice that the spaces on both sides of (4.3.6) have a natural measure: $\mathcal{M}_{g,n}^{\text{comb},\Gamma}(L)[\ell]$ is equipped with the disintegration of $\mu_{\mathbb{K}}^{\Gamma}$ along \mathcal{L}^{Γ} , and $\mathcal{M}_{\Gamma}^{\text{comb}}(\ell, \ell, L)$ with its Kontsevich measure. By construction of (4.3.4) and the property of disintegration,

$$\begin{aligned} \int_{\mathcal{M}_{g,n}^{\text{comb}}(L)} \Xi_{g,n}^{f,\Gamma} d\mu_{\mathbb{K}} &= \int_{\mathcal{M}_{g,n}^{\text{comb},\Gamma}(L)} (f \circ \mathcal{L}^{\Gamma}) \cdot (\Xi_{\Gamma} \circ \Pi) d\mu_{\mathbb{K}}^{\Gamma} \\ &= \int_{\mathbb{R}_+^k} f(L, \ell) \left(\int_{\mathcal{M}_{g,n}^{\text{comb},\Gamma}(L)[\ell]} (\Xi_{\Gamma} \circ \Pi) d\mu_{\mathbb{K}}^{\Gamma} \right) \prod_{j=1}^k d\ell_j. \end{aligned} \quad (4.3.7)$$

We can complete γ into a seamed pants decomposition and use the Fenchel–Nielsen coordinates to describe the space $\mathcal{M}_{g,n}^{\text{comb},\Gamma}(L)[\ell]$. The combinatorial Wolpert formula (4.2.1) implies that the measure $\mu_{\mathbb{K}}^{\Gamma}$ has a product structure with respect to the fibration Π . Besides, the fibre of $G' \in \mathcal{M}_{\Gamma}^{\text{comb}}(\ell, \ell, L)$ is identified with an open subset of full measure in $\prod_{j=1}^k \mathbb{R}/(2^{-t_j} \ell_j \mathbb{Z})$, where

$$t_j = \begin{cases} 1 & \text{if } \gamma_j \text{ separates off a torus with one boundary,} \\ 0 & \text{otherwise.} \end{cases}$$

This follows from the description of the image of the Fenchel–Nielsen coordinates in Theorem 3.4.5. The factor of 2^{-t_j} in the case when γ_j separates off a torus with one boundary is due to the fact that any element in $\mathcal{M}_{1,1}^{\text{comb}}(L)$ comes with an elliptic involution, so $\text{Stab}(\gamma_i)$ contains the half-twist along γ_j and the fundamental region of the twist coordinate in the

combinatorial moduli space becomes $[0, \ell_j/2]$ minus a measure-zero set. So, for any open set $U \subseteq \mathcal{M}_\Gamma^{\text{comb}}(\ell, \ell, L)$, we have

$$\prod_{j=1}^k 2^{-t_j} \int_{\Pi^{-1}(U)} (\Xi_\Gamma \circ \Pi) d\mu_K^\Gamma = \prod_{j=1}^k 2^{-t_j} \ell_j \int_U \Xi_\Gamma d\mu_K$$

whenever the functions we wish to integrate are integrable. We noticed that the integrals over $\mathcal{M}_{1,1}^{\text{comb}}$ have an extra factor of $\frac{1}{2}$ due to the presence of the elliptic involution, while such a factor is not present on $\mathcal{M}_{g,n}^{\text{comb},\Gamma}(L)[\ell]$. We must therefore include an extra factor 2^{-t_j} in the right-hand side, and this cancels the factor of 2^{-t_j} coming from the half-twist. By a partition of unity argument, we obtain

$$\int_{\mathcal{M}_{g,n}^{\text{comb},\Gamma}(L)[\ell]} (\Xi_\Gamma \circ \Pi) d\mu_K^\Gamma = \langle \Xi_\Gamma \rangle(\ell, \ell, L) \prod_{j=1}^k \ell_j,$$

which we insert in (4.3.7) to complete the proof. \square

Remark 4.3.2. In Mirzakhani’s work [Miro7a] there is an unnatural convention for the integral over $\mathcal{M}_{1,1}(L)$ which does not include the factor of $\frac{1}{2}$ coming from the elliptic involution. For this reason, she finds an extra factor $\prod_{j=1}^k 2^{-t_j}$.

Remark 4.3.3. As briefly mentioned before, an equivalent way to state Proposition 4.3.1 relies on the notion of stable graphs (see Definition 2.1.7). In this language, the set of connected components of Σ_γ is the set V_Γ of vertices of the stable graph with ordered edges $\Gamma = \text{Mod}_{\Sigma}^\partial \gamma$, the set of components of γ descends to the set E_Γ of edges of Γ , and the set of boundary component of Σ is the set Λ_Γ of leaves of Γ . With this notation, Equation (4.3.5) becomes

$$\int_{\mathcal{M}_{g,n}^{\text{comb}}(L)} \Xi_{g,n}^{f,\Gamma} d\mu_K = \int_{\mathbb{R}_+^{E_\Gamma}} f((L\lambda)_{\lambda \in \Lambda_\Gamma}, (\ell_e)_{e \in E_\Gamma}) \prod_{v \in V_\Gamma} \langle \Xi_{g(v),n(v)} \rangle((\ell_e)_{e \in E_v}, (L\lambda)_{\lambda \in \Lambda_v}) \prod_{e \in E_\Gamma} \ell_e d\ell_e. \quad (4.3.8)$$

If in the above arguments we do not suppose that the components of the primitive multicurve are ordered, then we have to include the automorphism factor $|\text{Aut}(\Gamma)|$ dividing the right-hand side of Equation (4.3.8). Moreover, if the multicurve is not primitive but comes with a weight $a \in \mathbb{Z}_+^{E_\Gamma}$, we rather have to consider the automorphism factor $|\text{Aut}(\Gamma, a)|$ of the weighted multicurve.

Remark 4.3.4. By applying Fubini–Tonelli, we can get rid of the integrability assumption for $\Xi_{g,n}^{f,\Gamma}$ if we suppose that each term in the series (4.3.4) is non-negative. In this case, the disintegration property still holds in (4.3.7), and the result of Proposition 4.3.1 becomes an equality between integrals taking values in $[0, +\infty]$.

CHAPTER 5 — FUNCTIONS FROM GEOMETRIC RECURSION

Like the Weil–Petersson volumes of $\mathcal{M}_{g,n}(L)$, the Kontsevich volumes of $\mathcal{M}_{g,n}^{\text{comb}}(L)$ satisfy topological recursion in the sense of Eynard–Orantin. In particular, they can be computed recursively in $2g - 2 + n$. We demonstrate that this type of recursive relation (and many others) can be obtained from a Mirzakhani-type identity (hence, before integration) on the combinatorial Teichmüller space.

More generally, in this chapter we set up the geometric recursion to construct mapping class group invariant functions on $\mathcal{T}_{\Sigma}^{\text{comb}}$. As usual, let us denote by P a pair of pants and by T a torus with one boundary component, and let A, B, C be measurable functions on $\mathcal{T}_P^{\text{comb}} \cong \mathbb{R}_+^3$, and D_T be a measurable function on $\mathcal{T}_T^{\text{comb}}$ which is mapping class group invariant. The following result is Theorem 5.1.4 in the main text.

THEOREM 5.A (Combinatorial GR is well-defined). *If (A, B, C, D) are “admissible”, then the following definitions are well-posed, and assign functorially to any bordered surface Σ a measurable function Ξ_{Σ} on $\mathcal{T}_{\Sigma}^{\text{comb}}$, called (combinatorial) geometric recursion amplitude.*

- $\Xi_P(\mathbb{G}) = A(\vec{\ell}_{\mathbb{G}}(\partial P))$, and $\Xi_T = D_T$.
- If $\Sigma = \Sigma_1, \dots, \Sigma_k$ is a disjoint union, $\Xi_{\Sigma_1 \sqcup \dots \sqcup \Sigma_k}(\mathbb{G}_1, \dots, \mathbb{G}_k) = \prod_{i=1}^k \Xi_{\Sigma_i}(\mathbb{G}_i)$.
- If Σ is connected and has Euler characteristic $\chi_{\Sigma} < -1$, define Ξ_{Σ} via geometric recursion:

$$\Xi_{\Sigma}(\mathbb{G}) = \sum_{m=2}^n \sum_{P \in \mathcal{B}_{\Sigma,m}} B(\vec{\ell}_{\mathbb{G}}(\partial P)) \Xi_{\Sigma-P}(\mathbb{G}|_{\Sigma-P}) + \frac{1}{2} \sum_{P \in \mathcal{C}_{\Sigma}} C(\vec{\ell}_{\mathbb{G}}(\partial P)) \Xi_{\Sigma-P}(\mathbb{G}|_{\Sigma-P}). \quad (5.0.1)$$

where $\mathcal{B}_{\Sigma,m}$ and \mathcal{C}_{Σ} are the sets of homotopy classes of embedded pairs of pants in Σ appearing in Mirzakhani’s identity (see Section 2.4), and $\mathbb{G}|_{\Sigma-P}$ is the result of cutting the combinatorial structure \mathbb{G} and restricting it to $\Sigma - P$.

Further, the function Ξ_{Σ} is invariant under mapping classes of Σ preserving $\partial_1 \Sigma$.

By means of the integration result (Proposition 4.3.1), we show that integrating combinatorial geometric recursion amplitudes automatically yields functions of boundary lengths that satisfy topological recursion. If Σ is a connected bordered surface of type (g, n) , let us denote by $\Xi_{g,n}$ the function induced by Ξ_{Σ} on $\mathcal{M}_{g,n}^{\text{comb}}$. The following result is Theorem 5.2.2 in the main text.

THEOREM 5.B (TR from GR). *Let (A, B, C, D) be “strongly admissible” initial data. Then the integrals*

$$\langle \Xi_{g,n} \rangle(L) = \int_{\mathcal{M}_{g,n}^{\text{comb}}(L)} \Xi_{g,n} d\mu_{\mathbb{K}} \quad (5.0.2)$$

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exist and define measurable functions on $L \in \mathbb{R}_+^n$ that satisfy topological recursion:

$$\begin{aligned}
\langle \Xi_{g,n} \rangle(L_1, \dots, L_n) &= \\
&= \sum_{m=2}^n \int_{\mathbb{R}_+} B(L_1, L_m, \ell) \langle \Xi_{g,n-1} \rangle(\ell, L_2, \dots, \widehat{L}_m, \dots, L_n) \ell d\ell \\
&\quad + \frac{1}{2} \int_{\mathbb{R}_+^2} C(L_1, \ell, \ell') \left(\langle \Xi_{g-1,n+2} \rangle(\ell, \ell', L_2, \dots, L_n) \right. \\
&\quad \left. + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2, \dots, n\}}} \langle \Xi_{g_1, 1+|I_1|} \rangle(\ell, L_{I_1}) \langle \Xi_{g_2, 1+|I_2|} \rangle(\ell', L_{I_2}) \right) \ell \ell' d\ell d\ell'.
\end{aligned} \tag{5.0.3}$$

A similar result holds if, instead of integrating against μ_K , we sum over the lattice $\mathcal{M}_{g,n}^{\text{comb}, \mathbb{Z}}(L) \subset \mathcal{M}_{g,n}^{\text{comb}}(L)$ consisting of metric ribbon graphs with integral edge lengths. This has no counterpart in the hyperbolic world. Due to the existence of pathological twists for the gluing which, although rare in the whole space, they could (and in fact sometimes do) hit the lattice, this is only possible under extra conditions for the initial data B and C .

THEOREM 5.C (Discrete TR from GR). *Let (A, B, C, D) be such that B and C are “supported on small pairs of pants”. Then the lattice sums*

$$\langle \Xi_{g,n}^{\mathbb{Z}} \rangle(L) = \sum_{G \in \mathcal{M}_{g,n}^{\text{comb}, \mathbb{Z}}(L)} \frac{\Xi_{g,n}(G)}{|\text{Aut}(G)|} \tag{5.0.4}$$

define functions of $L \in \mathbb{Z}_+^n$ which are zero whenever $\sum_{i=1}^n L_i$ is odd, and otherwise satisfy the discrete topological recursion:

$$\begin{aligned}
\langle \Xi_{g,n}^{\mathbb{Z}} \rangle(L_1, \dots, L_n) &= \\
&= \sum_{m=2}^n \sum_{\ell \geq 1} \ell B^{\mathbb{Z}}(L_1, L_m, \ell) \langle \Xi_{g,n-1}^{\mathbb{Z}} \rangle(\ell, L_2, \dots, \widehat{L}_m, \dots, L_n) \\
&\quad + \frac{1}{2} \sum_{\ell, \ell' \geq 1} \ell \ell' C^{\mathbb{Z}}(L_1, \ell, \ell') \left(\langle \Xi_{g-1,n+1}^{\mathbb{Z}} \rangle(\ell, \ell', L_2, \dots, L_n) \right. \\
&\quad \left. + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2, \dots, n\}}} \langle \Xi_{g_1, 1+|I_1|}^{\mathbb{Z}} \rangle(\ell, L_{I_1}) \langle \Xi_{g_2, 1+|I_2|}^{\mathbb{Z}} \rangle(\ell', L_{I_2}) \right),
\end{aligned} \tag{5.0.5}$$

where $X^{\mathbb{Z}}(L_1, L_2, L_3)$ is equal to $X(L_1, L_2, L_3)$ if $L_1 + L_2 + L_3$ is an even integer and 0 otherwise.

We also prove the combinatorial analogue of Theorem 2.4.26 and Corollary 2.4.28: the combinatorial length statistics of primitive multicurves is computed by geometric recursion for twisted initial data, and its average over the moduli space is computed by topological recursion and by a sum over stable graphs (cf. Theorem 5.4.1 and Corollary 5.4.3 in the main text). This fact will be the starting point of Part III, where we analyse the enumeration of multicurves in both the hyperbolic and combinatorial settings.

As applications of this general theory we can re-prove known results in a completely geometric and uniform way, as well as obtaining new results. A key role for applications is played by the combinatorial analogue of the Mirzakhani–McShane identity, whose proof transposes the original strategy of Mirzakhani [Miro7a] to the combinatorial world (where it is much simpler).

THEOREM 5.D (Combinatorial Mirzakhani identity). *Denote by $[x]_+ = \max\{x, 0\}$ and define the Kontsevich initial data*

$$\begin{aligned} A^K(L_1, L_2, L_3) &= 1, \\ B^K(L_1, L_2, \ell) &= \frac{1}{2L_1} ([L_1 - L_2 - \ell]_+ - [-L_1 + L_2 - \ell]_+ + [L_1 + L_2 - \ell]_+), \\ C^K(L_1, \ell, \ell') &= \frac{1}{L_1} [L_1 - \ell - \ell']_+, \\ D_T^K(\mathbb{G}) &= \sum_{\gamma \in \mathcal{S}_T} C^K(\ell_{\mathbb{G}}(\partial T), \ell_{\mathbb{G}}(\gamma), \ell_{\mathbb{G}}(\gamma)), \end{aligned} \tag{5.0.6}$$

where \mathcal{S}_T is the set of homotopy classes of essential simple closed curves in T . The corresponding geometric recursion amplitudes are the constant function 1 on $\mathcal{T}_{\Sigma}^{\text{comb}}$ for any bordered surface Σ :

$$1 = \sum_{m=2}^n \sum_{P \in \mathcal{B}_{\Sigma, m}} B^K(\vec{\ell}_{\mathbb{G}}(\partial P)) + \frac{1}{2} \sum_{P \in \mathcal{C}_{\Sigma}} C^K(\vec{\ell}_{\mathbb{G}}(\partial P)). \tag{5.0.7}$$

Combining this result with Theorem 5.B gives a new proof of the topological recursion for Kontsevich volumes, whereas Theorem 5.C gives a new proof of the topological recursion for the lattice point count. The former is equivalent to a proof of Witten’s conjecture. The latter is a result known since Norbury [Nor10].

We conclude by giving a geometric description via the spine map of certain functions on Teichmüller space that computes ψ -classes intersection, once integrated over the moduli space against the Weil–Petersson measure.

5.0.1 — RELATION WITH PREVIOUS WORKS

The above theorems are highly inspired by geometric recursion in the hyperbolic setting [ABO17], which in turn is inspired by Mirzakhani’s identity [Miro7a] and its relation with topological recursion.

As of Witten–Kontsevich result, there are many known proofs. To the best of our knowledge, the only geometric proof (i.e. a proof based on the geometry of the combinatorial moduli space only) was proposed by Bennett–Cochran–Safnuk–Woskoff in [BCSW12]. In this regard, the novel element of our work is firstly to make evident the connection of the partition of unity of [BCSW12, Section 4] with a Mirzakhani–McShane identity. Then the mechanism of integration in [BCSW12], which relies on a *local* torus action and was valid only for functions of restricted support such as the Kontsevich initial data, gets realised as a special case of the more general Theorem 5.A, by means of the *global* combinatorial Fenchel–Nielsen coordinates and of Theorem 5.B.

Regarding the enumeration of lattice points in $\mathcal{M}_{g,n}^{\text{comb}, \mathbb{Z}}(L)$ has been connected to matrix integrals in the early works of Chekhov and Makeenko [Che93; CM92b; CM92a] and further related to enumeration of chord diagrams in [ACNP15a; ACNP15b]. At that point,

Schwinger–Dyson equations for such models give rise to equations that are eventually (but non-obviously) equivalent to Norbury’s recursion [Nor10].

The scheme of proofs we put forward transcends the algebraic manipulations, whose geometric meaning is unclear, pertaining to the realm of matrix integrals and which were necessary in both Kontsevich’s original proof and in Chekhov–Makeenko’s works.

5.0.2 — ORGANISATION OF THE CHAPTER

The chapter is organised as follows.

- In Section 5.1 we set up geometric recursion on the combinatorial Teichmüller, proving Theorem 5.A for admissible initial data.
- Section 5.2 is devoted to the integration of geometric recursion amplitudes for strongly admissible initial data (Theorem 5.B), and the discrete integration (sum over lattice points) for initial data supported on small pairs of pants (Theorem 5.B).
- Section 5.3 contains the main example of the above setup, namely a combinatorial analogue of the Mirzakhani–McShane identity, and its (discrete) integration.
- In Section 5.4 we prove that combinatorial length statistics are computed by geometric recursion for twisted initial data, and the associated average by topological recursion and a sum over stable graphs.
- To conclude, in Section 5.5 we show how certain hyperbolic geometric recursion amplitudes that compute ψ -classes intersections have a geometric interpretation in terms of a random process involving the spine construction.

5.1 — GEOMETRIC RECURSION IN THE COMBINATORIAL SETTING

Recall from Section 2.4.2 that geometric recursion starts with a functor E from the category \mathbf{B}_1 of bordered surfaces to the category $\mathbf{TVect}_{\mathbb{R}}$ of topological vector spaces and aims at constructing E -valued functorial assignments

$$\Sigma \mapsto \Omega_{\Sigma} \in E(\Sigma) \tag{5.1.1}$$

starting from some appropriate initial data.

Let us describe here in concrete terms the geometric recursion for the functor $E(\Sigma) = \text{Mes}(\mathcal{T}_{\Sigma}^{\text{comb}})$ of \mathbb{C} -valued measurable functions on the combinatorial Teichmüller space of Σ .

DEFINITION 5.1.1. Combinatorial geometric recursion *initial data* consist of a quadruple (A, B, C, D) where

- A, B, C are measurable functions on $\mathcal{T}_P^{\text{comb}} \cong \mathbb{R}_+^3$,
- D is an assignment $T \mapsto D_T \in \text{Mes}(\mathcal{T}_T^{\text{comb}})$, for each T torus with one boundary component,

satisfying the following axioms.

- $A(L_1, L_2, L_3) = A(L_1, L_3, L_2)$ and $C(L_1, L_2, L_3) = C(L_1, L_3, L_2)$.
- The assignment $T \mapsto D_T$ is functorial, and in particular D_T is a mapping class group invariant function. We also denote by D the induced function on $\mathcal{M}_{1,1}^{\text{comb}}(L)$.

DEFINITION 5.1.2. We recursively construct an assignment $\Sigma \mapsto \Xi_\Sigma \in \text{Mes}(\mathcal{T}_\Sigma^{\text{comb}})$ as follows. We let

$$\Xi_\emptyset = 1, \quad \Xi_P(\mathbb{G}) = A(\vec{\ell}_\mathbb{G}(\partial P)), \quad \Xi_T = D_T, \quad (5.1.2)$$

where $\vec{\ell}_\mathbb{G}(\partial P)$ is the ordered triple of combinatorial lengths of the boundary components of P . For disconnected surfaces we set

$$\Xi_{\Sigma_1 \sqcup \dots \sqcup \Sigma_k}(\mathbb{G}_1, \dots, \mathbb{G}_k) = \prod_{i=1}^k \Xi_{\Sigma_i}(\mathbb{G}_i). \quad (5.1.3)$$

For connected surfaces with Euler characteristic $\chi_\Sigma < -1$, we define Ξ_Σ inductively on χ_Σ by *geometric recursion*:

$$\Xi_\Sigma(\mathbb{G}) = \sum_{m=2}^n \sum_{P \in \mathcal{B}_{\Sigma, m}} B(\vec{\ell}_\mathbb{G}(\partial P)) \Xi_{\Sigma-P}(\mathbb{G}|_{\Sigma-P}) + \frac{1}{2} \sum_{P \in \mathcal{C}_\Sigma} C(\vec{\ell}_\mathbb{G}(\partial P)) \Xi_{\Sigma-P}(\mathbb{G}|_{\Sigma-P}). \quad (5.1.4)$$

Here $\mathcal{B}_{\Sigma, m}$ and \mathcal{C}_Σ are the sets of homotopy classes of embedded pairs of pants bounding $\partial_1 \Sigma$ introduced in Equation (2.4.21) and appearing in Mirzakhani's recursion. Moreover, $\mathbb{G}|_{\Sigma-P}$ have been defined by the cutting procedure in Section 3.3, and to define the labelling of the boundary components of $\Sigma - P$, we say that the (labelled) boundary components of P that appear in $\Sigma - P$ are put first, followed by the (labelled) boundary components of Σ that appear in $\Sigma - P$.

As in the hyperbolic setting, convergence of the series (5.1.4) should be discussed. Denote by $\mathcal{T}_\Sigma^{\text{comb}, (\epsilon)} \subset \mathcal{T}_\Sigma^{\text{comb}}$ the ϵ -thick part of the combinatorial Teichmüller space, i.e. the set of $\mathbb{G} \in \mathcal{T}_\Sigma^{\text{comb}}$ such that $\ell_\mathbb{G}(\gamma) \geq \epsilon$ for any simple closed curve γ (including boundary components).

DEFINITION 5.1.3. We say that the initial data (A, B, C, D) are *admissible* if they satisfy the same conditions appearing in Definition 2.4.20, except we use $\mathcal{T}_T^{\text{comb}, (\epsilon)}$ in the condition for D_T .

THEOREM 5.1.4. *If (A, B, C, D) are admissible initial data, then for any bordered surface Σ*

- *the series (5.1.4) converges absolutely and uniformly on any compact of $\mathcal{T}_\Sigma^{\text{comb}}$,*
- *$\Sigma \mapsto \Xi_\Sigma \in \text{Mes}(\mathcal{T}_\Sigma^{\text{comb}})$ is a well-defined functorial assignment (in particular, Ξ_Σ is Mod_Σ^0 -invariant),*
- *there exists $u \geq 0$ depending only on the topological type of Σ , such that for any $\epsilon > 0$ we have*

$$\sup_{\mathbb{G} \in \mathcal{T}_\Sigma^{\text{comb}, (\epsilon)}} |\Xi_\Sigma(\mathbb{G})| \leq K_\epsilon \prod_{b \in \pi_0(\partial \Sigma)} (1 + \ell_\mathbb{G}(b))^u \quad (5.1.5)$$

for some constant K_ϵ depending only on ϵ and the topological type of Σ .

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Although the spaces $\text{Mes}(\mathcal{T}_\Sigma)$ and $\text{Mes}(\mathcal{T}_\Sigma^{\text{comb}})$ can be identified via the spine homeomorphism of Theorem 3.1.11, the way we measure lengths and we restrict to P and $\Sigma - P$ is different and as a result the hyperbolic/combinatorial structure in (2.4.33) and (5.1.4) are completely different. So, for identical initial data, the hyperbolic and the combinatorial geometric recursion do not produce the same functions (even after identification of their domains). Geometrically, this is due to the fact that the spine map is not compatible with cutting and gluing. The relation between the hyperbolic and combinatorial geometric recursion is elucidated in Section 6.3.

Proof. The result follows from the general theory of [ABO17] after proving that $\text{Mes}(\mathcal{T}_\Sigma^{\text{comb}})$ is a target theory. We present a self-contained proof which does not rely on these general notions, by specialising the strategy of [ABO17] to this simpler setting.

It is enough to prove the result for connected surfaces. By definition, the result holds for connected surfaces of Euler characteristic -1 (pairs of pants and one-holed tori). Let us assume it holds for all surfaces of Euler characteristic strictly greater than χ . Let Σ be a bordered surface of type (g, n) with $2 - 2g - n = \chi$, take $\epsilon > 0$ and fix $\mathbb{G} \in \mathcal{T}_\Sigma^{\text{comb},(\epsilon)}$. For any $P \in \mathcal{P}_\Sigma$, we have as well $\mathbb{G}|_{\Sigma-P} \in \mathcal{T}_{\Sigma-P}^{\text{comb},(\epsilon)}$. Therefore, by induction hypothesis, there exist $u \geq 0$ and $K_\epsilon > 0$ which we can choose to depend only on ϵ and the topological type of Σ , such that

$$|\Xi_{\Sigma-P}(\mathbb{G}_{\Sigma-P})| \leq K_\epsilon \prod_{b \in \pi_0(\partial(\Sigma-P))} (1 + \ell_{\mathbb{G}}(b))^u.$$

We now study the absolute convergence of the geometric recursion series (5.1.4). We use the notation X_P for the function B when $P \in \mathcal{B}_{\Sigma,m}$ and for the function $\frac{1}{2}C$ when $P \in \mathcal{C}_\Sigma$. We first isolate the sum over \mathbb{G} -small pairs of pants. Using the fact that there are at most $2(6g - 6 + 3n)$ \mathbb{G} -small pairs of pants (Remark 3.2.10), together with the admissibility conditions on X_P (Definition 5.1.3) and the inequality $(1 + L_1 + L_2)^t \leq (1 + L_1)^t (1 + L_2)^t$ for any $L_1, L_2 > 0$, we get

$$\sum_{\substack{P \in \mathcal{P}_\Sigma \\ \mathbb{G}\text{-small}}} |X_P(\vec{\ell}_{\mathbb{G}}(\partial P)) \Xi_{\Sigma-P}(\mathbb{G}|_{\Sigma-P})| \leq 2(6g - 6 + 3n) M_{\epsilon,0} K_\epsilon \prod_{i=1}^n (1 + \ell_{\mathbb{G}}(\partial_i \Sigma))^{\max\{u,t\}}. \quad (5.1.6)$$

We now turn to the contributions of the \mathbb{G} -big pairs of pants in $\mathcal{B}_{\Sigma,m}$. We have for any $s > 0$

$$\begin{aligned} & \sum_{\substack{P \in \mathcal{B}_{\Sigma,m} \\ \mathbb{G}\text{-big}}} |B(\vec{\ell}_{\mathbb{G}}(\partial P)) \Xi_{\Sigma-P}(\mathbb{G}|_{\Sigma-P})| \leq \\ & \leq M_{\epsilon,s} K_\epsilon \left(\prod_{i \neq 1,m} (1 + \ell_{\mathbb{G}}(\partial_i \Sigma))^u \right) \\ & \quad \times \left(\sum_{L \in \ell_{\mathbb{G}}(\partial_1 \Sigma) + \ell_{\mathbb{G}}(\partial_m \Sigma) + \mathbb{N}} \frac{(2+L)^t |\{\gamma \in \mathcal{S}_\Sigma \mid L \leq \ell_{\mathbb{G}}(\gamma) < L+1\}|}{(1+L - \ell_{\mathbb{G}}(\partial_1 \Sigma) - \ell_{\mathbb{G}}(\partial_m \Sigma))^s} \right) \\ & \leq M_{\epsilon,s} K_\epsilon m^\epsilon \left(\prod_{i \neq 1,m} (1 + \ell_{\mathbb{G}}(\partial_i \Sigma))^u \right) \left(\sum_{L \geq 1} (1+L + \ell_{\mathbb{G}}(\partial_1 \Sigma) + \ell_{\mathbb{G}}(\partial_m \Sigma))^{t+6g-6+2n} L^{-s} \right). \end{aligned}$$

In the last line, we invoked the polynomial growth of the number of multicurves with respect to combinatorial length, justified later in Proposition 8.1.7. Specialising to $s = (6g - 6 + 2n + t) + 2$

makes the sum in brackets converging to a polynomial of degree $t' = t + 6g - 6 + 2n$ in the variable $\ell_{\mathbb{G}}(\partial_1 \Sigma) + \ell_{\mathbb{G}}(\partial_m \Sigma)$ and, together with (5.1.6), it implies the existence of a constant $K'_\epsilon > 0$ such that

$$\sum_{P \in \mathcal{B}_{\Sigma, m}} |B(\vec{\ell}_{\mathbb{G}}(\partial P)) \Xi_{\Sigma-P}(\mathbb{G}|_{\Sigma-P})| \leq K'_\epsilon \prod_{i=1}^n (1 + \ell_{\mathbb{G}}(\partial_i \Sigma))^{\max\{u, t'\}}.$$

A similar argument shows that

$$\sum_{P \in \mathcal{C}_{\Sigma}} |C(\vec{\ell}_{\mathbb{G}}(\partial P)) \Xi_{\Sigma-P}(\mathbb{G}|_{\Sigma-P})| \leq K'_\epsilon \prod_{i=1}^n (1 + \ell_{\mathbb{G}}(\partial_i \Sigma))^{\max\{u, t'\}}$$

for a perhaps larger constant K'_ϵ . Consequently, the series

$$\sum_{P \in \mathcal{P}_{\Sigma}} X_P(\vec{\ell}_{\mathbb{G}}(\partial P)) \Xi_{\Sigma-P}(\mathbb{G}|_{\Sigma-P})$$

converges absolutely and uniformly on any compact of $\mathcal{T}_{\Sigma}^{\text{comb}, (\epsilon)}$, to a limit that we denote Ξ_{Σ} . Further, the bounds that we just proved imply that this limit satisfies

$$\forall \mathbb{G} \in \mathcal{T}_{\Sigma}^{\text{comb}, (\epsilon)}, \quad |\Xi_{\Sigma}(\mathbb{G})| \leq K''_\epsilon \prod_{i=1}^n (1 + \ell_{\mathbb{G}}(\partial_i \Sigma))^{u'}$$

for some constant $K''_\epsilon > 0$ and $u' = \max\{u, t'\}$. The proof is then completed by induction. \square

5.2 — (DISCRETE) INTEGRATION AND TOPOLOGICAL RECURSION

Since the functions produced by the combinatorial geometric recursion are pure mapping class group invariant, they descend to functions on the corresponding moduli spaces. For a connected surface Σ of type (g, n) and geometric recursion amplitudes Ξ_{Σ} , we denote by $\Xi_{g, n}$ the functions induced on the combinatorial moduli space. For the initial datum $T \mapsto D_T$, we denote by D the induced function $\mathcal{M}_{1,1}^{\text{comb}}$.

In the first part of this section we discuss how to integrate combinatorial geometric recursion amplitudes against the the Kontsevich measure, in parallel to Theorem 2.4.23 proved by Andersen–Borot–Orantin. In the second part, which belongs exclusively to the combinatorial setting, we discuss how to define discrete integration on the combinatorial moduli space via sums over integral metric ribbon graphs.

5.2.1 — INTEGRATION AND TOPOLOGICAL RECURSION

Since geometric recursion amplitudes are mapping class group invariant, they descend to function on the combinatorial moduli space, denoted by $\Xi_{g, n}$. Thanks to the combinatorial Wolpert formula, functions obtained by by excision of pairs of pants, i.e. by geometric recursion, can be integrated over the moduli space with fixed boundary lengths (cf. Proposition 4.3.1) producing functions on \mathbb{R}_+^n that also satisfy a recursion on the Euler characteristic, namely

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topological recursion. In order to guarantee integrability we are going to introduce stronger assumptions on the initial data. In the following, we will denote by

$$\langle f \rangle(L) = \int_{\mathcal{M}_{g,n}^{\text{comb}}(L)} f d\mu_K \quad (5.2.1)$$

the average over the combinatorial moduli space of any integrable function f on $(\mathcal{M}_{g,n}^{\text{comb}}(L), \mu_K)$.

DEFINITION 5.2.1. We say that the combinatorial initial data (A, B, C, D) are *strongly admissible* if they satisfy the same conditions appearing in the hyperbolic setting Definition 2.4.2.2, except for (2.4.3.8) which gets substituted by

$$|\langle D \rangle(L)| = \left| \int_{\mathcal{M}_{1,1}^{\text{comb}}(L)} D d\mu_K \right| \leq M_0 (1+L)^t. \quad (5.2.2)$$

THEOREM 5.2.2. Let (A, B, C, D) be strongly admissible combinatorial initial data and Ξ_Σ be the resulting functions. Then, $\Xi_{g,n}$ is integrable against μ_K on $\mathcal{M}_{g,n}^{\text{comb}}(L)$ for any $L \in \mathbb{R}_+^n$, and the integrals satisfy the following recursion on $2g - 2 + n > 1$

$$\begin{aligned} \langle \Xi_{g,n} \rangle(L_1, \dots, L_n) &= \\ &= \sum_{m=2}^n \int_{\mathbb{R}_+} B(L_1, L_m, \ell) \langle \Xi_{g,n-1} \rangle(\ell, L_2, \dots, \widehat{L_m}, \dots, L_n) \ell d\ell \\ &+ \frac{1}{2} \int_{\mathbb{R}_+^2} C(L_1, \ell, \ell') \left(\langle \Xi_{g-1,n+2} \rangle(\ell, \ell', L_2, \dots, L_n) \right. \\ &\quad \left. + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2, \dots, n\}}} \langle \Xi_{g_1, 1+|I_1|} \rangle(\ell, L_{I_1}) \langle \Xi_{g_2, 1+|I_2|} \rangle(\ell', L_{I_2}) \right) \ell \ell' d\ell d\ell' \end{aligned} \quad (5.2.3)$$

with the conventions $\langle \Xi_{0,1} \rangle = \langle \Xi_{0,2} \rangle = 0$, and the base cases

$$\langle \Xi_{0,3} \rangle(L_1, L_2, L_3) = A(L_1, L_2, L_3) \quad \text{and} \quad \langle \Xi_{1,1} \rangle(L) = \langle D \rangle(L). \quad (5.2.4)$$

Proof. We first note that the initial data $\langle \Xi_{0,3} \rangle$ is well-defined as A is, and that $\langle \Xi_{1,1} \rangle$ is well-defined by strong admissibility.

Now, for a connected surface Σ of type (g, n) with $2g - 2 + n > 1$, apply the integration over $\mathcal{M}_{g,n}^{\text{comb}}(L)$ with respect to μ_K to both sides of the combinatorial geometric recursion (5.1.4). We analyse the integration of the sum over $P \in \mathcal{B}_{\Sigma,m}$. Let Γ be the Mod_Σ^g -orbit of a simple closed curve bounding a pair of pants together with $\partial_1 \Sigma$ and $\partial_m \Sigma$. We have

$$\sum_{P \in \mathcal{B}_{\Sigma,m}} B(\vec{\ell}_\mathbb{G}(\partial P)) \Xi_{\Sigma-P}(\mathbb{G}|_{\Sigma-P}) = \sum_{\alpha \in \Gamma} B(L_1, L_m, \ell_\mathbb{G}(\alpha)) \Xi_{\Sigma-P_\alpha}(\mathbb{G}|_{\Sigma-P_\alpha}),$$

where P_α is the pair of pants bounded by $\partial_1 \Sigma$, $\partial_m \Sigma$ and α . Now applying Proposition 4.3.1 to

$f(L, \ell) = B(L_1, L_m, x)$ and the assignment $\Sigma_\alpha \mapsto \Xi_{\Sigma-P_\alpha}$, we find

$$\begin{aligned} & \int_{\mathcal{M}_{g,n}^{\text{comb}}(L)} \sum_{\alpha \in \Gamma} B(L_1, L_m, \ell_*(\alpha)) \Xi_\Gamma d\mu_K = \\ &= \int_{\mathbb{R}_+} B(L_1, L_m, \ell) \left(\int_{\mathcal{M}_{g,n-1}^{\text{comb}}(\ell, L_2, \dots, \widehat{L_m}, \dots, L_n)} \Xi_{g,n-1} d\mu_K \right) \ell d\ell \\ &= \int_{\mathbb{R}_+} B(L_1, L_m, \ell) \langle \Xi_{g,n-1} \rangle(\ell, L_2, \dots, \widehat{L_m}, \dots, L_n) \ell d\ell. \end{aligned}$$

The treatment of the C summands is similar, the main difference being that the excised pair of pants has two simple closed curves in Σ° , whose lengths are part of the combinatorial Fenchel–Nielsen coordinates over which we need to integrate. \square

5.2.2 — DISCRETE INTEGRATION AND TOPOLOGICAL RECURSION

As the combinatorial moduli spaces have an integral structure, we can also study discrete integration of geometric recursion amplitudes. Again, the geometric recursion here is the key property that guarantees a topological recursion for the discrete integrals, i.e. the sum over lattice points.

DEFINITION 5.2.3. For $2g - 2 + n > 0$, let

$$\mathcal{M}_{g,n}^{\text{comb},\mathbb{Z}} = \left\{ G \in \mathcal{M}_{g,n}^{\text{comb}} \mid \text{all edge lengths are in } \mathbb{Z}_+ \right\}. \quad (5.2.5)$$

and by $\mathcal{M}_{g,n}^{\text{comb},\mathbb{Z}}(L)$ the subset with fixed perimeter $L \in \mathbb{Z}_+^n$. We denote likewise $\mathcal{T}_\Sigma^{\text{comb},\mathbb{Z}}$ and $\mathcal{T}_\Sigma^{\text{comb},\mathbb{Z}}(L)$ their the set of combinatorial structures on a bordered surface Σ with integral edge lengths.

Since for any $G \in \mathcal{M}_{g,n}^{\text{comb},\mathbb{Z}}(L)$ we have $\sum_{i=1}^n L_i = \sum_{e \in E_G} 2\ell_e$, the set $\mathcal{M}_{g,n}^{\text{comb},\mathbb{Z}}(L)$ is finite for any fixed L , and it is empty if $\sum_{i=1}^n L_i$ is odd. For instance,

$$\mathcal{M}_{0,3}^{\text{comb},\mathbb{Z}} \cong \left\{ (L_1, L_2, L_3) \in \mathbb{Z}_+^3 \mid L_1 + L_2 + L_3 \text{ is even} \right\}. \quad (5.2.6)$$

It is useful to characterise the integral points in $\mathcal{M}_{g,n}^{\text{comb}}(L)$ in terms of integral points in $\mathcal{M}_{g,n}^{\text{comb}}$. Let G be a ribbon graph of type (g, n) , not necessarily integral, and recall that the collection of multiplicity $A_{i,e} \in \{0, 1, 2\}$ of edges $e \in E_G$ around the i -th face defines the *adjacency matrix* A of size $n \times (6g - 6 + 3n)$. Consider now a set $S \subset E_G$ such that $|S| = n$ and the dual graph G_S^* of S (considered as a subgraph of G) is connected and has a single cycle of odd length. We label the edges so that $S = \{e_1, \dots, e_n\}$ and $E_G = \{e_1, \dots, e_{6g-6+3n}\}$. If G is a metric structure on G , we call $\ell_i = \ell_G(e_i)$.

LEMMA 5.2.4. *With the above notation:*

- the restriction \hat{A} of the adjacency matrix to the first n columns is invertible, and $|\det(\hat{A})| = 2$ (see e.g. [DE14, Theorem 2.2]),
- $\ell_{n+1}, \dots, \ell_{6g-6+3n} \in \mathbb{Z}$ and $\sum_{i=1}^n L_i \in 2\mathbb{Z}$ hold if and only if $\ell_1, \dots, \ell_{6g-6+3n} \in \mathbb{Z}$.

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Proof. We refer to [DE14, Theorem 2.2] for the first point. However, we recall here the explicit construction of \hat{A}^{-1} . By hypothesis, G_S^* is the union of trees rooted at a cycle with $2p + 3$ edges for some $p \geq 0$, and its vertices are labelled from 1 to n . For $e \in S$ and $i \in \{1, \dots, n\}$, let $d_{e,i}$ be the graph distance in G_S^* between e^* (the dual of e) and the vertex i . If e is adjacent to the face i , then $d_{e,i} = 0$. There is a natural notion of descendent vertices of an edge belonging to a tree of G_S^* , given that the trees are rooted at the cycles of G_S^* . The inverse of \hat{A} can now be made explicit.

- If e^* does not belong to the cycle of G_S^*

$$\hat{A}_{e,i}^{-1} = \begin{cases} (-1)^{d_{e,i}} & \text{if } i \text{ is a descendent of } e^* \\ 0 & \text{otherwise.} \end{cases}$$

- If e^* belongs to the cycle of G_S^* ,

$$\hat{A}_{e,i}^{-1} = \frac{(-1)^{d_{e,i}}}{2}.$$

We are now ready to prove the second point of the lemma. It is sufficient to prove the implication, since the converse is obvious. Thus, suppose that $\ell_{n+1}, \dots, \ell_{6g-6+3n}$ are integers and $\sum_{i=1}^n L_i \equiv 0 \pmod{2}$. If e^* does not belong to the cycle of G_S^* , for all $i \in \{1, \dots, n\}$ we have $\hat{A}_{e,i}^{-1} \in \{-1, 0, 1\}$, so

$$\ell_e = \sum_{i=1}^n \hat{A}_{e,i}^{-1} \left(L_i - \sum_{k=n+1}^{6g-6+3n} A_{i,e_k} \ell_k \right)$$

belongs to \mathbb{Z} . If e^* belongs to the cycle of G_S^* , for all $i \in \{1, \dots, n\}$ we have $2\hat{A}_{e,i}^{-1} \equiv 1 \pmod{2}$. This implies the following:

$$\begin{aligned} 2\ell_e &= \sum_{i=1}^n 2\hat{A}_{e,i}^{-1} \left(L_i - \sum_{k=n+1}^{6g-6+3n} A_{i,e_k} \ell_k \right) \\ &\equiv \sum_{i=1}^n L_i - \sum_{k=n+1}^{6g-6+3n} \left(\sum_{i=1}^n A_{i,e_k} \right) \ell_k \pmod{2} \\ &\equiv 0 \pmod{2}, \end{aligned}$$

where the second to last line comes from the hypothesis $\sum_{i=1}^n L_i \equiv 0 \pmod{2}$ and $\sum_{i=1}^n A_{i,e} = 2$. The result of this calculation is that $\ell_e \in \mathbb{Z}$. \square

Since $\mathcal{M}_{g,n}^{\text{comb},\mathbb{Z}}(L)$ is empty for $\sum_{i=1}^n L_i$ odd, it is useful to introduce the following notation: for $X: \mathbb{R}_+^n \rightarrow \mathbb{R}$, define

$$X^{\mathbb{Z}}(L) = \begin{cases} X(L) & \text{if } L \in \mathbb{Z}_+^n \text{ and } \sum_{i=1}^n L_i \text{ is even,} \\ 0 & \text{otherwise.} \end{cases} \quad (5.2.7)$$

Further, for any function $\Xi_{g,n}$ on $\mathcal{M}_{g,n}^{\text{comb}}$, set

$$\langle \Xi_{g,n}^{\mathbb{Z}} \rangle(L) = \sum_{G \in \mathcal{M}_{g,n}^{\text{comb},\mathbb{Z}}(L)} \frac{\Xi_{g,n}(G)}{|\text{Aut}(G)|} = \sum_{G \in \mathcal{R}_{g,n}} \frac{1}{|\text{Aut}(G)|} \sum_{x \in P_G(L) \cap \mathbb{Z}_+^n} \Xi_{g,n}(x), \quad (5.2.8)$$

where we recall that $P_G(L)$ is the set of metrics on G with perimeter L .

THEOREM 5.2.5. *Let A, B, C be three functions on \mathbb{R}_+^3 such that A and C are symmetric under exchange of their last two variables, and supported on small pairs of pants:*

$$\begin{cases} \ell > L + L' & \implies & B(L, L', \ell) = 0, \\ \ell + \ell' > L & \implies & C(L, \ell, \ell') = 0. \end{cases} \quad (5.2.9)$$

Let D be Mod_T -invariant function on $\mathcal{T}_T^{\text{comb}}$. Denote by Ξ_{Σ} the corresponding combinatorial geometric recursion amplitudes. We have the following recursion on $2g - 2 + n > 1$.

$$\begin{aligned} \langle \Xi_{g,n}^{\mathbb{Z}} \rangle(L_1, \dots, L_n) &= \\ &= \sum_{m=2}^n \sum_{\ell \geq 1} \ell B^{\mathbb{Z}}(L_1, L_m, \ell) \langle \Xi_{g,n-1}^{\mathbb{Z}} \rangle(\ell, L_2, \dots, \widehat{L}_m, \dots, L_n) \\ &+ \frac{1}{2} \sum_{\ell, \ell' \geq 1} \ell \ell' C^{\mathbb{Z}}(L_1, \ell, \ell') \left(\langle \Xi_{g-1, n+1}^{\mathbb{Z}} \rangle(\ell, \ell', L_2, \dots, L_n) \right. \\ &\quad \left. + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2, \dots, n\}}} \langle \Xi_{g_1, 1+|I_1|}^{\mathbb{Z}} \rangle(\ell, L_{I_1}) \langle \Xi_{g_2, 1+|I_2|}^{\mathbb{Z}} \rangle(\ell', L_{I_2}) \right) \end{aligned} \quad (5.2.10)$$

with conventions $\langle \Xi_{0,1}^{\mathbb{Z}} \rangle = 0$ and $\langle \Xi_{0,2}^{\mathbb{Z}} \rangle = 0$, and base cases

$$\langle \Xi_{0,3}^{\mathbb{Z}} \rangle(L_1, L_2, L_3) = A^{\mathbb{Z}}(L_1, L_2, L_3) \quad \text{and} \quad \langle \Xi_{1,1}^{\mathbb{Z}} \rangle(L) = \langle D^{\mathbb{Z}} \rangle(L). \quad (5.2.11)$$

Remark 5.2.6. Since cutting combinatorial structures preserve their integrality, in order to obtain the geometric recursion amplitudes in Theorem 5.2.5, it is sufficient to have (A, B, C) defined on $\mathcal{T}_P^{\text{comb}, \mathbb{Z}}$ and D defined on $\mathcal{T}_T^{\text{comb}, \mathbb{Z}}$. When the vanishing condition (5.2.9) hold, the geometric recursion sums have only finitely many non-zero terms: they are always well-defined, without the need of admissibility conditions for (A, B, C, D) .

Before proving Theorem 5.2.5, let us explore some of its consequences. First of all, we can recover the continuous integration of Theorem 5.2.2 as a limit where we rescale the mesh of the lattice down to 0. If $k > 0$, we let $\mathcal{M}_{g,n}^{\text{comb}, \mathbb{Z}/k}$ be the set of metric ribbon graphs in $\mathcal{M}_{g,n}^{\text{comb}}$ whose edge lengths all become integral after dilation by k .

PROPOSITION 5.2.7. *Assume that (A, B, C, D) are continuous functions on their respective combinatorial Teichmüller spaces, satisfying the vanishing conditions (5.2.9), and such that for any fixed $L, L' > 0$, the functions $\ell \mapsto B(L, L, \ell)$ and $(\ell, \ell') \mapsto C(L, \ell, \ell')$ are bounded, and the function D_T is bounded on $\mathcal{T}_T^{\text{comb}}(L)$. Then (A, B, C, D) is strongly admissible. Moreover, for any positive integer d and $L \in (\mathbb{Z}_+/d)^n$, we have for $2g - 2 + n > 0$*

$$\lim_{\substack{k \rightarrow \infty \\ k \in d\mathbb{Z}_+}} \left(\frac{1}{k^{6g-6+2n}} \sum_{\mathbf{G} \in \mathcal{M}_{g,n}^{\text{comb}, \mathbb{Z}/k}(L)} \frac{\Xi_{g,n}(\mathbf{G})}{|\text{Aut}(\mathbf{G})|} \right) = \begin{cases} 2^{-2g+3-n} \langle \Xi_{g,n} \rangle(L) & \text{if } \sum_{i=1}^n d \cdot L_i \text{ is even,} \\ 0 & \text{otherwise.} \end{cases} \quad (5.2.12)$$

The continuous integration also appears in the asymptotics of the lattice count for large boundary lengths.

COROLLARY 5.2.8. *Assume that (A, B, C, D) are continuous functions on their respective combinatorial Teichmüller spaces and satisfy the vanishing conditions (5.2.9). We further assume the existence of $b, c \in \mathbb{R}$ for which, for any $M > 0$ and $L_1, L_2, L_3, \ell, \ell' \in (0, M]$*

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- $k^{-2b-c} A(kL_1, kL_2, kL_3)$ converges uniformly to $\hat{A}(L_1, L_2, L_3)$ as $k \rightarrow \infty$,
- $k^{-b} B(kL_1, kL_2, k\ell)$ converges uniformly to $\hat{B}(L_1, L_2, \ell)$ as $k \rightarrow \infty$,
- $k^{-c} C(kL_1, k\ell, k\ell')$ converges uniformly to $\hat{C}(L_1, \ell, \ell')$ as $k \rightarrow \infty$,
- $k^{-b} D_T(k\mathbb{G})$ converges uniformly to $\hat{D}_T(\mathbb{G})$ as $k \rightarrow \infty$, for $\mathbb{G} \in \bigcup_{\ell \in (0, M]} \mathcal{T}_T^{\text{comb}}(\ell)$, where $k\mathbb{G}$ is obtained from \mathbb{G} by dilation of the metric by a factor k ,
- $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ satisfy the assumptions of Proposition 5.2.7.

Denoting by $\hat{\Xi}_\Sigma$ the geometric recursion amplitudes associated to the initial data $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$, we have for any $d \in \mathbb{Z}_+$, $L \in (\mathbb{Z}_+/d)^n$ and $2g - 2 + n > 0$

$$\lim_{\substack{k \rightarrow \infty \\ k \in d\mathbb{Z}_+}} \frac{\langle \Xi_{g,n}^{\mathbb{Z}} \rangle(kL)}{k^{(g-1)(b+c)+nb} \cdot k^{6g-6+2n}} = \begin{cases} 2^{-2g+3-n} \langle \hat{\Xi}_{g,n} \rangle(L) & \text{if } \sum_{i=1}^n d \cdot L_i \text{ even,} \\ 0 & \text{otherwise.} \end{cases} \quad (5.2.13)$$

Proof of Theorem 5.2.5. When $\sum_{i=1}^n L_i$ is odd, both sides vanish, so we only need to prove the result when $\sum_{i=1}^n L_i$ is even, which we now assume. The vanishing conditions for B and C imply that for each $L \in \mathbb{Z}_+^n$, the sums in the right-hand side have finitely many terms.

Let us substitute the geometric recursion sum for Ξ_Σ in (5.2.8). For $m \in \{2, \dots, n\}$, we examine in detail how to handle the term

$$\Xi_\Sigma^{B,m}(\mathbb{G}) = \sum_{P \in \mathcal{B}_{\Sigma,m}} B(\vec{\ell}_\mathbb{G}(\partial P)) \Xi_{g,n-1}(\mathbb{G}|_{\Sigma-P}).$$

Let Γ be the $\text{Mod}_\Sigma^\partial$ -orbit of a simple closed curve that bounds some $P \in \mathcal{B}_{\Sigma,m}$. Adapting the notation of Section 4.3, we denote by $\mathcal{M}_{g,n}^{\text{comb},\Gamma,\mathbb{Z}}(L)$ the integral points in $\mathcal{M}_{g,n}^{\text{comb},\Gamma}(L)$. Then

$$\begin{aligned} \langle \Xi_{g,n}^{B,m,\mathbb{Z}} \rangle(L) &= \sum_{\mathbf{G} \in \mathcal{M}_{g,n}^{\text{comb},\mathbb{Z}}(L)} \frac{\Xi_{g,n}^{B,m}(\mathbf{G})}{|\text{Aut}(\mathbf{G})|} \\ &= \sum_{(\mathbf{G}, \alpha) \in \mathcal{M}_{g,n}^{\text{comb},\Gamma,\mathbb{Z}}(L)} \frac{1}{|\text{Aut}(\mathbf{G})|} B(L_1, L_m, \mathcal{L}^\Gamma(\mathbf{G}, \alpha)) \Xi_{g,n-1}(\Pi(\mathbf{G})) \\ &= \sum_{\ell \geq 1} \sum_{(\mathbf{G}, \alpha) \in \mathcal{M}_{g,n}^{\text{comb},\Gamma,\mathbb{Z}}(L)[\ell]} \frac{1}{|\text{Aut}(\mathbf{G})|} B(L_1, L_m, \ell) \Xi_{g,n-1}(\Pi(\mathbf{G})), \end{aligned}$$

where we recall from Section 4.3 the map

$$\mathcal{L}^\Gamma: \mathcal{M}_{g,n}^{\text{comb},\Gamma,\mathbb{Z}}(L) \longrightarrow \mathbb{Z}_+$$

assigning to (\mathbf{G}, α) the combinatorial length with respect to \mathbf{G} of α . It has fibres $(\mathcal{L}^\Gamma)^{-1}(\ell) = \mathcal{M}_{g,n}^{\text{comb},\Gamma,\mathbb{Z}}(L)[\ell]$. Moreover, we have the projection map

$$\Pi: \mathcal{M}_{g,n}^{\text{comb},\Gamma,\mathbb{Z}}(L)[\ell] \longrightarrow \mathcal{M}_{g,n-1}^{\text{comb},\mathbb{Z}}(\ell, L_2, \dots, \widehat{L}_m, \dots, L_n).$$

We want to cluster this sum according to the fibres of the map Π . We first notice that, as $\mathcal{M}_{g,n}^{\text{comb},\Gamma,\mathbb{Z}}(L)[\ell]$ is empty when ℓ and $L_1 + L_m$ have different parity, we can replace B by

$B^{\mathbb{Z}}$, and now only consider ℓ that has same parity as $L_1 + L_m$. In this case, for any $\mathbf{G}' \in \mathcal{M}_{g,n-1}^{\text{comb},\mathbb{Z}}(\ell, L_2, \dots, \widehat{L_m}, \dots, L_n)$ and any $\mathbf{G} \in \Pi^{-1}(\mathbf{G}')$, we remark that $|\text{Aut}(\mathbf{G}')| = |\text{Aut}(\mathbf{G})|$. Thus

$$\frac{1}{|\text{Aut}(\mathbf{G})|} B^{\mathbb{Z}}(L_1, L_m, \ell) \Xi_{g,n-1}(\Pi(\mathbf{G}))$$

is constant on the fibres of Π . Due to the vanishing conditions (5.2.9), the points (\mathbf{G}, α) with non-trivial contribution are associated to small pairs of pants P_α . Therefore from Corollary 3.3.3, $\Pi^{-1}(\mathbf{G}')$ is in bijection with the set of $[\tau] \in \mathbb{R}/\ell\mathbb{Z}$ such that the gluing of \mathbf{G}' to the combinatorial structure of the pair of pants with boundary lengths (L_1, L_m, ℓ) after a twist τ yields a combinatorial structure with integral edge lengths. In order to satisfy the latter condition, the twists must belong to the set $(\tau_\gamma(\mathbb{G}) + \mathbb{Z})/\ell\mathbb{Z} \simeq \mathbb{Z}/\ell\mathbb{Z} \simeq \Pi^{-1}(\mathbf{G}')$, whose cardinality is ℓ . Therefore, our sum becomes

$$\begin{aligned} \langle \Xi_{g,n}^{B,m,\mathbb{Z}} \rangle(L) &= \sum_{\ell \geq 1} \sum_{\mathbf{G}' \in \mathcal{M}_{g,n-1}^{\text{comb},\mathbb{Z}}(\ell, L_2, \dots, \widehat{L_m}, \dots, L_n)} \frac{1}{|\text{Aut}(\mathbf{G}')|} B^{\mathbb{Z}}(L_1, L_m, \ell) \Xi_{g,n-1}(\mathbf{G}') \\ &= \sum_{\ell \geq 1} \ell B^{\mathbb{Z}}(L_1, L_m, \ell) \langle \Xi_{g,n-1}^{\mathbb{Z}} \rangle(\ell, L_2, \dots, \widehat{L_m}, \dots, L_n). \end{aligned}$$

The treatment of the C -summand is similar, except that we should be cautious about automorphism factors. For each of the finitely many $\text{Mod}_\Sigma^\partial$ -orbit Γ of a simple closed curve bounding some $P \in C_\Sigma$, and we observe that

$$|\text{Aut}(\Gamma)| = \begin{cases} 2 & \text{if } \Sigma - P \text{ is connected,} \\ 1 & \text{otherwise.} \end{cases}$$

Since in the C -analogue of $\Xi_{g,n}^B(L)$ we have for any $\mathbf{G} \in \Pi^{-1}(\mathbf{G}')$

$$|\text{Aut}(\Gamma)| \cdot |\text{Aut}(\mathbf{G})| = |\text{Aut}(\mathbf{G}')|,$$

the automorphism factors are again naturally included in $N\Xi_\Gamma$, and we get the C -terms in (5.2.10) without extra automorphism factors (as the $\frac{1}{2}$ is already present in C). \square

Remark 5.2.9. The vanishing assumptions (5.2.9) are essential to allow the use of Corollary 3.3.3. If they did not hold, the fibres $\Pi^{-1}(\mathbf{G}')$ of the gluing fibration could, and do, meet integral non-admissible twists (take for instance the example of Figure 3.17 with integer lengths). Hence their cardinality could be smaller than ℓ or $\ell\ell'$ and depend on \mathbf{G}' . It would then not be possible to derive a recursion for the weighted sum over lattice points. This problem did not arise for the integration against μ_K as the set of non-admissible twists has zero measure with respect to μ_K .

Before turning to the proof of Proposition 5.2.7, we need two preliminary results.

Proof of Proposition 5.2.7. The case $(g, n) = (0, 3)$ is obvious, as there is equality before taking the limit. This initial case is special with respect to the other topologies since the moduli space is reduced to a point: in the rest of the proof, we suppose $(g, n) \neq (0, 3)$. In general, when $\sum_{i=1}^n d \cdot L_i$ is not even, the set $\mathcal{M}_{g,n}^{\text{comb},\mathbb{Z}/k}(L)$ with $k \in d\mathbb{Z}_+$ is empty, so the left-hand side of Equation (5.2.12) vanishes, which proves half of the result. Hereafter we assume that $k \in d\mathbb{Z}_+$, and fix $L \in (\mathbb{Z}_+/d)^n$ such that $\sum_{i=1}^n d \cdot L_i$ is even.

II. The combinatorial model of the moduli space of curves

The thesis follows now from a general discussion. Consider a bounded function f defined on $\mathcal{M}_{g,n}^{\text{comb}}(L)$. The sum over rescaled lattice points is, by definition,

$$\sum_{G \in \mathcal{M}_{g,n}^{\text{comb}, \mathbb{Z}/k}(L)} \frac{f(G)}{|\text{Aut}(G)|} = \sum_{G \in \mathcal{R}_{g,n}} \frac{1}{|\text{Aut}(G)|} \sum_{x \in P_G(L) \cap \mathbb{Z}_+^n/k} f(x),$$

where we recall that $P_G(L) \subset \mathbb{R}_+^{E_G}$ is the set of metrics on G with perimeters L . We first estimate the sum over non-trivalent graphs by

$$\left| \sum_{\substack{G \in \mathcal{R}_{g,n} \\ \text{non-trivalent}}} \frac{1}{|\text{Aut}(G)|} \sum_{x \in P_G(L) \cap \mathbb{Z}_+^n/k} f(x) \right| \leq \left(\sup_{\mathcal{M}_{g,n}^{\text{comb}}(L)} |f| \right) \sum_{\substack{G \in \mathcal{R}_{g,n} \\ \text{non-trivalent}}} \frac{|P_G(L) \cap \mathbb{Z}_+^n/k|}{|\text{Aut}(G)|}.$$

By dimensional reasons, the right-hand side is $O(k^{6g-7+2n})$ as $k \rightarrow +\infty$. Hence

$$\lim_{\substack{k \rightarrow \infty \\ k \in d\mathbb{Z}_+}} \left(\frac{1}{k^{6g-6+2n}} \sum_{G \in \mathcal{M}_{g,n}^{\text{comb}, \mathbb{Z}/k}(L)} \frac{f(G)}{|\text{Aut}(G)|} \right) = \lim_{\substack{k \rightarrow \infty \\ k \in d\mathbb{Z}_+}} \left(\frac{1}{k^{6g-6+2n}} \sum_{\substack{G \in \mathcal{R}_{g,n} \\ \text{trivalent}}} \frac{1}{|\text{Aut}(G)|} \sum_{x \in P_G(L) \cap \mathbb{Z}_+^n/k} f(x) \right).$$

By Lemma 5.2.4 and by definition of the Riemann integral, we have

$$\lim_{\substack{k \rightarrow \infty \\ k \in d\mathbb{Z}_+}} \left(\frac{1}{k^{6g-6+2n}} \sum_{x \in P_G(L) \cap \mathbb{Z}_+^n/k} f(x) \right) = \int_{P_G(L)} f d\ell_{n+1} \cdots d\ell_{6g-6+3n}$$

provided f is continuous. Let G be a trivalent ribbon graph, and let $\iota_L: P_G(L) \hookrightarrow \mathbb{R}_+^{E_G}$ be the inclusion map. The cell $P_G(L)$ is naturally equipped with the measure μ_{Leb} defined as $\iota_L^* \left(\prod_{e \in E_G} d\ell_e \right)$. Since the restriction of the adjacency matrix has determinant 2, we see that $\prod_{j=n+1}^{6g-6+3n} d\ell_j = 2\mu_{\text{Leb}}$. Besides, we know from Lemma 4.1.6 that $\mu_{\text{Leb}} = 2^{2-2g-n}\mu_{\mathbb{K}}$. As a consequence, we find

$$2^{3-2g-n} \int_{\mathcal{M}_{g,n}^{\text{comb}}(L)} f d\mu_{\mathbb{K}} = \lim_{\substack{k \rightarrow \infty \\ k \in d\mathbb{Z}_+}} \left(\frac{1}{k^{6g-6+2n}} \sum_{\substack{G \in \mathcal{R}_{g,n} \\ \text{trivalent}}} \frac{1}{|\text{Aut}(G)|} \sum_{x \in P_G(L) \cap \mathbb{Z}_+^n/k} f(x) \right).$$

□

Proof of Corollary 5.2.8. Let $\Xi_{\Sigma}^{(k)}$ be the geometric recursion amplitudes for the initial data

$$\begin{aligned} A^{(k)}(L_1, L_2, L_3) &= k^{-2b-c} A(kL_1, kL_2, kL_3), \\ B^{(k)}(L_1, L_2, L_3) &= k^{-b} B(kL_1, kL_2, k\ell), \\ C^{(k)}(L_1, L_2, L_3) &= k^{-c} C(kL_1, k\ell, k\ell'), \\ D_T^{(k)}(\mathbb{G}) &= k^{-b} D_T(k\mathbb{G}). \end{aligned}$$

Tracking the powers of k^{-1} in the geometric recursion sum (5.1.4), one can show by induction that $\Xi_{\Sigma}(k\mathbb{G}) = k^{(g-1)(b+c)+bn} \Xi_{\Sigma}^{(k)}(\mathbb{G})$ and thus

$$\frac{\langle \Xi_{g,n}^{\mathbb{Z}} \rangle(kL_1, \dots, kL_n)}{k^{6g-6+2n} \cdot k^{(g-1)(b+c)+bn}} = \frac{1}{k^{6g-6+2n}} \left(\sum_{G \in \mathcal{M}_{g,n}^{\text{comb}, \mathbb{Z}/k}(L)} \frac{\Xi_{\Sigma}^{(k)}(G)}{|\text{Aut}(G)|} \right).$$

The vanishing conditions (5.2.9) have two consequences for us.

- (i) The amplitude $\Xi_\Sigma^{(k)}(\mathbb{G})$ is given by a finite sum of products with $2g - 2 + n$ factors that can be either $A^{(k)}, B^{(k)}, C^{(k)}, D^{(k)}$.
- (ii) If we fix $L \in \mathbb{R}_+^n$, for any $\mathbb{G} \in \mathcal{T}_\Sigma^{\text{comb}}(L)$, the factors of $A^{(k)}, B^{(k)}, C^{(k)}$ are evaluated on triples of lengths that are smaller than $M = \sum_{i=1}^n L_i$, and D is evaluated on elements of $\bigcup_{\ell \in (0, M]} \mathcal{T}_T^{\text{comb}}(\ell)$.

Hence $\Xi_\Sigma^{(k)}$ converges uniformly to $\hat{\Xi}_\Sigma$ on $\mathcal{T}_\Sigma^{\text{comb}}(L)$. We can then replace $\Xi_\Sigma^{(k)}$ with $\hat{\Xi}_\Sigma$ up to an error that tends to 0 when $k \rightarrow \infty$, and we conclude by using Proposition 5.2.7 for $\hat{\Xi}_\Sigma$. \square

5.2.3 — REMARK: INDUCING D FROM C

As explained in Lemma 2.4.24 in the hyperbolic setting, there is a natural way to complete (A, B, C) into an initial data (A, B, C, D) , satisfying all the assumptions that we may desire to impose. The same holds in the combinatorial setting.

LEMMA 5.2.10. *If we are only given (A, B, C) satisfying the conditions in Definition 2.4.20, the series*

$$D_T(\mathbb{G}) = \sum_{\gamma \in \mathcal{S}_T} C(\ell_{\mathbb{G}}(\partial T), \ell_{\mathbb{G}}(\gamma), \ell_{\mathbb{G}}(\gamma)) \quad (5.2.14)$$

converges absolutely on any compact of $\mathcal{T}_T^{\text{comb}}$ to a Mod_T -invariant function, and (A, B, C, D) are admissible initial data. Furthermore, if (A, B, C) satisfy the conditions in Definition 5.2.1, where in the bound for C one assumes $0 \leq \eta < 1$, then (A, B, C, D) are strongly admissible and

$$\langle D \rangle(L) = \frac{1}{2} \int_{\mathbb{R}_+} C(L, \ell, \ell) \ell d\ell. \quad (5.2.15)$$

In the following, when we say that (A, B, C) are admissible initial data, we implicitly assume that they should be completed by the choice (5.2.14) of D_T .

We also remark that when $C(L, \ell, \ell')$ vanishes for $L < \ell + \ell'$, the sum (5.2.14) is finite. It is thus well-defined without admissibility conditions and satisfies

$$\langle D^{\mathbb{Z}} \rangle(L) = \frac{1}{2} \sum_{\ell=1}^{L/2} \ell C(L, \ell, \ell), \quad L \in 2\mathbb{Z}_+. \quad (5.2.16)$$

5.3 — A COMBINATORIAL MIRZAKHANI–MC SHANE IDENTITY

Mirzakhani's identity (see Theorem 2.4.13) is a recursion for the constant function 1 on the Teichmüller space of bordered surface, and it can be reformulated by saying that the constant function 1 can be obtained from geometric recursion. The basic idea behind Mirzakhani's identity is to express the length of the first boundary component of a bordered surface as a sum of geodesics arcs characterized by the behavior of the corresponding orthogeodesic emanating from it, as explained in Section 2.4.2. The lengths of such geodesic arcs are captured by the function B^M and C^M appearing in Mirzakhani's identity.

We are now going to prove a recursion for the constant function 1 on the combinatorial Teichmüller space, using the same strategy of Mirzakhani. The functions B^M and C^M are

replaced by

$$\begin{aligned} B^K(L, L', \ell) &= \frac{1}{2L} \left([L - L' - \ell]_+ - [-L + L' - \ell]_+ + [L + L' - \ell]_+ \right), \\ C^K(L, \ell, \ell') &= \frac{1}{L} [L - \ell - \ell']_+. \end{aligned} \quad (5.3.1)$$

Notice that B^K and C^K were already introduced in (3.2.8) while discussing homotopy classes of embedded pairs of pants in this combinatorial setting.

THEOREM 5.3.1. *For any bordered surface Σ such that $\chi_\Sigma < -1$ and any $\mathbb{G} \in \mathcal{T}_\Sigma^{\text{comb}}$, we have*

$$1 = \sum_{m=2}^n \sum_{P \in \mathcal{B}_{\Sigma, m}} B^K(\vec{\ell}_{\mathbb{G}}(\partial P)) + \frac{1}{2} \sum_{P \in \mathcal{C}_\Sigma} C^K(\vec{\ell}_{\mathbb{G}}(\partial P)), \quad (5.3.2)$$

and for a torus T with one boundary component and any $\mathbb{G} \in \mathcal{T}_T^{\text{comb}}$, we have

$$1 = \sum_{\gamma \in \delta_T} C^K(\ell_{\mathbb{G}}(\partial T), \ell_{\mathbb{G}}(\gamma), \ell_{\mathbb{G}}(\gamma)). \quad (5.3.3)$$

Proof. Consider the case of connected Σ with $\chi_\Sigma < -1$. For simplicity, set X_P^K equal to B^K or $\frac{1}{2}C^K$ depending on the type of $P \in \mathcal{P}_\Sigma$. The basic idea to prove such an identity is to write $\ell_{\mathbb{G}}(\partial_1 \Sigma)$ as a sum of lengths of the edges around $\partial_1 \Sigma$. Recall that in the proof of Theorem 3.2.3, we introduced a map that assigns to a combinatorial structure \mathbb{G} a functional on the set of arc $\mathfrak{A}_\Sigma^{\text{all}}$ between boundary components. Here we are going to restrict the functionals to $\mathfrak{A}_\Sigma = \mathfrak{A}_{\Sigma, 1}$, i.e. to arcs with initial point in $\partial_1 \Sigma$. Moreover, we do not fix the boundary lengths $L \in \mathbb{R}_+^n$ (this does not affect the definition of the map in Theorem 3.2.3). This said, we obtain a map $\mathcal{T}_\Sigma^{\text{comb}} \rightarrow \mathbb{R}_{\geq 0}^{\mathfrak{A}_\Sigma}$ that assign to a combinatorial structure \mathbb{G} the functional

$$\alpha \mapsto \begin{cases} \ell_{\mathbb{G}}(\partial_1 \Sigma) \left(B^K(\vec{\ell}_{\mathbb{G}}(\partial P_\alpha)) - C^K(\vec{\ell}_{\mathbb{G}}(\partial P_\alpha)) \right) & \text{if } \alpha \in \mathfrak{B}_{\Sigma, m} \\ \frac{1}{2} \ell_{\mathbb{G}}(\partial_1 \Sigma) C^K(\vec{\ell}_{\mathbb{G}}(\partial P_\alpha)) & \text{if } \alpha \in \mathfrak{C}_\Sigma \end{cases}$$

where $P_\alpha = Q(\alpha)$ is the homotopy class of pair of pants determined by the arc α . Fix once and for all $\mathbb{G} \in \mathcal{T}_\Sigma^{\text{comb}}$, and denote by $l_{\mathbb{G}}: \mathfrak{A}_\Sigma \rightarrow \mathbb{R}_{\geq 0}$ the value of above map at \mathbb{G} . This function has finite support, and the arcs with non-zero contribution are in bijection with the edges of \mathbb{G} around $\partial_1 \Sigma$. Furthermore, for α dual to an edge e around $\partial_1 \Sigma$, the value $l_{\mathbb{G}}(\alpha)$ is the combinatorial length of e (cf. Lemma 3.2.13). As a consequence,

$$\ell_{\mathbb{G}}(\partial_1 \Sigma) = \sum_{\alpha \in \mathfrak{A}_\Sigma} l_{\mathbb{G}}(\alpha).$$

Now from Remark 3.2.7, we know that the map $Q: \mathfrak{A}_\Sigma \rightarrow \mathcal{P}_\Sigma$ is not injective, but has finite fibers. More precisely, we have the following situation.

- If $P \in \mathcal{C}_\Sigma$, then $Q^{-1}(P)$ consists of a single arc $\alpha_0 \in \mathfrak{C}_\Sigma$. Then

$$\sum_{\alpha \in Q^{-1}(P)} l_{\mathbb{G}}(\alpha) = l_{\mathbb{G}}(\alpha_0) = \frac{1}{2} \ell_{\mathbb{G}}(\partial_1 \Sigma) C^K(\vec{\ell}_{\mathbb{G}}(\partial P)).$$

- If $P \in \mathcal{B}_{\Sigma, m}$, then $Q^{-1}(P)$ consists of three arcs: $\alpha_0 \in \mathfrak{B}_{\Sigma, m}$, α' and its inverse $-\alpha'$ in \mathfrak{C}_{Σ} . Then

$$\begin{aligned} \sum_{\alpha \in Q^{-1}(P)} l_{\mathbb{G}}(\alpha) &= l_{\mathbb{G}}(\alpha_0) + l_{\mathbb{G}}(\alpha') + l_{\mathbb{G}}(-\alpha') \\ &= l_{\mathbb{G}}(\partial_1 \Sigma) \left(B^K(\vec{\ell}_{\mathbb{G}}(\partial P)) - C^K(\vec{\ell}_{\mathbb{G}}(\partial P)) + \frac{1}{2} C^K(\vec{\ell}_{\mathbb{G}}(\partial P)) + \frac{1}{2} C^K(\vec{\ell}_{\mathbb{G}}(\partial P)) \right) \\ &= l_{\mathbb{G}}(\partial_1 \Sigma) B^K(\vec{\ell}_{\mathbb{G}}(\partial P)). \end{aligned}$$

Finally, as Q is surjective

$$l_{\mathbb{G}}(\partial_1 \Sigma) = \sum_{P \in \mathcal{P}_{\Sigma}} \sum_{\alpha \in Q^{-1}(P)} l_{\mathbb{G}}(\alpha) = l_{\mathbb{G}}(\partial_1 \Sigma) \sum_{P \in \mathcal{P}_{\Sigma}} X_P^K(\vec{\ell}_{\mathbb{G}}(P)).$$

Dividing by $l_{\mathbb{G}}(\partial_1 \Sigma)$ concludes the proof of the identity (5.3.2). The case of the torus with one boundary component can be handled in a similar way and leads to (5.3.3). \square

A notable difference with the original Mirzakhani–McShane identity is that in (5.3.2) there is a finite number of non-zero terms; this reflects the much simpler dynamics of geodesics in combinatorial surfaces compared to the hyperbolic ones. As in the hyperbolic case, the identity can be reformulated in terms of geometric recursion.

COROLLARY 5.3.2. *The initial data (5.3.1) are admissible, and lead by geometric recursion to $\Xi_{\Sigma}^K(\mathbb{G}) = 1$ for any Σ and $\mathbb{G} \in \mathcal{T}_{\Sigma}^{\text{comb}}$.*

Proof. The only thing left to check is the admissibility condition for (A^K, B^K, C^K, D^K) , which follows from the fact that the functions B^K and C^K are supported on small pairs of pants. \square

We can now apply the integration results of Section 5.2 to the combinatorial Mirzakhani–McShane identity. The continuous integration computes the Kontsevich volumes and Theorem 2.4.23 establishes again Witten’s conjecture [Wit90], which was first proved by the combination of [Kon92] and [DVV91]: the generating function of ψ -class intersection numbers satisfies Virasoro constraints/topological recursion. The discrete integration calculates the number of integral points in $\mathcal{M}_{g,n}^{\text{comb}}$ and Theorem 5.2.5 together with Corollary 5.2.8 with $b = c = 0$, gives a new proof of Norbury’s result [Nor10], i.e. the discrete topological recursion for these counts and their connection with Kontsevich volumes.

COROLLARY 5.3.3 (Witten’s conjecture/Kontsevich theorem). *The Kontsevich volumes $V_{g,n}^K(L)$ equal $\langle 1 \rangle(L)$ and satisfy the topological recursion of Theorem 5.2.2 with initial data (5.3.1).*

COROLLARY 5.3.4 (Norbury’s theorem). *The lattice point counting functions*

$$N_{g,n}(L) = \sum_{G \in \mathcal{M}_{g,n}^{\text{comb}, \mathbb{Z}}(L)} \frac{1}{|\text{Aut}(G)|} \tag{5.3.4}$$

equal $\langle 1^{\mathbb{Z}} \rangle(L)$ and satisfy the discrete topological recursion Theorem 5.2.5 with initial data (5.3.1). Moreover,

$$\lim_{\substack{k \rightarrow \infty \\ k \in d\mathbb{Z}_+}} \frac{N_{g,n}(kL)}{k^{6g-6+2n}} = \begin{cases} 2^{3-2g-n} V_{g,n}^K(L) & \text{if } \sum_{i=1}^n d \cdot L_i \text{ is even,} \\ 0 & \text{otherwise.} \end{cases} \tag{5.3.5}$$

The recursive formula for Kontsevich volumes in this integral form firstly appeared in [BCSW12], where it is derived by constructing a partition of unity similar to the combinatorial Mirzakhani–McShane identity and exploiting the local torus symmetries on the combinatorial moduli space, whose symplectic quotients are also combinatorial moduli space of higher Euler characteristics. Compared to [BCSW12], the new element of the proof that we propose is to make it a *complete* analogue to Mirzakhani’s proof of the recursion for Weil–Petersson volumes, by means of Theorem 5.3.1. Our perspective stresses that, at a general level, recursions between volumes (or more generally, between integrals over the moduli spaces) arise from finer recursions that hold at the geometric level (here between functions on Teichmüller spaces).

