

VERTEX ALGEBRAS AND FACTORIZATION ALGEBRAS

DISSERTATION

zur

Erlangung des Doktorgrades (Dr. rer. nat.)

der

Mathematisch-Naturwissenschaftlichen Fakultät

der

Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von

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Syracuse (New York), USA

Bonn 2020

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen
Fakultät der Rheinischen Friedrich-Wilhelms-Universität Bonn

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Tag der Promotion: 25.3.2021
Erscheinungsjahr: 2021

Summary

This thesis is about the relationship between vertex algebras and Costello-Gwilliam factorization algebras, two mathematical approaches to chiral conformal field theory. Many vertex algebras have already been constructed. Some of these are known to arise from holomorphic factorization algebras on the plane of complex numbers. We prove that every \mathbf{Z} -graded vertex algebra arises from such a factorization algebra.

First, we show that a \mathbf{Z} -graded vertex algebra is the same thing as a geometric vertex algebra. Geometric vertex algebras serve as an intermediary between \mathbf{Z} -graded vertex algebras and factorization algebras. Our factorization algebras take values in the symmetric monoidal category of complete bornological vector spaces. We describe how to obtain geometric vertex algebras from certain prefactorization algebras with values in the symmetric monoidal category of complete bornological vector spaces. Second, we attach a prefactorization algebra $\mathbf{F}\mathcal{V}$ to every geometric vertex algebra \mathcal{V} and show that the geometric vertex algebra associated with $\mathbf{F}\mathcal{V}$ is isomorphic to \mathcal{V} . Third, we prove that $\mathbf{F}\mathcal{V}$ is in fact a factorization algebra.

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1 Introduction

Vertex algebras and factorization algebras are two approaches to chiral conformal field theory. Chiral conformal field theory is a version of quantum field theory in which holomorphic functions of a single complex variable feature most prominently, as opposed to functions of several real variables. The subject of vertex algebras is well-developed. Some, but not all, aspects of it have been reformulated in terms of the geometry of \mathbf{C} , the plane of complex numbers, or other Riemann surfaces. Factorization algebras as developed by Costello and Gwilliam [2] are a more general approach to quantum field theory which applies to all kinds of geometries, including higher dimensional manifolds. This thesis compares vertex algebras and factorization algebras on \mathbf{C} .

Many models of chiral conformal field theory have been constructed as vertex algebras. Some of these have been constructed as factorization algebras, too. Costello and Gwilliam describe a procedure to obtain vertex algebras from suitable factorization algebras on \mathbf{C} . I provide a one-sided inverse to this procedure.

Theorem. *If \mathcal{V} is a vertex algebra, then there is a factorization algebra \mathbf{FV} on \mathbf{C} whose associated vertex algebra is isomorphic to \mathcal{V} .*

In particular, every vertex algebra arises from a factorization algebra. This was known for the universal affine vertex algebras [2] and the Virasoro vertex algebra [19], but not, for example, for the simple affine vertex algebras, the irreducible quotients of the universal affine vertex algebras.

This construction of a factorization algebra starting from a vertex algebra was suggested to the author by André Henriques who was inspired by work of Huang. In [12, 13], Huang studies locally convex completions of the underlying vector space of a vertex algebra \mathcal{V} . These locally convex completions are algebras over the E_2 -operad of little discs and their multiplication maps are related to the vertex operators of \mathcal{V} .

A factorization algebra F on \mathbf{C} assigns a vector space $F(U)$ to every open subset $U \subseteq \mathbf{C}$ and extension maps $F(U) \rightarrow F(V)$ for inclusions $U \subseteq V$ of open subsets. These extension maps assemble into a precosheaf. It is part of the definition of a factorization algebra that this precosheaf is a cosheaf for certain open covers called *Weiss covers*. Furthermore, a factorization algebra on \mathbf{C} has isomorphisms

$$F(U) \otimes F(V) \cong F(U \sqcup V)$$

for U, V disjoint open subsets of \mathbf{C} . So far, we have described factorization algebras on \mathbf{C} with values in the symmetric monoidal category of vector spaces. We have found it convenient to consider factorization algebras with values in the symmetric monoidal category of complete bornological vector spaces with the symmetric monoidal product given by the completed tensor product of bornological spaces.

Let \mathfrak{g} be a finite-dimensional Lie algebra over \mathbf{C} and κ a symmetric invariant form on \mathfrak{g} . If $V_{\mathfrak{g}, \kappa}$ denotes the corresponding universal affine vertex algebra, it is not known if $\mathbf{FV}_{\mathfrak{g}, \kappa}$ is isomorphic to the universal affine factorization algebra $F_{\mathfrak{g}, \kappa}$ from [2], called the Kac-Moody factorization algebra there and denoted \mathcal{F}^κ . These factorization algebras take values in the category of chain complexes of differentiable vector spaces. However, on open subsets of \mathbf{C} ,

as opposed to arbitrary Riemann surfaces, \mathcal{F}^κ is concentrated in degree zero by the arguments of [2]. We conjecture that $\mathbf{FV}_{\mathfrak{g},\kappa} \cong F_{\mathfrak{g},\kappa}$ as prefactorization algebras of differentiable chain complexes. Alternatively, we may form the zeroth homology of their construction as a complete bornological space and hope for an isomorphism of factorization algebras with values in the category of bornological spaces.

We now summarize the contents of this document's sections. The introduction of each section contains further remarks on other people's related work.

In Section 2, we recall the definitions of \mathbf{Z} -graded vertex algebras and geometric vertex algebras. Given a \mathbf{Z} -graded vector space, the set of vertex algebra structures on it is in bijection with the set of geometric vertex algebra structures. This bijection summarizes a number of basic facts about vertex algebras. Geometric vertex algebras are more similar to factorization algebras than vertex algebras, and our construction of a factorization algebra from a vertex algebra is phrased entirely in terms of its geometric vertex algebra.

In Section 3, we define holomorphic prefactorization algebras taking values in the category of complete bornological vector spaces. To do so, we first summarize various basic facts about complete bornological vector spaces. We then describe the constructions \mathbf{F} and \mathbf{V} going between geometric vertex algebras and holomorphic prefactorization algebras with discrete weight spaces and meromorphic operator product expansion. We check that \mathbf{FV} is in fact such a prefactorization algebra and that $\mathbf{VFV} \cong \mathcal{V}$ for every geometric vertex algebra \mathcal{V} .

In Section 4, we recall the definition of a factorization algebra and prove that \mathbf{FV} is a factorization algebra.

Acknowledgments. I would like to thank my advisors Peter Teichner and André Henriques for their advice, support, suggestions, and encouragement. Thank you, André, for the very large number of online conversations we had over the last years. Furthermore, thanks go to Bertram Arnold, Christian Blohmann, Kevin Costello, Josua Faller, Vassili Gorbounov, Owen Gwilliam, Malte Lackmann, Jack Kelly, Achim Krause, Mathoverflow user user48958, Mathoverflow user user131781, Eugene Rabinovich, Ingo Runkel, Claudia Scheimbauer, Jan Steinebrunner, Christoph Weis, Katrin Wendland, Jochen Wengenroth, Brian Williams, Mahmoud Zeinalian, and Tomas Zeman for helpful conversations or correspondence. Thanks to Eugene Rabinovich for proofreading a draft. Thanks to Prof. Stefan Schwede and Prof. Michael Köhl for agreeing to be on my thesis committee. During my studies, I visited UC Berkeley and the University of Oxford, and I would like to thank my advisors, these institutions, and their staff for making this possible. Most of this work was carried out at the Max Planck Institute for Mathematics in Bonn and financially supported by its IMPRS on Moduli Spaces, and I would like to thank MPIM and its wonderful staff. I thank my friends and family, and in particular my parents, for their support and encouragement.

2 Geometric Vertex Algebras

A theorem of Huang in [11] establishes the equivalence of vertex operator algebras and geometric vertex operator algebras. We give a self-contained account of this theorem in the simplified setting of vertex algebras and geometric vertex algebras, that is, without the infinitesimal conformal symmetries given by the Virasoro algebra. Geometric vertex algebras are equivalent to \mathbf{Z} -graded vertex algebras.

Theorem 2.0.1. *The set of \mathbf{Z} -graded vertex algebra structures on a fixed \mathbf{Z} -graded vector space is in natural bijection with the set of geometric vertex algebra structures.*

Both geometric vertex algebras and \mathbf{Z} -graded vertex algebras have an underlying \mathbf{Z} -graded vector space \mathcal{V} over \mathbf{C} . In a geometric vertex algebra, the multiplication maps μ takes elements $a_1, \dots, a_n \in \mathcal{V}$ placed at pairwise distinct point $z_1, \dots, z_n \in \mathbf{C}$, and the product $\mu(a)(z)$ is an element of $\bar{\mathcal{V}} := \prod_{k \in \mathbf{Z}} \mathcal{V}_k$ which contains \mathcal{V} as a subspace.

Related work. The meaning of the above theorem is very close to that of Theorem 2.12 in Runkel's lecture notes [17]. The holomorphic integral scale covariant field theories of [17] have essentially the same data as geometric vertex algebras. The axioms only differ in the kind of convergence in the infinite sum in the associativity property. The definition of geometric vertex algebras in this article uses locally normal convergence in its associativity axiom, as opposed to pointwise convergence as in [17], seen to imply pointwise absolute convergence there. We show that the geometric vertex algebra constructed from a \mathbf{Z} -graded vertex algebra always has locally normal convergence in its associativity property.

The analogous theorem of Huang is Theorem 5.4.5 in [11]. Costello and Gwilliam construct \mathbf{Z} -graded vertex algebras from certain holomorphic factorization algebras on \mathbf{C} . Our treatment of obtaining a vertex algebra from a geometric vertex algebra is modeled on their work, geometric vertex algebras serve as an intermediary between vertex algebras and factorization algebras.

2.1 Definitions and the Bijection

Unless stated otherwise, all vector spaces are over the field \mathbf{C} of complex numbers. Geometric vertex algebras have an n -ary operation parametrized by distinct points in \mathbf{C} . Let $z = (z_1, \dots, z_n) \in \mathbf{C}^n \setminus \Delta$ where

$$\Delta = \Delta_n = \{(z_1, \dots, z_n) \in \mathbf{C}^n \mid z_i = z_j \text{ for some } i, j \in \{1, \dots, n\} \text{ with } i \neq j\}.$$

for every natural number $n \geq 0$, we write Δ instead of Δ_n when n is apparent from the context. The result of the multiplying a_1, \dots, a_n at z is an element $\mu(a_1, z_1, \dots, a_n, z_n)$ of $\bar{\mathcal{V}} := \prod_{k \in \mathbf{Z}} \mathcal{V}_k$ and not necessarily its subspace \mathcal{V} , and depends holomorphically on $z \in \mathbf{C}^n \setminus \Delta$. Thus, in order to formulate associativity, we evaluate the multiplication map on infinitely many elements of \mathcal{V} , the projections to \mathcal{V}_k for $k \in \mathbf{Z}$ of the intermediate result, and then re-assemble the result of multiplying with some additional elements by using an infinite sum. Before giving the definition of a geometric vertex algebra, we explain the relevant notions of holomorphicity and convergence. Both are based on considering finite-dimensional subspaces of the \mathcal{V}_k for all $k \in \mathbf{Z}$, a reflection of the rather algebraic nature of vertex algebras.

Definition 2.1.1. Let $U \subseteq \mathbf{C}^n$ and X be a vector space. A map $f : U \rightarrow X$ is *holomorphic* if f is locally a holomorphic function with values in a finite-dimensional subspace X . This means that every point $p \in U$ has an open neighborhood V together with a finite-dimensional subspace $Y \subseteq X$ with $f(V) \subseteq Y$ and $f|_V : V \rightarrow Y$ holomorphic. If \mathcal{V} is a \mathbf{Z} -graded vector space, then $\mathcal{O}(U; \bar{\mathcal{V}})$ denotes the vector space of $\bar{\mathcal{V}}$ -valued functions on U each of whose components is holomorphic.

The condition that $f|_V$ be holomorphic is independent of the choice of Y . It can be shown that $f(U) \subseteq Y$ for all Y as above if U is connected. In particular, a holomorphic function on a connected set like $\mathbf{C}^n \setminus \Delta_n$ globally takes values in some finite-dimensional subspace.

For the associativity axiom of a geometric vertex algebra, we recall that a series $\sum_{i \in I} f_i$ of holomorphic functions on an open $U \subseteq \mathbf{C}^n$ with values in a finite-dimensional normed space is called *normal* if $\sum_{i \in I} \|f_i\| < \infty$ where $\|\cdot\|$ is the supremum norm. Such a series is called *locally normal* if every point in the domain has an open neighborhood V s.t. $\sum_{i \in I} f_i|_V$ is normal, equivalently if $\sum_{i \in I} \|f_i\|_K < \infty$ for each compact subset K of the domain. The notions of normal and locally normal series are the same for every norm. Therefore, it makes sense to say that a series of holomorphic functions with values in a vector space X is locally normal:

Definition 2.1.2. Let $U \subseteq \mathbf{C}^n$ and X be a vector space. A series $\sum_{i \in I} f_i$ of holomorphic X -valued functions on an open $U \subseteq \mathbf{C}^n$ is *locally normal* if, for every $p \in U$, there is an open neighborhood V of p and a finite-dimensional subspace Y s.t. $f(V) \subseteq Y$ and $\sum_i \|f_i|_V\| < \infty$, where the supremum norm is defined with respect to a norm on Y .

Definition 2.1.3. A *geometric vertex algebra* consists of:

- A \mathbf{Z} -graded vector space $\mathcal{V} = \bigoplus_{k \in \mathbf{Z}} \mathcal{V}_k$ over \mathbf{C} .

- Linear maps $\mu : \mathcal{V}^{\otimes n} \rightarrow \mathcal{O}(\mathbf{C}^n \setminus \Delta; \bar{\mathcal{V}})$ for $n \geq 0$ where $\bar{\mathcal{V}} = \prod_{k \in \mathbf{Z}} \mathcal{V}_k$. For all n , we write

$$\mu(a)(z) = \mu(a, z) = \mu(a_1, z_1, \dots, a_n, z_n)$$

for the value of the function corresponding to $a = a_1 \otimes \dots \otimes a_n \in \mathcal{V}^{\otimes n}$ at $z \in \mathbf{C}^n \setminus \Delta$.

The axioms of a geometric vertex algebra are:

- (permutation invariance)

$$\mu(a^\sigma, z^\sigma) = \mu(a, z)$$

for $a \in \mathcal{V}^n$, $z \in \mathbf{C}^n \setminus \Delta$ and every permutation $\sigma \in \Sigma_n$.

- (insertion at zero)

$$\mu(a, 0) = a$$

for all $a \in \mathcal{V}$, where a is viewed as element of $\bar{\mathcal{V}}$ via the embedding

$$\mathcal{V} = \bigoplus_n \mathcal{V}_n \hookrightarrow \prod_n \mathcal{V}_n = \bar{\mathcal{V}}.$$

- (associativity) For $k \in \mathbf{Z}$ let

$$p_k : \bar{\mathcal{V}} \rightarrow \mathcal{V}_k$$

denote the projection. For all $a_1, \dots, a_m \in \mathcal{V}$, $b_1, \dots, b_n \in \mathcal{V}$ and $z \in \mathbf{C}^{m+1} \setminus \Delta$, $w \in \mathbf{C}^n \setminus \Delta$ with $\max_i |w_i| < \min_{1 \leq j \leq m} |z_j - z_{m+1}|$ we demand that the sum

$$\sum_{k \in \mathbf{Z}} \mu(a_1, z_1, \dots, a_m, z_m, p_k \mu(b_1, w_1, \dots, b_n, w_n), z_{m+1}) \quad (1)$$

converge in the sense that, for each $l \in \mathbf{Z}$, the components of the summands form a locally normal sum of \mathcal{V}_l -valued functions on $\mathbf{C}^n \setminus \Delta$. This limit defines the left hand side of the equality

$$\begin{aligned} & \mu(a_1, z_1, \dots, a_m, z_m, \mu(b_1, w_1, \dots, b_n, w_n), z_{m+1}) \\ &= \mu(a_1, z_1, \dots, a_m, z_m, b_1, w_1 + z_{m+1}, \dots, b_n, w_n + z_{m+1}) \end{aligned} \quad (2)$$

which we require to hold. Here $m \geq 0$ and $n \geq 0$.

- (\mathbf{C}^\times -equivariance) For all $z \in \mathbf{C}^\times$, $a_1, \dots, a_n \in \mathcal{V}$ and $w \in \mathbf{C}^n \setminus \Delta$

$$z \cdot \mu(a_1, w_1, \dots, a_n, w_n) = \mu(z \cdot a_1, z w_1, \dots, z \cdot a_n, z w_n)$$

where $z \in \mathbf{C}^\times$ acts on $\bar{\mathcal{V}}$ and \mathcal{V} via multiplication by z^l on \mathcal{V}_l for all $l \in \mathbf{Z}$.

- (meromorphicity) For all $a, b \in \mathcal{V}$, there exists an N s.t. the function $z^N \mu(a, z, b, 0)$ of $z \in \mathbf{C} \setminus \{0\}$ extends holomorphically to $z \in \mathbf{C}$.

In the associativity axiom of a geometric vertex algebra, the condition that

$$\max_i |w_i| < \min_{1 \leq j \leq m} |z_j - z_{m+1}|$$

is equivalent to saying that all $w_j + z_{m+1}$, $j = 1, \dots, n$, are contained in the largest open ball around z_{m+1} not containing any of the z_i for $i = 1, \dots, m$.

By permutation invariance the multiplication μ only depends on the set

$$\{(a_1, z_1), \dots, (a_m, z_m)\}$$

for $a_1, \dots, a_m \in \mathcal{V}$ and $z \in \mathbf{C}^m \setminus \Delta$. The image $\mu(\emptyset)$ of $1 \in \mathbf{C}$ under the multiplication map

$$\mathbf{C} = \mathcal{V}^{\otimes 0} \rightarrow \mathcal{O}(\text{pt}; \bar{\mathcal{V}}) \cong \bar{\mathcal{V}}$$

for $n = 0$ is called the *vacuum vector* $|0\rangle$ of \mathcal{V} , or unit. It is actually an element of $\mathcal{V}_0 \subseteq \mathcal{V}$ because it is invariant under the action of \mathbf{C}^\times . The case $n = 0$ of the associativity axiom implies that

$$\mu(a_1, z_1, \dots, a_m, z_m, |0\rangle, z_{m+1}) = \mu(a_1, z_1, \dots, a_m, z_m)$$

for $a_1, \dots, a_m \in \mathcal{V}$ and $z \in \mathbf{C}^{m+1} \setminus \Delta$.

The next proposition says that the action of \mathbf{C}^\times on \mathcal{V} extends to an action of the group $\mathbf{C}^\times \times \mathbf{C}$ of affine transformations of \mathbf{C} on the subspace

$$\bar{\mathcal{V}}_{bb} = \{x \in \bar{\mathcal{V}} \mid \exists C \in \mathbf{Z} \ \forall k < C : x_k = 0\}$$

of bounded below vectors. More generally, when constructing geometric vertex algebras from vertex algebras, we will show that $\mu(a_1, z_1, \dots, a_m, z_m)$ vanishes in sufficiently low degree, i. e., is an element of $\bar{\mathcal{V}}_{bb}$. We do not need the next proposition to obtain a vertex algebra; rather it explains how the $m = 0$ case of associativity can be thought of as translation invariance of μ by making translations act on $\bar{\mathcal{V}}_{bb}$ by using μ .

Proposition 2.1.4. *The vector space $\bar{\mathcal{V}}_{bb}$ is a representation of $G = \mathbf{C}^\times \times \mathbf{C}$ where $w \in \mathbf{C}$ acts as $w.x = \sum_{k \in \mathbf{Z}} \mu(p_k(x), w)$ and $\lambda \in \mathbf{C}^\times$ acts as $(\lambda.x)_k = \lambda^k x_k$ for $x \in \bar{\mathcal{V}}_{bb}$.*

Proof. For $w_1, w_2 \in \mathbf{C}$ and $x \in \bar{\mathcal{V}}_{bb}$

$$\begin{aligned} w_1.(w_2.x) &= \sum_{k_1 \in \mathbf{Z}} \mu \left(p_{k_1} \left(\sum_{k_2 \in \mathbf{Z}} \mu(p_{k_2}(x), w_2) \right), w_1 \right) \\ &= \sum_{k_1 \in \mathbf{Z}} \sum_{k_2 \in \mathbf{Z}} \mu(p_{k_1}(\mu(p_{k_2}(x), w_2)), w_1) \\ &= \sum_{k_2 \in \mathbf{Z}} \sum_{k_1 \in \mathbf{Z}} \mu(p_{k_1}(\mu(p_{k_2}(x), w_2)), w_1) \\ &= \sum_{k_2 \in \mathbf{Z}} \mu(p_{k_2}(x), w_2 + w_1) \\ &= (w_1 + w_2).x, \end{aligned} \tag{3}$$

where the sums are exchangeable because they are finite in each component of $\overline{\mathcal{V}}$ and line (3) uses the associativity of \mathcal{V} for $m = 0$. Furthermore

$$0.x = \sum_{k \in \mathbf{Z}} \mu(p_k(x), 0) = \sum_{k \in \mathbf{Z}} p_k(x) = x.$$

These actions of \mathbf{C}^\times and \mathbf{C} assemble to an action of $\mathbf{C}^\times \ltimes \mathbf{C}$ because

$$\begin{aligned} \lambda.(w.(\lambda^{-1}.x)) &= \lambda. \sum_{k \in \mathbf{Z}} \mu(p_k(\lambda.x), w) \\ &= \sum_{k \in \mathbf{Z}} \lambda.\mu(\lambda^{-1}.p_k(x), w) \\ &= \sum_{k \in \mathbf{Z}} \mu(p_k(x), \lambda w) \\ &= (\lambda w).x \end{aligned}$$

because of \mathbf{C}^\times -equivariance. \square

If \mathcal{V} is a \mathbf{Z} -graded vector space and $v \in \mathcal{V}$ is homogeneous, then $|v|$ denotes the degree of v .

Definition 2.1.5. A \mathbf{Z} -graded *vertex algebra* consists of:

- a \mathbf{Z} -graded vector space $\mathcal{V} = \bigoplus_{l \in \mathbf{Z}} \mathcal{V}_l$ over \mathbf{C} ,
- a linear map

$$\begin{aligned} Y : \mathcal{V} &\longrightarrow \text{End}\mathcal{V}[[x^{\pm 1}]] \\ a &\longmapsto Y(a, x) = \sum_{k \in \mathbf{Z}} a_{(k)} x^{-k-1}, \end{aligned}$$

where the k -th mode $a_{(k)}$ is a homogeneous endomorphism of \mathcal{V} of degree $|a| - k - 1$ for homogeneous $a \in \mathcal{V}$,

- a degree 1 endomorphism T of \mathcal{V} ,
- a vector $|0\rangle \in \mathcal{V}_0$ called the *vacuum*,

such that:

- (locality) For all $a, b \in \mathcal{V}$, there exists a natural number N such that

$$(x - y)^N [Y(a, x), Y(b, y)] = 0$$

in $\text{End}(\mathcal{V})[[x^{\pm 1}, y^{\pm 1}]]$.

- (translation) $T|0\rangle = 0$ and $[T, Y(a, x)] = \partial_x Y(a, x)$ for all $a \in \mathcal{V}$.
- (creation) $Y(a, x)|0\rangle \in a + x\mathcal{V}[[x]]$ for all $a \in \mathcal{V}$.
- (vacuum) $Y(|0\rangle, x) = \text{id}_{\mathcal{V}}$.

2.1.6. In terms of the modes of $a \in \mathcal{V}$, the translation axiom for $a \in \mathcal{V}$ is equivalent to demanding that $[T, a_{(k)}] = -ka_{(k-1)}$ for all $k \in \mathbf{Z}$. The creation axiom for a is equivalent to the equation $a_{(-1)}|0\rangle = a$ and $a_{(k)}|0\rangle = 0$ for $k \geq 0$. The vacuum axiom says that for all $a \in \mathcal{V}$ we have $|0\rangle_{(k)}a = 0$ for $k \neq -1$ and $|0\rangle_{(-1)}a = a$.

In a vertex algebra, the translation operator T and the vacuum $|0\rangle$ are uniquely determined by the vertex operators $Y(a, x)$.

Proposition 2.1.7. *Let $(\mathcal{V}, Y, T, |0\rangle)$ be a vertex algebra. Then $Ta = a_{(-2)}|0\rangle$ for all $a \in \mathcal{V}$. If $a \in \mathcal{V}$ satisfies $Y(a, x) = \text{id}_{\mathcal{V}}$, then $a = |0\rangle$.*

Proof. Let $a \in \mathcal{V}$. It follows that

$$Ta = Ta_{(-1)}|0\rangle = [T, a_{(-1)}]|0\rangle + aT|0\rangle = a_{(-2)}|0\rangle + 0$$

by the creation axiom and the translation axiom. If $Y(a, x) = \text{id}_{\mathcal{V}}$, then

$$a = Y(a, x)|0\rangle|_{x=0} = \text{id}(|0\rangle)|_{x=0} = |0\rangle$$

by the creation axiom. \square

Proposition 2.1.8. *Let \mathcal{V} be a geometric vertex algebra. Let $a \in \mathcal{V}$ and $k \in \mathbf{Z}$. The k -th mode of a is a well-defined linear map*

$$a_{(k)} : \mathcal{V} \longrightarrow \mathcal{V}$$

$$b \longmapsto a_{(k)}b := \frac{1}{2\pi i} \int_{S^1} z^k \mu(a, z; b, 0) dz.$$

If $a \in \mathcal{V}$ is homogeneous, then $a_{(k)}$ is homogeneous of degree $|a| - k - 1$. A priori, the map $a_{(k)}$ is well-defined as a map $\mathcal{V} \rightarrow \bar{\mathcal{V}}$. Recall that we identify $\mathcal{V} = \bigoplus_k \mathcal{V}_k$ with its image in $\bar{\mathcal{V}} = \prod_k \mathcal{V}_k$ under the natural injection.

Proof. Let $a, b \in \mathcal{V}$ be homogeneous. Note that $z.(a_{(k)}b) = z^{|a|-k-1+|b|}a_{(k)}b$ for all $z \in \mathbf{C}^\times$ because

$$\begin{aligned} 2\pi i z.(a_{(k)}b) &= z. \int_{S^1} w^k \mu(a, w; b, 0) dw \\ &= \int_{S^1} w^k z.\mu(a, w; b, 0) dw \\ &= \int_{S^1} w^k \mu(z.a, zw; z.b, 0) dw \\ &= z^{|a|+|b|} \int_{S^1} w^k \mu(a, zw; b, 0) dw \\ &= z^{|a|-k-1+|b|} \int_{zS^1} w^k \mu(a, w; b, 0) dw \quad (\text{substituted } w/z) \\ &= z^{|a|-k-1+|b|} \int_{S^1} w^k \mu(a, w; b, 0) dw \quad (\text{holomorphic}) \\ &= 2\pi i z^{|a|-k-1+|b|} a_{(k)}b. \end{aligned}$$

This implies that $p_l(a_{(k)}b)$ is zero if $l \neq |a| - k - 1 + |b|$. Thus $a_{(k)}b \in \mathcal{V}_{|a|-k-1+|b|} \subseteq \mathcal{V}$. Since every element of \mathcal{V} is a finite sum of homogeneous elements, it follows that $a_{(k)}b \in \mathcal{V}$ for all $a, b \in \mathcal{V}$. \square

Definition 2.1.9. Let \mathcal{V} be a \mathbf{Z} -graded vector space over \mathbf{C} . Let $A \subseteq \mathbf{C}$ be an annulus with center 0. If $f : A \rightarrow \bar{\mathcal{V}}$ is a holomorphic function, meaning in particular that it takes values in a finite-dimensional subspace in each degree, then we define the Laurent expansion $L(f)(x) \in \prod_{k \in \mathbf{Z}} (\mathcal{V}_k[[x^{\pm 1}]])$ of f on A componentwise.

For \mathcal{V} a \mathbf{Z} -graded vector space, we identify $\mathcal{V}[[x^{\pm 1}]]$ with a subspace of

$$\prod_{k \in \mathbf{Z}} (\mathcal{V}_k[[x^{\pm 1}]]) = \left(\prod_{k \in \mathbf{Z}} \mathcal{V}_k \right) [[x^{\pm 1}]] = \overline{\mathcal{V}}[[x^{\pm 1}]]$$

via the injective linear map

$$\sum_l A_l x^l \mapsto \left(\sum_l p_k(A_l) x^l \right)_{k \in \mathbf{Z}}.$$

Proposition 2.1.10. *Let \mathcal{V} be a geometric vertex algebra. For all $a, b \in \mathcal{V}$, the Laurent expansion $L[z \mapsto \mu(a, z, b, 0)](x)$ is an element of $\mathcal{V}[[x^{\pm 1}]]$ and is equal to $\sum_l a_{(l)} b x^{-l-1}$.*

Proof. By the integral formula for the coefficients of the Laurent expansion

$$\begin{aligned} L[z \mapsto \mu(a, z, b, 0)](x)_k &= \sum_{l \in \mathbf{Z}} \frac{1}{2\pi i} \int_{S^1} z^l p_k(\mu(a, z, b, 0)) dz \cdot x^{-l-1} \\ &= \sum_{l \in \mathbf{Z}} p_k \left(\frac{1}{2\pi i} \int_{S^1} z^l \mu(a, z, b, 0) dz \right) x^{-l-1} \\ &= \sum_{l \in \mathbf{Z}} p_k(a_{(l)} b) x^{-l-1}. \end{aligned}$$

For fixed $l \in \mathbf{Z}$, the sum $\sum_k p_k(a_{(l)} b)$ is finite and equals $a_{(l)} b$, so $\sum_l a_{(l)} b x^{-l-1}$ is the desired preimage. \square

We are now ready to state a more detailed version of Theorem 2.0.1.

Theorem 2.1.11. *Let \mathcal{V} be a \mathbf{Z} -graded vector space over \mathbf{C} . There is a bijective map from the set of geometric vertex algebra structures on \mathcal{V} to the set of \mathbf{Z} -graded vertex algebra structures on \mathcal{V} defined by*

$$\begin{aligned} Y(a, x)b &= L[z \mapsto \mu(a, z, b, 0)](x) \quad (\in \mathcal{V}[[x^{\pm 1}]]) \\ Ta &= \partial_z \mu(a, z)|_{z=0} \\ |0\rangle &= \mu(\emptyset) \end{aligned}$$

for all $a, b \in \mathcal{V}$. The inverse to this bijection is uniquely determined by the equation

$$\mu(a_1, z_1, \dots, a_m, z_m) = \sum_{k_1 \in \mathbf{Z}} (a_1)_{(k_1)} z^{-k_1-1} \dots \sum_{k_m \in \mathbf{Z}} (a_m)_{(k_m)} z^{-k_m-1} |0\rangle$$

for $a_1, \dots, a_m \in \mathcal{V}$ and $z \in \mathbf{C}^m \setminus \Delta$ with $|z_1| > \dots > |z_m|$.

The r. h. s. above converges absolutely in each component of $\overline{\mathcal{V}}$ by Proposition 2.3.2.

2.2 From Geometric Vertex Algebras to Vertex Algebras

Given a geometric vertex algebra, we construct a \mathbf{Z} -graded vertex algebra with the same underlying vector space.

Proposition 2.2.1. *Let \mathcal{V} be a geometric vertex algebra. Let $a_1, \dots, a_m \in \mathcal{V}$ and $i, j \in \{1, \dots, m\}$ with $i < j$. For all $(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_m) \in \mathbf{C}^{m-1} \setminus \Delta$ and $\varepsilon > 0$ with $\varepsilon < |z_l - z_j|$ for all $l \neq i, j$,*

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\partial B_\varepsilon(z_j)} (z_i - z_j)^k \mu(a, z) dz_i \\ &= \mu(a_1, z_1, \dots, \widehat{a_i, z_i}, \dots, a_{j-1}, z_{j-1}, a_{i(k)} a_j, z_j, a_{j+1}, z_{j+1}, \dots, a_m, z_m). \end{aligned}$$

Proof. Using permutation invariance and associativity

$$\mu(a, z) = \sum_{l \in \mathbf{Z}} \mu(\dots, \widehat{a_i, z_i}, \dots, p_l \mu(a_i, z_i - z_j, a_j, 0), z_j, \dots)$$

and convergence is normal as functions of z_i on $\partial B_\varepsilon(z_j)$ in each component of $\bar{\mathcal{V}}$ since it is locally normal by the associativity axiom and $\partial B_\varepsilon(z_j)$ is compact. It follows that we can exchange integration over $\partial B_\varepsilon(z_j)$ with summation to get

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\partial B_\varepsilon(z_j)} (z_i - z_j)^k \mu(a, z) dz_i \\ &= \sum_{l \in \mathbf{Z}} \mu(\dots, \widehat{a_i, z_i}, \dots, p_l \frac{1}{2\pi i} \int_{\partial B_\varepsilon(z_j)} (z_i - z_j)^k \mu(a_i, z_i - z_j, a_j, 0) dz_i, z_j, \dots) dz_i. \end{aligned}$$

Here, we may move the integral into the argument of μ and p_l because these maps are linear and the relevant functions take values in finite-dimensional subspaces. Shifting the contour integral to zero and noting that it does not matter whether we integrate around a circle of radius ε or 1 as in the definition of the modes, we get

$$\begin{aligned} \frac{1}{2\pi i} \int_{\partial B_\varepsilon(z_j)} (z_i - z_j)^k \mu(a_i, z_i - z_j, a_j, 0) dz_i &= \frac{1}{2\pi i} \int_{\partial B_\varepsilon(0)} w^k \mu(a_i, w, a_j, 0) dw \\ &= a_{i(k)} a_j \end{aligned}$$

and thus

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\partial B_\varepsilon(z_j)} (z_i - z_j)^k \mu(a, z) dz_i = \sum_{l \in \mathbf{Z}} \mu(\dots, \widehat{a_i, z_i}, \dots, p_l a_{i(k)} a_j, z_j, \dots) \\ &= \mu(\dots, \widehat{a_i, z_i}, \dots, \sum_{l \in \mathbf{Z}} p_l a_{i(k)} a_j, z_j, \dots) = \mu(\dots, \widehat{a_i, z_i}, \dots, p_l a_{i(k)} a_j, z_j, \dots) \end{aligned}$$

because the sum is finite. \square

For $i, j \in \{1, \dots, m\}$ with $i < j$, let U_{ij} be the set of $z \in \mathbf{C}^m \setminus \Delta$ with $|z_i - z_j| < \min_{l \neq i, j} |z_l - z_j|$. The following proposition is the analogue of Proposition 5.3.6 from [2, p. 167] for geometric vertex algebras. It expresses products for z_i close to z_j in terms of a series in $z_i - z_j$ and other vertex algebra elements inserted at z_j and goes by the name of *operator product expansion (OPE)*.

Proposition 2.2.2. *Let \mathcal{V} be a geometric vertex algebra and let $a_1, \dots, a_m \in \mathcal{V}$. For $z \in U_{ij}$*

$$\mu(a_1, z_1, \dots, a_m, z_m) = \sum_{k \in \mathbf{Z}} \mu(\dots, \widehat{a_i, z_i}, \dots, a_{i(k)} a_j, z_j, \dots) (z_i - z_j)^{-k-1}$$

with locally normal convergence.

Proof. For fixed $z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_m$ the l. h. s. is a holomorphic function of z_i such that $z \in U_{ij}$. The integral formula for the coefficients of the locally normal Laurent expansion and Proposition 2.2.1 imply

$$\begin{aligned} \mu(a, z) &= \sum_{k \in \mathbf{Z}} \frac{1}{2\pi i} \int_{\partial B_\varepsilon(z_j)} (w - z_j)^k \mu(\dots, a_i, w, \dots, a_j, z_j, \dots) dw (z_i - z_j)^{-k-1} \\ &= \sum_{k \in \mathbf{Z}} \mu(\dots, \widehat{a_i, z_i}, \dots, a_{i(k)} a_j, z_j, \dots) (z_i - z_j)^{-k-1}. \end{aligned}$$

□

Proposition 2.2.3. *Assume that the underlying datum (\mathcal{V}, μ) of a geometric vertex algebra satisfies all the axioms of a geometric vertex algebra except meromorphicity. If \mathcal{V} is bounded from below, then \mathcal{V} is meromorphic.*

Proof. If \mathcal{V} is bounded from below, then meromorphicity follows because

$$\mu(b, w, c, 0) = \sum_{k \in \mathbf{Z}} \mu(b_{(k)} c, 0) w^{-k-1} = \sum_k b_{(k)} c w^{-k-1}$$

and $b_{(k)} c$ is zero for k large enough for degree reasons. □

The map of Theorem 2.1.11 is well-defined:

Proposition 2.2.4. *If \mathcal{V} is a geometric vertex algebra, then \mathcal{V} forms a \mathbf{Z} -graded vertex algebra with the above Y, T and $|0\rangle$.*

Proof. Fix μ such that (\mathcal{V}, μ) is a geometric vertex algebra. Our goal is to check that $(\mathcal{V}, Y, T, |0\rangle)$ as defined above is a vertex algebra. We determine the degrees of T and $|0\rangle$ using the action of \mathbf{C}^\times and the equivariance axiom similarly to how we determined the degrees of the modes in Proposition 2.1.8: For all $z \in \mathbf{C}^\times$,

$$z \cdot |0\rangle = z \cdot \mu(\emptyset) = \mu(\emptyset) = |0\rangle$$

so $|0\rangle \in \mathcal{V}_0 \subseteq \bar{\mathcal{V}}$ since this shows that $p_l |0\rangle = 0$ for $l \neq 0$. For all $z \in \mathbf{C}^\times$ and $a \in \mathcal{V}$ homogeneous, we have

$$\begin{aligned} z \cdot T a &= z \cdot \partial_w \mu(a, w)|_{w=0} = \partial_w z \cdot \mu(a, w)|_{w=0} = \partial_w \mu(z \cdot a, z w)|_{w=0} \\ &= \partial_w \mu(z^{|a|} a, z w)|_{w=0} = z^{|a|+1} \partial_w \mu(a, w)|_{w=0} = z^{|a|+1} T a \end{aligned}$$

so $T a$ is concentrated in degree $|a| + 1$ and T has degree 1. It follows that the image of T is a subset of \mathcal{V} .

Locality: Let $a, b, c \in \mathcal{V}$. It suffices to treat the case of a, b, c homogeneous. Applying Proposition 2.2.2 twice, we find

$$\mu(a, z, b, w, c, 0) = \sum_l \mu(a, z, b_{(l)}c w^{-l-1}, 0) = \sum_l \sum_k a_{(k)}b_{(l)}c z^{-k-1}w^{-l-1}$$

with locally normal convergence for $|z| > 1 > |w| > 0$. Let

$$A_{r,R} = \{z \in \mathbf{C} \mid r < |z| < R\}$$

for $0 \leq r, R \leq \infty$. Using a similar notation for the Laurent expansion of functions of several variables on products of annuli, we get

$$L_{(z,w) \in A_{1,2} \times A_{0,1}} \mu(a, z, b, w, c, 0)(x, y) = Y(a, x)Y(b, y)c$$

and analogously

$$L_{(z,w) \in A_{0,1} \times A_{1,2}} \mu(a, z, b, w, c, 0)(x, y) = Y(b, y)Y(a, x)c.$$

For all $N \in \mathbf{N}$

$$\begin{aligned} L_{(z,w) \in A_{1,2} \times A_{0,1}} (z-w)^N \mu(a, z, b, w, c, 0)(x, y) &= (x-y)^N Y(a, x)Y(b, y)c \\ L_{(z,w) \in A_{0,1} \times A_{1,2}} (z-w)^N \mu(a, z, b, w, c, 0)(x, y) &= (x-y)^N Y(b, y)Y(a, x)c \end{aligned}$$

because Laurent expansion intertwines the action of polynomials as functions with the action of polynomials as formal polynomials. For N large enough, the function $(z-w)^N \mu(a, z, b, w, c, 0)$ of $(z, w) \in A_{0,2} \times A_{0,2} \setminus \Delta$ extends to a holomorphic function of $(z, w) \in A_{0,2} \times A_{0,2}$ as a consequence of Proposition 2.2.2 about the OPE and meromorphicity. If $F \in \mathcal{O}(A_{0,2} \times A_{0,2})$, then $L_{A_{1,2} \times A_{0,1}} F = L_{A_{0,1} \times A_{1,2}} F$ in $\mathbf{C}[[x^{\pm 1}, y^{\pm 1}]]$, and therefore

$$(x-y)^N Y(a, x)Y(b, y)c = (x-y)^N Y(b, y)Y(a, x)c$$

in $\mathcal{V}[[x^{\pm 1}, y^{\pm 1}]]$.

Vacuum: Since

$$\mu(|0\rangle, z, a, 0) = \mu(\mu(\emptyset), z, a, 0) = \mu(a, 0) = a$$

by the definition of $|0\rangle$, associativity, and insertion at zero, we have that

$$|0\rangle_{(k)} a = \frac{1}{2\pi i} \int_{S^1} z^k \mu(|0\rangle, z, a, 0) dz = \frac{1}{2\pi i} \int_{S^1} z^k a dz = \delta_{k,-1} a$$

for all $a \in \mathcal{V}$ and $k \in \mathbf{Z}$.

Creation: Let $a \in \mathcal{V}$. By associativity $\mu(a, z, |0\rangle, 0) = \mu(a, z)$ and this is a holomorphic function of $z \in \mathbf{C}$. Therefore its Laurent expansion has no negative powers and its zeroth term is $\mu(a, 0) = a$ because of the axiom about insertion at zero.

Translation: Let $T : \bar{\mathcal{V}} \rightarrow \bar{\mathcal{V}}$ be defined by $Tx = \sum_k Tp_kx$. This sum is finite in each degree, and T is linear. Let $a, b \in \mathcal{V}$. The identity

$$[T, Y(a, x)]b = TY(a, x)b - Y(a, x)Tb = \partial_x Y(a, x)b.$$

is implied by

$$T\mu(a, z, b, 0) - \mu(a, z, Tb, 0) = \partial_z \mu(a, z, b, 0) \quad (4)$$

because Laurent expansion is compatible with linear maps and with differentiation. Recall that $Tc = \partial_w \mu(c, w)|_{w=0}$ for $c \in \mathcal{V}$. We may assume that w is close to zero to compute the w -derivative.

$$\mu(a, z, Tb, 0) = \mu(a, z, \sum_k p_k Tb, 0) \quad (5)$$

$$= \sum_k \mu(a, z, p_k \partial_w \mu(b, w)|_{w=0}, 0) \quad (6)$$

$$= \sum_k \partial_w \mu(a, z, p_k \mu(b, w), 0)|_{w=0} \quad (7)$$

$$= \left[\partial_w \sum_k \mu(a, z, p_k \mu(b, w), 0) \right]_{w=0} \quad (8)$$

$$= \partial_w \mu(a, z, b, w)|_{w=0} \quad (9)$$

Equation (9) follows from associativity which implies that the sum in (8) is locally normally convergent, and this implies that we can commute the sum and the derivative in Equation (8), so the sum in (7) is locally normal. Equation (7) uses p_k is linear and that μ is linear in each argument from \mathcal{V} . The sums in (5) and (6) are finite. Similarly,

$$\begin{aligned} T\mu(a, z, b, 0) &= \sum_k \partial_w \mu(p_k \mu(a, z, b, 0), w)|_{w=0} \\ &= \partial_w \sum_k \mu(p_k \mu(a, z, b, 0), w)|_{w=0} \\ &= \partial_w \mu(a, z + w, b, w)|_{w=0} \end{aligned}$$

and thus

$$\begin{aligned} T\mu(a, z, b, 0) - \mu(a, z, Tb, 0) &= \partial_w \mu(a, z + w, b, w)|_{w=0} - \partial_w \mu(a, z, b, w)|_{w=0} \\ &= \partial_w \mu(a, z + w, b, 0)|_{w=0} \\ &= \partial_z \mu(a, z, b, 0), \end{aligned}$$

which is Equation (4). □

2.3 From Vertex Algebras to Geometric Vertex Algebras

Proposition 2.3.1. *Let \mathcal{V} be a \mathbf{Z} -graded vertex algebra. For all $a, b \in \mathcal{V}$, there exists a number N s. t. $a_{(n)}b = 0$ if $n \geq N$.*

This follows for degree reasons in the bounded below case. The proof given below does not use this assumption and serves a warm-up for the construction of μ .