5.4 — COMBINATORIAL LENGTH STATISTICS OF MULTICURVES

Following [ABO17, Theorem 10.1] in the hyperbolic world (see Section 2.4.2 for a short overview), we can generalise the combinatorial Mirzakhani–McShane identity to obtain statistics of combinatorial lengths of primitive multicurves via geometric recursion.

THEOREM 5.4.1 (Combinatorial length statistics of multicurves). *Let (A, B, C, D) be admissible initial data and denote by Ξ_Σ the associated geometric recursion amplitudes. Let $f: \mathbb{R}_+ \rightarrow \mathbb{R}$ be a measurable function such that for any $\epsilon > 0$ and $s \geq 0$, there exists $M_{s,\epsilon}$ such that $\sup_{\ell \geq \epsilon} |f(\ell)| \ell^s \leq M_{s,\epsilon}$. Then, the following initial data are admissible:*

$$\begin{aligned} A[f](L_1, L_2, L_3) &= A(L_1, L_2, L_3), \\ B[f](L_1, L_2, \ell) &= B(L_1, L_2, \ell) + A(L_1, L_2, \ell) f(\ell), \\ C[f](L_1, \ell, \ell') &= C(L_1, \ell, \ell') + B(L_1, \ell, \ell') f(\ell) + B(L_1, \ell', \ell) f(\ell') + A(L_1, \ell, \ell') f(\ell) f(\ell'), \\ D_T[f](\mathbb{G}) &= D_T(\mathbb{G}) + \sum_{\gamma \in \delta_T} A(\ell_{\mathbb{G}}(\partial T), \ell_{\mathbb{G}}(\gamma), \ell_{\mathbb{G}}(\gamma)) f(\ell_{\mathbb{G}}(\gamma)). \end{aligned} \tag{5.4.1}$$

Denote by $\Xi_\Sigma[f]$ the corresponding geometric recursion amplitudes. If for all Σ , Ξ_Σ is invariant under all braidings of boundary components of Σ , we have

$$\Xi_\Sigma[f](\mathbb{G}) = \sum_{c \in \mathcal{M}_\Sigma} \Xi_{\Sigma_c}(\mathbb{G}|_{\Sigma_c}) \prod_{\gamma \in \pi_0(c)} f(\ell_{\mathbb{G}}(\gamma)), \tag{5.4.2}$$

where Σ_c is the bordered surface obtained by cutting Σ along a primitive multicurve c (the choice of the first boundary component is irrelevant due to the assumed invariance).

Remark 5.4.2. As explained in Remark 2.4.27, a useful version of the above result can be stated for combinatorial length statistics of multicurves (not only primitive ones). Namely, if $F: \mathbb{R}_+ \rightarrow \mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ is a measurable function with values in the unit disk such that

$$f(x) = \sum_{k \geq 1} F(x)^k = \frac{F(x)}{1 - F(x)} \tag{5.4.3}$$

satisfies the conditions of Theorem 5.4.1, then the geometric recursion amplitudes $\Xi_\Sigma[f]$ associated to the initial data Equation (5.4.1) are given by the combinatorial length statistic of multicurves weighted by F :

$$\Xi_\Sigma[f](\mathbb{G}) = \sum_{c \in \mathcal{M}_\Sigma} \Xi_{\Sigma_c}(\mathbb{G}|_{\Sigma_c}) \prod_{\gamma \in \pi_0(c)} F(\ell_{\mathbb{G}}(\gamma)). \tag{5.4.4}$$

Sketch of the proof of Theorem 5.4.1. The proof repeats the one of [ABO17, Theorem 10.1], with combinatorial lengths instead of hyperbolic ones. We only sketch the induction step here. Notice that, as all series are absolutely convergent, we can apply Fubini's theorem and interchange the summations:

$$\begin{aligned} \sum_{c \in \mathcal{M}'_\Sigma} \Xi_{\Sigma_c}(\mathbb{G}|\Sigma_c) \prod_{\gamma \in \pi_0(c)} f(\ell_{\mathbb{G}}(\gamma)) &= \sum_{c \in \mathcal{M}'_\Sigma} \sum_{P \in \mathcal{P}_{\Sigma_c}} X_P(\mathbb{G}) \Xi_{\Sigma_c - P}(\mathbb{G}|\Sigma_c - P) \prod_{\gamma \in \pi_0(c)} f(\ell_{\mathbb{G}}(\gamma)) \\ &= \sum_{P \in \mathcal{P}_\Sigma} X_P[f](\mathbb{G}) \left(\sum_{\gamma \in \mathcal{M}'_{\Sigma - P}} \Xi_{(\Sigma - P)_\gamma}(\mathbb{G}|\Sigma - P)_\gamma \prod_{c \in \pi_0(\gamma)} f(\ell_{\mathbb{G}}(\gamma_c)) \right) \\ &= \sum_{P \in \mathcal{P}_\Sigma} X_P[f](\mathbb{G}) \Xi_{\Sigma - P}[f](\mathbb{G}|\Sigma - P), \end{aligned}$$

where X_P is either B or $\frac{1}{2}C$ depending on the type of P . The second to last equality follows from a case discussion: to interchange the summands we must also sum over the possible ways in which ∂P and c can have homotopic components, weighted by their contributions. For example, for $P \in \mathcal{C}_\Sigma$ with ∂P and c sharing one component, say $\partial_2 P = \gamma_0$, then $P \in \mathcal{P}_{\Sigma_c, \gamma_0}$. This means that we get a contribution of

$$B(L_1, \ell_{\mathbb{G}}(\gamma_0), \ell_{\mathbb{G}}(\partial_3 P)) f(\ell_{\mathbb{G}}(\gamma_0)) \Xi_{(\Sigma - P)_c}(\mathbb{G}|\Sigma - P)_c \prod_{\gamma \in \pi_0(c - \gamma_0)} f(\ell_{\mathbb{G}}(\gamma)),$$

which matches one of the terms in the expression for $C[f]$. We refer to [ABO17] for the complete proof. \square

We can then integrate this identity over the combinatorial moduli space with respect to μ_K to express combinatorial length statistics of multicurves as a sum over stable graphs.

COROLLARY 5.4.3. *Assume that (A, B, C, D) are strongly admissible and fix a measurable function f for which there exists $\eta \in [0, 2)$ such that $\sup_{\ell > 0} |f(\ell)| \ell^\eta < +\infty$. Then the twisted initial data $(A[f], B[f], C[f], D[f])$ are strongly admissible and $\langle \Xi_{g,n}[f] \rangle(L)$ satisfy the topological recursion of Equation (5.2.3) with these initial data. Besides, we have*

$$\begin{aligned} \langle \Xi_{g,n}[f] \rangle(L_1, \dots, L_n) &= \\ &= \sum_{\Gamma \in \mathcal{G}_{g,n}} \frac{1}{|\text{Aut}(\Gamma)|} \int_{\mathbb{R}_+^{E_\Gamma}} \prod_{v \in V_\Gamma} \langle \Xi_{g(v), n(v)} \rangle((\ell_e)_{e \in E(v)}, (L_\lambda)_{\lambda \in \Lambda(v)}) \prod_{e \in E_\Gamma} f(\ell_e) \ell_e d\ell_e. \end{aligned} \quad (5.4.5)$$

Proof. The topological recursion follows from Theorem 5.2.2 and from the second part from the integration formula of Proposition 4.3.1. \square

We can also obtain an analogous result for the discrete integration, substituting $\langle \Xi_{g(v), n(v)} \rangle$ with $\langle \Xi_{g(v), n(v)}^{\mathbb{Z}} \rangle$ and the integral over $\mathbb{R}_+^{E_\Gamma}$ with the sum over edge decorations of the form $\ell: E_\Gamma \rightarrow \mathbb{Z}_+$. However, we cannot directly apply Theorem 5.2.5, as the twisted initial data are in general not supported on small pairs of pants. Instead, we introduce an (a priori) different lattice count, for $L \in \mathbb{Z}_+^n$

$$\begin{aligned} \langle \Xi_{g,n}^{\mathbb{Z}}[f] \rangle(L_1, \dots, L_n) &= \\ &= \sum_{\Gamma \in \mathcal{G}_{g,n}} \frac{1}{|\text{Aut}(\Gamma)|} \sum_{\ell: E_\Gamma \rightarrow \mathbb{Z}_+} \prod_{v \in V_\Gamma} \langle \Xi_{g(v), n(v)}^{\mathbb{Z}} \rangle((\ell_e)_{e \in E(v)}, (L_\lambda)_{\lambda \in \Lambda(v)}) \prod_{e \in E_\Gamma} \ell_e f(\ell_e). \end{aligned} \quad (5.4.6)$$

We omit the proof of the next proposition: it follows the same scheme as the proof of Theorem 5.4.1, in the simpler situation where multicurves are replaced by their mapping class group orbits (stable graphs), and with integrals replaced by discrete sums. The key principle is that topological recursion is preserved under the twisting operation.

PROPOSITION 5.4.4. *Let (A, B, C, D) and $\langle \Xi_{g,n}^{\mathbb{Z}} \rangle$ as in Theorem 5.2.5. Let $f: \mathbb{Z}_+ \rightarrow \mathbb{R}$ be such that for any $s > 0$, $\sup_{\ell \in \mathbb{Z}_+} |f(\ell)| \ell^s < +\infty$. Then $\langle \Xi_{g,n}^{\mathbb{Z}}[f] \rangle(L)$ is finite and, for $2g - 2 + n > 1$, it is calculated by the discrete topological recursion formula*

$$\begin{aligned} \langle \Xi_{g,n}^{\mathbb{Z}}[f] \rangle(L_1, \dots, L_n) &= \sum_{m=2}^n \sum_{\ell \geq 1} \ell B^{\mathbb{Z}}[f](L_1, L_m, \ell) \langle \Xi_{g,n-1}^{\mathbb{Z}}[f] \rangle(\ell, L_2, \dots, \widehat{L_m}, \dots, L_n) \\ &\quad + \frac{1}{2} \sum_{\ell, \ell' \geq 1} \ell \ell' C^{\mathbb{Z}}[f](L_1, \ell, \ell') \left(\langle \Xi_{g-1, n+1}^{\mathbb{Z}}[f] \rangle(\ell, \ell', L_2, \dots, L_n) \right. \\ &\quad \left. + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2, \dots, n\}}} \langle \Xi_{g_1, 1+|I_1|}^{\mathbb{Z}}[f] \rangle(\ell, L_{I_1}) \langle \Xi_{g_2, 1+|I_2|}^{\mathbb{Z}}[f] \rangle(\ell', L_{I_2}) \right), \end{aligned} \quad (5.4.7)$$

with conventions $\langle \Xi_{0,1}^{\mathbb{Z}}[f] \rangle = 0$ and $\langle \Xi_{0,2}^{\mathbb{Z}}[f] \rangle = 0$ and base cases

$$\begin{aligned} \langle \Xi_{0,3}^{\mathbb{Z}}[f] \rangle(L_1, L_2, L_3) &= \langle \Xi_{0,3}^{\mathbb{Z}} \rangle(L_1, L_2, L_3) = A^{\mathbb{Z}}(L_1, L_2, L_3), \\ \langle \Xi_{1,1}^{\mathbb{Z}}[f] \rangle(L) &= \langle \Xi_{1,1}^{\mathbb{Z}} \rangle(L) + \frac{1}{2} \sum_{\ell \geq 1} \ell A^{\mathbb{Z}}(L, \ell, \ell) f(\ell). \end{aligned} \quad (5.4.8)$$

5.5 — KONTSEVICH AMPLITUDES AND THE SPINE CONSTRUCTION

On the one hand, by Corollary 5.3.2, the combinatorial geometric amplitudes $\Xi_{\Sigma}^{\mathbb{K}}$ for the initial data $(A^{\mathbb{K}}, B^{\mathbb{K}}, C^{\mathbb{K}})$ coincide with the constant function 1 on $\mathcal{T}_{\Sigma}^{\text{comb}}$. On the other hand, we can consider the hyperbolic geometric recursion amplitudes $\Omega_{\Sigma}^{\mathbb{K}}$ associated to the *same* initial data. They are rather non-trivial functions of $\sigma \in \mathcal{T}_{\Sigma}$, but since the topological recursion formulae are the same in the combinatorial and hyperbolic setting, we have

$$V_{g,n}^{\mathbb{K}}(L) = \int_{\mathcal{M}_{g,n}^{\text{comb}}(L)} \Xi_{g,n}^{\mathbb{K}} d\mu_{\mathbb{K}} = \int_{\mathcal{M}_{g,n}(L)} \Omega_{g,n}^{\mathbb{K}} d\mu_{\text{WP}} = \int_{\overline{\mathcal{M}}_{g,n}} \exp\left(\sum_{i=1}^n \frac{L_i^2}{2} \psi_i\right) \quad (5.5.1)$$

for any $L = (L_1, \dots, L_n)$. Here we propose a geometric interpretation of $\Omega_{\Sigma}^{\mathbb{K}}$ and give some of its basic properties.

5.5.1 — A GEOMETRIC INTERPRETATION

In the following we discuss the combinatorial analogue of the geometric reasoning at the core of Mirzakhani's proof of the Mirzakhani–McShane identities. Consider a bordered surface Σ . Recall from Section 3.1.2 the construction of the spine as a subset $\text{sp}_{\sigma}(\Sigma) \subset \Sigma$ that depends on a hyperbolic marking (X, f) representing $\sigma \in \mathcal{T}_{\Sigma}$. We also denote by $\text{sp}'_{\sigma}(\Sigma) \subset \text{sp}_{\sigma}(\Sigma)$ the complement of the set of vertices of the spine.

For a given $\sigma \in \mathcal{T}_{\Sigma}$, we equip $\partial_1 \Sigma$ with the curvilinear measure μ_{σ} induced by σ .

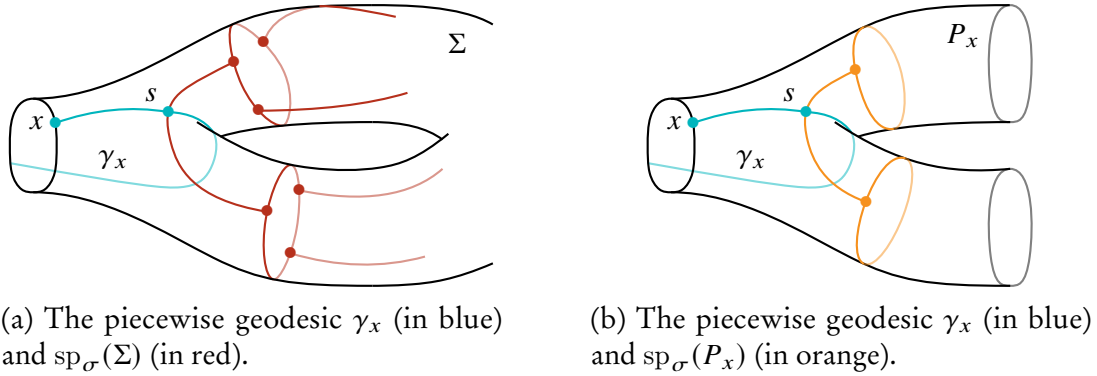


Figure 5.1: Example of a successful first step of the process.

LEMMA 5.5.1. *For all but finitely many $x \in \partial_1 \Sigma$, the orthogeodesic from x intersects the spine for the first time at a point $s_\Sigma(x) \in \text{sp}'_\sigma(\Sigma)$.*

Proof. Cutting out the spine, we have a cylinder around each boundary component of the surface. Consider the one around $\partial_1 \Sigma$ and take a geodesic that realises the distance between the two boundaries of the cylinder. Cutting along this, we obtain a hyperbolic polygon. As there are no hyperbolic triangles with two right angles, every geodesic shot orthogonally from $x \in \partial_1 \Sigma$ must reach the boundary corresponding to the spine at a certain point $s_\Sigma(x)$, that is not a vertex for all but finitely many x that are rib-ends on $\partial_1 \Sigma$. \square

Let $x \in \partial_1 \Sigma$ be such that $s_\Sigma(x)$ exists and is not a vertex of the spine. By definition of the spine and the vertices, there exists a unique second geodesic joining $s_\Sigma(x)$ to $\partial_i \Sigma$ for some $i \in \{1, \dots, n\}$. The union of these two geodesics form a piecewise geodesic arc, denoted by γ_x . This arc determines a unique homotopy class of embedded pair of pants, and we denote by P_x its representative that has geodesic boundaries in Σ . This pair of pants has a spine $\text{sp}_\sigma(P_x)$ of its own, and we can ask whether $s_\Sigma(x)$ is still part of $\text{sp}_\sigma(P_x)$. With these notations, we define the following process.

DEFINITION 5.5.2. Choose a random point $x \in \partial_1 \Sigma$ uniformly on $\partial_1 \Sigma$ – for the measure coming from the curvilinear measure induced by σ .

- If the geodesic shot from x orthogonally to $\partial_1 \Sigma$ hits for the first time $\text{sp}_\sigma(\Sigma)$ at a vertex, or if $s_\Sigma(x) \notin \text{sp}_\sigma(P_x)$, quit the process.
- Otherwise, consider the bordered hyperbolic surface $\Sigma - P_x$. If it is empty, we have finished the process successfully; if not, we repeat it with $\Sigma - P_x$, each connected component of $\Sigma - P_x$ being treated independently.

Denote by $\Pi_\Sigma(\sigma)$ the probability that the process ends successfully, i.e. by giving a pants decomposition. It is clear that $\Pi_\Sigma(\sigma)$ only depends on the projection $[\sigma] \in \mathcal{M}_\Sigma$ and the process makes no reference to a marking.

PROPOSITION 5.5.3. *We have $\Pi_\Sigma(\sigma) = \Omega_\Sigma^K(\sigma)$. In particular, $\Omega_\Sigma^K(\sigma) \in [0, 1]$, and if we consider the above process for a random hyperbolic surface of type (g, n) with fixed boundary lengths $L \in \mathbb{R}_+^n$ (with respect to the Weil–Peterson measure on $\mathcal{M}_\Sigma(L)$), we have*

$$\mathbb{E}[\Pi_\Sigma(\sigma) \mid \ell_\sigma(\partial \Sigma) = L] = \frac{V_{g,n}^K(L)}{V_{g,n}^{\text{WP}}(L)}, \quad (5.5.2)$$

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where $V_{g,n}^{\text{WP}}(L)$ are the Weil–Petersson volumes of $\mathcal{M}_{g,n}(L)$.

Proof. We prove the result by induction on $2g - 2 + n > 0$. The base case $(g, n) = (0, 3)$ is trivial: for any pair of pants P , we have $\Pi_P \equiv 1$. Furthermore, the following argument can be adjusted to prove that for a one-holed torus T ,

$$\Pi_T(\sigma) = \sum_{\gamma \in \mathcal{S}_T} C^K(\ell_\sigma(\partial T), \ell_\sigma(\gamma), \ell_\sigma(\gamma)) = D_T(\sigma).$$

Consider Σ of type (g, n) and suppose now by induction that the proposition holds for surfaces of Euler characteristic $\chi > -(2g - 2 + n)$. By definition of the process, we can write $\Pi_\Sigma(\sigma)$ as a sum over $\cup_{x \in \partial_1 \Sigma} P_x$ (where $P_x = \emptyset$ if x does not define a pair of pants):

$$\Pi_\Sigma(\sigma) = \sum_{P \in \cup_{x \in \partial_1 \Sigma} P_x} Y_P(\sigma) \Pi_{\Sigma-P}(\sigma|_{\Sigma-P}).$$

Here, we identify P with its representative $P \subseteq \Sigma$ having geodesic boundaries, and we set

$$Y_P(\sigma) = \frac{\mu_\sigma(\mathcal{Y}_P(\sigma))}{\mu_\sigma(\partial_1 \Sigma)}, \quad \mathcal{Y}_P(\sigma) = \{x \in \partial_1 \Sigma \mid s_\Sigma(x) \in \text{sp}_\sigma(P)\}. \quad (5.5.3)$$

By induction hypothesis, we have $\Pi_{\Sigma-P}(\sigma|_{\Sigma-P}) = \Omega_{\Sigma-P}^K(\sigma|_{\Sigma-P})$. So, we only need to show that $Y_P(\sigma) = X_P^K(\vec{\ell}_\sigma(\partial P))$ and that the embedded pairs of pants outside $\cup_{x \in \partial_1 \Sigma} P_x$ do not contribute to the geometric recursion sum defining Ω_Σ^K .

For the latter, we observe that if $P \notin \cup_{x \in \partial_1 \Sigma} P_x$, then all points of $\text{sp}_\sigma(P)$ incident to $\partial_1 \Sigma$ are equidistant to an internal boundary of P . Therefore we see that the spine of $\sigma|_P$ has the whole of the first boundary adjacent to an internal boundary, and therefore from Lemma 3.2.II we see that $X_P^K(\vec{\ell}_\sigma(\partial P)) = 0$. So, we can indeed restrict the range of the geometric recursion sum defining Ω_Σ^K to $P \in \cup_{x \in \partial_1 \Sigma} P_x$.

For the former, we first give a description of X_P^K similar to (5.5.3). Let $s_P(x)$ be the first intersection point (if it exists) between the geodesic shot from x orthogonally to $\partial_1 \Sigma$ and the spine of P – considered as a subset of Σ . We introduce the sets

$$\text{sp}_\sigma(P) = \begin{cases} \{s \in \text{sp}'_\sigma(P) \mid s \text{ is incident to } \partial_m \Sigma \text{ and } \partial_1 \Sigma \text{ or only to } \partial_1 \Sigma\} & \text{if } P \in \mathcal{B}_{\Sigma, m} \\ \{s \in \text{sp}'_\sigma(P) \mid s \text{ is incident to } \partial_1 \Sigma \text{ on both sides}\} & \text{if } P \in \mathcal{C}_\Sigma \end{cases}$$

$$\mathcal{X}_P(\sigma) = \{x \in \partial_1 \Sigma \mid s_P(x) \in \text{sp}_\sigma(P)\}$$

where it is understood that if $s_P(x)$ does not exist, x is not in $\mathcal{X}_P(\sigma)$. The properties of B^K and C^K stressed in Lemma 3.2.II show that

$$X_P^K(\vec{\ell}_\sigma(\partial P)) = \frac{\mu_\sigma(\mathcal{X}_P(\sigma))}{\mu_\sigma(\partial_1 \Sigma)}.$$

It remains to justify that $\mathcal{X}_P(\sigma) = \mathcal{Y}_P(\sigma)$.

(\subseteq) Consider $x \in \mathcal{X}_P(\sigma)$. From the definition of $\text{sp}_\sigma(P)$, it is clear that $\text{sp}_\sigma(P) \subseteq \text{sp}'_\sigma(\Sigma)$ and $s_P(x)$ is equidistant to $\partial \Sigma$ in exactly two ways. In particular,

$$\text{dist}_\sigma(s_P(x), \partial_1 \Sigma) = \text{dist}_\sigma(s_P(x), \partial_i \Sigma) < \text{dist}_\sigma(s_P(x), \partial \Sigma - (\partial_1 \Sigma \cup \partial_i \Sigma))$$

for some $i \in \{1, \dots, n\}$. Thus, $s_\Sigma(x) = s_P(x)$ and $s_P(x) \in \text{sp}_\sigma(P)$.

(\supseteq) Consider $x \in \mathcal{Y}_P(\sigma)$. Then we have $s_P(x) \in \text{sp}_\sigma(\Sigma)$, so that $s_P(x) = s_\Sigma(x)$ and $s_P(x) \in \text{sp}_\sigma(P)$.

□

5.5.2 — PROPERTIES OF THE KONTSEVICH AMPLITUDES

The geometric interpretation of Ω_Σ^K already shows the non-trivial fact that such functions take values in $[0, 1] \subset \mathbb{R}$. In the remaining part of this section we are going to show two other properties of Ω_Σ^K .

NON-INVARIANCE UNDER ALL BRAIDINGS

From their definition, the geometric recursion amplitudes are a priori only invariant under mapping classes that preserve the first boundary; the invariance under braidings of $\partial_1 \Sigma$ with some $\partial_m \Sigma$ for $m \neq 1$ is not guaranteed. This full invariance turns out to hold for the hyperbolic geometric recursion amplitudes Ω_Σ^M and the combinatorial geometric recursion amplitudes Ξ_Σ^K , for the obvious reason that they are identically 1. One could wonder if the full invariance also holds for Ω_Σ^K , but we show this is already not the case for four-holed spheres.

PROPOSITION 5.5.4. *Let X be a four-holed sphere, and take $\gamma \in \mathcal{S}_X$ separating $\partial_1 X$ and $\partial_2 X$ from $\partial_3 X$ and $\partial_4 X$. Consider $\rho = [\phi: X \rightarrow X] \in \text{Mod}_X$ the involution that fixes γ and such that $\phi(\partial_1 X) = \partial_3 X$ and $\phi(\partial_2 X) = \partial_4 X$. Then $\rho \cdot \Omega_X^K \neq \Omega_X^K$.*

Proof. The curve γ together with an arc from $\partial_1 X$ to $\partial_3 X$, determine a seamed pants decomposition of X , and we denote by $(L, \ell, \tau) \in \mathbb{R}_+^4 \times \mathbb{R}_+ \times \mathbb{R}$ the corresponding combinatorial Fenchel–Nielsen coordinates. If we choose $\sigma \in \mathcal{T}_X$ such that $\ell_\sigma(\gamma)$ is small enough, then by the collar lemma we can make the length of any simple closed geodesic intersecting γ greater than $\max\{L_1 + L_2, L_3 + L_4\}$. As $B^K(L_i, L_j, \ell)$ vanishes for $\ell \geq L_i + L_j$, we have

$$\Omega_X^K(L_1, L_2, L_3, L_4, \ell, \tau) = B^K(L_1, L_2, \ell)$$

and

$$\rho \cdot \Omega_X^K(L_1, L_2, L_3, L_4, \ell, \tau) = \Omega_X^K(L_3, L_4, L_1, L_2, \ell, \tau) = B^K(L_3, L_4, \ell).$$

Choosing $(L_1, \dots, L_4) \in \mathbb{R}_+^4$ such that $B^K(L_1, L_2, \ell) \neq B^K(L_3, L_4, \ell)$, we obtain the thesis. \square

This is the first example where the non-invariance of some geometric recursion amplitudes can be established. Nevertheless, we know that after integration over the moduli space against μ_{WP} , we obtain the symmetric polynomials (5.5.1). This is clear by the definition of the Kontsevich volumes, but it can also be directly proved from the theory of the topological recursion, as (A^K, B^K, C^K) obey a set of quadratic relations (IHX relations) that guarantee the invariance of the corresponding topological recursion amplitudes under permutations of L_1, \dots, L_n – see e.g. [Bor20].

SUPPORT WITH NON-EMPTY COMPLEMENT

We prove that the Kontsevich amplitudes are actually zero on some open subset. In order to achieve this, we use geometric recursion to construct an auxiliary function which has the same support as the Kontsevich amplitude and takes integer values.

Consider the following geometric recursion initial data

$$\begin{aligned} A(L_1, L_2, L_3) &= 1, \\ B(L, L', \ell) &= H(L + L' - \ell), \\ C(L, \ell, \ell') &= H(L - \ell - \ell'), \end{aligned} \tag{5.5.4}$$

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where $H(x)$ is the Heaviside function. Let $\Omega_\Sigma^H \in \text{Mes}(\mathcal{T}_\Sigma)$ be the geometric recursion amplitude associated to Σ computed with respect to hyperbolic lengths. The value $\Omega_\Sigma^H(\sigma) \in \mathbb{N}$ is counting the number of σ -small pairs of pants decompositions¹ of Σ . In particular, these are piecewise constant functions on \mathcal{T}_Σ with values in non-negative integers and, for Σ connected of type (g, n) , we see that Ω_Σ^H is bounded by $\prod_{k=1}^{2g-2+n} 6k$. The following lemma easily follows by induction on $2g - 2 + n$.

LEMMA 5.5.5. *The geometric recursion amplitudes Ω_Σ^H have the same support in \mathcal{T}_Σ as Ω_Σ^K , and the topological recursion amplitudes $\langle \Omega_{g,n}^H \rangle(L) = \int_{\mathcal{M}_{g,n}(L)} \Omega_\Sigma^H d\mu_{\text{WP}}$ are homogeneous polynomials of degree $6g - 6 + 2n$.*

COROLLARY 5.5.6. *For Σ not of type $(0, 3)$, the support of Ω_Σ^K has non-empty complement.*

Proof. The Weil–Petersson volumes $V_{g,n}^{\text{WP}}(L)$ are polynomials in $L = (L_1, \dots, L_n)$ with non-zero constant term, while $\langle \Omega_{g,n}^H \rangle(L)$ are homogeneous of positive degree for $(g, n) \neq (0, 3)$. Hence

$$\lim_{L \rightarrow 0} \frac{\langle \Omega_{g,n}^H \rangle(L)}{V_{g,n}^{\text{WP}}(L)} = 0.$$

This shows that $\langle \Omega_{g,n}^H \rangle(L)$ is strictly less than $V_{g,n}^{\text{WP}}(L)$ for small L and we deduce there exists an open set of $\mathcal{T}_\Sigma^{\text{comb}}(L)$ for small L on which $\Omega_\Sigma^H < 1$ and therefore on which $\Omega_\Sigma^H = 0$, hence Ω_Σ^K vanishes. \square

Remark 5.5.7. To conclude, we remark that the Laplace transform of the topological recursion amplitudes $\langle \Omega_{g,n}^H \rangle(L)$, namely

$$\omega_{g,n}(z_1, \dots, z_n) = \left(\int_{\mathbb{R}_+^n} \prod_{i=1}^n dL_i L_i e^{-z_i L_i} \langle \Omega_{g,n}^H \rangle(L_1, \dots, L_n) \right) dz_1 \cdots dz_n \quad (5.5.5)$$

satisfy a topological recursion à la Eynard–Orantin:

$$\begin{aligned} \omega_{g,n}(z_1, \dots, z_n) = \text{Res}_{z \rightarrow 0} K(z_1, z) & \left(\omega_{g-1, n+1}(z, -z, z_2, \dots, z_n) \right. \\ & \left. + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2, \dots, n\}}} \omega_{g_1, 1+|I_1|}(z, z_{I_1}) \omega_{g_2, 1+|I_2|}(-z, z_{I_2}) \right), \end{aligned} \quad (5.5.6)$$

where

$$K(z_1, z) = \frac{1}{2z(z-z_1)^2} \frac{dz_1}{dz} \quad \text{and} \quad \omega_{0,2}(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}. \quad (5.5.7)$$

However, in this case the recursion kernel $K(z_1, z)$ does not have the usual structure of Equation (2.3.7) and the multidifferentials $\omega_{g,n}(z_1, \dots, z_n)$ are not symmetric under permutation of their n variables. It seems to be the first known example where a non-symmetric topological recursion yields a geometrically meaningful quantity.

¹Such pants decomposition are “rooted”, in the sense that they always bound the first boundary component of the surface at each step (cf. [ABO17, Section 3.6])

CHAPTER 6 — RESCALING FLOW: FROM HYPERBOLIC TO COMBINATORIAL GEOMETRY

The hyperbolic and combinatorial Teichmüller spaces can be identified via the spine homeomorphism $\text{sp}: \mathcal{T}_\Sigma \rightarrow \mathcal{T}_\Sigma^{\text{comb}}$ of Penner and Bowditch–Epstein (see Definition 3.1.10). In this chapter, we consider a flow that interpolates between their respective geometries, coming from the work of Mondello and Do [Mon09; Do10].

DEFINITION 6.0.1. Let $\mathbb{G} \in \mathcal{T}_\Sigma^{\text{comb}}(L)$ and $\beta \in \mathbb{R}_+$. Define $\beta\mathbb{G} \in \mathcal{T}_\Sigma^{\text{comb}}(\beta L)$ to be the combinatorial structure represented by the same marked ribbon graph, but with all lengths multiplied by β . This defines a flow, called the *rescaling flow*, on $\mathcal{T}_\Sigma^{\text{comb}}$ which preserves the strata and is continuous. It can be lifted to $\Phi_\beta: \mathcal{T}_\Sigma(L) \rightarrow \mathcal{T}_\Sigma(\beta L)$ by the spine map: for $\sigma \in \mathcal{T}_\Sigma(L)$, set

$$\Phi_\beta(\sigma) = \sigma^\beta = \text{sp}^{-1}(\beta \text{sp}(\sigma)) \in \mathcal{T}_\Sigma(\beta L). \quad (6.0.1)$$

The maps Φ_β and $\beta \cdot$ are Mod_Σ -equivariant, and thus descend to the moduli spaces $\mathcal{M}_{g,n}(L)$ and $\mathcal{M}_{g,n}^{\text{comb}}(L)$. We also define the map $R_\beta: \mathcal{T}_\Sigma^{\text{comb}}(L) \rightarrow \mathcal{T}_\Sigma(\beta L)$ by postcomposing $\beta \cdot$ with sp^{-1} .

$$\begin{array}{ccc} \mathcal{T}_\Sigma(L) & \xrightarrow{\text{sp}} & \mathcal{T}_\Sigma^{\text{comb}}(L) \\ \Phi_\beta \downarrow & \swarrow R_\beta & \downarrow \beta \cdot \\ \mathcal{T}_\Sigma(\beta L) & \xrightarrow{\text{sp}^{-1}} & \mathcal{T}_\Sigma^{\text{comb}}(\beta L) \end{array} \quad (6.0.2)$$

The large β asymptotic of the rescaling flow has been previously studied pointwise on \mathcal{T}_Σ .

THEOREM 6.A ([Mon09; Do10]). *As $\beta \rightarrow \infty$:*

- *the metric space $(\Sigma, \beta^{-1}\sigma^\beta)$ converges in the Gromov–Hausdorff topology to the metric ribbon graph $\text{sp}(\sigma)$ for any bordered surface Σ (cf. [Do10, Theorem 1]);*
- *the Poisson structure $\beta^2 R_\beta^* \pi_{\text{WP}}$ converges pointwise to π_{K} on the top-dimensional strata of $\mathcal{T}_\Sigma^{\text{comb}}$ (cf. [Mon09, Theorem 4.3] or [Do10, Theorem 2]).*

In this chapter, we shall complete this description by giving effective bounds on the thick part of the Teichmüller space for lengths, and on compacts for twist parameters, showing convergence of the Fenchel–Nielsen coordinates.

THEOREM 6.B. *Let Σ be a bordered surface of type (g, n) . For any γ simple closed curve in Σ and $\sigma \in \mathcal{T}_\Sigma$, we have that*

$$\lim_{\beta \rightarrow \infty} \frac{\ell_{\sigma^\beta}(\gamma)}{\beta} = \ell_{\text{sp}(\sigma)}(\gamma), \quad (6.0.3)$$

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and the convergence is uniform on the thick parts of \mathcal{T}_Σ . Moreover, given a seamed pants decomposition, we have $3g - 3 + n$ hyperbolic and combinatorial twist parameter functions τ_i . Then for any $\sigma \in \mathcal{T}_\Sigma$, we have that

$$\lim_{\beta \rightarrow \infty} \frac{\tau_i(\sigma^\beta)}{\beta} = \tau_i(\text{sp}(\sigma)), \quad (6.0.4)$$

and the convergence is uniform on every compact of \mathcal{T}_Σ .

Thanks to this uniformity, we can flow quite systematically results in hyperbolic geometry to results in combinatorial geometry, and natural functions on \mathcal{T}_Σ to natural functions on $\mathcal{T}_\Sigma^{\text{comb}}$. In this regard, $\mathcal{T}_\Sigma^{\text{comb}}$ behaves like a differential-geometric tropicalisation of \mathcal{T}_Σ . We use this method to re-derive (an equivalent expression of) Penner’s formulae for the action of the mapping class group on Dehn–Thurston coordinates [Pen82] from their hyperbolic analogue [Oka93]. An immediate consequence of Penner’s formulae is a piecewise linear structure on the combinatorial Teichmüller space.

THEOREM 6.C. *The combinatorial Fenchel–Nielsen coordinates equip $\mathcal{T}_\Sigma^{\text{comb}}$ with the structure of a piecewise linear manifold.*

As a second application, we show that the flow in the $\beta \rightarrow \infty$ limit takes the hyperbolic geometric recursion amplitudes to the combinatorial counterpart, and does the same for topological recursion after integration. See Theorems 6.3.5 and 6.4.1 for more precise statements.

THEOREM 6.D. *Let $(A_\beta, B_\beta, C_\beta)_{\beta \geq 1}$ be a 1-parameter family of initial data that are “uniformly admissible” and converging uniformly on any compact to a limit $(\hat{A}, \hat{B}, \hat{C})$ as $\beta \rightarrow \infty$. Denote by $\Omega_{\Sigma; \beta}$ the GR amplitudes on \mathcal{T}_Σ , which results from the initial data $(A_\beta, B_\beta, C_\beta)$, and by $\widehat{\Xi}_\Sigma$ the GR amplitudes on $\mathcal{T}_\Sigma^{\text{comb}}$, which results from the initial data $(\hat{A}, \hat{B}, \hat{C})$.*

- For any bordered surface Σ and $\sigma \in \mathcal{T}_\Sigma$,

$$\lim_{\beta \rightarrow \infty} \Omega_{\Sigma; \beta}(\sigma^\beta) = \widehat{\Xi}_\Sigma(\text{sp}(\sigma)), \quad (6.0.5)$$

and the convergence is uniform on any compact of \mathcal{T}_Σ .

- If $(A_\beta, B_\beta, C_\beta)_{\beta \geq 1}$ are “uniformly strongly admissible”, and the converge is uniform on any subset of the form $(0, M]^3 \subset \mathbb{R}_+^3$, then for $2g - 2 + n > 0$ and any $L \in \mathbb{R}_+^n$

$$\lim_{\beta \rightarrow \infty} \frac{\langle \Omega_{g, n; \beta} \rangle(\beta L)}{\beta^{6g-6+2n}} = \langle \widehat{\Xi}_{g, n} \rangle(L), \quad (6.0.6)$$

and the convergence is uniform for L in any set of the form $(0, M]^n$ with $M > 0$.

In particular, this gives a *second proof* of the combinatorial Mirzakhani–McShane identity by applying the flow to the original hyperbolic Mirzakhani–McShane identity.

6.0.1 — RELATION WITH PREVIOUS WORKS AND OPEN PROBLEMS

As already mentioned, the rescaling flow originates from the work of Mondello and Do [Mon09; Do10], using the spine construction of Penner and Bowditch–Epstein [Pen87; BE88]. Here we promote some of their results from pointwise to uniform convergence. Geometrically, it formalises the idea that hyperbolic surfaces “flows uniformly” to their spine.

We also remark that a rescaling flow previously appeared in a similar form in the work of Papadopoulos–Penner [PP93], building on the fact that measured foliations appears as Thurston’s boundary of the Teichmüller space. More precisely, they considered the decorated Teichmüller space $\widetilde{\mathcal{T}}_{\Sigma}(L)$ of a *punctured* surface Σ , parametrising hyperbolic structures with cusps and horocycles of length L around the punctures, which can be equipped with the pullback of the Weil–Petersson symplectic form. Using a rescaling flow, Papadopoulos and Penner shows that the Weil–Petersson form on $\widetilde{\mathcal{T}}_{\Sigma}$ approaches Thurston symplectic form in this limit.

As of Penner’s formulae [Pen82] for the action of the mapping class group on Dehn–Thurston coordinates, he proved this result by direct combinatorial methods. We note that our proof is different, as we obtain it from the hyperbolic case via the rescaling flow.

We also observe that Theorem 6.D “lifts” Mirzakhani’s proof of Witten’s conjecture to the geometric level of functions on Teichmüller space. More precisely, she was able to show Equation (6.0.6) for the particular case of Weil–Petersson volumes and Kontsevich volumes by analysing the rescale at the level of intersection numbers. Here, we deduce such limit (and more general ones) as a consequence of the same type of scaling, but at the level of functions (i.e. from Equation (6.0.5)).

We conclude with an open question, which would be interesting for applications discussed in Chapter 8.

QUESTION 6.E. Consider the Jacobian of the rescaling flow $R_{\beta}: \mathcal{T}_{\Sigma}^{\text{comb}}(L) \rightarrow \mathcal{T}_{\Sigma}(\beta L)$, $R_{\beta}(\mathbb{G}) = \text{sp}^{-1}(\beta\mathbb{G})$:

$$J_{\beta} = \frac{1}{\beta^{6g-6+2n}} \frac{R_{\beta}^* d\mu_{\text{WP}}}{d\mu_{\text{K}}}, \quad (6.0.7)$$

viewed as a function on $\mathcal{T}_{\Sigma}^{\text{comb}}(L) \times \mathbb{R}_+$. We know by Theorem 6.A due to Mondello that J_{β} converges pointwise to 1. Is it possible to improve such result, and get an “integrable enough” bound independent of β ?

6.0.2 — ORGANISATION OF THE CHAPTER

The chapter is organised as follows.

- In Section 6.1 we discuss the uniform convergence of hyperbolic lengths and twists to their combinatorial counterparts.
- Using the aforementioned convergence, in Section 6.2 we re-prove Penner’s formulae for the action of the mapping class group on Dehn–Thurston coordinates (in this setting, combinatorial Fenchel–Nielsen). This endows the combinatorial Teichmüller space with the structure of a piecewise linear manifold.
- In Sections 6.3 and 6.4 we show how the rescaling flow behaves naturally with respect to geometric recursion and topological recursion.

6.1 — CONVERGENCE OF LENGTHS AND TWISTS

CONVERGENCE OF LENGTHS

We first obtain an effective comparison between lengths of simple closed curves along the flow, by refining the arguments of [Doio]. Recall first that, for any bordered surface Σ , the *combinatorial systole* with respect to $\mathbb{G} \in \mathcal{T}_\Sigma^{\text{comb}}$ is defined as the shortest essential simple closed curve:

$$\text{sys}_{\mathbb{G}} = \inf_{\gamma \in \mathcal{S}_\Sigma} \ell_{\mathbb{G}}(\gamma). \quad (6.1.1)$$

Moreover, we define the ϵ -thick part of $\mathcal{T}_\Sigma^{\text{comb}}$, denoted $\mathcal{T}_\Sigma^{\text{comb}, \geq \epsilon}$, as the subset of combinatorial structures \mathbb{G} for which $\text{sys}_{\mathbb{G}} \geq \epsilon$ and $\ell_{\mathbb{G}}(\partial_i \Sigma) \geq \epsilon$ for all $i = 1, \dots, n$. Analogous definition holds for \mathcal{T}_Σ .

PROPOSITION 6.1.1. *Let Σ be a bordered surface of topology (g, n) . For any $\mathbb{G} \in \mathcal{T}_\Sigma^{\text{comb}}$, we denote $\sigma = \text{sp}^{-1}(\mathbb{G})$ and $\sigma^\beta = \text{sp}^{-1}(\beta\mathbb{G})$. Then for all $\beta \geq 1$, we have*

$$\forall \gamma \in \mathcal{S}_\Sigma \cup \partial\Sigma, \quad \ell_{\mathbb{G}}(\gamma) \leq \frac{\ell_{\sigma^\beta}(\gamma)}{\beta}. \quad (6.1.2)$$

Moreover, for any $\epsilon > 0$ there exists $\beta_\epsilon \geq 1$ and $\kappa_\epsilon > 0$ depending only on ϵ and the topological type of Σ , such that for any $\beta \geq \beta_\epsilon$ and $\mathbb{G} \in \mathcal{T}_\Sigma^{\text{comb}, \geq \epsilon}$ we have

$$\forall \gamma \in \mathcal{S}_\Sigma \cup \partial\Sigma, \quad \frac{\ell_{\sigma^\beta}(\gamma)}{\beta} \leq \left(1 + \frac{\kappa_\epsilon}{\beta}\right) \ell_{\mathbb{G}}(\gamma). \quad (6.1.3)$$

We can take κ_ϵ and β_ϵ increasing with $2g - 2 + n$.

An immediate consequence of the above bounds is the following convergence result for length of simple closed curves.

COROLLARY 6.1.2. *For any $\gamma \in \mathcal{S}_\Sigma \cup \partial\Sigma$ and $\sigma \in \mathcal{T}_\Sigma$, we have*

$$\lim_{\beta \rightarrow \infty} \frac{\ell_{\sigma^\beta}(\gamma)}{\beta} = \ell_{\text{sp}(\sigma)}(\gamma), \quad (6.1.4)$$

and the limit is uniform when $\text{sp}(\sigma)$ belongs to the thick part of $\mathcal{T}_\Sigma^{\text{comb}}$.

Proof of Proposition 6.1.1. If γ is a boundary curve, there is nothing to prove since $\ell_{\sigma^\beta}(\gamma) = \beta \ell_\sigma(\gamma)$. We now assume $\gamma \in \mathcal{S}_\Sigma$ and start from the last inequality in the proof of [Doio, Lemma 11]:

$$\forall \gamma \in \mathcal{S}_\Sigma \quad \ell_{\mathbb{G}}(\gamma) \leq \frac{\ell_{\sigma^\beta}(\gamma)}{\beta} \leq \ell_{\mathbb{G}}(\gamma) + \frac{2E_{\mathbb{G}}(\gamma) r_\beta}{\beta}, \quad (6.1.5)$$

where $E_{\mathbb{G}}(\gamma)$ is the number (with multiplicity) of edges along which the non-backtracking representative of γ in \mathbb{G} travels, and r_β is the maximal of rib lengths with respect to σ^β . This already gives the lower bound.

Recall that \mathbb{G} contains at most $6g - 6 + 3n$ edges. We say that an edge is ϵ -big if its length in \mathbb{G} is larger or equal to $\epsilon/(6g - 6 + 3n)$, and that it is ϵ -short otherwise. Denote by $E_{\mathbb{G}}^{(\epsilon)}(\gamma)$ the number (with multiplicity) of ϵ -big edges along which γ travels. If we assume that $\text{sys}_{\mathbb{G}} \geq \epsilon$, the union of the ϵ -short edges appearing in γ must be a forest. As γ is a closed loop, it has

to exit each tree it passes through via an ϵ -big edge. Hence $E_{\mathbb{G}}(\gamma) \leq (6g - 6 + 3n)E_{\mathbb{G}}^{(\epsilon)}(\gamma)$. Observing that $\frac{\epsilon}{6g-6+3n} E_{\mathbb{G}}^{(\epsilon)}(\gamma) \leq \ell_{\mathbb{G}}(\gamma)$, we obtain

$$E_{\mathbb{G}}(\gamma) \leq \frac{(6g - 6 + 3n)^2}{\epsilon} \ell_{\mathbb{G}}(\gamma).$$

We must now bound uniformly the maximum of rib lengths for σ^β . For this purpose, we bound the distance between any point of the surface to the boundary for the metric σ^β . Recall that the injectivity radius at a point $q \in \Sigma$ for the hyperbolic structure σ , here denoted $r_\sigma(q)$, is the supremum of all $\rho > 0$ such that there is a locally isometric embedding of an open hyperbolic disk of radius ρ . Since the area of (Σ, σ^β) is $2\pi(2g - 2 + n)$, which must be greater or equal to the area of such disks, we have

$$r_\sigma(q) \leq \sqrt{2(2g - 2 + n)}. \quad (6.1.6)$$

Besides, it is clear that

$$r_\sigma(q) = \min \left\{ \frac{1}{2} \text{sys}_\sigma(q), \text{dist}_\sigma(q, \partial\Sigma) \right\}, \quad (6.1.7)$$

where $\text{sys}_\sigma(q)$ is defined to be the infimum over the lengths of non-constant geodesic loops based at q . We apply this to the hyperbolic structure σ^β . Using the lower bound (6.1.5), we remark that

$$\beta\epsilon \leq \text{sys}_{\beta\mathbb{G}} \leq \text{sys}_{\sigma^\beta} \leq \text{sys}_{\sigma^\beta}(q).$$

If $\beta \geq \beta_\epsilon = \frac{2\sqrt{2(2g-2+n)}}{\epsilon}$, we deduce from (6.1.6) and (6.1.7) that

$$\forall q \in \Sigma, \quad \text{dist}_{\sigma^\beta}(q, \partial\Sigma) \leq \sqrt{2(2g - 2 + n)}.$$

In particular, choosing for q the vertices of \mathbb{G} , we find $r_\beta \leq \sqrt{2(2g - 2 + n)}$. Together with (6.1.5), it shows that

$$\forall \beta \geq \beta_\epsilon \quad \forall \gamma \in \mathcal{S}_\Sigma, \quad \ell_{\mathbb{G}}(\gamma) \leq \frac{\ell_{\sigma^\beta}(\gamma)}{\beta} \leq \left(1 + \frac{18\sqrt{2}(2g - 2 + n)^{5/2}}{\beta\epsilon} \right) \ell_{\mathbb{G}}(\gamma),$$

which gives the thesis. \square

CONVERGENCE OF TWISTS

Studying the convergence of twists along the flow by a direct geometric method requires bounds on the distance between the hyperbolic and combinatorial seams. Instead of following this direction, we use the hyperbolic $(9g - 9 + 3n)$ -theorem to write the hyperbolic twists in terms of hyperbolic lengths of certain curves, than show that these formulae converge with help of Proposition 6.1.1, and compare the limit to the expressions for the combinatorial twists underlying the combinatorial $(9g - 9 + 3n)$ -theorem established in Section 3.5.

PROPOSITION 6.1.3. *Let Σ be a bordered surface of type (g, n) , fix a seamed pants decomposition and let $(\tau_i)_{i=1}^{3g-3+n}$ be the associated combinatorial twist parameters. For any compact $K \subset \mathcal{T}_\Sigma^{\text{comb}}$ and $\mathbb{G} \in K$, we denote $\sigma = \text{sp}^{-1}(\mathbb{G})$ and $\sigma^\beta = \text{sp}^{-1}(\beta\mathbb{G})$. There exists constants $\beta_K \geq 1$ and $c_K > 0$ depending only on K such that, for any $\beta \geq \beta_K$ and $i \in \{1, \dots, 3g - 3 + n\}$, we have*

$$\left| \frac{\tau_i(\sigma^\beta)}{\beta} - \tau_i(\mathbb{G}) \right| \leq \frac{c_K}{\beta}. \quad (6.1.8)$$

II. The combinatorial model of the moduli space of curves

An immediate consequence of the above bound is the convergence result for twist parameters.

COROLLARY 6.1.4. *In any fixed seamed pants decomposition, for $\sigma \in \mathcal{T}_\Sigma$, we have*

$$\lim_{\beta \rightarrow \infty} \frac{\tau_i(\sigma^\beta)}{\beta} = \tau_i(\text{sp}(\sigma)), \quad (6.1.9)$$

and the limit is uniform when σ belongs to an arbitrary compact of \mathcal{T}_Σ .

Before starting the proof of Proposition 6.1.3, we recall the formulae which allows us for four-holed spheres and one-holed tori to express the change of Fenchel–Nielsen coordinates under a flip of the pair of pants decomposition. They can be found in [Oka93, Theorems 1.i and 2.i], or deduced from [Busio, Sections 3.3 and 3.4].

Let X be a four-holed sphere and $\sigma \in \mathcal{T}_X(L_1, L_2, L_3, L_4)$. We place ourselves in the situation described in Section 3.5. Namely, we fix a coordinate system, which in turn defines a simple closed curve γ separating X into a pair of pants having boundary components $(\partial_1 X, \partial_4 X, \gamma)$ and another pair of pants having boundary components $(\partial_2 X, \partial_3 X, \gamma)$; we have δ a simple closed curve intersecting γ exactly twice and separating X into a pair of pants having boundary components $(\partial_1 X, \partial_2 X, \delta)$ and another pair of pants having boundary components $(\partial_3 X, \partial_4 X, \delta)$; finally, let η be the curve obtained from γ by applying a Dehn twist along δ . Let $\sigma \in \mathcal{T}_X(L_1, L_2, L_3, L_4)$ and denote $\ell = \ell_\sigma(\gamma)$, $\ell' = \ell_\sigma(\delta)$ and $\ell'' = \ell_\sigma(\eta)$, and define τ to be the hyperbolic twist determined by the seamed pants decomposition. If we define

$$C_{i,j}(\ell) = \cosh^2\left(\frac{\ell}{2}\right) + \cosh^2\left(\frac{L_i}{2}\right) + \cosh^2\left(\frac{L_j}{2}\right) + 2 \cosh\left(\frac{L_i}{2}\right) \cosh\left(\frac{L_j}{2}\right) \cosh\left(\frac{\ell}{2}\right) - 1, \quad (6.1.10)$$

the length of δ is then given in terms of Fenchel–Nielsen coordinates by

$$\begin{aligned} \cosh\left(\frac{\ell'(\ell, \tau)}{2}\right) \sinh^2\left(\frac{\ell}{2}\right) &= \cosh\left(\frac{L_1}{2}\right) \cosh\left(\frac{L_2}{2}\right) + \cosh\left(\frac{L_3}{2}\right) \cosh\left(\frac{L_4}{2}\right) \\ &+ \cosh\left(\frac{\ell}{2}\right) \left(\cosh\left(\frac{L_1}{2}\right) \cosh\left(\frac{L_3}{2}\right) + \cosh\left(\frac{L_2}{2}\right) \cosh\left(\frac{L_4}{2}\right) \right) \\ &+ \cosh(\tau) \sqrt{C_{1,4}(\ell) C_{2,3}(\ell)}, \end{aligned} \quad (6.1.11)$$

while the length of η is $\ell''(\ell, \tau) = \ell'(\ell, \tau + \ell)$.

Likewise, if T is a one-holed torus, $\sigma \in \mathcal{T}_T(L)$, and we are in the situation described in Section 3.5, the length of δ is given by

$$\cosh\left(\frac{\ell'(\ell, \tau)}{2}\right) = \frac{\cosh\left(\frac{\tau}{2}\right)}{\sinh\left(\frac{\ell}{2}\right)} \sqrt{\frac{\cosh\left(\frac{L}{2}\right) + \cosh(\ell)}{2}}, \quad (6.1.12)$$

while the length of η is $\ell''(\ell, \tau) = \ell'(\ell, \tau + \ell)$.

Proof of Proposition 6.1.3. Fix σ in a compact K of \mathcal{T}_Σ , and denote $\mathbb{G} = \text{sp}(\sigma)$. We use repeatedly Proposition 6.1.1, which implies that for any simple closed curve ν chosen in a fixed finite subset of $\mathcal{S}_\Sigma \cup \partial\Sigma$, we have

$$\ell_{\mathbb{G}}(\nu) \leq \frac{\ell_{\sigma^\beta}(\nu)}{\beta} \leq \ell_{\mathbb{G}}(\nu) + \frac{c_K}{\beta}$$

for any $\beta \geq \beta_K$ and some constant $c_K > 0$ depending only on the compact K and this finite set. In what follows, c_K, c'_K, \dots denote positive constants depending on K and whose value

may change from line to line. We denote lengths and twists with a superscript β to refer to the hyperbolic quantities measured with respect to σ^β , while lengths and twists without superscripts denote the combinatorial quantities measured with respect to \mathbb{G} .

As the twist parameters τ_i^β and τ_i are computed locally in each piece Σ_i of type $(0, 4)$ or $(1, 1)$ defined by the seamed pants decomposition as in Section 3.5, we can restrict our attention to each piece separately. On Σ_i we have the curve δ_i defined by the seamed pants decomposition, and for every $k \in \mathbb{Z}$ we consider the curve obtained as the image of δ_i after k Dehn twists along γ_i . Denote by $\ell_i^{(k),\beta}$ its hyperbolic length with respect to σ^β , and by $\ell_i^{(k)}$ its combinatorial length with respect to \mathbb{G} . Compared to the notation of Section 3.5,

$$\ell_i'^\beta = \ell_i^{(0),\beta}, \quad \ell_i''^\beta = \ell_i^{(1),\beta}, \quad \ell_i' = \ell_i^{(0)}, \quad \ell_i'' = \ell_i^{(1)}.$$

As we work with a single piece at a time, we in fact often omit the subscript i . Suppose $\Sigma_i = X$ is a four-holed sphere. With the labelling matching the one described in Section 3.5, we denote $L_i^\beta = \ell_{\sigma^\beta}(\partial_i X)$ for $i \in \{1, 2, 3, 4\}$. According to Lemma 3.5.2, we have

$$\ell^{(k)} = \max \{L_1 + L_3 - \ell, L_2 + L_4 - \ell, 2|\tau + k\ell| + M_{1,4}(\ell) + M_{2,3}(\ell)\} \quad (6.1.13)$$

where $M_{i,j}(\ell) = \max \{0, L_i - \ell, L_j - \ell, \frac{L_i + L_j - \ell}{2}\}$. The open sets

$$U^{(k)} = \left\{ \mathbb{G} \in \mathcal{T}_\Sigma^{\text{comb}} \mid 2\ell < \tau + k\ell < 4\ell \right\}, \quad k \in \mathbb{Z}.$$

cover the compact K , so we can select finitely many indices $k_1, \dots, k_N \in \mathbb{Z}$ to cover K . If $\mathbb{G} \in U^{(k)} \cap K$ for some $k \in \{k_1, \dots, k_N\}$, the maximum in (6.1.13) is given by the last argument, that is

$$\ell^{(k)} = 2(\tau + k\ell) + M_{1,4}(\ell) + M_{2,3}(\ell)$$

which we see as an expression of $\tau + k\ell$ in terms of other lengths. We would like to compare it, when β is large, to the expression of $\tau^\beta + k\ell^\beta$ which we can access via Equation (6.1.11), namely

$$\begin{aligned} \cosh(\tau^\beta + k\ell^\beta) &= \left[\cosh\left(\frac{\ell^{(k),\beta}}{2}\right) \sinh^2\left(\frac{\ell^\beta}{2}\right) - \cosh\left(\frac{L_1^\beta}{2}\right) \cosh\left(\frac{L_2^\beta}{2}\right) - \cosh\left(\frac{L_3^\beta}{2}\right) \cosh\left(\frac{L_4^\beta}{2}\right) \right. \\ &\quad \left. - \cosh\left(\frac{\ell^\beta}{2}\right) \left(\cosh\left(\frac{L_1^\beta}{2}\right) \cosh\left(\frac{L_3^\beta}{2}\right) + \cosh\left(\frac{L_2^\beta}{2}\right) \cosh\left(\frac{L_4^\beta}{2}\right) \right) \right] C_{1,4}^{-1/2}(\ell^\beta) C_{2,3}^{-1/2}(\ell^\beta). \end{aligned} \quad (6.1.14)$$

To obtain an upper bound for the right-hand side of (6.1.14), we can ignore the negative terms and use as upper bound for the numerator

$$\cosh\left(\frac{\ell^{(k),\beta}}{2}\right) \sinh^2\left(\frac{\ell^\beta}{2}\right) \leq \frac{1}{4} e^{\frac{\ell^{(k),\beta}}{2} + \ell^\beta},$$

and as lower bound for the factors in the denominator

$$C_{i,j}(\ell^\beta) \geq \frac{1}{4} e^{\max \left\{ \ell^\beta, L_i^\beta, L_j^\beta, \frac{L_i^\beta + L_j^\beta + \ell^\beta}{2} \right\}} \geq \frac{1}{4} e^{\max \left\{ L_i^\beta, L_j^\beta, \frac{L_i^\beta + L_j^\beta + \ell^\beta}{2} \right\}}.$$

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Combined with $\operatorname{arcosh}(x) \leq \log(2x)$ for $x \geq 1$, this results in

$$\begin{aligned} \frac{|\tau^\beta + k\ell^\beta|}{\beta} &\leq \frac{1}{\beta} \left(\log 2 + \frac{\ell^{(k),\beta}}{2} + \ell^\beta - \frac{1}{2} \max \{L_1^\beta, L_4^\beta, \frac{L_1^\beta + L_4^\beta + \ell^\beta}{2}\} - \frac{1}{2} \max \{L_2^\beta, L_3^\beta, \frac{L_2^\beta + L_3^\beta + \ell^\beta}{2}\} \right) \\ &\leq \frac{\ell^{(k)}}{2} + \ell - \frac{1}{2} \max \{L_1, L_4, \frac{L_1 + L_4 + \ell}{2}\} - \frac{1}{2} \max \{L_2, L_3, \frac{L_2 + L_3 + \ell}{2}\} + \frac{c'_K}{\beta} \\ &\leq \frac{\ell^{(k)}}{2} - \frac{1}{2} (M_{1,4}(\ell) + M_{2,3}(\ell)) + \frac{c'_K}{\beta} \end{aligned}$$

and thus

$$\frac{|\tau^\beta + k\ell^\beta|}{\beta} \leq \tau + k\ell + \frac{c'_K}{\beta}.$$

We now look for a bound from below for (6.1.14). We first observe that by Equation (6.1.2), we have $\ell^\beta \geq \ell$ for $\beta \geq 1$. Since on the compact K , ℓ is bounded from below by $\epsilon_K > 0$, we deduce that

$$\sinh^2\left(\frac{\ell^\beta}{2}\right) \geq m_K e^{\ell^\beta} \quad \text{with} \quad m_K = \frac{(1 - e^{-\epsilon_K})^2}{4} > 0.$$

This leads to a (rather crude) lower bound for the numerator of the right-hand side of (6.1.14)

$$\begin{aligned} &\cosh\left(\frac{\ell^{(k),\beta}}{2}\right) \sinh^2\left(\frac{\ell^\beta}{2}\right) - \cosh\left(\frac{L_1^\beta}{2}\right) \cosh\left(\frac{L_2^\beta}{2}\right) - \cosh\left(\frac{L_3^\beta}{2}\right) \cosh\left(\frac{L_4^\beta}{2}\right) \\ &\quad - \cosh\left(\frac{\ell^\beta}{2}\right) \left(\cosh\left(\frac{L_1^\beta}{2}\right) \cosh\left(\frac{L_2^\beta}{2}\right) + \cosh\left(\frac{L_2^\beta}{2}\right) \cosh\left(\frac{L_4^\beta}{2}\right) \right) \\ &\geq \frac{m_K}{2} e^{\frac{\ell^{(k),\beta}}{2} + \ell^\beta} - e^{\frac{L_1^\beta + L_2^\beta}{2}} - e^{\frac{L_3^\beta + L_4^\beta}{2}} - e^{\frac{\ell^\beta + L_1^\beta + L_3^\beta}{2}} - e^{\frac{\ell^\beta + L_2^\beta + L_4^\beta}{2}} \\ &\geq \frac{m_K}{4} e^{\frac{\ell^{(k),\beta}}{2} + \ell^\beta} + \left(\frac{m_K}{4} e^{\frac{\ell^{(k),\beta}}{2} + \ell^\beta} - 4 e^{\frac{1}{2} \max \{L_1^\beta + L_2^\beta, L_3^\beta + L_4^\beta, L_1^\beta + L_3^\beta + \ell^\beta, L_2^\beta + L_4^\beta + \ell^\beta\}} \right). \end{aligned} \tag{6.1.15}$$

We split the first term in order to exhibit positivity of our lower bound – which is therefore not useless. Indeed, using Proposition 6.1.1 and Equation (6.1.13) on $U^{(k)} \cap K$ we get

$$\frac{\ell^{(k),\beta}}{2} + \ell^\beta \geq \max \{L_1^\beta + L_2^\beta, L_3^\beta + L_4^\beta, L_1^\beta + L_3^\beta + \ell^\beta, L_2^\beta + L_4^\beta + \ell^\beta\} + \beta c''_K$$

for some constant $c''_K > 0$. Consequently, there exists $\beta_K \geq 1$ such that for any $\beta \geq \beta_K$ the expression inside the bracket in (6.1.15) is positive, and can be ignored in the lower bound. To obtain an upper bound for the denominator of (6.1.14), we write

$$C_{i,j}(\ell^\beta) \leq 5 e^{\max \{\ell^\beta, L_i^\beta, L_j^\beta, \frac{L_i^\beta + L_j^\beta + \ell^\beta}{2}\}}.$$

Combined with $\operatorname{arcosh}(x) \geq \log x$, this implies for $\beta \geq \beta_K$

$$\begin{aligned} \frac{|\tau^\beta + k\ell^\beta|}{\beta} &\geq \frac{1}{\beta} \left(\log\left(\frac{m_K}{20}\right) + \frac{1}{2} \ell^{(k),\beta} - \frac{1}{2} \max \{\ell^\beta, L_1^\beta, L_4^\beta, \frac{L_1^\beta + L_4^\beta + \ell^\beta}{2}\} \right. \\ &\quad \left. - \frac{1}{2} \max \{\ell^\beta, L_2^\beta, L_3^\beta, \frac{L_2^\beta + L_3^\beta + \ell^\beta}{2}\} \right) \\ &\geq \tau + k\ell - \frac{c'''_K}{\beta} \end{aligned}$$

using the arguments we already used for the upper bound. We deduce that on $U^{(k)} \cap K$ and for $\beta \geq \beta_K$

$$\left| \frac{|\tau^\beta + k\ell^\beta|}{\beta} - (\tau + k\ell) \right| \leq \frac{c_K}{\beta}. \quad (6.1.16)$$

A similar argument shows that on $U^{(k)} \cap K$ and for $\beta \geq \beta_K$, we have

$$\left| \frac{|\tau^\beta + (k+1)\ell^\beta|}{\beta} - (\tau + (k+1)\ell) \right| \leq \frac{c_K}{\beta}, \quad (6.1.17)$$

for perhaps larger constants $\beta_K, c_K > 0$. We can now conclude by using (6.1.16)–(6.1.17) to estimate

$$\frac{\tau^\beta}{\beta} = \frac{\beta}{2\ell^\beta} \left(\left| \frac{\tau^\beta + (k+1)\ell^\beta}{\beta} \right|^2 - \left| \frac{\tau^\beta + k\ell^\beta}{\beta} \right|^2 \right) - \frac{(2k+1)\ell^\beta}{2\beta}.$$

Since $\ell^\beta \geq \epsilon_K > 0$ on the compact K , we arrive on $U^{(k)} \cap K$ and for $\beta \geq \beta_K$ at the inequality

$$\left| \frac{\tau^\beta}{\beta} - \tau \right| \leq \frac{c_K}{\beta}, \quad (6.1.18)$$

for perhaps larger constants $\beta_K, c_K > 0$. These arguments were done for a fixed $k \in \{k_1, \dots, k_N\}$, and as $(U^{(k_i)})_{i=1}^N$ cover K , we can find constants β_K and c_K such that the same estimate (6.1.18) holds uniformly over K for $\beta \geq \beta_K$.

A similar argument can be carried out for Σ_i being a one-holed torus, in the situation described in Section 3.5. Instead of (6.1.14) we should estimate $\tau^\beta + k\ell^\beta$ via the formula

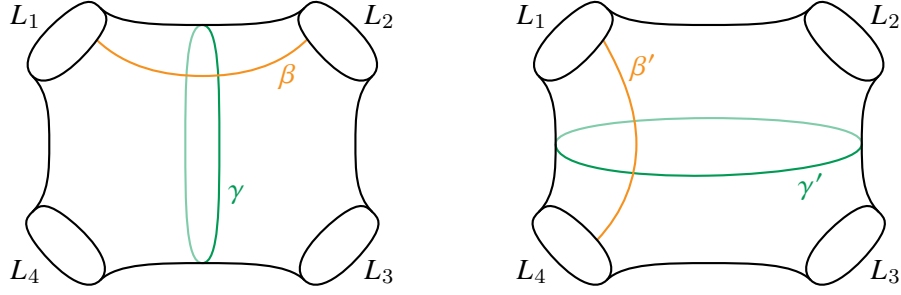
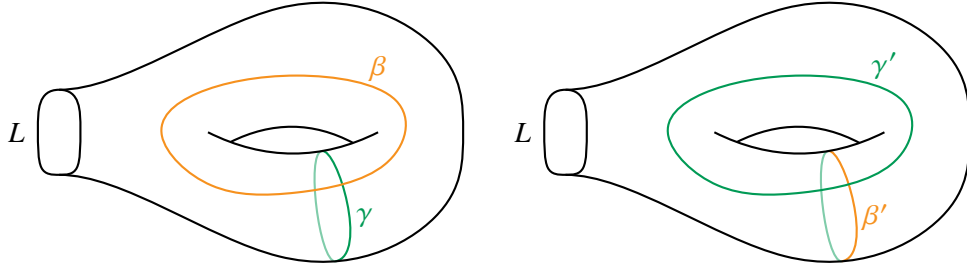
$$\cosh\left(\frac{\tau^\beta + k\ell^\beta}{2}\right) = \cosh\left(\frac{\ell^{(k),\beta}}{2}\right) \sinh\left(\frac{\ell^\beta}{2}\right) \sqrt{\frac{2}{\cosh\left(\frac{\ell^\beta}{2}\right) + \cosh(\ell^\beta)}}$$

coming from (6.1.12), and compare it to the expression of $\tau + k\ell$ deduced from Lemma 3.5.4. As the estimates in this case are much easier and without surprise, we omit them.

We conclude that the desired estimate (6.1.8) is valid for any fixed $i \in \{1, \dots, 3g - 3 + n\}$, and since this set is finite we can choose constants β_K and c_K independently of i : in the end, they only depend on the fixed seamed pants decomposition and on the compact K . \square

6.2 — PENNER'S FORMULAE

The combinatorial Fenchel–Nielsen coordinates on $\mathcal{T}_\Sigma^{\text{comb}}$ described in Theorem 3.4.5 depend on a seamed pants decomposition $(\mathcal{P}, \mathcal{S})$. Notice that changing \mathcal{S} amounts to changing $\tau_i(\mathbb{G}) \mapsto \tau_i(\mathbb{G}) + k_i \ell_i(\mathbb{G})$ by some $k_i \in \mathbb{Z}$, while $\ell_i(\mathbb{G})$ remains unchanged. The most interesting case occurs when we change the pair of pants decomposition \mathcal{P} . From [HT80], these changes are generated by local changes in four-holed spheres and one-holed tori and were firstly described by Penner in [Pen82; Pen84] for the case of multicurves. A generalisation of such action to measured laminations can be found in [PH92]. Notice that the same formulae apply to $\mathcal{T}_\Sigma^{\text{comb}}$, as the different behaviour of the foliations at the boundary do not affect the coordinates themselves (cf. Section 3.1.2).


 Figure 6.1: Change in the coordinate system of X .

 Figure 6.2: Change in the coordinate system of T .

In this section we give a new proof of Penner's result, by flowing their hyperbolic analogue found by Okai¹ [Ok93]. We now report on these hyperbolic formulae.

FOUR-HOLED SPHERE. For a four-holed sphere X , consider the hyperbolic Fenchel–Nielsen coordinates $(\ell, \tau) \in \mathcal{T}_X(L_1, L_2, L_3, L_4)$ relative to the system of curves $(\mathcal{P}, \mathcal{S})$ of Figure 6.1. Then the change of seamed pants decomposition to $(\mathcal{P}', \mathcal{S}')$ is given by Equation (6.1.11) and

$$\begin{aligned} \cosh(\tau'(\ell, \tau)) = & \left[\sinh^2\left(\frac{\ell'(\ell, \tau)}{2}\right) \cosh\left(\frac{\ell}{2}\right) - \cosh\left(\frac{L_1}{2}\right) \cosh\left(\frac{L_4}{2}\right) - \cosh\left(\frac{L_2}{2}\right) \cosh\left(\frac{L_3}{2}\right) \right. \\ & \left. - \cosh\left(\frac{\ell'(\ell, \tau)}{2}\right) \left(\cosh\left(\frac{L_1}{2}\right) \cosh\left(\frac{L_3}{2}\right) + \cosh\left(\frac{L_2}{2}\right) \cosh\left(\frac{L_4}{2}\right) \right) \right] C_{1,2}(\ell)^{-1/2} C_{3,4}(\ell)^{-1/2} \end{aligned} \quad (6.2.1)$$

with $\text{sgn}(\tau') = -\text{sgn}(\tau)$ and $C_{i,j}(\ell)$ has been defined in (6.1.10).

ONE-HOLED TORUS. For a one-holed torus T , consider the global coordinates $(\ell, \tau) \in \mathcal{T}_T(L)$ relative to $(\mathcal{P}, \mathcal{S})$ of Figure 6.2. Then the change of coordinate system to $(\mathcal{P}', \mathcal{S}')$ is given by Equation (6.1.12) and

$$\cosh\left(\frac{\tau'(\ell, \tau)}{2}\right) = \cosh\left(\frac{\ell}{2}\right) \sqrt{\frac{\cosh^2\left(\frac{\tau}{2}\right) (\cosh\left(\frac{L}{2}\right) + \cosh(\ell)) - 2 \sinh^2\left(\frac{\ell}{2}\right)}{\cosh^2\left(\frac{\tau}{2}\right) (\cosh\left(\frac{L}{2}\right) + \cosh(\ell)) + \sinh^2\left(\frac{\ell}{2}\right) (\cosh\left(\frac{L}{2}\right) - 1)}} \quad (6.2.2)$$

with $\text{sgn}(\tau') = -\text{sgn}(\tau)$.

Using the convergence under the rescaling flow of hyperbolic length and twist parameters to the combinatorial analogues, we can give a new proof of Penner's formulae.

¹In [Ok93, Theorem 2.ii] there is a misprint in the denominator of the formula giving τ' for the one-holed torus. More precisely, the first $\cosh\left(\frac{L}{2}\right)$ should not be squared, and the second $\cosh\left(\frac{L}{2}\right)$ should be replaced with $\cosh\left(\frac{L}{2}\right) - 1$. We report the correct formula (6.2.2) here.

PROPOSITION 6.2.1 (Penner's formulae).

SPHERE WITH FOUR BOUNDARY COMPONENTS. For a four-holed sphere X , consider the global coordinates $(\ell, \tau) \in \mathcal{T}_X^{\text{comb}}(L_1, L_2, L_3, L_4)$ relative to the system of curves $(\mathcal{P}, \mathcal{S})$ of Figure 6.1. Then the change of coordinate system to $(\mathcal{P}', \mathcal{S}')$ is given by Equation (3.5.2) and

$$|\tau'(\ell, \tau)| = \frac{1}{2} |2|\tau| + \ell + M_{1,4}(\ell) + M_{2,3}(\ell) - \ell'(\ell, \tau) - M_{1,2}(\ell') - M_{3,4}(\ell')| \quad (6.2.3)$$

with $\text{sgn}(\tau') = -\text{sgn}(\tau)$. We recall here that $M_{i,j}(\ell) = \max \{0, L_i - \ell, L_j - \ell, \frac{L_i + L_j - \ell}{2}\}$.

TORUS WITH ONE BOUNDARY COMPONENT. For a one-holed torus T , consider the global coordinates $(\ell, \tau) \in \mathcal{T}_T^{\text{comb}}(L)$ relative to the system of curves $(\mathcal{P}, \mathcal{S})$ of Figure 6.2. Then the change of coordinate system to $(\mathcal{P}', \mathcal{S}')$ is given by Equation (3.5.4) and

$$|\tau'(\ell, \tau)| = \left| \ell - \left[\frac{L}{2} - \ell'(\ell, \tau) \right]_+ \right| \quad (6.2.4)$$

with $\text{sgn}(\tau') = -\text{sgn}(\tau)$.

Proof. The formulae follow from Equations (6.2.1)–(6.2.2) by direct computation, together with the convergence of length and twist parameters and using relations of the form

$$\forall a, b \in \mathbb{R}_+, \quad \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \log \left(e^{\beta a} + e^{\beta b} \right) = \max \{ a, b \}.$$

The reader can check that the limit of Equations (6.1.11) and (6.1.12) coincide with Equations (3.5.2) and (3.5.4). \square

Remark 6.2.2. As stated in the introduction, Penner's original formulae for the action of the mapping class group on Dehn–Thurston coordinates are expressed in a different (but equivalent) way. We present here a comparison for the one-holed torus. In this case, Penner's formulae reads (cf. [Pen82, Theorem 6.1])

$$\begin{cases} \ell'_{11} = [r - |t_1|]_+ \\ r' = r + \ell_{11} - \ell'_{11} \\ \ell'_{23} = |t_1| - r + \ell'_{11} \end{cases} \quad \begin{cases} |t'_1| = |\ell_{23} + r - \ell'_{11}| \\ \text{sgn}(t'_1) = -\text{sgn}(t_1) \end{cases} \quad (6.2.5)$$

and the translation from his notation to ours (cf. [Pen82, Figure 6.4]) is

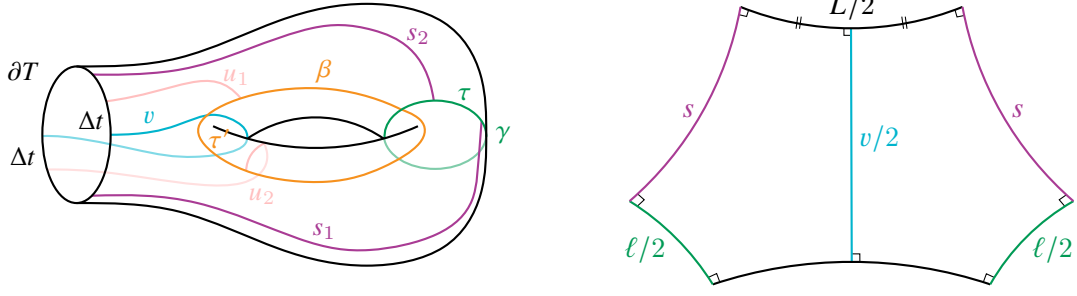
$$\begin{cases} L = 2(\ell_{11} + r) \\ \ell = r + \ell_{23} \\ \ell_{11} = \left[\frac{L - 2\ell}{2} \right]_+ \\ \tau = t_1 \end{cases} \quad (6.2.6)$$

and similarly for the primed quantities. We can now check that Penner's original formulae are equivalent to ours. First of all, we should find that the length of the boundary is fixed, i.e. $L' = L$. Indeed:

$$L' = 2(r' + \ell'_{11}) = 2(r + \ell_{11} - \ell'_{11} + \ell'_{11}) = 2(r + \ell_{11}) = L. \quad (6.2.7)$$

Let us check now that the change in the length coordinate agrees with Equation (3.5.4):

$$\ell' = r' + \ell'_{23} = r + \ell_{11} - \ell'_{11} + |t_1| - r + \ell'_{11} = |t_1| + \ell_{11} = |\tau| + \left[\frac{L - 2\ell}{2} \right]_+. \quad (6.2.8)$$


 Figure 6.3: Twist change at the boundary for T .