Proof. Let $a, b \in \mathcal{V}$. By the locality axiom, there exists an N such that

$$(x - y)^N Y(a, x)Y(b, y)|0\rangle = (x - y)^N Y(b, y)Y(a, x)|0\rangle. \quad (10)$$

By the creation axiom for b , we may set $y = 0$ on the l. h. s. and get

$$x^N Y(a, x)b = \sum_n a_{(n)}bx^{N-n-1}.$$

Using Equation (10) and the creation axiom for a on the r. h. s., it follows that the l. h. s. contains no negative powers of x , even after setting $y = 0$. This means that $a_{(n)}b = 0$ for $N - n - 1 < 0$, equivalently $n \geq N$. \square

Proposition 2.3.2. *Let \mathcal{V} be a \mathbf{Z} -graded vertex algebra. Fix an integer $m \geq 0$. For $a_1, \dots, a_m \in \mathcal{V}$ and $z \in \mathbf{C}^m \setminus \Delta$ the series*

$$Y(a_1, z_1) \dots Y(a_m, z_m)|0\rangle := \sum_{k_1 \in \mathbf{Z}} (a_1)_{(k_1)} z_1^{-k_1-1} \dots \sum_{k_m \in \mathbf{Z}} (a_m)_{(k_m)} z_m^{-k_m-1}|0\rangle$$

converges locally normally in each component of $\bar{\mathcal{V}}$ for $z \in D_m$, where

$$D_m := \{z \in \mathbf{C}^m \mid |z_1| > \dots > |z_m|\} \subseteq \mathbf{C}^m \setminus \Delta.$$

As a function of z , the value of this series extends to a unique holomorphic $\bar{\mathcal{V}}$ -valued function

$$\mu(a, z) = \mu(a_1, z_1, \dots, a_m, z_m)$$

of $z \in \mathbf{C}^m \setminus \Delta$. It satisfies $\mu(a^\sigma, z^\sigma) = \mu(a, z)$ for every permutation $\sigma \in \Sigma_m$. Furthermore, $\mu(a)$ is identically zero in sufficiently low degree for every $a \in \mathcal{V}^{\otimes m}$.

This proposition and its proof are very similar to what is found in [3, 1.2 and 4.5], [11, 5.3], [4, A.2], [5, 3.5.1], which treat the bounded below, degreewise finite-dimensional case in these parts.

Proof. We may assume that a_1, \dots, a_m are homogeneous. Let

$$f(a, x) \in \mathcal{V}[[x_1^{\pm 1}, \dots, x_m^{\pm 1}]]$$

denote the formal series corresponding to the sum in the claim. By the creation axiom for a_m , there are no negative powers of x_m in $f(a, x)$. The locality axiom for a_i and a_j implies that there is a natural number N_{ij} with

$$(x_i - x_j)^{N_{ij}} [Y(a_i, x_j), Y(a_j, x_j)] = 0$$

for $1 \leq i < j \leq m$. Let

$$g(x) = \prod_{i < j} (x_i - x_j)^{N_{ij}} \quad (11)$$

and it follows that

$$g(x)f(a^\sigma, x^\sigma) = g(x)f(a, x) \quad (12)$$

for all permutations $\sigma \in \Sigma_m$. This can be checked by considering the special case of σ a transposition. Equation 12 and the creation axiom for a_i imply that there are no negative powers of x_i in $g(x)f(a, x)$ for any $i = 1, \dots, m$. Let $l \in \mathbf{Z}$. The \mathcal{V}_l -component $p_l(g(x)f(a, x))$ has no negative powers of x_i for any $i = 1, \dots, n$. We claim that $p_l(g(x)f(a, x))$ is a polynomial in the x_i with coefficients in \mathcal{V}_l . Let $r_\beta \in \mathbf{Z}$ be the coefficient of x^β in $g(x)$. The coefficient of x^α in

$$p_l(g(x)f(a, x)) = \sum_{\alpha \in \mathbf{Z}^m} \left(\sum_{\beta+k=\alpha} r_\beta p_l((a_1)_{(k_1)} \dots (a_m)_{(k_m)} | 0) \right) x^\alpha \quad (13)$$

is zero if $\alpha_i < 0$ for some $i = 1, \dots, m$. Let $\alpha \in \mathbf{Z}^m$ be such that the coefficient of x^α is not zero. There is at least one pair (β, k) with $\beta + k = \alpha$ and

$$r_\beta p_l((a_1)_{(k_1)} \dots (a_m)_{(k_m)} | 0) \neq 0.$$

This implies that $r_\beta \neq 0$ and $p_l((a_1)_{(k_1)} \dots (a_m)_{(k_m)} | 0) \neq 0$. Thus β is in a fixed finite subset of \mathbf{Z}^m depending only on the N_{ij} , and k satisfies

$$l = |(a_1)_{(k_1)} \dots (a_m)_{(k_m)} | 0| = |a_1| - k_1 - 1 + \dots + |a_m| - k_m - 1, \quad (14)$$

so

$$\sum_{i=1}^m \alpha_i = \sum_{i=1}^m (\beta_i + k_i) = \sum_{i=1}^m (\beta_i + |a_i|) - n - l.$$

For fixed β , the set of such α is a hyperplane intersecting $\mathbf{Z}_{\geq 0}^m$ in finitely many points. It follows that α is in a finite subset of \mathbf{Z}^m depending only on the N_{ij} . In summary, each \mathcal{V}_l -component of $g(x)f(a, x)$ is a polynomial in the x_i with coefficients in \mathcal{V}_l . Thus

$$F_a(z) = [g(x)f(a, x)]_{x=z}$$

defines a holomorphic function $F_a : \mathbf{C}^m \rightarrow \bar{\mathcal{V}}$. Let $A \subseteq D_m$ be a product of annuli centered at 0. Let

$$L : \mathcal{O}(A; \bar{\mathcal{V}}) \longrightarrow \prod_k \mathcal{V}_k[[x_1^{\pm 1}, \dots, x_m^{\pm 1}]]$$

denote the Laurent expansion map on A . The Laurent expansion on A of holomorphic functions defined on D_m is independent of the choice of A . In particular, the Laurent expansion $L(F_a)$ of F_a on A is independent of the choice of A and satisfies $L(F_a)(x) = g(x)f(a, x)$. Then

$$\mu(a_1, z_1, \dots, a_m, z_m) = g(z)^{-1} F_a(z) \quad (z \in \mathbf{C}^m \setminus \Delta)$$

defines a holomorphic function on $\mathbf{C}^m \setminus \Delta$ with values in $\overline{\mathcal{V}}$ whose Laurent expansion on A , again independent of the choice of A , is equal to $f(a, x)$ because

$$\begin{aligned} L[z \mapsto \mu(a_1, z_1, \dots, a_n, z_n)](x) &= L(F_a/g)(x) = L(1/g)(x)L(F_a)(x) \\ &= L(1/g)(x)g(x)f(a, x) = L(1/g)(x)L(g)(x)f(a, x) = L(1)(x)f(a, x) = f(a, x). \end{aligned}$$

Thus $f(a, z)$ converges locally normally for $z \in A$ and f and μ agree on A . For every $z \in D_m$ there is an A as above with $z \in A$, so $f(a, z)$ converges locally normally for all $z \in D_m$ and f and μ agree on all of D_m . The uniqueness of μ follows by the identity theorem for analytic functions because $\mathbf{C}^m \setminus \Delta$ is connected, $D_m \subseteq \mathbf{C}^m$ is open and non-empty, and $\mu(a, z) = f(a, z)$ for $z \in D_m$. Note that the identity theorem for holomorphic functions with values in a finite-dimensional vector space applies here because there is a finite-dimensional subspace of \mathcal{V}_l for each $l \in \mathbf{Z}$ to which the function $z \mapsto p_l(\mu(a, z))$ maps holomorphically.

Proof of permutation invariance: Let $\sigma \in \Sigma_m$. We now write g_N for g to make the dependence of g on the N_{ij} explicit. Let \tilde{N} denote a choice like N for a^σ instead of a (e. g., $\tilde{N}_{ij} = N_{\sigma(i)\sigma(j)}$). Our goal is the equality of

$$\mu(a, z) = [g_N(x)f(a, x)]_{x=z}g_N(z)^{-1}$$

with

$$\begin{aligned} \mu(a^\sigma, z^\sigma) &= [g_{\tilde{N}}(x), f(a^\sigma, x)]_{x=z^\sigma}g_{\tilde{N}}(z^\sigma)^{-1} \\ &= [g_{\tilde{N}}(x^\sigma), f(a^\sigma, x^\sigma)]_{x=z}g_{\tilde{N}}(z^\sigma)^{-1} \end{aligned}$$

for $z \in \mathbf{C}^m \setminus \Delta$. Starting with Equation (12), it follows that

$$\begin{aligned} g_N(x)f(a, x) &= g_N(x)f(a^\sigma, x^\sigma) \\ g_{\tilde{N}}(x^\sigma)g_N(x)f(a, x) &= g_N(x)g_{\tilde{N}}(x^\sigma)f(a^\sigma, x^\sigma) \\ [g_{\tilde{N}}(x^\sigma)g_N(x)f(a, x)]_{x=z} &= [g_N(x)g_{\tilde{N}}(x^\sigma)f(a^\sigma, x^\sigma)]_{x=z} \\ g_{\tilde{N}}(z^\sigma)[g_N(x)f(a, x)]_{x=z} &= g_N(z)[g_{\tilde{N}}(x^\sigma)f(a^\sigma, x^\sigma)]_{x=z} \\ [g_N(x)f(a, x)]_{x=z}g_N(z)^{-1} &= [g_{\tilde{N}}(x^\sigma)f(a^\sigma, x^\sigma)]_{x=z}g_{\tilde{N}}(z^\sigma)^{-1} \\ \mu(a, z) &= \mu(a^\sigma, z^\sigma). \end{aligned}$$

We now prove that $\mu(a, z)$ is zero in sufficiently low degree. It suffices to prove that $p_l(g(x)f(a, x))$ is zero for l low enough. This formal series in the x_i with coefficients in \mathcal{V}_l has no negative powers of x_i for any i . Therefore, for a monomial in (13) with non-zero coefficient, the $-k_i - 1$ in Equation 14 are bounded from below, so l is as well, and the bound is independent of z . \square

Proposition 2.3.3. *The pair (\mathcal{V}, μ) satisfies the axioms of a geometric vertex algebra.*

Proof. Let $m \geq 0$. Both the multilinearity and \mathbf{C}^\times -equivariance may be checked on D_m by the uniqueness of analytic continuations from D_m to $\mathbf{C}^m \setminus \Delta$. On D_m , we can express μ in terms of Y and multilinearity follows from the fact that Y is linear and composition in $\text{End}(\mathcal{V})$ is bilinear. For \mathbf{C}^\times -equivariance, it suffices

to consider homogeneous $a_1, \dots, a_m \in \mathcal{V}$: for all $z \in \mathbf{C}^\times$ and $w \in D_m$,

$$\begin{aligned}
& z.\mu(a, w) \\
&= \sum_{k_1, \dots, k_m \in \mathbf{Z}} z \cdot \left((a_1)_{(k_1)} w_1^{-k_1-1} \dots (a_m)_{(k_m)} w_m^{-k_m-1} |0\rangle \right) \\
&= \sum_{k_1, \dots, k_m \in \mathbf{Z}} z^{|a_1|-k_1-1+\dots+|a_m|-k_m-1} (a_1)_{(k_1)} w_1^{-k_1-1} \dots (a_m)_{(k_m)} w_m^{-k_m-1} |0\rangle \\
&= \sum_{k_1, \dots, k_m \in \mathbf{Z}} (z.a_1)_{(k_1)} (zw_1)^{-k_1-1} \dots (z.a_m)_{(k_m)} (zw_m)^{-k_m-1} |0\rangle \\
&= \mu(z.a_1, zw_1, \dots, z.a_m, zw_m).
\end{aligned}$$

The axiom about insertion at zero follows from the creation axiom of the vertex algebra: for all $a \in \mathcal{V}$,

$$\mu(a, 0) = Y(a, 0)|0\rangle = a.$$

To see that the meromorphicity axiom is satisfied, let N be the maximum of the N_{ij} from the construction of μ , after fixing some elements of \mathcal{V} . By construction, we then have the following strong version of meromorphicity:

- For all $a_1, \dots, a_n \in \mathcal{V}$, there is a natural number N such that the function

$$z \mapsto \prod_{i < j} (z_i - z_j)^N \mu(a_1, z_1, \dots, a_n, z_n)$$

with values in $\bar{\mathcal{V}}$ has a holomorphic extension from $\mathbf{C}^n \setminus \Delta$ to \mathbf{C}^n .

Proposition 2.3.5, proven further below, says that μ satisfies the associativity axiom. \square

The next proposition describes how μ behaves w. r. t. translations. It is used in the proof of associativity to arrange a generic position. A translation by $z \in \mathbf{C}$ acts on $\bar{\mathcal{V}}_{bb}$ via the automorphism e^{zT} of $\bar{\mathcal{V}}_{bb}$ defined by

$$e^{zT} x = \sum_{k \in \mathbf{Z}} e^{zT} p_k(x)$$

which is finite in each component of $\bar{\mathcal{V}}_{bb}$ because T has degree 1 and $x \in \bar{\mathcal{V}}_{bb}$ is zero in sufficiently low degree..

Proposition 2.3.4. *Let $b_1, \dots, b_n \in \mathcal{V}$ and $w \in \mathbf{C}^n \setminus \Delta$. For all $z \in \mathbf{C}$,*

$$e^{zT} \mu(b_1, w_1, \dots, b_n, w_n) = \mu(b_1, w_1 + z, \dots, b_n, w_n + z). \quad (15)$$

In the case $n = 1$ and $w_1 = 0$, the proposition together with insertion at zero implies that $e^{zT} b = Y(b, z)|0\rangle$ for $b \in \mathcal{V}$ and $z \in \mathbf{C}$. This means that $a_{(k)}|0\rangle = \frac{1}{(-k-1)!} T^{-k-1} a$ for $k \leq -1$, which one could have also deduced directly from the translation axiom of the vertex algebra \mathcal{V} . In particular $Ta = a_{(-2)}|0\rangle$ and $\mu(a, w) = a + Taw + \dots$. Also, since $e^{zT} a = \mu(a, z)$ for all $a \in \mathcal{V}$ and $z \in \mathbf{C}$, the proposition is the special case $m = 0$ of the associativity axiom for μ .

Proof. Both sides of the equation are holomorphic functions of $(z, w) \in \mathbf{C} \times (\mathbf{C}^n \setminus \Delta)$. By the identity theorem we may assume

$$|w_1| > \dots > |w_n|,$$

and having fixed a choice of w , we may assume z to be in the non-empty, open, and convex region B of those z such that

$$|w_1 + z| > \dots > |w_n + z|.$$

For $z \in B$, the equation is equivalent to

$$e^{zT} Y(b_1, w_1) \dots Y(b_n, w_n) |0\rangle = Y(b_1, w_1 + z) \dots Y(b_n, w_n + z) |0\rangle.$$

Let $f(z)$ equal the l. h. s. and $g(z)$ equal the r. h. s. for $z \in B$. The function f is the unique solution of the (holomorphic) initial value problem

$$\partial_z \varphi(z) = T\varphi(z) \quad \varphi(0) = Y(b_1, w_1) \dots Y(b_n, w_n) |0\rangle \quad \varphi \in \mathcal{O}(B, \overline{\mathcal{V}}_{bb}) \quad (16)$$

as can be deduced from the fact that $\varphi(z) = e^{zA}x$ is the unique solution of

$$\partial_z \varphi(z) = A\varphi(z) \quad \varphi(0) = x \quad \varphi \in \mathcal{O}(B, X). \quad (17)$$

We prove by induction on n that g is a solution of (16), too. The base case $n = 0$ holds because then $g(z) = |0\rangle$. The induction hypothesis implies

$$\partial_z Y(b_2, w_2 + z) \dots Y(b_n, w_n + z) |0\rangle = TY(b_2, w_2 + z) \dots Y(b_n, w_n + z) |0\rangle.$$

Proposition 2.3.2 implies

$$g(z) = \sum_{k \in \mathbf{Z}} b_{1(k)} (w_1 + z)^{-k-1} Y(b_2, w_2 + z) \dots Y(b_m, w_m + z) |0\rangle$$

and thus

$$\begin{aligned} \partial_z g(z) &= \sum_{k \in \mathbf{Z}} b_{1(k)} (-k-1) (w_1 + z)^{-k-2} Y(b_2, w_2 + z) \dots Y(b_m, w_m + z) |0\rangle \\ &\quad + \sum_{k \in \mathbf{Z}} b_{1(k)} (w_1 + z)^{-k-1} \partial_z Y(b_2, w_2 + z) \dots Y(b_m, w_m + z) |0\rangle \\ &= \sum_{k \in \mathbf{Z}} b_{1(k-1)} (-k) (w_1 + z)^{-k-1} Y(b_2, w_2 + z) \dots Y(b_m, w_m + z) |0\rangle \\ &\quad + \sum_{k \in \mathbf{Z}} b_{1(k)} (w_1 + z)^{-k-1} TY(b_2, w_2 + z) \dots Y(b_m, w_m + z) |0\rangle \end{aligned}$$

after reindexing. The translation axiom implies that $[T, b_{1(k)}] = -kb_{1(k-1)}$. Therefore

$$\begin{aligned} \partial_z g(z) &= \sum_{k \in \mathbf{Z}} T b_{1(k)} (w_1 + z)^{-k-1} Y(b_2, w_2 + z) \dots Y(b_m, w_m + z) |0\rangle \\ &= TY(b_1, w_1 + z) \dots Y(b_m, w_m + z) |0\rangle = Tg(z), \end{aligned}$$

and this concludes the induction step. \square

Proposition 2.3.5. μ satisfies associativity.

Proof. By Proposition 2.3.4, it suffices to show that

$$\begin{aligned} & \sum_k \mu(a_1, z_1 - y, \dots, a_m, z_m - y, p_k(\mu(b_1, w_1, \dots, b_n, w_n)), z_{m+1} - y) \\ &= \mu(a_1, z_1 - y, \dots, a_m, z_m - y, b_1, w_1 + z_{m+1} - y, \dots, b_n, w_n + z_{m+1} - y) \end{aligned}$$

for some $y \in \mathbf{C}$. Locally normal convergence is preserved by the application of e^{yT} as this yields a finite sum in each component. We pick y so that

$$|z_{m+1} - y| < |z_{m+1} - z_i| \quad \text{and} \quad |w_j + z_{m+1} - y| < |z_i - y| \quad (18)$$

for all $i = 1, \dots, m$ and $j = 1, \dots, n$ and so that the $|z_i - y|$ are pairwise distinct and the $|w_j + z_{m+1} - y|$ are pairwise distinct. The second condition on the pairwise distinct norms holds for all y outside a finite union of lines. The first condition holds for all y close enough to z_{m+1} because $\max_j |w_j| < \min_{1 \leq i \leq m} |z_i - z_{m+1}|$ is a prerequisite in the associativity axiom. It follows that such a y exists.

Permuting the z_1, \dots, z_m and the w_1, \dots, w_n and using the permutation invariance of μ , we may assume that

$$|z_1 - y| > \dots > |z_m - y| \quad \text{and} \quad |w_1 + z_{m+1} - y| > \dots > |w_n + z_{m+1} - y|,$$

and we have $|z_m - y| > |w_1 + z_{m+1} - y|$ by Equation 18 so

$$|z_1 - y| > \dots > |z_m - y| > |w_1 + z_{m+1} - y| > \dots > |w_n + z_{m+1} - y|.$$

Without loss of generality, we may assume $y = 0$ by redefining z_i to $z_i - y$ and w_j to $w_j - y$ so that

$$|z_1| > \dots > |z_m| > |w_1 + z_{m+1}| > \dots > |w_n + z_{m+1}|.$$

Then, using the absolute convergence from Proposition 2.3.2,

$$\begin{aligned} & \mu(a_1, z_1, \dots, a_m, z_m, b_1, w_1 + z_{m+1}, \dots, b_n, w_n + z_{m+1}) \\ &= \sum_{i \in \mathbf{Z}^m, j \in \mathbf{Z}^n} a_{1(i_1)} z_1^{-i_1-1} \dots a_{m(i_m)} z_m^{-i_m-1} \\ & \quad b_{1(j_1)} (w_1 + z_{m+1})^{-j_1-1} \dots b_{n(i_n)} (w_n + z_{m+1})^{-j_n-1} |0\rangle \\ &= \sum_l \sum_{i \in \mathbf{Z}^m} a_{1(i_1)} z_1^{-i_1-1} \dots a_{m(i_m)} z_m^{-i_m-1} \\ & \quad p_l \sum_{j \in \mathbf{Z}^n} b_{1(j_1)} (w_1 + z_{m+1})^{-j_1-1} \dots b_{n(i_n)} (w_n + z_{m+1})^{-j_n-1} |0\rangle \\ &= \sum_l \sum_{i \in \mathbf{Z}^m} a_{1(i_1)} z_1^{-i_1-1} \dots a_{m(i_m)} z_m^{-i_m-1} p_l \mu(b_1, w_1 + z_{m+1}, \dots, b_n, w_n + z_{m+1}) \\ &= \sum_l \sum_{i \in \mathbf{Z}^m} a_{1(i_1)} z_1^{-i_1-1} \dots a_{m(i_m)} z_m^{-i_m-1} p_l \mu(b_1, w_1 + z_{m+1}, \dots, b_n, w_n + z_{m+1}), \end{aligned}$$

and, since

$$\begin{aligned}
& \mu(b_1, w_1 + z_{m+1}, \dots, b_n, w_n + z_{m+1}) \\
&= \sum_k e^{z_{m+1}T} p_k \mu(b, w) \\
&= \sum_k Y(p_k \mu(b, w), z_{m+1}) |0\rangle \\
&= \sum_k \sum_{i_{m+1}} (p_k \mu(b, w))_{(i_{m+1})} z^{-i_{m+1}-1} |0\rangle,
\end{aligned}$$

by Proposition 2.3.4, the former equals

$$\begin{aligned}
& \sum_l \sum_{i \in \mathbf{Z}^m} a_{1(i_1)} z_1^{-i_1-1} \dots a_{m(i_m)} z_m^{-i_m-1} p_l \sum_k \sum_{i_{m+1}} (p_k \mu(b, w))_{(i_{m+1})} z^{-i_{m+1}-1} |0\rangle \\
&= \sum_l \sum_{i \in \mathbf{Z}^{m+1}} a_{1(i_1)} z_1^{-i_1-1} \dots a_{m(i_m)} z_m^{-i_m-1} p_l \sum_k (p_k \mu(b, w))_{(i_{m+1})} z^{-i_{m+1}-1} |0\rangle \\
&= \sum_k \sum_l \sum_{i \in \mathbf{Z}^{m+1}} a_{1(i_1)} z_1^{-i_1-1} \dots a_{m(i_m)} z_m^{-i_m-1} p_l (p_k \mu(b, w))_{(i_{m+1})} z^{-i_{m+1}-1} |0\rangle \\
&= \sum_k \sum_{i \in \mathbf{Z}^{m+1}} a_{1(i_1)} z_1^{-i_1-1} \dots a_{m(i_m)} z_m^{-i_m-1} (p_k \mu(b, w))_{(i_{m+1})} z^{-i_{m+1}-1} |0\rangle \\
&= \sum_k \mu(a_1, z_1, \dots, a_m, z_m, p_k(\mu(b_1, w_1, \dots, b_n, w_n)), z_{m+1}),
\end{aligned}$$

where convergence is locally normal to begin with since power series converge locally normally and we may thus interchange the sums. \square

Proof of Theorem 2.1.11. Let Φ denote the map from geometric vertex algebra structures on \mathcal{V} to vertex algebra structures on \mathcal{V} as described in the statement of the theorem. Proposition 2.2.4 says that it is well-defined. Let Ψ denote the map from vertex algebra structures on \mathcal{V} to geometric vertex algebra structures defined using Proposition 2.3.2 and Proposition 2.3.3. It is clear from the construction that $\Phi \circ \Psi = \text{id}$ because it suffices to consider Y to equate two vertex algebra structures on the same \mathbf{Z} -graded vector space (see Proposition 2.1.7). To see that $\Psi \circ \Phi = \text{id}$, we repeatedly apply Proposition 2.2.2 to deduce that

$$\mu(a, z) = Y(a_1, z_1) \dots Y(a_m, z_m) |0\rangle$$

for all $a_1, \dots, a_m \in \mathcal{V}$ and $z \in \mathbf{C}^m \setminus \Delta$ with $|z_1| > \dots > |z_m|$, where μ is a given geometric vertex algebra structure and Y is part of $\Phi(\mu)$. Thus, for fixed a , the functions $\mu(a, z)$ and $\Psi(\Phi(\mu))(a, z)$ of $z \in \mathbf{C}^m \setminus \Delta$ are analytic continuations of the same function, and therefore agree by the uniqueness statement in Proposition 2.3.2, which characterizes $\Psi(\Phi(\mu))$. \square

3 Meromorphic Prefactorization Algebras

A prefactorization algebra F on a space X assigns a vector space $F(U)$ to every open $U \subseteq X$ and has multiplication maps

$$F(U) \otimes F(V) \longrightarrow F(W)$$

for all open subsets $U \sqcup V \subseteq W \subseteq X$. These maps are required to be associative, unital and symmetric in an appropriate sense. Prefactorization algebras have all of the data of a factorization algebra but not necessarily their local-to-global properties, which we treat in Section 4. In this section, we introduce the additional structures and properties of a prefactorization algebra on $X = \mathbf{C}$ which allow us to obtain a geometric vertex algebra $\mathbf{V}F$ from it. Roughly speaking, the underlying \mathbf{Z} -graded vector space of $\mathbf{V}F$ consists of the weight spaces for an action of $\mathbf{D}^\times = \{z \in \mathbf{C} \mid 0 < |z| < 1\}$ on a small disc around zero, and the multiplication map $\mu : \mathbf{V}F \otimes \mathbf{V}F \rightarrow \overline{\mathbf{V}F}$ for $(z, w) \in \mathbf{C}^2 \setminus \Delta$ corresponds to the multiplication map of F for the inclusion of two disjoint small discs centered at z and w into a large disc centered at zero. Here, the additional structure of F being affine-linearly invariant provides some of the isomorphisms between the $F(d)$ for the different discs $d \subseteq \mathbf{C}$. However, we also make use of the maps induced by inclusions to compare the values assigned to concentric discs of different sizes.

The first three subsections contain the relevant definitions and the verification of the axioms of the geometric vertex algebra $\mathbf{V}F$ associated with a holomorphic prefactorization algebra on \mathbf{C} with discrete weight spaces and meromorphic operator product expansion (OPE). In the fourth subsection, we take a geometric vertex algebra \mathcal{V} and describe $\mathbf{F}\mathcal{V}$ as a holomorphic affine-linearly invariant prefactorization algebra on \mathbf{C} . The fifth subsection establishes that $\mathbf{F}\mathcal{V}$ has discrete weight spaces and meromorphic operator product expansion, and that $\mathbf{V}\mathbf{F}\mathcal{V} \cong \mathcal{V}$ as geometric vertex algebras.

Related work. Huang [12] considers a locally convex completion H of a finitely generated degreewise finite-dimensional bounded-below \mathbf{Z} -graded vertex algebra \mathcal{V} . The construction of this locally convex completion uses the standard disc in \mathbf{C} . He constructs maps $H \otimes H \rightarrow H$ for pairs-of-pants and shows that they encode the vertex operators of \mathcal{V} . André Henriques suggested a variation of this construction for discs to the author, and how to extend it to open subsets of \mathbf{C} . Bornological vector spaces appear in the book [2] by Costello and Gwilliam, too. There, the bornologies of bornological vector spaces are assumed to arise from a locally convex Hausdorff topology. We have split their procedure to obtain vertex algebras from suitable factorization algebras into two steps by making a stop at geometric vertex algebras.

3.1 Functional-Analytic Preliminaries

Unless otherwise noted, all vector spaces are over \mathbf{C} . Recall that a linear map between semi-normed spaces is continuous if and only if it is bounded. Continuity can be phrased in terms of the collections of open subsets. Boundedness can be phrased in terms of the collections of bounded subsets: a map is bounded if and only if it sends bounded subsets to bounded subsets. The collection of bounded subsets of a semi-normed space form a *bornology*. These two perspectives in terms of topologies and bornologies lead to two generalizations of Banach spaces, namely complete locally convex topological vector spaces and complete convex bornological vector spaces, or complete bornological vector spaces for short. Our factorization algebras take values in the symmetric monoidal category of complete bornological vector spaces. A complete bornological vector space may be thought of as ascending union of Banach spaces. The author has made use of [14, 15, 9, 10, 16, 2] to learn about bornological spaces.

Definition 3.1.1. Let X be a set. A *bornology* \mathcal{B}_X on X is a set of subsets of X such that

- $A \subseteq B \in \mathcal{B}_X \Rightarrow A \in \mathcal{B}_X$,
- $A, B \in \mathcal{B}_X \Rightarrow A \cup B \in \mathcal{B}_X$, and
- $x \in X \Rightarrow \{x\} \in \mathcal{B}_X$.

We call the elements of \mathcal{B}_X the *bounded* subsets of X . A map $X \rightarrow Y$ of sets with bornologies \mathcal{B}_X and \mathcal{B}_Y is *bounded* if $f(\mathcal{B}_X) \subseteq \mathcal{B}_Y$. Now, let X be a vector space over \mathbf{C} . A subset A of X is called *absolutely convex* if $\lambda x + \mu y \in A$ for all $x, y \in A$ and $\lambda, \mu \in \mathbf{C}$ with $|\lambda| + |\mu| \leq 1$. We let $\langle A \rangle$ denote the absolutely convex hull of A , that is, the smallest absolutely convex subset of X containing A . A *bornological vector space* X is a vector space X together with a bornology such that

- $B \in \mathcal{B}_X, \lambda > 0 \Rightarrow \lambda B \in \mathcal{B}_X$ and
- $B \in \mathcal{B}_X \Rightarrow \langle B \rangle \in \mathcal{B}_X$.

If X is a bornological vector space, and $A, B \in \mathcal{B}_X$, then $A + B \in \mathcal{B}_X$, because $A + B \subseteq \langle 2(A \cup B) \rangle$.

Definition 3.1.2. Let BVS denote the category of bornological vector spaces and bounded linear maps.

The category BVS is additive but not abelian and has all colimits and limits. The direct sum $\bigoplus_{i \in I} X_i$ of a family of bornological vector spaces is the direct sum of the underlying vector spaces together with the following bornology. A subset $B \subseteq \bigoplus_{i \in I} X_i$ is bounded if there exists a family of bounded subsets $C_i \subseteq X_i$ for $i \in I$ with $B \subseteq \bigoplus_{i \in I} C_i$ such that all except finitely many C_i are zero. This bornology is the unique bornology turning the direct sum into the coproduct in BVS.

If X is a sub vector space of a bornological vector space Y , then Y/X is a bornological vector space where a subset is defined to be bounded if it is the image of a bounded subset under the quotient map $q : Y \rightarrow Y/X$:

$$\mathcal{B}_{Y/X} := \{C \subseteq Y/X \mid \exists B \in \mathcal{B}_Y : C = q(B)\}$$

This bornology is called the *quotient bornology*. If $f : X \rightarrow Y$ is a bounded linear map, then $Y/f(X)$ together with the quotient map from Y to X is a cokernel of f in the category of bornological vector spaces.

The product $\prod_{i \in I} X_i$ in the category of bornological vector spaces has the product of vector spaces as its underlying vector space. Its bornology consists of those sets which are contained in a set of the form $\prod_{i \in I} B_i$ for bounded subsets $B_i \subseteq X_i$ for $i \in I$. A sub vector space X of a bornological vector space Y is a bornological vector space by defining subsets to be bounded if they are bounded in Y ,

$$\mathcal{B}_X = \{A \subseteq X \mid A \in \mathcal{B}_Y\}.$$

This bornology is called the *subspace bornology*. If $f : X \rightarrow Y$ is a bounded linear map, then $f^{-1}(0)$ with the subspace bornology is a kernel of f in the category of bornological vector spaces. Our next goal is the definition of completeness for bornological spaces.

Definition 3.1.3. If X is a semi-normed space, then the set of all subsets of X bounded w.r.t. its norm defines a bornology on X and turns X into a bornological space. This defines a full embedding of the category of semi-normed spaces and bounded maps into BVS. The image of this functor consists of those bornological spaces whose bornology arises from some semi-norm and we call such bornological spaces *semi-normable*. Similarly, we call such bornological spaces *normable* if their bornology comes from a norm and *completely normable*, or *Banachable*, if their bornology comes from a complete norm.

Two semi-norms on a vector space are equivalent if and only if their bornologies are equal.

Definition 3.1.4. Let X be a bornological vector space with bornology \mathcal{B}_X . A bornological vector space Y with bornology \mathcal{B}_Y is a *subobject* of X if $Y \subseteq X$ is a sub vector space and $\mathcal{B}_Y \subseteq \mathcal{B}_X$.

The inclusion $Y \subseteq X$ of a subobject is bounded, but Y does not necessarily carry the subspace bornology inside X .

Definition 3.1.5. A bornological space is called *complete*, if every bounded subset is contained in some completely normable subobject and bounded there. A *completion* of X is an initial object in the full subcategory of $\text{BVS}_{X/}$ whose objects are maps from X to a complete bornological vector space.

This definition of completeness of bornological spaces is easily seen to be equivalent to the one given in [9, 3:2.1] using completant discs. If a map $f : X \rightarrow \bar{X}$ is a completion of X , we usually omit the map f and call \bar{X} a completion of X . If X is already complete, then id_X is a completion of X . Completions are unique up to unique isomorphism as initial objects in the category mentioned above. The category of complete bornological vector spaces has all colimits and limits, which are described further below. Completions exists because the completion of a semi-normable space is its usual completion and

$$\bar{X} = \underset{\substack{Y \subseteq X \\ \text{semi-norm.}}}{\text{colim}} \bar{Y}$$

is a completion of X where the colimit is taken inside the category of complete bornological vector spaces.

The direct sum of a family of complete bornological vector spaces is complete. In particular, such a sum is the coproduct in the category of complete bornological vector spaces. The analogous statements hold for products. We now define the topology of sequentially closed sets, or *b-closed* sets, in order to describe cokernels in the category of complete bornological spaces.

Definition 3.1.6. A sequence $(x_n)_n$ in a bornological vector space X *converges* to $y \in X$ if there is a semi-normable subobject $Y \subseteq X$ containing y and all x_n such that $(x_n)_n$ converges to y in Y .

Convergence of sequences in a semi-normable bornological space Y is independent of the choice of a semi-norm inducing the bornology of Y . If X is bornological vector space, then the set of sequentially closed sets form the closed sets of a topology on X . These closed sets are called *b-closed*. Bounded linear maps are continuous w. r. t. this topology. A subspace of a complete bornological space is *b-closed* if and only if it is complete, see [9, 3:2.3 Proposition 1]. If X is a *b-closed* subspace of a complete bornological space Y , then Y/X is complete, see [9, 3:2.3 Proposition 2]. Therefore, if $f : X \rightarrow Y$ is a bounded linear map, then $Y/\overline{f(X)}$ is a cokernel of f in the category of complete bornological vector spaces. The kernel of a bounded linear map $f : X \rightarrow Y$ between complete spaces in the category of complete bornological vector spaces is again $f^{-1}(0)$, since bounded linear maps are continuous w. r. t. the topology of *b-closed* sets and $\{0\}$ is *b-closed* in every complete bornological space. The quotient of a complete bornological space by the action of a finite group is again complete, as can be seen by identifying it with the complete bornological space of invariants.

Definition 3.1.7. Let $U \subseteq \mathbf{R}^k$ and X be a complete bornological space. A function $f : U \rightarrow X$ is *continuous* if $\lim_n x_n = x_\infty$ implies $\lim_n f(x_n) = f(x_\infty)$.

Continuity in the sense of Definition 3.1.7 is equivalent to the property that every compact $K \subseteq U$ is mapped into a completely normable subobject Y of X for which $f|_K$ is continuous as a map to Y in the usual sense.

Definition 3.1.8. Let U be an open subset of \mathbf{C}^n and let X be a complete bornological vector space. A map $f : U \rightarrow X$ of sets is called *holomorphic* if every $p \in U$ has an open neighborhood V s. t. $f(V) \subseteq Y$ for some completely normable subobject $Y \subseteq X$ and the map $f|_V : V \rightarrow Y$ is holomorphic.

Definition 3.1.9. Let X be a locally convex topological vector space. Let X' be its continuous *dual*, that is, the vector space of continuous linear functionals on X . Let $B \subseteq X'$ be bounded if and only if there exists a continuous semi-norm p on X with $|\alpha(x)| \leq p(x)$ for all $\alpha \in B$ and $x \in X$. This turns X' into a bornological vector space.

If X is a locally convex space, then X' is always complete as a bornological vector space. Our main example of a non-discrete bornological vector space is the complete bornological vector space $\mathcal{O}'(U) := \mathcal{O}(U)'$ of analytic functionals on an open subset $U \subseteq \mathbf{C}^k$. Here, $\mathcal{O}(U)$ is the locally convex space of holomorphic functions on U whose topology is defined by the supremum semi-norms $\|-\|_{\infty, K}$ on compact subsets of K . This means that a subset $B \subseteq \mathcal{O}'(U)$ is bounded if and only if there exists a compact subset $K \subseteq U$ and a number $C > 0$ s. t.

$$|\alpha(f)| \leq C \|f\|_K$$

for all $\alpha \in B$ and $f \in \mathcal{O}(U)$.

Definition 3.1.10. If $U \subseteq \mathbf{C}^n$ is open and $K \subseteq U$ is compact, then $\mathcal{O}'_K(U)$ is the subset of analytic functionals bounded w. r. t. $\|\cdot\|_{\infty, K}$, the supremum semi-norm on K .

Note that $\mathcal{O}'_K(U)$ is a Banach space and a completely normable subobject of $\mathcal{O}'(U)$. Another description of the complete bornological space $\mathcal{O}'(U)$ is

$$\mathcal{O}'(U) = \bigcup_{\substack{K \subseteq U \\ \text{compact}}} \mathcal{O}'_K(U)$$

where the bornology on $\mathcal{O}'(U)$ is the union of the bornologies on the $\mathcal{O}'_K(U)$, that is, a subset $B \subseteq \mathcal{O}'(U)$ is bounded if and only if there is a compact $K \subseteq U$ such that B is a bounded subset of $\mathcal{O}'_K(U)$.

Definition 3.1.11. Given bornological vector spaces X and Y , we define the bornological tensor product $X \otimes Y$ to be the algebraic tensor product $X \otimes Y$ with the smallest convex bornology for which all sets of the form $B \otimes C$ for $B \subseteq X$ and $C \subseteq Y$ are bounded.

Definition 3.1.12. For bornological vector spaces X and Y , we define the *completed tensor product* $X \bar{\otimes} Y$ to be the completion of $X \otimes Y$.

Bornological vector spaces form a symmetric monoidal category BVS whose structure maps are obtained from the symmetric monoidal category of vector spaces by forgetting the bornology. It is clear that these maps are bounded. They induce the structure of a symmetric monoidal category on the category CBVS of complete bornological spaces via the completion functor. The completed tensor product commutes with colimits in each variable separately. We take the liberty to denote the image of $x \otimes y$ in $X \bar{\otimes} Y$ by $x \otimes y$, too.

Every finite-dimensional vector space has a unique bornology turning it into a complete bornological vector space. If X is a vector space, then we can turn it into a bornological vector space by declaring a subset to be bounded if it is contained in a finite-dimensional subspace and is bounded there. These bornological vector spaces are called *discrete*. A bornological vector space is discrete if and only if it is a direct sum of one-dimensional bornological vector spaces. Discrete bornological vector spaces are complete. Taking the tensor product with a discrete bornological vector space preserves completeness. One way to see this is to use the fact that the tensor product is left adjoint to the internal hom in bornological spaces, defined next, and thus preserves colimits, and in particular direct sums.

Definition 3.1.13. Let Y and Z be bornological vector spaces. A subset $B \subseteq \text{BVS}(Y, Z)$ is bounded if, for every bounded $C \subseteq Y$ there exists a bounded $D \subseteq Z$ such that $f(C) \subseteq D$ for all $f \in B$. The collection of these bounded sets turns $\text{BVS}(Y, Z)$ into a bornological vector space.

Both CBVS and BVS are closed in the sense of having internal homs adjoint to the tensor product. If Z is complete in the above definition, then $\text{BVS}(Y, Z)$ is complete, so the internal hom in CBVS agrees with that in BVS.

Since the passage from factorization algebras on \mathbf{C} to \mathbf{Z} -graded geometric vertex algebras is via the direct sum of the weight spaces of the value of the

factorization algebra on a disc, we now consider \mathbf{Z} -graded complete bornological spaces.

Definition 3.1.14. A \mathbf{Z} -graded complete bornological vector space X is a complete bornological vector space X whose underlying vector space is equipped with a \mathbf{Z} -grading such that X is the direct sum of the X_k for $k \in \mathbf{Z}$ in the category of complete bornological spaces.

A \mathbf{Z} -graded complete bornological space X is the same thing as a \mathbf{Z} -graded vector space each of whose graded components X_k is additionally equipped with a bornology which turns X_k into a complete bornological space.

Proposition 3.1.15. *Let $(X_i)_{i \in I}$ be a family of complete bornological vector spaces. Let $U \subseteq \mathbf{C}^n$ be an open subset and let $f : U \rightarrow \prod_{i \in I} X_i$ be a map of sets. Then f is holomorphic if and only if each of its components is.*

Proof. A map g into a complete bornological space is called holomorphic if it is locally a holomorphic map to some completely normable subobject. Therefore, g holomorphically maps the interior of every compact subset $K \subseteq U$ to some completely normable subobject depending on K . Let each component f_i be holomorphic. Without loss of generality, $0 \in U$ and we show that f is holomorphic on a neighborhood of 0. Let $r > 0$ with $K := \overline{B_r(0)} \subseteq U$. For all $i \in I$, there is a completely normable subobject Y_i of X_i such that $f_i(K) \subseteq Y_i$ and f_i restricted to interior of K is holomorphic as a map with target Y_i . For all $i \in I$, the Taylor series $\sum_{k=0}^{\infty} f_{i,k} z^k$ of f_i converges absolutely at r in Y_i . For each $i \in I$, pick a norm p_i on Y_i so that

$$\sum_{k=0}^{\infty} p_i(f_{i,k}) r^k \leq 1.$$

Such a norm exists because absolute convergence means that there is some norm for which this sum is finite. Let Y be the completely normable subobject of $\prod_i X_i$ consisting of those x with finite norm $p(x) := \sup_i p_i(x_i)$. Let $f_k \in \prod_i X_i$ be given by $(f_k)_i = f_{i,k}$. Then $f_k \in Y$ because

$$p(f_k) = \sup_i p_i(f_{i,k}) \leq \frac{1}{r^k}$$

so

$$\sum_{k=0}^{\infty} p(f_k) |z|^k \leq \sum_{k=0}^{\infty} \frac{|z|^k}{r^k} < \infty$$

for $|z| < r$, meaning that $\sum_{k=0}^{\infty} f_k z^k$ absolutely converges on $B_r(0)$. Since the projection $\prod_j X_j \rightarrow X_i$ is bounded for all i and maps f_k to $f_{i,k}$, it follows that this sum converges to $f(z)$.