Similarly, the change in the twist parameter agrees with Equation (6.2.4): indeed, the signs agree and the modulus is given by

$$|\tau'| = |r + \ell_{23} - \ell'_{11}| = \left| \ell - \left[\frac{L - 2\ell}{2} \right]_+ \right|. \quad (6.2.9)$$

Remark 6.2.3. To give a complete description of the action of the mapping class group on combinatorial Fenchel–Nielsen coordinates, one should consider the effect of the change of the coordinate systems described in Figures 6.1 and 6.2 to the twisting numbers at the boundary components (cf. Equation (3.4.7)). Although Okai does not consider this, one can easily compute the change on the twisting numbers at the boundary components using hyperbolic trigonometry, and flow such formulae to obtain the analogous results in the combinatorial setting.

We present here the argument for a one-holed torus T , and remark that the same reasoning applies verbatim to a four-holed sphere. In order to talk about twisting number at the boundary, we fix an isotopy class of a hyperbolic metric on T , as well as geodesics representatives of curve as in Figure 6.3. In particular, we have well-defined Fenchel–Nielsen coordinates (ℓ, τ) and (ℓ', τ') relative to $(\mathcal{P}, \mathcal{S})$ and $(\mathcal{P}', \mathcal{S}')$ respectively, as in Proposition 6.2.1, and twisting numbers at the boundary t and t' .

In order to find the change in the twisting number at the boundary, we need to compute the distance between the seam s_1 and the seam u_1 , which is $t' - t = \frac{L}{4} + \Delta t$. In particular Δt is the signed distance between v and u_1 . First, notice that $\text{sgn}(\Delta t) = -\text{sgn}(\tau')$. On the other hand, from the orthogonal non-convex hyperbolic hexagon $\Delta t \rightarrow u_1 \rightarrow \tau' \rightarrow u_2 \rightarrow \Delta t \rightarrow v$, we get

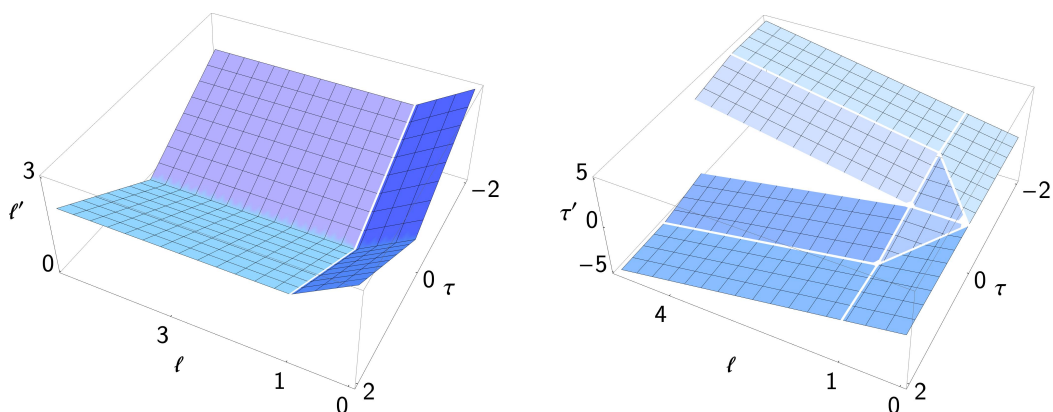
$$\cosh(\tau') = \sinh^2(\Delta t) \cosh(v) + \cosh^2(\Delta t). \quad (6.2.10)$$

The orthogonal pentagons on the right-hand side of Figure 6.3 give $\sinh(\frac{v}{2}) \sinh(\frac{L}{4}) = \cosh(\frac{\ell}{2})$, so that

$$\cosh(\tau') = 2 \sinh^2(\Delta t) \left(\frac{\cosh^2(\frac{\ell}{2})}{\sinh^2(\frac{L}{4})} + 1 \right) + 1, \quad \text{sgn}(\Delta t) = -\text{sgn}(\tau'). \quad (6.2.11)$$

Using the rescaling flow, we obtain the following formula for the combinatorial quantities:

$$|\Delta t| = \frac{|\tau'|}{2} - \frac{1}{2} \left[\ell - \frac{L}{2} \right]_+, \quad \text{sgn}(\Delta t) = -\text{sgn}(\tau'). \quad (6.2.12)$$


 Figure 6.4: The graphs of $\ell'(\ell, \tau)$ and $\tau'(\ell, \tau)$, with $L = 2$.

As in the case of measure foliations, completion of the set of multicurves, we can conclude a piecewise linear structure on the combinatorial Teichmüller space.

COROLLARY 6.2.4. *Let Σ be a bordered surface of type (g, n) with two seamed pants decompositions $(\mathcal{P}, \mathcal{S})$ and $(\mathcal{P}', \mathcal{S}')$, defining combinatorial Fenchel–Nielsen coordinates*

$$\Phi, \Phi': \mathcal{T}_\Sigma^{\text{comb}} \longrightarrow (\mathbb{R}_+ \times \mathbb{R})^{3g-3+n}. \quad (6.2.13)$$

Then the change of coordinate $\Phi' \circ \Phi^{-1}$ between open subsets of $(\mathbb{R}_+ \times \mathbb{R})^{3g-3+n}$ is piecewise linear. In particular, the combinatorial Teichmüller space $\mathcal{T}_\Sigma^{\text{comb}}$ is endowed with a canonical piecewise linear structure.

Remark 6.2.5. The above transformations are not continuous on the whole of $\mathbb{R}_+ \times \mathbb{R}$. The locus of discontinuity actually identifies a subset of the non-admissible twists, i.e. the creation of saddle connections in the measured foliation perspective (cf. Question 3.B). For instance, the plot of ℓ' and τ' on $\mathbb{R}_+ \times \mathbb{R} \supset \mathcal{T}_T^{\text{comb}}(L)$ for a one-holed torus is illustrated in Figure 6.4. Notice that along the line

$$I = \left[\frac{L}{2}, +\infty\right) \times \{0\},$$

the function ℓ' is identically zero, while τ' has a discontinuity. But I is not in the image of $\mathcal{T}_T^{\text{comb}}(L)$ under the map Φ of Theorem 3.4.5. Thus, having $\ell' = 0$ and τ' discontinuous along I is not contradictory.

6.3 — GEOMETRIC RECURSION IN THE FLOW

Geometric recursion in the hyperbolic and combinatorial settings produces functions respectively on the moduli spaces $\mathcal{M}_{g,n}(L)$ and $\mathcal{M}_{g,n}^{\text{comb}}(L)$. The main result of this section is that the flow in the $\beta \rightarrow \infty$ limit takes the hyperbolic geometric recursion to the combinatorial geometric recursion.

6.3.1 — RESCALING INITIAL DATA

Before considering the behaviour of geometric recursion amplitudes in the flow, we discuss the rescaling of initial data.

II. The combinatorial model of the moduli space of curves

DEFINITION 6.3.1. Let $(A_\beta, B_\beta, C_\beta)_{\beta \geq 1}$ be a family of triples of measurable functions on \mathbb{R}_+^3 . We say it is *uniformly (strongly) admissible* if the constants in the (strong) admissibility can be chosen to be independent of $\beta \geq 1$.

We remark that here, the triple (A, B, C) is completed by a natural choice of D , according to Section 5.2.3.

We introduce now the rescaling operator, acting on a function $\phi: \mathbb{R}_+^k \rightarrow \mathbb{R}$ by

$$\rho_{\beta*}\phi(L) = \phi(\beta L). \quad (6.3.1)$$

We notice two basic properties of this rescaling. Firstly, limits of uniformly admissible rescaled initial data (if they exist) are automatically admissible.

LEMMA 6.3.2. *Let $(A_\beta, B_\beta, C_\beta)_{\beta \geq 1}$ be initial data such that $\rho_{\beta*}(A_\beta, B_\beta, C_\beta)$ is uniformly admissible and converges to a pointwise limit $(\hat{A}, \hat{B}, \hat{C})$. Then $(\hat{A}, \hat{B}, \hat{C})$ is admissible.*

Proof. This is clear by taking the $\beta \rightarrow \infty$ limit in the inequalities specified by the uniform admissibility. \square

Secondly, if we rescale initial data that do not depend on β , the limit (if it exists) must be supported on small pairs of pants.

LEMMA 6.3.3. *If (A, B, C) is admissible, then $\rho_{\beta*}(A, B, C)$ is uniformly admissible, with same bounding constants. Besides, if $\rho_{\beta*}(B, C)$ has a pointwise limit (\hat{B}, \hat{C}) when $\beta \rightarrow \infty$, then*

$$\begin{cases} \ell > L + L' & \implies & \hat{B}(L, L', \ell) = 0, \\ \ell + \ell' > L & \implies & \hat{C}(L, \ell, \ell') = 0. \end{cases} \quad (6.3.2)$$

Proof. The bound on $\rho_{\beta*}A$ is clear. For the bound on B , we write

$$\begin{aligned} & \sup_{L, L', \ell \geq \epsilon} |B(\beta L, \beta L', \beta \ell)| (1 + [\ell - L - L']_+)^s \leq \\ & \leq \sup_{L, L', \ell \geq \beta \epsilon} |B(L, L', \ell)| (1 + [\ell - L - L']_+)^s \left(\frac{1 + [(\ell - L - L')/\beta]_+}{1 + [\ell - L - L']_+} \right)^s. \end{aligned}$$

Since $\beta \geq 1$, we can bound the last expression by the supremum over $L, L', \ell \geq \epsilon$. We also observe that $t \mapsto \frac{1+[t/\beta]_+}{1+[t]_+}$ is equal to 1 for $t \leq 0$ and is decreasing for $t > 0$, hence it is uniformly bounded by 1. Using the initial bound for B we get the desired bound for $\rho_{\beta*}B$. The bound for $\rho_{\beta*}C$ is proved similarly.

To establish the vanishing property for \hat{B} , we fix (L, L', ℓ) such that $\ell - (L + L') \geq \epsilon$ and $\min\{L, L', \ell\} \geq \epsilon > 0$ for some $\epsilon > 0$. Specialising the admissibility condition for B at $s = 1$ and rescaled lengths, we have

$$|B(\beta L, \beta L', \beta \ell)| \leq \frac{M_{\epsilon,1}}{1 + \beta \epsilon}.$$

Taking the limit $\beta \rightarrow \infty$ yields the claim. The vanishing property of \hat{C} is proved similarly. \square

Remark 6.3.4. If (A, B, C) are continuous functions on \mathbb{R}_+^3 , then $(\rho_{\beta*}A, \rho_{\beta*}B, \rho_{\beta*}C)_{\beta \geq 1}$ forms an equicontinuous family. By Arzelà–Ascoli theorem, for any fixed compact, it must admit uniformly converging subsequences, and the vanishing properties in Lemma 6.3.3 must hold for any limit point. Therefore, the assumption that $\rho_{\beta*}(A, B, C)$ converges is rather weak.

6.3.2 — RESCALING GEOMETRIC RECURSION AMPLITUDES

THEOREM 6.3.5. *Let $(A_\beta, B_\beta, C_\beta)_{\beta \geq 1}$ be initial data such that $\rho_{\beta^*}(A_\beta, B_\beta, C_\beta)$ is uniformly admissible and converges uniformly on any compact to a limit $(\hat{A}, \hat{B}, \hat{C})$. Let us denote by $\Omega_{\Sigma; \beta}$ the result of the $\text{Mes}(\mathcal{T}_\Sigma)$ -valued GR with initial data $(A_\beta, B_\beta, C_\beta)$, and by $\widehat{\Xi}_\Sigma$ the result of the $\text{Mes}(\mathcal{T}_\Sigma^{\text{comb}})$ -valued GR with initial data $(\hat{A}, \hat{B}, \hat{C})$. We have for any bordered surface Σ and $\sigma \in \mathcal{T}_\Sigma$*

$$\lim_{\beta \rightarrow \infty} \Omega_{\Sigma; \beta}(\sigma^\beta) = \widehat{\Xi}_\Sigma(\text{sp}(\sigma)), \quad (6.3.3)$$

and the convergence is uniform for σ in any compact of \mathcal{T}_Σ . Besides, there exists $t \geq 0$ depending only on the topology of Σ , such that, for any $\epsilon > 0$ there exists $M_\epsilon > 0$ for which we have, for any $\beta \geq 1$, any $\sigma \in \mathcal{T}_\Sigma$ such that $\text{sys}_{\text{sp}(\sigma)} \geq \epsilon$,

$$|\Omega_{\Sigma; \beta}(\sigma^\beta)| \leq M_\epsilon \prod_{b \in \pi_0(\partial\Sigma)} (1 + \ell_{\text{sp}(\sigma)}(b))^t \quad (6.3.4)$$

and the same inequality holds for the limit $\widehat{\Xi}_\Sigma(\text{sp}(\sigma))$.

Before proving the theorem, let us specialise $(A_\beta, B_\beta, C_\beta)$ to Mirzakhani's initial data (A^M, B^M, C^M) of Theorem 2.4.25, which can be conveniently rewritten as

$$\begin{aligned} A^M(L_1, L_2, L_3) &= 1, \\ B^M(L_1, L_2, \ell) &= \frac{1}{2L_1} (F(L_1 + L_2 - \ell) + F(L_1 - L_2 - \ell) \\ &\quad - F(-L_1 + L_2 - \ell) - F(-L_1 - L_2 - \ell)), \\ C^M(L_1, \ell, \ell') &= \frac{1}{L_1} (F(L_1 - \ell - \ell') - F(-L_1 - \ell - \ell')), \end{aligned} \quad (6.3.5)$$

for $F(x) = 2 \log(1 + e^{x/2})$. Then Theorem 6.3.5 gives another proof of the combinatorial Mirzakhani–McShane. Indeed, the hyperbolic Mirzakhani–McShane identities immediately show that $\Omega_\Sigma^M(\sigma^\beta) = 1$ for any $\sigma \in \mathcal{T}_\Sigma$ and $\beta \geq 1$, and the convergence

$$\lim_{\beta \rightarrow \infty} \frac{F(\beta x)}{\beta} = \lim_{\beta \rightarrow \infty} \frac{2 \log(1 + e^{\beta x/2})}{\beta} = [x]_+, \quad (6.3.6)$$

which is uniform on any compact of $\{-\infty\} \cup \mathbb{R}$, implies that

$$\lim_{\beta \rightarrow \infty} \rho_{\beta^*}(A^M, B^M, C^M) = (A^K, B^K, C^K) \quad (6.3.7)$$

uniformly on any compact of \mathbb{R}_+^3 . Thus, the $\beta \rightarrow \infty$ limit of $\Omega_\Sigma^M(\sigma^\beta)$ – which must be the constant function 1 – converges to the function $\Xi_\Sigma^K(\text{sp}(\sigma))$ satisfying the geometric recursion for the initial data (A^K, B^K, C^K) .

Proof of Theorem 6.3.5. It is enough to prove the result for connected surfaces, and we proceed by induction on the Euler characteristic. Throughout the proof, we denote by $\mathbb{G} = \text{sp}(\sigma)$. Recall that for any boundary component b , we have $\ell_{\sigma^\beta}(b) = \beta \ell_\sigma(b) = \beta \ell_{\mathbb{G}}(b)$. For a pair of pants P , we have

$$\Omega_{P; \beta}(\sigma^\beta) = A_\beta(\vec{\ell}_{\sigma^\beta}(\partial P)) = A_\beta(\beta \vec{\ell}_\sigma(\partial P)) = A_\beta(\beta \vec{\ell}_{\mathbb{G}}(\partial P)),$$

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which by assumption converges uniformly on any compact to $\hat{A}(\vec{\ell}_{\mathbb{G}}(\partial P)) = \widehat{\Xi}_P(\mathbb{G})$, and is bounded by a constant independent of β on any ϵ -thick part of $\mathcal{T}_P^{\text{comb}}$.

For a torus with one boundary component T , we have by the convention

$$\Omega_{T;\beta}(\sigma^\beta) = \sum_{\gamma \in \mathcal{S}_T} C_\beta(\beta \ell_{\mathbb{G}}(\partial T), \ell_{\sigma^\beta}(\gamma), \ell_{\sigma^\beta}(\gamma)). \quad (6.3.8)$$

Let K be a compact subset of $\mathcal{T}_\Sigma^{\text{comb}}$ and ϵ a lower bound of the systole on K . Let $\gamma \in \mathcal{S}_T$ and $\mathbb{G} \in K$. Since the function $\ell_{\mathbb{G}}(\gamma)$ is bounded from below and from above on K , Corollary 6.1.2 implies that $\beta^{-1} \ell_{\sigma^\beta}(\gamma)$ converges to $\ell_{\mathbb{G}}(\gamma)$ uniformly for $\sigma \in \text{sp}^{-1}(K)$. By uniform convergence of $\rho_{\beta*} C_\beta$ on that compact, we deduce that

$$\lim_{\beta \rightarrow \infty} C_\beta(\beta \ell_{\mathbb{G}}(\partial T), \ell_{\sigma^\beta}(\gamma), \ell_{\sigma^\beta}(\gamma)) = \hat{C}(\ell_{\mathbb{G}}(T), \ell_{\mathbb{G}}(\gamma), \ell_{\mathbb{G}}(\gamma))$$

uniformly for $\sigma \in K$. Next, we would like to bound each term in (6.3.8) by a summable (over γ) quantity depending only on K and not on β . If this holds, we can conclude by the (Banach-valued) dominated convergence theorem that $\Omega_{T;\beta}(\sigma^\beta)$ converges to $\widehat{\Xi}_T(\mathbb{G})$ when $\beta \rightarrow \infty$, uniformly for $\sigma \in K$.

To prove the bound, we notice that by uniform admissibility we have for any $s > 0$

$$\begin{aligned} |C_\beta(\beta \ell_{\mathbb{G}}(\partial T), \ell_{\sigma^\beta}(\gamma), \ell_{\sigma^\beta}(\gamma))| &\leq \frac{M_{\epsilon,s}}{(1 + [2\beta^{-1} \ell_{\sigma^\beta}(\gamma) - \beta^{-1} \ell_{\sigma^\beta}(\partial T)]_+)^s} \\ &\leq \frac{M_{\epsilon,s}}{(1 + [2\ell_{\mathbb{G}}(\gamma) - \ell_{\mathbb{G}}(\partial T)]_+)^s}, \end{aligned}$$

where the second inequality used the lower bound $\beta \ell_{\mathbb{G}}(\gamma) \leq \ell_{\sigma^\beta}(\gamma)$ of Proposition 6.1.1. Since the number of small pairs of pants is bounded by the number of oriented edges, there are at most 6 curves $\gamma \in \mathcal{S}_T$ for which the denominator is equal to 1. Hence

$$\sum_{\gamma \in \mathcal{S}_T} \frac{1}{(1 + [2\ell_{\mathbb{G}}(\gamma) - \ell_{\mathbb{G}}(\partial T)]_+)^s} \leq 6 + \sum_{L > \ell_{\mathbb{G}}(\partial T)/2} \frac{|\{\gamma \in \mathcal{S}_T \mid \ell_{\mathbb{G}}(\gamma) \leq L + 1\}|}{(1 + 2L - \ell_{\mathbb{G}}(\partial T))^s}.$$

Thanks to Proposition 8.1.7, the numerator is bounded by $m_\epsilon (L + 1)^2$ for some constant m_ϵ depending on ϵ only. Therefore

$$|\Omega_{T;\beta}(\sigma^\beta)| \leq M_{\epsilon,s} \left(6 + m_\epsilon \sum_{L \geq 0} \frac{(1 + \ell_{\mathbb{G}}(\partial T)/2 + L)^2}{(1 + 2L)^s} \right).$$

By choosing $s = 4$ we find that $|\Omega_{T;\beta}(\sigma^\beta)|$ is bounded by a polynomial of degree 2 in $\ell_{\mathbb{G}}(\partial T)$, whose coefficients are independent of β but may depend on ϵ . This proves the theorem for $\Sigma = T$.

Now let Σ be a bordered surface with $\chi_\Sigma < -1$, and assume the thesis for all Σ' such that $\chi_{\Sigma'} > \chi_\Sigma$. Let K be a compact subset of $\mathcal{T}_\Sigma^{\text{comb}}$ and ϵ a lower bound on the systole on K . The geometric recursion gives

$$\Omega_{\Sigma;\beta}(\sigma^\beta) = \sum_{P \in \mathcal{P}_\Sigma} X_{P;\beta}(\vec{\ell}_{\sigma^\beta}(\partial P)) \Omega_{\Sigma-P;\beta}(\sigma^\beta|_{\Sigma-P}).$$

For each $P \in \mathcal{P}_\Sigma$, we can repeat the previous arguments to show that $X_P(\vec{\ell}_{\sigma^\beta}(\partial P))$ is converging to $\hat{X}_{P,\beta}(\vec{\ell}_\mathbb{G}(\partial P))$ uniformly for $\mathbb{G} \in K$. If the initial data were the one of Mirzakhani (2.4.20), the factor $\Omega_{\Sigma-P;\beta}$ would be the constant function 1. Then, we could finish the proof very similarly to the case of $\Sigma = T$. However, in the general case, we only know by induction hypothesis that $\Omega_{\Sigma-P;\beta}(\vec{\sigma}^\beta)$ converges to $\widehat{\Xi}_{\Sigma-P}(\widehat{\mathbb{G}})$ uniformly for $\widehat{\mathbb{G}} = \text{sp}(\vec{\sigma})$ in any compact of $\mathcal{T}_{\Sigma-P}^{\text{comb}}$. To employ this information, we shall compare the two hyperbolic structures $\sigma^\beta|_{\Sigma-P}$ and $\vec{\sigma}^\beta := \text{sp}^{-1}(\beta\mathbb{G}|_{\Sigma-P})$, by means of their length functions. Since the combinatorial cutting does not decrease the systole, Proposition 6.1.1 on Σ and $\Sigma - P$ respectively imply for $\beta \geq \beta_\epsilon$ and $\gamma \in \mathcal{S}_{\Sigma-P}$

$$\begin{aligned} \ell_\mathbb{G}(\gamma) &\leq \frac{\ell_{\sigma^\beta}(\gamma)}{\beta} \leq \ell_\mathbb{G}(\gamma) \left(1 + \frac{\kappa_\epsilon}{\beta}\right), \\ \ell_{\mathbb{G}|_{\Sigma-P}}(\gamma) &\leq \frac{\ell_{\vec{\sigma}^\beta}(\gamma)}{\beta} \leq \ell_{\mathbb{G}|_{\Sigma-P}}(\gamma) \left(1 + \frac{\kappa_\epsilon}{\beta}\right), \end{aligned}$$

where we can use for $\kappa_\epsilon, \beta_\epsilon$ the constants provided by Proposition 6.1.1 for Σ . The combinatorial and hyperbolic cutting procedures are such that

$$\forall \gamma \in \mathcal{S}_{\Sigma-P} \quad \ell_{\mathbb{G}|_{\Sigma-P}}(\gamma) = \ell_\mathbb{G}(\gamma) \quad \text{and} \quad \ell_{\sigma^\beta|_{\Sigma-P}}(\gamma) = \ell_{\sigma^\beta}(\gamma).$$

Therefore

$$\forall \gamma \in \mathcal{S}_{\Sigma-P} \quad \left(1 + \frac{\kappa_\epsilon}{\beta}\right)^{-1} \ell_{\vec{\sigma}^\beta}(\gamma) \leq \ell_{\sigma^\beta|_{\Sigma-P}}(\gamma) \leq \ell_{\vec{\sigma}^\beta}(\gamma) \left(1 + \frac{\kappa_\epsilon}{\beta}\right).$$

Further, for any component of $\partial P \cap \Sigma^\circ$ and $\partial\Sigma \cap \partial(\Sigma - P)$, we have exactly $\ell_{\sigma^\beta|_{\Sigma-P}}(\gamma) = \ell_{\vec{\sigma}^\beta}(\gamma)$, so that the above bounds extend to all simple closed curves γ of $\Sigma - P$. From the description of the combinatorial Teichmüller spaces' topology (Theorem 3.2.3), we deduce that $\beta^{-1}\text{sp}(\sigma^\beta|_{\Sigma-P})$ remains in a compact of $\mathcal{T}_{\Sigma-P}^{\text{comb}}$ independent of β when $\mathbb{G} \in K$, and that it converges to $\mathbb{G}|_{\Sigma-P}$ uniformly for $\mathbb{G} \in K$. By the induction hypothesis, we have

$$\lim_{\beta \rightarrow \infty} \Omega_{\Sigma-P;\beta}(\sigma^\beta|_{\Sigma-P}) = \widehat{\Xi}_{\Sigma-P}(\mathbb{G}|_{\Sigma-P})$$

and the convergence is uniform for $\mathbb{G} \in K$. Supplemented with a summable (over $P \in \mathcal{P}_\Sigma$) bound whose derivation is similar to the case $\Sigma = T$ and therefore omitted, this proves the theorem for Σ , and thus in full generality by induction. \square

6.4 — TOPOLOGICAL RECURSION IN THE FLOW

Thanks to the result of the previous section, we can prove an analogue of Theorem 6.3.5 after integration over the moduli spaces. This result generalises Mirzakhani's argument in the proof of Witten's conjecture [Miro7b] to a large class of initial data.

THEOREM 6.4.1. *Let $(A_\beta, B_\beta, C_\beta)_{\beta \geq 1}$ be initial data such that $\rho_{\beta^*}(A_\beta, B_\beta, C_\beta)$ is uniformly strongly admissible and converges uniformly on any subset of the form $(0, M]^3 \subset \mathbb{R}_+^3$ to a limit $(\hat{A}, \hat{B}, \hat{C})$. Then, $(\hat{A}, \hat{B}, \hat{C})$ is strongly admissible and*

$$\lim_{\beta \rightarrow \infty} \frac{\langle \Omega_{g,n;\beta} \rangle(\beta L)}{\beta^{6g-6+2n}} = \langle \widehat{\Xi}_{g,n} \rangle(L), \quad (6.4.1)$$

with uniform convergence for L in any subset of the form $(0, M]^n \subset \mathbb{R}_+^n$.

II. The combinatorial model of the moduli space of curves

Proof. We prove the result by induction on $2g - 2 + n > 0$. For $(g, n) = (0, 3)$, we have $\langle \Omega_{0,3;\beta} \rangle(\beta L) = A_\beta(\beta L)$, which converges uniformly on any compact of \mathbb{R}^3 intersected with \mathbb{R}_+^3 to $\hat{A}(L) = \langle \hat{\Xi}_{0,3} \rangle(L)$. For $(g, n) = (1, 1)$ we have

$$\frac{\langle \Omega_{1,1;\beta} \rangle(\beta L_1)}{\beta^2} = \int_{\mathbb{R}_+} C_\beta(\beta L_1, \ell, \ell) \frac{\ell}{\beta} \frac{d\ell}{\beta} = \int_{\mathbb{R}_+} C_\beta(\beta L_1, \beta\ell, \beta\ell) \ell d\ell.$$

Note $C_\beta(\beta L_1, \beta\ell, \beta\ell)$ converges uniformly on any $(0, M]^2$ to $\hat{C}(L_1, \ell, \ell)$. Strong uniform admissibility means we can bound the integrand by an integrable function independent of β . Moreover, the uniformity of the convergence around zero implies that we can exchange the integral and the limit, so that

$$\lim_{\beta \rightarrow \infty} \frac{\langle \Omega_{1,1;\beta} \rangle(\beta L_1)}{\beta^2} = \lim_{\beta \rightarrow \infty} \int_{\mathbb{R}_+} C_\beta(\beta L_1, \beta\ell, \beta\ell) \ell d\ell = \int_{\mathbb{R}_+} \hat{C}(L_1, \ell, \ell) \ell d\ell = \langle \hat{\Xi}_{1,1} \rangle(L_1),$$

and the convergence is uniform on any $(0, M]$. This proves the two base cases.

The general argument follows along the same lines as the $(g, n) = (1, 1)$ case. Assume the result for (g', n') such that $2g' - 2 + n' < 2g - 2 + n$. The topological recursion for $\langle \Omega_{g,n;\beta} \rangle(\beta L)$ yields

$$\begin{aligned} \frac{\langle \Omega_{g,n;\beta} \rangle(\beta L_1, \dots, \beta L_n)}{\beta^{6g-6+2n}} &= \\ &= \sum_{m=2}^n \int_{\mathbb{R}_+} B_\beta(\beta L_1, \beta L_m, \ell) \frac{\langle \Omega_{g,n-1;\beta} \rangle(\ell, \beta L_2, \dots, \widehat{\beta L_m}, \dots, \beta L_n)}{\beta^{6g-6+2(n-1)}} \frac{\ell}{\beta} \frac{d\ell}{\beta} \\ &\quad + \frac{1}{2} \int_{\mathbb{R}_+^2} C_\beta(\beta L_1, \ell, \ell') \left(\frac{\langle \Omega_{g-1,n+1;\beta} \rangle(\ell, \ell', \beta L_2, \dots, \beta L_n)}{\beta^{6(g-1)-6+2(n+1)}} \right. \\ &\quad \left. + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2, \dots, n\}}} \frac{\langle \Omega_{g_1,1+|I_1|;\beta} \rangle(\ell, \beta L_{I_1})}{\beta^{6g_1-6+2(1+|I_1|)}} \frac{\langle \Omega_{g_2,1+|I_2|;\beta} \rangle(\ell', \beta L_{I_2})}{\beta^{6g_2-6+2(1+|I_2|)}} \right) \frac{\ell \ell'}{\beta^2} \frac{d\ell}{\beta} \frac{d\ell'}{\beta}. \end{aligned}$$

Now rescaling the integration variables, we get

$$\begin{aligned} \frac{\langle \Omega_{g,n;\beta} \rangle(\beta L_1, \dots, \beta L_n)}{\beta^{6g-6+2n}} &= \\ &= \sum_{m=2}^n \int_{\mathbb{R}_+} B_\beta(\beta L_1, \beta L_m, \beta\ell) \frac{\langle \Omega_{g,n-1;\beta} \rangle(\beta\ell, \beta L_2, \dots, \widehat{\beta L_m}, \dots, \beta L_n)}{\beta^{6g-6+2(n-1)}} \ell d\ell \\ &\quad + \frac{1}{2} \int_{\mathbb{R}_+^2} C_\beta(\beta L_1, \beta\ell, \beta\ell') \left(\frac{\langle \Omega_{g-1,n+1;\beta} \rangle(\beta\ell, \beta\ell', \beta L_2, \dots, \beta L_n)}{\beta^{6(g-1)-6+2(n+1)}} \right. \\ &\quad \left. + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{L_2, \dots, L_m\}}} \frac{\langle \Omega_{g_1,1+|I_1|;\beta} \rangle(\beta\ell, \beta I_{I_1})}{\beta^{6g_1-6+2(1+|I_1|)}} \frac{\langle \Omega_{g_2,1+|I_2|;\beta} \rangle(\beta\ell', \beta I_{I_2})}{\beta^{6g_2-6+2(1+|I_2|)}} \right) \ell \ell' d\ell d\ell'. \end{aligned}$$

By induction hypothesis, the topological recursion amplitudes $\beta^{-(6g'-6+2n')} \langle \Omega_{g',n';\beta} \rangle$ appearing on the right-hand side of the equation converge uniformly on any $(0, M]^n$. For $X_\beta \in \{B_\beta, C_\beta\}$, we assumed that $X_\beta(\beta L)$ converges to $\hat{X}(L)$ uniformly on any $(0, M]^3$ and moreover that $X_\beta(\beta L)$ is uniformly admissible. Therefore, using the strong uniform admissibility, we can bound the integrals around infinity by integrable functions independently of β and that can be uniformly chosen on compact sets of L . The uniformity in compacts around zero then implies that we can interchange the integral and the limit, which again by our induction assumption reproduces the topological recursion for $\langle \hat{\Xi}_{g,n} \rangle(L)$ uniformly on any $(0, M]^n$. \square

PART III

ENUMERATION OF MULTICURVES AND QUADRATIC DIFFERENTIALS

CHAPTER 7 — COUNTING MULTICURVES AND QUADRATIC DIFFERENTIALS

Recall from Section 2.4.1 that we denote by \mathcal{M}_Σ the set of multicurves in Σ (empty curve included). In [Miro8b], Mirzakhani defines a function on the hyperbolic Teichmüller space that counts the number of multicurves with hyperbolic length bounded by some parameter $t \in \mathbb{R}_+$:

$$\mathcal{N}_\Sigma : \mathcal{T}_\Sigma(L) \times \mathbb{R}_+ \rightarrow \mathbb{Z}_+, \quad \mathcal{N}_\Sigma(\sigma; t) = |\{c \in \mathcal{M}_\Sigma \mid \ell_\sigma(c) \leq t\}|. \quad (7.0.1)$$

Moreover, she calculates its average over the moduli space as a sum over stable graphs. In this chapter we show how this counting can be computed by geometric recursion, and its average by topological recursion, after a Laplace transform in the cutoff variable t .

THEOREM 7.A (Hyperbolic multicurve count). *Consider the Laplace transform of the hyperbolic multicurve count $\widehat{\mathcal{N}}_\Sigma(\sigma; s) = s \int_{\mathbb{R}_+} \mathcal{N}_\Sigma(\sigma; t) e^{-st} dt$.*

- *The function $\widehat{\mathcal{N}}_\Sigma$ is computed by geometric recursion for (s -dependent) initial data (A, B, C, D) given by Mirzakhani's initial data twisted by the function $f(\ell; s) = \frac{e^{-s\ell}}{1-e^{-s\ell}}$.*
- *Its average $\langle \widehat{\mathcal{N}}_{g,n} \rangle(L; s)$ over the moduli space $(\mathcal{M}_{g,n}(L), \mu_{\mathbb{WP}})$ is computed by topological recursion and by a sum over stable graphs.*

Driven by the parallelism between hyperbolic and combinatorial geometry developed in the previous chapters, we prove the corresponding results in the combinatorial setting, i.e. for the function

$$\mathcal{N}_\Sigma^{\text{comb}} : \mathcal{T}_\Sigma^{\text{comb}}(L) \times \mathbb{R}_+ \rightarrow \mathbb{Z}_+, \quad \mathcal{N}_\Sigma^{\text{comb}}(\mathbb{G}; t) = |\{c \in \mathcal{M}_\Sigma \mid \ell_\mathbb{G}(c) \leq t\}|. \quad (7.0.2)$$

THEOREM 7.B (Combinatorial multicurve count). *Consider the Laplace transform of the combinatorial multicurve count $\widehat{\mathcal{N}}_\Sigma^{\text{comb}}(\mathbb{G}; s) = s \int_{\mathbb{R}_+} \mathcal{N}_\Sigma^{\text{comb}}(\mathbb{G}; t) e^{-st} dt$.*

- *The function $\widehat{\mathcal{N}}_\Sigma^{\text{comb}}$ is computed by geometric recursion for (s -dependent) initial data $(A, B, C, D)^{\text{comb}}$ given by Kontsevich's initial data twisted by the function $f(\ell; s) = \frac{e^{-s\ell}}{1-e^{-s\ell}}$.*
- *Its average $\langle \widehat{\mathcal{N}}_{g,n}^{\text{comb}} \rangle(L; s)$ over the combinatorial moduli space $(\mathcal{M}_{g,n}^{\text{comb}}(L), \mu_{\mathbb{K}})$ is computed by topological recursion and by a sum over stable graphs.*

In particular, we can *effectively* compute the average number of multicurves with bounded hyperbolic/combinatorial length by topological recursion (cf. Tables 7.1 and 7.2).

We also note that the combinatorial multicurve count suggests a connection with quadratic differentials with double poles. Indeed, if we identify metric ribbon graphs with Jenkins–Strebel differentials and multicurves with core cylinders attached to such quadratic differentials,

we deduce a similar recursive structure for the counting of square-tiled surfaces weighted by their core area. More precisely, consider the generating function

$$N_{g,n}^\square(L; q) = \sum_S \frac{1}{|\text{Aut}(S)|} q^{\text{CoreArea}(S)}, \quad (7.0.3)$$

where the sum is taken over all quadratic differentials on a surface of genus g with n double poles of residue $L = (L_1, \dots, L_n)$ and integral periods (i.e. square-tiled surfaces).

THEOREM 7.C (Square-tiled surfaces count). *The generating function $N_{g,n}^\square(L; q)$ is computed by discrete topological recursion for (q -dependent) initial data $(A, B, C, D)^\square$ given by Kontsevich's initial data twisted by the function $f(\ell; q) = \frac{q^\ell}{1-q^\ell}$, and by a sum over stable graphs.*

7.0.1 — RELATION WITH PREVIOUS WORKS AND OPEN QUESTIONS

In [Miro8b], Mirzakhani introduced the above count of multicurves in the hyperbolic setting, and computed its average over the moduli space as a sum over stable graphs (in her language, as a sum over mapping class group orbits of primitive multicurves). The novelty of our work is the connection between Mirzakhani's counting and the length statistics of Andersen–Borot–Orantin [ABO17], allowing us to deduce geometric and topological recursion results for the (Laplace transform of such) counting and its average over the moduli space. Moreover, thanks to the complete parallelism between hyperbolic and combinatorial geometry developed in the previous chapters, we proved analogous results in the combinatorial setting.

One of the main properties of the multicurve counting is its connection with the Masur–Veech volumes of the principal stratum of the moduli space of quadratic differentials with *simple* poles, developed by Mirzakhani in [Miro8a] and discussed in the next chapter.

7.0.2 — ORGANISATION OF THE CHAPTER

The chapter is organised as follows.

- In Section 7.1 and (resp. Section 7.1) we prove that the count of multicurves with bounded hyperbolic (resp. combinatorial) length is computed by geometric recursion, and its average by topological recursion and by a sum over stable graphs.
- In Section 7.3 we introduce the moduli space of quadratic differentials with double poles, and prove that the number of integral points is computed by discrete topological recursion and by a sum over stable graphs.

7.1 — COUNTING MULTICURVES: THE HYPERBOLIC CASE

DEFINITION 7.1.1. Let Σ be a bordered surface of type (g, n) . Define the function counting multicurves with bounded hyperbolic length, i.e. $\mathcal{N}_\Sigma: \mathcal{T}_\Sigma(L) \times \mathbb{R}_+ \rightarrow \mathbb{Z}_+$ defined as

$$\mathcal{N}_\Sigma(\sigma; t) = |\{c \in \mathcal{M}_\Sigma \mid \ell_\sigma(c) \leq t\}|. \quad (7.1.1)$$

The function is invariant under the action of the pure mapping class group, and we denote by $N_{g,n}$ the induced function on the moduli space.

The counting function \mathcal{N}_Σ was studied by Mirzakhani in [Miro8b], where she proves that $\mathcal{N}_{g,n}$ is integrable with respect to the Weil–Petersson measure μ_{WP} and (the Laplace transform of) its average over the moduli space can be expressed as a certain sum over stable graphs. The following is a restatement of some of her results.

THEOREM 7.1.2 ([Miro8b] revisited). *The multicurve counting function $\mathcal{N}_{g,n}(X; t)$ is integrable over $(\mathcal{M}_{g,n}(L), \mu_{\text{WP}})$, and its average is a polynomial of degree $3g - 3 + n$ in L_1^2, \dots, L_n^2, t^2 , symmetric in the length variables. Furthermore, its Laplace transform in the cutoff variable t is given by the following sum over stable graphs*

$$\begin{aligned} \int_{\mathbb{R}_+} \left(\int_{\mathcal{M}_{g,n}(L)} \mathcal{N}_{g,n}(X; t) d\mu_{\text{WP}}(X) \right) e^{-st} dt &= \\ &= \frac{1}{s} \sum_{\Gamma \in \mathcal{G}_{g,n}} \frac{1}{|\text{Aut}(\Gamma)|} \int_{\mathbb{R}_+^{E_\Gamma}} \prod_{v \in V_\Gamma} V_{g(v),n(v)}^{\text{WP}}((\ell_e)_{e \in E_v}, (L_\lambda)_{\lambda \in \Lambda_v}) \prod_{e \in E_\Gamma} \frac{\ell_e e^{-s\ell_e} d\ell_e}{1 - e^{-s\ell_e}}. \end{aligned} \quad (7.1.2)$$

Proof. We explain how to derive the above formula from Mirzakhani’s results. We remark that she considered Teichmüller spaces and moduli spaces of punctured surfaces (i.e. $(L_1, \dots, L_n) = (0, \dots, 0)$), but her results generalise straightforwardly to bordered surfaces.

The integrability result is given in [Miro8b, Proposition 3.6]. In her notation, $\mathcal{N}_\Sigma(\sigma, t)$ corresponds to $b_X(L)$. For the integral of the counting function, she considers the class of a multicurve on Σ under the action of the mapping class group, which is determined by the data (Γ, a) of a stable graph $\Gamma \in \mathcal{G}_{g,n}$, together with an integral tuple $a \in \mathbb{Z}_+^{E_\Gamma}$ giving the multiplicity of each component ($\mathcal{G}_{g,n}$ corresponds to $\mathcal{S}_{g,n}$ in Mirzakhani’s notation, and a multicurve c is denoted by $\sum_{i=1}^k a_i \gamma_i$). She then considers the frequency of the multicurve in the mapping class orbit (Γ, a) with geodesic length bounded by t :

$$s_\Sigma(\sigma; \Gamma, a, t) = |\{c \in (\Gamma, a) \mid \ell_\sigma(c) \leq t\}|.$$

It is a non-negative mapping class group invariant function, which descend to a function $s_{g,n}$ on the moduli space ($s_{g,n}(X; \Gamma, a, t)$ corresponds to $s_X(L, a \cdot \tilde{\eta})$ in Mirzakhani’s notation of Theorem 5.3). Moreover, it is related to the counting function as

$$\mathcal{N}_{g,n}(X; t) = \sum_{\Gamma \in \mathcal{G}_{g,n}} \sum_{a \in \mathbb{Z}_+^{E_\Gamma}} \frac{|\text{Aut}(\Gamma, a)|}{|\text{Aut}(\Gamma)|} s_{g,n}(X; \Gamma, a, t).$$

The integral of $s_{g,n}$ over the moduli space is given in [Miro8b, Proposition 5.1]¹:

$$\begin{aligned} \int_{\mathcal{M}_{g,n}(L)} s_{g,n}(X; \Gamma, a, t) d\mu_{\text{WP}}(X) &= \\ &= \frac{1}{|\text{Aut}(\Gamma, a)|} \int_{\langle a, \ell \rangle \leq t} \prod_{v \in V_\Gamma} V_{g(v),n(v)}^{\text{WP}}((\ell_e)_{e \in E_v}, (L_\lambda)_{\lambda \in \Lambda_v}) \prod_{e \in E_\Gamma} \ell_e d\ell_e, \end{aligned}$$

where $\langle a, \ell \rangle = \sum_{e \in E_\Gamma} a_e \ell_e$. From this formula, one can easily obtain the polynomiality statement, knowing that the Weil–Petersson volumes $V_{g,n}^{\text{WP}}(L)$ are polynomials in the squared

¹We remark again that Mirzakhani [Miro7a; Miro8b] considers a different orbifold structure on $\mathcal{M}_{1,1}(L)$, ignoring the elliptic involution. Hence, our result differs from hers by factors of 2.

length variables L_1^2, \dots, L_n^2 . We now perform some new computations that lead to Equation (7.1.2). Notice that for any polynomial $p(\ell)$ in the variables $(\ell_e^2)_{e \in E_\Gamma}$, we can take the Laplace transform of its integral over $\langle a, \ell \rangle \leq t$. We claim that

$$\int_{\mathbb{R}_+} \left(\int_{\langle a, \ell \rangle \leq t} p(\ell) \prod_{e \in E_\Gamma} \ell_e d\ell_e \right) e^{-st} dt = \frac{1}{s} \int_{\mathbb{R}_+^{E_\Gamma}} p(\ell) \prod_{e \in E_\Gamma} e^{-sa_e \ell_e} \ell_e d\ell_e.$$

To prove such a formula, we can start by direct integration of a monomial over the standard k -simplex: $\int_{\Delta^k} \prod_{i=1}^k \ell_i^{\alpha_i} d\ell_i = \frac{\alpha_1! \cdots \alpha_k!}{(k + \alpha_1 + \cdots + \alpha_k)!}$. By linearity and a rescaling argument, one can infer the left-hand side of the above formula. The right-hand side can be easily computed by direct integration, and comparison yields the claim. From this, we find

$$\begin{aligned} & \int_{\mathbb{R}_+} \left(\int_{\mathcal{M}_{g,n}(L)} \mathcal{N}_{g,n}(X; t) d\mu_{\text{WP}}(X) \right) e^{-st} dt = \\ &= \sum_{\Gamma \in \mathcal{G}_{g,n}} \frac{1}{|\text{Aut}(\Gamma)|} \sum_{a \in \mathbb{Z}_+^{E_\Gamma}} \int_{\mathbb{R}_+} \left(\int_{\langle a, \ell \rangle \leq t} \prod_{v \in V_\Gamma} V_{g(v), n(v)}^{\text{WP}}((\ell_e)_{e \in E_v}, (L_\lambda)_{\lambda \in \Lambda_v}) \prod_{e \in E_\Gamma} \ell_e d\ell_e \right) e^{-st} dt \\ &= \frac{1}{s} \sum_{\Gamma \in \mathcal{G}_{g,n}} \frac{1}{|\text{Aut}(\Gamma)|} \sum_{a \in \mathbb{Z}_+^{E_\Gamma}} \int_{\mathbb{R}_+^{E_\Gamma}} \prod_{v \in V_\Gamma} V_{g(v), n(v)}^{\text{WP}}((\ell_e)_{e \in E_v}, (L_\lambda)_{\lambda \in \Lambda_v}) \prod_{e \in E_\Gamma} e^{-sa_e \ell_e} \ell_e d\ell_e \\ &= \frac{1}{s} \sum_{\Gamma \in \mathcal{G}_{g,n}} \frac{1}{|\text{Aut}(\Gamma)|} \int_{\mathbb{R}_+^{E_\Gamma}} \prod_{v \in V_\Gamma} V_{g(v), n(v)}^{\text{WP}}((\ell_e)_{e \in E_v}, (L_\lambda)_{\lambda \in \Lambda_v}) \prod_{e \in E_\Gamma} \frac{\ell_e e^{-s\ell_e} d\ell_e}{1 - e^{-s\ell_e}}. \end{aligned}$$

□

Notice how the above result resembles Corollary 2.4.28, computing the average over the moduli space of length statistics of multicurves weighted by a general function. The main difference is that, in that case, multicurves are weighed *multiplicatively* with respect to the lengths of each component: $\prod_{\gamma \in \pi_0(c)} F(\ell_\sigma(\gamma))$. On the other hand, in the counting function we rather consider multicurves weighted *additively* with respect to the lengths of each component: $\ell_\sigma(c) = \sum_{\gamma \in \pi_0(c)} \ell_\sigma(\gamma)$.

The above restatement of Mirzakhani's result, together with this simple remark, were the starting points that led us to a connection between the counting problem of Mirzakhani and the length statistics of multicurves in the sense of Andersen–Borot–Orantin. Indeed, the counting function \mathcal{N}_Σ can be rewritten as

$$\mathcal{N}_\Sigma(\sigma; t) = \sum_{c \in \mathcal{M}_\Sigma} H(t - \ell_\sigma(c)), \quad (7.1.3)$$

where H is the Heaviside function. Furthermore, the connection between multiplicative and additive statistics is taken into account precisely by the Laplace transform and a specific choice of F as a decaying exponential function:

$$\int_{\mathbb{R}_+} H\left(t - \sum_{\gamma \in \pi_0(c)} \ell_\sigma(\gamma)\right) e^{-st} dt = \frac{1}{s} \prod_{\gamma \in \pi_0(c)} e^{-s\ell_\sigma(\gamma)}, \quad \Re(s) > 0. \quad (7.1.4)$$

The main consequence of this observation is that we can compute the Laplace transform of the counting function by means of geometric recursion (Theorem 2.4.26). Moreover, the average number of multicurves is computed by topological recursion (Corollary 2.4.28).

THEOREM 7.1.3. *The Laplace transform of the counting function with respect to the cutoff variable t , namely $\widehat{N}_\Sigma(\sigma; s) = s \int_{\mathbb{R}_+} N_\Sigma(\sigma; t) e^{-st} dt$, is computed by geometric recursion:*

$$\widehat{N}_\Sigma(\sigma; s) = \sum_{m=2}^n \sum_{P \in \mathcal{B}_{\Sigma, m}} B(\ell_\sigma(\partial P); s) \widehat{N}_{\Sigma-P}(\sigma|_{\Sigma-P}; s) + \frac{1}{2} \sum_{P \in \mathcal{C}_\Sigma} C(\ell_\sigma(\partial P); s) \widehat{N}_{\Sigma-P}(\sigma|_{\Sigma-P}; s), \quad (7.1.5)$$

where B and C are Mirzakhani's kernels twisted by $f(\ell; s) = \frac{e^{-s\ell}}{1-e^{-s\ell}}$, i.e.

$$\begin{aligned} B(L, L', \ell; s) &= 1 - \frac{1}{L} \log \left(\frac{\cosh(\frac{L'}{2}) + \cosh(\frac{L+\ell}{2})}{\cosh(\frac{L'}{2}) + \cosh(\frac{L-\ell}{2})} \right) + \frac{e^{-s\ell}}{1-e^{-s\ell}}, \\ C(L, \ell, \ell'; s) &= \frac{2}{L} \log \left(\frac{e^{\frac{L}{2}} + e^{\frac{\ell+\ell'}{2}}}{e^{-\frac{L}{2}} + e^{\frac{\ell-\ell'}{2}}} \right) + \frac{e^{-s\ell} e^{-s\ell'}}{(1-e^{-s\ell})(1-e^{-s\ell'})} \\ &\quad + \frac{e^{-s\ell}}{1-e^{-s\ell}} \left(1 - \frac{1}{L} \log \left(\frac{\cosh(\frac{\ell}{2}) + \cosh(\frac{L+\ell'}{2})}{\cosh(\frac{\ell}{2}) + \cosh(\frac{L-\ell'}{2})} \right) \right) \\ &\quad + \frac{e^{-s\ell'}}{1-e^{-s\ell'}} \left(1 - \frac{1}{L} \log \left(\frac{\cosh(\frac{\ell'}{2}) + \cosh(\frac{L+\ell}{2})}{\cosh(\frac{\ell'}{2}) + \cosh(\frac{L-\ell}{2})} \right) \right), \end{aligned} \quad (7.1.6)$$

with initial conditions $\widehat{N}_P(\sigma; s) = 1$ for pairs of pants and $\widehat{N}_T(\sigma; s) = 1 + \sum_{\gamma \in \mathcal{S}_T} \frac{e^{-s\ell_\sigma(\gamma)}}{1-e^{-s\ell_\sigma(\gamma)}}$ for one-holed tori.

Proof. Notice that, combining (7.1.3) and (7.1.4) and applying Fubini's theorem (which requires $\Re(s) > 0$), we have

$$\widehat{N}_\Sigma(\sigma; s) = \sum_{c \in \mathcal{M}_\Sigma} e^{-s\ell_\sigma(c)}.$$

On the other hand, we can apply Theorem 2.4.26 with $f(\ell; s) = \frac{e^{-s\ell}}{1-e^{-s\ell}}$ to Mirzakhani's initial data (2.4.43) and obtain the following geometric recursion amplitudes:

$$\Omega_\Sigma[f] = \sum_{c \in \mathcal{M}'_\Sigma} \prod_{\gamma \in \pi_0(c)} \frac{e^{-s\ell_\sigma(\gamma)}}{1-e^{-s\ell_\sigma(\gamma)}} = \sum_{c \in \mathcal{M}_\Sigma} \prod_{\gamma \in \pi_0(c)} e^{-s\ell_\sigma(\gamma)}.$$

The second equality follows from rewriting the sum over all multicurves (and not only primitive ones, cf. Remark 2.4.27), together with the relation $\sum_{k \geq 1} e^{-sk\ell} = \frac{e^{-s\ell}}{1-e^{-s\ell}}$. Thus, \widehat{N}_Σ coincides with the geometric recursion amplitude $\Omega_\Sigma[f]$ computed from the initial data (7.1.6). To conclude, we remark that the amplitudes associated to one-holed tori can be written as

$$\widehat{N}_T(\sigma; s) = \sum_{c \in \mathcal{M}'_T} \frac{e^{-s\ell_\sigma(\gamma)}}{1-e^{-s\ell_\sigma(\gamma)}} = 1 + \sum_{\gamma \in \mathcal{S}_T} \frac{e^{-s\ell_\sigma(\gamma)}}{1-e^{-s\ell_\sigma(\gamma)}},$$

since for one-holed tori we have $\mathcal{M}'_T = \{\emptyset\} \sqcup \mathcal{S}_T$. □

COROLLARY 7.1.4. *The average over the moduli space of the Laplace transform of the multicurve counting function with respect to the cutoff variable t is computed by topological recursion:*

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(g, n)	$\langle \mathcal{N}_{g,n} \rangle(L; t)$
(0, 3)	1
(0, 4)	$\frac{1}{2}m_{(1)} + (2 + \frac{t^2}{4})\pi^2$
(0, 5)	$\frac{1}{8}m_{(2)} + \frac{1}{2}m_{(1^2)} + (3 + \frac{t^2}{4})\pi^2 m_{(1)} + (10 + \frac{5t^2}{3} + \frac{t^4}{32})\pi^4$
(1, 1)	$\frac{1}{48}m_{(1)} + (\frac{1}{12} + \frac{t^2}{24})\pi^2$
(1, 2)	$\frac{1}{192}m_{(2)} + \frac{1}{96}m_{(1^2)} + (\frac{1}{12} + \frac{t^2}{48})\pi^2 m_{(1)} + (\frac{1}{4} + \frac{13t^2}{144} + \frac{t^4}{384})\pi^4$
(1, 3)	$\frac{1}{1152}m_{(3)} + \frac{1}{192}m_{(2,1)} + \frac{1}{96}m_{(1^3)} + (\frac{1}{24} + \frac{13t^2}{2304})\pi^2 m_{(2)} + (\frac{1}{8} + \frac{t^2}{48})\pi^2 m_{(1^2)}$ $+ (\frac{13}{24} + \frac{13t^2}{96} + \frac{t^4}{384})\pi^4 m_{(1)} + (\frac{14}{9} + \frac{71t^2}{144} + \frac{61t^4}{3456} + \frac{11t^6}{69120})\pi^6$
(2, 1)	$\frac{1}{442368}m_{(4)} + (\frac{29}{138240} + \frac{t^2}{27648})\pi^2 m_{(3)} + (\frac{139}{23040} + \frac{49t^2}{27648} + \frac{119t^4}{3317760})\pi^4 m_{(2)}$ $+ (\frac{169}{2880} + \frac{5t^2}{216} + \frac{119t^4}{138240} + \frac{t^6}{138240})\pi^6 m_{(1)} + (\frac{29}{192} + \frac{115t^2}{1728} + \frac{4199t^4}{1244160} + \frac{t^6}{18432} + \frac{29t^8}{103219200})\pi^8$

Table 7.1: The average number of multicurves with geodesic length bounded by t on a hyperbolic bordered surface of perimeter (L_1, \dots, L_n) for low values of $2g - 2 + n$. Here m_λ is the monomial symmetric polynomial associated to the partition λ , evaluated at L_1^2, \dots, L_n^2 .

$$\begin{aligned}
\langle \widehat{\mathcal{N}}_{g,n} \rangle(L_1, \dots, L_n; s) &= \\
&= \sum_{m=2}^n \int_{\mathbb{R}_+} B(L_1, L_m, \ell; s) \langle \widehat{\mathcal{N}}_{g,n-1} \rangle(\ell, L_2, \dots, L_m, \dots, L_n; s) \ell d\ell \\
&+ \frac{1}{2} \int_{\mathbb{R}_+^2} C(L_1, \ell, \ell'; s) \left(\langle \widehat{\mathcal{N}}_{g-1,n+1} \rangle(\ell, \ell', L_2, \dots, L_n; s) \right. \\
&\quad \left. + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2, \dots, n\}}} \langle \widehat{\mathcal{N}}_{g_1, 1+|I_1|} \rangle(\ell, L_{I_1}; s) \langle \widehat{\mathcal{N}}_{g_2, 1+|I_2|} \rangle(\ell', L_{I_2}; s) \right) \ell \ell' d\ell d\ell'
\end{aligned} \tag{7.1.7}$$

with initial data $\langle \widehat{\mathcal{N}}_{0,3} \rangle(L; s) = 1$ and $\langle \widehat{\mathcal{N}}_{1,1} \rangle(L; s) = \frac{L^2}{48} + \frac{\pi^2}{12} + \frac{\pi^2}{12s^2}$. Moreover, $\langle \widehat{\mathcal{N}}_{g,n} \rangle(L; s)$ is given by the sum over stable graphs of Equation (7.1.2).

Remark 7.1.5. A consequence of the above corollary is an effective way to compute the average number of multicurves over $\mathcal{M}_{g,n}(L)$ with length bounded by t , i.e. $\langle \mathcal{N}_{g,n} \rangle(L; t)$. Indeed, one can use topological recursion to compute $\langle \widehat{\mathcal{N}}_{g,n} \rangle(L; s)$, then inverse Laplace to obtain the actual count. We remark that, in this specific case, the inverse Laplace is computationally easy, as $\langle \widehat{\mathcal{N}}_{g,n} \rangle(L; s)$ is a polynomial in s^{-1} . See Table 7.1 for some multicurve polynomials computed in this way.

The above topological recursion can be translated into the Eynard–Orantin residue form.

PROPOSITION 7.1.6. *Consider the 1-parameter family of spectral curves on \mathbb{P}^1 given by*

$$x(z) = \frac{z^2}{2}, \quad y(z) = -\frac{1}{2\pi} \sin(2\pi z), \quad B(z_1, z_2) = \left(\frac{1}{(z_1 - z_2)^2} + \frac{\pi^2}{s^2} \frac{1}{\sin^2(\pi \frac{z_1 - z_2}{s})} \right) \frac{dz_1 dz_2}{2}, \tag{7.1.8}$$

and define the projection operator (from meromorphic one-forms to meromorphic functions) and Laplace transform operator

$$\mathcal{P}[\omega](z_0) = \operatorname{Res}_{z=0} \frac{\omega(z)}{z_0 - z}, \quad \mathcal{L}[f](z) = \int_{\mathbb{R}_+} f(L) e^{-zL} L dL. \quad (7.1.9)$$

Then the Laplace transform with respect to the cutoff variable and boundary lengths of the average number of hyperbolic multicurves is computed by topological recursion à la Eynard–Orantin on the above spectral curve:

$$\mathcal{P}^{\otimes n}[\omega_{g,n}](z_1, \dots, z_n; s) = \mathcal{L}^{\otimes n}[\langle \widehat{\mathcal{N}}_{g,n} \rangle](z_1, \dots, z_n; s). \quad (7.1.10)$$

Proof. The twisting of geometric recursion amplitudes, described by Theorem 2.4.26 and Corollary 2.4.28, corresponds to a shift in the bidifferential in the spectral curve formalism, as described in Theorem 2.3.11. The precise relation is given in [ABO17, Section 10.3]. In this specific case, twisting Mirzakhani’s initial data by a certain function f corresponds to shifting the standard bidifferential of Mirzakhani’s curve (given in Proposition 2.4.17) by

$$\frac{dz_1 dz_2}{(z_1 - z_2)^2} \mapsto \left(\frac{1}{(z_1 - z_2)^2} + \int_{\mathbb{R}_+} f(\ell) \frac{e^{-(z_1 - z_2)\ell} + e^{-(z_2 - z_1)\ell}}{2} \ell d\ell \right) dz_1 dz_2,$$

The integral on the right-hand side is considered as $z_1, z_2 \rightarrow 0$ (we are then allowed to analytically continue the expression obtained in this way). With the particular choice $f(\ell) = \frac{e^{-s\ell}}{1 - e^{-s\ell}}$, we have

$$\int_{\mathbb{R}_+} \frac{e^{-(z+s)\ell}}{1 - e^{-s\ell}} \ell d\ell = \frac{1}{s^2} \int_{\mathbb{R}_+} \frac{e^{-\frac{z+s}{s}\ell}}{1 - e^{-\ell}} \ell d\ell = \frac{1}{s^2} \psi_1\left(\frac{z+s}{s}\right),$$

where $\psi_1(u) = \frac{d^2}{du^2} \log \Gamma(u)$ is the trigamma function. Using the recurrence relation and the reflection formula for ψ_1 , given by

$$\psi_1(u+1) = \psi_1(u) - \frac{1}{u^2}, \quad \psi_1(1-u) + \psi_1(u) = \frac{\pi^2}{\sin^2(\pi u)},$$

we obtain the thesis. □

7.2 — COUNTING MULTICURVES: THE COMBINATORIAL CASE

As in the spirit of the previous chapters, we develop similar results in the combinatorial setting by replacing hyperbolic length of multicurves by their combinatorial analogue, and the Weil–Petersson measure by the Kontsevich measure.

DEFINITION 7.2.1. Let Σ be a bordered surface of type (g, n) . Define the function counting multicurves with bounded combinatorial length, i.e. $\mathcal{N}_{\Sigma}^{\text{comb}}: \mathcal{T}_{\Sigma}^{\text{comb}}(L) \times \mathbb{R}_+ \rightarrow \mathbb{Z}_+$ defined as

$$\mathcal{N}_{\Sigma}^{\text{comb}}(\mathbb{G}; t) = |\{c \in \mathcal{M}_{\Sigma} \mid \ell_{\mathbb{G}}(c) \leq t\}|. \quad (7.2.1)$$

The function is invariant under the action of the pure mapping class group, and we denote by $\mathcal{N}_{g,n}^{\text{comb}}$ the induced function on the combinatorial moduli space.

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THEOREM 7.2.2. *The Laplace transform of the combinatorial counting function with respect to the cutoff variable t , namely $\widehat{\mathcal{N}}_{\Sigma}^{\text{comb}}(\mathbb{G}; s) = s \int_{\mathbb{R}_+} \mathcal{N}_{\Sigma}^{\text{comb}}(\mathbb{G}; t) e^{-st} dt$, is computed by geometric recursion:*

$$\begin{aligned} \widehat{\mathcal{N}}_{\Sigma}^{\text{comb}}(\mathbb{G}; s) &= \sum_{m=2}^n \sum_{P \in \mathcal{B}_{\Sigma, m}} B^{\text{comb}}(\ell_{\mathbb{G}}(\partial P); s) \widehat{\mathcal{N}}_{\Sigma-P}^{\text{comb}}(\mathbb{G}|_{\Sigma-P}; s) \\ &\quad + \frac{1}{2} \sum_{P \in \mathcal{C}_{\Sigma}} C^{\text{comb}}(\ell_{\mathbb{G}}(\partial P); s) \widehat{\mathcal{N}}_{\Sigma-P}^{\text{comb}}(\mathbb{G}|_{\Sigma-P}; s), \end{aligned} \quad (7.2.2)$$

where B^{comb} and C^{comb} are Kontsevich's kernels twisted by $f(\ell; s) = \frac{e^{-s\ell}}{1-e^{-s\ell}}$, i.e.

$$\begin{aligned} B^{\text{comb}}(L, L', \ell; s) &= \frac{1}{2L} \left([L - L' - \ell]_+ - [-L + L' - \ell]_+ + [L + L' - \ell]_+ \right) + \frac{e^{-s\ell}}{1 - e^{-s\ell}}, \\ C^{\text{comb}}(L, \ell, \ell'; s) &= \frac{1}{L} [L - \ell - \ell']_+ + \frac{e^{-s\ell} e^{-s\ell'}}{(1 - e^{-s\ell})(1 - e^{-s\ell'})} \\ &\quad + \frac{e^{-s\ell}}{1 - e^{-s\ell}} \frac{1}{2L} \left([L - \ell - \ell']_+ - [-L + \ell - \ell']_+ + [L + \ell - \ell']_+ \right) \\ &\quad + \frac{e^{-s\ell'}}{1 - e^{-s\ell'}} \frac{1}{2L} \left([L - \ell' - \ell]_+ - [-L + \ell' - \ell]_+ + [L + \ell' - \ell]_+ \right), \end{aligned} \quad (7.2.3)$$

with initial conditions $\widehat{\mathcal{N}}_P^{\text{comb}}(\mathbb{G}; s) = 1$ for pairs of pants and $\widehat{\mathcal{N}}_T^{\text{comb}}(\mathbb{G}; s) = 1 + \sum_{\gamma \in \mathcal{S}_T} \frac{e^{-s\ell_{\mathbb{G}}(\gamma)}}{1 - e^{-s\ell_{\mathbb{G}}(\gamma)}}$ for one-holed tori.

Proof. A straightforward adaptation of the proof of Theorem 7.1.3: it is a combination of the combinatorial length statistics (Theorem 5.4.1) with the choice $f(\ell) = \frac{e^{-s\ell}}{1 - e^{-s\ell}}$ and the Kontsevich initial data Equation (5.3.1) generating the constant function 1 on the combinatorial Teichmüller space. \square

COROLLARY 7.2.3. *The combinatorial multicurve counting function $\mathcal{N}_{g,n}^{\text{comb}}(X; t)$ is integrable over $(\mathcal{M}_{g,n}^{\text{comb}}(L), \mu_{\mathbb{K}})$, and its average is a homogeneous polynomial of degree $3g - 3 + n$ in L_1^2, \dots, L_n^2, t^2 , symmetric in the length variables. Furthermore, its Laplace transform in the cutoff variable t , namely*

$$\langle \widehat{\mathcal{N}}_{g,n}^{\text{comb}} \rangle(L; s) = s \int_{\mathbb{R}_+} \left(\int_{\mathcal{M}_{g,n}^{\text{comb}}(L)} \mathcal{N}_{g,n}^{\text{comb}}(\mathbb{G}; t) d\mu_{\mathbb{K}}(\mathbb{G}) \right) e^{-st} dt, \quad \Re(s) > 0, \quad (7.2.4)$$

is computed by topological recursion:

$$\begin{aligned} \langle \widehat{\mathcal{N}}_{g,n}^{\text{comb}} \rangle(L_1, \dots, L_n; s) &= \\ &= \sum_{m=2}^n \int_{\mathbb{R}_+} B^{\text{comb}}(L_1, L_m, \ell; s) \langle \widehat{\mathcal{N}}_{g,n-1}^{\text{comb}} \rangle(\ell, L_2, \dots, \widehat{L}_m, \dots, L_n; s) \ell d\ell \\ &\quad + \frac{1}{2} \int_{\mathbb{R}_+^2} C^{\text{comb}}(L_1, \ell, \ell'; s) \left(\langle \widehat{\mathcal{N}}_{g-1, n+1}^{\text{comb}} \rangle(\ell, \ell', L_2, \dots, L_n; s) \right. \\ &\quad \left. + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2, \dots, n\}}} \langle \widehat{\mathcal{N}}_{g_1, 1+|I_1|}^{\text{comb}} \rangle(\ell, L_{I_1}; s) \langle \widehat{\mathcal{N}}_{g_2, 1+|I_2|}^{\text{comb}} \rangle(\ell', L_{I_2}; s) \right) \ell \ell' d\ell d\ell' \end{aligned} \quad (7.2.5)$$

(g, n)	$\langle \mathcal{N}_{g,n}^{\text{comb}} \rangle(L; t)$
(0, 3)	1
(0, 4)	$\frac{1}{2}m(1) + \frac{t^2\pi^2}{4}$
(0, 5)	$\frac{1}{8}m(2) + \frac{1}{2}m(1^2) + \frac{t^2\pi^2}{4}m(1) + \frac{t^4\pi^4}{32}$
(0, 6)	$\frac{1}{48}m(3) + \frac{3}{16}m(2,1) + \frac{3}{4}m(1^3) + \frac{5t^2\pi^2}{48}m(2) + \frac{3t^2\pi^2}{8}m(1^2) + \frac{3t^4\pi^4}{64}m(1) + \frac{t^6\pi^6}{384}$
(1, 1)	$\frac{1}{48}m(1) + \frac{t^2\pi^2}{24}$
(1, 2)	$\frac{1}{192}m(2) + \frac{1}{96}m(1^2) + \frac{t^2\pi^2}{48}m(1) + \frac{t^4\pi^4}{384}$
(1, 3)	$\frac{1}{1152}m(3) + \frac{1}{192}m(2,1) + \frac{1}{96}m(1^3) + \frac{13t^2\pi^2}{2304}m(2) + \frac{t^2\pi^2}{48}m(1^2) + \frac{t^4\pi^4}{384}m(1) + \frac{11t^6\pi^6}{69120}$
(1, 4)	$\frac{1}{9216}m(4) + \frac{1}{768}m(3,1) + \frac{1}{384}m(2^2) + \frac{1}{128}m(2,1^2) + \frac{1}{64}m(1^4) + \frac{5t^2\pi^2}{4608}m(3)$ $+ \frac{13t^2\pi^2}{1536}m(2,1) + \frac{t^2\pi^2}{32}m(1^3) + \frac{61t^4\pi^4}{55296}m(2) + \frac{t^4\pi^4}{256}m(1^2) + \frac{11t^6\pi^6}{46080}m(1) + \frac{t^8\pi^8}{122880}$
(2, 1)	$\frac{1}{442368}m(4) + \frac{t^2\pi^2}{27648}m(3) + \frac{119t^4\pi^4}{3317760}m(2) + \frac{t^6\pi^6}{138240}m(1) + \frac{29t^8\pi^8}{103219200}$
(2, 2)	$\frac{1}{4423680}m(5) + \frac{1}{294912}m(4,1) + \frac{29}{2211840}m(3,2) + \frac{t^2\pi^2}{221184}m(4) + \frac{t^2\pi^2}{18432}m(3,1) + \frac{49t^2\pi^2}{442368}m(2^2)$ $+ \frac{17t^4\pi^4}{2211840}m(3) + \frac{119t^4\pi^4}{2211840}m(2,1) + \frac{t^6\pi^6}{294912}m(2) + \frac{t^6\pi^6}{92160}m(1^2) + \frac{29t^8\pi^8}{68812800}m(1) + \frac{337t^{10}\pi^{10}}{33443020800}$
(3, 1)	$\frac{1}{53508833280}m(7) + \frac{t^2\pi^2}{1274019840}m(6) + \frac{227t^4\pi^4}{76441190400}m(5) + \frac{8203t^6\pi^6}{2407897497600}m(4) + \frac{8107t^8\pi^8}{5350883328000}m(3)$ $+ \frac{1157t^{10}\pi^{10}}{4586471424000}m(2) + \frac{23t^{12}\pi^{12}}{1648072065024}m(1) + \frac{4111t^{14}\pi^{14}}{23138931792936960}$

Table 7.2: The average number of multicurves with combinatorial length bounded by t on an embedded metric ribbon graph of perimeter (L_1, \dots, L_n) for low values of $2g - 2 + n$. Here m_λ is the monomial symmetric polynomial associated to the partition λ , evaluated at L_1^2, \dots, L_n^2 .

with initial data $\langle \widehat{\mathcal{N}}_{0,3}^{\text{comb}} \rangle(L; s) = 1$ and $\langle \widehat{\mathcal{N}}_{1,1}^{\text{comb}} \rangle(L; s) = \frac{L^2}{48} + \frac{\pi^2}{12s^2}$. Moreover, $\langle \widehat{\mathcal{N}}_{g,n}^{\text{comb}} \rangle(L; s)$ is given by the following sum over stable graphs:

$$\sum_{\Gamma \in \mathcal{G}_{g,n}} \frac{1}{|\text{Aut}(\Gamma)|} \int_{\mathbb{R}_+^{E_\Gamma}} \prod_{v \in V_\Gamma} V_{g(v),n(v)}^{\text{K}}((\ell_e)_{e \in E_v}, (L_\lambda)_{\lambda \in \Lambda_v}) \prod_{e \in E_\Gamma} \frac{\ell_e e^{-s\ell_e} d\ell_e}{1 - e^{-s\ell_e}}. \quad (7.2.6)$$

Again, we can consider the topological recursion à la Eynard–Orantin for the combinatorial counting function.

PROPOSITION 7.2.4. Consider the 1-parameter family of spectral curves on \mathbb{P}^1 given by

$$x(z) = \frac{z^2}{2}, \quad y(z) = -z, \quad B(z_1, z_2) = \left(\frac{1}{(z_1 - z_2)^2} + \frac{\pi^2}{s^2} \frac{1}{\sin^2(\pi \frac{z_1 - z_2}{s})} \right) \frac{dz_1 dz_2}{2}, \quad (7.2.7)$$

and define the projection operator and Laplace transform operator

$$\mathcal{P}[\omega](z_0) = \text{Res}_{z=0} \frac{\omega(z)}{z_0 - z}, \quad \mathcal{L}[f](z) = \int_{\mathbb{R}_+} f(L) e^{-zL} L dL. \quad (7.2.8)$$

Then the Laplace transform with respect to the cutoff variable and boundary lengths of the average number of combinatorial multicurves is computed by topological recursion à la Eynard–Orantin on the above spectral curve:

$$\mathcal{P}^{\otimes n}[\omega_{g,n}](z_1, \dots, z_n; s) = \mathcal{L}^{\otimes n}[\langle \widehat{\mathcal{N}}_{g,n}^{\text{comb}} \rangle](z_1, \dots, z_n; s). \quad (7.2.9)$$

7.3 — QUADRATIC DIFFERENTIALS WITH DOUBLE POLES

An alternative description of the count of multicurves in the combinatorial Teichmüller space can be depicted as follows. For any combinatorial structure \mathbb{G} and multicurve c on Σ , consider the non-backtracking representative of the primitive components γ_i of c along \mathbb{G} . If γ_i has multiplicity m_i , we can imagine to cut \mathbb{G} along γ_i and glue a flat cylinder of height m_i . The resulting structure is not a combinatorial structure anymore (embedded ribbon graphs do not form cylinders except for the ones around boundary components). However, we can realise such structure as a quadratic differential with double poles.

Following this intuition, in this section we introduce quadratic differentials with double poles and their counting. We refer to [Str67; Zvo02; BK20] for further readings.

DEFINITION 7.3.1. Let $g, n \geq 0$ such that $2g - 2 + n > 0$, and fix a tuple $L = (L_1, \dots, L_n) \in \mathbb{R}_+^n$. We consider the moduli space $\mathcal{MQ}_{g,n}(L)$ parametrising *quadratic differential with double poles*, i.e. the moduli space of tuples $S = (C, x_1, \dots, x_n, \varphi)$ where

- C is a compact Riemann surface of genus g with n labeled marked points x_1, \dots, x_n ,
- φ is a meromorphic quadratic differential on C , holomorphic on $C - \{x_1, \dots, x_n\}$ and with double poles at x_i of residue²

$$\operatorname{Res}_{x_i} \varphi = -\left(\frac{L_i}{2\pi}\right)^2. \quad (7.3.1)$$

When we do not fix the residue at the double poles, which can be any complex number, we denote the associated moduli space of quadratic differentials by $\mathcal{MQ}_{g,n}$.

The local geometry induced by a quadratic differential φ as above can be described as follows.

- If x is a point on C that is neither a zero nor a double pole, then φ has a square root in a small neighbourhood of x . That is, there exists a holomorphic differential, unique up to a sign, such that $\omega^2 = \varphi$. Then the integral $\int_x^z \omega$ is a biholomorphic mapping from a neighborhood of x in C to a neighbourhood of 0 in \mathbb{C} , and the preimages of the horizontal (resp. vertical) lines in \mathbb{C} under this mapping are called *horizontal* (resp. *vertical*) *trajectories* of the quadratic differential φ .
- If x_0 is a zero of φ of order m , then there are $m + 2$ horizontal trajectories issuing from x_0 , also called *saddle connections*.
- If x_i is a double pole, then x_i is surrounded by closed horizontal trajectories forming a semi-infinite cylinder.

From this analysis, it should be clear that a quadratic differential determines a horizontal and a vertical foliation on C . Moreover, the saddle connections are precisely the singular leaves of

²If x_i is a double pole for φ , then in a neighborhood of x_i the quadratic differential φ has an expansion

$$\varphi = a_i \frac{dz^2}{(z - x_i)^2} + \dots$$

The number $a_i \in \mathbb{C}$, which does not depend on the choice of the local coordinate, is called the residue of φ at x_i .

the horizontal foliation, joining the singular points of the foliation that correspond to zeros of the differential.

Another description of the geometry induced on C comes from the fact that φ induces a flat metric on $C - \{x_1, \dots, x_n\}$ with conical singularities at the zeros of φ . The geometry of the flat metric in a neighbourhood of a double pole x_i of residue $-(\frac{L_i}{2\pi})^2$ is that of a semi-infinite cylinder as described above, whose circumference (or width) is precisely L_i . The point x_i itself is at infinite distance from the rest of the surface. For each double pole, there is a maximal such semi-infinite cylinder that avoids the zeros of φ . We call the *convex core* of $(C, x_1, \dots, x_n, \varphi)$ the surface obtained by removing the union of the maximal open semi-infinite cylinders around each pole (the surface might be degenerate, i.e. a union of straight line segments; see Figure 7.1b). The convex core with the metric induced from φ is still a flat metric and has n horizontal boundaries of lengths L_1, \dots, L_n which is the union of saddle connections bounding the maximal half-infinite cylinder around x_1, \dots, x_n respectively. The *core area* of $(C, x_1, \dots, x_n, \varphi)$ denoted $\text{CoreArea}(C, x_1, \dots, x_n, \varphi)$ is the area of the convex core of $(C, x_1, \dots, x_n, \varphi)$. It is a finite non-negative real number (contrarily to the area of $C - \{x_1, \dots, x_n\}$).