Conversely, if f is holomorphic, then each component of f is holomorphic, because the projections $\prod X \rightarrow Y_i$ are bounded and the composition of a holomorphic map with a bounded linear map to another complete bornological space is again holomorphic. \square

Definition 3.1.16. Let X be a complete bornological space and let $U \subseteq \mathbf{C}^n$ be open. A subset $B \subseteq \mathcal{O}(U; X)$ is bounded if and only if there exists a map $\beta :$

$\mathcal{P}_c(U) \rightarrow \mathcal{B}_X$ from the set of compact subsets of U to the set of bounded subsets of X such that $f(K) \subseteq \beta(K)$ for all $f \in B$ and $K \subseteq U$ compact. With these bounded sets $\mathcal{O}(U; X)$ is a complete bornological space.

Let X and Y be complete bornological spaces. For $U \subseteq \mathbf{C}^m$ open, the pointwise tensor product of holomorphic functions $f : U \rightarrow X$ and $g : U \rightarrow Y$ is the holomorphic function $fg(z) = f(z) \otimes g(z) \in X \bar{\otimes} Y$ of $z \in \mathbf{C}^m$. This defines a bounded linear map

$$\mathcal{O}(U; X) \otimes \mathcal{O}(U; Y) \longrightarrow \mathcal{O}(U; X \bar{\otimes} Y).$$

Similarly, we have the exterior product $f \times g(z, w) = f(z) \otimes g(w)$ of $(z, w) \in U \times V$ for $V \subseteq \mathbf{C}^n$ open and $g : V \rightarrow Y$. The exterior product defines a bounded linear map

$$\mathcal{O}(U; X) \otimes \mathcal{O}(V; Y) \longrightarrow \mathcal{O}(U \times V; X \bar{\otimes} Y).$$

For $\alpha \in \mathcal{O}'(U)$ and $\beta \in \mathcal{O}'(V)$, the exterior product $\alpha \times \beta \in \mathcal{O}'(U \times V)$ is defined by

$$\alpha \times \beta(f) = \alpha(z \mapsto \beta(w \mapsto f(z, w))). \quad (19)$$

This defines a bounded linear map

$$\mathcal{O}'(U) \otimes \mathcal{O}'(V) \rightarrow \mathcal{O}'(U \times V),$$

called the exterior product of analytic functionals.

3.2 Holomorphic Affine-Linearly Invariant Prefactorization Algebras

We define prefactorization algebras on a topological space X with values in a symmetric monoidal category \mathcal{C} . For X a G -space for a group G , there is also the notion of a G -invariant prefactorization algebra on X . Being G -invariant is not just a property, but comes with isomorphisms $F(U) \cong F(gU)$ for $g \in G$ and $U \subseteq X$ open. If G is complex-analytic, we also define what it means for a prefactorization algebra with values in CBVS to be holomorphic. Our case of interest is $X = \mathbf{C}$ with the group of affine-linear isomorphisms $\mathbf{C}^\times \times \mathbf{C}$.

Definition 3.2.1. Let X be a topological space and $(\mathcal{C}, \otimes, \mathbf{1}_{\mathcal{C}})$ a symmetric monoidal category. A (unital) prefactorization algebra on X with values in \mathcal{C} consists of

- an object $F(U)$ of \mathcal{C} for every open $U \subseteq X$,
- multiplication maps

$$M_W^{U,V} : F(U) \otimes F(V) \longrightarrow F(W)$$

in \mathcal{C} for all disjoint open subsets U, V of some open subset $W \subseteq X$, and

- unit maps

$$\mathbf{1}_{\mathcal{C}} \rightarrow F(\emptyset)$$

such that:

- (associativity) For $W \subseteq X$ open, $V_1, V_2 \subseteq W$ open, and $U_1, U_2, U_3 \subseteq X$ open and pairwise disjoint with $U_1, U_2 \subseteq V_1$, $U_2, U_3 \subseteq V_2$, and U_1 disjoint from V_2 and V_1 disjoint from U_3 , the diagram

$$\begin{array}{ccc}
 (F(U_1) \otimes F(U_2)) \otimes F(U_3) & \cong & F(U_1) \otimes (F(U_2) \otimes F(U_3)) \\
 \downarrow M_{V_1}^{U_1, U_2} \otimes \text{id} & & \downarrow \text{id} \otimes M_{V_2}^{U_2, U_3} \\
 F(V_1) \otimes F(U_3) & & F(U_1) \otimes F(V_2) \\
 \searrow M_W^{V_1, U_3} & & \swarrow M_W^{U_1, V_2} \\
 & F(W) &
 \end{array}$$

commutes where the unlabeled isomorphism is the associator in \mathcal{C} .

- (unitality) For $U \subseteq X$ open, the composition

$$F(U) \cong F(U) \otimes \mathbf{1}_{\mathcal{C}} \longrightarrow F(U) \otimes F(\emptyset) \longrightarrow F(U)$$

is the identity of $F(U)$.

- (compatibility with braiding) For disjoint, open $U, V \subseteq X$, the braiding $\beta_{F(U), F(V)}$ of $F(U)$ with $F(V)$ in \mathcal{C} is compatible with multiplication in the sense that

$$M_{U \sqcup V}^{V, U} \circ \beta_{F(U), F(V)} = M_{U \sqcup V}^{V, U}.$$

Remark 3.2.2. Let F be a prefactorization algebra on a topological space X . If $U_1 \sqcup \dots \sqcup U_m \subseteq V$ are pairwise disjoint open subsets of X , we get an m -ary multiplication map

$$M_V^{U_1, \dots, U_m} : F(U_1) \otimes \dots \otimes F(U_m) \longrightarrow F(V)$$

subject to associativity as in [2, 3.1.1]. The conditions involving the maps for $m = 0$ express unitality. Recall that a precosheaf on a topological space X is a functor from the category of opens of X and inclusions among them to some other category. For $m = 1$, the maps

$$M_V^U : F(U) \rightarrow F(V)$$

for $U \subseteq V$ open are the extension maps of the underlying precosheaf of F , which we also denote by F . If F is a precosheaf, we denote its extension maps by M_V^U for notational compatibility.

Definition 3.2.3. Let X be a topological space with an action of a group G . The category of open subsets of X has a left G -action which is compatible with taking disjoint unions. Therefore G acts on the category of prefactorization algebras. A prefactorization algebra is called *G -invariant* if it is equipped with the data of a fixed point for this action. This consists of maps

$$\sigma_g^U : F(U) \longrightarrow F(gU)$$

for $g \in G$ and $U \subseteq X$ open such that

- $\sigma_{\text{id}}^U = \text{id}_{F(U)}$ for $U \subseteq X$ open,
- $\sigma_g^{hU} \circ \sigma_h^U = \sigma_{gh}^U$ for $g, h \in G$ and $U \subseteq X$ open,
- $\sigma_g^\emptyset : F(\emptyset) \rightarrow F(\emptyset)$ is compatible with the unit map for all $g \in G$
- $\sigma_g^W \circ M_W^{U, V} = M_{gW}^{gU, gV} \circ (\sigma_g^U \otimes \sigma_g^V)$ for $g \in G$ and $U, V \subseteq W$ open and disjoint and $W \subseteq X$ open.

A prefactorization algebra on \mathbf{C} is called *affine-linearly invariant* if it is invariant w. r. t. the group $G = \mathbf{C}^\times \times \mathbf{C}$ of affine transformations of \mathbf{C} .

Remark 3.2.4. Note that the maps σ_g^U are isomorphisms. The m -ary versions of M constructed from the unit and the binary M , which are part of a prefactorization algebra F by definition, are also fixed by the action of G if F is G -invariant:

$$\sigma_g^V \circ M_V^{U_1, \dots, U_m} = M_{gV}^{gU_1, \dots, gU_m} \circ (\sigma_g^{U_1} \otimes \dots \otimes \sigma_g^{U_m})$$

for $U_1 \sqcup \dots \sqcup U_m \subseteq V$ open subsets of X and $g \in G$.

We make analogous definitions for precosheaves instead of prefactorization algebras. We denote the extension maps of a precosheaf F by

$$M_V^U : F(U) \rightarrow F(V)$$

to be compatible with the notation for prefactorization algebras. The extension maps of a G -invariant precosheaf satisfy

$$\sigma_g^V \circ M_V^U = M_{gV}^{gU} \circ \sigma_g^U$$

for $g \in G$ and $U \subseteq V \subseteq X$ open.

Definition 3.2.5. Let F be a \mathbf{C}^\times -invariant precosheaf on \mathbf{C} with values in the category of vector spaces. For $r > 0$, the semi-group $\mathbf{D}^\times = \{z \in \mathbf{C} \mid 0 < |z| < 1\}$ acts on $B_r(0)$ and hence $F(B_r(0))$ by

$$\begin{aligned} \rho : \mathbf{D}^\times &\longrightarrow \text{Hom}(F(B_r(0)), F(B_r(0))) \\ q &\longmapsto M_{B_r(0)}^{B_{qr}(0)} \circ \sigma_q^{B_r(0)}. \end{aligned}$$

For a character $\chi \in \text{Hom}(\mathbf{C}^\times, \mathbf{C}^\times)$, let $F(B_r(0))_\chi$ be the weight space for χ , that is,

$$F(B_r(0))_\chi = \{x \in F(B_r(0)) \mid \forall q \in \mathbf{D}^\times : \rho(q)(x) = \chi(q)x\}.$$

Let F be a \mathbf{C}^\times -invariant precosheaf on \mathbf{C} . We write $F(B_r(0))_k$ for the k -th weight space for $k \in \mathbf{Z}$ which corresponds to $\chi(q) = q^k$. The maps

$$i_R^r := M_{B_R(0)}^{B_r(0)} : F(B_r(0)) \longrightarrow F(B_R(0))$$

induced by the inclusions for $0 < r \leq R \leq \infty$ are \mathbf{D}^\times -equivariant, so they induce maps

$$(i_R^r)_\chi : F(B_r(0))_\chi \longrightarrow F(B_R(0))_\chi$$

for all $\chi \in \text{Hom}(\mathbf{C}^\times, \mathbf{C}^\times)$.

Proposition 3.2.6. *The map $(i_R^r)_\chi$ is an isomorphism for all characters $\chi \in \text{Hom}(\mathbf{C}^\times, \mathbf{C}^\times)$ and $0 < r \leq R < \infty$.*

Proof. Let $q = r/R$. The map $(i_R^r)_\chi$ is inverse to $(\sigma_q^{B_R(0)})_\chi$ up to a non-zero scalar factor, namely $\chi(q)$.

$$(i_R^r)_\chi \circ (\sigma_q^{B_R(0)})_\chi = \chi(q) \text{id}_{F(B_R(0))_\chi} \quad (20)$$

by the definition of $F(B_R(0))_\chi$. To prove

$$(\sigma_q^{B_R(0)})_\chi \circ (i_R^r)_\chi = \chi(q) \text{id}_{F(B_r(0))_\chi}$$

we replace R with $qR = r$ and r with qr in Equation (20) to get

$$(i_r^{qr})_\chi \circ (\sigma_q^{B_r(0)})_\chi = \chi(q) \text{id}_{F(B_r(0))_\chi}$$

whose l. h. s. equals

$$(\sigma_q^{B_R(0)})_\chi \circ (i_R^r)_\chi$$

because

$$i_r^{qr} \circ \sigma_q^{B_r(0)} = \sigma_q^{B_R(0)} \circ i_R^r$$

because F is an invariant precosheaf. \square

Definition 3.2.7. Let F be a precosheaf on \mathbf{C} . Let

$$F(z) = \lim_{r>0} F(B_r(z))$$

denote the *costalk* of F at $z \in \mathbf{C}$.

We restrict attention to weights $\chi \in \text{Hom}(\mathbf{C}^\times, \mathbf{C}^\times)$ given by $\chi(q) = q^k$ for some $k \in \mathbf{Z}$ because vertex algebras are \mathbf{Z} -graded. The k -th weight space of the costalk is

$$F(z)_k = \left(\lim_{r>0} F(B_r(z)) \right)_k \cong \lim_{r>0} F(B_r(z))_k.$$

because limits commute. We suppress this isomorphism in our notation. By Proposition 3.2.6,

$$F(z)_k \cong F(B_R(z))_k$$

for all $R > 0$.

Definition 3.2.8. Let X be a topological space with the action of a complex-analytic group manifold G . For $U, V \subseteq X$, let

$$\mathcal{D}_{U,V} = \{g \in G \mid gU \subseteq V\}.$$

A G -invariant prefactorization algebra F on X with values in the symmetric monoidal category of complete bornological spaces is called *holomorphic* if the map

$$\begin{aligned} \rho_{U,V} : \mathcal{D}_{U,V} &\longrightarrow \text{BVS}(F(U), F(V)) \\ g &\longmapsto M_V^{gU} \circ \sigma_g^U \end{aligned}$$

is holomorphic on the interior of its domain. This condition only depends on the underlying G -invariant precosheaf of F and we define holomorphic G -invariant precosheaves using this condition. We abbreviate $\rho_{U,V}$ by ρ .

Proposition 3.2.9. *Let X be a complete bornological vector space with a holomorphic representation*

$$\rho : \mathbf{D}^\times \longrightarrow \text{End}(X) := \text{BVS}(X, X)$$

of the semi-group \mathbf{D}^\times . For $k \in \mathbf{Z}$ the map

$$\begin{aligned} l_k : X &\longrightarrow X_k \\ x &\longmapsto \frac{1}{2\pi i} \oint z^{-k-1} \rho(z)(x) dz \end{aligned}$$

to the k -th weight space of ρ is a well-defined bounded linear map and a \mathbf{D}^\times -equivariant splitting of the inclusion.

Proof. The contour integral may be taken over any of the circles tS^1 in \mathbf{D}^\times

with $0 < t < 1$. For $q \in \mathbf{D}^\times$,

$$\begin{aligned}
\rho(q)(l_k(x)) &= \frac{1}{2\pi i} \int_{tS^1} z^{-k-1} \rho(q)(\rho(z)(x)) dz \\
&= \frac{1}{2\pi i} \int_{tS^1} z^{-k-1} \rho(qz)(x) dz \\
&= \frac{1}{2\pi i} \int_{qtS^1} q^{k+1} z^{-k-1} \rho(z)(x) q^{-1} dz \\
&= q^k \frac{1}{2\pi i} \int_{qtS^1} z^{-k-1} \rho(z)(x) dz \\
&= q^k \frac{1}{2\pi i} \int_{tS^1} z^{-k-1} \rho(z)(x) dz \quad (\text{deform}) \\
&= q^k l_k(x),
\end{aligned}$$

so $l_k(x) \in X_k$ for $x \in X$. Equivariance uses the commutativity of \mathbf{D}^\times :

$$\begin{aligned}
\rho(q)(l_k(x)) &= \frac{1}{2\pi i} \oint z^{-k-1} \rho(qz)(x) dz \\
&= \frac{1}{2\pi i} \oint z^{-k-1} \rho(z)(\rho(q)(x)) dz \\
&= l_k(\rho(q)(x))
\end{aligned}$$

For $x \in X_k$

$$l_k(x) = \frac{1}{2\pi i} \oint z^{-k-1} z^k x dz = x,$$

so l_k is a splitting of the inclusion of X_k into X .

It remains to prove that l_k is bounded. Let $Y \subseteq X$ be a completely normable subobject of X . It suffices to prove that $l_k|_Y$ is bounded. By the compactness of tS^1 and the holomorphicity of ρ , there is a completely normable subspace Z of the endomorphisms of X with $\rho(tS^1) \subseteq Z$ bounded. If $B \subseteq Y$ is bounded, then $\rho(tS^1)(B)$ is bounded inside some completely normable subobject Y' of X , so the boundedness of $l_k(B)$ follows, since $l_k(B)$ is a subset of the closure inside Y' of the convex hull of $\rho(tS^1)(B)$. \square

Let F be a holomorphic \mathbf{C}^\times -invariant precosheaf on \mathbf{C} . For $r > 0$ and $k \in \mathbf{Z}$, let

$$l_k^r : F(B_r(0)) \longrightarrow F(B_r(0))_k$$

denote the weight space projection l_k for the \mathbf{D}^\times -representation $F(B_r(0))$. The costalk $F(0)$ is a holomorphic \mathbf{D}^\times -representation, too, as the limit of holomorphic representations. For $r > 0$, let

$$\pi_r : F(0) \rightarrow F(B_r(0))$$

denote the projection to the r -component of the costalk of F at zero. Of course, π_r is equivariant by construction and hence compatible with the weight space projections, i. e.

$$\begin{array}{ccc}
F(0) & \xrightarrow{l_k} & F(0)_k \\
\downarrow \pi_r & & \downarrow \pi_r \\
F(B_r(0)) & \xrightarrow{l_k^r} & F(B_r(0))_k
\end{array}$$

commutes. It can be shown that the \mathbf{D}^\times -representation on $F(0)$ extends to a holomorphic \mathbf{C}^\times -representation and its weight space projections l_k sum to the identity, $k \in \mathbf{Z}$. However, to prove associativity of the geometric vertex algebra $\mathbf{V}F$ associated with certain holomorphic prefactorization algebras F as in the next subsection, we need to sum up all weight space projections for F evaluated on a disc of medium size after including into a slightly larger disc.

Proposition 3.2.10. *Let F be a holomorphic \mathbf{C}^\times -invariant precosheaf on \mathbf{C} . For $R > r > 0$, we have*

$$i_R^r = \sum_{k \in \mathbf{Z}} i_R^r \circ l_k^r$$

in the space $\text{BVS}(F(B_r(0)), F(B_R(0)))$ with absolute convergence in a completely normable subobject.

Proof. Let $A = B_{R/r}(0) - 0$. The function

$$\begin{aligned}
\rho : A &\longrightarrow \text{BVS}(F(B_r(0)), F(B_R(0))) \\
q &\longmapsto M_{B_R(0)}^{B_{qR}(0)} \circ \sigma_q^{B_r(0)} = i_R^{qR} \circ \sigma_q^{B_r(0)}
\end{aligned}$$

is holomorphic by assumption. The Laurent expansion of ρ can be computed on \mathbf{D}^\times and is ($0 < t < 1$)

$$\rho(q) = \sum_{k \in \mathbf{Z}} \frac{1}{2\pi i} \int_{tS^1} z^{-k-1} \rho(z) dz q^k = \sum_{k \in \mathbf{Z}} (i_R^r \circ l_k^r) q^k$$

because

$$\begin{aligned}
\frac{1}{2\pi i} \int_{tS^1} z^{-k-1} \rho(z) dz &= \frac{1}{2\pi i} \int_{tS^1} z^{-k-1} i_R^{zR} \circ \sigma_z^{B_r(0)} dz \\
&= \frac{1}{2\pi i} \int_{tS^1} z^{-k-1} i_R^r \circ i_r^{zr} \circ \sigma_z^{B_r(0)} dz \\
&= i_R^r \circ \left(\frac{1}{2\pi i} \int_{tS^1} z^{-k-1} i_r^{zr} \circ \sigma_z^{B_r(0)} dz \right) \\
&= i_R^r \circ l_k^r.
\end{aligned}$$

We evaluate at $q = 1$ to get

$$i_R^r = \sum_{k \in \mathbf{Z}} i_R^r \circ l_k^r$$

with absolute convergence in $\text{BVS}(F(B_r(0)), F(B_R(0)))$ because the Laurent expansion of a holomorphic function with values in a complete bornological space converges absolutely in a completely normable subobject on any compact subset. \square

3.3 From Prefactorization Algebras to Geometric Vertex Algebras

We fix a holomorphic prefactorization algebra F and construct a geometric vertex algebra \mathbf{FV} from it under additional assumptions which we now describe. Let $\mathbf{V}F$ denote the \mathbf{Z} -graded vector space

$$\mathbf{V}F = \bigoplus_{k \in \mathbf{Z}} F(0)_k,$$

the direct sum of the weight spaces of the \mathbf{D}^\times -action on the costalk of F at zero. The homogeneous components $F(0)_k$ are complete bornological vector spaces and we assume these to be discrete, as is done in the setting of differentiable vector spaces in [2] for the weight spaces of F evaluated on a disc. In common examples, the homogeneous components are in fact finite-dimensional and $\mathbf{V}F$ is bounded from below or even concentrated in non-negative degrees. Assuming that the weight spaces of the costalk are discrete, we construct a \mathbf{Z} -graded vector space $\mathbf{V}F$ from F together with the underlying data of a geometric vertex algebra. We prove that these data satisfy the axioms of a geometric vertex algebra except possibly meromorphicity. We then say that a holomorphic prefactorization algebra has *meromorphic OPE* if $\mathbf{V}F$ satisfies the meromorphicity axiom, so that holomorphic prefactorization algebras with discrete weight spaces and meromorphic OPE yield geometric vertex algebras. The author expects that the discreteness assumption can be removed with more work.

The multiplication maps of F induce a map

$$M^{z_1, \dots, z_m} : F(z_1) \otimes \dots \otimes F(z_m) \longrightarrow \operatorname{colim}_{R>0} F(B_R(0))$$

for $z \in \mathbf{C}^m \setminus \Delta$ with $|z_i| < R$. We get a bounded linear map

$$\widetilde{M}^{z_1, \dots, z_m} : F(0)^{\otimes m} \longrightarrow \operatorname{colim}_{R>0} F(B_R(0))$$

by using the translation invariance of F . This is almost the multiplication map of the geometric vertex algebra associated with F ; it remains to precompose with a map to $\mathbf{V}F$ and postcompose with a map to $\overline{\mathbf{V}F}$. The target of \widetilde{M}^z maps to

$$\overline{\mathbf{V}F} := \prod_{k \in \mathbf{Z}} F(0)_k$$

as follows. For $R > 0$ and $k \in \mathbf{Z}$, the map $F(B_R(0)) \longrightarrow F(0)_k$ is the composite

$$F(B_R(0)) \xrightarrow{l_k^R} F(B_R(0))_k \xrightarrow{\cong} \lim_{r>0} F(B_r(0))_k \cong F(0)_k, \quad (21)$$

where the first isomorphism arises from Proposition 3.2.6 and the second from the fact that limits commute with each other. The source of \widetilde{M} receives a map from $\mathbf{V}F^{\otimes m}$ induced by the inclusion of $\mathbf{V}F$ into $F(0)$. Post- and precomposing with these maps results in a map

$$\mu^{z_1, \dots, z_m} : \mathbf{V}F^{\otimes m} \longrightarrow \overline{\mathbf{V}F}$$

still depending on $z \in \mathbf{C}^m \setminus \Delta$, as it should.

We now introduce a little bit more notation to write down the explicit formulas for μ^z used to check the axioms of a geometric vertex algebra. These formulas differ from the above more abstract description in that specific radii of discs appear. As in [2] and [12] and other places, we consider a moduli space of a fixed number of discs sitting inside a larger disc. For $r_1, \dots, r_m > 0$, let $D(r_1, \dots, r_m; R)$ denote the set of $z \in \mathbf{C}^m$ with

$$B_{r_1}(z_1) \sqcup \dots \sqcup B_{r_m}(z_m) \subseteq B_R(0).$$

We also denote this set by $D(\mathbf{r}; R)$ and can often restrict attention to the case of $r_1 = \dots = r_m = r$ for some small $r > 0$. Let

$$P_r : \mathbf{V}F = \bigoplus_{k \in \mathbf{Z}} F(0)_k \longrightarrow F(B_r(0))$$

be the map given by

$$F(0)_k = (\lim_{s>0} F(B_s(0)))_k \xrightarrow{(\pi_r)_k} F(B_r(0))_k \subseteq F(B_r(0))$$

on the k -th summand for $k \in \mathbf{Z}$. Let

$$L^R : F(B_R(0)) \longrightarrow \prod_{k \in \mathbf{Z}} F(0)_k = \overline{\mathbf{V}F}$$

be the map whose k -th component is given by (21). In terms of the maps L^R, P_{r_i} , the definition of μ^z is equivalent to the equation

$$\begin{aligned} \mu(a_1, z_1, \dots, a_m, z_m) &= \mu^z(a) \\ &= L^R(M_{B_R(0)}^{B_{r_1}(z_1), \dots, B_{r_m}(z_m)}(\sigma_{(+z_1)}^{B_{r_1}(0)}(P_{r_1}(a_1)), \dots, \sigma_{(+z_m)}^{B_{r_m}(0)}(P_{r_m}(a_m)))) \end{aligned} \quad (22)$$

in $\overline{\mathcal{V}}$ for $a = a_1 \otimes \dots \otimes a_m \in \mathbf{V}F^{\otimes m}$ and $r_1, \dots, r_m, R > 0$ with $z \in D(\mathbf{r}; R)$. This is because P_r is the lower left composite in the commutative square

$$\begin{array}{ccc} F(0)_k & \longrightarrow & F(0) \\ \downarrow (\pi_r)_k & & \downarrow \pi_r \\ F(B_r(0))_k & \longrightarrow & F(B_r(0)) \end{array}$$

in which the horizontal maps are the inclusions. Note that, for every $R > 0$, an element $x \in \overline{\mathbf{V}F}$ is uniquely determined by the $P_R p_k x$ for $k \in \mathbf{Z}$. Considering one degree $k \in \mathbf{Z}$ at a time,

$$\begin{aligned} &P_R(p_k(\mu^z(a))) \\ &= l_k^R(M_{B_R(0)}^{B_{r_1}(z_1), \dots, B_{r_m}(z_m)}(\sigma_{(+z_1)}^{B_{r_1}(0)}(P_{r_1}(a_1)), \dots, \sigma_{(+z_m)}^{B_{r_m}(0)}(P_{r_m}(a_m)))) \end{aligned} \quad (23)$$

in $F(B_R(0))_k$ because $P_R p_k L^R y = l_k^R y$ for all $y \in F(B_R(0))$.

So far, we know that the maps μ are linear maps from $\mathbf{V}F^{\otimes m}$ to the set of functions on $\mathbf{C}^m \setminus \Delta$ with values in $\overline{\mathbf{V}F}$. Our next goal is holomorphicity.

Proposition 3.3.1. *Let $a \in \mathcal{V}^{\otimes m}$. Then $\mu^z(a) = \mu(a, z)$ is a holomorphic function of $z \in \mathbf{C}^m \setminus \Delta$.*

Proof. A function $\mathbf{C}^m \setminus \Delta \rightarrow \overline{\mathbf{V}F}$ is holomorphic if and only if its components $p_k \circ f$ are holomorphic for all $k \in \mathbf{Z}$. Since $\mathbf{V}F_k$ is assumed to be discrete, $p_k \circ f$ is holomorphic if it is locally a holomorphic map to a finite-dimensional subspace of $\mathbf{V}F_k$. This notion of holomorphicity coincides with the notion of holomorphicity used in the definition of a geometric vertex algebra.

Let $z \in \mathbf{C}^m \setminus \Delta$. Pick $R > 0$ and $r > 0$ s. t. $z \in D(\mathbf{r}; R)$. Let $s = r/2$. It suffices to find a neighborhood U of z s. t. $\mu^{z'}(a)$ is a holomorphic function of $z' \in U$ for some given $a = a_1 \otimes \dots \otimes a_m$. We use $U = \prod_{i=1}^m B_s(z)$ and note that it suffices to prove the holomorphicity of

$$N(z') = M_{B_R(0)}^{B_s(z'_1), \dots, B_s(z'_m)}(\sigma_{(+z'_1)}^{B_s(0)}(x_1), \dots, \sigma_{(+z'_m)}^{B_s(0)}(x_m))$$

for $x_i \in F(B_s(0))$ because we can then plug in $x_i = P_s a_i$ and postcompose with L_R to get $\mu(a, z')$. By the associativity of M ,

$$N(z') = M_{B_R(0)}^{B_r(z_1), \dots, B_r(z_m)}(\rho_1(z'_1)(x_1), \dots, \rho_m(z'_m)(x_m)),$$

where

$$\begin{aligned} \rho_i(z'_i) : F(B_s(0)) &\longrightarrow F(B_r(z_i)) \\ y &\longmapsto M_{B_r(z_i)}^{B_s(z'_i)}(\sigma_{(+z'_i)}^{B_s(0)}(y)), \end{aligned}$$

so N is holomorphic as the composite of

- the map

$$\begin{aligned} U &\longrightarrow \overline{\bigotimes_{i=1}^m \text{BVS}(F(B_s(0)), F(B_r(z_i)))} \\ (z'_1, \dots, z'_m) &\longmapsto \rho_1(z'_1) \otimes \dots \otimes \rho_m(z'_m), \end{aligned}$$

which is holomorphic because F is holomorphically translation-invariant, and the pointwise tensor product of holomorphic functions is holomorphic,

- the bounded evaluation-at- a map to $\bigotimes_{i=1}^m F(B_r(z_i))$, and
- the bounded map $M_{B_R(0)}^{B_r(z_1), \dots, B_r(z_m)}$.

□

Remark 3.3.2. If F is holomorphic in the sense of Definition 3.2.8, then it is holomorphic pointwise, meaning that $\rho_{U,V}(z)(x)$ is a holomorphic function of $z \in \mathcal{D}_{U,V}$ for all $x \in F(U)$. The proof of Proposition 3.3.1 works if we only assume that F is holomorphic pointwise.

Proposition 3.3.3. $(\mathbf{V}F, \mu)$ satisfies the insertion-at-zero axiom.

Proof. Let $a \in \mathbf{V}F$. For all $R > 0$ and $k \in \mathbf{Z}$,

$$\begin{aligned} P_R p_k \mu(a, 0) &= l_k^R M_{B_R(0)}^{B_r(0)}(P_r a) = l_k^R P_R a = l_k^R P_R \sum_{l \in \mathbf{Z}} p_l a \\ &= \sum_{l \in \mathbf{Z}} l_k^R P_R p_l a = P_R p_k a, \end{aligned}$$

where the last equality holds because P_R is \mathbf{D}^\times -equivariant, so $P_R p_l a$ has degree l , and l_k^R is the identity on the k -th weight space and zero on the others. This implies $\mu(a, 0) = a$. □

Proposition 3.3.4. $(\mathbf{V}F, \mu)$ is \mathbf{C}^\times -equivariant.

Proof. For \mathbf{D}^\times instead of \mathbf{C}^\times , this follows since the maps are equivariant w. r. t. this semi-group in an appropriate sense: For all degrees $k \in \mathbf{Z}$, it suffices to consider some $R > 0$ with $z_i \in B_R(z_i)$ for $i = 1, \dots, m$ with $r > 0$ small enough so that $z \in D(r, \dots, r; R)$, and to show equality after application of $P_R \circ p_k$.

$$\begin{aligned}
& P_R p_k(q \cdot \mu(a_1, z_1, \dots, a_m, z_m)) \\
&= q \cdot P_R(p_k(\mu(a_1, z_1, \dots, a_m, z_m))) \\
&= q \cdot l_k^R(M_{B_R(0)}^{B_r(z_1), \dots, B_r(z_m)}(\sigma_{(+z_1)}^{B_r(0)}(P_r a_1), \dots, \sigma_{(+z_m)}^{B_r(0)}(P_r a_m))) \\
&= l_k^R(q \cdot M_{B_R(0)}^{B_r(z_1), \dots, B_r(z_m)}(\sigma_{(+z_1)}^{B_r(0)}(P_r a_1), \dots, \sigma_{(+z_m)}^{B_r(0)}(P_r a_m))) \\
&= l_k^R M_{B_R(0)}^{B_{qR}(0)} \sigma_q^{B_R(0)} M_{B_R(0)}^{B_r(z_1), \dots, B_r(z_m)}(\sigma_{(+z_1)}^{B_r(0)}(P_r a_1), \dots, \sigma_{(+z_m)}^{B_r(0)}(P_r a_m)) \\
&= l_k^R M_{B_R(0)}^{B_{qR}(0)} M_{B_{qR}(0)}^{B_{qr}(qz_1), \dots, B_{qr}(qz_m)}(\sigma_{(q)}^{B_r(z_1)}(\sigma_{(+z_1)}^{B_r(0)}(P_r a_1)), \dots, \\
&\quad \sigma_{(q)}^{B_r(z_m)}(\sigma_{(+z_m)}^{B_r(0)}(P_r a_m))) \\
&= l_k^R M_{B_R(0)}^{B_{qR}(0)} M_{B_{qR}(0)}^{B_{qr}(qz_1), \dots, B_{qr}(qz_m)}(\sigma_{(+qz_1)}^{B_{qr}(0)}(\sigma_{(q)}^{B_r(0)}(P_r a_1)), \dots, \\
&\quad \sigma_{(+qz_m)}^{B_{qr}(0)}(\sigma_{(q)}^{B_r(0)}(P_r a_m))) \\
&= l_k^R M_{B_R(0)}^{B_{qR}(0)} M_{B_{qR}(0)}^{B_{qr}(qz_1), \dots, B_{qr}(qz_m)}(\sigma_{(+qz_1)}^{B_{qr}(0)}(\sigma_{(q)}^{B_r(0)}(P_r a_1)), \dots, \\
&\quad \sigma_{(+qz_m)}^{B_{qr}(0)}(\sigma_{(q)}^{B_r(0)}(P_r a_m))) \\
&= l_k^R M_{B_R(0)}^{B_{qr}(qz_1), \dots, B_{qr}(qz_m)}(\sigma_{(+qz_1)}^{B_{qr}(0)}(\sigma_{(q)}^{B_r(0)}(P_r a_1)), \dots, \sigma_{(+qz_m)}^{B_{qr}(0)}(\sigma_{(q)}^{B_r(0)}(P_r a_m))) \\
&= l_k^R M_{B_R(0)}^{B_r(qz_1), \dots, B_r(qz_m)}(M_{B_r(qz_1)}^{B_{qr}(qz_1)}(\sigma_{(+qz_1)}^{B_{qr}(0)}(\sigma_{(q)}^{B_r(0)}(P_r a_1))), \dots, \\
&\quad M_{B_r(qz_m)}^{B_{qr}(qz_m)}(\sigma_{(+qz_m)}^{B_{qr}(0)}(\sigma_{(q)}^{B_r(0)}(P_r a_m)))) \\
&= l_k^R M_{B_R(0)}^{B_r(qz_1), \dots, B_r(qz_m)}(\sigma_{(+qz_1)}^{B_r(0)}(M_{B_r(0)}^{B_{qr}(0)}(\sigma_{(q)}^{B_r(0)}(P_r a_1))), \dots, \\
&\quad \sigma_{(+qz_m)}^{B_r(0)}(M_{B_r(0)}^{B_{qr}(0)}(\sigma_{(q)}^{B_r(0)}(P_r a_m)))) \\
&= l_k^R M_{B_R(0)}^{B_r(qz_1), \dots, B_r(qz_m)}(\sigma_{(+qz_1)}^{B_r(0)}(q \cdot P_r a_1), \dots, \sigma_{(+qz_m)}^{B_r(0)}(q \cdot P_r a_m)) \\
&= l_k^R M_{B_R(0)}^{B_r(qz_1), \dots, B_r(qz_m)}(\sigma_{(+qz_1)}^{B_r(0)}(P_r(q \cdot a_1)), \dots, \sigma_{(+qz_m)}^{B_r(0)}(P_r(q \cdot a_m))) \\
&= P_R p_k \mu(q \cdot a_1, qz_1, \dots, q \cdot a_m, qz_m)
\end{aligned}$$

Equivariance for \mathbf{C}^\times follows from the uniqueness of analytic continuation. \square

Proposition 3.3.5. μ satisfies the associativity axiom.

The map Φ in the following proof plays a very similar role as the map denoted by the same letter in Proposition 5.3.6 of [2].

Proof. Let $a_1, \dots, a_m \in \mathbf{V}F$, $b_1, \dots, b_n \in \mathbf{V}F$. Let

$$\mathcal{A}_{m,n} = \{(z, w) \in (\mathbf{C}^m \setminus \Delta) \times (\mathbf{C}^n \setminus \Delta) \mid \max_{1 \leq j \leq n} |w_j| < \min_{1 \leq i \leq m} |z_i - z_{m+1}|\}$$

denote the set of tuples of points in \mathbf{C} as in the associativity axiom of a geometric vertex algebra. The summands in the associativity axiom, viewed as functions of $(z, w) \in \mathcal{A}_{m,n}$, are elements of $\mathcal{O}(\mathcal{A}_{m,n}; \overline{\mathbf{V}F})$ and we prove absolute

convergence in a completely normable subobject. By the definition of the bornology on $\mathcal{O}(\mathcal{A}_{m,n}; \mathbf{V}F)$, this implies that, for every compact subset $K \subseteq \mathcal{A}_{m,n}$ and $k \in \mathbf{Z}$, there is a completely normable subobject $V_{K,k}$ of $\mathbf{V}F_k$ in which absolute convergence takes place. The discreteness of $\mathbf{V}F_k$ implies that the bounded unit disc of $V_{K,k}$ is contained in a finite-dimensional subspace of $\mathbf{V}F_k$ so $V_{K,k}$ is finite-dimensional.

It suffices to prove convergence of functions on a neighborhood U of (z, w) for every $(z, w) \in \mathcal{A}_{m,n}$. Let $S > T_2 > T_1 > 0$ all be smaller than $\min_{1 \leq i \leq m} |z_i - z_{m+1}|$, but only slightly smaller, so that

$$w_1, \dots, w_n \in B_{T_1}(0) \quad (\subset B_{T_2}(0) \subset B_S(0)).$$

Let $r > 0$ be small enough so that $w \in D(r, \dots, r; T_1)$ and the $B_r(z_i), i = 1, \dots, m$, are pairwise disjoint and disjoint from $B_S(z_{m+1})$. Let $R > 0$ be big enough so that

$$z \in D(r, \dots, r, S; R).$$

Let $s = r/2$ and

$$U = \prod_{i=1}^m B_s(z_i) \times B_{S-T_2}(z_{m+1}) \times \prod_{j=1}^n B_s(w_j)$$

which is a subset of $\mathcal{A}_{m,n}$ as we now check. Let $(z', w') \in U$. For all i, j with $1 \leq i \leq m$ and $1 \leq j \leq n$,

$$\begin{aligned} |w'_j| &< |w_j| + s = |w_j| + r - s < T_1 - s \\ &< S = S + r - 2s \leq |z_i - z_{m+1}| - 2s < |z'_i - z'_{m+1}| \end{aligned}$$

so $(z', w') \in \mathcal{A}_{m,n}$.

We show convergence of holomorphic functions on U with values in $\overline{\mathbf{V}F}$ by constructing a bounded linear map

$$\Phi : \text{BVS}(F(B_{T_1}(0)), F(B_{T_2}(0))) \longrightarrow \mathcal{O}(U; \overline{\mathbf{V}F})$$

with $\Phi(i_{T_2}^{T_1})$ given by the r. h. s. of the associativity axiom, see (2), and $\Phi(i_{T_2}^{T_1} \circ l_k^{T_1})$ given by the k -th summand of the associativity axiom for all $k \in \mathbf{Z}$, see (1). The existence of such a map implies the associativity axiom on U because bounded linear maps map absolutely convergent sums to absolutely convergent sums and $i_{T_2}^{T_1} = \sum_{k \in \mathbf{Z}} i_{T_2}^{T_1} \circ l_k^{T_1}$ by Proposition 3.2.10. Let

$$\begin{aligned} &(\Phi(f))(z', w') \\ &= L^R M_{B_R(0)}^{B_s(z'_1), \dots, B_s(z'_m), B_{T_2}(z'_{m+1})} (\sigma_{(+z'_1)}^{B_s(0)}(P_s a_1), \dots, \sigma_{(+z'_m)}^{B_s(0)}(P_s a_m), \\ &\quad \sigma_{(+z'_{m+1})}^{B_{T_2}(0)}(f(M_{B_{T_1}(0)}^{B_s(w'_1), \dots, B_s(w'_n)} (\sigma_{(+w'_1)}^{B_s(0)}(P_s b_1), \dots, \sigma_{(+w'_n)}^{B_s(0)}(P_s b_n))))). \end{aligned}$$

For $f = i_{T_2}^{T_1}$, we use associativity, translation invariance, and associativity again:

$$\begin{aligned}
& (\Phi(f))(z', w') \\
&= L^R M_{B_R(0)}^{B_s(z'_1), \dots, B_s(z'_m), B_{T_2}(z'_{m+1})} (\sigma_{(+z'_1)}^{B_s(0)}(P_s a_1), \dots, \sigma_{(+z'_m)}^{B_s(0)}(P_s a_m), \\
&\quad \sigma_{(+z'_{m+1})}^{B_{T_2}(0)} (M_{B_{T_2}(0)}^{B_s(w'_1), \dots, B_s(w'_n)} (\sigma_{(+w'_1)}^{B_s(0)}(P_s b_1), \dots, \sigma_{(+w'_n)}^{B_s(0)}(P_s b_n)))) \\
&= L^R M_{B_R(0)}^{B_s(z'_1), \dots, B_s(z'_m), B_{T_2}(z'_{m+1})} (\sigma_{(+z'_1)}^{B_s(0)}(P_s a_1), \dots, \sigma_{(+z'_m)}^{B_s(0)}(P_s a_m), \\
&\quad M_{B_{T_2}(z'_{m+1})}^{B_s(w'_1+z'_{m+1}), \dots, B_s(w'_n+z'_{m+1})} (\sigma_{(+w'_1+z'_{m+1})}^{B_s(0)}(P_s b_1), \dots, \sigma_{(+w'_n+z'_{m+1})}^{B_s(0)}(P_s b_n))) \\
&= L^R M_{B_R(0)}^{B_s(z'_1), \dots, B_s(z'_m), B_{T_2}(z'_{m+1}), B_s(w'_1+z'_{m+1}), \dots, B_s(w'_n+z'_{m+1})} (\\
&\quad \sigma_{(+z'_1)}^{B_s(0)}(P_s a_1), \dots, \sigma_{(+z'_m)}^{B_s(0)}(P_s a_m), \sigma_{(+w'_1+z'_{m+1})}^{B_s(0)}(P_s b_1), \dots, \sigma_{(+w'_n+z'_{m+1})}^{B_s(0)}(P_s b_n)) \\
&= \mu(a_1, z'_1, \dots, a_m, z'_m, b_1, w'_1 + z'_{m+1}, \dots, b_n, w'_n + z'_{m+1})
\end{aligned}$$

For $k \in \mathbf{Z}$ and $f = i_{T_2}^{T_1} \circ l_k^{T_1}$, we consider the argument of $\sigma_{(+z'_{m+1})}^{B_{T_2}(0)}$ which is

$$i_{T_2}^{T_1} \circ l_k^{T_1} (M_{B_{T_1}(0)}^{B_s(w'_1), \dots, B_s(w'_n)} (\sigma_{(+w'_1)}^{B_s(0)}(P_s b_1), \dots, \sigma_{(+w'_n)}^{B_s(0)}(P_s b_n)))$$

and, because of $i_{T_2}^{T_1} \circ l_k^{T_1} = l_k^{T_2} \circ i_{T_2}^{T_1}$ and associativity, equal to

$$P_{T_2}(p_k(\mu(b_1, w'_1, \dots, b_n, w'_n))).$$

Plugging in, this implies that

$$\begin{aligned}
& (\Phi(f))(z', w') \\
&= L^R M_{B_R(0)}^{B_s(z'_1), \dots, B_s(z'_m), B_{T_2}(z'_{m+1})} (\sigma_{(+z'_1)}^{B_s(0)}(P_s a_1), \dots, \sigma_{(+z'_m)}^{B_s(0)}(P_s a_m), \\
&\quad \sigma_{(+z'_{m+1})}^{B_{T_2}(0)} (P_{T_2}(p_k(\mu(b_1, w'_1, \dots, b_n, w'_n)))))) \\
&= \mu(a_1, z_1, \dots, a_m, z_m, p_k(\mu(b_1, w'_1, \dots, b_n, w'_n)), z_{m+1})
\end{aligned}$$

which is the k -th summand in the associativity axiom.