The space $\mathcal{MQ}_{g,n}(L)$ admits a stratification with respect to the degree of the zeros. More precisely, fix a partition (μ, ν) of $4g - 4 + 2n$, where $\mu = (2\alpha_a)_{a=1}^r$ are the even part and $\nu = (2\beta_b - 1)_{b=1}^s$ are the odd parts, with $m_a, n_b > 0$. Define the stratum $\mathcal{MQ}_{g,n}^{\mu,\nu}(L)$ parametrising quadratic differential whose associated divisor is of the form

$$(\varphi) = \sum_{a=1}^r 2\alpha_a z_a + \sum_{b=1}^s (2\beta_b - 1) z'_b - \sum_{i=1}^n 2x_i. \quad (7.3.2)$$

Notice that $\deg(\varphi) = 4g - 4$, and in particular s is even.

When we do not fix the residue at the double poles, the anti-invariant relative homology of the canonical double cover provides a convenient coordinate system on each stratum $\mathcal{MQ}_{g,n}^{\mu,\nu}$, called *period coordinates*, that we now describe.

For each point $(C, x_1, \dots, x_n, \varphi) \in \mathcal{MQ}_{g,n}^{\mu,\nu}$, consider the canonical double cover $\pi: \hat{C} \rightarrow C$, ramified over the odd zeros of φ , which comes with a canonical (up to sign) meromorphic one-form ω such that $\omega^2 = \pi^*\varphi$ and a canonical involution $\tau: \hat{C} \rightarrow \hat{C}$ interchanging the sheets. The genus of \hat{C} is computed by Riemann–Hurwitz as $\hat{g} = 2g - 1 + s/2$. The even zeros and double poles of φ , whose associate divisors are denoted by Z and P respectively, have two preimages under π , and we denote by \hat{Z} and \hat{P} the associated divisors on \hat{C} . In particular, we can consider the τ -anti-invariant part of the relative homology group $H_1^-(\hat{C} - \hat{P}, \hat{Z}; \mathbb{Z})$, which has rank $2g - 2 + n + r + s$. Indeed, we can realise

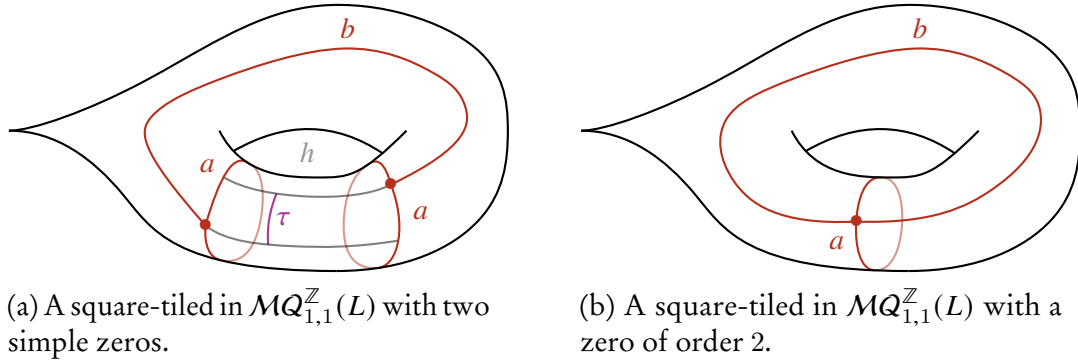
$$H_1^-(\hat{C} - \hat{P}, \hat{Z}; \mathbb{Z}) = H_1(\hat{C} - \hat{P}, \hat{Z}; \mathbb{Z}) / \pi^* H_1(C - P, Z; \mathbb{Z}), \quad (7.3.3)$$

from which we get the rank

$$\begin{aligned} h_1^-(\hat{C} - \hat{P}, \hat{Z}; \mathbb{Z}) &= h_1(\hat{C} - \hat{P}, \hat{Z}; \mathbb{Z}) - h_1(C - P, Z; \mathbb{Z}) \\ &= (2\hat{g} + 2n + 2r - 1) - (2g + n + r - 1) \\ &= 2g - 2 + n + r + s. \end{aligned} \quad (7.3.4)$$

Fix now a point $(C_0, x_0, \varphi_0) \in \mathcal{MQ}_{g,n}^{\mu,\nu}$ with a neighbourhood U and a basis $\{\gamma_{0,i}\}_{i=1}^{2g-2+n+r+s}$ of $H_1^-(\hat{C}_0 - \hat{P}_0, \hat{Z}_0; \mathbb{Z})$, providing a homeomorphism

$$H_1^-(\hat{C}_0 - \hat{P}_0, \hat{Z}_0; \mathbb{Z}) \cong \mathbb{Z}^{2g-2+n+r+s}. \quad (7.3.5)$$


 Figure 7.1: Square-tiled surface of type $(1, 1)$.

For $(C, x, \varphi) \in U$, there exists $\pi: \hat{C} \rightarrow C$ and τ with the corresponding properties, since the canonical double cover can be constructed in families. In particular, we can parallel transport the basis $\{\gamma_{0,i}\}_i$ from $H_1^-(\hat{C}_0 - \hat{P}_0, \hat{Z}_0; \mathbb{Z})$ to a basis $\{\gamma_i\}_i$ of $H_1^-(\hat{C} - \hat{P}, \hat{Z}; \mathbb{Z})$. Integration of ω along this basis provides a map from

$$U \longrightarrow \mathbb{C}^{2g-2+n+r+s}, \quad (C, x, q) \longmapsto \left(\oint_{\gamma_i} \omega \right)_{i=1}^{2g-2+n+r+s}, \quad (7.3.6)$$

and this is the desired coordinate system. Notice that, for the principal stratum parametrising quadratic differential with simple zeros (i.e. $\mu = \emptyset$ and $\nu = 1^{4g-4+2n}$), the dimension is $6g-6+2n$. To summarise, we have the following result.

PROPOSITION 7.3.2. *The moduli space $\mathcal{MQ}_{g,n}$ is a complex orbifold of dimension $6g-6+3n$. The stratum $\mathcal{MQ}_{g,n}^{\mu,\nu}$ parametrising quadratic differentials with zeros of order (μ, ν) has dimension $2g-2+n+\ell(\mu)+\ell(\nu)$.*

7.3.1 — COUNTING SQUARE TILED-SURFACES

The moduli space $\mathcal{MQ}_{g,n}(L)$ has a natural integral structure, denoted by $\mathcal{MQ}_{g,n}^{\mathbb{Z}}(L)$: when all the L_i are integral and all periods are in $\mathbb{Z} \oplus i\mathbb{Z}$, we say that the surface is *square-tiled* (notice that the set $\mathcal{MQ}_{g,n}^{\mathbb{Z}}(L)$ is empty if $\sum_{i=1}^n L_i$ is odd). A square-tiled surface can be obtained by gluing side by side as many squares as the core area (which is integral) and leaving open some of the horizontal sides forming n topological circles of length $L = (L_1, \dots, L_n)$, on which we can attach semi-infinite cylinders.

Contrary to the combinatorial moduli space, the set of square-tiled surfaces with fixed residue is infinite. However, once the core area is fixed, there are only finitely many square-tiled surfaces. In particular, this means that we can count integral points in $\mathcal{MQ}_{g,n}^{\mathbb{Z}}(L)$ weighting them by their core area:

$$N_{g,n}^{\square}(L_1, \dots, L_n; q) = \sum_{S \in \mathcal{MQ}_{g,n}^{\mathbb{Z}}(L)} \frac{q^{\text{CoreArea}(S)}}{|\text{Aut}(S)|}, \quad (7.3.7)$$

EXAMPLE 7.3.3. Let us describe the set $\mathcal{MQ}_{1,1}^{\mathbb{Z}}(L)$. Its intersection with the principal stratum can be parametrised as follows (see Figure 7.1a).

- There are two positive integers a and b parametrising the length of the saddle connections of φ . If we fix the residue at the double pole to be $-(\frac{L}{2\pi})^2$, then we find $2(a+b) = L$.
- We have a non-negative integer h parametrising the height of the core cylinder separating the two zeros of φ .
- There is a “twist” parameter $\tau \in \mathbb{Z} \cap [0, a)$, measuring the distance between two vertical trajectories issuing from the zeros of φ .

Similarly, the intersection of $\mathcal{MQ}_{1,1}^{\mathbb{Z}}(L)$ with the codimension 1 stratum can be parametrised by two positive integers a and b satisfying $2(a+b) = L$, corresponding to the length of the saddle connections of φ (see Figure 7.1b).

In order to count square-tiled surfaces in $\mathcal{MQ}_{1,1}^{\mathbb{Z}}(L)$, let us distinguish between zero and non-zero core area. In the former case, it should be clear from the analysis above that there is an homeomorphism of 0-dimensional orbifolds

$$\left\{ S \in \mathcal{MQ}_{1,1}^{\mathbb{Z}}(L) \mid \text{CoreArea}(S) = 0 \right\} \cong \mathcal{M}_{1,1}^{\text{comb}, \mathbb{Z}}(L).$$

In particular, there are $N_{1,1}(L)$ square-tiled surface of type $(1, 1)$ with vanishing core area (here $N_{g,n}(L)$ denotes the number of lattice points in the combinatorial moduli space, cf. Corollary 5.3.4). If the core area is non-vanishing (see Figure 7.1a with $h \geq 1$), we can count such square-tiled surfaces by fixing an integral metric ribbon graph in $\mathcal{M}_{0,3}^{\text{comb}, \mathbb{Z}}(L, a, a)$, parametrising the horizontal trajectories issuing from the zeros of the quadratic differential, and glue a core cylinder of height h with a twist $\tau \in \mathbb{Z} \cap [0, a)$. In this case we obtain the count

$$\frac{1}{2} \sum_{a \geq 1} N_{0,3}(L, a, a) \sum_{h \geq 1} \sum_{0 \leq \tau < a} q^{ah} = \frac{1}{2} \sum_{a \geq 1} N_{0,3}(L, a, a) \frac{aq^a}{1 - q^a}.$$

Here the $\frac{1}{2}$ takes into account the non-trivial automorphism of such square-tiled surfaces (the elliptic involution). All in all, we obtain the q -series

$$N_{1,1}^{\square}(L; q) = N_{1,1}(L) + \frac{1}{2} \sum_{a \geq 1} N_{0,3}(L, a, a) \frac{aq^a}{1 - q^a}.$$

We can generalise the above argument as follows. First, note that square-tiled surfaces with vanishing core area are exactly the Jenkins–Strebel differentials [Jen57; Str67], i.e. differentials all of whose period coordinates are purely real. Hence, the constant coefficient of $N_{g,n}^{\square}(L; q)$ (seen as a q -series) is the number of lattice points in the combinatorial moduli space:

$$N_{g,n}^{\square}(L; q) = N_{g,n}(L) + O(q). \tag{7.3.8}$$

Moreover, if the core area is non-vanishing, we can obtain such square-tiled surfaces by gluing core cylinders on integral metric ribbon graphs by specifying the heights of the cylinders and a twist. As a result, we obtain the counting of square-tiled surfaces as a sum over stable graphs, with vertex weights given by the integral metric ribbon graphs counting $N_{g,n}(L_1, \dots, L_n)$ and edge weights given by the function $\frac{\ell q^{\ell}}{1 - q^{\ell}}$, where ℓ is the width of the glued cylinder.

III. Enumeration of multicurves and quadratic differentials

PROPOSITION 7.3.4. *Let $g, n \geq 0$ be non-negative integers such that $2g - 2 + n > 0$ and let $(L_1, \dots, L_n) \in \mathbb{Z}_+^n$. We have*

$$N_{g,n}^\square(L; q) = \sum_{\Gamma \in \mathcal{G}_{g,n}} \frac{1}{|\text{Aut}(\Gamma)|} \sum_{\ell: E_\Gamma \rightarrow \mathbb{Z}_+} \prod_{v \in V_\Gamma} N_{g(v), n(v)}((\ell_e)_{e \in E(v)}, (L_\lambda)_{\lambda \in \Lambda(v)}) \prod_{e \in E_\Gamma} \frac{\ell_e q^{\ell_e}}{1 - q^{\ell_e}}, \quad (7.3.9)$$

where $N_{g,n}(\ell_1, \dots, \ell_n)$ are the number of lattice points in the combinatorial moduli space computed by Norbury's recursion (cf. Corollary 5.3.4).

Proof. Each square-tiled surface admits a decomposition into horizontal cylinders and saddle connections between the zeros of the differential q . The union of all saddle connections forms a union of ribbon graphs that we call the *singular layer* of the square-tiled surface. To such decomposition, we associate a stable graph Γ by the following rule.

- A vertex in Γ corresponds to a connected component of the singular layer, where the genus and number of half-edges are respectively the genus and the number of faces of the associated ribbon graph.
- An edge of Γ between two vertices correspond to a core cylinder, whose ends belong to the components of the singular layer corresponding to the two vertices. Note that each end is a face of the corresponding ribbon graph.

Let us now fix a stable graph Γ in $\mathcal{G}_{g,n}$. We claim that the term

$$\frac{1}{|\text{Aut}(\Gamma)|} \sum_{\ell: E_\Gamma \rightarrow \mathbb{Z}_+} \prod_{v \in V_\Gamma} N_{g(v), n(v)}((\ell_e)_{e \in E(v)}, (L_\lambda)_{\lambda \in \Lambda(v)}) \prod_{e \in E_\Gamma} \frac{\ell_e q^{\ell_e}}{1 - q^{\ell_e}}$$

appearing in the right-hand side of Equation (7.3.9) is the generating series of square-tiled surfaces whose associated stable graph is Γ . Indeed, to reconstruct the singular layer one needs to choose a ribbon graph for each vertex $v \in V_\Gamma$ and fix the lengths of each edge. This is counted by the term $N_{g(v), n(v)}((\ell_e)_{e \in E(v)}, (L_\lambda)_{\lambda \in \Lambda(v)})$. Next, one needs to reconstruct the core cylinders. The cylinders are glued on faces of ribbon graphs and have a height parameter h (which is a positive integer) and a twist parameter τ (a non-negative integer strictly smaller than ℓ_i). The generating series for this cylinder is just

$$\sum_{h \geq 1} \sum_{0 \leq \tau < \ell_e} q^{\ell_e h} = \frac{\ell_e q^{\ell_e}}{1 - q^{\ell_e}}.$$

This concludes the proof. □

The expression for the q -enumeration of square-tiled surfaces is an example of the discrete twisting procedure presented in Proposition 5.4.4, with the function $f(\ell) = \frac{q^\ell}{1 - q^\ell}$. In particular, we can compute it by discrete topological recursion.

COROLLARY 7.3.5. *The counting of square-tiled surfaces satisfies the discrete topological recursion:*

$$\begin{aligned}
 N_{g,n}^\square(L_1, \dots, L_n; q) &= \sum_{m=2}^n \sum_{\ell \geq 1} \ell B^\square(L_1, L_m, \ell; q) N_{g,n-1}^\square(\ell, L_2, \dots, \widehat{L_m}, \dots, L_n; q) \\
 &\quad + \frac{1}{2} \sum_{\ell, \ell' \geq 1} \ell \ell' C^\square(L_1, \ell, \ell'; q) \left(N_{g-1, n+1}^\square(\ell, \ell', L_2, \dots, L_n; q) \right. \\
 &\quad \left. + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2, \dots, n\}}} N_{g_1, 1+|I_1|}^\square(\ell, L_{I_1}; q) N_{g_2, 1+|I_2|}^\square(\ell', L_{I_2}; q) \right), \tag{7.3.10}
 \end{aligned}$$

where we have $N_{0,1}^\square = N_{0,2}^\square = 0$ by convention, the recursion kernels are defined as

$$\begin{aligned}
 B^\square(L, L', \ell; q) &= \frac{1}{2L} \left([L - L' - \ell]_+ - [-L + L' - \ell]_+ + [L + L' - \ell]_+ \right) + \frac{q^\ell}{1 - q^\ell}, \\
 C^\square(L, \ell, \ell'; q) &= \frac{1}{L} [L - \ell - \ell']_+ + \frac{q^{\ell+\ell'}}{(1 - q^\ell)(1 - q^{\ell'})} + \\
 &\quad + \frac{q^\ell}{1 - q^\ell} \frac{1}{2L} \left([L - \ell - \ell']_+ - [-L + \ell - \ell']_+ + [L + \ell - \ell']_+ \right) \\
 &\quad + \frac{q^{\ell'}}{1 - q^{\ell'}} \frac{1}{2L} \left([L - \ell' - \ell]_+ - [-L + \ell' - \ell]_+ + [L + \ell' - \ell]_+ \right), \tag{7.3.11}
 \end{aligned}$$

for integral lengths summing up to an even integer and zero otherwise, and initial conditions

$$\begin{aligned}
 N_{0,3}^\square(L_1, L_2, L_3; q) &= N_{0,3}(L_1, L_2, L_3), \\
 N_{1,1}^\square(L) &= N_{1,1}(L) + \frac{1}{2} \sum_{\ell \geq 1} N_{0,3}(L, \ell, \ell) \frac{\ell q^\ell}{1 - q^\ell} \tag{7.3.12}
 \end{aligned}$$

This result can also be brought in the residue formalism of Eynard–Orantin. To this end, let us recall the spectral curve computing lattice points on the combinatorial moduli space.

THEOREM 7.3.6 ([Nor13]). *Consider the spectral curve on \mathbb{P}^1 given by*

$$x(z) = z + \frac{1}{z}, \quad y(z) = -z, \quad B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}. \tag{7.3.13}$$

Then the associated topological recursion differentials $\omega_{g,n}(z)$ compute the number of lattice points on $\mathcal{M}_{g,n}^{\text{comb}, \mathbb{Z}}(L)$: for any $L \in \mathbb{Z}_+^n$,

$$N_{g,n}(L_1, \dots, L_n) = (-1)^n \left(\prod_{i=1}^n \text{Res}_{z_i = \infty} \frac{z_i^{L_i}}{L_i} \right) \omega_{g,n}(z_1, \dots, z_n). \tag{7.3.14}$$

By shifting the bidifferential B , we can obtain a topological recursion for $N_{g,n}^\square$ via Theorem 2.3.11.

PROPOSITION 7.3.7. *Consider the spectral curve differing from (7.3.13) only for the choice of*

$$B^\square(z_1, z_2; q) = \frac{1}{2} \frac{dz_1 dz_2}{(z_1 - z_2)^2} + \frac{1}{2} \left(\wp(u_1 - u_2; q) + \frac{\pi^2 E_2(q)}{3} \right) du_1 du_2, \tag{7.3.15}$$

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where $z_k = \exp(2\pi i u_k)$, $\wp(u; q)$ is the Weierstrass function for the elliptic curve $\mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ where $q = e^{2\pi i \tau}$, and $E_2(q)$ is the second Eisenstein series

$$E_2(q) = 1 - 24 \sum_{\ell \geq 1} \frac{\ell q^\ell}{1 - q^\ell}. \quad (7.3.16)$$

Then the associated topological recursion differentials $\omega_{g,n}^\square(z; q)$ compute the number of square-tiled surfaces: for any $L \in \mathbb{Z}_+^n$,

$$N_{g,n}^\square(L_1, \dots, L_n; q) = (-1)^n \left(\prod_{i=1}^n \operatorname{Res}_{z_i=\infty} \left(1 + \frac{q^{L_i}}{2(1-q^{L_i})} \right)^{-1} \frac{z_i^{L_i}}{L_i} \right) \omega_{g,n}^\square(z_1, \dots, z_n; q). \quad (7.3.17)$$

Proof. We first make some preliminary computations. First, the vector space \mathcal{V} from Theorem 2.3.11 can be identified with the space of meromorphic 1-forms ϕ on \mathbb{P}^1 whose poles are located at ± 1 and such that $\phi(z) + \phi(1/z) = 0$. Moreover, we claim that the operator $\mathcal{O}: \operatorname{Sym}^2 \mathcal{V} \rightarrow \mathbb{C}[[q]]$ from Theorem 2.3.11 can be realised as

$$\mathcal{O}[\varpi] = \sum_{\ell \geq 1} \frac{\ell q^\ell}{1 - q^\ell} \operatorname{Res}_{z_1=\infty} \operatorname{Res}_{z_2=\infty} \frac{z_1^\ell z_2^\ell}{\ell^2} \varpi(z_1, z_2).$$

Indeed, since elements of \mathcal{V} are odd under the involution $z \mapsto 1/z$, we can write

$$\mathcal{O}[\varpi] = -\frac{1}{2} \left(\operatorname{Res}_{z_1=\infty} \operatorname{Res}_{z_2=0} O_q \left(\frac{z_1}{z_2} \right) \varpi(z_1, z_2) + \operatorname{Res}_{z_1=0} \operatorname{Res}_{z_2=\infty} O_q \left(\frac{z_2}{z_1} \right) \varpi(z_1, z_2) \right),$$

where $O_q(z) = \sum_{\ell \geq 1} \frac{q^\ell z^\ell}{\ell(1-q^\ell)} \in \mathbb{C}[z][[q]]$. Recall the expansion of the Weierstrass function when $u \rightarrow 0$, expressed in terms of the $(2m)$ -th Eisenstein series:

$$\wp(u; q) = \frac{1}{u^2} + \sum_{k \geq 1} 2(2k+1)G_{2k+2}(q)u^{2k}, \quad G_{2m}(q) = \zeta(2m) + \frac{(2\pi i)^{2m}}{(2m-1)!} \sum_{\ell \geq 1} \frac{\ell^{2m-1} q^\ell}{1 - q^\ell}.$$

From the identity $\sum_{k \geq 0} \zeta(2k)u^{2k} = -\frac{\pi u}{2} \cotan(\pi u)$, we deduce that

$$\sum_{k \geq 0} 2(2k+1)\zeta(2k+2)u^{2k} = \frac{\pi^2}{\sin^2 \pi u} - \frac{1}{u^2}.$$

Adding/subtracting the $k = 0$ term in the expansion of the Weierstrass function and computing separately the contribution of the Riemann zeta values yields

$$\begin{aligned} \wp(u; q) &= \frac{\pi^2}{\sin^2 \pi u} - 2G_2(q) + \sum_{\ell \geq 1} \frac{q^\ell}{1 - q^\ell} \sum_{k \geq 0} \frac{2(2\pi i)^{2k+2} \ell^{2k+1}}{(2k)!} u^{2k} \\ &= \frac{\pi^2}{\sin^2 \pi u} - 2G_2(q) + (2\pi i)^2 \sum_{\ell \geq 1} \frac{\ell q^\ell (z^\ell + z^{-\ell})}{1 - q^\ell}, \end{aligned}$$

where we have set $z = e^{2\pi i u}$. Since $E_2(q) = \frac{6}{\pi^2} G_2(q)$, setting $z_k = e^{2\pi i u_k}$ yields

$$\begin{aligned} &\left(\wp(u_1 - u_2; q) + \frac{\pi^2}{3} E_2(q) \right) du_1 du_2 = \\ &= \left(\frac{1}{(z_1 - z_2)^2} + \frac{1}{z_1 z_2} \sum_{\ell \geq 1} \frac{\ell q^\ell ((z_1/z_2)^\ell + (z_2/z_1)^\ell)}{1 - q^\ell} \right) dz_1 dz_2 \\ &= \left(\frac{1}{(z_1 - z_2)^2} - \operatorname{Res}_{z'_1=\infty} \operatorname{Res}_{z'_2=0} \frac{O_q(z'_1/z'_2) dz'_1 dz'_2}{(z_1 - z'_1)^2 (z_2 - z'_2)^2} - \operatorname{Res}_{z'_1=0} \operatorname{Res}_{z'_2=\infty} \frac{O_q(z'_2/z'_1) dz'_1 dz'_2}{(z_1 - z'_1)^2 (z_2 - z'_2)^2} \right) dz_1 dz_2. \end{aligned}$$

Hence, we find

$$\begin{aligned} B(z_1, z_2) &- \frac{1}{2} \left(\operatorname{Res}_{z_1=\infty} \operatorname{Res}_{z_2=0} O_q \left(\frac{z'_1}{z'_2} \right) B(z_1, z'_1) B(z_2, z'_2) + \operatorname{Res}_{z_1=0} \operatorname{Res}_{z_2=\infty} O_q \left(\frac{z'_2}{z'_1} \right) B(z_1, z'_1) B(z_2, z'_2) \right) \\ &= \frac{1}{2} \frac{dz_1 dz_2}{(z_1 - z_2)^2} + \frac{1}{2} \left(\wp(u_1 - u_2; q) + \frac{\pi^2 E_2(q)}{3} \right) du_1 du_2, \end{aligned}$$

which we took as definition for $B^\square(z_1, z_2; q)$. This proves the above claim. We then apply Theorem 2.3.11, which expresses $\omega_{g,n}^\square(z; q)$ as a sum over stable graphs, with vertex weights given by $\omega_{g(v),n(v)}$, the operator \odot acting on each edge, and the operator

$$\tilde{\mathcal{P}}[\phi](z_0) = \sum_{\alpha \in \{-1, 1\}} \operatorname{Res}_{z=\alpha} \left(\int^z B^\square(\cdot, z_0; q) \right) \phi(z), \quad \phi \in \mathcal{V}$$

acting on each leaf. Here we can chose an arbitrary primitive of $B^\square(\cdot, z_0; q)$, and the final result does not depend on this choice since it is not changing the residue. We also remark that this expression should be considered an equality of q -series. It remains to compute the expansion of $\omega_{g,n}^\square$ near $z_i \rightarrow \infty$ (more precisely, one should expand as a q -series, and then expand each term when $z_i \rightarrow \infty$). For $\phi \in \mathcal{V}$, we find

$$\begin{aligned} - \operatorname{Res}_{z_0=\infty} z_0^L \tilde{\mathcal{P}}[\phi](z_0) &= - \sum_{\alpha \in \{-1, 1\}} \operatorname{Res}_{z=\alpha} \operatorname{Res}_{z_0=\infty} z_0^L \left(\int^z B^\square(\cdot, z_0; q) \right) \phi(z) \\ &= - \sum_{\alpha \in \{-1, 1\}} \operatorname{Res}_{z=\alpha} \phi(z) \operatorname{Res}_{z_0=\infty} z_0^L \left(\frac{1}{z_0 - z} + \frac{1}{2} \sum_{\ell \geq 1} \frac{q^\ell}{1 - q^\ell} (-z_0^{\ell-1} z^{-\ell} + z^\ell z_0^{-(\ell+1)}) \right) \\ &= \sum_{\alpha \in \{-1, 1\}} \operatorname{Res}_{z=\alpha} \phi(z) z^L \left(1 + \frac{q^L}{2(1 - q^L)} \right) \\ &= - \operatorname{Res}_{z=\infty} \phi(z) z^L \left(1 + \frac{q^L}{2(1 - q^L)} \right). \end{aligned}$$

Recalling (7.3.14), we deduce that

$$(-1)^n \left(\prod_{i=1}^n \operatorname{Res}_{z_i=\infty} \left(1 + \frac{q^{L_i}}{2(1 - q^{L_i})} \right)^{-1} \frac{z_i^{L_i}}{L_i} \right) \omega_{g,n}^\square(z_1, \dots, z_n; q)$$

coincides with the right-hand side of (7.3.17) and this concludes the proof. \square

CHAPTER 8 — ASYMPTOTIC COUNTING AND MASUR–VEECH VOLUMES

In [Miro8b] Mirzakhani consider not only the problem of counting multicurves of length $\leq t$ on a hyperbolic surfaces, but also its asymptotic as $t \rightarrow +\infty$. In [Miro8a] she shows that the average number over the moduli space of this limiting value gives the Masur–Veech volume of the moduli space of quadratic differentials with *simple* poles (as opposed to the previous section, where we considered quadratic differentials with *double* poles).

In this chapter we are going to review Mirzakhani’s analysis of the asymptotic number of multicurves in the hyperbolic setting, and perform a similar analysis in the combinatorial one:

$$\mathcal{B}_\Sigma(\sigma) = \lim_{t \rightarrow \infty} \frac{\mathcal{N}_\Sigma(\sigma; t)}{t^{6g-6+2n}}, \quad \mathcal{B}_\Sigma^{\text{comb}}(\mathbb{G}) = \lim_{t \rightarrow \infty} \frac{\mathcal{N}_\Sigma^{\text{comb}}(\mathbb{G}; t)}{t^{6g-6+2n}}. \quad (8.0.1)$$

Surprisingly, we find that such asymptotic value does not depend on the boundary lengths, is the same in both the hyperbolic and combinatorial setting, and (thanks to Mirzakhani’s result) it coincides with the Masur–Veech volume $V_{g,n}^{\text{MV}}$ of the principal stratum of the moduli space of quadratic differentials.

THEOREM 8.A (Asymptotics of multicurves and Masur–Veech volumes). *The following equalities hold:*

$$V_{g,n}^{\text{MV}} \sim \int_{\mathcal{M}_{g,n}(L)} \mathcal{B}_{g,n} d\mu_{\text{WP}} = \int_{\mathcal{M}_{g,n}^{\text{comb}}(L)} \mathcal{B}_{g,n}^{\text{comb}} d\mu_{\mathbb{K}}, \quad (8.0.2)$$

where \sim means “equality up to a combinatorial factor”. Moreover, both quantities on the right-hand side are independent of $L \in \mathbb{R}_+^n$.

In particular, thanks to the analysis of the previous chapter, we can express the Masur–Veech volumes as a sum over stable graph and as the constant term of a family of polynomials computed by topological recursion. Employing the computational power of topological recursion, we were able to formulate a series of conjectures regarding the polynomial structure of Masur–Veech volumes for fixed genus and varying number of poles (see Table 8.2 for a list of such polynomials for $g \leq 8$).

CONJECTURE 8.B (Polynomial structure of Masur–Veech volumes). *For any $g \geq 0$, there exist polynomials $a_g, b_g \in \mathbb{Q}[n]$ such that, for any $n \geq 0$,*

$$V_{g,n}^{\text{MV}} = 2^{4g-2+n} \pi^{6g-6+2n} \frac{(2g-3+n)!(4g-4+n)!}{(6g-7+2n)!} \left(a_g(n) + \frac{1}{2^{4g-6+2n}} \binom{4g-6+2n}{2g-3+n} b_g(n) \right). \quad (8.0.3)$$

Thanks to Goujard’s work [Gou15], we also derive a similar conjecture for area Siegel–Veech constants, and get explicit asymptotics as $n \rightarrow \infty$ for both Masur–Veech volumes and Siegel–Veech constants.

We conclude by connecting the asymptotic count of square-tiled surfaces with double poles to Masur–Veech volumes.

THEOREM 8.C (Asymptotics of square-tiles and Masur–Veech volumes). *Let $L \in \mathbb{Z}_+^n$ such that $L_1 + \dots + L_n \in 2\mathbb{Z}$. The limit*

$$\lim_{\substack{T \rightarrow \infty \\ T \in 2\mathbb{Z}_+}} \frac{N_{g,n}^\square(TL_1, \dots, TL_n; e^{-1/T})}{T^{6g-6+2n}} \quad (8.0.4)$$

exists, is a polynomial in L_1^2, \dots, L_n^2 , and its constant term is the Masur–Veech volumes $V_{g,n}^{\text{MV}}$ (up to a combinatorial factor).

8.0.1 — RELATION WITH OTHER WORKS AND OPEN QUESTIONS

As we will discuss in the next chapter, Conjecture 8.B was proved shortly after its formulation by Chen–Möller–Sauvaget [CMS+19] using algebraic-geometric tools.

Notice also that the theorems above are the realisation of Masur–Veech volumes as limits of multicurve counts in the hyperbolic and combinatorial setting, as well as a limit of the square-tiled surface count. For all such results, we relay here on the Mirzakhani’s work concerning the hyperbolic multicurve count. It would be interesting to prove such results in an independent way.

QUESTION 8.D. *Prove independently that the Masur–Veech volumes of the principal stratum of the moduli space of quadratic differentials with simple poles are realised through:*

- *the asymptotic count of combinatorial multicurves,*
- *the asymptotic count of square-tiled surfaces with double poles.*

A hint towards this can be found in the work of Delecroix–Goujard–Zograf–Zorich [DGZZ20, Proposition 2.4]: the counting of integral metric ribbon graphs of genus g with n faces of length $L = (L_1, \dots, L_n)$ and p univalent vertices are computed by evaluation of Norbury’s polynomial of type $(g, n + p)$, and in the limit by Kontsevich’s volume of type $(g, n + p)$:

$$N_{g,n+p}(L_1, \dots, L_n, 0, \dots, 0) = V_{g,n+p}^{\text{K}}(L_1, \dots, L_n, 0, \dots, 0) + \text{l.o.t.} \quad (8.0.5)$$

Such evaluations are exactly the ones appearing in the asymptotic counts of combinatorial multicurves and square-tiled surfaces with double poles. In particular, the univalent vertices appearing here should corresponds to simple poles of quadratic differentials considered for the Masur–Veech volumes. Moreover, this approach would suggest a connection between Masur–Veech volumes and Kontsevich volumes of lower (odd) strata. Such a connection is currently under exploration by Duryev and Goujard.

To conclude, it would be interesting to have a better understanding of the functions $\mathcal{B}_\Sigma(\sigma)$ and $\mathcal{B}_\Sigma^{\text{comb}}(\mathbb{G})$. Integrability properties of the latter are currently under investigation in a joint work with Borot, Charbonnier, Delecroix and Wheeler [Bor+]. In particular, we found the range of $s \in \mathbb{R}$ for which $(\mathcal{B}_\Sigma^{\text{comb}})^s$ is integrable over $(\mathcal{M}_{g,n}^{\text{comb}}(L), \mu_{\text{K}})$: surprisingly, it depends on the topology of the surface Σ . This contrasts with the situation for hyperbolic surfaces where Arana-Herrera and Athreya recently proved square-integrability [AA20].

8.0.2 — ORGANISATION OF THE CHAPTER

The chapter is organised as follows.

- In Section 8.1 we discuss the asymptotic of hyperbolic and combinatorial counts of multicurves.
- In Section 8.2 we recall the definition of Masur–Veech volumes, as well as Mirzakhani’s result. We then employ the topological recursion results for multicurve counts to formulate our main conjecture regarding the polynomial structure of Masur–Veech volumes of the principal strata. We also deduce similar results for Siegel–Veech constants.
- In Section 8.3 we discuss the connection between square-tiles with double poles and Masur–Veech volumes.

8.1 — UNIT BALLS IN MEASURED FOLIATIONS

Let Σ be a bordered surface of type (g, n) . In Section 3.1.2, we considered the space $\mathcal{MF}_\Sigma^\star$ parametrising measured foliations on Σ up to Whitehead equivalence, allowing all types of boundary behaviours. We consider now the subset $\mathcal{MF}_\Sigma \subset \mathcal{MF}_\Sigma^\star$ that contains only those measured foliations whose boundary is made up of a union of singular leaves (that is, internal points are described as in Figures 3.7a and 3.7d and leaves at the boundary are of parallel type, cf. Figures 3.7c and 3.7f). In other words, on each boundary component there is at least one singularity, and the singularities on the boundary are connected by singular leaves. For convenience, we include in \mathcal{MF}_Σ the empty foliation. Notice that \mathcal{MF}_Σ is disjoint from the image of $\mathcal{T}_\Sigma^{\text{comb}}$ in $\mathcal{MF}_\Sigma^\star$.

The space \mathcal{MF}_Σ has (real) dimension $6g - 6 + 2n$ and admits a canonical piecewise linear structure. It also admits an integral structure given by \mathcal{M}_Σ , the set of multicurves in which components are not allowed to be homotopic to boundary components of Σ (see Figure 8.1 for an example). We denote by μ_{Th} the Thurston measure on \mathcal{MF}_Σ associated to this piecewise linear integral structure¹. That is, for an open set $A \subset \mathcal{MF}_\Sigma$ we have

$$\mu_{\text{Th}}(A) = \lim_{t \rightarrow +\infty} \frac{|(t \cdot A) \cap \mathcal{M}_\Sigma|}{t^{6g-6+2n}}. \quad (8.1.1)$$

The Thurston measure allows one to define a counting of multicurves associated to any length function on \mathcal{MF}_Σ . In fact, \mathcal{MF}_Σ is a completion of the set of formal \mathbb{Q}_+ -linear combinations of simple closed curves.

8.1.1 — THE HYPERBOLIC CASE

In the hyperbolic context, for any $\sigma \in \mathcal{T}_\Sigma$ the hyperbolic length function $\ell_\sigma : \mathcal{M}_\Sigma \rightarrow \mathbb{R}_+$ has a unique continuous extension $\ell_\sigma : \mathcal{MF}_\Sigma \rightarrow \mathbb{R}_{\geq 0}$ that is compatible with the piecewise linear structure. Mirzakhani [Miro8b] introduces a function measuring the volume of the unit ball in \mathcal{MF}_Σ with respect to ℓ_σ .

DEFINITION 8.1.1. The *Mirzakhani function* $\mathcal{B}_\Sigma : \mathcal{T}_\Sigma \rightarrow (0, +\infty]$ is defined by

$$\mathcal{B}_\Sigma(\sigma) = \mu_{\text{Th}}(\{ \mathcal{F} \in \mathcal{MF}_\Sigma \mid \ell_\sigma(\mathcal{F}) \leq 1 \}). \quad (8.1.2)$$

It is manifestly Mod_Σ -invariant, hence descends to a function $\mathcal{B}_{g,n}$ on $\mathcal{M}_{g,n}(L)$.

¹On \mathcal{MF}_Σ , the normalisation by counting lattice points differs by a power of 2 from the normalisation obtained by taking the top power of the Thurston symplectic form.

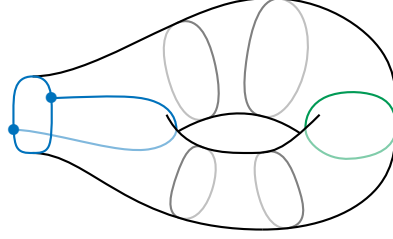


Figure 8.1: Example of a foliation associated to a simple closed curve (in green) on a one-holed torus. The singular leaves are depicted in blue. Note that the boundary of the torus is the union of two singular leaves.

By definition of the Thurston measure, this function describes the asymptotic growth of the number of multicurves of length $\leq t$ when $t \rightarrow \infty$, namely

$$\mathcal{B}_\Sigma(\sigma) = \lim_{t \rightarrow +\infty} \frac{|\{c \in \mathcal{M}_\Sigma \mid \ell_\sigma(c) \leq t\}|}{t^{6g-6+2n}} = \lim_{t \rightarrow +\infty} \frac{\mathcal{N}_\Sigma(\sigma; t)}{t^{6g-6+2n}}. \quad (8.1.3)$$

The main properties of \mathcal{B}_Σ established by Mirzakhani for punctured surfaces – i.e. on $\mathcal{M}_{g,n}(0)$ – can easily be generalised to the case of bordered surfaces.

THEOREM 8.1.2 ([Miro8b, Proposition 3.2 and Theorem 3.3]). *The function \mathcal{B}_Σ takes values in \mathbb{R}_+ , is continuous on \mathcal{T}_Σ , and $\mathcal{B}_{g,n}$ is integrable on $\mathcal{M}_{g,n}(L)$ with respect to μ_{WP} .*

We remark that finiteness of \mathcal{B}_Σ is non-trivial, and it comes from a the following bound on the number of simple closed curves, known since [Rivo1] for punctured surfaces and extended to bordered surfaces in [ABO17, Theorem 7.2]:

$$|\{\gamma \in \mathcal{S}_\Sigma \mid \ell_\sigma(\gamma) \leq t\}| \leq m_\epsilon t^{6g-6+2n} \quad (8.1.4)$$

for all $\sigma \in \mathcal{T}_\Sigma$ such that $\text{sys}_\sigma \geq \epsilon$, for some positive constant m_ϵ depending on ϵ and the topology (g, n) of the surface.

As a consequence of Theorem 7.1.2, using the final value theorem for the Laplace transform, we obtain an expression of the integral of $\mathcal{B}_{g,n}$ over $\mathcal{M}_{g,n}(L)$ as a sum over stable graphs. We attribute this result to Mirzakhani [Miro8b, Theorem 5.3], and restate it here following our conventions and notations.

THEOREM 8.1.3 ([Miro8b, Theorem 5.3]). *The integral of $\mathcal{B}_{g,n}$ is computed as*

$$\begin{aligned} (6g - 6 + 2n)! \int_{\mathcal{M}_{g,n}(L)} \mathcal{B}_{g,n} d\mu_{\text{WP}} &= \\ &= \sum_{\Gamma \in \mathcal{G}_{g,n}} \frac{1}{|\text{Aut}(\Gamma)|} \int_{\mathbb{R}_+^{E_\Gamma}} \prod_{v \in V_\Gamma} V_{g(v), n(v)}^{\text{K}}((\ell_e)_{e \in E_v}, (0_\lambda)_{\lambda \in \Lambda_v}) \prod_{e \in E_\Gamma} \frac{\ell_e d\ell_e}{e^{\ell_e} - 1}. \end{aligned} \quad (8.1.5)$$

Notice that we have Kontsevich volumes at the vertices, rather than Weil–Peterson volumes.

Proof. From Theorem 7.1.2, with the change of variable $\ell_e \mapsto \ell_e/s$, we have

$$\begin{aligned} & s \int_{\mathbb{R}_+} \left(\int_{\mathcal{M}_{g,n}(L)} \mathcal{N}_{g,n}(\sigma; t) d\mu_{\text{WP}}(\sigma) \right) e^{-st} dt = \\ &= \sum_{\Gamma \in \mathcal{G}_{g,n}} \frac{1}{|\text{Aut}(\Gamma)|} \int_{\mathbb{R}_+^{E_\Gamma}} \prod_{v \in V_\Gamma} V_{g(v),n(v)}^{\text{WP}} \left((s^{-1}\ell_e)_{e \in E_v}, (L\lambda)_{\lambda \in \Lambda_v} \right) \prod_{e \in E_\Gamma} \frac{s^{-2}\ell_e d\ell_e}{e^{\ell_e} - 1} \\ &= \frac{1}{s^{6g-6+2n}} \left(\sum_{\Gamma \in \mathcal{G}_{g,n}} \frac{1}{|\text{Aut}(\Gamma)|} \int_{\mathbb{R}_+^{E_\Gamma}} \prod_{v \in V_\Gamma} V_{g(v),n(v)}^{\text{K}} \left((\ell_e)_{e \in E_v}, (0\lambda)_{\lambda \in \Lambda_v} \right) \prod_{e \in E_\Gamma} \frac{\ell_e d\ell_e}{e^{\ell_e} - 1} + O(s) \right). \end{aligned}$$

Here we used the relation $|E_\Gamma| + \sum_{v \in V_\Gamma} (3g(v) - 3 + n(v)) = 3g - 3 + n$ and the observation that the rescaled Weil–Petersson volumes are given by Kontsevich volumes, up to lower order term:

$$V_{g,n}^{\text{WP}} \left(\frac{\ell_1}{s}, \dots, \frac{\ell_k}{s}, L_{k+1}, \dots, L_n \right) = \frac{1}{s^{6g-6+2n}} \left(V_{g,n}^{\text{K}}(\ell_1, \dots, \ell_k, 0, \dots, 0) + O(s) \right).$$

From properties of the Laplace transform, we find

$$\begin{aligned} & s \int_{\mathbb{R}_+} \left(\int_{\mathcal{M}_{g,n}(L)} \frac{\mathcal{N}_{g,n}(\sigma; t)}{t^{6g-6+2n}} d\mu_{\text{WP}}(\sigma) \right) e^{-st} dt = \\ &= \frac{1}{(6g-6+2n)!} \sum_{\Gamma \in \mathcal{G}_{g,n}} \frac{1}{|\text{Aut}(\Gamma)|} \int_{\mathbb{R}_+^{E_\Gamma}} \prod_{v \in V_\Gamma} V_{g(v),n(v)}^{\text{K}} \left((\ell_e)_{e \in E_v}, (0\lambda)_{\lambda \in \Lambda_v} \right) \prod_{e \in E_\Gamma} \frac{\ell_e d\ell_e}{e^{\ell_e} - 1} + O(s), \end{aligned}$$

and from the final value theorem for the Laplace transform, we have the thesis. \square

On the one hand, the integral of Mirzakhani’s function can be computed by taking the $t \rightarrow \infty$ limit of the stable graph sum of Theorem 7.1.2. On the other hand, the average number of multicurves can also be computed using the topological recursion of Corollary 7.1.4. As a consequence, we obtain a topological recursion statement for the integral of Mirzakhani’s function. We will come back to this in the next section, after a brief discussion of the above asymptotic in the combinatorial setting.

8.1.2 — THE COMBINATORIAL CASE

For $\mathbb{G} \in \mathcal{T}_\Sigma^{\text{comb}}$ the combinatorial length function $\ell_{\mathbb{G}}: \mathcal{M}_\Sigma \rightarrow \mathbb{R}_+$ has a unique continuous extension $\ell_{\mathbb{G}}: \mathcal{MF}_\Sigma \rightarrow \mathbb{R}_{\geq 0}$ that is compatible with the piecewise linear structure. We can now introduce a combinatorial analogue of Mirzakhani’s function measuring the volume of the unit ball in \mathcal{MF}_Σ with respect to $\ell_{\mathbb{G}}$.

DEFINITION 8.1.4. The *combinatorial Mirzakhani function* $\mathcal{B}_\Sigma^{\text{comb}}: \mathcal{T}_\Sigma^{\text{comb}} \rightarrow (0, +\infty]$ is defined by

$$\mathcal{B}_\Sigma^{\text{comb}}(\mathbb{G}) = \mu_{\text{Th}}(\{ \mathcal{F} \in \mathcal{MF}_\Sigma \mid \ell_{\mathbb{G}}(\mathcal{F}) \leq 1 \}). \quad (8.1.6)$$

It is manifestly Mod_Σ -invariant, hence descends to a function $\mathcal{B}_{g,n}^{\text{comb}}$ on $\mathcal{M}_{g,n}^{\text{comb}}(L)$.

By definition of the Thurston measure, this function describes the asymptotic growth of the number of multicurves of combinatorial length $\leq t$ when $t \rightarrow \infty$, namely

$$\mathcal{B}_\Sigma^{\text{comb}}(\mathbb{G}) = \lim_{t \rightarrow +\infty} \frac{|\{ c \in \mathcal{M}_\Sigma \mid \ell_{\mathbb{G}}(c) \leq t \}|}{t^{6g-6+2n}} = \lim_{t \rightarrow +\infty} \frac{\mathcal{N}_\Sigma(\mathbb{G}; t)}{t^{6g-6+2n}}. \quad (8.1.7)$$

III. Enumeration of multicurves and quadratic differentials

In order to prove that $\mathcal{B}_\Sigma^{\text{comb}}$ takes value in \mathbb{R}_+ , rather than $(0, +\infty]$, we need a parametrisation of multicurves adapted to different cells of the combinatorial Teichmüller space.

Consider an open cell $\mathfrak{Z}_{\Sigma, G}$ on $\mathcal{T}_\Sigma^{\text{comb}}$, that is a trivalent ribbon graph G together with a marking $f: \Sigma \rightarrow |G|$. Assigning to each edge $e \in E_G$ the number of times a non-backtracking representative on G of $c \in \mathcal{M}_\Sigma$ passes through it, we obtain a map

$$\mathfrak{m}_{\mathfrak{Z}_{\Sigma, G}}: \mathcal{M}_\Sigma \longrightarrow \mathbb{N}^{E_G}. \quad (8.1.8)$$

We show that in fact this map gives a parametrisation of \mathcal{M}_Σ . Let us first introduce some notation to describe its image.

DEFINITION 8.1.5. Given a trivalent ribbon graph G , a *corner* is an ordered triple $\Delta = (e, e', e'')$ where e, e', e'' are edges incident to a vertex in the cyclic order. Equivalently, a corner consists of a vertex v together with the choice of an incident edge e . We say that a corner belongs to a face $\mathfrak{f} \in F_G$ if e' and e'' are edges around that face. We denote by $\mathfrak{C}(\mathfrak{f})$ the set of corners belonging to \mathfrak{f} , and by \mathfrak{C}_G the set of all corners. If we have an assignment of real numbers $(x_e)_{e \in E_G}$ and $\Delta = (e, e', e'')$ is a corner, we set $x_\Delta = x_{e''} + x_{e'} - x_e$.

LEMMA 8.1.6. *The map $\mathfrak{m}_{\mathfrak{Z}_{\Sigma, G}}$ is a bijection between the set of multicurves \mathcal{M}_Σ and the set*

$$Z_G = \left\{ m \in \mathbb{N}^{E_G} \left| \begin{array}{l} \forall \Delta \in \mathfrak{C}_G, m_\Delta \in 2\mathbb{N} \\ \forall \mathfrak{f} \in F_G, \exists \Delta \in \mathfrak{C}(\mathfrak{f}) \text{ s.t. } m_\Delta = 0 \end{array} \right. \right\}. \quad (8.1.9)$$

Proof. As $\mathfrak{m}_{\mathfrak{Z}_{\Sigma, G}}$ is additive under union, it is enough to prove that for any simple closed curve $\gamma \in \mathcal{S}_\Sigma$, $\mathfrak{m}_{\mathfrak{Z}_{\Sigma, G}}(\gamma) \in Z_G$ and there is a unique multicurve corresponding to each $m \in Z_G$.

For the first part, we decompose the geometric realisation $|G|$ into strips S_e for each edge e and small triangular neighbourhoods U_v of each vertex as in Figure 8.2a, and pullback this structure to Σ via f . If $\gamma \in \mathcal{S}_\Sigma$ is a simple closed curve in Σ , we can isotope γ to a non-backtracking simple representative γ that has m_e parallel paths in the strip corresponding to $e \in E_G$. At each vertex v , it is possible to draw pairwise non-intersecting arcs connecting inside U_v the endpoints of γ in ∂U_v in a non-backtracking way if and only if $m_\Delta \in 2\mathbb{N}$ for each corner Δ . When these conditions hold, there is in fact a unique way (up to isotopy relative to ∂U_v) to draw such arcs as in Figure 8.2b. Namely, we can label the points in $\partial U_v \cap \gamma \cap S_e$ by $p_{e,1}, \dots, p_{e,m_e}$ following the cyclic order around v . Then, $p_{e,i}$ must be connected to

- $p_{e'', m_{e''}+1-i}$ for $1 \leq i \leq \frac{1}{2}m_{\Delta'}$,
- $p_{e', \frac{1}{2}m_{\Delta'}+1-i}$ for $\frac{1}{2}m_{\Delta'} < i \leq \frac{1}{2}(m_{\Delta'} + m_{\Delta''}) = m_e$.

This proves that $\mathfrak{m}_{\mathfrak{Z}_{\Sigma, G}}$ is injective on \mathcal{S}_Σ and its image is included in $\{m \in \mathbb{N}^{E_G} \mid m_\Delta \in 2\mathbb{N}\}$.

Let $(e_i)_{i \in \mathbb{Z}/N\mathbb{Z}}$ be the sequence of edges around a face \mathfrak{f} , and Δ_i be the corner containing both e_i and e_{i+1} . Then, the $\frac{1}{2}m_{\Delta_i}$ arcs in $\gamma \cap S_{e_i}$ which are closest to \mathfrak{f} are connected to the $\frac{1}{2}m_{\Delta_i}$ arcs in $\gamma \cap S_{e_{i+1}}$ which are the closest to \mathfrak{f} . In particular, the $\frac{1}{2} \min_i \{m_{\Delta_i}\}$ arcs which are (in each strip around \mathfrak{f}) the closest to \mathfrak{f} are connected and form loops, which are homotopic to the boundary component of Σ that \mathfrak{f} represents. By definition of \mathcal{S}_Σ , we must have $\min_i \{m_{\Delta_i}\} = 0$. This proves that $\mathfrak{m}_{\mathfrak{Z}_{\Sigma, G}}(\mathcal{S}_\Sigma) \subset Z_G$ and the first part of the claim.

Conversely, if we are given $m \in Z_G$, we draw m_e parallel arcs in S_e , and connect them inside each U_v in the unique non-intersecting and non-backtracking (as explained above) way. We obtain a collection of simple closed curves, none of them being homotopic to a boundary component of Σ . \square

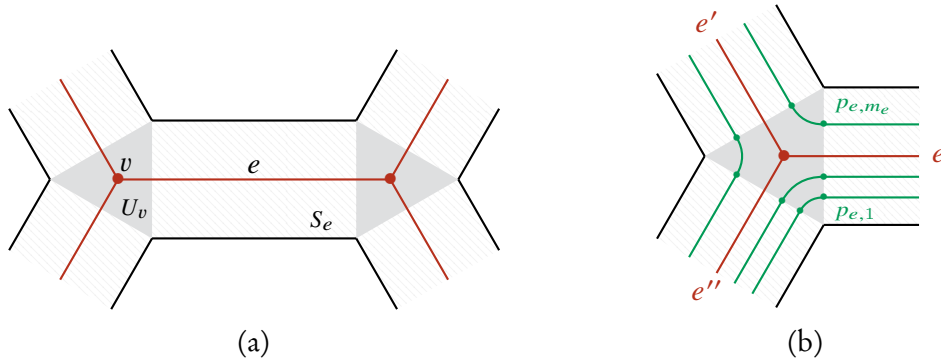


Figure 8.2: Decomposition of Σ into strips S_e around edges and triangular neighbourhoods U_v around vertices.

As a consequence, we obtain a polynomial growth of the number of essential simple closed curve of bounded combinatorial length.

PROPOSITION 8.1.7. *Let Σ be a connected bordered surface of type (g, n) and $\epsilon > 0$. For any $\mathbb{G} \in \mathcal{T}_\Sigma^{\text{comb}}$ such that $\text{sys}_\mathbb{G} \geq \epsilon$, and any $t > 0$, we have*

$$|\{\gamma \in \mathcal{S}_\Sigma \mid \ell_\mathbb{G}(\gamma) \leq t\}| \leq m_\epsilon t^{6g-6+2n}, \quad (8.1.10)$$

for some positive constant m_ϵ depending on ϵ and the topology (g, n) of the surface.

Proof. Let $\gamma \in \mathcal{S}_\Sigma$ and denote by $E_\mathbb{G}(\gamma)$ the number (with multiplicity) of edges γ passes through. We recall from the proof of Proposition 6.1.1 that for $\text{sys}_\mathbb{G} \geq \epsilon$ we have

$$E_\mathbb{G}(\gamma) \leq \frac{(6g - 6 + 3n)^2}{\epsilon} \ell_\mathbb{G}(\gamma).$$

If the underlying graph G is trivalent, from this bound and Lemma 8.1.6 we deduce

$$|\{\gamma \in \mathcal{S}_\Sigma \mid \ell_\mathbb{G}(\gamma) \leq t\}| \leq \left| \left\{ x \in Z_G \mid \sum_{e \in E_G} x_e \leq \frac{(6g - 6 + 3n)^2}{\epsilon} t \right\} \right|$$

and the claim follows, as the latter is the set of integer points with bounded L^1 -norm in a polytope of dimension $6g - 6 + 2n$. If G is not trivalent, we resolve it in an arbitrary way into a trivalent ribbon graph G' and the same argument works with G' . \square

Remark 8.1.8. A second proof of Proposition 8.1.7 can be obtained by invoking the corresponding result on \mathcal{T}_Σ with respect to hyperbolic lengths, namely Equation (8.1.4), and flowing it to the combinatorial setting using the fact that hyperbolic length of multicurves flows to their combinatorial analogue (cf. Proposition 6.1.1).

We first notice that $\mathcal{B}_\Sigma^{\text{comb}}$ takes values in \mathbb{R}_+ , as a consequence of Proposition 8.1.7, and is continuous since the length function $\ell_* : \mathcal{T}_\Sigma^{\text{comb}} \rightarrow \mathbb{R}_+^{\mathcal{S}_\Sigma}$ is continuous (and, in particular, measurable). The integrability result is more delicate. We derive it from the explicit expression as a sum over stable graphs.

THEOREM 8.1.9. *The integral of $\mathcal{B}_{g,n}^{\text{comb}}$ is computed as*

$$\begin{aligned} (6g-6+2n)! \int_{\mathcal{M}_{g,n}^{\text{comb}}(L)} \mathcal{B}_{g,n}^{\text{comb}} d\mu_{\mathbb{K}} &= \\ &= \sum_{\Gamma \in \mathcal{G}_{g,n}} \frac{1}{|\text{Aut}(\Gamma)|} \int_{\mathbb{R}_+^{E_\Gamma}} \prod_{v \in V_\Gamma} V_{g(v),n(v)}^{\mathbb{K}}((\ell_e)_{e \in E_v}, (0_\lambda)_{\lambda \in \Lambda_v}) \prod_{e \in E_\Gamma} \frac{\ell_e d\ell_e}{e^{\ell_e} - 1}. \end{aligned} \quad (8.1.11)$$

Proof. As in the hyperbolic case, we start from the sum over stable graphs (7.2.6) expressing the combinatorial counting function $\mathcal{N}_{g,n}^{\text{comb}}$ and perform change of variable $\ell_e \mapsto \ell_e/s$. The only difference is in the presence of Kontsevich volumes at the vertices, for which we have

$$V_{g,n}^{\mathbb{K}}\left(\frac{\ell_1}{s}, \dots, \frac{\ell_k}{s}, L_{k+1}, \dots, L_n\right) = \frac{1}{s^{6g-6+2n}} \left(V_{g,n}^{\mathbb{K}}(\ell_1, \dots, \ell_k, 0, \dots, 0) + O(s) \right).$$

We then conclude similarly. \square

8.1.3 — HYPERBOLIC VS COMBINATORIAL

By comparing the asymptotic counts in the hyperbolic and combinatorial settings, we get that the two agree and are independent of boundary lengths.

COROLLARY 8.1.10. *The following equality holds:*

$$\int_{\mathcal{M}_{g,n}(L)} \mathcal{B}_{g,n} d\mu_{\text{WP}} = \int_{\mathcal{M}_{g,n}^{\text{comb}}(L)} \mathcal{B}_{g,n}^{\text{comb}} d\mu_{\mathbb{K}}. \quad (8.1.12)$$

Furthermore, both quantities are independent of $L \in \mathbb{R}_+^n$.

Using the rescaling flow, we can actually say more: the combinatorial Mirzakhani function can be obtained as a limit of the hyperbolic one.

PROPOSITION 8.1.11. *For any connected bordered surface Σ of type (g, n) , we have*

$$\lim_{\beta \rightarrow \infty} \beta^{6g-6+2n} \mathcal{B}_\Sigma(\sigma^\beta) = \mathcal{B}_\Sigma^{\text{comb}}(\text{sp}(\sigma)) \quad (8.1.13)$$

and the limit is uniform for $\text{sp}(\sigma)$ in any thick part of $\mathcal{T}_\Sigma^{\text{comb}}$.

Proof. Let $\epsilon > 0$ and $\mathbb{G} \in \mathcal{T}_\Sigma^{\text{comb}}$ such that $\text{sys}_{\mathbb{G}} \geq \epsilon$. We denote $\sigma = \text{sp}^{-1}(\mathbb{G})$ and $\sigma^\beta = \text{sp}^{-1}(\beta\mathbb{G})$. Since the length of a multicurve is the sum of lengths of its connected components, Proposition 6.1.1 implies that for any $c \in \mathcal{M}_\Sigma$, we have $\beta\ell_{\mathbb{G}}(c) \leq \ell_{\sigma^\beta}(c) \leq (\beta + \kappa_\epsilon)\ell_{\mathbb{G}}(c)$. Therefore

$$\left\{ c \in \mathcal{M}_\Sigma \mid \ell_{\mathbb{G}}(c) \leq \frac{t}{\beta + \kappa_\epsilon} \right\} \subseteq \left\{ c \in \mathcal{M}_\Sigma \mid \ell_{\sigma^\beta}(c) \leq t \right\} \subseteq \left\{ c \in \mathcal{M}_\Sigma \mid \ell_{\sigma^\beta}(c) \leq \frac{t}{\beta} \right\}$$

and thus

$$\frac{\mathcal{B}_\Sigma^{\text{comb}}(\mathbb{G})}{(\beta + \kappa_\epsilon)^{6g-6+2n}} \leq \mathcal{B}_\Sigma(\sigma^\beta) \leq \frac{\mathcal{B}_\Sigma^{\text{comb}}(\mathbb{G})}{\beta^{6g-6+2n}}.$$

Multiplying by $\beta^{6g-6+2n}$ and taking the limit $\beta \rightarrow \infty$ yields the result. \square

We remark that the above proposition is not enough to derive Corollary 8.1.10. Indeed, we can certainly write

$$\int_{\mathcal{M}_{g,n}(\beta L)} \mathcal{B}_{g,n} d\mu_{\text{WP}} = \int_{\mathcal{M}_{g,n}^{\text{comb}}(L)} \beta^{6g-6+2n} (\mathcal{B}_{g,n} \circ R_\beta) J_\beta d\mu_{\mathbb{K}}, \quad J_\beta = \frac{1}{\beta^{6g-6+2n}} \frac{R_\beta^* d\mu_{\text{WP}}}{d\mu_{\mathbb{K}}}, \quad (8.1.14)$$

where $R_\beta: \mathcal{T}_\Sigma^{\text{comb}}(L) \rightarrow \mathcal{T}_\Sigma(\beta L)$, $R_\beta(\mathbb{G}) = \text{sp}^{-1}(\beta\mathbb{G})$ is the rescaling flow. We know by Theorem 6.A due to Mondello that J_β converges pointwise to 1, and by Proposition 8.1.11 that $\beta^{6g-6+2n} (\mathcal{B}_{g,n} \circ R_\beta)$ converges to $\mathcal{B}_{g,n}^{\text{comb}}$ uniformly on compacts. This is however not sufficient, as we would need an effective and integrable enough bound independent of β to conclude that the integral of $\mathcal{B}_{g,n}$ coincide with the integral of its combinatorial analogue by dominated convergence, and such a bound is not currently available. Note that one cannot hope for a bound by constant, because $\mathcal{B}_\Sigma^{\text{comb}}(\mathbb{G})$ can diverge when $\text{sys}_\mathbb{G} \rightarrow 0$. Getting effective and uniform bounds for the Jacobian J_β over $\mathcal{T}_\Sigma^{\text{comb}}$ is a question of broader interest (cf. Question 6.E): it would allow to study the behaviour for large boundary lengths L of the integral on $\mathcal{M}_{g,n}(L)$ of a larger class of functions.

Remark 8.1.12. It is convenient to notice that the integral of $\mathcal{B}_{g,n}$ or $\mathcal{B}_{g,n}^{\text{comb}}$ coincides with the (Laplace transform of the) average number of combinatorial multicurves $\widehat{N}_{g,n}^{\text{comb}}(L; s)$ evaluated at $s = 1$ and $L = 0$ (cf. Equation (8.1.11) with Equation (7.2.6)). In particular, one can use topological recursion at $s = 1$ to compute such integrals. We will come back to this observation in the next section.

8.2 — QUADRATIC DIFFERENTIALS, MASUR–VEECH VOLUMES AND TOPOLOGICAL RECURSION

In this section, we review the connection between the integral of Mirzakhani’s function and the Masur–Veech volumes of the principal stratum of the moduli space of quadratic differentials with simple poles.

Fix an integer partition (μ, ν) of $4g - 4$, where $\mu = (2\alpha_a)_{a=1}^r$ and $\nu = (2\beta_b - 1)_{b=1}^s$ are the even and odd parts respectively, with $\alpha_a, \beta_b \geq 0$. Consider the moduli space $\mathcal{Q}_g(\mu, \nu)$ parametrising *quadratic differential with simple poles and zeros of prescribed order*, i.e. the moduli space of tuples (C, φ) where

- C is a compact Riemann surface of genus g ,
- φ is a meromorphic quadratic differential on C that is not a global square of a one-form and with associated divisor

$$(\varphi) = \sum_{a=1}^r 2\alpha_a z_a + \sum_{b=1}^s (2\beta_b - 1) z'_b. \quad (8.2.1)$$

Contrary to the previous section, double poles are not allowed, but we consider simple poles when some $\beta_b = 0$, which are regarded as “zeros of order -1 ”. Similarly quadratic differentials are allowed to have ordinary marked points, i.e. when some $\alpha_a = 0$, which are regarded as “zeros of order 0 ”. There is a natural \mathbb{C}^\times action on $\mathcal{Q}_g(\mu, \nu)$ given by rescaling of the quadratic differential, and we denote by $\mathbb{P}\mathcal{Q}_g(\mu, \nu)$ the quotient space.

Similarly to Section 7.3, the anti-invariant relative homology groups $H_1^-(\hat{C}, \hat{Z}; \mathbb{Z})$ of the canonical double cover $\pi: \hat{C} \rightarrow C$ with the one-form ω provide period coordinates for $\mathcal{Q}_g(\mu, \nu)$. Here Z is the preimage of the even zeros of the differential. In particular, integration of ω along a basis of $H_1^-(\hat{C}, \hat{Z}; \mathbb{Z})$ provides a the period coordinates map

$$U \longrightarrow \mathbb{C}^{2g-2+r+s}, \quad (C, \varphi) \longmapsto \left(\oint_{\gamma_i} \omega \right)_{i=1}^{2g-2+r+s}. \quad (8.2.2)$$

In particular, $\mathcal{Q}_g(\mu, \nu)$ is a complex orbifold of dimension $2g - 2 + r + s$.

Such local coordinates provides $\mathcal{Q}_g(\mu, \nu)$ with a natural volume form by considering a linear volume element in the vector spaces $\mathbb{C}^{2g-2+r+s}$ normalised in such way that a fundamental domain of the lattice $(\mathbb{Z} \oplus i\mathbb{Z})^{2g-2+r+s}$ has area one. It can be proved that the induced measure does not depend on the choice of local coordinates used in the construction, so the volume element μ_{MV} , called the *Masur–Veech measure*, is defined canonically.

On the other hand, we have a well-defined area function $\text{Area}: \mathcal{Q}_g(\mu, \nu) \rightarrow \mathbb{R}_+$ given by

$$\text{Area}(C, \varphi) = \int_C |\varphi| = \frac{i}{4} \int_{\hat{C}} \omega \wedge \bar{\omega} = \frac{1}{2} \text{Area}(\hat{C}, \omega), \quad (8.2.3)$$

and the measure μ_{MV} defines a canonical measure ν_{MV} (also called Masur–Veech measure) on the unit hyperboloid

$$\mathcal{Q}_g^1(\mu, \nu) = \left\{ (C, \varphi) \in \mathcal{Q}_g(\mu, \nu) \mid \text{Area}(\hat{C}, \omega) = 1 \right\}, \quad d\nu_{\text{MV}} = \frac{d\mu_{\text{MV}}}{d\text{Area}}. \quad (8.2.4)$$

The following result, proved independently by Masur and Veech, states the volume of the unit hyperboloid is finite.

THEOREM 8.2.1 ([Mas82; Vee82]). *The total volume of any stratum $\mathcal{Q}_g^1(\mu, \nu)$ of meromorphic quadratic differentials with at most simple poles with respect to the Masur–Veech measure is finite:*

$$V_g^{\text{MV}}(\mu, \nu) = \nu_{\text{MV}}(\mathcal{Q}_g^1(\mu, \nu)) < +\infty. \quad (8.2.5)$$

In the following, we will be mainly interested in the principal stratum of the moduli space, parametrising quadratic differentials with only simple zeros and simple poles: $\mu = \emptyset$ and $\nu = (1^{4g-4+n}, -1^n)$. To simplify the notation, we denote such volumes simply by $V_{g,n}^{\text{MV}}$:

$$V_{g,n}^{\text{MV}} = V_g^{\text{MV}}(1^{4g-4+n}, -1^n). \quad (8.2.6)$$

The main connection between Masur–Veech volumes and the asymptotic counting of multicurves is the following result, due to Mirzakhani, who proved it in the case of hyperbolic surfaces with punctures.

THEOREM 8.2.2 ([Miro8a]). *The Masur–Veech volumes of the principal strata of the moduli space of quadratic differentials are computed by the average of the asymptotic number of hyperbolic multicurves:*

$$V_{g,n}^{\text{MV}} = 2^{4g-2+n} (4g - 4 + n)! \cdot (6g - 6 + 2n) \int_{\mathcal{M}_{g,n}(L)} \mathcal{B}_{g,n} d\mu_{\text{WP}}. \quad (8.2.7)$$

Moreover, it is computed by the following sum over stable graphs:

$$\begin{aligned}
 V_{g,n}^{\text{MV}} &= 2^{4g-2+n} \frac{(4g-4+n)!}{(6g-7+2n)!} \times \\
 &\times \sum_{\Gamma \in \mathcal{G}_{g,n}} \frac{1}{|\text{Aut}(\Gamma)|} \int_{\mathbb{R}_+^{E_\Gamma}} \prod_{v \in V_\Gamma} V_{g(v),n(v)}^{\text{K}}((\ell_e)_{e \in E_v}, (0_\lambda)_{\lambda \in \Lambda_v}) \prod_{e \in E_\Gamma} \frac{\ell_e d\ell_e}{e^{\ell_e} - 1}.
 \end{aligned} \tag{8.2.8}$$

Remark 8.2.3. Let us remark that Mirzakhani did not compute the normalisation factor in Equation (8.2.7), which was recently and independently computed by Monin–Teplukhovskiy [MT19], Arana-Herrera [Ara20], and Delecroix–Goujard–Zograf–Zorich [DGZZ20]. The expression of Masur–Veech volumes as a sum over stable graphs was also proved in [DGZZ20] by purely combinatorial methods. We also give a short remark about normalisation conventions (cf. [AEZ16, Remark 1.2] or [DGZZ20, Section 2.1]). In this context, it is customary to consider quadratic differentials with labelled zeros. This convention is responsible for the normalisation factor $(4g-4+n)!$ in Equation (8.2.7). Moreover, the Masur–Veech volumes are defined as the volumes of the hyperboloid of quadratic differentials of area $\frac{1}{2}$. This second convention is responsible for the factor 2^{4g-2+n} in Equation (8.2.7).

Combining Mirzakhani’s result with the topological recursion computing the integral of $\mathcal{B}_{g,n}$, we obtain our major contribution from [And+19] to the theory of quadratic differentials: a topological recursion formula for $V_{g,n}^{\text{MV}}$.

THEOREM-DEFINITION 8.2.4. Define the *Masur–Veech polynomials* by the following topological recursion: for $2g-2+n > 1$

$$\begin{aligned}
 V_{g,n}^{\text{MV}}(L_1, \dots, L_n) &= \\
 &= \sum_{m=2}^n \int_{\mathbb{R}_+} B^{\text{MV}}(L_1, L_m, \ell) V_{g,n-1}^{\text{MV}}(\ell, L_2, \dots, \widehat{L_m}, \dots, L_n) \ell d\ell \\
 &+ \frac{1}{2} \int_{\mathbb{R}_+^2} C^{\text{MV}}(L_1, \ell, \ell') \left(V_{g-1,n+1}^{\text{MV}}(\ell, \ell', L_2, \dots, L_n) \right. \\
 &\quad \left. + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2, \dots, n\}}} V_{g_1, 1+|I_1|}^{\text{MV}}(\ell, L_{I_1}) V_{g_2, 1+|I_2|}^{\text{MV}}(\ell', L_{I_2}) \right) \ell \ell' d\ell d\ell'
 \end{aligned} \tag{8.2.9}$$

where we have $V_{0,1}^{\text{MV}} = V_{0,2}^{\text{MV}} = 0$ by convention, recursion kernels

$$\begin{aligned}
 B^{\text{MV}}(L, L', \ell) &= \frac{1}{2L} \left([L - L' - \ell]_+ - [-L + L' - \ell]_+ + [L + L' - \ell]_+ \right) + \frac{1}{e^\ell - 1}, \\
 C^{\text{MV}}(L, \ell, \ell') &= \frac{1}{L} [L - \ell - \ell']_+ + \frac{1}{(e^\ell - 1)(e^{\ell'} - 1)} + \\
 &+ \frac{1}{e^\ell - 1} \frac{1}{2L} \left([L - \ell - \ell']_+ - [-L + \ell - \ell']_+ + [L + \ell - \ell']_+ \right) \\
 &+ \frac{1}{e^{\ell'} - 1} \frac{1}{2L} \left([L - \ell' - \ell]_+ - [-L + \ell' - \ell]_+ + [L + \ell' - \ell]_+ \right),
 \end{aligned} \tag{8.2.10}$$

and initial conditions $V_{0,3}^{\text{MV}}(L_1, L_2, L_3) = 1$ and $V_{1,1}^{\text{MV}}(L) = \frac{L^2}{48} + \frac{\pi^2}{12}$. They are computed by the

following sum over stable graphs:

$$\begin{aligned} V_{g,n}^{\text{MV}}(L_1, \dots, L_n) &= \\ &= \sum_{\Gamma \in \mathcal{G}_{g,n}} \frac{1}{|\text{Aut}(\Gamma)|} \int_{\mathbb{R}_+^{E_\Gamma}} \prod_{v \in V_\Gamma} V_{g(v),n(v)}^{\text{K}}((\ell_e)_{e \in E_v}, (L_\lambda)_{\lambda \in \Lambda_v}) \prod_{e \in E_\Gamma} \frac{\ell_e d\ell_e}{e^{\ell_e} - 1}. \end{aligned} \quad (8.2.11)$$

Moreover, the Masur–Veech volumes $V_{g,n}^{\text{MV}}$ of the principal strata of the moduli space of quadratic differentials coincide with the constant term (up to a normalisation constant) of the Masur–Veech polynomials:

$$V_{g,n}^{\text{MV}} = 2^{4g-2+n} \frac{(4g-4+n)!}{(6g-7+2n)!} V_{g,n}^{\text{MV}}(0, \dots, 0) \quad (8.2.12)$$

Proof. As explained in Remark 8.1.12, the integral of $\mathcal{B}_{g,n}$ (or equivalently $\mathcal{B}_{g,n}^{\text{comb}}$) coincides with the Laplace transform of the average number of combinatorial multicurves $\widehat{\mathcal{N}}_{g,n}^{\text{comb}}(L; s)$ evaluated at $s = 1$ and $L = 0$. \square

By recursion, it is easy to prove that $V_{g,n}^{\text{MV}}(L)$ is a homogeneous polynomial in $\pi^2, L_1^2, \dots, L_n^2$ of degree $3g - 3 + n$ with rational positive coefficients. Hence, the Masur–Veech volumes $V_{g,n}^{\text{MV}} \in \pi^{6g-6+2n} \mathbb{Q}_+$. See Table 8.1 for some Masur–Veech volumes computed by topological recursion.

Remark 8.2.5. So far we only considered bordered surfaces, so that $g \geq 0, n > 0$ and $2g - 2 + n > 0$. However, Masur–Veech volumes makes sense for $n = 0$ too, and a natural question is whether the above results extend to the $n = 0$ case. The answer is affirmative, and in particular Theorem-definition 8.2.4 can be used to compute Masur–Veech volumes of the principal stratum of holomorphic quadratic differentials, i.e. for $n = 0$, using the dilaton equation:

$$\left[\frac{L_{n+1}^2}{2} \right] V_{g,n+1}^{\text{MV}}(L_1, \dots, L_n, L_{n+1}) = (2g - 2 + n) V_{g,n+1}^{\text{MV}}(L_1, \dots, L_n). \quad (8.2.13)$$

Here $\left[\frac{L_{n+1}^2}{2} \right]$ extract the coefficient of $\frac{L_{n+1}^2}{2}$ on the polynomial on its right-hand side. In particular, we can obtain the Masur–Veech volumes of the principal stratum of holomorphic quadratic differentials as

$$V_{g,0}^{\text{MV}} = 2^{4g-2} \frac{(4g-4)!}{(2g-2)(6g-7)!} \left[\frac{L^2}{2} \right] V_{g,1}^{\text{MV}}(L). \quad (8.2.14)$$

We refer to [And+19, Lemma 3.4] for the proof.

By computing a large quantity of Masur–Veech volumes, we were able to formulate some conjectures [And+19, Conjecture 5.4] about their behaviour for fixed genus and varying number of poles. The conjecture in genus zero was previously formulated by Kontsevich and proved by Athreya–Eskin–Zorich [AEZ16].

CONJECTURE 8.2.6. *For any $g \geq 0$, there exist polynomials $a_g, b_g \in \mathbb{Q}[n]$ of degrees*

$$\deg a_g = \left\lfloor \frac{g-1}{2} \right\rfloor \quad \text{and} \quad \deg b_g = \left\lfloor \frac{g}{2} \right\rfloor \quad (8.2.15)$$

$n \setminus g$	0	1	2	3	4	5	6
0	-	-	$\frac{1}{15}$	$\frac{115}{33264}$	$\frac{2106241}{11548293120}$	$\frac{7607231}{790778419200}$	$\frac{51582017261473}{101735601235107840000}$
1	-	$\frac{2}{3}$	$\frac{29}{840}$	$\frac{4111}{2223936}$	$\frac{58091}{592220160}$	$\frac{35161328707}{6782087854080000}$	$\frac{1725192578138153}{6307607276576686080000}$
2	-	$\frac{1}{3}$	$\frac{337}{18144}$	$\frac{77633}{77837760}$	$\frac{160909109}{3038089420800}$	$\frac{27431847097}{9796349122560000}$	$\frac{236687293214441}{1601932006749634560000}$
3	4	$\frac{11}{60}$	$\frac{29}{2880}$	$\frac{207719}{384943104}$	$\frac{14674841399}{512424415641600}$	$\frac{5703709895459}{3767985230929920000}$	$\frac{37679857842043}{471817281090355200000}$
4	2	$\frac{1}{10}$	$\frac{919}{168480}$	$\frac{16011391}{54854392320}$	$\frac{9016171639}{582300472320000}$	$\frac{143368101519407}{175211313238241280000}$	$\frac{13237209152580169}{306665505466027868160000}$
5	1	$\frac{163}{3024}$	$\frac{653}{221760}$	$\frac{6208093}{39382640640}$	$\frac{442442475179}{52900285261824000}$	$\frac{259645860580231}{587375069141532672000}$	$\frac{6359219722433607397}{272686967460391980367872000}$
6	$\frac{1}{2}$	$\frac{29}{1008}$	$\frac{88663}{56010240}$	$\frac{5757089}{67781007360}$	$\frac{1537940628689}{340912949465088000}$	$\frac{229686916047007}{962777317187911680000}$	$\frac{43310941179948284069}{3440050974115714213871616000}$
7	$\frac{1}{4}$	$\frac{1255}{82368}$	$\frac{295133}{348281856}$	$\frac{2598992519}{56936046182400}$	$\frac{643391778377}{264869710110720000}$	$\frac{11267167909498433}{87618715847436533760000}$	$\frac{74408487930504838727}{10957199399035237866405888000}$
8	$\frac{1}{8}$	$\frac{2477}{308880}$	$\frac{1835863}{4063288320}$	$\frac{1769539}{720943441920}$	$\frac{127802659622551}{97895844856922112000}$	$\frac{2762333771707}{39907473380632166400}$	$\frac{76034947449385560773}{20780895411963382160424960000}$
9	$\frac{1}{16}$	$\frac{39203}{9335040}$	$\frac{12653167}{52718561280}$	$\frac{6756335603}{516534771916800}$	$\frac{76170641989903}{108773160952135680000}$	$\frac{46331482996262911}{1245354014578231266508800}$	$\frac{7583038108310022233611}{3850996789771271334071894016000}$
10	$\frac{1}{32}$	$\frac{1363}{622336}$	$\frac{5219989}{41079398400}$	$\frac{2863703603}{410578921267200}$	$\frac{364975959330977}{973541287193739264000}$	$\frac{110488317513510709}{5533939090421837306265600}$	$\frac{1597788327762805352162251}{1509590741590338362956182454272000}$
11	$\frac{1}{64}$	$\frac{308333}{270885888}$	$\frac{644710519}{9612579225600}$	$\frac{28221517763}{7606514751897600}$	$\frac{26274127922961227}{131162562511011053568000}$	$\frac{39074093749702556551}{3652399799678412622135296000}$	$\frac{32893791972666409219189}{57890914971883041214200545280000}$

Table 8.1: Masur–Veech volumes $\pi^{-(6g-6+2n)}V_{g,n}^{MV}$. We display in black the values that were honestly computed from the recursion, and in grey the values that the conjecture predicts. The polynomials a_g and b_g appearing in Conjecture 8.2.6 have been determined by the values $V_{g,n}^{MV}$ with $n = 1, \dots, g + 1$.

III. Enumeration of multicurves and quadratic differentials

g	$a_g(n)$	$b_g(n)$
0	0	1
1	$\frac{1}{24}$	$\frac{1}{24}$
2	$\frac{5}{2304}$	$\frac{7n}{2160} + \frac{7}{1152}$
3	$\frac{245n}{3981312} + \frac{643}{1990656}$	$\frac{223n}{1088640} + \frac{6523}{8709120}$
4	$\frac{1757n}{382205952} + \frac{95413}{3185049600}$	$\frac{37079n^2}{12009254400} + \frac{5110337n}{112086374400} + \frac{5951381}{37623398400}$
5	$\frac{38213n^2}{880602513408} + \frac{4218671n}{4403012567040} + \frac{63657059}{12842119987200}$	$\frac{1306751n^2}{4899775795200} + \frac{3134026741n}{685968611328000} + \frac{63849553}{3310859059200}$

Table 8.2: Polynomials conjecturally appearing in the Masur–Veech volumes for $g \leq 5$.

(with the convention that $a_0 = 0$) such that, for any $n \geq 0$,

$$V_{g,n}^{\text{MV}} = 2^{4g-2+n} \pi^{6g-6+2n} \frac{(2g-3+n)!(4g-4+n)!}{(6g-7+2n)!} \left(a_g(n) + \frac{1}{2^{4g-6+2n}} \binom{4g-6+2n}{2g-3+n} b_g(n) \right). \quad (8.2.16)$$

The first few polynomials are displayed on Table 8.2.

The notation here slightly differ from [And+19, Conjecture 5.4], where we have $2^{4g-2}(a_g, b_g) = (p_g, q_g)$. From the above conjecture, we can examine the asymptotic behaviour of Masur–Veech volumes of principal strata with a large number of poles (i.e. when $n \rightarrow \infty$).

CONDITIONAL THEOREM 8.2.7. Assuming Conjecture 8.2.6 to hold, the Masur–Veech volumes of the principal strata of the moduli space of quadratic differentials have the following asymptotic behaviour as $n \rightarrow \infty$:

$$V_{g,n}^{\text{MV}} \sim \frac{\pi^{6g-6+2n}}{2^{2g-5+n}} \pi^{\epsilon(g)/2} n^{g/2} \sigma_g \quad \epsilon(g) = \begin{cases} 0 & \text{if } g \text{ is even} \\ 1 & \text{if } g \text{ is odd} \end{cases} \quad (8.2.17)$$

where $\sigma_g \in \mathbb{Q}$ is the top coefficient of a_g if g is odd and the top coefficient of b_g if g is even.

Proof. From Stirling’s approximation $k! \sim \sqrt{2\pi k} \left(\frac{k}{e}\right)^k$ as $k \rightarrow \infty$, we find

$$2^{4g-2+n} \pi^{6g-6+2n} \frac{(2g-3+n)!(4g-4+n)!}{(6g-7+2n)!} \sim 2^{4g-2+n} \pi^{6g-6+2n} \frac{\sqrt{\pi n}}{2^{6g-7+2n}} = \frac{\pi^{6g-6+2n}}{2^{2g-5+n}} \sqrt{\pi n}.$$

On the other hand, applying Stirling’s approximation again, we find

$$\frac{1}{2^{4g-6+2n}} \binom{4g-6+2n}{2g-3+n} \sim \frac{1}{\sqrt{\pi n}}.$$

As a consequence, considering the degree of a_g and b_g , we obtain that

$$\begin{aligned} a_g(n) + \frac{1}{2^{4g-6+2n}} \binom{4g-6+2n}{2g-3+n} b_g(n) &\sim \begin{cases} a_g \sim \sqrt{n^{g-1}} \sigma_g & \text{if } g \text{ is odd} \\ \frac{1}{2^{4g-6+2n}} \binom{4g-6+2n}{2g-3+n} b_g \sim \sqrt{\frac{n^{g-1}}{\pi}} \sigma_g & \text{if } g \text{ is even} \end{cases} \\ &= \sqrt{\pi^{\epsilon(g)-1} n^{g-1}} \sigma_g. \end{aligned}$$

Altogether, this gives the thesis. \square

After we formulated the above conjecture, Chen, Möller and Sauvaget [CMS+19] found a new formula to compute Masur–Veech volumes via intersection theory, from which they could prove our conjecture. Their approach will be discussed in the next chapter.

8.2.1 — SIEGEL–VEECH CONSTANTS

Let us move our attention to some applications to Siegel–Veech constants. The area Siegel–Veech constant $c_{g,n}^{\text{SV}}$ of $\mathcal{Q}_{g,4g-4+2n}(1^{4g-4+n}, -1^n)$ is a positive real number related to the asymptotic number of flat cylinders of a generic quadratic differential. More precisely, given an element $(C, \varphi) \in \mathcal{Q}_{g,4g-4+2n}(1^{4g-4+n}, -1^n)$ we define

$$n_{\text{area}}(C, \varphi; t) = \frac{1}{\text{Area}(C, \varphi)} \sum_{\substack{c \subset \subset \varphi \\ w(c) \leq t}} \text{Area}(c), \quad (8.2.18)$$

where the sum is over flat cylinders c of φ whose width $w(c)$ (or circumference) is less or equal to t and Area refers to the total mass of the measure induced by the flat metric of φ . By a theorem of Veech and Vorobets [Vee98; Vor05], the number

$$c_{g,n}^{\text{SV}} = \frac{1}{V_{g,n}^{\text{MV}}} \frac{1}{\pi t^2} \int_{\mathcal{Q}_{g,4g-4+2n}(1^{4g-4+n}, -1^n)} n_{\text{area}}(C, \varphi; t) dv_{\text{MV}}(C, \varphi) \quad (8.2.19)$$

exists and is independent of $t > 0$. It is called the (area) *Siegel–Veech constant* of the principal stratum of the moduli space of quadratic differentials.

Remark 8.2.8. As shown by Eskin and Masur, the asymptotic count of cylinders is almost everywhere constant over the moduli space: for a.e. $(C, \varphi) \in \mathcal{Q}_{g,4g-4+2n}(1^{4g-4+n}, -1^n)$ the limit

$$\lim_{t \rightarrow \infty} \text{Area}(C, \varphi) \frac{n_{\text{area}}(C, \varphi; t)}{\pi t^2} \quad (8.2.20)$$

exists and is independent of (C, φ) . In particular the constant value is the Siegel–Veech constant $c_{g,n}^{\text{SV}}$, which can be recovered by integration over the moduli space.

In [Gou15], Goujard showed how to compute $c_{g,n}^{\text{SV}}$ in terms of the Masur–Veech volumes. Her result is in fact more general, as it deals with all strata of the moduli space of quadratic differentials, while the present dissertation is only concerned with the principal stratum. To state her result cleanly, consider the rescaled Masur–Veech volumes

$$V_{g,n}^{\text{MV}} = 2^{4g-2+n} \frac{(4g-4+n)!}{(6g-7+2n)!} \widetilde{V}_{g,n}^{\text{MV}}. \quad (8.2.21)$$

THEOREM 8.2.9 ([Gou15, Section 4.2, Corollary 1]). *For $g, n \geq 0$ such that $2g-2+n > 1$, we have*

$$c_{g,n}^{\text{SV}} \widetilde{V}_{g,n}^{\text{MV}} = \frac{1}{4} \left(\widetilde{V}_{g-1, n+2}^{\text{MV}} + \frac{1}{2} \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n}} \frac{n!}{n_1!n_2!} \widetilde{V}_{g_1, 1+n_1}^{\text{MV}} \widetilde{V}_{g_2, 1+n_2}^{\text{MV}} \right). \quad (8.2.22)$$

In [Gou15] the contribution of $V_{0,3}^{\text{MV}} V_{g,n-1}^{\text{MV}}$ was written separately, but this term can be included in the sum if we remark that $V_{0,3}^{\text{MV}} = 4$ (see Table 8.1) and

$$\frac{(2n-5)!}{(n-3)!} \Big|_{n=2} = \lim_{n \rightarrow 2} \frac{\Gamma(2n-4)}{\Gamma(n-2)} = \frac{1}{2}. \quad (8.2.23)$$

Using topological recursion for Masur–Veech volumes, together with Goujard’s formula, we were able to compute a large amount of Siegel–Veech constants (see Table 8.3). Moreover, combining Goujard’s formula with the conjectural form of Masur–Veech volumes, we can obtain formulas for the area Siegel–Veech constants. In genus zero, the conjecture reduces to a result of Eskin–Kontsevich–Zorich [EKZ14].

CONDITIONAL THEOREM 8.2.10. Assuming Conjecture 8.2.6 to hold, there exist polynomials $a_g^*, b_g^* \in \mathbb{Q}[n]$ with degrees

$$\deg a_g^* = \left\lfloor \frac{g}{2} \right\rfloor \quad \text{and} \quad \deg b_g^* = \left\lfloor \frac{g+1}{2} \right\rfloor \quad (8.2.24)$$

such that, for any $n \geq 0$ such that $2g - 2 + n > 1$

$$c_{g,n}^{\text{SV}} = \frac{n+5-5g}{6\pi^2} + \frac{1}{(2g-3+n)\pi^2} \frac{a_g^*(n) + \frac{2g-3+n}{2^{4g-6+2n}} \binom{4g-6+2n}{2g-3+n} b_g^*(n)}{a_g(n) + \frac{1}{2^{4g-6+2n}} \binom{4g-6+2n}{2g-3+n} b_g(n)}. \quad (8.2.25)$$

Again, the notation here slightly differ from [And+19], and we refer to the original article for a complete proof which goes via generating series. Indeed Conjecture 8.2.6 is equivalent to a certain polynomial behaviour for the generating series of Masur–Veech volumes, while Goujard’s result can be rewritten as a relation between the generating series of Masur–Veech volumes and that of Siegel–Veech constants. Combining the two result, one can deduce the above polynomial structure for Siegel–Veech constants.

As for Masur–Veech volumes, we were able to deduce the asymptotic behaviour of Siegel–Veech constants as $n \rightarrow \infty$.

CONDITIONAL THEOREM 8.2.11. Assuming Conjecture 8.2.6 to hold, the Siegel–Veech constants of the principal strata of the moduli space of quadratic differentials have the following asymptotic behaviour as $n \rightarrow \infty$:

$$c_{g,n}^{\text{SV}} \sim \frac{n+5-5g}{6\pi^2} + \frac{1}{\pi^{\epsilon(g)+3/2} n^{1/2}} \frac{\sigma_g^*}{\sigma_g} + O(n^{-1}) \quad (8.2.26)$$

where σ_g^* is the top coefficient of a_g^* if g is even and the top coefficient of b_g^* if g is odd.

A consequence of the above result is a refined version of Fougeron’s conjecture [Fou18] for the $n \rightarrow \infty$ asymptotic behaviour of $L_{g,n}^+$, the sum of the g Lyapunov exponents of the Hodge bundle along the Teichmüller flow on the principal stratum of the moduli space of quadratic differentials. Based on extensive numerical experiments, Fougeron conjectured that for each g we have $L_{g,n}^+ = O(n^{-1/2})$ as $n \rightarrow \infty$. On the other hand, by a result of Eskin–Kontsevich–Zorich [EKZ14, Theorem 2], $L_{g,n}^+$ are related to Siegel–Veech constants by

$$\frac{\pi^2}{3} c_{g,n}^{\text{SV}} = \frac{n+5-5g}{18} + L_{g,n}^+. \quad (8.2.27)$$

The above formula, together with Equation (8.2.17), gives the following asymptotic refinement of Fougeron’s conjecture:

$$L_{g,n}^+ \sim \frac{3}{\pi^{\epsilon(g)-1/2} n^{1/2}} \frac{\sigma_g^*}{\sigma_g} + O(n^{-1}). \quad (8.2.28)$$

We remark again that, thanks to the work of Chen–Möller–Sauvaget [CMS+19], the above conditional theorems are now theorems.

$n \setminus g$	0	1	2	3	4	5	6
0	-	-	$\frac{19}{6}$	$\frac{24199}{8625}$	$\frac{283794163}{105312050}$	$\frac{180693680}{68465079}$	$\frac{806379495590975}{309492103568838}$
1	-	-	$\frac{230}{87}$	$\frac{529239}{205550}$	$\frac{14053063}{5518645}$	$\frac{533759417507}{210967972242}$	$\frac{4346055982466800}{1725192578138153}$
2	-	$\frac{7}{3}$	$\frac{8131}{3370}$	$\frac{2843354}{1164495}$	$\frac{11842209371}{4827273270}$	$\frac{606925117339}{246886623873}$	$\frac{122318875814791931}{49704331575032610}$
3	-	$\frac{47}{22}$	$\frac{11041}{4785}$	$\frac{73870699}{31157850}$	$\frac{35221419482}{14674841399}$	$\frac{82681229028041}{34222259372754}$	$\frac{5057811587495459887}{2085014933689449405}$
4	$\frac{3}{2}$	$\frac{44}{21}$	$\frac{688823}{303270}$	$\frac{187549387}{80056955}$	$\frac{1414826039249}{595067328174}$	$\frac{1031120131654286}{430104304558221}$	$\frac{1339844245835171101}{555962784408367098}$
5	$\frac{5}{3}$	$\frac{2075}{978}$	$\frac{96716}{42445}$	$\frac{87365995}{37248558}$	$\frac{15788133716389}{6636637127685}$	$\frac{1245335246460801}{519291721160462}$	$\frac{321899861240823487478}{133543614171105755337}$
6	$\frac{11}{6}$	$\frac{697}{319}$	$\frac{8622217}{3723846}$	$\frac{1433623484}{604494345}$	$\frac{7380284015613}{3075881257378}$	$\frac{18305424406953487}{7579668229551231}$	$\frac{3150765025310943712637}{1299328235398448522070}$
7	2	$\frac{17101}{7530}$	$\frac{10506949}{4426995}$	$\frac{12557689333}{5197985038}$	$\frac{32906433038620}{13511227345917}$	$\frac{165332043184123111}{67603007456990598}$	$\frac{1276869600669686371105}{520859415513533871089}$
8	$\frac{13}{6}$	$\frac{17630}{7431}$	$\frac{44927707}{18358630}$	$\frac{3273823127}{1322965425}$	$\frac{1905176709014543}{766815957735306}$	$\frac{931701551880070892}{374503401099176525}$	$\frac{32923598627691820002839}{13230080856193087574502}$
9	$\frac{7}{3}$	$\frac{194829}{78406}$	$\frac{480821458}{189797505}$	$\frac{515867741141}{202690068090}$	$\frac{3294839869674121}{1294900913828351}$	$\frac{13416096198217292533}{5281789061573971854}$	$\frac{403660475951758341605956}{159243800274510466905831}$
10	$\frac{5}{2}$	$\frac{202415}{77691}$	$\frac{905804827}{344519274}$	$\frac{488680850166}{186140734195}$	$\frac{658216299971112017}{251833411938374130}$	$\frac{2586449275763662283}{994394857621596381}$	$\frac{57921215793035879725637191}{22369036588679274930271514}$
11	$\frac{8}{3}$	$\frac{5054467}{1849998}$	$\frac{1761936475}{644710519}$	$\frac{2297552653219}{846645532890}$	$\frac{212103557000574050}{78822383768883681}$	$\frac{208627514502680586639}{78148187499405113102}$	$\frac{9156519282251402538004459}{3453848157129972968014845}$

Table 8.3: Siegel–Veech constants $\pi^2 c_{g,n}^{SV}$. They are computed from Masur–Veech volumes of Table 8.2 using Goujard’s formula. In grey, the values computed from the predicted Masur–Veech volumes.

8.3 — ASYMPTOTIC OF SQUARE-TILED SURFACES AND MASUR–VEECH POLYNOMIALS

In the previous section, we described how Masur–Veech volumes can be computed as the constant term of some polynomials $V_{g,n}^{\text{MV}}(L_1, \dots, L_n)$. In this section we show how such polynomials can be recovered by considering a certain limit of the square-tiled surface count presented in Section 7.3.

THEOREM 8.3.1. *Let $L \in \mathbb{Z}_+^n$ such that $L_1 + \dots + L_n \in 2\mathbb{Z}$. We have*

$$\lim_{\substack{T \rightarrow \infty \\ T \in 2\mathbb{Z}_+}} \frac{N_{g,n}^\square(TL_1, \dots, TL_n; e^{-1/T})}{T^{6g-6+2n}} = 2^{3-2g-n} V_{g,n}^{\text{MV}}(L_1, \dots, L_n). \quad (8.3.1)$$

Proof. Let T be an even integer and set $q = e^{-1/T}$. The strategy of the proof is to start from Equation (7.3.9), expressing $N_{g,n}^\square$ as a sum over stable graphs, substitute $q = e^{-1/T}$ and analyse the limit as $T \rightarrow \infty$. To this end, fix a stable graph Γ of type (g, n) . We want to compute the large T behavior of

$$\begin{aligned} & \sum_{\ell: E_\Gamma \rightarrow \mathbb{Z}_+} \prod_{v \in V_\Gamma} N_{g(v), n(v)}((\ell_e)_{e \in E(v)}, (TL_\lambda)_{\lambda \in \Lambda(v)}) \prod_{e \in E_\Gamma} \frac{\ell_e q^{\ell_e}}{1 - q^{\ell_e}} \\ &= \sum_{\hat{\ell}: E_\Gamma \rightarrow T^{-1}\mathbb{Z}_+} \prod_{v \in V_\Gamma} N_{g(v), n(v)}((T\hat{\ell}_e)_{e \in E(v)}, (TL_\lambda)_{\lambda \in \Lambda(v)}) \prod_{e \in E_\Gamma} \frac{T \hat{\ell}_e}{e^{\hat{\ell}_e} - 1}. \end{aligned}$$

Let $\mathbb{L} \subseteq \mathbb{Z}^{E_\Gamma}$ be the sublattice defined by the congruences

$$\forall v \in V_\Gamma, \quad \sum_{e \in E(v)} \ell_e + \sum_{\lambda \in E(\lambda)} L_\lambda \in 2\mathbb{Z}.$$

From Norbury’s result (cf. Corollary 5.3.4), we know that the vertex weights $N_{g(v), n(v)}$ vanish unless $(T\hat{\ell}_e)_e \in \mathbb{L}_+ = \mathbb{Z}_+^{E_\Gamma} \cap \mathbb{L}$. In this case, they limit to the Kontsevich volumes, up to powers of 2. More precisely,

$$N_{g(v), n(v)}((T\hat{\ell}_e)_{e \in E(v)}, (TL_\lambda)_{\lambda \in \Lambda(v)}) \sim \frac{T^{6g(v)-6+2n(v)}}{2^{2g(v)-3+n(v)}} V_{g(v), n(v)}^{\text{K}}((\hat{\ell}_e)_{e \in E(v)}, (L_\lambda)_{\lambda \in \Lambda(v)})$$

up to an error that will produce subleading terms when $T \rightarrow \infty$. We are left with analysing the large T behaviour of the following sum:

$$\sum_{\hat{\ell} \in T^{-1}\mathbb{L}_+} \prod_{v \in V_\Gamma} \frac{T^{6g(v)-6+2n(v)}}{2^{2g(v)-3+n(v)}} V_{g(v), n(v)}^{\text{K}}((\hat{\ell}_e)_{e \in E(v)}, (L_\lambda)_{\lambda \in \Lambda(v)}) \prod_{e \in E_\Gamma} \frac{T \hat{\ell}_e}{e^{\hat{\ell}_e} - 1}.$$

Now, since $V_{g,n}^{\text{K}}(L_1, \dots, L_n)$ are polynomial in L_1, \dots, L_n , the function of $\hat{\ell}$ appearing in the summands is a continuous function of $\hat{\ell} \in \mathbb{R}_+^{E_\Gamma}$, which is Riemann-integrable due to the exponential decay in the edge weights. Taking into account the relation between the Kontsevich measure and the Lebesgue measure (cf. Lemma 4.1.6) as in Proposition 5.2.7, we obtain that the above sum is asymptotically equivalent to

$$2^{1-|V_\Gamma|} \int_{\mathbb{R}_+^{E_\Gamma}} \prod_{v \in V_\Gamma} \frac{T^{6g(v)-6+2n(v)}}{2^{2g(v)-3+n(v)}} V_{g(v), n(v)}^{\text{K}}((\hat{\ell}_e)_{e \in E(v)}, (L_\lambda)_{\lambda \in \Lambda(v)}) \prod_{e \in E_\Gamma} \frac{T^2 \hat{\ell}_e d\hat{\ell}_e}{e^{\hat{\ell}_e} - 1}$$

when T is large. The overall powers of 2 and T can be easily computed as

$$1 - |V_\Gamma| - \sum_{v \in V_\Gamma} (2g(v) - 3 + n(v)) = -(2g - 3 + n),$$

$$2|E_\Gamma| + \sum_{v \in V_\Gamma} (6g(v) - 6 + 2n(v)) = 6g - 6 + 2n,$$

which are independent of Γ . Performing the (finite) sum over all stable graphs of type (g, n) weighted by automorphisms, and dividing the result by $T^{-(6g-6+2n)}$, one finds exactly the sum over stable graphs computing the Masur–Veech polynomials in Equation (8.2.11). \square

Remark 8.3.2. The scaling TL_i of the boundary term L_i is of strange nature. Indeed q^T is of order 1, suggesting that the typical contribution in $N_{g,n}^\square$ comes from surfaces with core area $O(T)$, but scaling the area with T usually rescale the boundary by \sqrt{T} . So the limit in Theorem 8.3.1 somehow reflects a blowup of the contribution coming from the boundaries of the square-tiled surface, that is necessary in order to obtain the Masur–Veech polynomials.

CHAPTER 9 — AN INTERSECTION THEORETIC APPROACH

In this chapter we present some results of Chen–Möller–Sauvaget [CMS+19] concerning the Masur–Veech volumes of the moduli spaces of quadratic differentials in term of intersection numbers on the moduli space of curve involving the Segre class $s(\mathcal{E}^{(2)}; u) = \sum_{k=0}^{3g-3+n} u^k s_k$ of the quadratic Hodge bundle $\mathcal{E}^{(2)} \rightarrow \overline{\mathcal{M}}_{g,n}$:

$$\langle s_{k_0} \tau_{k_1} \cdots \tau_{k_n} \rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} s_{k_0} \prod_{i=1}^n \psi_i^{k_i}, \quad k_0 + k_1 + \cdots + k_n = 3g - 3 + n. \quad (9.0.1)$$

We then show how topological recursion can compute such intersection numbers. In turn, we find a new topological recursion for computing Masur–Veech volumes.

THEOREM 9.A (Topological recursion for $\langle s_{k_0} \tau_{k_1} \cdots \tau_{k_n} \rangle_g$). *The above intersection numbers are computed by topological recursion on the spectral curve on \mathbb{P}^1 given by*

$$x(z) = \log(z) - z, \quad y(z) = z^2, \quad B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2} \quad (9.0.2)$$

by expanding the correlators on the basis of differentials determined by $\theta^k(z) = \left(\frac{z}{1-z} \frac{d}{dz}\right)^k \frac{z}{1-z}$:

$$\omega_{g,n}(z_1, \dots, z_n) = 2^{-(2g-2+n)} \sum_{\substack{k_0, \dots, k_n \geq 0 \\ k_0 + \dots + k_n = 3g-3+n}} \langle s_{k_0} \tau_{k_1} \cdots \tau_{k_n} \rangle_g \prod_{i=1}^n d\theta^{k_i}(z_i), \quad (9.0.3)$$

In particular, the Masur–Veech volumes of the principal strata of quadratic differentials are computed as

$$V_{g,n}^{\text{MV}} = (-1)^{3g-3+n} \frac{2^{3-n} \pi^{6g-6+2n}}{(6g-7+2n)!} \left(\prod_{i=1}^n \text{Res}_{z_i=1} \right) \omega_{g,n}(z_1, \dots, z_n). \quad (9.0.4)$$

The Segre class of the quadratic Hodge bundle is a specialisation of Chiodo’s class, namely $s(\mathcal{E}^{(2)}) = C_{g,n}^{1,2}(1^n)$ in the notation of Section 2.2.2. Indeed, we obtain the above spectral curve by specialising the topological recursion for Chiodo’s class weighted by a degree parameter. This result generalises the topological recursion of [LPSZ17] (where no degree parameter appears) and the topological recursion of [CMS+19, Appendix A] (where only Chiodo classes for $r = 1$ are considered).

To conclude, we show how the Chern class of the same quadratic Hodge bundle capture the Euler characteristic of the moduli space of curves, and present an intersection-theoretic proof of the Harer–Zagier formula [HZ86] via Hodge integrals.

THEOREM 9.B (Harer–Zagier formula via Hodge integrals). *The orbifold Euler characteristic of $\mathcal{M}_{g,n}$ is given by*

$$\chi_{g,n} = \sum_{\ell \geq 0} \frac{(-1)^\ell}{\ell!} \sum_{\mu_1, \dots, \mu_\ell \geq 1} \int_{\overline{\mathcal{M}}_{g,n+\ell}} \Lambda(-1) \prod_{j=1}^{\ell} \psi_{n+j}^{\mu_j+1}. \quad (9.0.5)$$

Moreover, the above linear combination of Hodge integrals is evaluated as

$$\chi_{g,n} = \begin{cases} (-1)^{n-3}(n-3)! & g=0, n \geq 3, \\ (-1)^n \frac{(n-1)!}{12} & g=1, n \geq 1, \\ (-1)^n (2g-3+n)! \frac{B_{2g}}{2g(2g-2)!} & g \geq 2, n \geq 0. \end{cases} \quad (9.0.6)$$

9.0.1 — RELATION WITH PREVIOUS WORKS AND OPEN QUESTIONS

The strategy of Chen–Möller–Sauvaget [CMS+19] uses a good metric h (in the sense of Mumford) induced by the area function on the compactified strata, and prove that the corresponding volume form ν_h coincide with the Masur–Veech volume form ν_{MV} . On the principal strata, the top Segre class of the quadratic Hodge bundle corresponds to ν_h . A natural question would be to generalise this argument to moduli spaces of higher differentials.

QUESTION 9.C. *Generalise the result of Chen–Möller–Sauvaget to (principal strata of) moduli spaces of higher differentials.*

It is interesting to notice that, as of now, there are two ways of computing Masur–Veech volumes of the principal strata of quadratic differentials via topological recursion.

1. The first recursion [And+19; And+20], explained in the previous chapter, computes the average number of multicurves (in the hyperbolic or combinatorial setting), i.e. the polynomials

$$\langle N_{g,n} \rangle(L; t) \quad \text{or} \quad \langle N_{g,n}^{\text{comb}} \rangle(L; t). \quad (9.0.7)$$

We then recover the Masur–Veech volumes as the coefficient of $t^{6g-6+2n}$, corresponding to the asymptotic number of multicurves [Miro8a].

2. The second recursion [CMS+19, Appendix A], explained in Section 9.2, computes the intersection of the Segre class of the quadratic Hodge bundle with ψ -classes. We then recover the Masur–Veech volumes by taking a specific residue extracting the top Segre class intersection.

A similar situation occurs for the Euler characteristic of the moduli space of curves.

1. The first recursion [Nor10, Theorem 2] computes the number of lattice points on $\mathcal{M}_{g,n}^{\text{comb}}(L)$, i.e. the quasi-polynomials

$$N_{g,n}(L) = \sum_{G \in \mathcal{M}_{g,n}^{\text{comb}, \mathbb{Z}}(L)} \frac{1}{|\text{Aut}(G)|}. \quad (9.0.8)$$

Norbury recovers the Euler characteristic of $\mathcal{M}_{g,n}$ as the constant term of $N_{g,n}(L)$.

2. The second recursion, developed in Section 9.3, computes the intersection of the Chern class of the quadratic Hodge bundle with ψ -classes. We then recover the Euler characteristic by taking a specific residue extracting the top Chern class intersection.

9.0.2 — ORGANISATION OF THE CHAPTER

The chapter is organised as follows.

- In Section 9.1 we present the original work of Chen–Möller–Sauvaget, expressing Masur–Veech volumes via intersection theory.
- In Section 9.2 we find the spectral curve computing Chiodo’s class intersection numbers with a degree parameter. We then specialise it to recover the Segre class of the quadratic Hodge bundle.
- To conclude, in Section 9.3 we compute the Euler character sic of $\mathcal{M}_{g,n}$ via the Chern class of the quadratic Hodge bundle, expressing it in terms of Hodge classes, and we prove the Harer–Zagier formula via intersection theory.

9.1 — MASUR–VEECH VOLUMES AND INTERSECTION THEORY

The new major contribution of Chen–Möller–Sauvaget is a conjectural expression for the Masur–Veech volumes of the moduli space of quadratic differentials via intersection theory, proved for the case of strata with odd zeros only (see also [CMSZ20] for analogous results on the moduli space of holomorphic differentials). In order to talk about intersection numbers, a compactification of the moduli space is required. To this end, Bainbridge–Chen–Gendron–Grushevsky–Möller [Bai+19] introduced a compactification $\mathbb{P}\overline{\mathcal{Q}}_g(\mu, \nu)$ of the projectivised strata of quadratic differentials, called the *incidence variety compactification*. Let ζ be the first Chern class of the universal line bundle $\mathcal{O}(1)$ on $\mathbb{P}\overline{\mathcal{Q}}_g(\mu, \nu)$, and denote by Ψ_{μ_i} and Ψ_{ν_j} the pullbacks to the strata $\mathbb{P}\overline{\mathcal{Q}}_g(\mu, \nu)$ under the map $\mathbb{P}\overline{\mathcal{Q}}_g(\mu, \nu) \rightarrow \overline{\mathcal{M}}_{g,r+s}$ forgetting the quadratic differential (here $r = \ell(\mu)$ and $s = \ell(\nu)$).

CONJECTURE 9.1.1 ([CMS+19, Conjecture 1.1]). *The Masur–Veech volumes of strata of quadratic differentials are expressed as the following intersection numbers:*

$$V_g^{\text{MV}}(\mu, \nu) = \frac{2^{r-s+3}(2\pi i)^{2g-2+s}}{(2g-3+r+s)!} \int_{\mathbb{P}\overline{\mathcal{Q}}_g(\mu, \nu)} \zeta^{2g+s-3} \Psi_{\mu_1} \dots \Psi_{\mu_r}, \quad (9.1.1)$$

where $\Psi_{\mu_1}, \dots, \Psi_{\mu_r}$ are the classes associated to the r even order zeros.

THEOREM 9.1.2 ([CMS+19, Theorem 1.2]). *Conjecture 9.1.1 holds for strata with odd zeros only (i.e. $\mu = \emptyset$):*

$$V_g^{\text{MV}}(\nu) = \frac{2^{3-s}(2\pi i)^{2g-2+s}}{(2g-3+s)!} \int_{\mathbb{P}\overline{\mathcal{Q}}_g(\nu)} \zeta^{2g+s-3}. \quad (9.1.2)$$

The above theorem has a particularly nice expression for the principal strata, which we recall correspond to quadratic differentials with only simple zeros and simple poles: $\mu = \emptyset$ and $\nu = (1^{4g-4+n}, -1^n)$. In this case, the projectivisation of the quadratic Hodge bundle $\mathcal{E}^{(2)} \rightarrow \overline{\mathcal{M}}_{g,n}$ provides an alternative compactification for $\mathbb{P}\overline{\mathcal{Q}}_g(1^{4g-4+n}, -1^n)$, where the top self-intersection of the ζ -class corresponds to the top Segre class of the quadratic Hodge bundle $\mathcal{E}^{(2)}$. Here we recall that the quadratic Hodge bundle corresponds to Chiodo’s construction (see Section 2.2.2) for $r = 1, s = 2$ and all $a_i = 1$. In other words, the fiber over

a stable curve (C, x_1, \dots, x_n) is given by the vector space $H^0(C, \omega_C^{\otimes 2}(x_1 + \dots + x_n))$, hence parametrising quadratic differentials with at worst simple poles at the marked points. In this case, Chiodo's class coincide with the Segre class, i.e. $C_{g,n}^{1,2}(1^n) = s(\mathcal{E}^{(2)})$, since by degree reasons $H^1(C, \omega_C^{\otimes 2}(x_1 + \dots + x_n)) = 0$, and thus $R^\bullet \pi_* \mathcal{L} = \mathcal{E}^{(2)}$.

Thanks to Chiodo's formula for the Chern character of $R^\bullet \pi_* \mathcal{L}$, we can express the Masur–Veech volumes of the principal strata of quadratic differentials in terms of Hodge integrals:

$$\langle \lambda_m \tau_{k_1} \cdots \tau_{k_n} \rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \lambda_m \psi_1^{k_1} \cdots \psi_n^{k_n}. \quad (9.1.3)$$

As in the previous section, denote by $V_{g,n}^{\text{MV}}$ the Masur–Veech volumes of the principal stratum $\mathcal{Q}_g(1^{4g-4+n}, -1^n)$.

THEOREM 9.1.3 ([CMS+19, Theorem 1.3]). *The Masur–Veech volumes of the principal strata of quadratic differentials are expressed as the following linear combination of Hodge integrals:*

$$\begin{aligned} V_{g,n}^{\text{MV}} &= (-1)^{3g-3+n} \frac{2^{2g+1} \pi^{6g-6+2n}}{(6g-7+2n)!} \int_{\overline{\mathcal{M}}_{g,n}} s(\mathcal{E}^{(2)}) \\ &= 2^{2g+1+n} \pi^{6g-6+2n} \frac{(4g-4+n)!}{(6g-7+2n)!} \sum_{k=0}^g \binom{5g-5-k}{2}_n \frac{\langle \lambda_k \tau_2^{3g-3-k} \rangle_g}{(3g-3-k)!}. \end{aligned} \quad (9.1.4)$$

Here $(x)_n = x(x+1) \cdots (x+n-1)$ denotes the Pochhammer symbol.

As a consequence of the above expression, Chen–Möller–Sauvaget were able to prove Conjecture 8.2.6 regarding the polynomial structure of Masur–Veech volumes for fixed genus and varying number of poles (cf. [CMS+19, Theorem 1.4]), and consequently the conditional results 8.2.7, 8.2.10 and 8.2.11. In particular, they found a closed expression for the polynomials appearing in Conjecture 8.2.6 and Conditional theorem 8.2.10 in terms of Hodge integrals.

It is worth mentioning that Kazarian [Kaz21] and Yang–Zagier–Zhang [YZZ20] independently proved an effective recursion for the Masur–Veech volumes using techniques from integrable systems, based on the above formula for $V_{g,n}^{\text{MV}}$. This allowed them to conjecture the *large genus* asymptotics of Masur–Veech volumes and Siegel–Veech constants, later proved by Aggarwal [Agg20].

9.2 — ANOTHER TOPOLOGICAL RECURSION FOR MASUR–VEECH VOLUMES

In the previous section, we saw how Masur–Veech volumes are expressed in terms of intersection theory of the Segre class of the quadratic Hodge bundle. In this section, we prove that their intersection theory is computed by topological recursion on the spectral curve on \mathbb{P}^1 given by

$$x(z) = \log(z) - z, \quad y(z) = z^2, \quad B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}. \quad (9.2.1)$$

Notice that the above spectral curve differ from the one in [CMS+19] by a minus sign in front of the logarithm, and agrees with the analysis of [LPSZ17]. However, we emphasize that there is no contradiction in this: the two spectral curves store the same intersection numbers, and

the only difference is in the multidifferentials multiplying them. In order to give a complete overview of this fact, we consider the more general case of the Chern *polynomial* of the derived pushforward $-R^\bullet \pi_* \mathcal{L}$ on the moduli space $\overline{\mathcal{M}}_{g,a}^{r,s}$ of twisted spin curves (cf. Section 2.2.2), for which we compute the associated spectral curve.

9.2.1 — TOPOLOGICAL RECURSION FOR THE CHIODO POLYNOMIALS

Fix a positive integer r , and integers s, a_1, \dots, a_n satisfying the modular constraint

$$\sum_{i=1}^n a_i \equiv s(2g - 2 + n) \pmod{r}. \quad (9.2.2)$$

Recall from Section 2.2.2 the definition of the moduli space¹ $\overline{\mathcal{M}}_{g,a}^{r,s}$ of twisted spin curves: it parametrises stable curves (C, x_1, \dots, x_n, L) of genus g with n marked points and a line bundle $L \rightarrow C$ satisfying $L^{\otimes r} \cong \omega_{\log}^{\otimes s}(-\sum_i a_i x_i)$. It has a universal curve and a universal line bundle

$$\pi: \overline{\mathcal{C}}_{g,a}^{r,s} \rightarrow \overline{\mathcal{M}}_{g,a}^{r,s}, \quad \mathcal{L} \rightarrow \overline{\mathcal{C}}_{g,a}^{r,s}, \quad (9.2.3)$$

and it comes with a forgetful map $\epsilon: \overline{\mathcal{M}}_{g,a}^{r,s} \rightarrow \overline{\mathcal{M}}_{g,n}$.

DEFINITION 9.2.1. Define the *Chiodo polynomial* as

$$C_{g,n}^{r,s}(a_1, \dots, a_n; \tau) = \epsilon_* c(-R^\bullet \pi_* \mathcal{L}; \tau) \in H^\bullet(\overline{\mathcal{M}}_{g,n})[\tau], \quad (9.2.4)$$

where

$$c(-E^\bullet; \tau) = \exp \left(\sum_{d \geq 1} (-\tau)^d (d-1)! \text{ch}_d(E^\bullet) \right). \quad (9.2.5)$$

Here $R^\bullet \pi_* \mathcal{L}$ is the derived pushforward of \mathcal{L} , and $c(\cdot; \tau)$ is its Chern polynomial.

The aim of this section is to prove, thorough the Eynard–DOSS correspondence, that the intersection theory of the Chiodo polynomial $C_{g,n}^{r,s}(a; \tau)$ is computed by the spectral curve on \mathbb{P}^1 given by

$$x(z) = t \log(z) - z^r, \quad y(z) = z^s, \quad B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2} \quad (9.2.6)$$

for $t = \tau^{-1} \in \mathbb{C}^\times$. These computations generalises those in [SSZ15; LPSZ17], where only $t = 1$ was considered, and those in [CMS+19, Appendix A], where only $r = 1$ was considered.

We first choose an arbitrary determination of the logarithm whose branchcut is away from the ramification points, and we point out that the choice will not affect our discussion. The same holds for choices of an r -th root and a square-root of t . The ramification points of the spectral curve are given by

$$a_i = J^i \left(\frac{t}{r} \right)^{\frac{1}{r}}, \quad \text{where } J = e^{\frac{2\pi i}{r}}, i = 0, 1, \dots, r-1, \quad (9.2.7)$$

¹Contrary to Section 2.2.2, here we use the notation s instead of k to avoid confusion with indices.

III. Enumeration of multicurves and quadratic differentials

Let us choose $c[k] = \frac{1}{\sqrt{2r}}$ for all $k = 0, \dots, r-1$ and $c = \frac{\sqrt{t}}{s} \left(\frac{t}{r}\right)^{-\frac{s}{r}-1}$, so that we have local coordinates

$$x(z) = -\frac{\zeta_i^2(z)}{2r} + x(a_i). \quad (9.2.8)$$

We choose the following local expansion for z in the local coordinate ζ_i :

$$z = a_i + \frac{a_i}{\sqrt{tr}} \zeta_i(z) + O(\zeta_i^2(z)), \quad (9.2.9)$$

which in turn determines the local expansion of y as

$$y(z) = a_i^s + \frac{sa_i^s}{\sqrt{tr}} \zeta_i(z) + O(\zeta_i^2(z)). \quad (9.2.10)$$

With this choice, $\Delta^i = \frac{sa_i^s}{r\sqrt{t}}$ and $t^i = \frac{J^{is}}{tr}$ in the notations of Section 2.3.1. Moreover, the underlying topological field theory on $V = \mathbb{C}\langle e_0, \dots, e_{r-1} \rangle$ is given by

$$\eta(e_i, e_j) = \delta_{i,j}, \quad \mathbb{1} = \sum_{i=0}^{r-1} \frac{J^{is}}{tr} e_i, \quad \varpi_{g,n}(e_{i_1} \otimes \dots \otimes e_{i_n}) = \delta_{i_1, \dots, i_n} \left(\frac{J^{is}}{tr}\right)^{-(2g-2+n)}. \quad (9.2.11)$$

Let us compute now the other ingredients for the Eynard–DOSS formula. In the following lemma, the translation is expressed in terms of the associated \hat{T} (cf. Equation (2.3.30)).

LEMMA 9.2.2. *For $t \notin \mathbb{R}_-$, the auxiliary functions, R -matrix and translation associated to the spectral curve (9.2.6) are given by*

$$\xi^i(z) = \frac{a_i}{\sqrt{tr}} \frac{1}{a_i - z}, \quad (9.2.12)$$

$$R^{-1}(u)_i^j = \frac{1}{r} \sum_{k=0}^{r-1} J^{k(j-i)} \exp\left(-\sum_{m=1}^{\infty} \frac{B_{m+1}\left(\frac{k}{r}\right)}{m(m+1)t^m} (-u)^m\right), \quad (9.2.13)$$

$$\hat{T}^i(u) = \sum_{m=1}^{\infty} \frac{B_{m+1}\left(\frac{s}{r}\right)}{m(m+1)t^m} (-u)^m, \quad (9.2.14)$$

for $i, j = 0, \dots, r-1$.

Proof. For the auxiliary functions, we simply have

$$\xi^i(z) = \int^z \frac{B(\zeta_i(w), \cdot)}{d\zeta_i(w)} \Big|_{w=a_i} = \frac{a_i}{\sqrt{tr}} \int^z \frac{dz}{(a_i - z)^2} = \frac{a_i}{\sqrt{tr}} \frac{1}{a_i - z}.$$

For the R -matrix, inserting the expression for ξ^i in the definition of R and integrating by parts, we get

$$\begin{aligned} R^{-1}(u)_i^j &= -\sqrt{\frac{u}{2\pi}} \int_{\mathbb{R}} d\xi^i(\zeta_j) e^{-\frac{1}{2u} \zeta_j^2} \\ &= -\frac{1}{r} \frac{1}{\sqrt{2\pi tu}} \int_{\mathbb{R}} \frac{1}{1 - \frac{z(\zeta_j)}{a_i}} e^{-\frac{1}{2u} \zeta_j^2} \zeta_j d\zeta_j. \end{aligned}$$

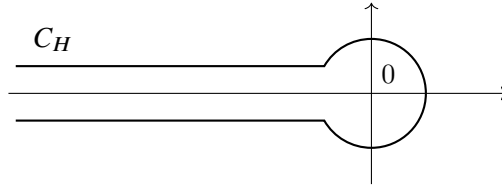


Figure 9.1: The Hankel contour.

We perform now the change of variable $\zeta_j \mapsto w$ determined by $z = J^j(\frac{w}{r})^{\frac{1}{r}}$, so that

$$-\frac{\zeta_j^2}{2r} = x - x(a_j) = \frac{w}{r} - \frac{w}{r} + \frac{t}{r} \log\left(\frac{w}{t}\right),$$

and differentiating both sides we get $\zeta d\zeta = (1 - \frac{w}{t})dw$. Note also that w runs along the Hankel contour C_H when ζ_j runs from $-\infty$ to $+\infty$ (see Figure 9.1 for a representation of the Hankel contour). As a consequence,

$$\begin{aligned} R^{-1}(u)_i^j &= -\frac{1}{r} \frac{1}{\sqrt{2\pi t u}} \int_{C_H} \frac{1}{1 - J^{j-i}(\frac{w}{t})^{\frac{1}{r}}} e^{\frac{1}{u}(w-w+t \log(\frac{w}{t}))} \left(1 - \frac{w}{t}\right) dw \\ &= \frac{e^{\frac{t}{u}}}{r} \frac{1}{\sqrt{2\pi t u}} \int_{C_H} \left(\frac{w}{t}\right)^{\frac{t}{u}-1} \frac{1 - \frac{w}{t}}{1 - J^{j-i}(\frac{w}{t})^{\frac{1}{r}}} e^{-\frac{w}{u}} dw. \end{aligned}$$

On the other hand, we have the geometric progression formula

$$\frac{1 - \frac{w}{t}}{1 - J^{j-i}(\frac{w}{t})^{\frac{1}{r}}} = \sum_{k=0}^{r-1} J^{k(j-i)} \left(\frac{w}{t}\right)^{\frac{k}{r}}$$

and the integral expression for the reciprocal of the Gamma function, together with its asymptotic expansion as $v \rightarrow 0$ (valid for $|\arg(-v^{-1})| < \pi$) involving Bernoulli polynomials:

$$e^{\frac{1}{v}} \sqrt{2\pi} \frac{(-v)^{\frac{1}{v}+a+\frac{1}{2}}}{\Gamma(a-v^{-1})} = e^{\frac{1}{v}} \sqrt{\frac{v}{2\pi}} \int_{C_H} (vw)^{\frac{1}{v}-a} e^{-w} dw \sim \exp\left(\sum_{m=1}^{\infty} \frac{B_{m+1}(a)}{m(m+1)} v^m\right). \quad (9.2.15)$$

Thus, for $t \notin \mathbb{R}_-$ we get

$$\begin{aligned} R^{-1}(u)_i^j &= \frac{e^{\frac{t}{u}}}{r} \frac{1}{\sqrt{2\pi t u}} \sum_{k=0}^{r-1} J^{k(j-i)} \int_{C_H} \left(\frac{w}{t}\right)^{\frac{t}{u}-1+\frac{k}{r}} e^{-\frac{w}{u}} dw \\ &= \frac{e^{\frac{t}{u}}}{r} \sqrt{\frac{u}{2\pi t}} \sum_{k=0}^{r-1} J^{k(j-i)} \int_{C_H} \left(\frac{u}{t} w\right)^{\frac{t}{u}-1+\frac{k}{r}} e^{-w} dw \quad \text{rescaling } w \mapsto uw \\ &\sim \frac{1}{r} \sum_{k=0}^{r-1} J^{k(j-i)} \exp\left(\sum_{m=1}^{\infty} \frac{B_{m+1}(1-\frac{k}{r})}{m(m+1)} \left(\frac{u}{t}\right)^m\right). \end{aligned}$$

We conclude using the property of Bernoulli polynomials $B_{m+1}(1-a) = (-1)^{m+1} B_{m+1}(a)$. For

the translation: performing the change of variable $\zeta_j \mapsto w$, we find

$$\begin{aligned}
 \exp(-\hat{T}^i(u)) &= \frac{r}{sa_i^s} \sqrt{\frac{t}{2\pi u}} \int_{\mathbb{R}} dy(\zeta_i) e^{-\frac{\zeta_i^2}{2u}} \\
 &= \frac{r}{sa_i^s} \sqrt{\frac{t}{2\pi u}} \int_{\mathbb{R}} sz^{s-1}(\zeta_i) e^{-\frac{\zeta_i^2}{2u}} \frac{dz(\zeta_i)}{d\zeta_i} d\zeta_i \\
 &= \frac{e^{\frac{t}{u}}}{\sqrt{2\pi tu}} \int_{C_H} \left(\frac{w}{t}\right)^{\frac{t}{u}-1+\frac{s}{r}} e^{-\frac{w}{u}} dw \\
 &= e^{\frac{t}{u}} \sqrt{\frac{u}{2\pi t}} \int_{C_H} \left(\frac{u}{t}w\right)^{\frac{t}{u}-1+\frac{s}{r}} e^{-w} dw \quad \text{rescaling } w \mapsto uw \\
 &\sim \exp\left(\sum_{m=1}^{\infty} \frac{B_{m+1}(1-\frac{s}{r})}{m(m+1)} \left(\frac{u}{t}\right)^m\right).
 \end{aligned}$$

We conclude again using the aforementioned property of Bernoulli polynomials. \square

Notice that the computations for the R -matrix and the translation required the technical assumption $t \notin \mathbb{R}_-$, that we used in order to apply the asymptotic expansion of the Gamma function. We can work around it thanks to the following polynomiality result. As we need to stress the dependence on t , we momentarily denote by $\omega_{g,n}(z_1, \dots, z_n; t)$ the multidifferentials associated to the spectral curve (9.2.6), by $x(z; t) = t \log(z) - z^r$, and by $d\xi^{k,i}(z; t)$ the differential forms given by

$$d\xi^{k,i}(z; t) = d\left(\left(\frac{z}{t-rz^r} \frac{d}{dz}\right)^k \xi^i(z; t)\right), \quad \xi^i(z; t) = \frac{J^i}{\sqrt{tr}} \left(\frac{t}{r}\right)^{\frac{1}{r}} \frac{1}{J^i\left(\frac{t}{r}\right)^{\frac{1}{r}} - z}. \quad (9.2.16)$$

We also set $I = \mathbb{N} \times \{0, \dots, r-1\}$.

LEMMA 9.2.3. *Consider the expansion of the correlators $\omega_{g,n}(z; t)$ in the basis of differentials $d\xi^\alpha$:*

$$\omega_{g,n}(z_1, \dots, z_n; t) = \sum_{\alpha_1, \dots, \alpha_n \in I} F_{g; \alpha_1, \dots, \alpha_n}(t) \prod_{i=1}^n d\xi^{\alpha_i}(z_i; t). \quad (9.2.17)$$

Then, denoting by $\mathbb{C}_d[\tau]$ the space of polynomials with complex coefficients in the variable τ of degree $\leq d$, we have

$$F_{g; \alpha_1, \dots, \alpha_n}(t) \in (-t)^{-\left(\frac{s}{r}-\frac{1}{2}\right)(2g-2+n)} \mathbb{C}_{3g-3+n}[t^{-1}]. \quad (9.2.18)$$

Proof. We first claim that

$$\omega_{g,n}(z_1, \dots, z_n; t) = (-t)^{-\left(\frac{s}{r}+1\right)(2g-2+n)} \omega_{g,n}\left((-t)^{-\frac{1}{r}} z_1, \dots, (-t)^{-\frac{1}{r}} z_n; -1\right).$$

Indeed, one can simply check that $x(z; t) = (-t)x\left((-t)^{-\frac{1}{r}} z; -1\right) + c$ and $y(z) = (-t)^{\frac{s}{r}} \left((-t)^{-\frac{1}{r}} z\right)^s$ for some c independent of z , so that

$$\begin{aligned}
 \omega_{0,1}(z; t) &= (-t)^{\frac{s}{r}+1} \omega_{0,1}\left((-t)^{-\frac{1}{r}} z; -1\right), \\
 \omega_{0,2}(z_1, z_2; t) &= \omega_{0,2}\left((-t)^{-\frac{1}{r}} z_1, (-t)^{-\frac{1}{r}} z_2; -1\right).
 \end{aligned}$$

Thanks to the homogeneity property of topological recursion (see Theorem 2.3.6), we obtain the claim. Notice that the claim holds for $n = 0$ too (cf. Definition 2.3.7).

On the other hand, by induction on k one can check that

$$d\xi^{k,i}(z; t) = (-t)^{-k-\frac{1}{2}} d\xi^{k,i}((-t)^{-\frac{1}{r}}z; -1).$$

We deduce that, denoting $\alpha_j = (k_j, i_j)$,

$$\begin{aligned} F_{g;\alpha_1,\dots,\alpha_n}(t) &= (-t)^{-\left(\frac{s}{r}+1\right)(2g-2+n)+\sum_{i=1}^n(k_i+\frac{1}{2})} F_{g;\alpha_1,\dots,\alpha_n}(-1) \\ &= (-t)^{-\left(\frac{s}{r}-\frac{1}{2}\right)(2g-2+n)+\sum_{i=1}^n k_i-(3g-3+n)} F_{g;\alpha_1,\dots,\alpha_n}(-1). \end{aligned}$$

Since $\sum_i k_i \leq 3g - 3 + n$, we have the thesis. \square

COROLLARY 9.2.4. *The R-matrix and translation formulae of Lemma 9.2.2 hold for $t \in \mathbb{C}^\times$.*

Proof. In the notation of the previous lemma, we have that for $t \notin \mathbb{R}_-$:

$$F_{g;\alpha_1,\dots,\alpha_n}(t) = \left(\frac{\sqrt{t}}{s} \left(\frac{t}{r} \right)^{-\frac{s}{r}-1} \right)^{2g-2+n} \int_{\mathcal{M}_{g,n}} \Omega_{g,n}(e_{i_1} \otimes \cdots \otimes e_{i_n}; t) \prod_{j=1}^n \psi_j^{k_j},$$

where $\alpha_j = (k_j, i_j)$ and $\Omega_{g,n} = RT\overline{\omega}_{g,n}$ is the CohFT obtained from the data of Equations (9.2.11), (9.2.13) and (9.2.14). The CohFT is a polynomial in t^{-1} , so that the right-hand side is a polynomial in t^{-1} up to the prefactor $\left(\frac{\sqrt{t}}{s} \left(\frac{t}{r} \right)^{-\frac{s}{r}-1} \right)^{2g-2+n}$. Thus, the equality holds for $t \in \mathbb{C}^\times$. \square

Consider now the change of basis on the vector space underlying the cohomological field theory, from (e_0, \dots, e_{r-1}) to (v_1, \dots, v_r) :

$$v_a = \sum_{i=0}^{r-1} \frac{J^{ai}}{tr} e_i, \quad e_i = t \sum_{a=1}^r J^{-ai} v_a. \quad (9.2.19)$$

In the following lemma, the indices of the Kronecker deltas are taken modulo r .

LEMMA 9.2.5. *In the basis (v_1, \dots, v_r) , the following holds.*

- *The underlying topological field theory is given by*

$$\eta(v_a, v_b) = \frac{1}{t^{2r}} \delta_{a+b}, \quad \overline{\omega}_{g,n}(v_{a_1} \otimes \cdots \otimes v_{a_n}) = (tr)^{2g-2} r \delta_{a_1+\dots+a_n-s(2g-2+n)}. \quad (9.2.20)$$

Moreover, the unit is given by v_s .

- *The R-matrix and translation are given by*

$$R^{-1}(u) = \exp \left(- \sum_{m=1}^{\infty} \frac{\text{diag}_{a=1}^r (B_{m+1}(\frac{a}{r}))}{m(m+1)t^m} (-u)^m \right), \quad (9.2.21)$$

$$T(u) = u \left(1 - \exp \left(- \sum_{m=1}^{\infty} \frac{B_{m+1}(\frac{s}{r})}{m(m+1)t^m} (-u)^m \right) \right) v_s. \quad (9.2.22)$$

III. Enumeration of multicurves and quadratic differentials

- The auxiliary functions $\theta^a = t \sum_{i=0}^{r-1} J^{-ai} \xi^i$ are given by

$$\theta^a(z) = \sqrt{t} \left(\frac{r}{t}\right)^{1-\frac{a}{r}} \sum_{m=0}^{\infty} \frac{\left(\frac{rm+r-a}{t}\right)^m}{m!} e^{(rm+r-a)x(z)}. \quad (9.2.23)$$

Proof. The pairing is given by a simple computation:

$$\eta(v_a, v_b) = \frac{1}{(tr)^2} \sum_{i,j=0}^{r-1} J^{ai+bj} \eta(e_i, e_j) = \frac{1}{(tr)^2} \sum_{i=0}^{r-1} J^{(a+b)i} = \frac{1}{t^2 r} \delta_{a+b}.$$

Similarly for the TFT, the R -matrix elements, and the translation. To conclude, let us compute the basis of auxiliary functions after the change of variable.

$$\theta^a(z) = t \sum_{i=0}^{r-1} J^{-ai} \xi^i(z) = \frac{\sqrt{t}}{1 - \frac{r}{t} z^r} \sum_{i,k=1}^{r-1} \frac{J^{-(a+k)i}}{r} \left(\frac{r}{t}\right)^{\frac{k}{r}} z^k = \sqrt{t} \left(\frac{r}{t}\right)^{1-\frac{a}{r}} \frac{z^{r-a}}{1 - \frac{r}{t} z^r}.$$

On the other hand, we can express z in terms of x through the Lambert W -function:

$$z = \left(-\frac{t}{r} W\left(-\frac{r}{t} e^{\frac{r}{t}x}\right)\right)^{\frac{1}{r}}.$$

In particular, one can compute $\frac{dz^\alpha}{dx} = \frac{\alpha}{t} \frac{z^\alpha}{1 - \frac{r}{t} z^r}$, so that setting $\alpha = r - a$ we find

$$\theta^a = \sqrt{t} \left(\frac{r}{t}\right)^{1-\frac{a}{r}} \frac{t}{r-a} \frac{dz^{r-a}}{dx} = \frac{t\sqrt{t}}{r-a} \frac{d}{dx} \left(-W\left(-\frac{r}{t} e^{\frac{r}{t}x}\right)\right)^{1-\frac{a}{r}}.$$

We can now use the expansion of the Lambert function (see [Cor+96, Equation (2.36)] as a reference)

$$\left(\frac{W(-t)}{-t}\right)^\alpha = \sum_{m=0}^{\infty} \frac{\alpha(m+\alpha)^{m-1}}{m!} t^m \quad (9.2.24)$$

to finally get

$$\begin{aligned} \theta^a &= \sqrt{t} \frac{t}{r} \frac{d}{dx} \sum_{m=0}^{\infty} \frac{(m+1-\frac{a}{r})^{m-1}}{m!} \left(\frac{r}{t} e^{\frac{r}{t}x}\right)^{m+1-\frac{a}{r}} \\ &= \sqrt{t} \left(\frac{r}{t}\right)^{1-\frac{a}{r}} \sum_{m=0}^{\infty} \frac{\left(\frac{rm+r-a}{t}\right)^m}{m!} e^{(rm+r-a)x}. \end{aligned}$$

□

THEOREM 9.2.6. For every $t \in \mathbb{C}^\times$, the CohFT $\Omega_{g,n} = RT\omega_{g,n}$ coincides with the Chiodo polynomial:

$$\Omega_{g,n}(v_{a_1} \otimes \cdots \otimes v_{a_n}) = t^{2g-2} C_{g,n}^{r,s}(a_1, \dots, a_n; t^{-1}). \quad (9.2.25)$$

Moreover, the correlators $\omega_{g,n}$ are expressed as

$$\begin{aligned} \omega_{g,n}(z_1, \dots, z_n) &= \\ &= t^{3g-2+n} \left(\frac{r}{ts}\right)^{2g-2+n} \sum_{\mu_1, \dots, \mu_n \geq 1} \binom{r}{t}^b \left(\prod_{i=1}^n \frac{\left(\frac{\mu_i}{r}\right)^{[\mu_i]}}{[\mu_i]!} e^{\mu_i x(z_i)}\right) \int_{\mathcal{M}_{g,n}} \frac{C_{g,n}^{r,s}(\langle \mu \rangle; t^{-1})}{\prod_{i=1}^n (1 - \frac{\mu_i}{r} \psi_i)}, \end{aligned} \quad (9.2.26)$$

where $b = \frac{(2g-2+n)s+|\mu|}{r}$ and the integers $[\mu_i], \langle \mu_i \rangle$ are uniquely determined by $\mu_i = [\mu_i]r + r - \langle \mu_i \rangle$ for $1 \leq \langle \mu_i \rangle \leq r$.

Proof. From Lemma 9.2.5 and the definition of R -matrix and translation action, we get the following formula for $\Omega_{g,n}$:

$$\begin{aligned} & \sum_{\Gamma \in \mathcal{G}_{g,n}} \sum_{w \in W_{\Gamma}^{r,s}(a)} \frac{1}{|\text{Aut}(\Gamma)|} \xi_{\Gamma,*} \prod_{v \in V_{\Gamma}} (tr)^{2g(v)-2} r \cdot \exp \left(\sum_{m \geq 1} \frac{(-1)^m B_{m+1}(\frac{s}{r})}{m(m+1)t^m} \kappa_m(v) \right) \\ & \times \prod_{\substack{e \in E_{\Gamma} \\ e=(h,h')}} (t^2 r) \cdot \frac{1 - \exp \left(- \sum_{m \geq 1} \frac{(-1)^m B_{m+1}(\frac{w(h)}{r})}{m(m+1)t^m} ((\psi_h)^m - (-\psi_{h'})^m) \right)}{\psi_h + \psi_{h'}} \\ & \times \prod_{\lambda_i \in \Lambda_{\Gamma}} \exp \left(- \sum_{m \geq 1} \frac{(-1)^m B_{m+1}(\frac{a_i}{r})}{m(m+1)t^m} \psi_{\lambda_i}^m \right). \end{aligned}$$

if $\sum_{i=1}^n a_i \equiv (2g-2+n)s \pmod{r}$, and zero otherwise. Here $W_{\Gamma}^{r,s}(a)$ is the set of s -weightings modulo r are simply keeping track of the Kronecker deltas in the TFT and the R -matrix (see [JPPZ17, Subsection 1.3] for more details). Collecting the powers of r , we get the exponent

$$|E_{\Gamma}| + \sum_{v \in V_{\Gamma}} (2g(v) - 1) = 2g - 1 - h^1(\Gamma),$$

and collecting the powers of t , we find the exponent

$$2|E_{\Gamma}| + \sum_{v \in V_{\Gamma}} (2g(v) - 2) = 2g - 2.$$

Comparing it to Chiodo's formula for the Chern character of $R^{\bullet} \pi_* \mathcal{L}$, we see that the resulting sum over stable graphs coincides with pushforward of the Chern polynomial $c(R^{\bullet} \pi_* \mathcal{L}; \tau)$, with $\tau = t^{-1}$ and up to the global factor t^{2g-2} .

For the second part of the theorem, we apply the Eynard–DOSS correspondence of Theorem 2.3.12 to get

$$\begin{aligned} \omega_{g,n}(z_1, \dots, z_n) &= \\ &= \left(\frac{\sqrt{t}}{s} \left(\frac{t}{r} \right)^{-\frac{s}{r}-1} \right)^{2g-2+n} \sum_{a_1, \dots, a_n=1}^r \int_{\mathcal{M}_{g,n}} \Omega_{g,n}(v_{a_1} \otimes \dots \otimes v_{a_n}) \prod_{i=1}^n \sum_{k_i \geq 0} \psi_i^{k_i} d\theta^{k_i, a_i}(z_i). \end{aligned}$$

Taking into account that

$$\theta^{k,a}(z) = \left(\frac{1}{r} \frac{d}{dx} \right)^k \theta^a(z) = \sqrt{t} \left(\frac{r}{t} \right)^{1-\frac{a}{r}-k} \sum_{m=0}^{\infty} \frac{(\frac{rm+r-a}{t})^{m+k}}{m!} e^{(rm+r-a)x(z)},$$

setting $\mu_i = rm_i + r - a_i$, and converting the double sum over (m_i, a_i) into a unique sum over μ_i , we find the thesis (see [LPSZ17] for more details on a similar computation). Notice that $b = \frac{(2g-2+n)s+|\mu|}{r}$ is an integer, thanks to the modular constraint in the definition of the moduli space of twisted spin curves. \square

9.2.2 — SPECIALISATION TO THE SEGRE CLASS OF THE QUADRATIC HODGE BUNDLE

Let us consider the moduli space of quadratic differentials. The previous theorem specialised to $r = 1$ and $s = 2$ gives the following result.

THEOREM 9.2.7. *Consider the intersection numbers*

$$\langle s_{k_0} \tau_{k_1} \cdots \tau_{k_n} \rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} s_{k_0} \prod_{i=1}^n \psi_i^{k_i}, \quad k_0 + \cdots + k_n = 3g - 3 + n, \quad (9.2.27)$$

where $s(\mathcal{E}^{(2)}) = 1 + s_1 + \cdots + s_{3g-3+n}$ is the homogeneous decomposition with respect to the complex degree. They are computed by topological recursion on the spectral curve (9.2.1) as

$$\omega_{g,n}(z_1, \dots, z_n) = 2^{-(2g-2+n)} \sum_{\substack{k_0, \dots, k_n \geq 0 \\ k_0 + \cdots + k_n = 3g-3+n}} \langle s_{k_0} \tau_{k_1} \cdots \tau_{k_n} \rangle_g \prod_{i=1}^n d\theta^{k_i}(z_i), \quad (9.2.28)$$

where $\theta^k(z) = \left(\frac{z}{1-z} \frac{d}{dz}\right)^k \frac{z}{1-z}$. In particular, the Masur–Veech volumes of the principal strata of quadratic differentials are computed as

$$V_{g,n}^{\text{MV}} = (-1)^{3g-3+n} \frac{2^{3-n} \pi^{6g-6+2n}}{(6g-7+2n)!} \left(\prod_{i=1}^n \text{Res}_{z_i=1} \right) \omega_{g,n}(z_1, \dots, z_n). \quad (9.2.29)$$

9.3 — THE EULER CHARACTERISTIC OF THE MODULI SPACE OF CURVE

In this last section, we show an application of Chiodo’s formula to compute the Euler characteristic of the moduli space of curves. Although it has nothing to do with the enumeration of multicurves, it uses the Chern class of the quadratic Hodge bundle $\mathcal{E}^{(2)}$ (as opposed to the Segre class for Masur–Veech volumes).

Firstly, let us recall a generalised Gauss–Bonnet formula, expressing the orbifold Euler characteristic of certain open orbifolds as integrals of the Chern class of the logarithmic cotangent bundle. A proof of the formula can be found in [CMZ20], and we refer to it for the precise definitions.

PROPOSITION 9.3.1. *Let \overline{M} be a compact smooth m -dimensional orbifold and $D \subset M$ be a normal crossing divisor. Set $\overline{M} = M \setminus D$. Then the orbifold Euler characteristic of M can be computed as*

$$\chi(M) = (-1)^m \int_{\overline{M}} c_m(\Omega_{\overline{M}}^1(\log D)), \quad (9.3.1)$$

where $c_m(\Omega_{\overline{M}}^1(\log D))$ is the top Chern class of the logarithmic cotangent bundle.

Let us apply the above proposition to compute the Euler characteristic $\chi_{g,n}$ of the (open) moduli space of curves $\mathcal{M}_{g,n} = \overline{\mathcal{M}}_{g,n} \setminus \partial \overline{\mathcal{M}}_{g,n}$. The logarithmic cotangent bundle of $\overline{\mathcal{M}}_{g,n}$ is the quadratic Hodge bundle $\mathcal{E}^{(2)}$. On the other hand, consider Chiodo’s class with parameter $r = 1$, $s = -1$ and $a_1 = \cdots = a_n = 0$. It is the Chern class of the (*a priori* virtual) bundle over $\overline{\mathcal{M}}_{g,n}$, whose fiber over a curve (C, x_1, \dots, x_n) is

$$H^1(C, L) - H^0(C, L), \quad L \cong (\omega_{C, \log})^{-1}. \quad (9.3.2)$$

By degree considerations, $H^0(C, L) = 0$, while by Serre duality $H^1(C, L) \cong H^0(C, \omega_C^{\otimes 2}(\sum_i x_i))^\vee$. Thus, we find that

$$\chi_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} C_{g,n}^{1,-1}(0^n). \quad (9.3.3)$$

Specialising Chiodo's formula, we get $C_{g,n}^{1,-1}(0^n) = \Lambda(-1) \exp(-\sum_{m \geq 1} \frac{1}{m} \kappa_m)$. This is a simple consequence of the identity $B_m(-1) = B_m + (-1)^m m$, together with Mumford's formula for Hodge classes (cf. Equation (2.2.26) for the Hodge class and Proposition 2.2.19 for Chiodo's class). We can convert the evaluation of the above class into a combination of simple Hodge integrals:

$$\begin{aligned} \chi_{g,n} &= \int_{\overline{\mathcal{M}}_{g,n}} \Lambda(-1) \exp\left(-\sum_{m \geq 1} \frac{1}{m} \kappa_m\right) \\ &= \int_{\overline{\mathcal{M}}_{g,n}} \Lambda(-1) + \sum_{\ell \geq 1} \frac{(-1)^\ell}{\ell!} \sum_{\mu_1, \dots, \mu_\ell \geq 1} \int_{\overline{\mathcal{M}}_{g,n+\ell}} \Lambda(-1) \prod_{j=1}^{\ell} \psi_{n+j}^{\mu_j+1}. \end{aligned} \quad (9.3.4)$$

Notice that the sum over ℓ terminates at $\ell = 3g - 3 + n$, and the sum over μ 's is also finite. Moreover, the summand corresponding to $\ell = 0$ vanishes for degree reasons, unless $(g, n) = (0, 3)$ or $(1, 1)$. In these cases,

$$\int_{\overline{\mathcal{M}}_{0,3}} \Lambda(-1) = \int_{\overline{\mathcal{M}}_{0,3}} 1 = 1, \quad \int_{\overline{\mathcal{M}}_{1,1}} \Lambda(-1) = -\int_{\overline{\mathcal{M}}_{1,1}} \lambda_1 = -\frac{1}{24}. \quad (9.3.5)$$

Thus, we find the following intersection-theoretic expression for the Euler characteristic of the moduli space.

PROPOSITION 9.3.2. *The orbifold Euler characteristic of $\mathcal{M}_{g,n}$ is given by*

$$\chi_{g,n} = \delta_{g,0} \delta_{n,3} - \frac{1}{24} \delta_{g,1} \delta_{n,1} + \sum_{\ell \geq 1} \frac{(-1)^\ell}{\ell!} \sum_{\mu_1, \dots, \mu_\ell \geq 1} \int_{\overline{\mathcal{M}}_{g,n+\ell}} \Lambda(-1) \prod_{j=1}^{\ell} \psi_{n+j}^{\mu_j+1}. \quad (9.3.6)$$

Thanks to the above intersection-theoretic expression of the Euler characteristic, together with an explicit formula for Hodge integrals due to Dubrovin–Yang–Zagier (see [DYZ17, Section 1.3]), we are able to give a new proof of the Harer–Zagier formula [HZ86].

THEOREM 9.3.3 (*Harer–Zagier formula*). *The orbifold Euler characteristic of $\mathcal{M}_{g,n}$ is given by*

$$\chi_{g,n} = \begin{cases} (-1)^{n-3} (n-3)! & g = 0, n \geq 3, \\ (-1)^n \frac{(n-1)!}{12} & g = 1, n \geq 1, \\ (-1)^n (2g-3+n)! \frac{B_{2g}}{2g(2g-2)!} & g \geq 2, n \geq 0. \end{cases} \quad (9.3.7)$$

Proof. We first write the Euler characteristic as

$$\chi_{g,n} = \sum_{\ell \geq 0} \frac{(-1)^\ell}{\ell!} m_{g,n,\ell}, \quad m_{g,n,\ell} = \sum_{\mu_1, \dots, \mu_\ell \geq 1} \int_{\overline{\mathcal{M}}_{g,n+\ell}} \Lambda(-1) \prod_{j=1}^{\ell} \psi_{n+j}^{\mu_j+1},$$

with the convention that $m_{g,n,0} = \delta_{g,0} \delta_{n,3} - \frac{1}{24} \delta_{g,1} \delta_{n,1}$.

Claim 1. For every (g, n, ℓ) such that $2g - 2 + n > 0$ and $\ell \geq 1$, we claim that

$$m_{g,n+1,\ell} = \ell \left(m_{g,n,\ell} + (2g - 2 + n + \ell - 1) m_{g,n,\ell-1} \right).$$

III. Enumeration of multicurves and quadratic differentials

Such relation is a consequence of the string and dilaton equations for Hodge integrals [GJV01, Equations (8) and (10)]: applying the string equation, we find

$$\begin{aligned}
m_{g,n+1,\ell} &= \sum_{\mu_1, \dots, \mu_\ell \geq 1} \int_{\overline{\mathcal{M}}_{g,n+\ell+1}} \Lambda(-1) \prod_{j=1}^{\ell} \psi_{n+j}^{\mu_j+1} \\
&= \sum_{i=1}^{\ell} \sum_{\mu_1, \dots, \mu_\ell \geq 1} \int_{\overline{\mathcal{M}}_{g,n+\ell}} \Lambda(-1) \psi_{n+i}^{\mu_i} \prod_{j \neq i} \psi_{n+j}^{\mu_j+1} \\
&= \ell \sum_{\mu_1, \dots, \mu_\ell \geq 1} \int_{\overline{\mathcal{M}}_{g,n+\ell}} \Lambda(-1) \prod_{j=1}^{\ell} \psi_{n+j}^{\mu_j+1} + \sum_{i=1}^{\ell} \sum_{\mu_1, \dots, \mu_i, \dots, \mu_\ell \geq 1} \int_{\overline{\mathcal{M}}_{g,n+\ell}} \Lambda(-1) \psi_{n+i} \prod_{j \neq i} \psi_{n+j}^{\mu_j+1}.
\end{aligned}$$

Notice that, for $\ell = 1$, the last factor vanishes by degree reasons, unless $(g, n) = (0, 3)$ or $(1, 1)$. In such cases, we find

$$\int_{\overline{\mathcal{M}}_{0,4}} \Lambda(-1) \psi_4 = \int_{\overline{\mathcal{M}}_{0,3}} \Lambda(-1) = 1, \quad \int_{\overline{\mathcal{M}}_{1,2}} \Lambda(-1) \psi_2 = \int_{\overline{\mathcal{M}}_{1,1}} \Lambda(-1) = -\frac{1}{24}.$$

Thus, we have

$$m_{g,n+1,1} = m_{g,n,1} + \delta_{g,0} \delta_{n,3} - \frac{1}{24} \delta_{g,1} \delta_{n,1} = m_{g,n,1} + (2g - 2 + n) m_{g,n,0}$$

with the above convention for $m_{g,n,0}$. For $\ell > 1$, we can safely use the dilaton equation in the second factor:

$$\begin{aligned}
m_{g,n+1,\ell} &= \ell \sum_{\mu_1, \dots, \mu_\ell \geq 1} \int_{\overline{\mathcal{M}}_{g,n+\ell}} \Lambda(-1) \prod_{j=1}^{\ell} \psi_{n+j}^{\mu_j+1} \\
&\quad + \ell(2g - 2 + n + \ell - 1) \sum_{\mu_1, \dots, \mu_{\ell-1} \geq 1} \int_{\overline{\mathcal{M}}_{g,n+\ell-1}} \Lambda(-1) \prod_{j=1}^{\ell-1} \psi_{n+j}^{\mu_j+1} \\
&= \ell \left(m_{g,n,\ell} + (2g - 2 + n + \ell - 1) m_{g,n,\ell-1} \right).
\end{aligned}$$

This proves the first claim.