Now, we check that Φ is a bounded linear map and actually maps to the space of holomorphic functions $\mathcal{O}(U; \overline{VF})$ by writing Φ as a composite of bounded linear maps. This argument is similar to the proof of Proposition 3.3.1 about holomorphicity. By associativity,

$$\begin{aligned}
& (\Phi(f))(z', w') \\
&= L^R M_{B_R(0)}^{B_r(z_1), \dots, B_r(z_m), B_s(z_{m+1})} (\rho_1(z'_1)(P_s a_1), \dots, \rho_m(z'_m)(P_s a_m), \\
&\quad \rho_{m+1}(z'_{m+1})(f(M_{B_{T_1}(0)}^{B_r(w_1), \dots, B_r(w_n)} (\tau_1(w'_1)(P_s b_1), \dots, \tau_n(w'_n)(P_s b_n))))),
\end{aligned}$$

where the map

$$\begin{aligned}
\rho_i : B_s(z_i) &\longrightarrow \text{BVS}(F(B_s(0)), F(B_r(z_i))) \\
z'_i &\longmapsto M_{B_r(z_i)}^{B_s(z'_i)} \circ \sigma_{(+z'_i)}^{B_s(0)}
\end{aligned}$$

is holomorphic for $i = 1, \dots, m$, and so are

$$\begin{aligned} \rho_{m+1} : B_{S-T_2}(z_{m+1}) &\longrightarrow \text{BVS}(F(B_{T_2}(0)), F(B_S(z_i))) \\ z'_{m+1} &\longmapsto M_{B_S(z_{m+1})}^{B_{T_2}(z'_{m+1})} \circ \sigma_{(+z'_{m+1})}^{B_{T_2}(0)} \end{aligned}$$

and

$$\begin{aligned} \tau_j : B_s(w_j) &\longrightarrow \text{BVS}(F(B_s(0)), F(B_r(w_j))) \\ w'_j &\longmapsto M_{B_r(w_j)}^{B_s(w'_j)} \circ \sigma_{(+w'_j)}^{B_s(0)} \end{aligned}$$

for $j = 1, \dots, n$. Thus the map

$$\begin{aligned} \prod_{j=1}^n B_s(w_j) &\longrightarrow \overline{\bigotimes}_{j=1}^n F(B_r(w_j)) \\ (w'_1, \dots, w'_n) &\longmapsto \tau_1(w'_1)(P_s b_1) \otimes \dots \otimes \tau_n(w'_n)(P_s b_n) \end{aligned}$$

is holomorphic. Postcomposing with $M_{B_{T_1}(0)}^{B_r(w_1), \dots, B_r(w_n)}$ and f defines a bounded linear map

$$\text{BVS}(F(B_{T_1}(0)), F(B_{T_2}(0))) \longrightarrow \mathcal{O} \left(\prod_{j=1}^n B_s(w_j); F(B_{T_2}(0)) \right).$$

Taking the exterior product with

$$\rho_{m+1} \in \mathcal{O}(B_{S-T_2}(z_{m+1}); \text{BVS}(F(B_{T_2}(0)), F(B_S(z_i))))$$

is a bounded linear map with target

$$\mathcal{O} \left(B_{S-T_2}(z_{m+1}) \times \prod_{j=1}^n B_s(w_j); \text{BVS}(F(B_{T_2}(0)), F(B_S(z_i))) \bar{\otimes} F(B_{T_2}(0)) \right)$$

from which we can postcompose with the evaluation pairing to get an element of

$$\mathcal{O}(B_{S-T_2}(z_{m+1}) \times \prod_{j=1}^n B_s(w_j); F(B_S(z_i))).$$

Again using the holomorphicity of F , the map

$$\begin{aligned} \prod_{i=1}^m B_s(z_i) &\longrightarrow \overline{\bigotimes}_{i=1}^m F(B_r(z_i)) \\ (z'_1, \dots, z'_m) &\longmapsto \rho_1(z'_1)(P_s b_1) \otimes \dots \otimes \rho_m(z'_m)(P_s b_m) \end{aligned}$$

is holomorphic, and taking the exterior product with it is a bounded linear map with target

$$\mathcal{O}(U; \overline{\bigotimes}_{i=1}^m F(B_r(z_i)) \otimes F(B_S(z_i))).$$

Postcomposing with $L^R \circ M_{B_R(0)}^{B_r(z_1), \dots, B_r(z_m), B_S(z_{m+1})}$ gives $\Phi(f) \in \mathcal{O}(U; \overline{\mathbf{V}F})$. \square

Definition 3.3.6. Let F be a holomorphic affine-linearly invariant factorization algebra on \mathbf{C} . We say that F has discrete weight spaces if $\mathbf{V}F_k$ is discrete for all $k \in \mathbf{Z}$. If F has discrete weight spaces, we say that F has *meromorphic operator product expansion (OPE)* if $(\mathbf{V}F, \mu)$ satisfies the meromorphicity axiom of a geometric vertex algebra, where μ is the sequence of multiplication maps for $\mathbf{V}F$ constructed above.

We sometimes call holomorphic prefactorization algebras with meromorphic OPE *meromorphic prefactorization algebras*, with the caveat that, in contrast to the language for functions, a meromorphic prefactorization algebra is a special kind of holomorphic prefactorization algebra.

Proposition 3.3.7. *If F is a holomorphic prefactorization algebra on \mathbf{C} with discrete weight spaces and meromorphic OPE, then $(\mathbf{V}F, \mu)$ is a geometric vertex algebra.*

Proof. Permutation invariance holds because the multiplication maps in a prefactorization algebra are compatible with the braiding. We have already checked all the other axioms except meromorphicity which we now assume. \square

Recall from Proposition 2.2.3 that $\mathbf{V}F$ is meromorphic if it is bounded from below. Being bounded from below is the assumption made in [2] when constructing a vertex algebra from a prefactorization algebra. While less general, this assumption has the pleasing feature that it does not make reference to the multiplication maps of the prefactorization algebra.

3.4 From Geometric Vertex Algebras to Prefactorization Algebras

Let \mathcal{V} be a geometric vertex algebra. We construct a holomorphic prefactorization algebra $\mathbf{F}\mathcal{V}$ on \mathbf{C} . In Section 3.5, we prove that $\mathbf{F}\mathcal{V}$ has meromorphic OPE and discrete weight spaces by relating it to \mathcal{V} . We prove that $\mathbf{F}\mathcal{V}$ is a factorization algebra in Section 4.

The precosheaf $\mathbf{F}\mathcal{V}$ is a quotient of the precosheaf $\mathbf{E}\mathcal{V}$ of *expressions* with labels in \mathcal{V} , which we define first. It only makes use of the underlying \mathbf{Z} -graded vector space of \mathcal{V} and not of its multiplication maps.

Definition 3.4.1. Let \mathcal{V} be a \mathbf{Z} -graded vector space. We define a precosheaf on \mathbf{C} with values in the category of complete bornological spaces. Let $U \subseteq \mathbf{C}$ be open. The precosheaf $\mathbf{E}\mathcal{V}$ of *expressions* is defined by

$$\mathbf{E}\mathcal{V}(U) = \bigoplus_{m \geq 0} (\mathcal{O}'(U^m \setminus \Delta) \otimes \mathcal{V}^{\otimes m})_{\Sigma_m}$$

which is complete by the following proposition. Here, the action of $\sigma \in \Sigma_m$ on $\mathcal{O}'(U^m \setminus \Delta) \otimes \mathcal{V}^{\otimes m}$ is given by

$$\sigma.(\alpha \otimes a_1 \otimes \dots \otimes a_m) = \sigma_*\alpha \otimes a_{\sigma(1)} \otimes \dots \otimes a_{\sigma(m)}.$$

The extension maps of $\mathbf{E}\mathcal{V}$ are the pushforward of analytic functionals, dual to restriction of functions, and the identity on the $\mathcal{V}^{\otimes m}$.

If we have a fixed \mathcal{V} in mind, we also write E for $\mathbf{E}\mathcal{V}$.

Proposition 3.4.2. $E(U)$ is complete for all open $U \subseteq \mathbf{C}$.

Proof. Analytic functionals are complete as the dual of a locally convex topological vector space. The tensor products are again complete because taking the bornological tensor product with a discrete space preserves completeness and \mathcal{V} is defined to be discrete. The quotient of a complete space by the action of a finite group is complete, too, so $(\mathcal{O}'(U^n \setminus \Delta) \otimes \mathcal{V}^{\otimes n})_{\Sigma_n}$ is complete. Infinite direct sums of complete spaces are again complete, finishing our argument that $E(U)$ is complete. \square

We now turn E into a prefactorization algebra. To multiply $\alpha \in \mathcal{O}'(U^m \setminus \Delta)$ and $\beta \in \mathcal{O}'(V^n \setminus \Delta)$ for $U \sqcup V \subseteq W$, we use the exterior product of functionals, see Equation 19, to get $\alpha \times \beta \in \mathcal{O}'((U^m \setminus \Delta) \times (V^n \setminus \Delta))$ which we can then pushforward to an element $\alpha\beta \in \mathcal{O}'((U \sqcup V)^{m+n} \setminus \Delta)$ along the embedding of $(U^m \setminus \Delta) \times (V^n \setminus \Delta)$ into $(U \sqcup V)^{m+n} \setminus \Delta$ induced by the associativity isomorphism $\mathbf{C}^m \times \mathbf{C}^n \cong \mathbf{C}^{m+n}$. This embedding is well-defined because U and V are disjoint.

Proposition 3.4.3. *If \mathcal{V} is a \mathbf{Z} -graded vector space, then $\mathbf{E}\mathcal{V}$ is a prefactorization algebra on \mathbf{C} with values in the symmetric monoidal category of complete bornological spaces with the underlying precosheaf of Definition 3.4.1 whose multiplication maps are*

$$\begin{aligned} \mu_W^{U,V} : \mathbf{E}\mathcal{V}(U) \otimes \mathbf{E}\mathcal{V}(V) &\longrightarrow \mathbf{E}\mathcal{V}(W) \\ [\alpha \otimes a] \otimes [\beta \otimes b] &\longmapsto [\alpha\beta \otimes a \otimes b] \end{aligned}$$

where $\alpha \otimes a \in \mathcal{O}'(U^m \setminus \Delta) \otimes \mathcal{V}^{\otimes m}$ and $\beta \otimes b \in \mathcal{O}'(V^n \setminus \Delta) \otimes \mathcal{V}^{\otimes n}$. Its unit is 1 in the summand corresponding to $m = 0$ which can be identified with \mathbf{C} in a natural way and which is present for $U = \emptyset$, too.

Proof. The exterior product of functionals is compatible with pushforwards along complex-analytic embeddings. This implies that the multiplication maps for E are well-defined w. r. t. the permutation actions. Taking exterior product of functionals is a bounded linear map and the braiding of CBVS is as well, so the multiplication maps of E are bounded linear maps. Unitality and associativity follow from the behavior of the exterior product of functionals.

Compatibility with the braiding is enforced by modding out the action of the permutation groups, let $\tau \in \Sigma_{m+n}$ be the permutation moving the first m elements past the last n elements: We want to show that

$$\beta\alpha(f) = \beta(y \mapsto \alpha(x \mapsto f(y, x)))$$

is equal to

$$\tau_*(\alpha\beta)(f) = \alpha\beta(\tau^*f) = \alpha(x \mapsto \beta(y \mapsto \tau^*f(x, y))) = \alpha(x \mapsto \beta(y \mapsto f(y, x)))$$

for

$$\begin{aligned} f &\in \mathcal{O}((U \sqcup V)^{m+n} \setminus \Delta) \\ \alpha &\in \mathcal{O}'(U^m \setminus \Delta), \quad \beta \in \mathcal{O}'(V^n \setminus \Delta). \end{aligned}$$

The functionals $\beta\alpha$ and $\tau_*(\alpha\beta)$ are the pushforwards of

$$\beta \times \alpha, \tau_*(\alpha \times \beta) \in \mathcal{O}'((V^n \setminus \Delta) \times (U^m \setminus \Delta)).$$

Thus, it suffices to check the desired equality on the dense subset of holomorphic functions f which are a finite sum of functions $h(y)g(x)$ of $x \in U^m \setminus \Delta$ and $y \in V^n \setminus \Delta$ for g and h holomorphic. For $f(y, x) = h(y)g(x)$,

$$\beta \times \alpha(f) = \alpha(g)\beta(h) = (\tau_*(\alpha \times \beta))(f).$$

To identify the underlying precosheaf of E with the precosheaf defined earlier, we note that multiplying with the unit in $\mathbf{C} \cong \mathcal{O}'(\text{pt}) = \mathcal{O}'(\emptyset^0 \setminus \Delta_0)$ is the pushforward of analytic functionals. \square

Now using the multiplication maps of \mathcal{V} , we define the *evaluation map*

$$\text{ev}_U : \mathbf{E}\mathcal{V}(U) \rightarrow \overline{\mathcal{V}}_{bb}$$

for $U \subseteq \mathbf{C}$ open. Let $\alpha \otimes a \in \mathcal{O}'(U^m \setminus \Delta) \otimes \mathcal{V}^m$. As part of the structure of a geometric vertex algebra, we have a holomorphic function $\mu(a) : U^m \setminus \Delta \rightarrow \overline{\mathcal{V}}$. We would like to evaluate α on it to obtain an element of $\overline{\mathcal{V}}$. Let $k \in \mathbf{Z}$. The holomorphic function $p_k \circ \mu(a)$ takes values in a finite-dimensional subspace $F \subseteq \mathcal{V}_k$. The evaluation pairing tensored with F gives a map ($X := U^m \setminus \Delta$)

$$\mathcal{O}'(X) \otimes \mathcal{O}(X; F) \cong \mathcal{O}'(X) \otimes \mathcal{O}(X) \otimes F \rightarrow \mathbf{C} \otimes F \cong F$$

so we define the evaluation of α on $p_k \circ \mu(a)$ to be the image of $\alpha \otimes (p_k \circ \mu(a))$ under this map to get an element of \mathcal{V}_k for all $k \in \mathbf{Z}$. Let $\alpha(\mu(a))$ denote the resulting element of $\overline{\mathcal{V}}$. What we have just described more generally applies to any open $X \subseteq \mathbf{C}^m$ and any holomorphic function $f \in \mathcal{O}(X; \overline{\mathcal{V}})$ and $\alpha \in \mathcal{O}'(X)$ and defines $\alpha(f) \in \overline{\mathcal{V}}$, where \mathcal{V} is a \mathbf{Z} -graded vector space. It is clear that $\alpha(f)$ depends linearly and boundedly on $\alpha \otimes f$, by the universal property of the

product $\overline{\mathcal{V}}$ this means linearity and boundedness in each component. It follows that $\alpha(\mu(a))$ depends linearly and boundedly on $\alpha \otimes a$, recall that $\mathcal{V}^{\otimes m}$ is given the discrete bornology. If $(e_i)_i$ is a basis of a finite-dimensional subspace F in which $p_k \circ f$ takes values for some $k \in \mathbf{Z}$, then

$$\alpha(f)_k = \sum_i \alpha(e_i^*(p_k \circ f))e_i.$$

The permutation invariance of μ and the universal property of the direct sum give a well-defined and bounded linear map

$$\begin{aligned} \text{ev}_U : \mathbf{E}\mathcal{V}(U) &\longrightarrow \overline{\mathcal{V}} \\ [\alpha \otimes a] &\longmapsto \alpha(\mu(a)). \end{aligned}$$

The collection of the ev_U for $U \subseteq \mathbf{C}$ open defines a map of precosheaves from E to the constant precosheaf $\text{const } \overline{\mathcal{V}}$.

Definition 3.4.4. The precosheaf R on \mathbf{C} is defined as the kernel of $\text{ev} : E(U) \rightarrow \text{const } \overline{\mathcal{V}}$, that is,

$$R(U) = \ker \text{ev}_U \subseteq E(U)$$

for $U \subseteq \mathbf{C}$ open. We call the elements of $R(U)$ *relations* on U .

Remark 3.4.5. Note that $R(U) \subseteq E(U)$ is b-closed as the kernel of a bounded linear map since such maps are continuous w.r.t. the b-topology.

The multiplication on E does not induce a multiplication on E/R in general.

Example 3.4.6. We consider the case that there are $a, b \in \mathcal{V}$ s.t. $a_{(m)}b \neq 0$ for some $m \geq 0$. For example, this is the case for \mathcal{V} the free boson, $a = b$ the generator and $m = 1$. Let

$$\begin{aligned} U_1 &= B_2(0) \setminus \overline{B_1(0)} \\ U_2 &= B_1(0) \\ W &= B_2(0) \end{aligned}$$

so that we have $U_1 \cap U_2 = \emptyset$ and $U_1, U_2 \subseteq W$. Our goal is to show that there is no dotted arrow making

$$\begin{array}{ccc} E(U_1) \otimes E(U_2) & \longrightarrow & E(W) \\ \downarrow & & \downarrow \\ E(U_1)/R(U_1) \otimes E(U_2)/R(U_2) & \dashrightarrow & E(W)/R(W) \end{array}$$

commute, where the left map is the tensor product of the quotient maps and the right map is the quotient map.

Let $\alpha \in \mathcal{O}'(U_1)$ and $\beta \in \mathcal{O}'(U_2)$ be the analytic functionals

$$\alpha : h \in \mathcal{O}(U_1) \mapsto \int_{\frac{3}{2}S^1} z^m h(z) dz$$

resp.

$$\beta : h \in \mathcal{O}(U_2) \mapsto h(0).$$

Then $\alpha \otimes a$ and $\beta \otimes b$ represent classes $x \in E(U_1)$ resp. $y \in E(U_2)$. The image of $x \otimes y$ in $E(W)$ under $M_W^{U_1, U_2}$ is represented by $\alpha\beta \otimes a \otimes b$, where

$$\alpha\beta : h \in \mathcal{O}(W^2 \setminus \Delta) \mapsto \alpha[z \mapsto \beta[w \mapsto h(z, w)]] = \int_{\frac{3}{2}S^1} z^m h(z, 0) dz.$$

The image of this class in $E(W)/R(W)$ is non-zero because this space maps injectively to $\bar{\mathcal{V}}$ by construction and the image there is

$$\text{ev}_W(M_W^{U_1, U_2}([x \otimes y])) = \int_{\frac{3}{2}S^1} z^m \mu(a, z, b, 0) dz = a_{(m)} b \neq 0.$$

If there were a dotted arrow making the diagram commute, then the lower composition would give zero since the image of x in $E(U_1)/R(U_1)$ is zero because

$$\text{ev}_{U_1}(x) = \int_{\frac{3}{2}S^1} z^m \mu(a, z) dz = 0$$

since the integrand is defined on \mathbf{C} and holomorphic.

Nevertheless, the multiplication maps of E induce a multiplication on $F = E/\overline{R^{disc}}$; this is Proposition 3.4.9, where R^{disc} are the relations visible on round, open discs. A (*round, open disc*) is a subset $D \subseteq \mathbf{C}$ which is equal to

$$B_r(z) = \{w \in \mathbf{C} \mid |z - w| < r\}$$

for some $z \in \mathbf{C}$ and some real number $r > 0$.

Definition 3.4.7. The multiplication maps of E assemble to give a map

$$\bigoplus_{d, W} R(d) \otimes E(W) \longrightarrow E(U)$$

where the direct sum runs over all discs $d \subseteq U$ and open subsets $W \subseteq U$ s. t. $d \cap W = \emptyset$. Let $R^{disc}(U) \subseteq E(U)$ be the image of this map,

$$R^{disc}(U) = \text{im} \left(\bigoplus_{d, W} R(d) \otimes E(W) \longrightarrow E(U) \right).$$

We call R^{disc} the precosheaf of *relations on discs*. Let F be the precosheaf of complete bornological vector spaces defined by

$$F(U) = E(U)/\overline{R^{disc}(U)}$$

for $U \subseteq \mathbf{C}$ open. If we want to make the dependence on \mathcal{V} explicit, we use the notation $\mathbf{F}\mathcal{V}$ for F .

Note that $F(U)$ is complete because $E(U)$ is complete by Proposition 3.4.2 and because we are modding out by the b-closure of $R^{disc}(U)$. The author does not know if $R^{disc}(U)$ is b-closed or not. If U is a disc, then $R^{disc}(U) = R(U)$ by Lemma 3.5.1 further below, and $R(U)$ is b-closed as the kernel of a bounded map. To prove that F inherits multiplication maps from E , we slightly rephrase

its definition so that it only refers to complete bornological spaces. Let $U \subseteq \mathbf{C}$ open. The source of the map

$$\bigoplus_{d,W} R(d) \otimes E(W) \longrightarrow E(U)$$

as in the definition of $R^{disc}(U)$ is not complete, but

$$S^{disc}(U) := \bigoplus_{d,W} R(d) \bar{\otimes} E(W),$$

with d, W as above, is a completion. The space of relations $R(U) \subseteq E(U)$ is b-closed, because it is the intersection of the kernels of the components of ev_U , each of which is a bounded map, see its construction. Since $E(d)$ is complete, its b-closed subspace $R(d)$ is complete, too.

Proposition 3.4.8. *The map $E(U) \rightarrow F(U)$ is a cokernel of the map*

$$S^{disc}(U) \rightarrow E(U)$$

given by the sum of the multiplication maps.

Proof. This amounts to $\overline{R^{disc}(U)} = \bar{I}$ where I is the image of $S^{disc}(U)$ in $E(U)$. Let d, W be as above. The algebraic tensor product $R(d) \otimes E(W)$ maps to its completion $R(d) \bar{\otimes} E(W)$, so $R^{disc}(U) \subseteq I$, hence $\overline{R^{disc}(U)} \subseteq \bar{I}$. Every algebraic tensor product has dense image in its completion, implying that $\overline{R^{disc}(U)} \supseteq I$ and $\overline{R^{disc}(U)} \supseteq \bar{I}$. \square

Proposition 3.4.9. *F is a prefactorization algebra on \mathbf{C} with its multiplication maps induced from those of E . The multiplication maps of E induce unique multiplication maps on F , also denoted M , such that the square*

$$\begin{array}{ccc} E(U) \bar{\otimes} E(V) & \xrightarrow{M_X^{U,V}} & E(X) \\ \downarrow & & \downarrow \\ F(U) \bar{\otimes} F(V) & \xrightarrow{M_X^{U,V}} & F(X) \end{array}$$

commutes for all $X \subseteq \mathbf{C}$ open and $U, V \subseteq X$ open and disjoint, where the vertical maps are the completed tensor product of the quotient maps resp. the quotient map.