Claim 2. The Euler characteristic satisfies $\chi_{g,n+1} = -(2g - 2 + n) \chi_{g,n}$. Indeed, Claim 1 implies

$$\chi_{g,n+1} = \sum_{\ell \geq 1} \frac{(-1)^\ell}{\ell!} m_{g,n+1,\ell} = \sum_{\ell \geq 1} \frac{(-1)^\ell}{\ell!} \ell \left(m_{g,n,\ell} + (2g - 2 + n + \ell - 1) m_{g,n,\ell-1} \right).$$

Relabelling the index in the second sum, we obtain

$$\chi_{g,n+1} = \sum_{\ell \geq 1} \frac{(-1)^\ell}{\ell!} \left(\ell m_{g,n,\ell} - (2g - 2 + n + \ell) m_{g,n,\ell} \right) = -(2g - 2 + n) \chi_{g,n}.$$

Claim 3. The Harer–Zagier relation holds true. Indeed, as a consequence of Claim 2, we just have to compute $\chi_{0,3}$, $\chi_{1,1}$ and $\chi_{g,0}$ for $g \geq 2$. Clearly, $\chi_{0,3} = 1$, so that $\chi_{0,n} = (-1)^{n-3} (n-3)!$. In genus one we compute

$$\chi_{1,1} = -\frac{1}{24} - \int_{\overline{\mathcal{M}}_{1,2}} \Lambda(-1) \psi_2^2 = -\frac{1}{12}.$$

Thus, the relation $\chi_{1,n} = (-1)^n \frac{(n-1)!}{12}$. Finally, in genus $g \geq 2$, we can use the explicit formula of [DYZ17, Section 1.3], namely

$$\sum_{\ell \geq 1} \frac{1}{\ell!} \sum_{\mu_1, \dots, \mu_\ell \geq 1} \int_{\mathcal{M}_{g,\ell}} \Lambda(-1) \prod_{i=1}^{\ell} \psi_i^{\mu_i+1} = \frac{B_{2g}}{2g(2g-2)}.$$

As the left-hand side equals $\chi_{g,0}$ by Proposition 9.3.2, we have the thesis. \square

Thanks to the above expression of the Euler characteristic of the moduli space of curve as Chiodo integrals, we find a new spectral curve computing the Euler characteristic of the moduli space of curves by specialising Theorem 9.2.6.

PROPOSITION 9.3.4. *The topological recursion applied to the spectral curve on \mathbb{P}^1 given by*

$$x(z) = \log(z) - z, \quad y(z) = z^{-1}, \quad B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}, \quad (9.3.8)$$

computes the Euler characteristic of the moduli space of curves as

$$\chi_{g,n} = (-1)^n \left(\prod_{i=1}^n \operatorname{Res}_{z_i=1} (z_i - 1) \right) \omega_{g,n}(z_1, \dots, z_n). \quad (9.3.9)$$

Remark 9.3.5. We remark that a compact way to restate the Harer–Zagier formula is via a generating series

$$X(\hbar) = \sum_{\substack{g \geq 0, n > 0 \\ 2g-2+n > 0}} \chi_{g,n} \frac{\hbar^{2g-2+n}}{n!}. \quad (9.3.10)$$

Then $X(t)$ is the asymptotic expansion as $\hbar \rightarrow 0$ of the function

$$X(\hbar) \sim \log \left(\sqrt{\frac{\hbar}{2\pi}} \hbar^{\hbar-1} e^{\hbar-1} \Gamma(1 + \hbar^{-1}) \right). \quad (9.3.11)$$

See for instance [Pen88; Kon92].

From the works of Norbury [Nor10; Nor13], it is known that another spectral curve on \mathbb{P}^1 computes the Euler characteristic of the moduli space of curves:

$$x(z) = z + \frac{1}{z}, \quad y(z) = z, \quad B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}. \quad (9.3.12)$$

The computation of the Euler characteristics is somehow indirect, though. Indeed, the above spectral curve computes the number of lattice points on the combinatorial moduli space, and it capture the Euler characteristic of the moduli space as a residue at ∞ . On the other hand, the spectral curve from the theorem above computes the intersection of the Chern class of the log tangent bundle to the moduli space and ψ -classes, so that the Euler characteristic is given by the top intersection numbers.

As noted in the introduction, a similar situation occurs for Masur–Veech volumes: the spectral curves counting multicurves of bounded hyperbolic and combinatorial length somehow capture the Masur–Veech volumes, while the spectral curve (9.2.1) computes the intersection of the Segre class of a compactification of the principal stratum of the moduli space of quadratic differentials with ψ -classes, so that the Masur–Veech volumes is given by the top intersection numbers.

PART IV
SPIN HURWITZ THEORY

CHAPTER 10 — PRELIMINARIES ON SPIN HURWITZ NUMBERS

As explained in Section 2.6, double and single Hurwitz numbers with completed cycles, enumerating branched covers of \mathbb{P}^1 satisfying given conditions on their ramifications, enjoy many interesting properties and connections with different areas of mathematics.

1. Through the monodromy representation, they count certain decompositions of the identity in the symmetric group and can be computed via Burnside’s character formula.
2. They can be expressed as vacuum expectation values on Fock space, from which one can deduce (among other things):
 - 2.i. connections with the KP and $2d$ Toda hierarchies, as well as a cut-and-join equation,
 - 2.ii. chamber polynomiality and wall-crossing formulae for Hurwitz numbers as functions of partitions specifying the ramification profiles over 0 and ∞ on \mathbb{P}^1 .
3. They are computed via topological recursion.
4. Through ELSV-type formulae, they are expressed as intersection numbers on $\overline{\mathcal{M}}_{g,n}$.
5. They compute the stationary sector of the Gromov–Witten theory of \mathbb{P}^1 via the celebrated Gromov–Witten/Hurwitz (GW/H) correspondence.

In this last part of the dissertation, we consider a type of Hurwitz numbers called *spin Hurwitz numbers*, introduced by Eskin–Okounkov–Pandharipande in [EOP08]. The defining feature of these numbers is the presence of a spin structure (or theta characteristic) on the source, and the count is weighted by the parity of this theta characteristic. For these type of Hurwitz numbers, some of the above properties are already known.

1. Through the monodromy representation, they count decompositions of the identity in the *Sergeev group* and can be computed via Gunningham’s character formula [Guni16].
2. They can be expressed as certain vacuum expectation values on the *neutral* Fock space of [DKM81; DJKM82].

The aim of this chapter is to recall the above facts, and in particular give a precise definition of spin completed cycles, already appearing in some form in [MMN20] (cf. Definition 2.6.6 for the non-spin version) and their connection with the neutral Fock space. It contains almost no new results, but it collects various facts scattered in literature in a way that facilitates the development of Chapters 11 and 12, where we are going to obtain the following results.

- 2’. Introduce a spin version of the Okounkov–Pandharipande operators, to deduce:
 - 2.i. a generating series expression for the spin cut-and-join operators (the connections with the BKP hierarchy of the Kyoto school and existence of a spin cut-and-join equation was already known from [Lee19; MMN20; MMNO20]).

IV. Spin Hurwitz theory

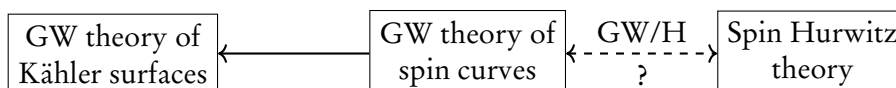
- 2.ii. chamber polynomiality and wall-crossing formulae for spin Hurwitz as functions of partitions specifying the ramification profiles over 0 and ∞ on \mathbb{P}^1 .
3. Conjecture a spectral curve computing spin Hurwitz numbers via topological recursion, now proved by Alexandrov–Shadrin [AS21].
4. Prove that topological recursion on this spectral curve is equivalent to an ELSV-type formula involving Chiodo classes and Witten 2-spin class.

10.0.1 — RELATION WITH OTHER WORKS AND OPEN QUESTIONS

As already mentioned, spin Hurwitz covers were introduced in [EOP08]. The main motivation was the connection with the Masur–Veech volumes of the corresponding strata of holomorphic differentials [EO01; KZ03]. The completed cycle version of these numbers were indirectly considered in [MMN20], and we give here a definition that is parallel to the non-spin version. Another important feature of spin Hurwitz numbers is the conjectural connection with the Gromov–Witten theory of Kähler surfaces [LP07; MP08]. More precisely, let X be a Kähler surface with a smooth canonical divisor D , so that each component of (D, N) is a spin curve. Under certain assumptions, the Gromov–Witten invariants of X can be expressed in terms of “local” Gromov–Witten invariants of the (irreducible components of the) spin curve (D, N) :

$$\mathrm{GW}_{g,n}(X, \beta) = \sum_i \mathrm{GW}_{g,n}^{\mathrm{loc}}(D_i, N_i, \beta_i), \quad (10.0.1)$$

We remark, however, that an explicit algebraic construction of a Gromov–Witten theory of spin curves does not exist yet. Assuming the foundational issues will be solved, one expects such a theory to be related to spin Hurwitz numbers via a spin analogue of the GW/H correspondence. The base cases of this correspondence have been proved for degree 1 and 2, and conjectured for $d \geq 3$ [LP13].



QUESTION 10.A. *Properly formulate a Gromov–Witten theory of spin curves, and prove the spin Gromov–Witten correspondence.*

An easier version of the above question is to formulate a Gromov–Witten theory of $(\mathbb{P}^1, \mathcal{O}(-1))$. As the original ELSV formula represents a key ingredient of Okounkov–Pandharipande’s approach to the GW/H correspondence on \mathbb{P}^1 , we hope that the result of Chapter 12 would contribute towards a spin version of this correspondence.

10.0.2 — ORGANISATION OF THE CHAPTER

The chapter is organised as follows.

- In Section 10.1 we review the theory of spin-symmetric groups, and their relations with the Sergeev algebra and strict/odd partitions.
- In Section 10.2 we recall some basic facts about the neutral fermion formalism, and explain its connection with supersymmetric functions and the BKP hierarchy.

- To conclude, in Section 10.3 we review the definition of spin Hurwitz numbers, recall Gunningham's character formula, and define spin Hurwitz numbers with completed cycles.

10.1 — SPIN REPRESENTATIONS

In order to fix the notation, consider the following presentation of the symmetric group:

$$\mathfrak{S}_d = \langle \sigma_1, \dots, \sigma_{d-1} \mid \sigma_i^2 = 1, (\sigma_i \sigma_{i+1})^3 = 1, (\sigma_i \sigma_j)^2 = 1 \text{ for } |i - j| > 1 \rangle. \quad (10.1.1)$$

DEFINITION 10.1.1. The *spin-symmetric group* of order d is the (unique) non-trivial central extension of \mathfrak{S}_d :

$$1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \tilde{\mathfrak{S}}_d \rightarrow \mathfrak{S}_d \rightarrow 1. \quad (10.1.2)$$

Explicitly, it can be presented as follows:

$$\tilde{\mathfrak{S}}_d = \langle s_1, \dots, s_{d-1}, \epsilon \mid \epsilon^2 = 1, s_i^2 = \epsilon, (s_i s_{i+1})^3 = \epsilon, (s_i s_j)^2 = \epsilon \text{ for } |i - j| > 1 \rangle, \quad (10.1.3)$$

and the map $\tilde{\mathfrak{S}}_d \rightarrow \mathfrak{S}_d$ is given by $s_i \mapsto \sigma_i, \epsilon \mapsto 1$. It has a natural $\mathbb{Z}/2\mathbb{Z}$ grading given by $\deg(\epsilon) = 0$ and $\deg(s_i) = 1$. The representations of $\tilde{\mathfrak{S}}_d$ that do not factor through \mathfrak{S}_d are called *spin representations*.

LEMMA 10.1.2. *Spin representations are exactly the representations of the twisted group algebra*

$$\mathcal{S}_d = \mathbb{C}[\tilde{\mathfrak{S}}_d]/(\epsilon + 1), \quad (10.1.4)$$

where ϵ is the added central element. It inherits a $\mathbb{Z}/2\mathbb{Z}$ grading, and hence is a superalgebra.

For many explicit computations, it is easier not to work with the twisted symmetric group algebra, but with the Sergeev algebra, which we now introduce.

DEFINITION 10.1.3. Let D be a set of $2d$ elements and ι a fixed-point free involution on D . We define the *hyperoctahedral group* \mathfrak{H}_d of order d to be the centraliser in \mathfrak{S}_{2d} of ι . The *Sergeev group* of order d is the (unique) non-trivial central extension of \mathfrak{H}_d :

$$1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \tilde{\mathfrak{H}}_d \rightarrow \mathfrak{H}_d \rightarrow 1. \quad (10.1.5)$$

It has a natural $\mathbb{Z}/2\mathbb{Z}$ grading, with the added central element ϵ of degree 0.

Remark 10.1.4. Alternatively, one can realise the hyperoctahedral group and the Sergeev group as

$$\mathfrak{H}_d = \mathfrak{S}_d \ltimes (\mathbb{Z}/2\mathbb{Z})^d, \quad \tilde{\mathfrak{H}}_d = \mathfrak{S}_d \ltimes \mathfrak{C}\mathfrak{I}_d. \quad (10.1.6)$$

Here $\mathfrak{C}\mathfrak{I}_d$ is the Clifford group (the unique non-trivial central extension of $(\mathbb{Z}/2\mathbb{Z})^d$), and \mathfrak{S}_d acts on $(\mathbb{Z}/2\mathbb{Z})^d$ and $\mathfrak{C}\mathfrak{I}_d$ by permuting factors. The Clifford group is explicitly presented as

$$\mathfrak{C}\mathfrak{I}_d = \langle \xi_1, \dots, \xi_d, \epsilon \mid \xi_i^2 = \epsilon^2 = 1, \xi_i \xi_j = \epsilon \xi_j \xi_i \text{ for } i \neq j \rangle. \quad (10.1.7)$$

DEFINITION 10.1.5. The *Sergeev algebra* is the twisted group algebra of the Sergeev group:

$$\mathcal{H}_d = \mathbb{C}[\tilde{\mathfrak{H}}_d]/(\epsilon + 1), \quad (10.1.8)$$

where ϵ is the added central element. It inherits a $\mathbb{Z}/2\mathbb{Z}$ grading, and hence is a superalgebra.

The link between the representation theory of \mathcal{S}_d and that of \mathcal{H}_d is realised through the Clifford algebra, i.e. the twisted group algebra of the Clifford group

$$\mathcal{C}\ell_d = \mathbb{C}[\mathfrak{C}\ell_d]/(\epsilon + 1). \quad (10.1.9)$$

It coincided with the Clifford algebra from Example 2.5.2.

THEOREM 10.1.6 ([Yam99, Theorem 3.2]). *There is an isomorphism of superalgebras*

$$\mathcal{S}_d \otimes \mathcal{C}\ell_d \simeq \mathcal{H}_d. \quad (10.1.10)$$

In particular, there is a bijection between simple supermodules for \mathcal{S}_d and simple supermodules for \mathcal{H}_d , given by tensoring with the unique simple supermodule $\mathcal{C}\ell_d$ for $\mathcal{C}\ell_d$.

The classification of simple supermodules for \mathcal{S}_d and \mathcal{H}_d had both been done before, respectively by Schur [Sch11] and by Sergeev [Ser84], recast in the language of superalgebras by Józefiak [Józ89; Józ90]. To give it, we will need some preliminary definitions.

DEFINITION 10.1.7. A partition is *odd* if all its parts are odd and *strict* if all its parts are distinct. We write \mathcal{OP}_d , and \mathcal{SP}_d for, respectively, the set of odd partitions and the set of strict partitions of d . Also write $\mathcal{OP} = \bigcup_{d \geq 0} \mathcal{OP}_d$ and $\mathcal{SP} = \bigcup_{d \geq 0} \mathcal{SP}_d$ for the sets of all odd and strict partitions (including the empty partition \emptyset).

By a classical result of Euler, the set of odd partitions and the set of strict partitions of d are of equal size: $|\mathcal{OP}_d| = |\mathcal{SP}_d|$.

PROPOSITION 10.1.8 ([Ser84]).

- *There is a bijection between \mathcal{SP}_d and irreducible supermodules V_λ of \mathcal{H}_d .*
- *Let $\mu \in \mathcal{OP}_d$. The conjugacy class $\tilde{\mathcal{O}}_\mu$ in $\tilde{\mathfrak{S}}_d$ of a permutation of cycle type μ has cardinality $|\tilde{\mathcal{O}}_\mu| = 2^{d-\ell(\mu)} \frac{d!}{z_\mu}$. Moreover, the elements*

$$\tilde{C}_\mu = \sum_{\eta \in \tilde{\mathcal{O}}_\mu} \eta \in \mathcal{H}_d \quad (10.1.11)$$

forms a linear basis of $\tilde{\mathcal{Z}}_d$, the even part of the centre of \mathcal{H}_d . We call \tilde{C}_μ the spin conjugacy class element of type μ , and $\tilde{\mathcal{Z}}_d$ the spin class algebra.

As a consequence, we can talk about characters $\tilde{\chi}_\lambda(\mu)$ of irreducible supermodules V_λ evaluated at \tilde{C}_μ , with $\mu \in \mathcal{OP}_d$ and $\lambda \in \mathcal{SP}_d$.

The basic correspondence between the representation theory of the symmetric group and its spin counterpart are summarised in the table below.

	Ordinary	Spin
Main algebra	$\mathbb{C}[\mathfrak{S}_d]$	$\mathbb{C}[\tilde{\mathfrak{S}}_d]/(\epsilon + 1)$
Class algebra	\mathcal{Z}_d	$\tilde{\mathcal{Z}}_d$
Index set class algebra	\mathcal{P}_d	\mathcal{OP}_d
Index set irreps	\mathcal{P}_d	\mathcal{SP}_d
Characters	$\chi_\lambda(\mu)$	$\tilde{\chi}_\lambda(\mu)$

10.2 — NEUTRAL FERMION FORMALISM

As explained in Section 2.5.1, the Fock space is the highest weight module of a certain infinite-dimensional Clifford algebra. Moreover, it can be realised as the space of semi-infinite wedges, or alternatively as the space of Maya diagrams. In the spin case, a Fock space can still be defined as the highest weight module of another infinite-dimensional Clifford algebra, or alternatively as the space of half-line Maya diagrams (cf. [FWZ09]), although there is no description in terms of wedges. As the non-spin and spin theories are related to A- and B-type Dynkin diagrams respectively, we also refer to them by these letters. The material of this section follows the exposition from [DKM81; DJKM82; You89; Ale21].

DEFINITION 10.2.1. Let \mathcal{V}^B be the infinite-dimensional complex vector space with basis ϕ_k for $k \in \mathbb{Z}$ integers. Consider the bilinear form

$$(\phi_k, \phi_l) = \frac{(-1)^k}{2} \delta_{k+l}, \quad (10.2.1)$$

and define the *space of neutral fermions* as $\mathcal{B} = \mathcal{Cl}(\mathcal{V}^B, (\cdot, \cdot))$. It has a $\mathbb{Z}/2\mathbb{Z}$ grading $\mathcal{B}_0 \oplus \mathcal{B}_1$, with \mathcal{B}_p spanned by products of m elements with $m \equiv p \pmod{2}$. Moreover, it has *canonical anticommutation relations* (CAR) given by

$$\{\phi_k, \phi_l\} = (-1)^k \delta_{k+l}. \quad (10.2.2)$$

DEFINITION 10.2.2. Consider the subspace \mathcal{L}^B of \mathcal{V}^B generated by ϕ_k for $k < 0$, which is maximal isotropic for $(\mathcal{V}^B, (\cdot, \cdot))$. Define the (*fermionic*) *Fock space of type B* as the unique graded highest-weight left module of \mathcal{B} :

$$\mathfrak{F}^B = \mathcal{B}/(\mathcal{B} \cdot \mathcal{L}^B). \quad (10.2.3)$$

We write $|0\rangle$ for the class of 1, also called the *vacuum state*, and $|1\rangle = \sqrt{2}\phi_0|0\rangle$. The space \mathfrak{F}^B inherits the $\mathbb{Z}/2\mathbb{Z}$ grading: $\mathfrak{F}^B = \mathfrak{F}_0^B \oplus \mathfrak{F}_1^B$, and $|p\rangle \in \mathfrak{F}_p^B$.

With the dual construction, (i.e. considering the unique graded highest-weight right module) we define the dual Fock space $\mathfrak{F}^{B,*}$ and the covectors $\langle 0|$ and $\langle 1|$. In particular, we have a pairing $\mathfrak{F}^{B,*} \times \mathfrak{F}^B \rightarrow \mathbb{C}$ denoted by

$$\langle \omega | \eta \rangle = (\langle \omega |, |\eta \rangle). \quad (10.2.4)$$

Moreover, for any $O \in \mathcal{B}$ we can define its *vacuum expectation value* $\langle O \rangle$ as $\langle 0|O|0\rangle$. Since the (right) action of \mathcal{B} on the dual Fock space is the adjoint of the (left) action on the Fock space, there is no ambiguity in the notation.

LEMMA 10.2.3.

1. The vacuum expectation values of quadratic expressions in the ϕ 's are

$$\langle \phi_k \phi_l \rangle = (-1)^k \delta_{k+l} u[l], \quad u[l] = \begin{cases} 1 & \text{if } l > 0, \\ \frac{1}{2} & \text{if } l = 0, \\ 0 & \text{if } l < 0. \end{cases} \quad (10.2.5)$$

2. For $\lambda \in \mathcal{SP}$, define $|\lambda\rangle = \phi_{\lambda_1} \cdots \phi_{\lambda_{\ell(\lambda)}} |p(\lambda)\rangle$, where

$$p(\lambda) \equiv \ell(\lambda) \pmod{2} \quad (10.2.6)$$

is the parity of λ . Then $\{|\lambda\rangle \mid \lambda \in \mathcal{SP}\}$ form a basis of \mathfrak{F}_0^B .

We can now define an infinite dimensional Lie algebra acting on \mathfrak{F}_0^B . Recall from Definition 2.5.10 the bi-infinite general linear algebra $\mathfrak{gl}(\infty)$ spanned by band matrices $(a_{m,n})_{m,n \in \mathbb{Z}}$. It has a basis $\{E_{k,l} \mid k, l \in \mathbb{Z}\}$ such that $(E_{k,l})_{m,n} = \delta_{k,m} \delta_{l,n}$.

DEFINITION 10.2.4. Consider the involution $\iota: E_{k,l} \mapsto (-1)^{k+l} E_{-l,-k}$, and define the bi-infinite orthogonal linear algebra

$$\mathfrak{go}(\infty) = \{g \in \mathfrak{gl}(\infty) \mid \iota(g) = -g\}. \quad (10.2.7)$$

It has a standard basis given by $\{E_{k,l}^B = (-1)^l E_{k,l} - (-1)^k E_{-l,-k}\}_{k+l>0}$, and commutation relations

$$[E_{i,j}^B, E_{k,l}^B] = (-1)^j \delta_{j,k} E_{i,l}^B - (-1)^i \delta_{i,l} E_{k,j}^B + (-1)^j \delta_{j+l} E_{k,-i}^B - (-1)^i \delta_{i+k} E_{-j,l}^B. \quad (10.2.8)$$

PROPOSITION 10.2.5. There is a representation of the central extension $\widehat{\mathfrak{go}}(\infty) = \mathfrak{go}(\infty) \oplus \mathbb{C}$ to the space of neutral fermions \mathcal{B} , defined on the central factor by $1 \mapsto 1$ and on $\mathfrak{go}(\infty)$ by

$$E_{k,l}^B \mapsto \hat{E}_{k,l}^B = : \phi_k \phi_{-l} :, \quad (10.2.9)$$

where $: \phi_i \phi_j := \phi_i \phi_j - \langle \phi_i \phi_j \rangle$ is the normal ordered product. Moreover, the following parity relation holds

$$\hat{E}_{k,l}^B = -\hat{E}_{-l,-k}^B, \quad (10.2.10)$$

and the commutation relation between basis elements is given by

$$\begin{aligned} [\hat{E}_{i,j}^B, \hat{E}_{k,l}^B] &= (-1)^j \delta_{j,k} \hat{E}_{i,l}^B - (-1)^i \delta_{i,l} \hat{E}_{k,j}^B + (-1)^j \delta_{j+l} \hat{E}_{k,-i}^B - (-1)^i \delta_{i+k} \hat{E}_{-j,l}^B \\ &\quad + (-1)^{i+j} (\delta_{j,k} \delta_{i,l} - \delta_{j+l} \delta_{i+k}) (u[j] - u[i]). \end{aligned} \quad (10.2.11)$$

Remark 10.2.6. In [GKL21], we used a different choice of basis elements, namely $\hat{F}_{k,l} = (-1)^k : \phi_k \phi_l :$. Here we changed the convention, following [DKM81; DJKM82; Ale21]. Moreover, with this convention most of the formulae are completely parallel to the A setting of Section 2.5.1. See also the table at the end of this section.

EXAMPLE 10.2.7. Examples of elements in $\widehat{\mathfrak{go}}(\infty)$ are given as follows. They are the B-type analogue of the operators \mathcal{F}_m and J_n acting on the Fock space \mathfrak{F} of A-type. The main difference, as often in this chapter, is the introduction of signs and factors of 2.

- For any positive odd integer $m \in \mathbb{Z}_+^{\text{odd}}$, we have the diagonal operators

$$\mathcal{F}_m^B = \frac{1}{2} \sum_{k \in \mathbb{Z}} (-1)^k k^m \hat{E}_{k,k}^B,$$

called (fermionic) completed cut-and-join operators of type B. They can be defined for even m , but they would vanish due to the parity relation (10.2.10).

- For any odd integer $n \in \mathbb{Z}^{\text{odd}}$, we have the elements

$$J_n^B = \frac{1}{2} \sum_{k \in \mathbb{Z}} (-1)^k \hat{E}_{k-n, k}^B,$$

also called *currents of type B*. Again, they can be defined for even m , but they would vanish. They form an Heisenberg subalgebra of $\widehat{\mathfrak{go}}(\infty)$, i.e. they satisfy the *canonical commutation relation* (CCR):

$$[J_m^B, J_n^B] = \frac{m}{2} \delta_{m+n}.$$

We also define the generating series $J^B(t) = \sum_{n \in \mathbb{Z}_+^{\text{odd}}} t_n J_n^B$, that will play an important role in the neutral boson-fermion correspondence.

- In the next section, we will introduce for the first time the B-analogue of the Okounkov–Pandharipande operators.

As explained in [Ale21] one can introduce the B-analogue of the (big cell of the) Sato Grassmannian, also called *orthogonal Sato Grassmannian*. In this case, the Plücker relations can be expressed in terms of the generating series of neutral fermions

$$\phi(z) = \sum_{k \in \mathbb{Z}} \phi_k z^k \tag{10.2.12}$$

as the following quadratic relations for elements of $\mathbb{P}\mathfrak{F}_0^B$:

$$\text{Res}_{z=\infty} \phi(z) |\omega\rangle \otimes \phi(-z) |\omega\rangle \frac{dz}{z} = 0. \tag{10.2.13}$$

See [DJKM82; Ale21] for further details.

10.2.1 — SUPERSYMMETRIC FUNCTIONS AND NEUTRAL BOSON-FERMION CORRESPONDENCE

The bosonic counterpart of the fermionic Fock space of type B is given by the algebra of supersymmetric functions, that we now introduce.

DEFINITION 10.2.8. Define the *algebra of supersymmetric functions* (or bosonic Fock space of type B), denoted by Γ , as the free algebra on the odd power-sum symmetric functions:

$$\Gamma = \mathbb{C}[p_1, p_3, \dots]. \tag{10.2.14}$$

Another important basis is the one consisting of *Q-Schur functions* (see [McD98, Section III.8]). They are indexed by strict partitions, and the following lemma expresses the duality between Q-Schur and odd power-sum symmetric functions. For an odd partition μ , define $p_\mu = \prod_{i=1}^{\ell(\mu)} p_{\mu_i}$.

LEMMA 10.2.9. *The change of basis from Q-Schur functions to odd power-sum symmetric functions is given by the irreducible characters of the Sergeev group:*

$$Q_\lambda = 2^{[\ell(\lambda)/2]} \sum_{\mu \in \mathcal{OP}_d} \frac{\tilde{\chi}_\lambda(\mu)}{\mathfrak{z}_\mu} p_\mu, \quad p_\mu = \sum_{\lambda \in \mathcal{SP}_d} \frac{\tilde{\chi}_\lambda(\mu)}{2^{\ell(\mu)+[\ell(\lambda)/2]}} Q_\lambda. \tag{10.2.15}$$

Here $\mathfrak{z}_\mu = \prod_{i=1}^{\ell(\mu)} \mu_i \prod_{m>0} |\{i \mid \mu_i = m\}|!$ is the order of the centraliser of an element of cycle type μ in the symmetric group.

IV. Spin Hurwitz theory

As in the charged case, we can now relate the fermionic and bosonic Fock spaces of type B.

THEOREM 10.2.10 (Neutral boson-fermion correspondence). *Let $t_n = \frac{p_n}{n}$. The map*

$$\sigma^B: \mathfrak{F}^B \longrightarrow \Gamma[\zeta]/(\zeta^2 - 1), \quad |\omega\rangle \longmapsto \langle 0|e^{J^B(t)}|\omega\rangle + \langle 1|e^{J^B(t)}|\omega\rangle \zeta \quad (10.2.16)$$

is an isomorphism, called the neutral boson-fermion correspondence. Moreover, the action of the currents is given by

$$\sigma^B(J_n^B |\omega\rangle) = \begin{cases} \partial_{t_n} \sigma^B(|\omega\rangle) & \text{if } n > 0, \\ -\frac{n}{2} t_{-n} \sigma^B(|\omega\rangle) & \text{if } n < 0, \end{cases} \quad n \in \mathbb{Z}^{\text{odd}}, \quad (10.2.17)$$

and that of the neutral fermions by

$$\sigma^B(\phi(z) |\omega\rangle) = \frac{\zeta}{\sqrt{2}} \exp\left(\sum_{k \in \mathbb{Z}_+^{\text{odd}}} z^{2k+1} t_{2k+1}\right) \exp\left(-2 \sum_{k \in \mathbb{Z}_+^{\text{odd}}} \frac{z^{-k}}{k} \partial_{t_k}\right) \sigma^B(|\omega\rangle). \quad (10.2.18)$$

Via the boson-fermion correspondence, one can compute the action of the completed cut-and-join operators and that of currents on the fermionic Fock space as follows.

PROPOSITION 10.2.11.

- *For any strict partition λ and odd integer m , the action of the completed cut-and-join operator \mathcal{F}_m^B of type B is given by*

$$\mathcal{F}_m^B |\lambda\rangle = p_m(\lambda) |\lambda\rangle, \quad (10.2.19)$$

where p_m is the usual symmetric power-sum of odd index.

- *For an odd partition μ , set $J_{\pm\mu}^B = J_{\pm\mu_1}^B \cdots J_{\pm\mu_n}^B$. Then*

$$J_{-\mu}^B |0\rangle = \sum_{\lambda \in \mathcal{SP}(|\mu|)} \frac{\tilde{\chi}_\lambda(\mu)}{2^{p(\lambda)/2 + \ell(\mu)}} |\lambda\rangle, \quad J_\mu^B |\lambda\rangle = \frac{\tilde{\chi}_\lambda(\mu)}{2^{p(\lambda)/2 + \ell(\mu)}} |0\rangle. \quad (10.2.20)$$

Here $p(\lambda)$ is the parity of λ , defined by Equation (10.2.6).

Again, one can translate the quadratic relations satisfied by elements in the big cell of the orthogonal Sato Grassmannian into an infinite collection of non-linear PDEs, known as the *BKP hierarchy*. The first equation of the hierarchy, the BKP equation, reads

$$(D_{t_1}^6 - 5D_{t_1}^3 D_{t_3} - 5D_{t_3}^2 + 9D_{t_1} D_{t_5})[\tau, \tau] = 0. \quad (10.2.21)$$

Moreover, one can consider a B-type analogue of the Toda hierarchy, known as the 2-BKP hierarchy. We refer to [DKM81; DJKM82; You89] for further readings, and to [Ale21] for a modern account.

A short comparison between the A- and B-type theories is given in the following table.

	A type	B type
Index set	$\mathbb{Z}' = \mathbb{Z} + \frac{1}{2}$	\mathbb{Z}
Vector space	$\mathcal{V}^A = \bigoplus_{s \in \mathbb{Z}'} (\mathbb{C}\psi_s \oplus \mathbb{C}\psi_s^\dagger)$	$\mathcal{V}^B = \bigoplus_{k \in \mathbb{Z}} \mathbb{C}\phi_k$
Inner product	$(\psi_r, \psi_s^\dagger) = \frac{\delta_{r+s}}{2} \quad (\psi_r, \psi_s) = (\psi_r^\dagger, \psi_s^\dagger) = 0$	$(\phi_k, \phi_l) = (-1)^k \frac{\delta_{k+l}}{2}$
Isotropic subspace	$\mathcal{L}^A = \bigoplus_{\mathbb{Z}_-} (\mathbb{C}\psi_s \oplus \mathbb{C}\psi_{-s}^\dagger)$	$\mathcal{L}^B = \bigoplus_{\mathbb{Z}_-} \mathbb{C}\phi_k$
Clifford algebra	$\mathcal{A} = \text{Cl}(\mathcal{V}^A, (\cdot, \cdot))$	$\mathcal{B} = \text{Cl}(\mathcal{V}^B, (\cdot, \cdot))$
CAR	$\{\psi_r, \psi_s^\dagger\} = \delta_{r+s} \quad \{\psi_r^\dagger, \psi_s^\dagger\} = \{\psi_r, \psi_s\} = 0$	$\{\phi_k, \phi_l\} = (-1)^k \delta_{k+l}$
Fermionic Fock space	$\mathfrak{F}^A = \mathcal{A}/(\mathcal{A} \cdot \mathcal{L}^A) \supset \mathfrak{F}_0^A$	$\mathfrak{F}^B = \mathcal{B}/(\mathcal{B} \cdot \mathcal{L}^B) \supset \mathfrak{F}_0^B$
Basis of \mathfrak{F}_0	$ \lambda\rangle = \psi_{\lambda_1} \cdots \psi_{\lambda_{\ell(\lambda)}} 0\rangle \quad (\lambda \in \mathcal{P})$	$ \lambda\rangle = \phi_{\lambda_1} \cdots \phi_{\lambda_{\ell(\lambda)}} p(\lambda)\rangle \quad (\lambda \in \mathcal{SP})$
Lie algebras	$\mathfrak{gl}(\infty), \widehat{\mathfrak{gl}}(\infty)$	$\mathfrak{go}(\infty), \widehat{\mathfrak{go}}(\infty)$
Currents	$J_m^A = \sum_{s \in \mathbb{Z}'} \hat{E}_{s-n, s}^A \quad (m \in \mathbb{Z})$ $J^A(t) = \sum_{n \in \mathbb{Z}_+} J_n^A t_n$	$J_m^B = \frac{1}{2} \sum_{k \in \mathbb{Z}} (-1)^k \hat{E}_{k-n, k}^B \quad (m \in \mathbb{Z}^{\text{odd}})$ $J^B(t) = \sum_{n \in \mathbb{Z}_+^{\text{odd}}} J_n^B t_n$
CCR	$[J_m^A, J_n^A] = m\delta_{m+n}$	$[J_m^B, J_n^B] = \frac{m}{2}\delta_{m+n}$
Bosonic Fock space	$\Lambda = \mathbb{C}[t_1, t_2, t_3, \dots]$	$\Gamma = \mathbb{C}[t_1, t_3, t_5, \dots]$
B-F correspondence	$\sigma^A: \mathfrak{F}_0^A \simeq \Lambda, \omega\rangle \mapsto \langle 0 e^{J^A(t)} \omega\rangle$	$\sigma^B: \mathfrak{F}_0^B \simeq \Gamma, \omega\rangle \mapsto \langle 0 e^{J^B(t)} \omega\rangle$
Fermionic C'n'J operators	$\mathcal{F}_m^A = \sum_{s \in \mathbb{Z}'} s^m \hat{E}_{s, s}^A \quad (m \in \mathbb{Z}_+)$ $\mathcal{F}_m^A \lambda\rangle = p_m(\lambda) \lambda\rangle \quad (\lambda \in \mathcal{P})$	$\mathcal{F}_m^B = \frac{1}{2} \sum_{k \in \mathbb{Z}'} (-1)^k k^m \hat{E}_{s, s}^A \quad (m \in \mathbb{Z}_+^{\text{odd}})$ $\mathcal{F}_m^B \lambda\rangle = p_m(\lambda) \lambda\rangle \quad (\lambda \in \mathcal{SP})$
Integrable hierarchy	KP and 2d Toda lattice	BKP and 2-BKP

10.3 — INTRODUCTION TO SPIN HURWITZ NUMBERS

Spin Hurwitz numbers are weighted counts of covers of a curve equipped with a spin structure or theta characteristic, where the weight includes a sign taking into account the parity of the spin structure [EOP08]. This is captured in the following definitions. For more background on spin Hurwitz numbers in relation to integrable hierarchies and supersymmetric functions, see also [Lee19; MMN20; MMNO20].

DEFINITION 10.3.1. A *spin structure* or *theta characteristic* on a curve C is a line bundle $\vartheta \rightarrow C$ such that $\vartheta^{\otimes 2} \cong \omega_C$. A *spin curve* is a pair (C, ϑ) of a curve C with a spin structure ϑ on it. Define the *parity* of a spin structure $\vartheta \rightarrow C$ as

$$p(\vartheta) \equiv h^0(C, \vartheta) \pmod{2}. \quad (10.3.1)$$

Thus, we can talk about *even* and *odd* theta characteristics on C .

The parity is a deformation invariant of (C, ϑ) , a fact proved for smooth curves by Riemann in the language of theta functions, or more abstractly by Mumford [Mum71] in the algebraic and Atiyah [Ati71] in the analytic settings. Mumford’s proof was extended to nodal spin curves by Cornalba [Cor89, Section 6]. The same authors also proved that, for a genus g curve, there are $2^{g-1}(2^g + 1)$ even theta characteristics and $2^{g-1}(2^g - 1)$ odd ones.

Spin structures can be pulled back along branched covers, as long as all ramifications are odd: in that case the ramification divisor is even.

DEFINITION 10.3.2. Let (B, ϑ) be a spin curve and $f: C \rightarrow B$ a branched cover with only odd ramifications. Denote by R_f its ramification divisor. Then the *twisted pullback* of ϑ along f is defined as

$$N_{f, \vartheta} = f^* \vartheta \otimes \mathcal{O}(\tfrac{1}{2}R_f). \quad (10.3.2)$$

It is a spin structure on C .

In this dissertation, we will focus on ramified covers of the spin curve $(\mathbb{P}^1, \mathcal{O}(-1))$, counted with respect to the parity of the twisted pullback.

DEFINITION 10.3.3. Let p_1, \dots, p_k be distinct points on \mathbb{P}^1 , and μ^1, \dots, μ^k odd partitions of $d > 0$. Define the *spin Hurwitz number* as

$$H_d^\vartheta(\mu^1, \dots, \mu^k) = \sum_{[f]} \frac{(-1)^{N_{f, \mathcal{O}(-1)}}}{|\text{Aut}(f)|}, \quad (10.3.3)$$

where the sum runs over all isomorphism classes of Hurwitz covers $f: C \rightarrow \mathbb{P}^1$ of degree d and ramification data μ^1, \dots, μ^k . As usual, when dealing with disconnected covers, we add a superscript \bullet .

In the spin setting, the analogue of the Burnside character formula is expressed in terms of characters of the Sergeev group. In this form, it was proved by Eskin–Okounkov–Pandharipande [EOP08, Theorem 2], and later generalised by Gunningham [Gun16] to arbitrary covers of a spin curve (B, ϑ) . Following [Lee19], we will refer to it as Gunningham’s formula.

THEOREM 10.3.4 (Gunningham character formula [Guni6]). *Disconnected spin Hurwitz numbers are given by*

$$H_d^{\bullet, \vartheta}(\mu^1, \dots, \mu^k) = 2^{\frac{\sum_i (\ell(\mu^i) - d) - 2d}{2}} \sum_{\lambda \in \mathcal{SP}_d} \left(\frac{\dim(\lambda)}{2^{p(\lambda)/2} d!} \right)^2 \prod_{i=1}^k f_{\mu^i}(\lambda). \quad (10.3.4)$$

Here $f_{\mu}(\lambda) = |\tilde{\mathcal{O}}_{\mu}| \frac{\tilde{\chi}_{\lambda}(\mu)}{\dim(\lambda)}$, where $\tilde{\mathcal{O}}_{\mu}$ is the conjugacy class in $\tilde{\mathfrak{S}}_d$ of a permutation of cycle type μ , $\tilde{\chi}_{\mu}(\lambda)$ are the irreducible characters of the Sergeev group, and $\dim(\lambda)$ is the dimension of the irreducible representation of the Sergeev group labeled by λ .

As in the case of ordinary Hurwitz numbers, the character formula suggests to replace the functions $f_{\mu}(\lambda)$ with power-sums, so to get an expression of the corresponding Hurwitz numbers as vacuum expectation values in the neutral Fock space. In particular, one can extend the definition of f_{μ} as function on the set \mathcal{SP} of all strict partitions (cf. Section 2.6), and consider the map

$$\tilde{\phi}: \bigoplus_{d \geq 0} \tilde{\mathcal{Z}}_d \longrightarrow \mathbb{C}^{\mathcal{SP}}, \quad \tilde{C}_{\mu} \longmapsto f_{\mu}. \quad (10.3.5)$$

DEFINITION 10.3.5. For $\mu \in \mathcal{OP}_d$, define the *spin completed conjugacy class elements* as

$$\widehat{C}_{\mu} = \frac{1}{\prod_i \mu_i} \tilde{\phi}^{-1}(p_{\mu}) \in \bigoplus_{m=0}^d \tilde{\mathcal{Z}}_m. \quad (10.3.6)$$

For $\mu = (d)$ with d odd, we call the associated element $(\widehat{d}) = \widehat{C}_{(d)}$ *spin completed cycle*.

We can now define the spin analogue of double and simple Hurwitz numbers with completed cycles through Gunningham's character formula.

DEFINITION 10.3.6. Let μ, ν be odd partitions of d , r a positive even integer. Define the *disconnected spin double Hurwitz numbers with $(r+1)$ -completed cycles* as

$$\begin{aligned} h_{g; \mu, \nu}^{\bullet, r, \vartheta} &= \frac{|\text{Aut}(\mu)| |\text{Aut}(\nu)|}{b!} H_d^{\bullet, \vartheta} \left(\mu, (\widehat{(r+1)})^b, \nu \right) \\ &= \frac{|\text{Aut}(\mu)| |\text{Aut}(\nu)|}{b!} 2^{1-g-2d} \sum_{\lambda \in \mathcal{SP}_d} \left(\frac{\dim(\lambda)}{2^{p(\lambda)/2} d!} \right)^2 f_{\mu}(\lambda) \left(\frac{p_{r+1}(\lambda)}{r+1} \right)^b f_{\nu}(\lambda). \end{aligned} \quad (10.3.7)$$

The value b is determined by the Riemann–Hurwitz formula: $rb = 2g - 2 + \ell(\mu) + \ell(\nu)$.

Similarly, define the *spin single Hurwitz numbers with $(r+1)$ -completed cycles* as

$$\begin{aligned} h_{g; \mu}^{\bullet, r, \vartheta} &= \frac{|\text{Aut}(\mu)|}{b!} H_d^{\bullet, \vartheta} \left(\mu, (\widehat{(r+1)})^b \right) \\ &= \frac{|\text{Aut}(\mu)|}{b!} 2^{1-g-2d} \sum_{\lambda \in \mathcal{SP}_d} \left(\frac{\dim(\lambda)}{2^{p(\lambda)/2} d!} \right)^2 f_{\mu}(\lambda) \left(\frac{p_{r+1}(\lambda)}{r+1} \right)^b. \end{aligned} \quad (10.3.8)$$

Again, the value b is determined by the Riemann–Hurwitz formula: $rb = 2g - 2 + \ell(\mu) + d$.

Remark 10.3.7. As stated in Remark 2.6.8, certain papers (e.g. [MSS13; SSZ15; Bor+21; KLPS19; DKPS19]) on the non-spin version of these numbers use the term r -spin Hurwitz numbers for what would be called $(r+1)$ -completed cycles Hurwitz numbers here, to emphasise the relation to r -spin structures on the moduli spaces of curves.

Combining the definition of spin Hurwitz numbers with completed cycles and the action of the operators $\mathcal{F}_{r+1}^B, J_n^B$ (cf. Proposition 10.2.11), we obtain the following expression for spin Hurwitz numbers in terms of vacuum expectation values. The case $r = 2$ can be found in [Lee19, Equation (3.10)], and the case $r > 2$ can be deduced from [MMN20; MMNO20].

THEOREM 10.3.8. *The disconnected spin double and single Hurwitz numbers with $(r + 1)$ -completed cycles can be expressed as the following vacuum expectation values on the Fock space of type B:*

$$h_{g;\mu,\nu}^{\bullet,r,\vartheta} = \frac{2^{1-g}}{b!} \left\langle \frac{J_\mu^B}{\prod_{i=1}^{\ell(\mu)} \mu_i} \left(\frac{\mathcal{F}_{r+1}^B}{r+1} \right)^b \frac{J_{-\nu}^B}{\prod_{j=1}^{\ell(\nu)} \nu_j} \right\rangle, \quad (10.3.9)$$

$$h_{g;\mu}^{\bullet,r,\vartheta} = \frac{2^{1-g}}{b!} \left\langle \frac{J_\mu^B}{\prod_{i=1}^{\ell(\mu)} \mu_i} \left(\frac{\mathcal{F}_{r+1}^B}{r+1} \right)^b \frac{(J_{-1}^B)^d}{d!} \right\rangle. \quad (10.3.10)$$

These formulae will be the starting point to study spin Hurwitz numbers in the following chapters.

CHAPTER 11 — ON SPIN DOUBLE HURWITZ NUMBERS WITH COMPLETED CYCLES

In the non-spin case, double Hurwitz numbers with completed cycles satisfy several properties, such as an evolution equation via explicit bosonic cut-and-join operators, chamber polynomiality with an explicit expression within each chamber, and they exhibit explicit wall-crossing formulae. The crucial tool in the proof of such properties is the Okounkov–Pandharipande algebra.

The goal of this chapter is two-fold. Firstly, we develop the B-type analogue of the Okounkov–Pandharipande operators.

THEOREM 11.A (B-Okounkov–Pandharipande operators). *The operators*

$$\mathcal{E}_n^B(z) = \frac{1}{2} \sum_{k \in \mathbb{Z}} (-1)^k e^{(k-\frac{n}{2})z} \hat{E}_{k-n,k}^B + \frac{\delta_n}{4} \coth\left(\frac{z}{2}\right), \quad (11.0.1)$$

acting on the Fock space of type B, satisfy the following commutation relation:

$$[\mathcal{E}_m^B(z), \mathcal{E}_n^B(w)] = \frac{1}{2} \zeta\left(\det \begin{bmatrix} m & z \\ n & w \end{bmatrix}\right) \mathcal{E}_{n+m}^B(z+w) + \frac{(-1)^n}{2} \zeta\left(\det \begin{bmatrix} m & -z \\ n & -w \end{bmatrix}\right) \mathcal{E}_{n+m}^B(z-w). \quad (11.0.2)$$

Here $\zeta(z) = 2 \sinh\left(\frac{z}{2}\right)$.

Since $\mathcal{E}_n^B(z)$ specialises to the fermionic completed cut-and-join in the expansion around $z = 0$, we were able to express the bosonic version of the latter in terms of an explicit generating series.

THEOREM 11.B (Spin cut-and-join equation). *Define the bosonic completed cut-and-join operators of type B as the operators corresponding to the fermionic completed cut-and-join operator of type B via the boson-fermion correspondence:*

$$\sigma^B\left(\frac{\mathcal{F}_m^B}{m} |\omega\rangle\right) = \mathcal{W}_m^B \sigma^B(|\omega\rangle), \quad \text{for all } |\omega\rangle \in \mathfrak{F}_0^B. \quad (11.0.3)$$

Then the partition function of spin double Hurwitz numbers with $(r+1)$ -completed cycles $Z^{r,\vartheta}(\beta; \mathbf{p}, \mathbf{q})$ satisfies the cut-and-join equation

$$\partial_\beta Z^{r,\vartheta} = \mathcal{W}_{r+1}^B Z^{r,\vartheta}, \quad (11.0.4)$$

and the cut-and-join operators can be effectively computed via an explicit generating series.

We remark that the spin cut-and-join equation was already proved in [MMN20]. Our contribution is an explicit generating series for \mathcal{W}_{r+1}^B (Proposition 11.1.6 in the main text).

In the second part of the chapter, we employ the relation between the operators $\mathcal{E}_m^B(z)$, the fermion cut-and-join operators \mathcal{F}_m^B , and the currents J_n^B to study chamber properties of spin double Hurwitz numbers. The following theorem summarises (without details) the results of Theorems 11.2.6, 11.2.10 and 11.2.19.

THEOREM 11.C (Chamber properties). *Consider the space*

$$\mathcal{H}(m, n) = \left\{ (\mu, \nu) \in \mathcal{OP} \times \mathcal{OP} \left| \begin{array}{l} \ell(\mu) = m, \ell(\nu) = n, \\ \sum_{i=1}^m \mu_i = \sum_{j=1}^n \nu_j \end{array} \right. \right\} \quad (11.0.5)$$

which has a chamber structure defined by walls $\mathcal{W}_{I,J} = \{ (\mu, \nu) \in \mathcal{H}(m, n) \mid \sum_{i \in I} \mu_i = \sum_{j \in J} \nu_j \}$ for $I \subset \llbracket m \rrbracket, J \subset \llbracket n \rrbracket$. Then spin double Hurwitz numbers $h_{g;\mu,\nu}^{r,\vartheta}$, considered as functions of $(\mu, \nu) \in \mathcal{H}(m, n)$, satisfy the following properties.

- Within each chamber, $h_{g;\mu,\nu}^{r,\vartheta}$ is computed by an explicit generating series.
- Within each chamber, $h_{g;\mu,\nu}^{r,\vartheta}$ is a polynomial in (μ, ν) with a specific homogeneous decomposition.
- The difference between values of $h_{g;\mu,\nu}^{r,\vartheta}$ on neighbouring chambers is computed by an explicit generating series.

11.0.1 — RELATION WITH PREVIOUS WORKS AND OPEN QUESTIONS

In the non-spin setting, Bloch–Okounkov defined in [BO00] an algebra that interpolates between the fermionic cut-and-join operators and the currents, later packed into generating functions by Okounkov–Pandharipande in [OP06] (cf. Equation (2.5.22)). Here we adapted their definitions to the spin case.

As of the chamber polynomiality of double Hurwitz numbers, it was originally conjectured for simple Hurwitz numbers by Goulden–Jackson–Vakil [GJV05], later proved by Johnson [Joh15] and generalised to the completed cycle case by Shadrin–Spitz–Zvonkine [SSZ12]. In fact, once the relations between the various concepts and constructions are in place, the proofs of our results are simple adaptations of the arguments of Shadrin–Spitz–Zvonkine.

On the integrability side, Lee [Lee19, Theorem 1.1] has found that a generating series for 3-completed spin double Hurwitz numbers squares to the generating series of 3-completed non-spin double Hurwitz numbers after a proper *tuning* of the weights. A natural question would be whether this generalises to higher r .

QUESTION 11.D. *For every even r , find (and justify) the right linear combination of fermionic completed cut-and-join operators of type A and B of lower order, i.e.*

$$\mathcal{F}^A = \sum_{0 \leq s \leq r+1} a_{r+1,s} \mathcal{F}_{s+1}^A, \quad \mathcal{F}^B = \sum_{\substack{0 \leq s \leq r+1 \\ s \text{ odd}}} b_{r+1,s} \mathcal{F}_{s+1}^A, \quad (11.0.6)$$

such that the associated 2d Toda and 2-BKP tau functions

$$\tau^A = \left\langle e^{J_+^A(t_+)} e^{\mathcal{F}^A} e^{-J_-^A(t_-)} \right\rangle, \quad \tau^B = \left\langle e^{J_+^B(t_+)} e^{\mathcal{F}^B} e^{-J_-^B(t_-)} \right\rangle \quad (11.0.7)$$

satisfy $(\tau^B)^2 = \tau^A|_{t_{+2k}=0}$. Lee’s result for $(r+1) = 3$ corresponds to $(a_{3,3}, a_{3,2}, a_{3,1}) = (\frac{1}{3}, \frac{1}{2}, \frac{11}{12})$ and $(b_{3,3}, b_{3,1}) = (\frac{1}{3}, \frac{2}{3})$.

Note that in general, any BKP tau function squares to a KP tau function (see for instance [You89, Proposition 1]). The content of Lee’s result is that both of these tau functions have a geometric interpretation in terms of spin and ordinary Hurwitz numbers.

11.0.2 — ORGANISATION OF THE CHAPTER

The chapter is organised as follows.

- In Section 11.1 we introduce the B-Okounkov–Pandharipande operators, and prove some of their basic properties. We also introduce the bosonic cut-and-join operators, and express their generating series.
- Section 11.2 is devoted to the study of spin double Hurwitz numbers, in particular their chamber and wall-crossing structure.

11.1 — B-OKOUNKOV–PANDHARIPANDE AND CUT-AND-JOIN OPERATORS

In the neutral boson-fermion correspondence, Theorem 10.2.10, we translate the action of the neutral fermions ϕ_k into the action of a certain operator on the bosonic Fock space. A similar statement holds for quadratic terms in the ϕ 's, i.e. elements of the Lie algebra $\widehat{\mathfrak{g}}(\infty)$. To express the correspondence, we introduce the generating series

$$:\phi(z)\phi(w): = \sum_{k,\ell \in \mathbb{Z}} :\phi_k \phi_\ell: z^k w^\ell. \quad (11.1.1)$$

THEOREM 11.1.1 ([DKM81]). *Under the boson-fermion correspondence, the normal ordered products of neutral fermions act as*

$$\sigma^B(:\phi(z)\phi(w):|\omega\rangle) = \hat{Y}^B(z,w) \sigma^B(|\omega\rangle), \quad (11.1.2)$$

where $\hat{Y}^B(z,w)$, called the (regularised) vertex operator, is given by

$$\hat{Y}^B(z,w) = \frac{1}{2} \frac{z-w}{z+w} \left(\exp \left(\sum_{k \in \mathbb{Z}_+^{\text{odd}}} (z^k + w^k) t_k \right) \exp \left(-2 \sum_{k \in \mathbb{Z}_+^{\text{odd}}} \left(\frac{z^{-k}}{k} + \frac{w^{-k}}{k} \right) \partial_{t_k} \right) - 1 \right). \quad (11.1.3)$$

Here $\frac{1}{z+w}$ is interpreted as the Laurent series expansion in the region $|z| > |w|$.

We are now armed to define an algebra that interpolates between the fermionic cut-and-join operators \mathcal{F}^B and the currents J^B .

DEFINITION 11.1.2. The B-Okounkov–Pandharipande operators are defined for any $n \in \mathbb{Z}$ by

$$\mathcal{E}_n^B(z) = \frac{1}{2} \sum_{k \in \mathbb{Z}} (-1)^k e^{(k-\frac{n}{2})z} \hat{E}_{k-n,k}^B + \frac{\delta_n}{4} \coth\left(\frac{z}{2}\right). \quad (11.1.4)$$

We also define the non-corrected operators

$$\hat{\mathcal{E}}_n^B(z) = \frac{1}{2} \sum_{k \in \mathbb{Z}} (-1)^k e^{(k-\frac{n}{2})z} \hat{E}_{k-n,k}^B. \quad (11.1.5)$$

Notice that $\mathcal{E}_n^B(0) = J_n^B$ for $n \in \mathbb{Z}^{\text{odd}}$, and $\mathcal{F}_m^B = m![z^m] \hat{\mathcal{E}}_0^B(z)$ for $m \in \mathbb{Z}_+^{\text{odd}}$.

Remark 11.1.3. The motivation behind the hyperbolic cotangent correction term has the same origin as in [OPo6]: we would like $\mathcal{E}_0^B(z)$ to be

$$\mathcal{E}_0^B(z) \text{ “=”} \frac{1}{2} \sum_{k \in \mathbb{Z}} (-1)^k e^{kz} \phi_k \phi_{-k} \quad (11.1.6)$$

without normal ordering, but its vacuum expectation value would be

$$\frac{1}{2} \left(\sum_{k < 0} e^{kz} + \frac{1}{2} \right) = \frac{1}{4} \coth\left(\frac{z}{2}\right), \quad \Re(z) > 0, \quad (11.1.7)$$

which is well-defined for $\Re(z) > 0$ only. Thus, we define $\mathcal{E}_0^B(z)$ using normal ordering and add the vacuum expectation value term by hand.

We collect here some other useful properties of the B-Okounkov–Pandharipande operators.

LEMMA 11.1.4. *The following results hold.*

1. *The operators \mathcal{E}_n^B and $\hat{\mathcal{E}}_n^B$ obey the parity relations*

$$\mathcal{E}_n^B(-z) = (-1)^{n+1} \mathcal{E}_n^B(z), \quad \hat{\mathcal{E}}_n^B(-z) = (-1)^{n+1} \hat{\mathcal{E}}_n^B(z). \quad (11.1.8)$$

2. *Under the boson-fermion correspondence, the operators $\hat{\mathcal{E}}_n^B$ act as*

$$\sigma^B(\hat{\mathcal{E}}_n^B(z) |\omega\rangle) = \frac{1}{2} [x^{-n}] \hat{Y}^B(xe^{\frac{z}{2}}, -xe^{-\frac{z}{2}}) \sigma^B(|\omega\rangle). \quad (11.1.9)$$

3. *The subspace of $\mathfrak{go}(\infty)$ spanned by the coefficients $[z^k] \mathcal{E}_m^B(z)$ is a Lie subalgebra. Explicitly,*

$$[\mathcal{E}_m^B(z), \mathcal{E}_n^B(w)] = \frac{1}{2} \mathcal{S}(\det \begin{bmatrix} m & z \\ n & w \end{bmatrix}) \mathcal{E}_{n+m}^B(z+w) + \frac{(-1)^n}{2} \mathcal{S}(\det \begin{bmatrix} m & -z \\ n & w \end{bmatrix}) \mathcal{E}_{n+m}^B(z-w). \quad (11.1.10)$$

Proof. For the parity property, use that $\hat{E}_{k,l}^B = -\hat{E}_{-l,-k}^B$. For the second property, we simply note that

$$:\phi(xe^{\frac{z}{2}})\phi(-xe^{-\frac{z}{2}}): = \sum_{k,l \in \mathbb{Z}} (-1)^l x^{k-l} e^{-\frac{k+l}{2}z} \hat{E}_{k,l}^B.$$

Taking the coefficient of x^{-n} yields $2\hat{\mathcal{E}}_n^B$ under the neutral boson-fermion correspondence via Theorem 11.1.1. To conclude, the commutation relation follows from an explicit calculation: using the expression

$$\mathcal{E}_n^B(z) = \frac{1}{2} \sum_{k \in \mathbb{Z}} (-1)^k e^{(k-\frac{n}{2})z} E_{k-n,k}^B,$$

where $E_{k,l}^B = \phi_k \phi_{-l}$, together with the commutation relations (10.2.8) in $\mathfrak{go}(\infty)$, we find

$$\begin{aligned}
 [\mathcal{E}_m^B(z), \mathcal{E}_n^B(w)] &= \frac{1}{4} \sum_{k,l \in \mathbb{Z}} (-1)^{k+l} e^{(k-\frac{m}{2})z+(l-\frac{n}{2})w} [E_{k-m,k}^B, E_{l-n,l}^B] \\
 &= \frac{1}{4} \sum_{k,l \in \mathbb{Z}} (-1)^{k+l} e^{(k-\frac{m}{2})z+(l-\frac{n}{2})w} \left((-1)^k \delta_{k,l-n} E_{k-m,l}^B - (-1)^{k-m} \delta_{k-m,l} E_{l-n,k}^B \right. \\
 &\quad \left. + (-1)^k \delta_{k+l} E_{l-n,-k+m}^B - (-1)^{k-m} \delta_{k-m+l-n} E_{-k,l}^B \right) \\
 &= \frac{1}{4} \sum_{l \in \mathbb{Z}} (-1)^l e^{(l-n-\frac{m}{2})z+(l-\frac{n}{2})w} E_{l-n-m,l}^B - \frac{1}{4} \sum_{k \in \mathbb{Z}} (-1)^k e^{(k-\frac{m}{2})z+(k-m-\frac{n}{2})w} E_{k-m-n,k}^B \\
 &\quad + \frac{1}{4} \sum_{l \in \mathbb{Z}} (-1)^l e^{(-l-\frac{m}{2})z+(l-\frac{n}{2})w} E_{l-n,l+m}^B - \frac{1}{4} \sum_{k \in \mathbb{Z}} (-1)^{n-k} e^{(k-\frac{m}{2})z+(m-k+\frac{n}{2})w} E_{-k,m+n-k}^B \\
 &= \frac{1}{4} \sum_{k \in \mathbb{Z}} (-1)^k \left(e^{(k-n-\frac{m}{2})z+(k-\frac{n}{2})w} - e^{(k-\frac{m}{2})z+(k-m-\frac{n}{2})w} \right) E_{k-m-n,k}^B \\
 &\quad + \frac{(-1)^m}{4} \sum_{k \in \mathbb{Z}} (-1)^k \left(e^{(-k+\frac{m}{2})z+(k-m-\frac{n}{2})w} - e^{(n-k+\frac{m}{2})z+(k+\frac{n}{2})w} \right) E_{k-m-n,k}^B \\
 &= \frac{1}{4} \sum_{k \in \mathbb{Z}} (-1)^k \left(e^{-\frac{n}{2}z+\frac{m}{2}w} - e^{\frac{n}{2}z-\frac{m}{2}w} \right) e^{(k-\frac{m+n}{2})(z+w)} E_{k-m-n,k}^B \\
 &\quad + \frac{(-1)^m}{4} \sum_{k \in \mathbb{Z}} (-1)^k \left(e^{-\frac{n}{2}z-\frac{m}{2}w} - e^{\frac{n}{2}z+\frac{m}{2}w} \right) e^{(k-\frac{m+n}{2})(w-z)} E_{k-m-n,k}^B \\
 &= \sinh\left(\frac{mw-nz}{2}\right) \mathcal{E}_{n+m}^B(z+w) - (-1)^m \sinh\left(\frac{nz+mw}{2}\right) \mathcal{E}_{n+m}^B(w-z).
 \end{aligned}$$

Now use the parity property. □

We can now shift our attention towards the bosonic completed cut-and-join operators. In the non-spin case, they are the analogues of multiplication by the completed cycles in the symmetric algebra, or equivalently the fermionic cut-and-join operators under the boson-fermion correspondence.

DEFINITION 11.1.5. Let $m \in \mathbb{Z}_+^{\text{odd}}$. Define the m -th (bosonic) completed cut-and-join operator of type B , denoted \mathcal{W}_m^B , as the operator corresponding to the m -th (fermionic) completed cut-and-join operator of type B :

$$\sigma^B\left(\frac{\mathcal{F}_m^B}{m} \mid \omega\right) = \mathcal{W}_m^B \sigma^B(\mid \omega) \tag{11.1.11}$$

for all $\mid \omega\rangle \in \mathfrak{F}_0^B$.

We can explicitly compute their generating series as follows.

PROPOSITION 11.1.6. *The generating series of spin completed cut-and-join operator is given by*

$$\mathcal{W}^B(z) = \sum_{m \in \mathbb{Z}_+^{\text{odd}}} \frac{\mathcal{W}_m^B}{(m-1)!} z^m = \frac{\coth(\frac{z}{2})}{4} \sum_{n=1}^{\infty} \sum_{\substack{k_1+\dots+k_n=0 \\ k_i \text{ odd}}} \frac{2^n}{n!} : \prod_{i=1}^n \varsigma(k_i z) \frac{a_{k_i}}{k_i} :, \tag{11.1.12}$$

where

$$a_k = \begin{cases} k \partial_{t_k} & k > 0, \\ \frac{1}{2} t_{-k} & k < 0. \end{cases} \quad (\text{II.I.I3})$$

As usual, the normal product of the a_k 's moves all derivatives to the right.

Proof. From Equation (II.I.9), we find $\mathcal{W}^B(z) = \frac{1}{2} [x^0] \hat{Y}^B(xe^{\frac{z}{2}}, -xe^{-\frac{z}{2}})$. Computing the right-hand side, we find

$$\mathcal{W}^B(z) = [x^0] \frac{\coth(\frac{z}{2})}{4} \left(\exp \left(\sum_{k \in \mathbb{Z}_+^{\text{odd}}} \varsigma(kz) t_k x^k \right) \exp \left(2 \sum_{k \in \mathbb{Z}_+^{\text{odd}}} \varsigma(kz) \frac{\partial_{t_k}}{k} x^{-k} \right) - 1 \right),$$

and expanding the exponentials finishes the proof. \square

II.2 — PROPERTIES OF SPIN DOUBLE HURWITZ NUMBERS

In this section we employ the algebra of B-Okounkov–Pandharipande operators to analyse and derive several structural properties of spin double Hurwitz numbers. These properties will be described and referred to as:

1. Vacuum expectation in terms of the algebra of $\mathcal{E}_n^B(z)$;
2. Integrability and cut-and-join equation;
3. Chamber structure and wall-crossing formulae.

Each of these results have been observed and proved for several non-spin Hurwitz enumerative problems over the past years, by developing new techniques via the Fock space formalism, and represent major advancements in the field of Hurwitz theory. By now these techniques are more consolidated, and we can prove analogous results by a suitable adaptation of these methods. We therefore derive the results and refer to the original proofs, only pointing out the necessary adaptations.

II.2.1 — VACUUM EXPECTATION IN TERMS OF THE ALGEBRA OF $\mathcal{E}_n^B(z)$

Point (1) is a simple restatement of Theorem 10.3.8. We include it here, since it is the starting point to obtain (2) and (3).

PROPOSITION II.2.1. *Disconnected spin double Hurwitz numbers with $(r+1)$ -completed cycles are given by*

$$h_{g;\mu,\nu}^{\bullet,r,\vartheta} = \frac{2^{1-g}(r!)^b}{b!} [z_1^{r+1} \dots z_b^{r+1}] \left\langle \prod_{i=1}^{\ell(\mu)} \frac{\mathcal{E}_{\mu_i}^B(0)}{\mu_i} \prod_{k=1}^b \hat{\mathcal{E}}_0^B(z_k) \prod_{j=1}^{\ell(\nu)} \frac{\mathcal{E}_{-\nu_j}^B(0)}{\nu_j} \right\rangle, \quad (\text{II.2.1})$$

where $b = \frac{2g-2+\ell(\mu)+\ell(\nu)}{r}$ is given by the Riemann–Hurwitz formula.

11.2.2 — INTEGRABILITY AND CUT-AND-JOIN EQUATION

Integrability properties of spin double Hurwitz numbers were investigated in [Lee19] and further generalised in [MMN20; MMNO20]. Such results can also be easily obtained from Section 11.1, and we include them below for completeness.

DEFINITION 11.2.2. Define the generating series of spin double Hurwitz numbers with $(r+1)$ -completed cycles as

$$Z^{r,\vartheta}(\beta; \mathbf{p}, \mathbf{q}) = \sum_{g,d} \sum_{\mu, \nu \in \mathcal{OP}_d} h_{g;\mu,\nu}^{\bullet, r, \vartheta} 2^{g-1} \beta^b \left(\frac{1}{\ell(\mu)!} \prod_{i=1}^{\ell(\mu)} p_{\mu_i} \right) \left(\frac{1}{\ell(\nu)!} \prod_{i=1}^{\ell(\nu)} q_{\nu_i} \right). \quad (11.2.2)$$

The summands with $b = \frac{2g-2+\ell(\mu)+\ell(\nu)}{r} \notin \mathbb{N}$ are set to 0.

THEOREM 11.2.3 ([Lee19; MMN20; MMNO20]).

1. The generating series $Z^{r,\vartheta}$ can be expressed as the following vacuum expectation value

$$Z^{r,\vartheta}(\beta; \mathbf{p}, \mathbf{q}) = \left\langle \exp \left(\sum_{m \in \mathbb{Z}_+^{\text{odd}}} J_m^B \frac{p_m}{m} \right) \exp \left(\beta \frac{\mathcal{F}_{r+1}^B}{r+1} \right) \exp \left(\sum_{n \in \mathbb{Z}_+^{\text{odd}}} J_{-n}^B \frac{q_n}{n} \right) \right\rangle, \quad (11.2.3)$$

and it is a hypergeometric tau function of the 2-BKP hierarchy (identically in β). Its expansion in terms of Schur Q -functions is given by

$$Z^{r,\vartheta}(\beta; \mathbf{p}, \mathbf{q}) = \sum_{\lambda \in \mathcal{SP}} 2^{-\ell(\lambda)} \exp \left(\beta \frac{p_{r+1}(\lambda)}{r+1} \right) Q_\lambda \left(\frac{1}{2} \mathbf{p} \right) Q_\lambda \left(\frac{1}{2} \mathbf{q} \right). \quad (11.2.4)$$

More generally, define the generating series

$$\begin{aligned} Z^\vartheta(\mathbf{t}; \mathbf{p}, \mathbf{q}) &= \left\langle \exp \left(\sum_{m \in \mathbb{Z}_+^{\text{odd}}} J_m^B \frac{p_m}{m} \right) \exp \left(\sum_{k \in \mathbb{Z}_+^{\text{odd}}} \mathcal{F}_k^B \frac{t_k}{k} \right) \exp \left(\sum_{n \in \mathbb{Z}_+^{\text{odd}}} J_{-n}^B \frac{q_n}{n} \right) \right\rangle \\ &= \sum_{\lambda \in \mathcal{SP}} 2^{-\ell(\lambda)} \exp \left(\sum_{k \in \mathbb{Z}_+^{\text{odd}}} p_k(\lambda) \frac{t_k}{k} \right) Q_\lambda \left(\frac{1}{2} \mathbf{p} \right) Q_\lambda \left(\frac{1}{2} \mathbf{q} \right). \end{aligned} \quad (11.2.5)$$

Then $Z^\vartheta(\mathbf{t}; \mathbf{p}, \mathbf{q})$ is a hypergeometric 2-BKP tau function in \mathbf{p} and \mathbf{q} (identically in \mathbf{t}). Furthermore, $Z^\vartheta(\mathbf{t}; \mathbf{p} = \delta_{m,1}, \mathbf{q} = \delta_{n,1})$ is a KdV tau function in \mathbf{t} and $Z^\vartheta(\mathbf{t}; \mathbf{p} = \delta_{m,1}, \mathbf{q})$ is a 2-BKP tau function in \mathbf{q} and \mathbf{t} .

2. The generating series Z^ϑ satisfies the the partial differential equation (the spin cut-and-join equation):

$$\partial_{t_{r+1}} Z^\vartheta = \mathcal{W}_{r+1}^B Z^\vartheta \quad (11.2.6)$$

for any even r . Consequently,

$$\partial_\beta Z^{r,\vartheta} = \mathcal{W}_{r+1}^B Z^{r,\vartheta}. \quad (11.2.7)$$

The essential ingredient for this theorem is Gunningham's formula. Most results then follow almost immediately from the definitions and various isomorphisms, such as the boson-fermion correspondence. However, the fact that $Z^\vartheta(t; \delta_{m,1}, \delta_{n,1})$ is a KdV tau function in t is not just a formal consequence, and we refer to [MMN20] for a complete discussion.

Furthermore, as mentioned in the introduction, [Lee19, Theorem 1.1] finds a particular linear combination of completed 1-, 2-, and 3-cycles for both the spin and non-spin case, such that the generating series for the spin case squares to the generating series of the non-spin case. It would be interesting to develop such a square formula for higher r as well.

11.2.3 — CHAMBER STRUCTURE

It is known for the non-spin case that some rich structure underlies the double Hurwitz numbers $h_{g;\mu,\nu}^{r,\vartheta}$, when considered as functions of μ and ν for fixed lengths m and n . In fact, the entries of the odd partitions μ and ν can be seen as coordinates in an affine space that can be divided into chambers: in each chamber the Hurwitz numbers can be represented by a chamber-dependent polynomial, and its homogeneous decomposition also satisfies several constraints.

DEFINITION 11.2.4. Let us define the subspace

$$\mathcal{H}(m, n) = \left\{ (\mu, \nu) \in (\mathbb{Z}_+^{\text{odd}})^m \times (\mathbb{Z}_+^{\text{odd}})^n \mid \sum_{i=1}^m \mu_i = \sum_{j=1}^n \nu_j \right\} \quad (11.2.8)$$

and we view spin double Hurwitz numbers as a function in the following sense:

$$h_{g;\mu,\nu}^{r,\vartheta} : \mathcal{H}(m, n) \longrightarrow \mathbb{Q}, \quad (\mu, \nu) \longmapsto h_{g;\mu,\nu}^{r,\vartheta}. \quad (11.2.9)$$

DEFINITION 11.2.5. Let $I \subseteq \llbracket m \rrbracket$ and $J \subseteq \llbracket n \rrbracket$ be non-empty proper subsets. Define the hyperplane (or *wall*) indexed by (I, J) as the set

$$\mathcal{W}_{I,J} = \left\{ (\mu, \nu) \in \mathcal{H}(m, n) \mid \sum_{i \in I} \mu_i = \sum_{j \in J} \nu_j \right\}. \quad (11.2.10)$$

Define the *hyperplane arrangement* $\mathcal{W}(m, n) \subset \mathcal{H}(m, n)$ to be the union of all walls $\mathcal{W}_{I,J}$. A connected component of $\mathcal{H}(m, n) \setminus \mathcal{W}(m, n)$ is called a *chamber*.

THEOREM 11.2.6 (Strong piecewise polynomiality). *Let g be a non-negative integer and let m, n be positive integers such that $(g, n+m) \neq (0, 2)$ and $b = \frac{2g-2+m+n}{r}$ is a positive integer. Then within each chamber \mathfrak{c} of the hyperplane arrangement $\mathcal{W}(m, n)$ there exists a polynomial $P_g^{r,\vartheta,\mathfrak{c}}$ such that*

$$h_{g;\mu,\nu}^{r,\vartheta} = P_g^{r,\vartheta,\mathfrak{c}}(\mu, \nu), \quad \text{for all } (\mu, \nu) \in \mathfrak{c}. \quad (11.2.11)$$

Moreover, $P_g^{r,\vartheta,\mathfrak{c}}$ has the homogeneous degree decomposition

$$P_g^{r,\vartheta,\mathfrak{c}}(\mu, \nu) = \sum_{k=0}^g P_{g,k}^{r,\vartheta,\mathfrak{c}}(\mu, \nu), \quad \deg_{\mu,\nu}(P_{g,k}^{r,\vartheta,\mathfrak{c}}) = 2g - 1 + b - 2k. \quad (11.2.12)$$

We are going to prove the strong piecewise polynomiality from Proposition 11.2.1 and from an explicit generating series of the vacuum expectations involved within each chamber. The method used to compute such generating series is based on a simple commutation procedure: positive-energy operators (i.e. \mathcal{E}_n^B with $n > 0$) are commuted to the right until they hit and annihilate the vacuum. This refines an algorithm by Johnson for double Hurwitz numbers [Joh15]. We start by defining a particular subclass of operators \mathcal{E}^B that play a special role in this commutation procedure.

DEFINITION 11.2.7. For $I \subseteq \llbracket m \rrbracket$, $J \subseteq \llbracket n \rrbracket$, $K \subseteq \llbracket b \rrbracket$, and $\epsilon \in \{\pm 1\}^K$, define

$$\mathcal{E}(I, J, K, \epsilon) = \mathcal{E}_{|\mu_I| - |\nu_J|}^B(z_{K, \epsilon}), \quad (11.2.13)$$

where $\mu_I = \sum_{i \in I} \mu_i$, $\nu_J = \sum_{j \in J} \nu_j$, and $z_{K, \epsilon} = \sum_{k \in K} \epsilon_k z_k$. Moreover for any collection of disjoint pairs $I, M \subseteq \llbracket m \rrbracket$, $J, N \subseteq \llbracket n \rrbracket$, $K, L \subseteq \llbracket b \rrbracket$, and for $\epsilon \in \{\pm 1\}^K$, $\delta \in \{\pm 1\}^L$, define

$$\mathcal{S} \left(\begin{array}{ccc|c} I & J & K & \epsilon \\ M & N & L & \delta \end{array} \right) = \mathcal{S} \left(\det \begin{bmatrix} |\mu_I| - |\nu_J| & z_{K, \epsilon} \\ |\mu_M| - |\nu_N| & z_{L, \delta} \end{bmatrix} \right). \quad (11.2.14)$$

The commutation relation (11.1.10) expressed in the new notation turns into:

$$\begin{aligned} [\mathcal{E}(I, J, K, \epsilon), \mathcal{E}(M, N, L, \delta)] &= \frac{1}{2} \mathcal{S} \left(\begin{array}{ccc|c} I & J & K & \epsilon \\ M & N & L & \delta \end{array} \right) \mathcal{E}(I \cup M, J \cup N, K \cup L, \epsilon \cup \delta) \\ &\quad - \frac{(-1)^{|I|+|J|}}{2} \mathcal{S} \left(\begin{array}{ccc|c} I & J & K & -\epsilon \\ M & N & L & \delta \end{array} \right) \mathcal{E}(I \cup M, J \cup N, K \cup L, (-\epsilon) \cup \delta). \end{aligned} \quad (11.2.15)$$

On the other hand, spin double Hurwitz numbers are expressed in terms of the \mathcal{E} 's as

$$\begin{aligned} h_{g; \mu, \nu}^{\bullet, r, \vartheta} &= \frac{2^{1-g} (r!)^b}{b!} [z_1^{r+1} \dots z_b^{r+1}] \\ &\quad \left\langle \prod_{i=1}^n \frac{\mathcal{E}(\{i\}, \emptyset, \emptyset, \emptyset)}{\mu_i} \prod_{k=1}^b \hat{\mathcal{E}}(\emptyset, \emptyset, \{k\}, \{+1\}) \prod_{j=1}^m \frac{\mathcal{E}(\emptyset, \{j\}, \emptyset, \emptyset)}{\nu_j} \right\rangle, \end{aligned} \quad (11.2.16)$$

where the $\hat{\mathcal{E}}$ symbol refers as usual to the absence of the correction term.

DEFINITION 11.2.8. A *commutation pattern* P is a set of tuples

$$\left\{ (I_t^P, J_t^P, K_t^P, \epsilon_t^P; M_t^P, N_t^P, L_t^P, \delta_t^P) \right\}_{t \in \llbracket m+n+b-1 \rrbracket} \quad (11.2.17)$$

where $I_t^P, M_t^P \subseteq \llbracket m \rrbracket$, $J_t^P, N_t^P \subseteq \llbracket n \rrbracket$, $K_t^P, L_t^P \subseteq \llbracket b \rrbracket$, $\epsilon_t^P \in \{\pm 1\}^{K_t^P}$, $\delta_t^P \in \{\pm 1\}^{L_t^P}$ are such that we get a non-vanishing contribution to the vacuum expectation value (11.2.16) when we go through the algorithm that commutes the rightmost positive-energy operator to the right, in such a way that the t -th commutator computed is

$$[\mathcal{E}(I_t^P, J_t^P, K_t^P, \epsilon_t^P), \mathcal{E}(M_t^P, N_t^P, L_t^P, \delta_t^P)]. \quad (11.2.18)$$

A fundamental point is that commutation patterns do not depend on the specific partitions μ and ν , but only on the signs of the expressions $|\mu_I| - |\nu_J|$ (that is, the energies of the operators produced during the commutation process), and therefore only on the chamber \mathfrak{c} considered. This consideration allows to define the set $\text{CP}^{\mathfrak{c}}$ of commutation patterns relative to a chamber \mathfrak{c} as independent of (μ, ν) .

Remark 11.2.9. Within a chamber \mathfrak{c} of $\mathcal{H}(m, n)$ there are no disconnected covers. In fact, in order to have one of a certain degree d , a splitting $\llbracket m \rrbracket = I \sqcup I'$ and $\llbracket n \rrbracket = J \sqcup J'$ such that $|\mu_I| = |\nu_J| = d$ is needed, for the ramifications indices of μ_I and of ν_J to lie on the same connected component of the cover. Therefore within a chamber \mathfrak{c} the notations $h_{g;\mu,\nu}^{\bullet,r,\vartheta}$ and $h_{g;\mu,\nu}^{\circ,r,\vartheta}$ coincide and we refer to them simply as $h_{g;\mu,\nu}^{r,\vartheta}$.

THEOREM 11.2.10 (Chamber generating series). *Within a chamber \mathfrak{c} of $\mathcal{H}(m, n)$ we have:*

$$h_{g;\mu,\nu}^{r,\vartheta} = \frac{2^{1-g-\tau} (r!)^b}{b! \cdot \prod_i \mu_i \prod_j \nu_j} [z_1^{r+1} \cdots z_b^{r+1}] \sum_{P \in \text{CP}^{\mathfrak{c}}} (-1)^{\iota(P)} \frac{\coth\left(\frac{z \llbracket b \rrbracket, \epsilon_t^P \cup \delta_t^P}{2}\right)}{4} \prod_{t=1}^{\tau} \mathcal{S} \left(\begin{matrix} I_t^P & J_t^P & K_t^P & \epsilon_t^P \\ M_t^P & N_t^P & K_t^P & \delta_t^P \end{matrix} \right), \quad (11.2.19)$$

where $\tau = m + n + b - 1$ is the total number of commutations for each commutation pattern, and

$$\iota(P) = |S| + \sum_{s \in S} |I_s^P| + |J_s^P|, \quad (11.2.20)$$

where $S \subseteq \llbracket \tau \rrbracket$ is the set indexing the times the summand $-\epsilon_t^P$ was chosen (as opposed to the first summand, involving ϵ_t^P with no minus sign), out of the t -th commutator.

Proof. The proof is a straightforward adaptation of the argument in [SSZ12]. \square

Remark 11.2.11. This statement can be slightly generalised to disconnected Hurwitz numbers on the walls, see [SSZ12, Theorem 4.6].

Proof of Theorem 11.2.6. The proof is an adaptation of the argument in [SSZ12]. The only differences consist in the introduction of an extra signed summand at each commutator and the introduction of the $\coth(z)$ function replacing the $1/\sinh(z)$ function. The latter does not spoil the polynomial argument, as $\coth(z) = \cosh(z)/\sinh(z)$, so that multiplying by $\cosh(z)$ does not introduce new poles (and therefore, already proved to be removable) and also does not spoil the parity argument in the degrees.

The introduction of new summands does not change the fact that the sum over $\text{CP}^{\mathfrak{c}}$ and the product over t are finite, and the degrees in the μ_i and in the ν_j are coupled to degrees in the z_k as in [SSZ12], therefore collecting the coefficient of $z_1^{r+1} \cdots z_b^{r+1}$ again guarantees a polynomial in the μ_i and ν_j . The parity of the functions involved determines again even jumps in number g for what concerns the homogeneous degrees, again starting at the same top degree $2g - 1 + b$. \square

Remark 11.2.12. The only part of the statement [SSZ12, Theorem 6.4] that is in principle not guaranteed anymore by Theorem 11.2.6 is the positivity of the polynomials in the homogeneous decomposition. Such a statement could still exist in some form, but it falls beyond the scope of the current dissertation.

In the special case of one-part spin double Hurwitz numbers, that is the case of a generic ramification $\mu \in \mathcal{OP}_d$ over zero and a total ramification $\nu = (d)$ over infinity, the chamber structure is trivial. Thus, we can get a closed formula that does not depend on the choice of a chamber. We also derive two specialisations of this formula. Following [OP06], we denote

$$\mathcal{S}(z) = \frac{\mathcal{S}(z)}{z}. \quad (11.2.21)$$

COROLLARY 11.2.13. *Let $\mu \in \mathcal{OP}_d$. The one-part spin double Hurwitz numbers are given by:*

$$h_{g;\mu,(d)}^{r,\vartheta} = \frac{2^{1-g-b} d^{b-1} (r!)^b}{b!} [z_1^r \cdots z_b^r] \sum_{\epsilon \in \{\pm 1\}^{\llbracket b \rrbracket}} \frac{\coth\left(\frac{z_{\llbracket b \rrbracket}, \epsilon}}{2}\right)}{4} \prod_{k=1}^b \mathcal{S}(dz_k) \prod_{i=1}^{\ell(\mu)} \frac{\mathcal{S}(\mu_i z_{\llbracket b \rrbracket}, \epsilon)}{\mu_i}. \quad (11.2.22)$$

Proof. It is a specialisation of Theorem 11.2.10: since there is a single negative energy operator within the vacuum expectation, that operator has to be involved in every commutations until the end of the commutation procedure, therefore the sum over commutation patterns collapses. \square

We further derive two specialisations for spin single Hurwitz numbers, having in mind conjectural applications in Gromov–Witten theory of $(\mathbb{P}^1, \mathcal{O}(-1))$.

COROLLARY 11.2.14. *Let d be an odd integer. The one-part spin single Hurwitz numbers are given by:*

$$h_{g;(d)}^{r,\vartheta} = \frac{2^{1-g-b} d^{b-1} (r!)^b}{b! d!} [z_1^r \cdots z_b^r] \sum_{\epsilon \in \{\pm 1\}^{\llbracket b \rrbracket}} \frac{\coth\left(\frac{z_{\llbracket b \rrbracket}, \epsilon}}{2}\right)}{4} \mathcal{S}(z_{\llbracket b \rrbracket}, \epsilon)^d \prod_{k=1}^b \mathcal{S}(dz_k). \quad (11.2.23)$$

Proof. From Definition 10.3.6, we see that $h_{g;(d)}^{r,\vartheta} = \frac{1}{|\text{Aut}(1^d)|} h_{g;(1^d),(d)}^{r,\vartheta}$. Using that $|\text{Aut}(1^d)| = d!$, the result follows from Corollary 11.2.13. \square

COROLLARY 11.2.15. *For $b = 1$ we have:*

$$h_{g;(d)}^{r,\vartheta} = \frac{2^{1-g} r!}{\mu!} [z^{2g}] \frac{\cosh\left(\frac{z}{2}\right)}{2} \mathcal{S}(z)^{d-1} \mathcal{S}(dz), \quad g = \frac{r - \mu + 1}{2}. \quad (11.2.24)$$

The formula above is conjecturally related to the spin Gromov–Witten correlator of $(\mathbb{P}^1, \mathcal{O}(-1))$ with a single point insertion relative to the partition μ via the conjectural spin GW/H correspondence (cf. Question 10.A).

We are now armed to define and derive wall-crossing formulae for double spin Hurwitz numbers.

DEFINITION 11.2.16. *A wall-crossing formula for spin double Hurwitz numbers is an expression for the quantity*

$$\text{WC}_{g,I,J}^{r,\vartheta}(\mu, \nu) = h_g^{r,\vartheta}|_{\mathfrak{c}_1} - h_g^{r,\vartheta}|_{\mathfrak{c}_2} \quad (11.2.25)$$

for two neighbouring chambers \mathfrak{c}_1 and \mathfrak{c}_2 separated by the wall $\mathcal{W}_{I,J} = \{(\mu, \nu) \mid |\mu_I| - |\nu_J| = 0\}$, where we fix \mathfrak{c}_1 as chamber with $|\mu_I| < |\nu_J|$ and \mathfrak{c}_2 the chamber with $|\mu_I| > |\nu_J|$.

DEFINITION 11.2.17. *Define the generating series for multi-completed cycles as:*

$$H_{\mu,\nu}(z_1, \dots, z_b) = \frac{2^{1-g}}{b!} \left\langle \prod_{i=1}^m \frac{\mathcal{E}_{\mu_i}^B(0)}{\mu_i} \prod_{k=1}^b \hat{\mathcal{E}}_0^B(z_k) \prod_{j=1}^n \frac{\mathcal{E}_{\nu_j}^B(0)}{\nu_j} \right\rangle. \quad (11.2.26)$$

Moreover, within a chamber \mathfrak{c} , let $H_{\mu,\nu}^\sigma(z_1, \dots, z_b)$ be the sum of the contributions in the statement of Theorem 11.2.10 of all those commutation patterns $P \in \text{CP}^{\mathfrak{c}}$ whose last sign vector $\epsilon_\tau^P \cup \delta_\tau^P$ is σ .

Remark 11.2.18. Piecewise polynomiality results extend to multi-completed cycles generating series, in the sense that collecting the coefficient of arbitrary $z_1^{r_1+1} \cdots z_b^{r_b+1}$ (as opposed to along the diagonal $r_i = r$) imposes completed cycles of different sizes on the ramifications, but the piecewise polynomiality structure still stands.

What is peculiar about wall-crossing formulae is that the vast majority of the terms arising from the vacuum expectations defining $\text{WC}_{g,I,J}$ cancel out, and what remains can be expressed as a finite sum of quadratic terms in the generating series H .

THEOREM 11.2.19 (Wall-crossing formula). *Let $I \subseteq \llbracket m \rrbracket$ and $J \subseteq \llbracket n \rrbracket$ be non-empty proper subsets, and denote their complements by I' and J' respectively. Let $\Delta = |\mu_I| - |\nu_J|$. We have:*

$$\begin{aligned} \text{WC}_{g,I,J}^{r,\vartheta}(\mu, \nu) &= \Delta^2 [z_1^{r+1} \cdots z_b^{r+1}] \\ &\sum_{\substack{K \sqcup K' = \llbracket b \rrbracket \\ \epsilon \in \{\pm 1\}^K \\ \epsilon' \in \{\pm 1\}^{K'}}} \frac{H_{\mu_I, \nu_J \cup \Delta}^\epsilon(z_K)}{\langle \mathcal{E}(I, J, K, \epsilon) \mathcal{E}_{-\Delta}(0) \rangle} \langle \mathcal{E}(I, J, K, \epsilon) \mathcal{E}(I', J', K', \epsilon') \rangle \frac{H_{\mu_{I' \cup \Delta}, \nu_{J'}}^{\epsilon'}(z_{K'})}{\langle \mathcal{E}_\Delta(0) \mathcal{E}(I', J', K', \epsilon') \rangle}. \end{aligned} \tag{11.2.27}$$

Remark 11.2.20. The formula above suggests the appearance of two poles arising from the vacuum expectations in the denominator, but these poles are removable. For example, the first vacuum expectation gives

$$\frac{1}{\langle \mathcal{E}(I, J, K, \epsilon) \mathcal{E}_{-\Delta}(0) \rangle} = \frac{4}{\coth(z_{K, \epsilon}) \mathcal{G}(\Delta z_{K, \epsilon})} \frac{2}{((-1)^\Delta - 1)},$$

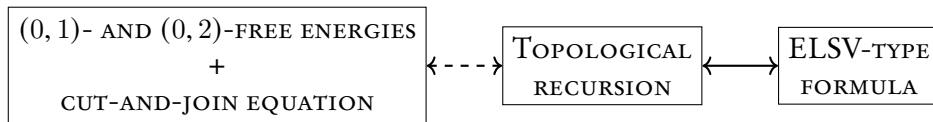
giving a pole for even Δ . In fact, the entire commutator $\langle \mathcal{E}(I, J, K, \epsilon) \mathcal{E}_{-\Delta}(0) \rangle$ simplifies against each term of the expansion of $H_{\mu_I, \nu_J \cup \Delta}^\epsilon(z_K)$ by definition (we inserted the operator $\mathcal{E}_{-\Delta}(0)$ to the right and we ask the linear combination of the variables to be $z_{K, \epsilon}$, so it is always possible to run the commutation process in such a way that $\langle \mathcal{E}(I, J, K, \epsilon) \mathcal{E}_{-\Delta}(0) \rangle$ is the last step of it, for each commutation pattern). The same reasoning leads to the simplification of $\langle \mathcal{E}_\Delta(0) \mathcal{E}(I', J', K', \epsilon') \rangle$ against $H_{\mu_{I' \cup \Delta}, \nu_{J'}}^{\epsilon'}(z_{K'})$.

Proof. The proof is a straightforward adaptation of the argument in [SSZ12, Theorem 6.6]. \square

CHAPTER 12 — A SPIN BOUCHARD–MARIÑO CONJECTURE AND ELSV FORMULA

We now move our attention to *single spin* Hurwitz numbers. In the non-spin case, single Hurwitz numbers are known to satisfy topological recursion: this was first conjecture by Bouchard and Mariño [BM08], proved in [BEMSI1; EMSI1], and later generalised to completed cycles, orbifold and monotone variations.

As explained in Section 2.3.1, topological recursion is a way to compute enumerative quantities from $(0, 1)$ and $(0, 2)$ data (the spectral curve), recursively on $2g - 2 + n$. In Hurwitz theory, the cut-and-join equation is, roughly speaking, a recursive procedure to compute Hurwitz numbers. Thus, one would expect topological recursion to hold and, applying the Eynard–DOSS correspondence of Theorem 2.3.12, to obtain an ELSV-type formula.



To actually prove topological recursion for Hurwitz problems, an extra property quasi-polynomiality is required. However, the actual shape of the spectral curve can be obtained by looking at the $(0, 1)$ - and $(0, 2)$ -free energies.

In this chapter, we compute such unstable free energies for single spin Hurwitz numbers via the fermion formalism, and we conjecture that the stable free energies can be computed by topological recursion on a specific spectral curve. We then give evidence for this conjecture by proving it in genus zero and for $(g, n) = (1, 1)$.

CONJECTURE 12.A. *Let r be a positive even integer. The spectral curve on \mathbb{P}^1 given by*

$$x(z) = \log(z) - z^r, \quad y(z) = z, \quad B(z_1, z_2) = \frac{1}{2} \left(\frac{1}{(z_1 - z_2)^2} + \frac{1}{(z_1 + z_2)^2} \right) dz_1 dz_2. \quad (12.0.1)$$

generates via topological recursion spin single Hurwitz numbers with $(r + 1)$ -completed cycles: for $2g - 2 + n > 0$

$$\omega_{g,n}^{r,\vartheta}(z_1, \dots, z_n) = \sum_{\mu_1, \dots, \mu_n \in \mathbb{Z}_+^{\text{odd}}} h_{g;\mu}^{r,\vartheta} \prod_{i=1}^n \mu_i e^{\mu_i x(z_i)} dx(z_i). \quad (12.0.2)$$

In a recent paper, Alexandrov and Shadrin [AS21] prove topological recursion for a wide class of hypergeometric BKP tau-functions, using methods similar to [BDKS20]. In particular, they confirm our conjecture.

We remark that the above spectral curve does not satisfy one canonical requirement imposed by the theory of topological recursion in the classical sense: the $(0, 2)$ -correlator $B(z_1, z_2)$

has poles when z_1 and z_2 approach two *distinct* ramification points of x . However, the way these extra poles arise encode in a beautiful way some extra structure of this curve. First of all, these new poles are of order two and with the same bi-residue. More importantly, the number of critical points is even and they come in pairs because of an underlying $\mathbb{Z}/2\mathbb{Z}$ -action: the double poles only arise for z_2 approaching either the same critical point z_1 , or its conjugate with respect to the group action. We then realise the quotient of the conjectural spectral curve modulo the group action, reducing by half the number of ramification points. Surprisingly, we find that the correlators $\omega_{g,n}$ of the quotient spectral curve differ by the initial correlators just by some simple prefactor. We export this principle to the more general setting of spectral curves with a finite group acting on them.

In the second part of this chapter we apply the Eynard–DOSS correspondence to compute the cohomological field theory representing spin Hurwitz numbers, and realise such CohFT as a Chiodo class (also appearing in the non-spin case) twisted by Witten 2-spin class.

THEOREM 12.B (Spin r -ELSV formula). *Conjecture 12.A is equivalent to the following ELSV-type formula: for every $\mu = (\mu_1, \dots, \mu_n) \in \mathcal{OP}_d$,*

$$h_{g;\mu}^{r,\vartheta} = qr^{\frac{(r+1)(2g-2+n)+d}{r}} \left(\prod_{i=1}^n \frac{\binom{\mu_i}{r}^{[\mu_i]}}{[\mu_i]!} \right) \int_{\overline{\mathcal{M}}_{g,n}} \frac{C_{g,n}^{r,1,\vartheta}(\langle \mu_1 \rangle, \dots, \langle \mu_n \rangle)}{\prod_{i=1}^n (1 - \frac{\mu_i}{r} \psi_i)}, \quad (12.0.3)$$

where $\mu_i = r[\mu_i] + r - (2\langle \mu_i \rangle + 1)$, and $C_{g,n}^{r,1,\vartheta}$ is the Chiodo class twisted by the Witten 2-spin class (see Corollary 12.4.9 for the precise definition).

It is worth remarking that the product of the Chiodo class and the Witten 2-spin class has a particularly clean expression as a sum over stable graphs, and we use this expression in the proof.

12.0.1 — RELATION WITH OTHER WORKS AND OPEN QUESTIONS

In principle, Conjecture 12.A can be proved by adapting the well-established techniques that have provided in the past a proof of topological recursion for the analogous statement in the non-spin case. This procedure involves the fermion formalism and is rather technical. On the other hand, the recent work of Bychkov–Dunin-Barkowski–Kazarian–Shadrin [BDKS20] proves topological recursion for a large class of hypergeometric KP tau functions, including all known cases of non-spin Hurwitz numbers. While working on [GKL21], we learned that Alexandrov and Shadrin were adapting the techniques of [BDKS20] to the BKP setting. Their proof of Conjecture 12.A is an application of their general statement to the specific case of spin Hurwitz numbers.

It is also worth noting that the twist obtained by intersecting with Witten’s class in the Gromov–Witten potential for Kähler targets already appeared in [JKV01; CZ09]. However, both these appearances in a sense take place on the Gromov–Witten side of the GW/H correspondence, whereas our result reveals the Witten class on the Hurwitz theory side.

The class $C_{g,n}^{r,k,\vartheta}$ is also related to a forthcoming work by Costantini–Sauvaget–Schmitt [CSS]. For a fixed value of k and a satisfying $\sum_i m_i = k(2g - 2 + n)$, the usual Chiodo class $C_{g,n}^{r,k}(m)$ (after multiplying by a certain power of r) is a polynomial in r for large values of r . The constant term of this polynomial is denoted by $\text{DR}_{g,n}^k(m)$, and is called the (twisted) double

ramification cycle. The restriction to $\mathcal{M}_{g,n}$ of the Poincaré dual of this class is the locus

$$\begin{aligned} \mathcal{M}_{g,n}^k(m) &= \left\{ (C, x_1, \dots, x_n) \mid \omega_{C, \log}^{\otimes k} \cong \mathcal{O}_C(\sum_i m_i x_i) \right\} \\ &= \left\{ (C, x_1, \dots, x_n) \mid \begin{array}{l} \exists \eta \text{ a meromorphic } k\text{-differential on } C \\ \text{with } \operatorname{div}(\eta) = \sum_i (m_i - k)x_i \end{array} \right\} \end{aligned} \quad (12.0.4)$$

and intersection numbers of the form $\operatorname{DR}_{g,n}^k(m) \cdot \psi_1^{2g-3+n}$ appear in the computation of the orbifold Euler characteristic of $\mathcal{M}_{g,n}^k(m)$.

It is expected that the classes $C_{g,n}^{r,k,\vartheta}(a)$ satisfy the same polynomiality in r , thus allowing one to define a class $\operatorname{DR}_{g,n}^{k,\vartheta}(a)$. In particular, if $k = 2l+1$ and the $m_i = 2a_i+1$ are odd, then the space $\mathcal{M}_{g,n}^k(m)$ splits into components with a constant parity of the spin structure $\mathcal{O}(\sum_i a_i x_i) \otimes \omega_C^{-l}$. Costantini, Sauvaget, and Schmitt conjecture that the restriction of $\operatorname{DR}_{g,n}^{k,\vartheta}(a)$ to $\mathcal{M}_{g,n}$ is a sum of the classes of components of $\mathcal{M}_{g,n}^k(m)$ with a sign determined by the parity. Besides, they propose some conjectural properties of this class, allowing them to compute the intersection numbers $\operatorname{DR}_{g,n}^{k,\vartheta}(a) \cdot \psi_1^{2g-3+n}$.

12.0.2 — ORGANISATION OF THE CHAPTER

The chapter is organised as follows.

- Section 12.1 contains our main conjecture: single spin Hurwitz numbers are generated by topological recursion. We also give evidence for this conjecture by proving it in genus zero and for $(g, n) = (1, 1)$.
- Since the conjectural spectral curve differs from the usual definition, in Section 12.2 we define and analyse G -quotients of spectral curves, and reduce them to the usual setting of topological recursion.
- We then employ the correspondence between topological recursion and cohomological field theories in Section 12.3 to derive the representation of spin Hurwitz numbers as intersection numbers on $\overline{\mathcal{M}}_{g,n}$ involving an explicit CohFT.
- To conclude, in Section 12.4 we show that such CohFT is constructed from a twist of Chiodo’s class and Witten 2-spin class.

12.1 — THE SPECTRAL CURVE

Let r be a positive even integer. Consider the spectral curve on \mathbb{P}^1 given by

$$x(z) = \log(z) - z^r, \quad y(z) = z, \quad B(z_1, z_2) = \frac{1}{2} \left(\frac{1}{(z_1 - z_2)^2} + \frac{1}{(z_1 + z_2)^2} \right) dz_1 dz_2. \quad (12.1.1)$$

As noted in the introduction, this spectral curve does not satisfy the usual axioms of topological recursion, since B has poles when the two arguments approach different ramification points. Nevertheless, one can define the topological recursion correlators $\omega_{g,n}^{r,\vartheta}$ via the Eynard–Orantin topological recursion formula (2.3.9). Our main conjecture relates the multidifferentials $\omega_{g,n}^{r,\vartheta}$ and the spin Hurwitz numbers free energies.

IV. Spin Hurwitz theory

DEFINITION 12.1.1. The *free energies* for spin single Hurwitz numbers with $(r+1)$ -completed cycles are

$$F_{g,n}^{r,\vartheta}(e^{x_1}, \dots, e^{x_n}) = \sum_{\mu_1, \dots, \mu_n \in \mathbb{Z}_+^{\text{odd}}} h_{g;\mu}^{r,\vartheta} e^{\mu_1 x_1} \dots e^{\mu_n x_n}. \quad (12.1.2)$$

CONJECTURE 12.1.2. The coefficients obtained by expanding the correlators $\omega_{g,n}^{r,\vartheta}$ near $e^{x_i} = 0$ are exactly the $(r+1)$ -completed cycles spin single Hurwitz numbers:

$$\omega_{g,n}^{r,\vartheta}(z_1, \dots, z_n) - \delta_{g,0} \delta_{n,2} \omega_{0,2}^{r,\vartheta}(e^{x(z_1)}, e^{x(z_2)}) = d_1 \dots d_n F_{g,n}^{r,\vartheta}(e^{x_1}, \dots, e^{x_n}) \Big|_{x_i=x(z_i)}. \quad (12.1.3)$$

Alexandrov and Shadrin [AS21] recently proved topological recursion for a wide class of hypergeometric BKP tau functions, confirming our conjecture.

THEOREM 12.1.3 ([AS21]). Conjecture 12.1.2 is true.

The remaining part of the section is devoted to proving the conjecture for $g=0$ (fact that originally motivated our conjecture) and for $(g,n) = (1,1)$. In turn, we also prove that genus zero spin and non-spin Hurwitz numbers are related by a simple formula.

12.1.1 — SPIN HURWITZ NUMBERS IN GENUS ZERO

When the source of a spin Hurwitz cover is rational, two simplifications occur. First, \mathbb{P}^1 has a unique spin structure, and we see that in Definition 10.3.6

$$p(N_{f,O(-1)}) = p(O(-1)) = 0 \quad (12.1.4)$$

for any cover in the count. Therefore, for $g=0$, the spin Hurwitz numbers $h_{0,\mu}^{r,\vartheta}$ are actual counts, without any sign.

For the other simplification, recall the Riemann–Hurwitz formula: for a ramified cover of Riemann surfaces $f: S \rightarrow T$ of degree d with ramification profiles μ^i ,

$$2 - 2g(S) = d(2 - 2g(T)) - \sum_i (d - \ell(\mu^i)). \quad (12.1.5)$$

Looking at the definition of spin single Hurwitz numbers, Definition 10.3.6, we see that $b = \frac{2g-2+\ell(\mu)+d}{r}$ is chosen such that we do get a genus g source curve if we have one branch point with ramification profile μ and b branch points with ramification profile $(r+1, 1, \dots, 1)$ (recall that we always have $g(T) = 0$). However, the definition uses the spin completed cycles $\widehat{C}_{(r+1)}$. From Definition 10.3.5 and [Iva04, Proposition 6.4] we see that

$$\widehat{C}_{(r+1)} - \widetilde{C}_{(r+1)} \in \bigoplus_{d=0}^r \widetilde{Z}_d. \quad (12.1.6)$$

In particular, for any $\mu \in \mathcal{OP}_d$ with $d \leq r$, we have $r+1 - \ell(\mu \cup (1^{r+1-d})) = d - \ell(\mu) < r = r+1 - \ell((r+1))$. It follows that for any cover of \mathbb{P}^1 with ramification profiles μ and b choices of partitions occurring with non-zero coefficients in $\widehat{C}_{(r+1)}$, the genus of the source curve is at most g , with equality occurring exactly if we choose $(r+1)$ every time. This occurrence of a source curve whose genus is lower than expected is called *genus defect* in e.g. [SSZ15]. As a consequence, for $g=0$, there are no contributions from the completions $\widehat{C}_{(r+1)} - \widetilde{C}_{(r+1)}$,

as this would require a connected source curve with negative genus. This means that in the definition of $h_{0;\mu}^{r,\vartheta}$, we may replace the $\tilde{C}_{(r+1)}$ by the $\tilde{C}_{(r+1)}$.

The same argument holds for the non-spin case: there we may also replace the completed cycles $\tilde{C}_{(r+1)} \in \bigoplus_{d=0}^{r+1} \mathcal{Z}_d$ by the non-completed $C_{(r+1)}$ for $g = 0$ Hurwitz numbers (for more background on completed cycles for the non-spin case, see e.g. [OP06; SSZ12]). Hence, because $C_{(r+1)}$ and $\tilde{C}_{(r+1)}$ represent the same partition $(r+1)$, we get:

PROPOSITION 12.1.4. *Let r be a positive even integer and $\mu \in \mathcal{OP}$. Then the spin and non-spin Hurwitz numbers with these arguments are equal:*

$$h_{0;\mu}^{r,\vartheta} = h_{0;\mu}^r. \quad (12.1.7)$$

As a corollary, we obtain that the spin free energies are the anti-symmetrisations of the non-spin ones.

COROLLARY 12.1.5. *Let r be a positive even integer. The genus zero free energies for the spin case are the anti-symmetrisations in all arguments of those in the non-spin case:*

$$F_{0,n}^{r,\vartheta}(e^{x_1}, \dots, e^{x_n}) = \frac{1}{2^n} \sum_{\epsilon_1, \dots, \epsilon_n \in \{\pm 1\}} \left(\prod_{i=1}^n \epsilon_j \right) F_{0,n}^r(\epsilon_1 e^{x_1}, \dots, \epsilon_n e^{x_n}). \quad (12.1.8)$$

In particular, Conjecture 12.1.2 holds in genus zero:

$$\omega_{0,n}^{r,\vartheta}(z_1, \dots, z_n) - \delta_{n,2} \omega_{0,2}^{r,\vartheta}(e^{x(z_1)}, e^{x(z_2)}) = d_1 \cdots d_n F_{0,n}^{r,\vartheta}(e^{x_1}, \dots, e^{x_n}) \Big|_{x_i = x(z_i)}. \quad (12.1.9)$$

Proof. The first part follows directly from Proposition 12.1.4: antisymmetrising is the same as restricting to odd powers of e^{x_i} in the series expansion. For the second part, the unstable free energies for the non-spin case were computed in [MSS13; KLPS19]. In that case the $(0, 1)$ -free energy is already odd, so it must be equal to its spin counterpart $\omega_{0,1}^{r,\vartheta}$. For $\omega_{0,2}^{r,\vartheta}$, Equation (12.1.9) is the antisymmetrisation of the non-spin case. The case $n > 2$ follows by induction. \square

In the next part of this section, we will exploit the operator formalism to get a closed formula for the one-part free energies (i.e. $n = 1$ and general g).

12.1.2 — ONE-PART SPIN SINGLE HURWITZ NUMBERS

A GENERAL FORMULA FOR VEVs IN THE NEUTRAL FERMIONS FORMALISM

Following [KLPS19], our starting point is the following expression for spin single Hurwitz numbers

$$h_{g;\mu}^{\bullet,r,\vartheta} = 2^{1-g} [u^{rb}] \left\langle e^{J_1^B} e^{u^r \frac{\mathcal{F}_{r+1}^B}{r+1}} \prod_{i=1}^{\ell(\mu)} \frac{J_{-\mu_i}^B}{\mu_i} e^{-u^r \frac{\mathcal{F}_{r+1}^B}{r+1}} e^{-J_1^B} \right\rangle. \quad (12.1.10)$$

This can be easily obtained from Theorem 10.3.8 and the fact that J_1^B and \mathcal{F}_{r+1}^B annihilates the vacuum. The advantage of this formulation is that we now have $\ell(\mu)$ factors of the same shape, namely

$$h_{g;\mu}^{\bullet,r,\vartheta} = 2^{1-g} [u^{rb}] \left\langle e^{J_1^B} \prod_{i=1}^{\ell(\mu)} \frac{\mathcal{O}_{-\mu_i}^{B,r}(u)}{\mu_i} e^{-J_1^B} \right\rangle, \quad \mathcal{O}_{-\mu}^{B,r}(u) = e^{u^r \frac{\mathcal{F}_{r+1}^B}{r+1}} J_{-\mu}^B e^{-u^r \frac{\mathcal{F}_{r+1}^B}{r+1}} \quad (12.1.11)$$

which we can describe uniformly.

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LEMMA 12.1.6. *Let $\mu \in \mathbb{Z}_+^{\text{odd}}$. Then:*

$$\Theta_{-\mu}^{B,r}(u) = \sum_{l+\mu/2>0} (-1)^l e^{u^r \frac{(l+\mu)^{r+1}-l^{r+1}}{r+1}} E_{l+\mu,l}^B. \quad (12.1.12)$$

Proof. Using the symmetry of the base elements, we can express the fermionic cut-and-join operators and currents in terms of the elements $E_{i,j}^B = \phi_i \phi_{-j}$ as

$$\mathcal{F}_{r+1}^B = \sum_{k>0} (-1)^k k^{r+1} E_{k,k}^B \quad J_{-\mu}^B = \sum_{l+\mu/2>0} (-1)^l E_{l+\mu,l}^B.$$

More generally, write $J_{-\mu}^\varphi = \sum_{l+\mu/2>0} \varphi_l(\mu) E_{l+\mu,l}^B$ for any function φ of (l, μ) . Then, using the commutation relations (10.2.8), we find

$$\begin{aligned} [\mathcal{F}_{r+1}^B, J_{-\mu}^\varphi] &= \sum_{k>0} \sum_{l+\mu/2>0} (-1)^k k^{r+1} \varphi_l(\mu) [E_{k,k}^B, E_{l+\mu,l}^B] \\ &= \sum_{k>0} \sum_{l+\mu/2>0} k^{r+1} \varphi_l(\mu) (\delta_{k,l+\mu} E_{k,l}^B - \delta_{k,l} E_{l+\mu,k}^B + \delta_{k+l} E_{l+\mu,-k}^B - \delta_{k+l+\mu} E_{-k,l}^B). \end{aligned}$$

The last summand vanishes because of the conditions in the sum, and for the others, the k -sum gives certain restrictions on μ . Collecting the terms together, we find

$$[\mathcal{F}_{r+1}^B, J_{-\mu}^\varphi] = \sum_{l+\mu/2>0} ((l+\mu)^{r+1} - l^{r+1}) \varphi_l(\mu) E_{l+\mu,l}^B = J_{-\mu}^\psi,$$

where $\psi_l(\mu) = ((l+\mu)^{r+1} - l^{r+1}) \varphi_l(\mu)$. Applying this inductively, we get

$$(\text{ad}_{\mathcal{F}_{r+1}^B})^b J_{-\mu}^B = \sum_{l+\mu/2>0} (-1)^l ((l+\mu)^{r+1} - l^{r+1})^b E_{l+\mu,l}^B$$

and, to conclude,

$$\Theta_{-\mu}^{B,r}(u) = \sum_{b=0}^{\infty} \frac{u^{rb}}{b!(r+1)^b} (\text{ad}_{\mathcal{F}_{r+1}^B})^b J_{-\mu}^B = \sum_{l+\mu/2>0} (-1)^l e^{u^r \frac{(l+\mu)^{r+1}-l^{r+1}}{r+1}} E_{l+\mu,l}^B.$$

□

PROPOSITION 12.1.7. *Let $\mu \in \mathbb{Z}_+^{\text{odd}}$. Then:*

$$e^{J_1^B} \Theta_{-\mu}^{B,r}(u) e^{-J_1^B} = \sum_{t=0}^{\infty} \sum_{l \geq \frac{t+1-\mu}{2}} (-1)^l \frac{(\Delta^t f)(l)}{t!} E_{l+\mu-t,l}^B + \frac{1}{2} \frac{(\Delta^{\mu-1} f)(0)}{\mu!}, \quad (12.1.13)$$

where $f(l) = \exp(u^r \frac{(l+\mu)^{r+1}-l^{r+1}}{r+1})$, omitting its dependence on r, μ , and u , and Δ is the backward difference operator, i.e. $\Delta f(l) = f(l) - f(l-1)$.

Proof. We claim that

$$(\text{ad}_{J_1^B})^t \Theta_{-\mu}^{B,r}(u) = \sum_{l \geq \frac{t+1-\mu}{2}} (-1)^l (\Delta^t f)(l) E_{l+\mu-t,l}^B + \frac{\delta_{t,\mu}}{2} (\Delta^{\mu-1} f)(0),$$

which would prove the proposition by summing over t . We prove the claim by induction on t . For $t = 0$, this is Lemma 12.1.6. Suppose now the claim holds for t . Then

$$\begin{aligned} (\mathrm{ad}_{J_1^B})^{t+1} \mathcal{O}_{-\mu}^{B,r}(u) &= \sum_{l \geq \frac{t+1-\mu}{2}} (-1)^l (\Delta^t f)(l) \mathrm{ad}_{J_1^B} E_{l+\mu-t,l}^B \\ &= \sum_{k>0} \sum_{l \geq \frac{t+1-\mu}{2}} (-1)^{l+k} (\Delta^t f)(l) [E_{k-1,k}^B, E_{l+\mu-t,l}^B] \\ &= \sum_{k>0} \sum_{l \geq \frac{t+1-\mu}{2}} (-1)^l (\Delta^t f)(l) \left(\delta_{k,l+\mu-t} E_{k-1,l}^B + \delta_{k-1,l} E_{l+\mu-t,k}^B \right. \\ &\quad \left. + \delta_{k+l} E_{l+\mu-t,1-k}^B + \delta_{k-1+l+\mu-t} E_{-k,l}^B \right). \end{aligned}$$

A simple case analysis shows that the first and last Kronecker deltas give

$$\sum_{l \geq \frac{t+2-\mu}{2}} (-1)^l (\Delta^t f)(l) E_{l+\mu-t-1,l}^B + \delta_{t+1,\mu} (-1)^{\frac{\mu-t-1}{2}} (\Delta^t f)\left(\frac{t+1-\mu}{2}\right) E_{\frac{\mu-t-1}{2}, \frac{t+1-\mu}{2}},$$

while the second and third Kronecker deltas give

$$\sum_{l \geq \frac{t+2-\mu}{2}} (-1)^{l-1} (\Delta^t f)(l-1) E_{l+\mu-t-1,l}^B.$$

Using the fact that $E_{0,0} = \phi_0^2 = \frac{1}{2}$, we get the thesis. \square

ONE-PART SPIN SINGLE HURWITZ NUMBERS

In order to compute spin single Hurwitz numbers with $\ell(\mu) = 1$, also known as *one-part Hurwitz numbers*, we may first realise that in this case there is no difference between connected and disconnected counts: if the ramification profile over a point has length 1, clearly this connects the source. In particular, connected one-part spin single Hurwitz numbers are given by

$$h_{g;\mu}^{r,\vartheta} = 2^{1-g} [u^{rb}] \left\langle e^{J_1^B} e^{u^r \frac{J_{r+1}^B}{r+1}} \frac{J_{-\mu}^B}{\mu} e^{-u^r \frac{J_{r+1}^B}{r+1}} e^{-J_1^B} \right\rangle = 2^{1-g} \frac{[u^{rb}]}{\mu} \left\langle e^{J_1^B} \mathcal{O}_{-\mu}^{B,r}(u) e^{-J_1^B} \right\rangle, \quad (12.1.14)$$

for $\mu \in \mathbb{Z}_+^{\mathrm{odd}}$ and $rb = 2g - 1 + \mu$. First note that, as $\langle \phi_i \phi_j \rangle = (-1)^i \delta_{i+j} u [j]$, the vacuum expectation of each summand from Equation (12.1.13) including ϕ 's vanishes. Hence we get that

$$h_{g;\mu}^{r,\vartheta} = \frac{[u^{rb}]}{2^g \mu^2} \frac{(\Delta^{\mu-1} f)(0)}{(\mu-1)!}. \quad (12.1.15)$$

Let us set $P_{r,\mu}(l) = \frac{(l+\mu)^{r+1-l^{r+1}}}{r+1}$, so that $f(l) = \exp(u^r P_{r,\mu}(l))$. Because u only occurs in this formula in combinations $u^r P_{r,\mu}(l)$, we get

$$h_{g;\mu}^{r,\vartheta} = \frac{1}{2^g \mu^2 b!} \frac{(\Delta^{\mu-1} (P_{r,\mu})^b)(0)}{(\mu-1)!}. \quad (12.1.16)$$

Notice that $P_{r,\mu}$ is a polynomial in l of degree r :

$$P_{r,\mu}(l) = \sum_{a=0}^r \binom{r+1}{a+1} \frac{\mu^{a+1}}{r+1} l^{r-a}. \quad (12.1.17)$$

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Therefore, taking the b -th power, it gives a polynomial $(P_{r,\mu})^b(l) = \sum_{a=0}^{rb} C_{r,\mu}^{a,b} l^{rb-a}$ in l of degree rb with coefficients

$$C_{r,\mu}^{a,b} = \sum_{\lambda \vdash a} \binom{b}{\{\lambda_i^T - \lambda_{i+1}^T\}_{i \geq 1}} \left(\prod_{i=1}^{\ell(\lambda)} \frac{1}{r+1} \binom{r+1}{\lambda_i + 1} \right) \mu^{a+b}, \quad (12.1.18)$$

where the multinomial coefficient is

$$\binom{b}{\{\lambda_i^T - \lambda_{i+1}^T\}_{i \geq 1}} = \frac{b!}{(b - \ell(\lambda))! \prod_{i \geq 1} (\lambda_i^T - \lambda_{i+1}^T)!}. \quad (12.1.19)$$

Here λ^T is the conjugate partition of λ , obtained by transposing the associated Young tableau. Applying the backward difference operators, only few of these terms are going to contribute, as $\Delta^v . l^n = 0$ whenever $v > n$. For us it means that only the terms for $\mu - 1 \leq rb - a$ contribute non-trivially, i.e. $a = 0, \dots, 2g$. On the other hand, the expression $\Delta^v . l^n|_{l=0}$ can be explicitly computed in terms of Stirling numbers of the second kind (cf. [KLPS19, Lemma 4.5]):

$$\frac{\Delta^v . l^n|_{l=0}}{v!} = (-1)^{v+n} \left\{ \begin{matrix} n \\ v \end{matrix} \right\}. \quad (12.1.20)$$

Collecting all the ingredients and taking into account the parity conditions $r+1, \mu \in \mathbb{Z}_+^{\text{odd}}$, we find a closed formula for the one-part Hurwitz numbers.

PROPOSITION 12.1.8. *One-part spin single Hurwitz numbers with $(r+1)$ -completed cycles of genus g and degree μ have the following closed formula:*

$$h_{g;\mu}^{r,\vartheta} = \frac{1}{2^g \mu^2 b!} \sum_{a=0}^{2g} (-1)^a C_{r,\mu}^{a,b} \left\{ \begin{matrix} rb - a \\ \mu - 1 \end{matrix} \right\}, \quad (12.1.21)$$

where by Riemann–Hurwitz $b = \frac{2g+\mu-1}{r}$, $\left\{ \begin{matrix} p \\ q \end{matrix} \right\}$ are Stirling numbers of the second kind, and the coefficients $C_{r,\mu}^{a,b}$ are defined in (12.1.18).

EXAMPLE 12.1.9. For $g = 1$, we have $rb = \mu - 1$ and:

$$h_{1;\mu}^{r,\vartheta} = \frac{1}{2} \frac{1}{\mu^2 b!} \left(\mu^b \left\{ \begin{matrix} \mu + 1 \\ \mu - 1 \end{matrix} \right\} - \frac{br}{2} \mu^{b+1} \left\{ \begin{matrix} \mu \\ \mu - 1 \end{matrix} \right\} + \left(\frac{br(r-1)}{6} + \frac{b(b-1)r^2}{8} \right) \mu^{b+2} \left\{ \begin{matrix} \mu - 1 \\ \mu - 1 \end{matrix} \right\} \right).$$

Using the following expressions for Stirling numbers

$$\left\{ \begin{matrix} \mu + 1 \\ \mu - 1 \end{matrix} \right\} = \frac{\mu(\mu-1)(3\mu^2 + \mu - 2)}{24}, \quad \left\{ \begin{matrix} \mu + 1 \\ \mu - 1 \end{matrix} \right\} = \frac{\mu(\mu-1)}{2}, \quad \left\{ \begin{matrix} \mu - 1 \\ \mu - 1 \end{matrix} \right\} = 1,$$

together with $rb = \mu - 1$, we obtain the closed formula (where we denote $r = 2s$):

$$h_{1;\mu}^{r,\vartheta} = \frac{s^2}{12} \frac{\mu^{b-1}}{(b-1)!} \left(\mu + \frac{1}{s} \right). \quad (12.1.22)$$

In the following we compute the genus one spin Hurwitz numbers via topological recursion, implemented through the software `Mathematica` [Inc]. Employing the symmetric properties

of the spectral curve (12.1.1), the Taylor expansion of the Galois involutions around each ramification point (cf. Section 2.3) read:

$$\begin{aligned} \sigma_0(z) &= a_0 - (z - a_0) - (2s)^{\frac{1}{2s}} \frac{2s-3}{3} (z - a_0)^2 - (2s)^{\frac{2}{2s}} \frac{(2s-3)^2}{9} (z - a_0)^3 \\ &\quad - (2s)^{\frac{3}{2s}} \frac{76s^3 - 360s^2 + 525s - 270}{250} (z - a_0)^4 + O((z - a_0)^5), \\ \sigma_i(z) &= J^i \sigma_0(J^{-i}z). \end{aligned}$$

where $a_i = (2s)^{-\frac{1}{2s}} J^i$, $J = e^{\frac{i\pi}{s}}$. As a consequence, the $(1, 1)$ -correlators turn out to be

$$\omega_{1,1}^{r,\theta}(z) = d \sum_{i=0}^{s-1} \frac{1}{a_i} F\left(r; \frac{z}{a_i}\right), \quad F(r; z) = \frac{z(z^4 - sz^2 + 1 + s)}{24s(1 - z^2)^3}.$$

A simple computation shows that

$$\omega_{1,1}^{r,\theta}(z) = d \left(\left(\frac{sz}{1 - 2sz^{2s}} \frac{d}{dz} + 1 \right) \frac{sz^{2s-1}}{12(1 - 2sz^{2s})} \right) = d \sum_{b \geq 1} \frac{s^2}{12} \frac{\mu^{b-1}}{(b-1)!} \left(\mu + \frac{1}{s} \right) e^{\mu x(z)},$$

where again $\mu = 2sb - 1$. In the last equation, we used the expansion of the Lambert function, Equation (9.2.24), and some algebraic manipulations that we omit. In particular, this confirms Conjecture 12.1.2 for $(g, n) = (1, 1)$.

g	μ	$h_{g;\mu}^{2,\theta}$	g	μ	$h_{g;\mu}^{4,\theta}$	g	μ	$h_{g;\mu}^{6,\theta}$
0	1	1	0	1	1	0	1	1
	3	$\frac{1}{3}$		5	$\frac{1}{5}$		7	$\frac{1}{7}$
	5	$\frac{1}{2}$		9	$\frac{1}{2}$		13	$\frac{1}{2}$
	7	$\frac{7}{6}$		13	$\frac{13}{6}$		19	$\frac{19}{6}$
	9	$\frac{27}{8}$		17	$\frac{289}{24}$		25	$\frac{625}{24}$
1	1	$\frac{1}{6}$	1	3	$\frac{7}{6}$	1	5	4
	3	1		7	$\frac{35}{2}$		11	$\frac{187}{2}$
	5	$\frac{25}{4}$		11	$\frac{2783}{12}$		17	$\frac{3757}{2}$
	7	$\frac{343}{9}$		15	$\frac{11625}{4}$		23	$\frac{425845}{12}$
	9	$\frac{2645}{13}$		19	$\frac{1694173}{48}$		29	$\frac{7780091}{12}$
2	1	$\frac{1}{72}$	2	1	$\frac{1}{20}$	2	3	$\frac{49}{12}$
	3	$\frac{13}{8}$		5	$\frac{451}{8}$		9	$\frac{11109}{4}$
	5	$\frac{5975}{144}$		9	$\frac{84987}{20}$		15	$\frac{2134515}{8}$
	7	$\frac{1409387}{2160}$		13	$\frac{12793131}{80}$		21	14054082
	9	$\frac{2556603}{320}$		17	$\frac{416853311}{96}$		27	$\frac{88146516681}{160}$

Table 12.1: One-part spin single Hurwitz numbers computed via topological recursion. The numbers agree with the neutral fermion computations of Proposition 12.1.8.

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g	(μ_1, μ_2)	$h_{g;\mu}^{2,\vartheta}$	g	(μ_1, μ_2)	$h_{g;\mu}^{4,\vartheta}$	g	(μ_1, μ_2, μ_3)	$h_{g;\mu}^{2,\vartheta}$
0	(1, 1)	1	0	(3, 1)	3	0	(1, 1, 1)	4
	(3, 1)	$\frac{9}{4}$		(7, 1)	$\frac{49}{4}$		(3, 1, 1)	12
	(5, 1)	$\frac{125}{18}$		(5, 3)	$\frac{75}{4}$		(5, 1, 1)	50
	(3, 3)	$\frac{27}{4}$		(9, 3)	$\frac{243}{2}$		(3, 3, 1)	36
	(5, 3)	$\frac{375}{16}$		(7, 5)	$\frac{1225}{12}$		(3, 3, 3)	108
	(5, 5)	$\frac{3125}{36}$		(9, 7)	$\frac{11907}{16}$		(5, 3, 1)	150
1	(1, 1)	$\frac{5}{6}$	1	(1, 1)	$\frac{3}{2}$	0	(5, 3, 3)	450
	(3, 1)	$\frac{17}{2}$		(5, 1)	$\frac{115}{2}$		(5, 5, 1)	625
	(5, 1)	$\frac{925}{12}$		(3, 3)	40		(5, 5, 3)	1875
	(3, 3)	$\frac{99}{2}$		(7, 3)	784		(5, 5, 5)	$\frac{15625}{2}$
	(5, 3)	$\frac{1425}{4}$		(5, 5)	$\frac{4025}{6}$			
	(5, 5)	$\frac{53125}{24}$		(7, 7)	$\frac{30184}{3}$			

Table 12.2: More spin single Hurwitz numbers computed via topological recursion. In genus zero, the numbers agree with the non-spin Hurwitz numbers (see Proposition 12.1.4).

12.2 — EQUIVARIANT TOPOLOGICAL RECURSION

A central point in the original formulation of topological recursion by Eynard and Orantin is that the germ of the bidifferential B near the ramification points has a double pole along the diagonal and no other pole. The spectral curve defined by Equation (12.1.1) does not satisfy this constraint. However, the particular $\mathbb{Z}/2\mathbb{Z}$ symmetry realised by the map $z \mapsto -z$ will allow us to reduce the computation to half of the ramification points, where we can actually apply the original formulation of topological recursion.

More generally, in this section we develop a theory of topological recursion for spectral curves whose associated bidifferential has poles at different ramification points, but satisfying a certain symmetry. The spectral curve (12.1.1) is a specific example of such a curve with symmetries, and we will use the theory developed in this section to find the ELSV-type formula associated with this spectral curve in Section 12.3.

12.2.1 — GLOBAL TO LOCAL SPECTRAL CURVES WITH SYMMETRIES

In the following, we denote by G a finite group, and by $e \in G$ its unit.

DEFINITION 12.2.1. A G -equivariant spectral curve is the data $\mathcal{S} = (C, \phi, x, y, B, \chi, \nu, \beta)$ of

- a Riemann surface C , not necessarily compact nor connected, with a free action $\phi: G \times C \rightarrow C$, which we will often write $\phi(\gamma, z) = \phi_\gamma z = \gamma z$,
- a function $x: C \rightarrow \mathbb{C}$ such that its differential dx is meromorphic and has finitely many zeros a_1, \dots, a_r that are simple (called ramification points),
- a meromorphic function $y: C \rightarrow \mathbb{C}$ that is holomorphic at the ramification points and such that dy is non-zero at the ramification points,

- a symmetric bidifferential B on $C \times C$,
- three one-dimensional representations $\chi, \nu, \beta: G \rightarrow \mathbb{C}^\times$,

such that for any $\gamma \in G$

$$dx(\gamma z) = \chi_\gamma dx(z), \quad y(\gamma z) = \nu_\gamma y(z), \quad B(\gamma z_1, z_2) = \beta_\gamma B(z_1, z_2), \quad (12.2.1)$$

and $B(z_1, z_2) - B^{G,\beta}(z_1, z_2)$ is holomorphic as $z_1 \rightarrow \gamma z_2$, where

$$B^{G,\beta}(z_1, z_2) = \frac{1}{|G|} \sum_{\eta \in G} \beta_\eta^{-1} \frac{d(\eta z_1) dz_2}{(\eta z_1 - z_2)^2}. \quad (12.2.2)$$

The *topological recursion* is defined by the usual Equation (2.3.9).

EXAMPLE 12.2.2. Identifying $\mathbb{Z}/2\mathbb{Z}$ with $\{\pm\}$, then the spectral curve in Conjecture 12.1.2 is a $\mathbb{Z}/2\mathbb{Z}$ -equivariant spectral curve with $\phi_\pm z = \pm z$, χ the trivial representation, and $\nu = \beta$ the sign representation.

LEMMA 12.2.3. For any G -equivariant spectral curve as in Definition 12.2.1, we have $\beta^2 = 1$, i.e. $\beta: G \rightarrow \{\pm 1\}$.

Proof. It is sufficient to consider the polar part $B^{G,\beta}$ as z_1 approaches any element of the G -orbit of z_2 . Let $\gamma \in G$; then

$$\begin{aligned} \beta_\gamma B^{G,\beta}(z_1, z_2) &= B^{G,\beta}(\phi_\gamma z_1, z_2) = \frac{1}{|G|} \sum_{\eta \in G} \beta_\eta^{-1} \frac{d(\phi_\eta \phi_\gamma z_1) dz_2}{(\phi_\eta \phi_\gamma z_1 - z_2)^2} \\ &= \frac{1}{|G|} \sum_{\eta \in G} \beta_\eta^{-1} \frac{d(\phi_\gamma \phi_{\gamma^{-1} \eta \gamma} z_1) dz_2}{(\phi_\gamma \phi_{\gamma^{-1} \eta \gamma} z_1 - z_2)^2} \\ &\sim \frac{1}{|G|} \sum_{\eta' \in G} \beta_{\gamma \eta' \gamma^{-1}}^{-1} \frac{d(\phi_{\eta'} z_1) d(\phi_\gamma^{-1} z_2)}{(\phi_{\eta'} z_1 - \phi_\gamma^{-1} z_2)^2} \\ &= B^{G,\beta}(z_1, \phi_\gamma^{-1} z_2) = \beta_\gamma^{-1} B^{G,\beta}(z_1, z_2), \end{aligned}$$

using that β is one-dimensional, and where \sim means “equality up to holomorphic terms”. It follows that $\beta_\gamma^2 = 1$ for all $\gamma \in G$, so $\beta^2 = 1$. \square

Let a be a fixed ramification point. Then $dx(\gamma a) = \chi_\gamma dx(a) = 0$, so γa is a ramification point as well. Choose a local coordinate ζ_e such that $\zeta_e(a) = 0$ and $x(z) = \zeta_e(z)^2 + x(a)$ around $z = a$ (we call such a coordinate *adapted to x at a*). Also choose a square root $\sqrt{\chi}$, i.e. a function $\sqrt{\chi}: G \rightarrow \mathbb{C}^\times$ such that $(\sqrt{\chi})_\gamma^2 = \chi_\gamma$. We also require $(\sqrt{\chi})_e = 1$, but $\sqrt{\chi}$ is not necessarily an homeomorphism. Then near γa we have a local coordinate defined by $\zeta_\gamma(z) = (\sqrt{\chi})_\gamma \zeta_e(\gamma^{-1} z)$. Indeed, we get $\zeta_\gamma(\gamma a) = 0$ and, as $x(z) = \chi_\gamma x(\gamma^{-1} z) + C(z)$ for some locally constant C , around $z = \gamma a$ we find

$$x(z) = \chi_\gamma x(\gamma^{-1} z) + C(z) = \chi_\gamma \zeta_e(\gamma^{-1} z)^2 + x(a) + C(z) = \zeta_\gamma(z)^2 + x(\gamma a). \quad (12.2.3)$$

Therefore, ζ_γ is a local coordinate adapted to x at γa . Note that the ambiguity in the choice of $\sqrt{\chi}$ corresponds to the fact that $-\zeta_\gamma$ is also a local coordinate adapted to x at γa .

We define the *times* as the Taylor coefficients of y in these coordinates as $z \rightarrow \gamma a$:

$$y = \sum_{k \geq 0} t_{k,\gamma} \zeta_\gamma^k, \quad (12.2.4)$$

Then, around $z = \gamma a$,

$$\sum_{k \geq 0} t_{k,\gamma} \zeta_\gamma^k(z) = y(z) = v_\gamma y(\gamma^{-1}z) = v_\gamma \sum_{k \geq 0} t_{k,e} \zeta_e^k(\gamma^{-1}z) = \sum_{k \geq 0} (v_\gamma (\sqrt{\chi})_\gamma^{-k} h_k^e) \zeta_\gamma^k(z), \quad (12.2.5)$$

so $t_{k,\gamma} = v_\gamma (\sqrt{\chi})_\gamma^{-k} t_{k,e}$. Similarly, define *jumps* as the Taylor coefficients of B in these coordinates as $z_1 \rightarrow \gamma_1 a$, $z_2 \rightarrow \gamma_2 a$:

$$B = \frac{\beta_{\gamma_1 \gamma_2}}{|G|} \frac{d\zeta_{\gamma_1} d\zeta_{\gamma_2}}{(\zeta_{\gamma_1} - \zeta_{\gamma_2})^2} + \sum_{k_1, k_2 \geq 0} u_{(k_1, \gamma_1), (k_2, \gamma_2)} \zeta_{\gamma_1}^{k_1} \zeta_{\gamma_2}^{k_2} d\zeta_{\gamma_1} d\zeta_{\gamma_2}. \quad (12.2.6)$$

This expansion is justified by the assumption $B = B^{G,\beta}$ up to holomorphic terms. Moreover, we have $u_{(k_1, \gamma_1), (k_2, \gamma_2)} = u_{(k_2, \gamma_2), (k_1, \gamma_1)}$ by symmetry of B , and by expanding around $z_i = \gamma_i a$, we find $u_{(k_1, \gamma_1), (k_2, \gamma_2)} = \beta_{\gamma_1 \gamma_2} (\sqrt{\chi})_{\gamma_1}^{-k_1-1} (\sqrt{\chi})_{\gamma_2}^{-k_2-1} u_{(k_1, e), (k_2, e)}$.

This analysis is local around G -orbits of ramification points. If we have $s|G|$ ramification points $\{a_i\}_{i \in I}$ such that the index set I has a free G -action $G \times I \rightarrow I: (\gamma, i) \mapsto \gamma i$ and $\gamma a_i = a_{\gamma i}$, then in the same way we can choose a system of representatives $\{a_i\}_{i \in \bar{I}}$ of the G -action and local coordinates ζ_i adapted to x around a_i such that $\zeta_{\gamma i}(z) = (\sqrt{\chi})_\gamma \zeta_i(\phi_\gamma^{-1}z)$ for $i \in \bar{I}$. And then with the expansions

$$\begin{aligned} y &= \sum_{k \geq 0} t_{k,i} \zeta_i^k, \\ B &= \frac{\sum_{\gamma \in G} \beta_\gamma \delta_{i_1, \gamma i_2}}{|G|} \frac{d\zeta_{i_1} d\zeta_{i_2}}{(\zeta_{i_1} - \zeta_{i_2})^2} + \sum_{k_1, k_2 \geq 0} u_{(k_1, i), (k_2, i_2)} \zeta_{i_1}^{k_1} \zeta_{i_2}^{k_2} d\zeta_{i_1} d\zeta_{i_2}, \end{aligned} \quad (12.2.7)$$

we get the relations

$$t_{k,\gamma i} = v_\gamma (\sqrt{\chi})_\gamma^{-k} t_{k,i}, \quad \begin{cases} u_{(k_1, i_1), (k_2, i_2)} = u_{(k_2, i_2), (k_1, i_1)} \\ u_{(k_2, \gamma_2 i_2), (k_1, \gamma_1 i_1)} = \beta_{\gamma_1 \gamma_2} (\sqrt{\chi})_{\gamma_1}^{-k_1-1} (\sqrt{\chi})_{\gamma_2}^{-k_2-1} u_{(k_1, i_1), (k_2, i_2)} \end{cases} \quad (12.2.8)$$

for $i, i_1, i_2 \in \bar{I}$. This is the structure for the local G -equivariant spectral curve, as defined in the next subsection.

12.2.2 — THE QUOTIENT OF A LOCAL EQUIVARIANT CURVE

DEFINITION 12.2.4. For a fixed $s \geq 1$, a *local G -equivariant spectral curve* \mathcal{S}_G is given by the following data: we fix an index set I of size $s|G|$, with free G -action and system of representatives \bar{I} , and as above three one-dimensional representations χ, v, β of G such that $\beta^2 = 1$, along with a square root $\sqrt{\chi}$. The curve is $\bigsqcup_{i \in \bar{I}} \text{Spec } \mathbb{C}[[\zeta_i]]$, with $\zeta_{\gamma i} = (\sqrt{\chi})_\gamma \zeta_i$ for $i \in \bar{I}$, and

$$x(\zeta_i) = \zeta_i^2 + \alpha_i, \quad y(\zeta_i) = \sum_{k \geq 0} t_{k,i} \zeta_i^k, \quad (12.2.9)$$

and

$$B(\zeta_{i_1}, \zeta_{i_2}) = \frac{\sum_{\gamma \in G} \beta_\gamma \delta_{i_1, \gamma i_2}}{|G|} \frac{d\zeta_{i_1} d\zeta_{i_2}}{(\zeta_{i_1} - \zeta_{i_2})^2} + \sum_{k_1, k_2 \geq 0} u_{(k_1, i), (k_2, i_2)} \zeta_{i_1}^{k_1} \zeta_{i_2}^{k_2} d\zeta_{i_1} d\zeta_{i_2} \quad (12.2.10)$$

with coefficients satisfying (12.2.8).

To lighten the notation, we may write ζ for any ζ_i if the index is not important. For multi-differentials, we may also use an index in $\{0, \dots, n\}$ to denote which argument we mean. We define the topological recursion correlators as usual:

$$\begin{aligned} \omega_{g,n+1}(\zeta_0, \zeta_{\llbracket n \rrbracket}) &= \sum_{i \in I} \text{Res} K(\zeta_0, \zeta_i) \left(\omega_{g-1, n+2}(\zeta_i, -\zeta_i, \zeta_{\llbracket n \rrbracket}) \right. \\ &\quad \left. + \sum_{\substack{\text{no } (0,1) \\ g_1+g_2=g \\ J_1 \sqcup J_2 = \llbracket n \rrbracket}} \omega_{g_1, 1+|J_1|}(\zeta_i, \zeta_{J_1}) \omega_{g_2, 1+|J_2|}(-\zeta_i, \zeta_{J_2}) \right), \end{aligned} \quad (12.2.11)$$

where

$$K(\zeta_0, \zeta_i) = \frac{1}{2} \frac{\int_{-\zeta_i}^{\zeta_i} B(\zeta_0, \cdot)}{(y(\zeta_i) - y(-\zeta_i)) dx(\zeta_i)}. \quad (12.2.12)$$

LEMMA 12.2.5. *All of the $\omega_{g,n}$ constructed via (12.2.11) (i.e those with $2g - 2 + n > 0$) are G -equivariant in any of the individual variables: for any $1 \leq m \leq n$,*

$$\omega_{g,n}(\zeta_{\llbracket m-1 \rrbracket}, \zeta_{\gamma i, m}, \zeta_{\llbracket m+1, n \rrbracket}) = \beta_\gamma \omega_{g,n}(\zeta_{\llbracket m-1 \rrbracket}, \zeta_{i, m}, \zeta_{\llbracket m+1, n \rrbracket}). \quad (12.2.13)$$

Proof. This is a standard induction argument. Noting that $\omega_{0,1}$ does not occur in Equation (12.2.11), and the base case $\omega_{0,2} = B$ follows immediately from Equations (12.2.8) to (12.2.10).

For the induction step, there are two cases: if $m > 1$, it follows immediately from the induction hypothesis and the shape of (12.2.11). If $m = 1$, it follows from (12.2.11) together with

$$K(\zeta_{\gamma i, 0}, \zeta_j) = \frac{\int_{-\zeta_j}^{\zeta_j} B(\zeta_{\gamma i, 0}, \cdot)}{2(\omega_{0,1}(\zeta_j) - \omega_{0,1}(-\zeta_j))} = \frac{\int_{-\zeta_j}^{\zeta_j} \beta_\gamma B(\zeta_{i, 0}, \cdot)}{2(\omega_{0,1}(\zeta_j) - \omega_{0,1}(-\zeta_j))} = \beta_\gamma K(\zeta_{i, 0}, \zeta_j).$$

□

DEFINITION 12.2.6. Let \mathcal{S}_G be a G -equivariant spectral curve. The G -quotient spectral curve or reduced local spectral curve $\mathcal{S}_G/G = \mathcal{S}_{\text{red}}$ associated to \mathcal{S}_G is the curve $\bigsqcup_{i \in \bar{I}} \text{Spec } \mathbb{C} \llbracket \zeta_i \rrbracket$, with the same x, y , and B restricted to this curve. Then define the reduced correlators $\omega_{g,n}^{\text{red}}$ via the usual (non-equivariant) local topological recursion on \mathcal{S}^{red} . The reduced correlators can be extended to the full index range I by G -equivariance as in Lemma 12.2.5; we will denote these extended reduced correlators by the same symbols.

The main point of considering \mathcal{S}^{red} is that the bidifferential B , once restricted to the reduced curve, has double poles along the diagonal only. The next proposition states that correlators computed on the equivariant spectral curve \mathcal{S}_G coincide with the reduced correlators, up to a global power of $|G|$.

PROPOSITION 12.2.7. *For a local G -spectral curve \mathcal{S}_G , the correlators $\omega_{g,n}$ defined via Equation (12.2.11) are zero, unless $\chi \cdot v = \beta$, in which case they are equal to the extended reduced correlators $\omega_{g,n}^{\text{red}}$ defined via \mathcal{S}_{red} , up to powers of $|G|$:*

$$\omega_{g,n}(\zeta_{\llbracket n \rrbracket}) = |G|^{2g-2+n} \omega_{g,n}^{\text{red}}(\zeta_{\llbracket n \rrbracket}). \quad (12.2.14)$$

Proof. This is again proved by induction. The main step is the reduction of the sum over $s|G|$ ramification points in Equation (12.2.11) to a sum over s ramification points. For this, we need

$$K(\zeta_0, \zeta_{\gamma i}) = \beta_{\gamma} \chi_{\gamma}^{-1} \nu_{\gamma}^{-1} K(\zeta_0, \zeta_i).$$

By Lemma 12.2.5, we may as well prove the equality on \mathcal{S}_{red} , i.e. with indices restricted to \bar{I} . Then, from the definitions, we see

$$\omega_{0,2}(\zeta_1, \zeta_2) = \omega_{0,2}^{\text{red}}(\zeta_1, \zeta_2), \quad K(\zeta_0, \zeta) = K^{\text{red}}(\zeta_0, \zeta).$$

For the induction step, we calculate

$$\begin{aligned} \omega_{g,n+1}(\zeta_0, \zeta_{\llbracket n \rrbracket}) &= \sum_{i \in \bar{I}} \text{Res}_{\zeta_i=0} K(\zeta_0, \zeta_i) \left(\omega_{g-1,n+2}(\zeta_i, -\zeta_i, \zeta_{\llbracket n \rrbracket}) \right. \\ &\quad \left. + \sum_{\substack{g_1+g_2=g \\ J_1 \sqcup J_2 = \llbracket n \rrbracket}} \omega_{g_1,1+|J_1|}(\zeta_i, \zeta_{J_1}) \omega_{g_2,1+|J_2|}(-\zeta_i, \zeta_{J_2}) \right) \\ &= \sum_{\substack{\gamma \in G \\ i \in \bar{I}}} \text{Res}_{\zeta_i=0} K(\zeta_0, \zeta_i) \beta_{\gamma} \chi_{\gamma}^{-1} \nu_{\gamma}^{-1} \left(\beta_{\gamma}^2 \omega_{g-1,n+2}(\zeta_i, -\zeta_i, \zeta_{\llbracket n \rrbracket}) \right. \\ &\quad \left. + \sum_{\substack{g_1+g_2=g \\ J_1 \sqcup J_2 = \llbracket n \rrbracket}} \beta_{\gamma} \omega_{g_1,1+|J_1|}(\zeta_i, \zeta_{J_1}) \beta_{\gamma} \omega_{g_2,1+|J_2|}(-\zeta_i, \zeta_{J_2}) \right) \\ &= \sum_{i \in \bar{I}} |G| \delta_{\chi \nu, \beta} \text{Res}_{\zeta_i=0} K(\zeta_0, \zeta_i) \left(\omega_{g-1,n+2}(\zeta_i, -\zeta_i, \zeta_{\llbracket n \rrbracket}) \right. \\ &\quad \left. + \sum_{\substack{g_1+g_2=g \\ J_1 \sqcup J_2 = \llbracket n \rrbracket}} \omega_{g_1,1+|J_1|}(\zeta_i, \zeta_{J_1}) \omega_{g_2,1+|J_2|}(-\zeta_i, \zeta_{J_2}) \right) \\ &= |G|^{2g-2+(n+1)} \delta_{\chi \nu, \beta} \sum_{i \in \bar{I}} \text{Res}_{\zeta_i=0} K^{\text{red}}(\zeta_0, \zeta_i) \left(\omega_{g-1,n+2}^{\text{red}}(\zeta_i, -\zeta_i, \zeta_{\llbracket n \rrbracket}) \right. \\ &\quad \left. + \sum_{\substack{g_1+g_2=g \\ J_1 \sqcup J_2 = \llbracket n \rrbracket}} \omega_{g_1,1+|J_1|}^{\text{red}}(\zeta_i, \zeta_{J_1}) \omega_{g_2,1+|J_2|}^{\text{red}}(-\zeta_i, \zeta_{J_2}) \right) \\ &= \delta_{\chi \nu, \beta} |G|^{2g-2+(n+1)} \omega_{g,n+1}^{\text{red}}(\zeta_0, \zeta_{\llbracket n \rrbracket}). \end{aligned}$$

The first equality is the definition, the second applies equivariance, the third gathers characters. The fourth equality applies the induction hypothesis and gathers factors of $|G|$, and the last is the definition of the reduced correlators. \square

COROLLARY 12.2.8. *The correlators calculated on \mathcal{S}_G are invariant under permutation of the variables.*

Proof. This symmetry is well-known for usual Eynard–Orantin topological recursion [EO07a] (see also [ABCO17] for an algebraic proof). Therefore, it follows immediately from Proposition 12.2.7. \square

Because of Proposition 12.2.7, we propose the following definition:

DEFINITION 12.2.9. Let G be a finite group. A (local) G -spectral curve is a (local) G -equivariant spectral curve as in Definition 12.2.1, respectively Definition 12.2.4, such that $\chi\nu = \beta$.

Thus, G -spectral curves are those G -equivariant curves for which the stable correlators are not necessarily trivial due to Proposition 12.2.7.

12.3 — A COHFT FOR SPIN SINGLE HURWITZ NUMBERS

We can now apply the Eynard–DOSS correspondence to express spin single Hurwitz numbers with completed cycles as a certain CohFT on the moduli space of curves, intersected with powers of ψ -classes. Throughout this section, we consider $r = 2s$ to be an even, positive integer.

Recall the conjectural spectral curve on \mathbb{P}^1 for spin single Hurwitz numbers with $(r + 1)$ -completed cycles:

$$x(z) = \log(z) - z^{2s}, \quad y(z) = z, \quad B(z_1, z_2) = \frac{1}{2} \left(\frac{1}{(z_1 - z_2)^2} + \frac{1}{(z_1 + z_2)^2} \right) dz_1 dz_2. \quad (12.3.1)$$

It is $\mathbb{Z}/2\mathbb{Z}$ -spectral curve, with χ the trivial representation, and $\nu = \beta$ the sign representation. The ramification points of the full spectral curve are given by

$$a_i = \frac{J^i}{(2s)^{\frac{1}{2s}}}, \quad J = e^{\frac{i\pi}{s}}, \quad i = 0, 1, \dots, 2s - 1, \quad (12.3.2)$$

with $\mathbb{Z}/2\mathbb{Z}$ -action given by $(-1).i \equiv i + s \pmod{s}$. Here we identify $\mathbb{Z}/2\mathbb{Z}$ with $\{\pm 1\}$. Thus, we can choose the system of representative ramification points to be $\bar{I} = \{0, \dots, s - 1\}$. As a consequence of Proposition 12.2.7, together with a rescaling of B (cf. Theorem 2.3.6), we get the following result.

LEMMA 12.3.1. Consider the spectral curve on \mathbb{P}^1 given by

$$x(z) = \log(z) - z^{2s}, \quad y(z) = z, \quad \hat{B}(z_1, z_2) = \left(\frac{1}{(z_1 - z_2)^2} + \frac{1}{(z_1 + z_2)^2} \right) dz_1 dz_2. \quad (12.3.3)$$

Denote by $\hat{\omega}_{g,n}^{r,\vartheta}$ the multidifferentials obtained by summing over the ramification points indexed by $\{0, \dots, s - 1\}$. Then

$$\omega_{g,n}^{r,\vartheta}(z_1, \dots, z_n) = 2^{1-g-n} \hat{\omega}_{g,n}^{r,\vartheta}(z_1, \dots, z_n). \quad (12.3.4)$$

Proof. Reducing the set of ramification points to \bar{I} gives a factor of 2^{2g-2+n} , while the rescaling $B \mapsto \hat{B} = \frac{1}{2}B$ gives a factor of $2^{-3g+3-2n}$. \square

The main consequence of the above lemma is that we can safely apply the Eynard–DOSS correspondence, Theorem 2.3.12. Indeed, although we do not sum over all zeros of dx , one can easily check that this does not spoil the argument of [Eyn11; DOSS14]: the computation to express the multidifferentials as a sum over stable graphs is local, and the expression of the edge weights in terms of the R -matrix of Equation (2.3.25) only requires a compact curve and dx meromorphic.

IV. Spin Hurwitz theory

Let us choose $c[i] = -\frac{i}{\sqrt{4s}}$ and $c = (2s)^{\frac{1}{2s}+1}$, so that we have local expansions

$$x = -\frac{1}{2} \frac{\zeta_i^2}{2s} + x(a_i), \quad y = a_i + \frac{a_i}{2s} \zeta_i + O(\zeta_i^2). \quad (12.3.5)$$

With this choice, $\Delta^i = \frac{a_i}{2s}$ and $t^i = \frac{J^i}{2s}$. Moreover, the underlying TFT on $V = \mathbb{C} \langle e_0, \dots, e_{s-1} \rangle$ is given by

$$\eta(e_i, e_j) = \delta_{i,j}, \quad \mathbb{1} = \sum_{i=0}^{s-1} \frac{J^i}{2s} e_i, \quad \varpi_{g,n}^{r,\vartheta}(e_{i_1} \otimes \dots \otimes e_{i_n}) = \delta_{i_1, \dots, i_n} \left(\frac{J^i}{2s} \right)^{-2g+2-n}. \quad (12.3.6)$$

Let us compute now the other ingredients for the Eynard–DOSS formula (cf. [LPSZ17] for similar computations).

LEMMA 12.3.2. *The auxiliary functions, R-matrix and translation associated to the spectral curve (12.3.1) are given by*

$$\xi^i(z) = 2\Delta^i \frac{z}{a_i^2 - z^2}, \quad (12.3.7)$$

$$R^{-1}(u)_i^j = \frac{1}{s} \sum_{k=0}^{s-1} J^{(2k+1)(j-i)} \exp\left(-\sum_{m=1}^{\infty} \frac{B_{m+1}(\frac{2k+1}{2s})}{m(m+1)} (-u)^m\right), \quad (12.3.8)$$

$$\hat{T}^i(u) = \sum_{m=1}^{\infty} \frac{B_{m+1}(\frac{1}{2s})}{m(m+1)} (-u)^m. \quad (12.3.9)$$

for $i, j = 0, \dots, s-1$. Moreover y and B are compatible, in the sense that Equation (2.3.31) is satisfied.

Proof. For the auxiliary functions, we simply have

$$\xi^i(z) = \int^z \frac{\hat{B}(\zeta_i, z)}{d\zeta_i} \Big|_{\zeta_i=0} = \Delta^i \int^z \left(\frac{1}{(a_i - z)^2} + \frac{1}{(a_i + z)^2} \right) dz = 2\Delta^i \frac{z}{a_i^2 - z^2}.$$

For the R -matrix, inserting the expression for ξ^i in the definition of the R -matrix and integrating by parts, we get

$$\begin{aligned} R^{-1}(u)_i^j &= -\sqrt{\frac{u}{2\pi}} \int_{\mathbb{R}} d\xi^i(\zeta_j) e^{-\frac{1}{2u} \zeta_j^2} \\ &= -\frac{2\Delta^i}{\sqrt{2\pi u}} \int_{\mathbb{R}} \frac{z(\zeta_j)}{a_i^2 - z^2(\zeta_j)} e^{-\frac{1}{2u} \zeta_j^2} \zeta_j d\zeta_j. \end{aligned}$$

We perform now the change of variable $\zeta_j \mapsto w$ determined by $z = \frac{J^j}{(2s)^{\frac{1}{2s}}} w^{\frac{1}{2s}}$. We find

$$-\frac{1}{2} \frac{\zeta_j^2}{2s} = x - x(a_j) = \frac{1}{2s} (1 - w + \log(w)),$$

and differentiating both sides we get $-\zeta d\zeta = (\frac{1}{w} - 1)dw$. Note also that w runs along the Hankel contour C_H when ζ_j runs from $-\infty$ to $+\infty$. As a consequence,

$$\begin{aligned} R^{-1}(u)_i^j &= \frac{2\Delta^i}{\sqrt{2\pi u}} \int_{C_H} \frac{J^j (2s)^{-\frac{1}{2s}} w^{\frac{1}{2s}}}{J^{2i} (2s)^{-\frac{1}{s}} - J^{2j} (2s)^{-\frac{1}{s}} w^{\frac{1}{s}}} e^{\frac{1}{u}(1-w+\log(w))} \left(\frac{1}{w} - 1 \right) dw \\ &= \frac{J^{j-i}}{s\sqrt{2\pi u}} \int_{C_H} w^{\frac{1}{2s}-1} \frac{1-w}{1 - J^{2(j-i)} w^{\frac{1}{s}}} e^{\frac{1}{u}(1-w+\log(w))} dw. \end{aligned}$$

On the other hand, we have the geometric progression formula

$$\frac{1-w}{1-J^{2(j-i)}w^{\frac{1}{s}}} = \sum_{k=0}^{s-1} J^{2k(j-i)}w^{\frac{k}{s}}$$

and the asymptotic expansion for the reciprocal of the Gamma function, Equation (9.2.15), so that

$$\begin{aligned} R^{-1}(u)_i^j &= \frac{e^{\frac{1}{u}}}{s\sqrt{2\pi u}} \sum_{k=0}^{s-1} J^{(2k+1)(j-i)} \int_{C_H} w^{\frac{2k+1}{2s}-1+\frac{1}{u}} e^{-\frac{w}{u}} dw \\ &= \frac{e^{\frac{1}{u}}}{s} \sqrt{\frac{u}{2\pi}} \sum_{k=0}^{s-1} J^{(2k+1)(j-i)} \int_{C_H} (uw)^{\frac{2k+1}{2s}-1+\frac{1}{u}} e^{-w} dw \quad \text{rescaling } w \mapsto uw \\ &\sim \frac{1}{s} \sum_{k=0}^{s-1} J^{(2k+1)(j-i)} \exp\left(\sum_{m=1}^{\infty} \frac{B_{m+1}(1-\frac{2k+1}{2s})}{m(m+1)} u^m\right). \end{aligned}$$

We conclude using the property $B_{m+1}(1-a) = (-1)^{m+1}B_{m+1}(a)$. The translation is given by (2.3.30) and therefore does not depend on the $(0, 2)$ -correlator, but only on x and y . Therefore, we can use the result of [LPSZ17, Lemma 4.1] which, with the special case and change of notation $s = 1$ and $r = 2s$, computes the relevant integral for the spectral curve for non-spin Hurwitz number with completed cycles, with the same x and y . The result is

$$\frac{\Delta^i}{\sqrt{2\pi u}} \int_{\mathbb{R}} dy e^{-\frac{\xi_i^2}{2u}} \sim \exp\left(-\sum_{m=1}^{\infty} \frac{B_{m+1}(\frac{1}{2s})}{m(m+1)} (-u)^m\right)$$

and dividing by Δ^i gives the result. For the compatibility, we need to check that

$$\frac{1}{\sqrt{2\pi u}} \int_{\mathbb{R}} dy e^{-\frac{\xi_i^2}{2u}} \sim \sum_{i=0}^{s-1} R^{-1}(u)_i^j \Delta^i.$$

The left-hand side is given above, while the right-hand side follows from the formula for the R -matrix elements:

$$\begin{aligned} \sum_{i=0}^{s-1} R^{-1}(u)_i^j \Delta^i &\sim \frac{1}{s} \sum_{i=0}^{s-1} \sum_{k=0}^{s-1} J^{(2k+1)(j-i)} \exp\left(-\sum_{m=1}^{\infty} \frac{B_{m+1}(\frac{2k+1}{2s})}{m(m+1)} (-u)^m\right) \frac{J^i}{(2s)^{\frac{1}{2s}+1}} \\ &= \frac{1}{(2s)^{\frac{1}{2s}+1}} \sum_{k=0}^{s-1} \left(\frac{1}{s} \sum_{i=0}^{s-1} J^{-2ki}\right) J^{(2k+1)j} \exp\left(-\sum_{m=1}^{\infty} \frac{B_{m+1}(\frac{2k+1}{2s})}{m(m+1)} (-u)^m\right) \\ &= \frac{J^j}{(2s)^{\frac{1}{2s}+1}} \exp\left(-\sum_{m=1}^{\infty} \frac{B_{m+1}(\frac{1}{2s})}{m(m+1)} (-u)^m\right). \end{aligned}$$

So the two sides are indeed equal. \square

Consider now the change of basis on the vector space underlying the CohFT, from (e_0, \dots, e_{s-1}) to (v_0, \dots, v_{s-1}) :

$$v_a = \sum_{i=0}^{s-1} \frac{J^{(2a+1)i}}{2s} e_i, \quad e_i = 2 \sum_{a=0}^{s-1} J^{-(2a+1)i} v_a. \quad (12.3.10)$$

In the following lemma, the indices of the Kronecker deltas are taken modulo s .

LEMMA 12.3.3. *In the basis (v_0, \dots, v_{s-1}) , the following holds.*

- *The underlying topological field theory is given by*

$$\eta(v_a, v_b) = \frac{1}{4s} \delta_{a+b+1} \quad \varpi_{g,n}^{r,\theta}(v_{a_1} \otimes \dots \otimes v_{a_n}) = \frac{(2s)^{2g-1}}{2} \delta_{a_1+\dots+a_n-g+1}. \quad (12.3.11)$$

Moreover, the unit is given by v_0 .

- *The R-matrix is given by*

$$R^{-1}(u) = \exp\left(-\sum_{m=1}^{\infty} \frac{\text{diag}_{a=0}^{s-1}(B_{m+1}(\frac{2a+1}{2s}))}{m(m+1)} (-u)^m\right). \quad (12.3.12)$$

- *The auxiliary functions $\theta^a = 2 \sum_{i=0}^{s-1} J^{-(2a+1)i} \xi^i$ are given by*

$$\theta^a(z) = 2(2s)^{\frac{2s-2a-1}{2s}} \sum_{m=0}^{\infty} \frac{(2sm+2s-2a-1)^m}{m!} e^{(2sm+2s-2a-1)x(z)}. \quad (12.3.13)$$

Proof. The pairing is given by a simple computation:

$$\eta(v_a, v_b) = \frac{1}{(2s)^2} \sum_{i,j=0}^{s-1} J^{(2a+1)i+(2b+1)j} \eta(e_i, e_j) = \frac{1}{(2s)^2} \sum_{i=0}^{s-1} J^{2(a+b+1)i} = \frac{1}{4s} \delta_{a+b+1}.$$

Similarly for the TFT and the R-matrix elements. To conclude, let us compute the expansion of the auxiliary functions after the change of basis:

$$\theta^a(z) = 2 \sum_{i=0}^{s-1} J^{-(2a+1)i} \xi^i(z) = \frac{4z}{(2s)^{\frac{1}{2s}+1}} \sum_{i=0}^{s-1} J^{-2ai} \frac{1}{a_i^2 - z^2} = \frac{2((2s)^{\frac{1}{2s}} z)^{2s-2a-1}}{1 - 2sz^{2s}}.$$

On the other hand, the spectral curve equation express $y = z$ in terms of x through the Lambert W-function:

$$z = \left(\frac{W(-2se^{2sx})}{-2s} \right)^{\frac{1}{2s}}.$$

In particular, from the relation $\frac{dz^\alpha}{dx} = \frac{\alpha z^\alpha}{1-2sz^{2s}}$ with $\alpha = 2s - 2a - 1$, we find that

$$\theta^a = \frac{2(2s)^{\frac{2s-2a-1}{2s}}}{2s-2a-1} \frac{dz^{2s-2a-1}}{dx} = \frac{2}{(2s-2a-1)} \frac{d}{dx} (-W(-2se^{2sx}))^{\frac{2s-2a-1}{2s}}.$$

We can now use the expansion given in Equation (9.2.24) to finally get

$$\begin{aligned} \theta^a &= \frac{2}{s} \frac{d}{dx} \sum_{m=0}^{\infty} \frac{(m + \frac{2s-2a-1}{2s})^{m-1}}{m!} (2se^{2sx})^{m+\frac{2s-2a-1}{2s}} \\ &= 2(2s)^{\frac{2s-2a-1}{2s}} \sum_{m=0}^{\infty} \frac{(2sm+2s-2a-1)^m}{m!} e^{(2sm+2s-2a-1)x}. \end{aligned}$$

□

We can now express the cohomological field theory $\Omega_{g,n}^{r,\theta} = R.\varpi_{g,n}^{r,\theta}$ as a sum over stable graphs. Because of the simple expression of the underlying topological field theory and R -matrix, the result can be expressed in a concise way through weighted stable graphs. The following definition is an adaptation of [JPPZ17, Subsection 1.1] to our setting.

DEFINITION 12.3.4. Let $\Gamma \in \mathcal{G}_{g,n}$ be a stable graph of type (g, n) , and consider some weights $a = (a_1, \dots, a_n)$, $0 \leq a_i \leq s-1$, satisfying the modular constraint $\sum_{i=1}^n a_i \equiv g-1 \pmod{s}$. A *spin weighting modulo s* of Γ is a map $w: H_\Gamma \rightarrow \{0, \dots, s-1\}$ satisfying the following modular constraints.

- **VERTEX CONDITIONS.** For every vertex $v \in V_\Gamma$,

$$\sum_{h \in H_\Gamma(v)} w(h) \equiv g(v) - 1 \pmod{s}. \quad (12.3.14)$$

- **EDGE CONDITIONS.** For every edge $e = (h, h') \in E_\Gamma$,

$$w(h) + w(h') \equiv -1 \pmod{s}. \quad (12.3.15)$$

- **LEAF CONDITIONS.** For every leaf $\lambda_i \in \Lambda_\Gamma$ corresponding to the marking $i \in \{1, \dots, n\}$,

$$w(\lambda_i) \equiv a_i \pmod{s}. \quad (12.3.16)$$

Denote by $W_\Gamma^{s,\theta}(a)$ the set of spin weighting modulo s of Γ .

PROPOSITION 12.3.5. *The CohFT $\Omega_{g,n}^{r,\theta} = R.\varpi_{g,n}^{r,\theta}$ evaluated at $v_{a_1} \otimes \dots \otimes v_{a_n}$ is given by the following sum over stable graphs*

$$\begin{aligned} & 2^{2g-2} \sum_{\Gamma \in \mathcal{G}_{g,n}} \sum_{w \in W_\Gamma^{s,\theta}(a)} \frac{s^{2g-1-h^1(\Gamma)}}{|\text{Aut}(\Gamma)|} \xi_{\Gamma,*} \prod_{v \in V_\Gamma} \exp\left(\sum_{m \geq 1} \frac{(-1)^m B_{m+1}(\frac{1}{2s})}{m(m+1)} \kappa_m(v)\right) \\ & \times \prod_{\substack{e \in E_\Gamma \\ e=(h,h')}} \frac{1 - \exp\left(-\sum_{m \geq 1} \frac{(-1)^m B_{m+1}(\frac{2w(h)+1}{2s})}{m(m+1)} ((\psi_h)^m - (-\psi_{h'})^m)\right)}{\psi_h + \psi_{h'}} \\ & \times \prod_{\lambda_i \in \Lambda_\Gamma} \exp\left(-\sum_{m \geq 1} \frac{(-1)^m B_{m+1}(\frac{2a_i+1}{2s})}{m(m+1)} \psi_{\lambda_i}^m\right) \end{aligned} \quad (12.3.17)$$

if $\sum_{i=1}^n a_i \equiv g-1 \pmod{s}$, and zero otherwise.

Proof. From Lemma 12.3.3 and the definition of unit-preserving R -matrix action, we get the following expression for $\Omega_{g,n}^{r,\theta}(v_{a_1} \otimes \dots \otimes v_{a_n})$:

$$\begin{aligned} & \sum_{\Gamma \in \mathcal{G}_{g,n}} \sum_{w \in W_\Gamma^{s,\theta}(a)} \frac{1}{|\text{Aut}(\Gamma)|} \xi_{\Gamma,*} \prod_{v \in V_\Gamma} \frac{(2s)^{2g(v)-1}}{2} \exp\left(\sum_{m \geq 1} \frac{(-1)^m B_{m+1}(\frac{1}{2s})}{m(m+1)} \kappa_m(v)\right) \\ & \times \prod_{\substack{e \in E_\Gamma \\ e=(h,h')}} (4s) \cdot \frac{1 - \exp\left(-\sum_{m \geq 1} \frac{(-1)^m B_{m+1}(\frac{2w(h)+1}{2s})}{m(m+1)} ((\psi_h)^m - (-\psi_{h'})^m)\right)}{\psi_h + \psi_{h'}} \\ & \times \prod_{\lambda_i \in \Lambda_\Gamma} \exp\left(-\sum_{m \geq 1} \frac{(-1)^m B_{m+1}(\frac{2a_i+1}{2s})}{m(m+1)} \psi_{\lambda_i}^m\right). \end{aligned}$$

if $\sum_{i=1}^n a_i \equiv g-1 \pmod{s}$, and zero otherwise. The spin weightings modulo s are simply keeping track of the Kronecker deltas in the TFT and the R -matrix. Collecting the powers of s , we get the exponent

$$|E_\Gamma| + \sum_{v \in V_\Gamma} (2g(v) - 1) = 2g - 1 - h^1(\Gamma),$$

and collecting the powers of 2, we find the exponent

$$2|E_\Gamma| + 2 \sum_{v \in V_\Gamma} (g(v) - 1) = 2g - 2.$$

□

We have all the ingredients now to state the main result of this section.

THEOREM 12.3.6 (Spin ELSV formula). *Conjecture 12.1.2 is equivalent to the following statement. For $r = 2s$ and $\mu = (\mu_1, \dots, \mu_n) \in \mathcal{OP}_d$, the spin single Hurwitz numbers are given by*

$$h_{g;\mu}^{r,\vartheta} = 2^{1-g} r^{\frac{(r+1)(2g-2+n)+d}{r}} \left(\prod_{i=1}^n \frac{\binom{\mu_i}{r}^{[\mu_i]}}{[\mu_i]!} \right) \int_{\mathcal{M}_{g,n}} \frac{\Omega_{g,n}^{r,\vartheta}(v_{\langle \mu_1 \rangle} \otimes \cdots \otimes v_{\langle \mu_n \rangle})}{\prod_{i=1}^n (1 - \frac{\mu_i}{r} \psi_i)}. \quad (12.3.18)$$

Here we wrote $\mu_i = r[\mu_i] + r - (2\langle \mu_i \rangle + 1)$, with $0 \leq \langle \mu_i \rangle \leq s-1$.

Proof. Combining Lemma 12.3.1 and the Eynard–DOSS formula of Theorem 2.3.12 in the basis (v_0, \dots, v_{s-1}) , we find that the topological recursion amplitudes $\omega_{g,n}^{r,\vartheta}$ computed on the spin Hurwitz numbers spectral curve are given by

$$2^{1-g-n} (2s)^{\frac{(2s+1)(2g-2+n)}{2s}} \sum_{a_1, \dots, a_n=0}^{s-1} \int_{\mathcal{M}_{g,n}} \Omega_{g,n}^{r,\vartheta}(v_a) \prod_{i=1}^n \left(\sum_{k_i \geq 0} \psi_i^{k_i} d\theta^{k_i, a_i}(z_i) \right),$$

where $v_a = v_{a_1} \otimes \cdots \otimes v_{a_n}$. With our choice of constants $c[i]$, we have $-\frac{1}{\zeta_i} \frac{d}{d\zeta_i} = \frac{1}{2s} \frac{d}{dx}$. Thus, $\omega_{g,n}^{r,\vartheta}$ is given by

$$\begin{aligned} d_1 \cdots d_n \sum_{\substack{0 \leq a_1, \dots, a_n < s \\ k_1, \dots, k_n \geq 0}} \int_{\mathcal{M}_{g,n}} \Omega_{g,n}^{r,\vartheta}(v_a) 2^{1-g-n} (2s)^{\frac{(2s+1)(2g-2+n)}{2s}} \prod_{i=1}^n \left(\frac{\psi_i}{2s} \frac{d}{dx} \right)^{k_i} \theta^{a_i}(z_i) \\ = d_1 \cdots d_n \sum_{\substack{0 \leq a_1, \dots, a_n < s \\ k_1, \dots, k_n \geq 0}} \int_{\mathcal{M}_{g,n}} \Omega_{g,n}^{r,\vartheta}(v_a) 2^{1-g} (2s)^{\frac{(2s+1)(2g-2+n) + \sum_i (2s-2a_i-1)k_i}{2s}} \\ \times \prod_{i=1}^n \left(\frac{\psi_i}{2s} \right)^{k_i} \sum_{m_i \geq 0} \frac{(2sm_i + 2s - 2a_i - 1)^{m_i + k_i}}{m_i!} e^{(2sm_i + 2s - 2a_i - 1)x(z_i)}. \end{aligned}$$

For μ odd, let us write $\mu = 2s[\mu] + 2s - (2\langle \mu \rangle + 1)$. Then, after substituting $\mu_i = 2sm_i + 2s - 2a_i - 1$, the above formula reads

$$d_1 \cdots d_n \sum_{\substack{\mu_1, \dots, \mu_n > 0 \\ \mu_i \text{ odd}}} \int_{\mathcal{M}_{g,n}} \frac{\Omega_{g,n}^{r,\vartheta}(v_{\langle \mu \rangle})}{\prod_{i=1}^n (1 - \frac{\mu_i}{2s} \psi_i)} 2^{1-g} (2s)^{\frac{(2s+1)(2g-2+n) + |\mu|}{2s}} \prod_{i=1}^n \frac{\binom{\mu_i}{2s}^{[\mu_i]}}{[\mu_i]!} e^{\mu_i x(z_i)}.$$

Comparing the above expression for $\omega_{g,n}^{r,\vartheta}(z_1, \dots, z_n)$ with the statement of Conjecture 12.1.2, we get the thesis. □

EXAMPLE 12.3.7 (The case $(g, n) = (1, 1)$). Let us explicitly write the CohFT for $g = n = 1$. We have $\Omega_{1,1}^{r,\vartheta}(a) = \text{Cont}_\Gamma + \frac{\xi_{\Gamma',s}}{2} \text{Cont}_{\Gamma'}$, where the stable graphs are

$$\Gamma = \text{---} \bullet \quad \text{and} \quad \Gamma' = \text{---} \circ$$

and their contributions are given by

$$\begin{aligned} \text{Cont}_\Gamma &= \delta_{a,0} s \left(1 - \frac{B_2(\frac{1}{2s})}{2} \kappa_1 \right) \left(1 + \frac{B_2(\frac{1}{2s})}{2} \psi_1 \right) \in H^\bullet(\overline{\mathcal{M}}_{1,1}), \\ \text{Cont}_{\Gamma'} &= \delta_{a,0} \left(- \sum_{w=0}^{s-1} \frac{B_2(\frac{2w+1}{2s})}{2} \right) \in H^\bullet(\overline{\mathcal{M}}_{0,3}). \end{aligned}$$

Using the relations $\kappa_1 = \psi_1$ and $\frac{\xi_{\Gamma',s}}{2}[\text{pt}] = 12\psi_1$, together with the Bernoulli polynomial identity $-12s \sum_{w=0}^{s-1} B_2(\frac{2w+1}{2s}) = 1$, we find

$$\Omega_{1,1}^{r,\vartheta}(a) = \delta_{a,0} \left(s + \frac{\psi_1}{2s} \right).$$

In particular, the ELSV for $g = n = 1, \mu = 2sb - 1$ reads

$$\begin{aligned} h_{1;\mu}^{r,\vartheta} &= 4s^2 \frac{\mu^{b-1}}{(b-1)!} \int_{\overline{\mathcal{M}}_{1,1}} \left(s + \frac{\psi_1}{2s} \right) \left(1 + \frac{\mu}{2s} \psi_1 \right) \\ &= 4s^2 \frac{\mu^{b-1}}{(b-1)!} \left(\int_{\overline{\mathcal{M}}_{1,1}} \psi_1 \right) \left(\frac{\mu}{2} + \frac{1}{2s} \right) \\ &= \frac{s^2}{12} \frac{\mu^{b-1}}{(b-1)!} \left(\mu + \frac{1}{s} \right) \end{aligned}$$

which coincides with the expression of Equation (12.1.22) obtained via Fock space computations.

Remark 12.3.8. For the special case $r = 2$, i.e. $s = 1$, we can use Proposition 2.2.13 to write the (1-dimensional) CohFT $\Omega_{g,n}^{2,\vartheta}(\mathbb{1}^{\otimes n})$ in exponential form. Moreover, thanks to the Bernoulli polynomials property $B_{m+1}(\frac{1}{2}) = (2^{-m} - 1)B_{m+1}$ and Mumford’s formula (2.2.26), we can rewrite the CohFT $\Omega_{g,n}^{2,\vartheta}(\mathbb{1}^{\otimes n})$ as a product of two Hodge classes:

$$2^{2g-2} \exp\left(- \sum_{m \geq 1} \frac{B_{m+1}(\frac{1}{2})}{m(m+1)} \left(\kappa_m - \sum_{i=1}^n \psi_i^m + \delta_m \right)\right) = 2^{2g-2} \Lambda(1)\Lambda(-\frac{1}{2}). \quad (12.3.19)$$

In particular, the ELSV formula for $\mu_i = 2b_i - 1$ reads

$$h_{g;\mu}^{2,\vartheta} = 2^{4g-4+2n} \left(\prod_{i=1}^n \frac{\mu_i^{b_i-1}}{(b_i-1)!} \right) \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda(1)\Lambda(-\frac{1}{2})}{\prod_{i=1}^n (1 - \frac{\mu_i}{2} \psi_i)}, \quad (12.3.20)$$

expressing spin single Hurwitz numbers with 3-completed cycles in terms of double Hodge integrals.

12.4 — THE COHFT AS SPIN CHIDO CLASS

As explained in Section 2.6.1, the (non-spin) single Hurwitz numbers with $(r + 1)$ -completed cycles can be expressed in terms of intersection numbers of the Chiodo class [Chio8b] by Zvonkine’s r -ELSV formula [Zvo06], proved to be equivalent to topological recursion in [SSZ15]. In this section, we show that the cohomological field theory $\Omega^{r,\theta}$ of Proposition 12.3.5 can be interpreted in a similar way. The main difference, as more often in the spin setting, is the introduction of a sign and powers of 2.

Since we are going to modify Chiodo’s construction, let us recall the basic setup, which we specialise to our setting. It should be noted that there are three different, but equivalent, constructions of the compactified moduli space of spin bundles: by Jarvis [Jar98; Jar00] using curves with A_r -singularities at the nodes and relatively torsion-free sheaves, by Caporaso–Casagrande–Cornalba [CCCo7] using iterated blowups at nodes (bubbled curves), and by Abramovich–Jarvis and Chiodo [AJ03; Chio8a] using stacky curves with μ_r -automorphisms at the nodes. For an excellent introduction into this theory, see [CZ09].

We fix a positive integer r as above. In [Olso7; Chio8a], the authors construct alternative compactifications of $\mathcal{M}_{g,n}$ adapted to spin structures. In particular, we will need the moduli space of r -stable curves, for which the main result for us is stated below. The exact definition of an r -stable curve is given in [Chio8b, Definition 2.1.1]. Informally, an r -stable curve is a nodal one-dimensional stack, whose coarse space is stable, whose smooth locus is an algebraic space, and whose nodes are described by

$$[\mathrm{Spec}(A[z, w]/(zw - t))/\mu_r] \rightarrow \mathrm{Spec}, \tag{12.4.1}$$

for some $t \in A$, where μ_r is the group scheme of r -th roots of unity, and an element $\xi \in \mu_r$ acts by $(z, w) \mapsto (\xi z, \xi^{-1}w)$.

THEOREM 12.4.1 ([Olso7; Chio8a]).

1. The moduli functor $\overline{\mathcal{M}}_{g,n}(r)$ of r -stable n -pointed curves of genus g forms a proper, smooth, and irreducible Deligne–Mumford stack of dimension $3g - 3 + n$, and $\overline{\mathcal{M}}_{g,n}(1) = \overline{\mathcal{M}}_{g,n}$.
2. If $r = ls$, there is a natural surjective, finite, and flat morphism $f_s^r : \overline{\mathcal{M}}_{g,n}(r) \rightarrow \overline{\mathcal{M}}_{g,n}(s)$ that is invertible on the open dense substack of smooth n -pointed genus g curves and yields an isomorphism between coarse spaces (and hence on Chow and cohomology). The morphism is not injective: indeed, the restriction to the substack of singular r -stable curves has degree l as a morphism between stacks.

Fix $k \in \mathbb{Z}$ and n numbers $0 \leq m_i \leq r - 1$ satisfying the modular constraint

$$\sum_{i=1}^n m_i \equiv k(2g - 2 + n) \pmod{r}. \tag{12.4.2}$$

For a genus g curve with n markings, define

$$K = \omega_{\log}^{\otimes k} \left(- \sum_{i=1}^n m_i x_i \right). \tag{12.4.3}$$

As we briefly explained in Section 2.2.2, we will be interested in the moduli space of r -th roots of K : line bundles L such that $L^{\otimes r} \cong K$. In the smooth case, there is a natural torsor on $\overline{\mathcal{M}}_{g,n}$ of curves with an r -th root of K . There are different ways of compactifying these, but the most natural for us is the construction of [Chio8a]: its compactification parametrises r -th roots of K on r -stable curves. The main properties of this moduli stack are given below. We will write $m = (m_1, \dots, m_n)$.

THEOREM 12.4.2 ([Chio8a, Theorem 4.6]).

1. The moduli functor $\overline{\mathcal{M}}_{g,m}^{r,k}$ of r -th roots of K on r -stable curves forms a proper and smooth stack of Deligne–Mumford type of dimension $3g - 3 + n$. For $k = m_1 = \dots = m_n = 0$, the stack $\overline{\mathcal{M}}_{g,m}^{r,k}$ is a group stack \mathcal{G} on $\overline{\mathcal{M}}_{g,n}(r)$. In general $\overline{\mathcal{M}}_{g,m}^{r,k}$ is a finite torsor on $\overline{\mathcal{M}}_{g,n}(r)$ under \mathcal{G} .
2. The morphism $p: \overline{\mathcal{M}}_{g,m}^{r,k} \rightarrow \overline{\mathcal{M}}_{g,n}(r)$ is étale. It factors through a morphism locally isomorphic to the classifying stack $B\mu_r \rightarrow \text{Spec } \mathbb{C}$ (a μ_r -gerbe) and a representable étale r^{2g} -fold cover; therefore it has degree r^{2g-1} .

Note that we need not restrict the m_i to lie between 0 and $r-1$. However, for any $t = (t_1, \dots, t_n)$, there are canonical equivalences

$$i_t: \overline{\mathcal{M}}_{g,m+rt}^{r,k} \longrightarrow \overline{\mathcal{M}}_{g,m}^{r,k}, \quad (C, x, L) \longmapsto (C, x, L(\sum_{i=1}^n t_i x_i)). \quad (12.4.4)$$

We will use these maps implicitly to always reduce to $0 \leq m_i \leq r-1$. Similarly, there are canonical equivalences

$$\sigma: \overline{\mathcal{M}}_{g,m}^{r,k+r} \longrightarrow \overline{\mathcal{M}}_{g,m}^{r,k}, \quad (C, x, L) \longmapsto (C, x, L \otimes \omega^{-1}). \quad (12.4.5)$$

Moreover, if $r = ls$, there is a map $\epsilon_s^r: \overline{\mathcal{M}}_{g,m}^{r,k} \rightarrow \overline{\mathcal{M}}_{g,m}^{s,k \pmod{s}}$, which takes the l -th power of the line bundle and the $B\mu_r$ -structure at the nodes (and incorporates the index restricting from Equation (12.4.4)). This factors through a μ_l -gerbe and a representable l^{2g} -fold cover. We get $p \circ \epsilon_s^r = f_s^r \circ p$. In particular, for $s = 1$, we have $\epsilon_1^r = f_1^r \circ p$.

These moduli spaces have a universal curve and a universal r -th root

$$\pi: \overline{\mathcal{C}}_{g,m}^{r,k} \longrightarrow \overline{\mathcal{M}}_{g,m}^{r,k}, \quad \mathcal{L} \longrightarrow \overline{\mathcal{C}}_{g,m}^{r,k}. \quad (12.4.6)$$

Moreover, there is a natural substack $Z \subset \overline{\mathcal{C}}_{g,m}^{r,k}$ consisting of the nodes of the singular curves. This has a double cover, $j: Z' \rightarrow Z$, parametrising nodes of singular curves with a choice of branch at the node. We denote its deck transformation, i.e. the map sending a node on a curve with chosen branch to the same node with opposite branch, by $y \mapsto \bar{y}$. On Z' , there are two natural line bundles whose fibres are the cotangent lines at the two branches of the node; we denote their first Chern class by ψ and $\hat{\psi}$.

A node with chosen branch y has two natural lines over it: $\mathcal{L}|_y$ and $T_y^* \tilde{C}$, where \tilde{C} is the normalisation of C , both of which are μ_r -representations (by the local structure of r -stable curves at a node). Let $q(y) \in \mathbb{Z}/r\mathbb{Z}$ be the number defined by $\mathcal{L}|_y = (T_y^* \tilde{C})^{\otimes q(y)}$ as representations. This is locally constant on Z' , and we get a decomposition

$$Z' = \bigsqcup_{q \in \mathbb{Z}/r\mathbb{Z}} Z'_q, \quad j_q = j|_{Z'_q}: Z'_q \rightarrow \overline{\mathcal{M}}_{g,m}^{r,k}. \quad (12.4.7)$$

THEOREM 12.4.3 ([Chio8b, Theorem 1.1.1]). *With notation as above,*

$$(d+1)! \operatorname{ch}_d(R^\bullet \pi_* \mathcal{L}) = B_{d+1} \left(\frac{k}{r} \right) \kappa_d - \sum_{i=1}^n B_{d+1} \left(\frac{b_i}{r} \right) \psi_i^d + \frac{r}{2} \sum_{q=0}^{r-1} B_{d+1} \left(\frac{q}{r} \right) j_{q,*}(\gamma_{d-1}). \quad (12.4.8)$$

Here $\gamma_d = \sum_{i+j=d} (-\psi)^i \hat{\psi}^j$.

As stated in Proposition 2.2.19, Janda–Pixton–Pandharipande–Zvonkine concluded from the above formula an expression for the Chiodo class

$$\begin{aligned} C_{g,n}^{r,k}(m_1, \dots, m_n) &= (\epsilon_1^r)_* c(-R^\bullet \pi_* \mathcal{L}) \\ &= (\epsilon_1^r)_* \exp \left(\sum_{d \geq 1} (-1)^d (d-1)! \operatorname{ch}_d(R^\bullet \pi_* \mathcal{L}) \right) \in H^\bullet(\overline{\mathcal{M}}_{g,n}) \end{aligned} \quad (12.4.9)$$

as a sum over stable graphs, and thus its Givental–Teleman data (for this it is important to restrict the m_i , cf. Equation (12.4.4)).

In our current setting, we want to understand what the correct geometric adaptation to the construction of some Chern class should be in order to obtain the sum over stable graphs of (12.3.17). To this end, we have to introduce a sign, again given by the parity of a theta characteristic. For this, we assume $k = 2l + 1$ is odd, $r = 2s$ is even, and moreover $m_i = 2a_i + 1$ is odd for all i . Recall the notion of theta characteristic and its parity, Definition 10.3.1.

Theta characteristics on a nodal curve were considered by Cornalba [Cor89], and are a particular case of 2-spin curves in the bubbled curves framework of [CC07], with all the $m_i = 1$. In [Cor89, Section 6], Cornalba indicates how to extend the proof of deformation invariance of the parity to nodal curves (still defined as $h^0(C, \vartheta) \pmod{2}$) in this case. In our language, this shows that parity is a locally constant function on $\overline{\mathcal{M}}_{g,1^n}^{2,1}$.

DEFINITION 12.4.4. Let $g, n \geq 0$ such that $2g - 2 + n > 0$, $r = 2s \in \mathbb{Z}_+^{\text{even}}$, $k = 2l + 1 \in \mathbb{Z}^{\text{odd}}$, and $0 \leq m_1, \dots, m_n \leq r - 1$ such that $m_i = 2a_i + 1$ and

$$\sum_{i=1}^n (a_i - l) \equiv (2l + 1)(g - 1) \pmod{s}. \quad (12.4.10)$$

For an r -spin curve $(C, x, L) \in \overline{\mathcal{M}}_{g,m}^{r,k}$, we get that $\sigma^l \epsilon_2^r(L)$ is a theta characteristic on C , and we define its *parity* to be the parity of this theta characteristic. As the parity is locally constant, we get a decomposition

$$\overline{\mathcal{M}}_{g,m}^{r,k} = \overline{\mathcal{M}}_{g,m}^{r,k,+} \sqcup \overline{\mathcal{M}}_{g,m}^{r,k,-}. \quad (12.4.11)$$

Here, we identify $\mathbb{Z}/2\mathbb{Z} \cong \{\pm 1\}$ for clearer notation. We will use these superscripts more often to denote objects restricted to these subspaces. The *spin Chiodo class* is defined as

$$C_{g,n}^{r,k,\vartheta}(a_1, \dots, a_n) = (\epsilon_1^{r,+})_* c(-R^\bullet \pi_*^+ \mathcal{L}^+) - (\epsilon_1^{r,-})_* c(-R^\bullet \pi_*^- \mathcal{L}^-) \in H^\bullet(\overline{\mathcal{M}}_{g,n}). \quad (12.4.12)$$

Remark 12.4.5. Recall that we restricted the indices m_i to lie between 0 and $r - 1$. This is essential: the map ϵ_2^r “untwists” $L^{\otimes r}$ to a theta characteristic. In particular, it also “untwists” at the nodes: the Z'_q from Equation (12.4.7) is mapped to $Z'_q \pmod{2}$ in $\overline{\mathcal{M}}_{g,1^n}^{2,1}$. We may then extend the a_i to all of \mathbb{Z} in this definition, but we get canonical equivalences between the

spaces for a_i and $a_i + s$, which preserve the parity by construction. Note that we require $m_i = 2a_i + 1$ and $k = 2l + 1$ to be odd. This is necessary to obtain an honest theta characteristic, whose parity is well-behaved. For even m_i or k , there is no clear map to $\overline{\mathcal{M}}_{g,1^n}^{2,1}$, where the bundles with theta characteristic live.

As suggested by A. Chiodo, the class $C^{r,k,\vartheta}$ can be expressed as a product of the usual Chiodo class with Witten 2-spin class

$$\mathcal{W}_{g,n}^2 \in H^*(\overline{\mathcal{M}}_{g,1^n}^{2,1}). \quad (12.4.13)$$

We refer to [JKV01; Chio6] for the definition Witten’s spin class on the moduli space of spin curves.

PROPOSITION 12.4.6. *In the situation of Definition 12.4.4, the spin Chiodo class can be given by multiplying the usual Chiodo class with Witten 2-spin class on the moduli space of 2-spin curves:*

$$C_{g,n}^{r,k,\vartheta}(a_1, \dots, a_n) = (\epsilon_1^2)_* \left(\mathcal{W}_{g,n}^2 \cdot (\epsilon_2^r)_* c(-R^\bullet \pi_* \mathcal{L}) \right). \quad (12.4.14)$$

Proof. By [JKV01, Theorem 4.6], cf. also [Chio6, Section 6.3], the 2-spin Witten class on $\overline{\mathcal{M}}_{g,m}^{2,1}$ is non-zero if and only if m_i is odd, and in that case it is given by 1 on the component of even spin curves and -1 on the component of odd spin curves. Therefore

$$\begin{aligned} (\epsilon_1^2)_* \left(\mathcal{W}_{g,n}^2 \cdot (\epsilon_2^r)_* c(-R^\bullet \pi_* \mathcal{L}) \right) &= (\epsilon_1^2)_* (\epsilon_2^r)_* \left((\epsilon_2^r)^* \mathcal{W}_{g,n}^2 \cdot c(-R^\bullet \pi_* \mathcal{L}) \right) \\ &= (\epsilon_1^{r,+})_* c(-R^\bullet \pi_*^+ \mathcal{L}^+) - (\epsilon_1^{r,-})_* c(-R^\bullet \pi_*^- \mathcal{L}^-), \end{aligned}$$

which is the spin Chiodo class. \square

In other words, spin Chiodo classes naturally “live” on $\overline{\mathcal{M}}_{g,1^n}^{2,1}$, which makes some sense, as it is related via the ELSV formula (12.3.18) to spin Hurwitz numbers.

Remark 12.4.7. Proposition 12.4.6 allows us to extend the definition of the spin Chiodo class to even m_i by taking the right-hand side of Equation (12.4.14) as definition, as the class naturally vanishes in that case. It is also natural to define a “ l -spin Chiodo class” analogously, as was also suggested to us by A. Chiodo: if $r = ls$

$$C_{g,n}^{r,k,l\text{-spin}}(m_1, \dots, m_n) = (\epsilon_1^l)_* \left(\mathcal{W}_{g,n}^l \cdot (\epsilon_l^r)_* c(-R^\bullet \pi_* \mathcal{L}) \right), \quad (12.4.15)$$

which vanishes if any of the m_i is divisible by l . This class, or a related construction, could be useful for the local Gromov–Witten invariants of Lee–Parker [LP07; LP13], relating the Gromov–Witten invariants of Kähler surfaces to spin Hurwitz numbers. Note that for $l > 2$, Witten’s class is not zero-dimensional anymore. We will not pursue this in this dissertation.

As suggested by D. Zvonkine, the CohFT $\Omega^{r,\vartheta}$ coincides with the spin Chiodo class, up to powers of 2.

PROPOSITION 12.4.8. *Let r be a positive even integer. The CohFT of Proposition 12.3.5 is equal to the spin Chiodo class for $k = 1$:*

$$\Omega_{g,n}^{r,\vartheta}(v_{a_1} \otimes \dots \otimes v_{a_n}) = 2^{g-1} C_{g,n}^{r,1,\vartheta}(a_1, \dots, a_n). \quad (12.4.16)$$

Proof. In the non-spin case, the corresponding expansion in stable graphs is given by [JPPZ17, Corollary 4]. We emphasise here the changes coming from the inserted sign. Let us start from a slightly more refined formula for the Chern character of $R^\bullet \pi_* \mathcal{L}$, [Chio8b, Corollary 3.1.8]: $(d+1)! \text{ch}_d(R^\bullet \pi_* \mathcal{L})$ is given by

$$p^* \left(B_{d+1} \left(\frac{1}{r} \right) \kappa_d - \sum_{i=1}^n B_{d+1} \left(\frac{m_i}{r} \right) \psi_i^d + \frac{r}{2} \sum_{\substack{0 \leq h \leq g \\ I \subseteq \llbracket n \rrbracket}} B_{d+1} \left(\frac{q(h, I)}{r} \right) (i_{(h, I)})_* (\gamma_{d-1}) \right) + \frac{r}{2} \sum_{q=0}^{r-1} B_{d+1} \left(\frac{q}{r} \right) (j_{(\text{irr}, q)})_* (\gamma_{d-1}).$$

Let us explain the notation in this formula. Here the singular locus Z' has been decomposed more than in Equation (12.4.7): consider a spin curve (C, x, L) with a node y , denote by $\nu: \tilde{C} \rightarrow C$ the normalisation at y and $\tilde{L} = \nu^* L$.

If the node y is *separating*, denote $\tilde{C} = C_1 \sqcup C_2$ with the chosen first branch C_1 of genus h and marked points x_I for some $I \subseteq \llbracket n \rrbracket$. Then the value of $q(h, I)$ is determined by $2h - 2 + (|I| + 1) - \sum_{i \in I} m_i + q(h, I) \in r\mathbb{Z}$ (and in particular must be odd), so $Z'_{(h, I)} \subset Z'_{q(h, I)}$ and

$$\tilde{L}^{\otimes r}|_{C_1} \cong \omega_{C_1, \log} \left((q-r)y - \sum_{i \in I} m_i x_i \right).$$

We denote by $i_{(h, I)}: V'_{(h, I)} \rightarrow \overline{\mathcal{M}}_{g, n}(r)$ the double cover that fits into the following diagram.

$$\begin{array}{ccc} Z'_{(h, I)} & \xrightarrow{j_{(h, I)}} & \overline{\mathcal{M}}_{g, m}^{r, k} \\ p_{(h, I)} \downarrow & & \downarrow p \\ V'_{(h, I)} & \xrightarrow{i_{(h, I)}} & \overline{\mathcal{M}}_{g, n}(r) \end{array}$$

If the node y is *non-separating*, then the moduli point lies in $Z'_{(\text{irr}, q)} \subset Z'_q$ for some q , and we have

$$\tilde{L}^{\otimes r} \cong \omega_{\tilde{C}, \log} \left((q-r)y - q\bar{y} - \sum_{i=1}^n m_i x_i \right).$$

We denote by $j_{(\text{irr}, q)}: Z'_{(\text{irr}, q)} \rightarrow \overline{\mathcal{M}}_{g, m}^{r, k}$ the double cover of the relevant singular locus.

Now, let us start with the local picture, near a spin curve (C, x, L) with a node y , and let us write $\tilde{L} = \nu^* L$, $\vartheta = \epsilon_2^r L$ (which is a twist of $L^{\otimes r}$, recall Remark 12.4.5) and $\tilde{\vartheta} = \nu^* \vartheta = \epsilon_2^r \tilde{L}$. The analysis is similar to [Cor89, Examples 6.1 & 6.2], which is in the bubbled curve formalism. If y is separating, then $(C_1, x_I, \tilde{L}|_{C_1})$ and $(C_2, x_{I^c}, \tilde{L}|_{C_2})$ are r -spin curves, and have parities. Moreover, although the $\tilde{L}|_{C_i}$ do not determine L (there are r choices of glueing at the node), it is clear that $H^0(C, \vartheta) \rightarrow H^0(C_1, \tilde{\vartheta}|_{C_1}) \oplus H^0(C_2, \tilde{\vartheta}|_{C_2})$ is an isomorphism, so the parities add. If y is non-separating, and the arithmetic genus of C is g , then that of \tilde{C} is $g-1$. There are two distinct cases, given by the parity of q .

- If q is odd, then ϑ is a theta characteristic on C such that $\tilde{\vartheta}$ is a theta characteristic on \tilde{C} , and in the same way as the previous case, $h^0(C, \vartheta) = h^0(\tilde{C}, \tilde{\vartheta})$. Moreover, we can glue \tilde{L} on \tilde{C} in r ways.

- If q is even, we see that $\tilde{\theta}$ is not quite a theta characteristic: in fact $\tilde{\theta}^2 = \omega_{\tilde{C}}(y + \bar{y})$. By the argument of [Cor89, Example 6.2], such line bundles on \tilde{C} can be glued to theta characteristics on C in two ways, and the resulting theta characteristics have opposite parity.

Rewriting this, we get that if q is odd, then $Z'_{(\text{irr},q)} = (Z'_{(\text{irr},q)})^+ \sqcup (Z'_{(\text{irr},q)})^-$, and the maps $j_{(\text{irr},q)}$ preserve parity. However, if q is even, then $Z'_{(\text{irr},q)}$ does not decompose, and the maps

$j_{(\text{irr},q)}^\pm : Z_{(\text{irr},q)} \rightarrow \overline{\mathcal{M}}_{g,m}^{r,1,\pm}$ are such that $p_*^+(j_{(\text{irr},q)}^+)_* = p_*^-(j_{(\text{irr},q)}^-)_*$.

For the global calculation, we use [JPPZ17, Proposition 4]: the class $c(-R^* \pi_* \mathcal{L})$ equals

$$\sum_{\Gamma \in \mathcal{G}_{g,n}} \sum_{w \in W_{\Gamma}^{r,1}(m)} \frac{r^{|E(\Gamma)|}}{|\text{Aut}(\Gamma)|} (\xi_{\Gamma,w})_* \prod_{v \in V_{\Gamma}} C_V(v) \prod_{\lambda_i \in \Lambda_{\Gamma}} C_{\Lambda}(\lambda) \\ \times \prod_{\substack{e \in E_{\Gamma} \\ e=(h,h')}} \frac{1 - \exp\left(-\sum_{m \geq 1} \frac{(-1)^m B_{m+1}(\frac{w(h)}{r})}{m(m+1)} ((\psi_h)^m - (-\psi_{h'})^m)\right)}{\psi_h + \psi_{h'}},$$

where C_V and C_{Λ} are the same contributions as in Equation (12.3.17), and $W_{\Gamma}^{r,1}(m)$ is the set of 1-weightings modulo r of Γ (see Definition 2.2.18). The sum over weights w encodes the values of q at the different nodes. All of the terms of this formula are pulled back from the base, so in order to calculate the spin Chiodo class, we only need to analyse for each pair (Γ, w) which part of the contribution lies in $\overline{\mathcal{M}}_{g,n}^{r,1,+}$ and which part in $\overline{\mathcal{M}}_{g,n}^{r,1,-}$. We will use the local analysis for this.

If w takes an even value at a half-edge h (and therefore also at h'), then we know this can only be on a non-separating edge, and by our previous argument, it can be glued in two ways with opposite parity. Hence, all of these contributions cancel, and we may restrict to odd values.

If w only takes odd values, then each of the $\overline{\mathcal{M}}_{g(v),n(v)}^{r,1}$ actually splits according to parity, and taking parities $\{p(v) \mid v \in V_{\Gamma}\}$ results in a spin curve with parity $\prod_v p(v)$. So for fixed parities $\{p(v) \mid v \in V_{\Gamma}\}$, we get

$$\prod_{v \in V(\Gamma)} s^{2g(v)} 2^{g(v)-1} (2^{g(v)} + p(v))$$

curves. As each r -spin bundle on the stratum Γ has exactly $r^{|V_{\Gamma}|}$ automorphisms, subtracting the number of odd-parity curves from the number of even-parity curves, we get

$$\prod_{v \in V_{\Gamma}} \sum_{p(v) \in \{\pm 1\}} p(v) s^{2g(v)-1} 2^{g(v)-2} (2^{g(v)} + p(v)) = \prod_{v \in V_{\Gamma}} s^{2g(v)-1} 2^{g(v)-1}$$

as the degree for $(p^+ - p^-)$ on the stratum (Γ, w) . After pushforward, s and 2 occur a total of

$$|E_{\Gamma}| + \sum_{v \in V_{\Gamma}} (2g(v) - 1) = |E_{\Gamma}| - |V_{\Gamma}| + 2 \sum_{v \in V_{\Gamma}} g(v) = (h^1(\Gamma) - 1) + 2(g - h^1(\Gamma)) = 2g - 1 - h^1(\Gamma)$$

and

$$|E_{\Gamma}| + \sum_{v \in V_{\Gamma}} (g(v) - 1) = |E_{\Gamma}| - |V_{\Gamma}| + \sum_{v \in V_{\Gamma}} g(v) = (h^1(\Gamma) - 1) + (g - h^1(\Gamma)) = g - 1$$

times, respectively. Comparing this with Equation (12.3.17) yields the result. \square

IV. Spin Hurwitz theory

COROLLARY 12.4.9. *Conjecture 12.1.2 is equivalent to the following spin ELSV formula: for $r = 2s$ and for $\mu = (\mu_1, \dots, \mu_n) \in \mathcal{OP}_d$, the spin single Hurwitz numbers are given by*

$$\begin{aligned} h_{g;\mu}^{r,\vartheta} &= r^{\frac{(r+1)(2g-2+n)+d}{r}} \left(\prod_{i=1}^n \frac{\binom{\mu_i}{r}^{[\mu_i]}}{[\mu_i]!} \right) \int_{\overline{\mathcal{M}}_{g,n}} \frac{C_{g,n}^{r,1,\vartheta}(\langle \mu_1 \rangle, \dots, \langle \mu_n \rangle)}{\prod_{i=1}^n (1 - \frac{\mu_i}{r} \psi_i)} \\ &= r^{\frac{(r+1)(2g-2+n)+d}{r}} \left(\prod_{i=1}^n \frac{\binom{\mu_i}{r}^{[\mu_i]}}{[\mu_i]!} \right) \int_{\overline{\mathcal{M}}_{g,1}^{2,1}} \frac{\mathcal{W}_{g,n}^2 \cdot (\epsilon_2^r)_* \mathcal{C}(-R^\bullet \pi_* \mathcal{L})}{\prod_{i=1}^n (1 - \frac{\mu_i}{r} \psi_i)}. \end{aligned} \quad (12.4.17)$$

Here we wrote $\mu_i = r[\mu_i] + r - (2\langle \mu_i \rangle + 1)$, with $0 \leq \langle \mu_i \rangle \leq s - 1$.

Remark 12.4.10. Analogously to the non-spin case, cf. [KLPS19, Conjecture 6.1], [DKPS19, Theorem 1.18], these formulae may be generalised to spin q -orbifold Hurwitz numbers $h_{g;\mu}^{r,q,\vartheta}$ with $(r+1)$ -completed cycles, i.e. spin Hurwitz numbers with one ramification profile (q, q, \dots, q) , one given by a partition μ , and all other ramification profiles being spin completed $(r+1)$ -cycles. Then $q, r+1 = 2s+1$, and μ would need to be odd, and the cohomological field theory would be

$$C_{g,n}^{r,q,q,\vartheta}(\langle \mu_1 \rangle, \dots, \langle \mu_n \rangle), \quad (12.4.18)$$

with now $\mu_i = rq[\mu_i] + rq - (2\langle \mu_i \rangle + 1)$, and $0 \leq \langle \mu_i \rangle \leq sq - 1$. The spectral curve for this problem should be

$$x(z) = \log(z) - z^{rq}, \quad y(z) = z^q, \quad B(z_1, z_2) = \frac{1}{2} \left(\frac{1}{(z_1 - z_2)^2} + \frac{1}{(z_1 + z_2)^2} \right) dz_1 dz_2. \quad (12.4.19)$$

We do not pursue this generalisation here, although we do not expect any theoretical complications.

APPENDIX A — EXAMPLES OF CUTTING, GLUING, AND COMBINATORIAL FENCHEL–NIELSEN COORDINATES

CUTTING AND GLUING

In order to make the presentation of the cutting and gluing algorithms clearer, we present some neat examples of these procedures. The first two (Figures A.1 and A.2) cover the case of a sphere Σ with four boundary components, the last two (Figures A.3 and A.4) cover the case of a torus T with one boundary component.

In all examples, the cutting algorithm is presented in lexicographic order, i.e. the images are labelled by (a), (b), (c), et cetera, while the gluing algorithm is presented with the same images, but in reversed lexicographic order. The combinatorial structure is depicted in red, the associated measured foliation in blue (only the singular leaves are reported). The cutting curve γ is depicted in green, and it coincide with the curve obtain in the gluing algorithm after identification of two boundary components $\gamma_- \sim \gamma_+$. The component γ_- is always located on the left side of the figure, while γ_+ is located on the right side. The identification points $p_{\pm} \in \gamma_{\pm}$ are depicted in grey, and no twist is performed (that is, p_- is identified with p_+). Further, the letters a, b, c, \dots are referring to edge lengths of the cutting process, while the letters r, s, t, \dots are referring to edge lengths of the gluing process.

COMBINATORIAL FENCHEL–NIELSEN COORDINATES

We present two computations of combinatorial Fenchel–Nielsen coordinates on a torus T with one boundary component, relative to two different cells (Figure A.5 and Figure A.6). Furthermore, an illustration of Penner’s formulae (Proposition 6.2.1) is presented.

In all examples, the combinatorial structure is depicted in red, the associated measured foliation in blue (only the singular leaves are reported), the pants decompositions $\mathcal{P}, \mathcal{P}'$ in green and the collection $\mathcal{S}, \mathcal{S}'$ in yellow.

CUTTING. Consider $\mathbb{G} \in \mathcal{T}_\Sigma^{\text{comb}}$ and γ a cutting curve as in Figure A.1a. We have

$$\ell_{\mathbb{G}}(\gamma) = a + b + e + f.$$

After cutting, we obtain two pairs of pants P_\pm and two combinatorial structures $\mathbb{G}_\pm \in \mathcal{T}_{P_\pm}^{\text{comb}}$.

GLUING. Consider two combinatorial structures $\mathbb{G}_\pm \in \mathcal{T}_{P_\pm}^{\text{comb}}$ and two boundary components γ_\pm of P_\pm of the same length as in Figure A.1f. We have

$$\ell_{\mathbb{G}_-}(\gamma_-) = r + s, \quad \ell_{\mathbb{G}_+}(\gamma_+) = u + v.$$

After gluing (with a choice of p_\pm), we obtain a sphere Σ with four boundary components and a combinatorial structure $\mathbb{G} \in \mathcal{T}_\Sigma^{\text{comb}}$.

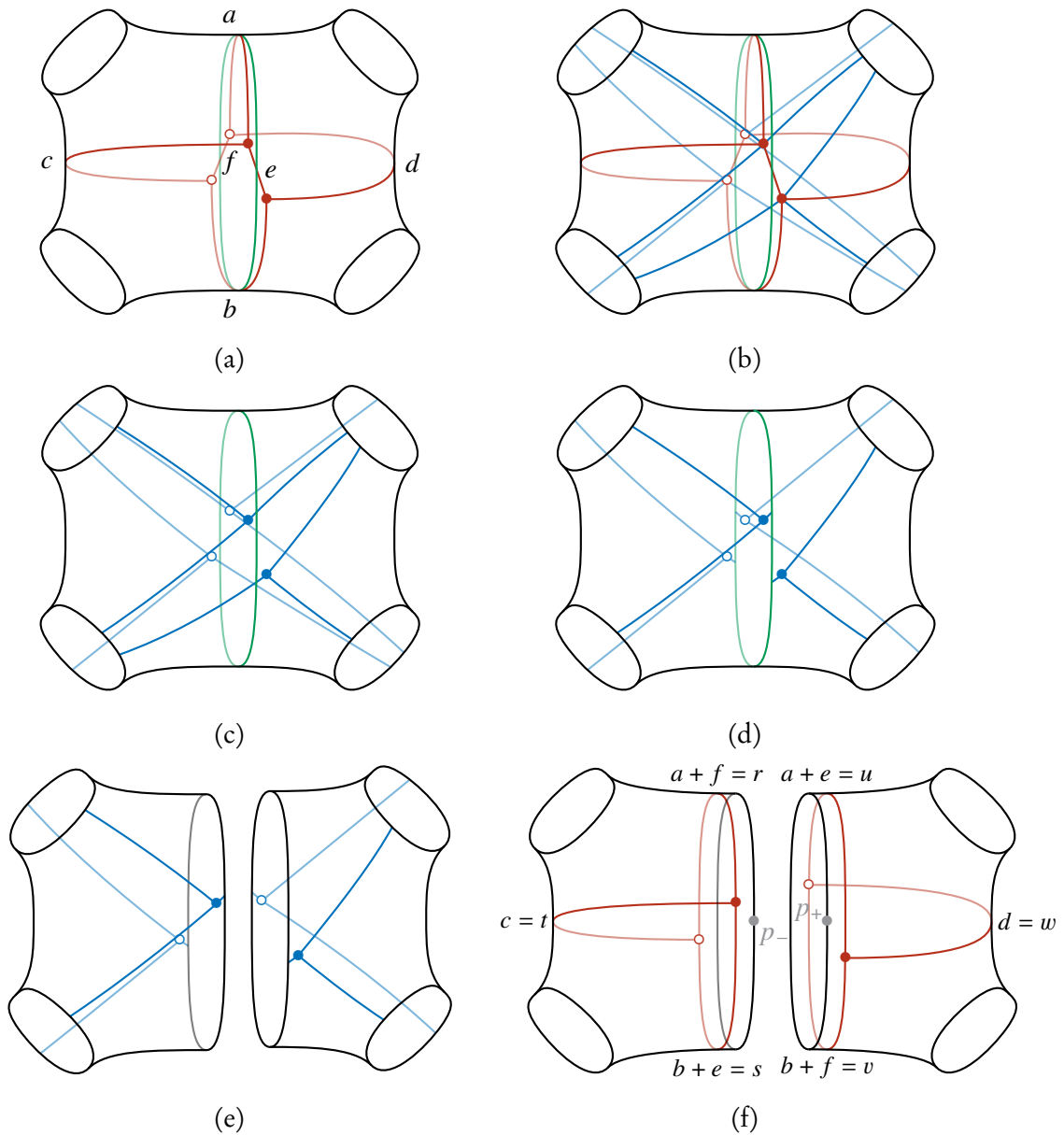


Figure A.1

CUTTING. Consider $\mathbb{G} \in \mathcal{T}_\Sigma^{\text{comb}}$ and γ a cutting curve as in Figure A.2a. We have

$$\ell_{\mathbb{G}}(\gamma) = 2a + b + 2c + 2d + e.$$

After cutting, we obtain two pairs of pants P_\pm and two combinatorial structures $\mathbb{G}_\pm \in \mathcal{T}_{P_\pm}^{\text{comb}}$. From Figure A.2c to Figure A.2d, a Whitehead move is performed.

GLUING. Consider two combinatorial structures $\mathbb{G}_\pm \in \mathcal{T}_{P_\pm}^{\text{comb}}$ and two boundary components γ_\pm of P_\pm of the same length as in Figure A.2g. We have

$$\ell_{\mathbb{G}_-}(\gamma_-) = r + s + 2t, \quad \ell_{\mathbb{G}_+}(\gamma_+) = u + v + 2w.$$

After gluing (with a choice of p_\pm), we obtain a sphere Σ with four boundary components and a combinatorial structure $\mathbb{G} \in \mathcal{T}_\Sigma^{\text{comb}}$. From Figure A.2d to Figure A.2c, a Whitehead move is performed.

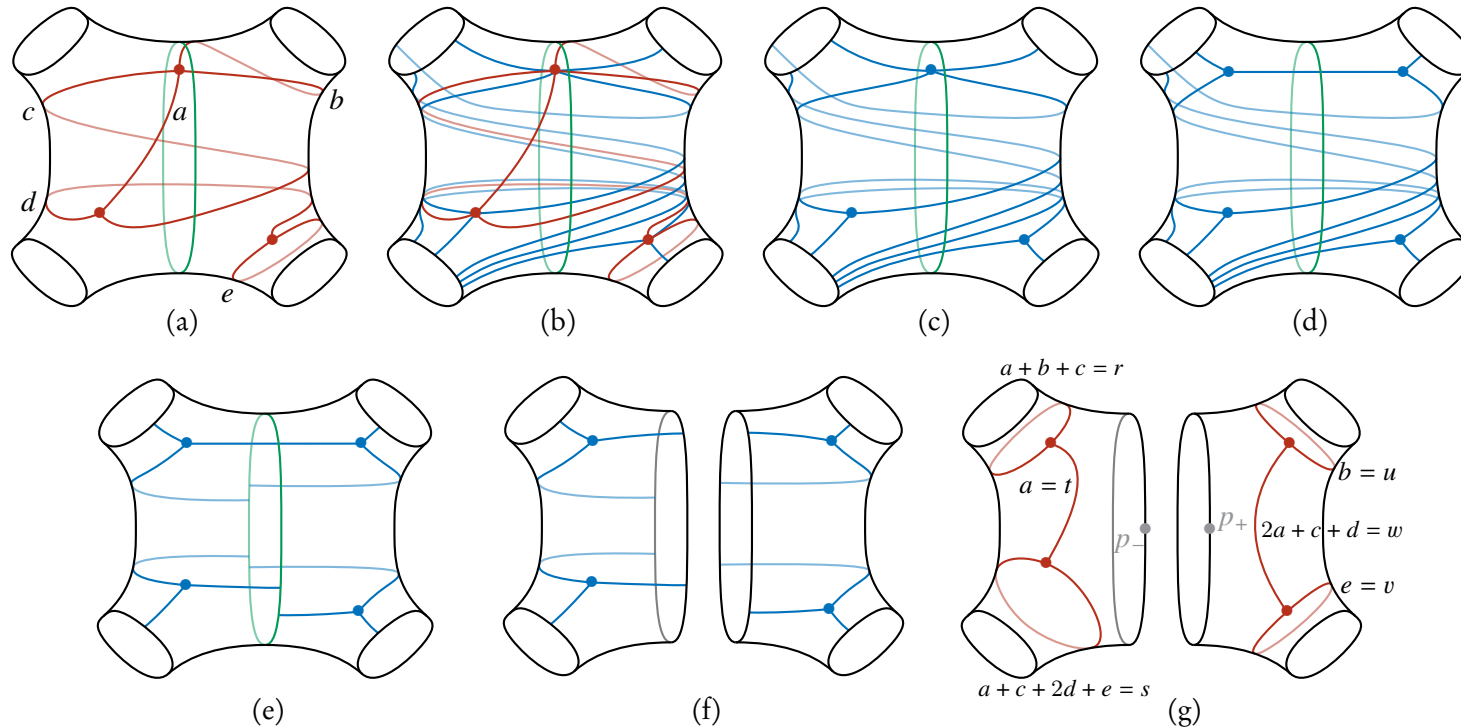


Figure A.2

CUTTING. Consider $\mathbb{G} \in \mathcal{T}_\Sigma^{\text{comb}}$ and γ a cutting curve as in Figure A.1a. We have

$$\ell_{\mathbb{G}}(\gamma) = a + c.$$

After cutting, we obtain a pair of pants P and a combinatorial structure $\mathbb{G}' \in \mathcal{T}_P^{\text{comb}}$.

GLUING. Consider a combinatorial structure $\mathbb{G}' \in \mathcal{T}_P^{\text{comb}}$ and two boundary components γ_\pm of P of the same length as in Figure A.1f. We have

$$\ell_{\mathbb{G}'}(\gamma_-) = r, \quad \ell_{\mathbb{G}'}(\gamma_+) = s.$$

After gluing (with a choice of p_\pm), we obtain a torus T with one boundary component and a combinatorial structure $\mathbb{G} \in \mathcal{T}_T^{\text{comb}}$.

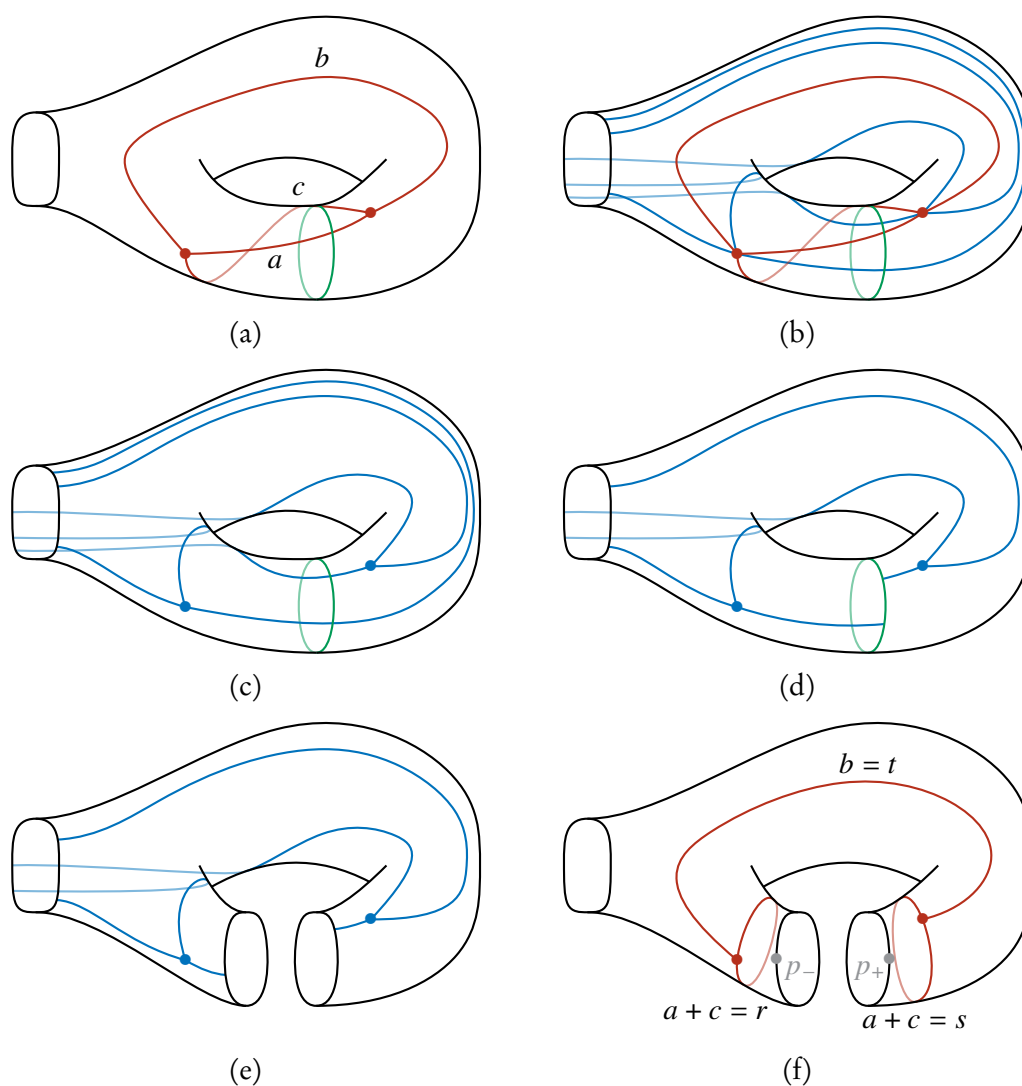


Figure A.3

CUTTING. Consider $\mathbb{G} \in \mathcal{T}_\Sigma^{\text{comb}}$ and γ a cutting curve as in Figure A.2a. We have

$$\ell_{\mathbb{G}}(\gamma) = 2a + b + c.$$

After cutting, we obtain a pair of pants P and a combinatorial structure $\mathbb{G}' \in \mathcal{T}_P^{\text{comb}}$.

GLUING. Consider a combinatorial structure $\mathbb{G}' \in \mathcal{T}_P^{\text{comb}}$ and two boundary components γ_\pm of P of the same length as in Figure A.2f. We have

$$\ell_{\mathbb{G}'}(\gamma_-) = r + s, \quad \ell_{\mathbb{G}'}(\gamma_+) = r + t.$$

After gluing (with a choice of p_\pm), we obtain a torus T with one boundary component and a combinatorial structure $\mathbb{G} \in \mathcal{T}_T^{\text{comb}}$.

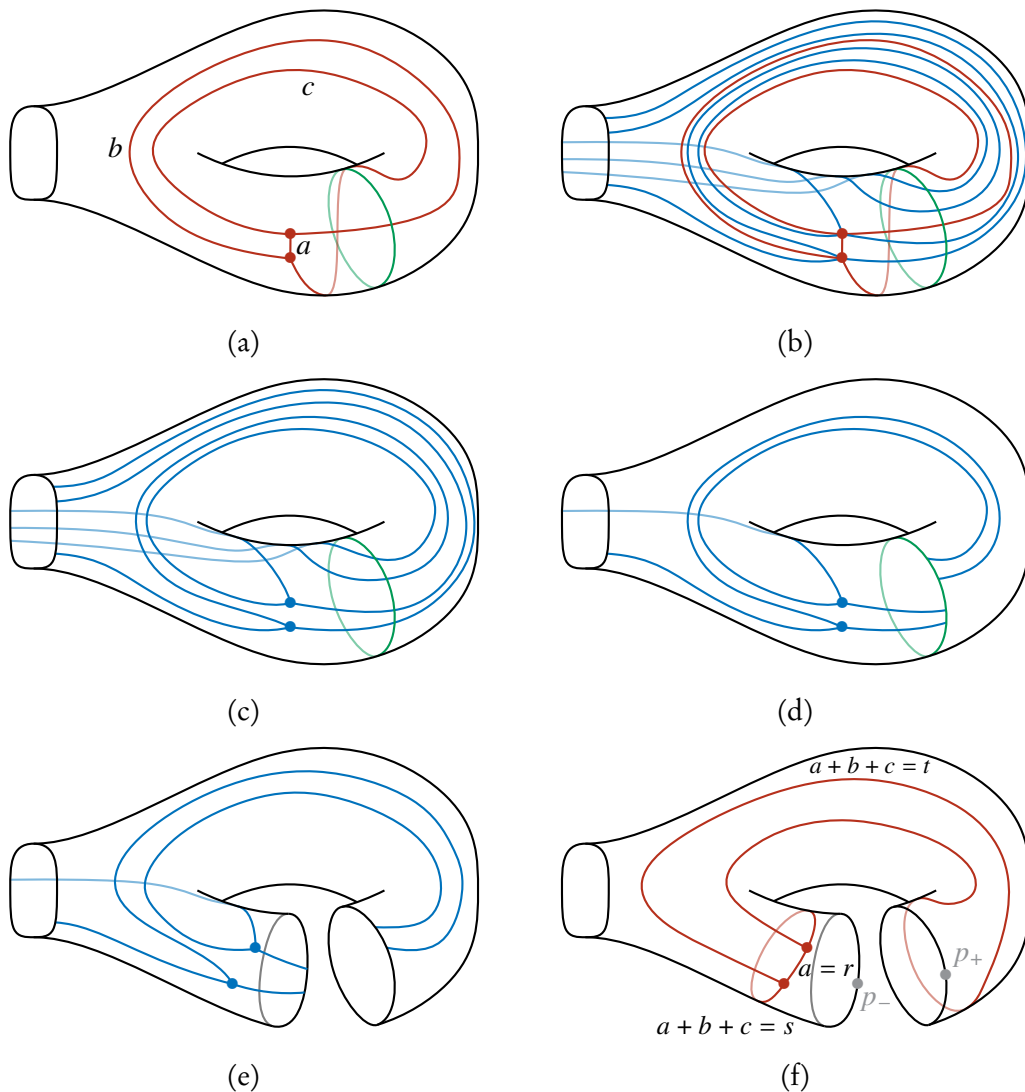


Figure A.4

COMBINATORIAL FENCHEL–NIELSEN COORDINATES. Consider the combinatorial structure $\mathbb{G} \in \mathcal{T}_T^{\text{comb}}$ on a torus T with one boundary component as in Figure A.5. We have

$$L = 2a + 2b + 2c.$$

Further, the combinatorial Fenchel–Nielsen coordinates with respect to the seamed pants decomposition $(\mathcal{P}, \mathcal{S}) = (\gamma, \beta)$ of Figure A.5a are computed with the help of Figure A.5b and are given by

$$\ell = a + c, \quad \tau = c,$$

while the combinatorial Fenchel–Nielsen coordinates with respect to the seamed pants decomposition $(\mathcal{P}', \mathcal{S}') = (\gamma', \beta')$ of Figure A.5c are computed with the help of Figure A.5d and are given by

$$\ell' = b + c, \quad \tau' = -c.$$

This is in accordance with Penner’s formulae:

$$\begin{aligned} \ell' &= |\tau| + \left[\frac{L-2\ell}{2} \right]_+ & \tau' &= -\text{sgn}(\tau) \left| \ell - \left[\frac{L-2\ell'(\ell, \tau)}{2} \right]_+ \right| \\ &= |c| + b & &= -|(a+c) - a| \\ &= b + c, & &= -c. \end{aligned}$$

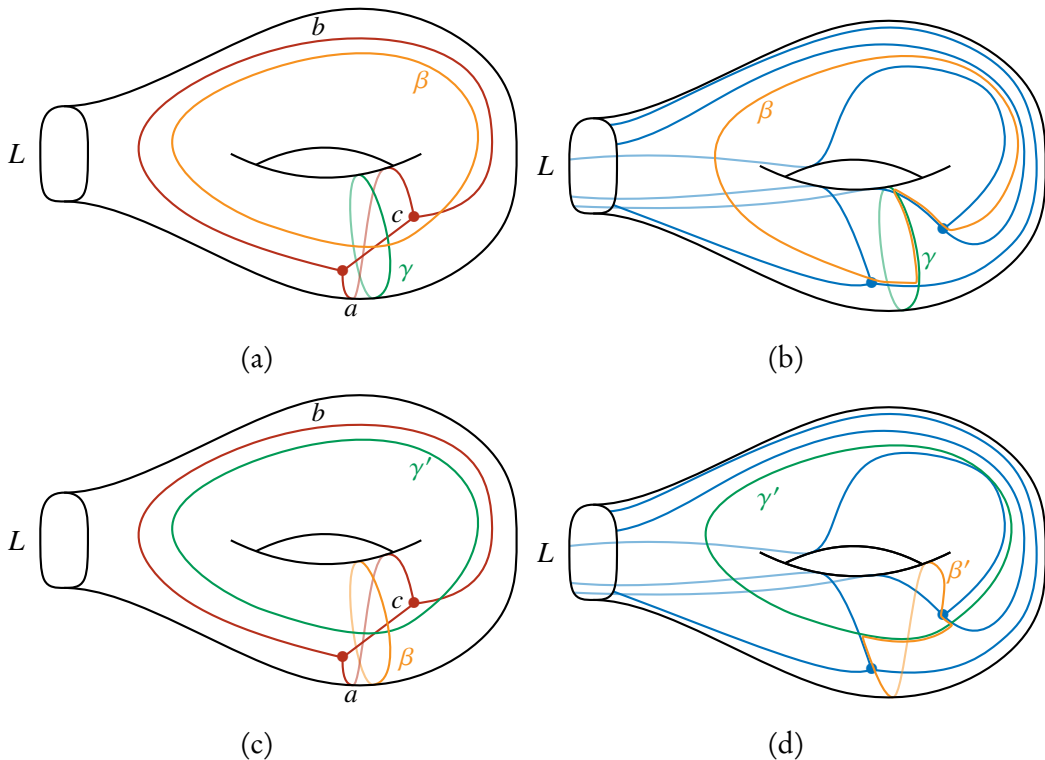


Figure A.5

COMBINATORIAL FENCHEL–NIELSEN COORDINATES. Consider the combinatorial structure $\mathbb{G} \in \mathcal{T}_T^{\text{comb}}$ on a torus T with one boundary component as in Figure A.6. We have

$$L = 2a + 2b + 2c.$$

Further, the combinatorial Fenchel–Nielsen coordinates with respect to the seamed pants decomposition $(\mathcal{P}, \mathcal{S}) = (\gamma, \beta)$ of Figure A.6a are computed with the help of Figure A.6b and are given by

$$\ell = 2a + b + c, \quad \tau = a + b,$$

while the combinatorial Fenchel–Nielsen coordinates with respect to the seamed pants decomposition $(\mathcal{P}', \mathcal{S}') = (\gamma', \beta')$ of Figure A.6c are computed with the help of Figure A.6d and are given by

$$\ell' = a + b, \quad \tau' = -2a - b.$$

This is in accordance with Penner’s formulae:

$$\begin{aligned} \ell' &= |\tau| + \left\lfloor \frac{L-2\ell}{2} \right\rfloor_+ & \tau' &= -\text{sgn}(\tau) \left| \ell - \left\lfloor \frac{L-2\ell'(\ell, \tau)}{2} \right\rfloor_+ \right| \\ &= |a + b| + 0 & &= -|(2a + b + c) - c| \\ &= a + b, & &= -2a - b. \end{aligned}$$

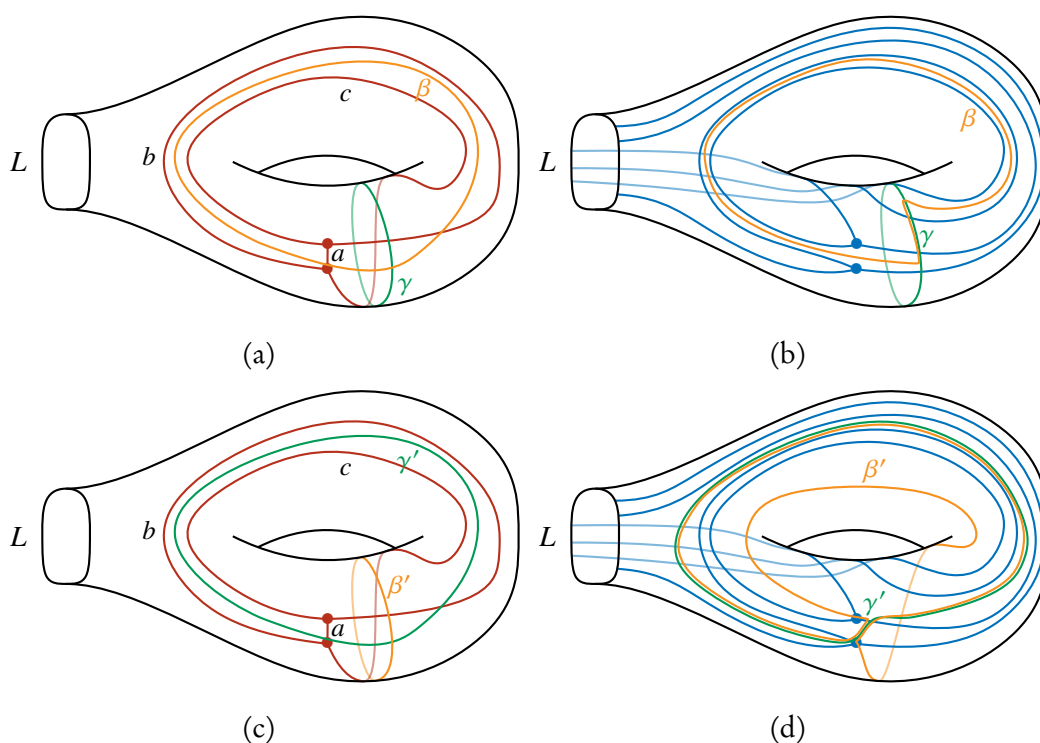


Figure A.6

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