Proof. Let U and V be open and disjoint and contained in an open $X \subseteq \mathbf{C}$. Since cokernels in CBVS commute with $\bar{\otimes}$ in both variables separately, Proposition 3.4.8 implies that the map $E(U) \bar{\otimes} E(V) \rightarrow F(U) \bar{\otimes} F(V)$ is the cokernel of the multiplication map

$$T := S^{disc}(U) \bar{\otimes} E(V) \oplus E(U) \bar{\otimes} S^{disc}(V) \rightarrow E(U) \bar{\otimes} E(V).$$

For $Y \subseteq \mathbf{C}$ open, let $q_Y : E(Y) \rightarrow F(Y)$ denote the quotient map. We want to see that T maps to zero under $q_X \circ M_X^{U,V}$, as it then follows that F is a prefactorization algebra because E is a prefactorization algebra. To show that $S^{disc}(U) \bar{\otimes} E(V)$ maps to zero, let $d, W \subseteq U$ be as in the definition

of $S^{disc}(U)$. Using the associativity of E , we see that the map on the summand $(R(d) \bar{\otimes} E(W)) \bar{\otimes} E(V)$ factors through the summand $R(d) \bar{\otimes} E(W \sqcup V)$ of $S^{disc}(X)$. To show that $E(U) \bar{\otimes} S^{disc}(V)$ maps to zero, we also use the compatibility of M with the braiding. This time $d, W \subseteq V$ and the map on the summand $E(U) \bar{\otimes} (R(d) \bar{\otimes} E(W))$ factors through the summand $R(d) \bar{\otimes} E(U \sqcup W)$ of $S^{disc}(X)$. \square

To describe E as an affine-linearly invariant prefactorization algebra on \mathbf{C} , let L_0 denote the grading operator of \mathcal{V} . This is the endomorphism of \mathcal{V} defined by $L_0 a = |a|a$ for a homogeneous, so that $\lambda^{L_0 v} = \lambda^{|a|}a$ for $a \in \mathcal{V}$ homogeneous and $\lambda \in \mathbf{C}^\times$. This defines an action of \mathbf{C}^\times on \mathcal{V} and also $\mathbf{C}^\times \ltimes \mathbf{C}$ because $(\lambda, w) \mapsto \lambda$ is a group homomorphism. Therefore $\mathbf{E}\mathcal{V}$ is an affine-linearly invariant prefactorization algebra where

$$\sigma_{(\lambda, w)}^U(\alpha \otimes a_1 \otimes \dots \otimes a_n) = (\lambda, w)_*(\alpha) \otimes \lambda^{L_0} a_1 \otimes \dots \otimes \lambda^{L_0} a_n$$

for $\alpha \in \mathcal{O}'(U^n \setminus \Delta)$, $a_1, \dots, a_n \in \mathcal{V}$, and $U \subseteq \mathbf{C}$ open. Here, the pushforward of an analytic functional $\alpha \in \mathcal{O}'(X)$ along a holomorphic map $g : X \rightarrow Y$ is

$$g_*\alpha := [f \in \mathcal{O}(Y) \mapsto \alpha(x \mapsto f(g(x)))].$$

We now proceed to show that $\mathbf{F}\mathcal{V}$ is affine-linearly invariant with the maps σ_g^U induced from those of $\mathbf{E}\mathcal{V}$. The evaluation maps are natural w.r.t. inclusions because they evaluate a function on $\mathbf{C}^n \setminus \Delta$ only depending on some vertex algebra elements which are not changed by the extension maps of $\mathbf{E}\mathcal{V}$.

Proposition 3.4.10. *Let \mathcal{V} be a geometric vertex algebra. The evaluation maps are $\mathbf{C}^\times \ltimes \mathbf{C}$ -equivariant in the sense that the square*

$$\begin{array}{ccc} \mathbf{E}\mathcal{V}(U) & \xrightarrow{\text{ev}_U} & \overline{\mathcal{V}}_{bb} \\ \downarrow \sigma_g^U & & \downarrow g \\ \mathbf{E}\mathcal{V}(gU) & \xrightarrow{\text{ev}_{gU}} & \overline{\mathcal{V}}_{bb} \end{array}$$

commutes for $U \subseteq \mathbf{C}$ open and $g \in \mathbf{C}^\times \ltimes \mathbf{C}$.

Recall that translations act on $\overline{\mathcal{V}}_{bb}$, the set of bounded-below vectors in $\overline{\mathcal{V}}$, by Proposition 2.1.4.

Proof. We check equivariance for translation and multiplication maps separately. Let

$$[\alpha \otimes a] \in (\mathcal{O}'(U^m \setminus \Delta) \otimes \mathcal{V}^m)_{\Sigma_m} \subseteq \mathbf{E}\mathcal{V}(U).$$

If $g \in \mathbf{C}^\times \ltimes \mathbf{C}$, then

$$\begin{aligned} \text{ev}_{gU}(\sigma_g^U([\alpha \otimes a])) &= \text{ev}_{gU}([g_*\alpha \otimes (\lambda.a)]) = (g_*\alpha)(\mu(\lambda.a)) \\ &= \alpha(z \mapsto \mu(g.a)(g.z)). \end{aligned}$$

If $g \in \mathbf{C}^\times \ltimes \mathbf{C}$ is multiplication with some $\lambda \in \mathbf{C}^\times$, then

$$\begin{aligned} &\text{ev}_{gU}(\sigma_g^U([\alpha \otimes a])) \\ &= \alpha(z \mapsto \mu(\lambda.a)(\lambda.z)) \\ &= \alpha(z \mapsto \lambda.\mu(a)(z)) && (\mathbf{C}^\times\text{-equivariance}) \\ &= \lambda.\alpha(\mu(a)) && (\text{linearity}) \\ &= \lambda.\text{ev}_U([\alpha \otimes a]), \end{aligned}$$

and thus

$$g.\text{ev}_U([\alpha \otimes a]) = \text{ev}_{gU}(\sigma_g([\alpha \otimes a])).$$

If $g \in \mathbf{C}^\times \times \mathbf{C}$ is translation by some $w \in \mathbf{C}$, then

$$\begin{aligned} & \text{ev}_{gU}(\sigma_g^U([\alpha \otimes a])) \\ &= \alpha(\mathbf{z} \mapsto \mu(a)(\mathbf{z} + w)) \\ &= \alpha(\mathbf{z} \mapsto g.\mu(a)(\mathbf{z})) && \text{(associativity for } m = 0) \\ &= g.\alpha(\mu(a)) && \text{(linearity)} \\ &= g.\text{ev}_U([\alpha \otimes a]). \end{aligned}$$

Equivariance follows because $\mathbf{C}^\times \times \mathbf{C}$ is generated by the union of the two subgroups corresponding to \mathbf{C}^\times and \mathbf{C} . \square

Proposition 3.4.11. *\mathbf{FV} is an affine-linearly invariant prefactorization algebra on \mathbf{C} with its invariance maps induced from those of \mathbf{EV} .*

Proof. We have to show that the maps σ_g^U for E pass to the quotient, their properties then follow. The image of a disc under an affine-linear map is again a disc. The induced map on E preserves membership in R and hence $\overline{R^{disc}}$ because $R(U)$ is the kernel of ev_U and the evaluation maps are compatible with the actions of the affine-linear group by Proposition 3.4.10. \square

The rest of this subsection consists of a sequence of propositions proving that F is holomorphic, i. e., F is a holomorphic affine-linearly invariant pre-sheaf on \mathbf{C} . We first prove this for \mathcal{O}' using the Cauchy integral formula and the first of these propositions, then proceed to \mathcal{V} , which is a $G = \mathbf{C}^\times$ -representation, then E , and finally F .

Proposition 3.4.12. *Let $D \subseteq \mathbf{C}$ be a closed disc. The map*

$$\begin{aligned} D^\circ &\longrightarrow C^0(\partial D) \\ w &\longmapsto \left[z \mapsto \frac{1}{z - w} \right] \end{aligned}$$

is holomorphic.

Proof. Let $w_0 \in D^\circ$ so $\varepsilon := \text{dist}(\partial D, w) > 0$. Let $\delta = \varepsilon/2$. For $w \in B_\delta(w_0) \subseteq D^\circ$ and $z \in \partial D$,

$$q(z) := \left| \frac{w - w_0}{z - w_0} \right| < 1$$

and thus

$$\begin{aligned} \frac{1}{z - w} &= \frac{1}{(z - w_0) - (w - w_0)} = \frac{1}{z - w_0} \frac{1}{1 - \frac{w - w_0}{z - w_0}} \\ &= \frac{1}{z - w_0} \sum_{k=0}^{\infty} \left(\frac{w - w_0}{z - w_0} \right)^k \\ &= \sum_{k=0}^{\infty} \frac{1}{(z - w_0)^{k+1}} (w - w_0)^k. \end{aligned}$$

We now prove uniform convergence of functions of $z \in \partial D$, that is,

$$\left[z \mapsto \frac{1}{z-w} \right] = \sum_{k=0}^{\infty} \left[z \mapsto \frac{1}{(z-w_0)^{k+1}} \right] (w-w_0)^k$$

in $C^0(\partial D)$. The remainder term

$$\begin{aligned} \left| \frac{1}{z-w} - \sum_{k=0}^N \frac{1}{(z-w_0)^{k+1}} (w-w_0)^k \right| &= \left| \sum_{k=N+1}^{\infty} \frac{1}{(z-w_0)^{k+1}} (w-w_0)^k \right| \\ &\leq \frac{1}{|z-w_0|} \sum_{k=N+1}^{\infty} \left| \frac{w-w_0}{z-w_0} \right|^k \\ &\leq \frac{1}{|z-w_0|} \frac{q(z)^{N+1}}{1-q(z)} \end{aligned}$$

converges to zero uniformly in $z \in \partial D$ because

$$q(z) = \left| \frac{w-w_0}{z-w_0} \right| \leq \frac{\text{dist}(w, w_0)}{\text{dist}(\partial D, w_0)} < 1/2.$$

□

Proposition 3.4.13. *If $X \subseteq \mathbf{C}^n$ is open, then*

$$\begin{aligned} \delta : X &\longrightarrow \mathcal{O}'(X) \\ x &\longmapsto \delta_x \end{aligned}$$

is holomorphic.

Proof. Let $x \in X$. There is a closed polydisc $D = D_1 \times \dots \times D_n \subseteq X$ with center x . Let $d = d_1 \times \dots \times d_n$ be an open polydisc concentric with D s. t. the radius of d_i is strictly than the radius of D_i for $i = 1, \dots, n$. We claim that $\delta(d) \subseteq \mathcal{O}'_D(X)$ and that $\delta|_d$ is holomorphic as a map to the Banach space $\mathcal{O}'_D(X)$. Let $z \in d$ and $f \in \mathcal{O}(X)$. We have $\delta_z \in \mathcal{O}'_D(X)$ since

$$|\delta_z(f)| = |f(z)| \leq \|f\|_D.$$

The map

$$\begin{aligned} F : d &\longrightarrow \mathbf{C}^0(\partial D_1 \times \dots \times \dots \partial D_n) \\ w &\longmapsto \left[z \mapsto \prod_{i=1}^n \frac{1}{z_i - w_i} \right] \end{aligned}$$

is holomorphic since the maps

$$\begin{aligned} d &\xrightarrow{\text{pr}_i} d_i \longrightarrow \mathbf{C}^0(\partial D_i) \\ w &\longmapsto \left[z_i \mapsto \frac{1}{z_i - w_i} \right] \end{aligned}$$

are holomorphic for $i = 1, \dots, n$ by Proposition 3.4.12 and the pointwise exterior product of holomorphic maps is again holomorphic. The map

$$G : C^0(\partial D_1 \times \dots \times \partial D_2) \longrightarrow \mathcal{O}'_D(X)$$

$$f \longmapsto \left[g \mapsto \frac{1}{(2\pi i)^n} \int_{\partial D_1 \times \dots \times \partial D_n} f(z)g(z)dz \right]$$

is bounded because

$$\left| \frac{1}{(2\pi i)^n} \int_{\partial D_1 \times \dots \times \partial D_n} f(z)g(z)dz \right| \leq \|f\| \|g\|_D.$$

The map $\delta|_d$ is analytic because it is equal to $G \circ F$ by the Cauchy integral formula. \square

Proposition 3.4.14. *Let G be a complex-analytic group manifold acting on a complex-analytic manifold X via an analytic map $G \times X \rightarrow X$. Then the precosheaf \mathcal{O}' of analytic functionals on X is a holomorphic G -invariant cosheaf.*

Proof. Let $U, V \subseteq X$ be open. We wish to show that the action map

$$\mathcal{D}_{U,V} \longrightarrow \text{BVS}(\mathcal{O}'(U), \mathcal{O}'(V))$$

$$g \longmapsto g_*$$

is holomorphic on the interior $\mathcal{D}_{U,V}^\circ$ of the domain. For each $g \in \mathcal{D}_{U,V}^\circ$, the map g_* is the composite of

$$(u \mapsto (g, u))_* : \mathcal{O}'(U) \rightarrow \mathcal{O}'(\mathcal{D}_{U,V}^\circ \times U) \quad (24)$$

and the bounded, linear pushforward map

$$\mathcal{O}'(\mathcal{D}_{U,V}^\circ \times U) \longrightarrow \mathcal{O}'(V)$$

along the action map

$$\mathcal{D}_{U,V}^\circ \times U \longrightarrow V.$$

The latter of these is independent of g and pointwise composition with a fixed bounded linear map preserves holomorphicity, so it suffices to show that the map

$$I : \mathcal{D}_{U,V}^\circ \longrightarrow \text{BVS}(\mathcal{O}'(U), \mathcal{O}'(\mathcal{D}_{U,V}^\circ \times U))$$

$$g \longmapsto (u \mapsto (g, u))_*$$

is holomorphic. Proposition 3.4.13 for $X = \mathcal{D}_{U,V}^\circ$ implies that

$$\mathcal{D}_{U,V}^\circ \longrightarrow \mathcal{O}'(\mathcal{D}_{U,V}^\circ)$$

$$g \longmapsto \delta_g$$

is holomorphic. It follows that

$$\mathcal{D}_{U,V}^\circ \longrightarrow \text{BVS}(\mathcal{O}'(U), \mathcal{O}'(\mathcal{D}_{U,V}^\circ)) \bar{\otimes} \mathcal{O}'(U)$$

$$g \longmapsto [\alpha \mapsto \delta_g \bar{\otimes} \alpha]$$

is holomorphic because the map

$$\begin{aligned}\mathcal{O}'(U) &\longrightarrow \mathcal{O}'(\mathcal{D}_{U,V}^\circ) \bar{\otimes} \mathcal{O}'(U) \\ \alpha &\longmapsto \delta_g \bar{\otimes} \alpha\end{aligned}$$

is bounded and linear. Postcomposing with the exterior product of functionals

$$\times : \mathcal{O}'(\mathcal{D}_{U,V}^\circ) \bar{\otimes} \mathcal{O}'(U) \longrightarrow \mathcal{O}'(\mathcal{D}_{U,V}^\circ \times U)$$

preserves holomorphicity. Overall, the map

$$\begin{aligned}J : \mathcal{D}_{U,V}^\circ &\longrightarrow \mathcal{O}'(\mathcal{D}_{U,V}^\circ \times U) \\ g &\longmapsto \delta_g \times \alpha\end{aligned}$$

is holomorphic. It remains to identify I with J . Let $g \in \mathcal{D}_{U,V}^\circ$. It suffices to show that $I(g)$ and $J(g)$ agree on the dense subset of product functions $p(z, w) = f(z)h(w)$, $(z, w) \in \mathcal{D}_{U,V}^\circ$ for $f \in \mathcal{O}(\mathcal{D}_{U,V}^\circ)$ and $h \in \mathcal{O}(U)$:

$$J(g)(p) = \delta_g(z \mapsto \alpha(w \mapsto f(z)h(w))) = \delta_g(f)\alpha(h) = f(g)\alpha(h),$$

which coincides with

$$\begin{aligned}I(g)(p) &= ((u \mapsto (g, u))_*(\alpha))(p) = \alpha((u \mapsto (g, u))^*p) = \alpha(u \mapsto p(g, u)) \\ &= \alpha(u \mapsto f(g)h(u)) = f(g)\alpha(h).\end{aligned}$$

□

Proposition 3.4.15. *If A and B are holomorphic invariant precosheaves, then their pointwise completed tensor product $A \bar{\otimes} B$ is a holomorphic invariant precosheaf in a natural way.*

Proof. The function $\rho_{U,V}^{A \bar{\otimes} B}$ is holomorphic because

$$\bar{\otimes} : \text{BVS}(A(U), A(V)) \bar{\otimes} \text{BVS}(B(U), B(V)) \rightarrow \text{BVS}(A(U) \bar{\otimes} B(U), A(V) \bar{\otimes} B(V))$$

is a bounded linear map so postcomposing with it preserves holomorphicity. □

Proposition 3.4.16. *Let F_1, F_2 be affine-linearly invariant precosheaves on \mathbf{C} of complete bornological vector spaces. If F_2 is a quotient of F_1 , the invariance isomorphisms of F_1 are induced from those of F_2 , and F_1 is holomorphic, then F_2 is holomorphic.*

Here, quotient means cokernel by some map of precosheaves of complete bornological spaces.

Proof. Let X, Y be complete bornological spaces and $A \subseteq X$ a sub vector space. The map

$$\{f \in \text{BVS}(X, Y) \mid f|_A = 0\} \longrightarrow \text{BVS}(X/\bar{A}, Y) \quad (25)$$

sending f to the map induced by f is a bounded linear map. Let $i : G \hookrightarrow F_1$ be a pointwise inclusion of precosheaves of which $q : F_1 \rightarrow F_2$ is a cokernel. For $U, V \subseteq \mathbf{C}$ open, we apply this to $X = F_1(U)$, $A = R(U)$, and $Y = F_2(V)$ to conclude that the action map $\rho_{U,V}^{F_2}$ for F_2 is holomorphic because it is the composite of $q_* \circ \rho_{U,V}^{F_1}$ with the bounded linear map from (25). □

Proposition 3.4.17. *Direct sums of holomorphic precosheaves are again holomorphic.*

Proof. This follows from the compatibility of direct sums with the enrichment of CBVS over itself and Proposition 3.1.15 that a product-valued function is holomorphic if and only if its components are. \square

The same argument proves that direct sums of holomorphic representations are holomorphic, so \mathcal{V} is holomorphic as the direct sum of holomorphic one-dimensional representations. Thus the constant precosheaf assigning \mathcal{V} to every open is holomorphic.

Proposition 3.4.18. *The precosheaves E and F on \mathbf{C} are holomorphic.*

Proof. The preceding propositions imply that, for $n \geq 0$ the precosheaves given by $\mathcal{O}'(U^n \setminus \Delta) \otimes \mathcal{V}^n$ on $U \subseteq \mathbf{C}$ are holomorphic. Proposition 3.4.16 and Proposition 3.4.17 about quotients resp. sums imply that E is holomorphic. Proposition 3.4.16 implies that F is holomorphic. \square

3.5 Back to Geometric Vertex Algebras

Having obtained a holomorphic affine-linearly invariant prefactorization \mathbf{FV} from a geometric vertex algebra \mathcal{V} in the previous subsection, we now show that we can recover \mathcal{V} using the procedure of Section 3.3. First, we identify \mathbf{VFV} with \mathcal{V} as a \mathbf{Z} -graded bornological vector space, proving the discreteness of \mathbf{VFV} . Second, we identify the multiplication maps of \mathbf{VFV} with those of \mathcal{V} . This implies that \mathbf{FV} has meromorphic OPE. In summary, this means that \mathbf{FV} is the sort of prefactorization algebra that gives rise to a (geometric) vertex algebra, and we show that this geometric vertex algebra \mathbf{VFV} is isomorphic to \mathcal{V} .

We will use the evaluation map ev_U from $E(U)$ to $\overline{\mathcal{V}}$ for U a disc to get a map $\mathbf{VFV} \rightarrow \overline{\mathcal{V}}$ which will turn out to have image in \mathcal{V} . In contrast to the case of E/R , it is not an immediate consequence of the definition that ev_U induces a map from $F(U)$ to $\overline{\mathcal{V}}$, not even if $U \subseteq \mathbf{C}$ is a disc. The following lemma implies that ev_U is in fact well-defined on $F(U)$. Its proof uses the associativity of \mathcal{V} which we have not used so far, except for the case $m = 0$ which is translation invariance.

Lemma 3.5.1. *If $D \subset \mathbf{C}$ is an open disc or $U = \mathbf{C}$, then $R(D) = R^{\text{disc}}(D)$.*

Proof. First, let $D \subseteq \mathbf{C}$ be an open disc. The inclusion $R(D) \subseteq R^{\text{disc}}(D)$ follows from the definition by considering $d = D$ and $W = \emptyset$. For the other inclusion, let $d \subseteq D$ be an open disc and $W \subseteq D$ be open and disjoint from d . Our goal is to show that $M_U^{d,W}(R(d) \otimes E(W)) \subseteq R(D)$ because sets of this form span $R^{\text{disc}}(D)$. We may restrict attention to $r \otimes e \in R(d) \otimes E(W)$ with $r = \sum_{i=0}^m \alpha^i \otimes a^i$ with $\alpha^i \in \mathcal{O}'(d^i \setminus \Delta)$ for some $m \geq 0$ and $a^i \in \mathcal{V}^{\otimes i}$ and $e = \beta \otimes b$ with $\beta \in \mathcal{O}'(W^n \setminus \Delta)$ and $b \in \mathcal{V}^{\otimes n}$ for some $n \geq 0$. Let p be the center of d . Since r is a relation, so is $\sigma_{-p}^d(r)$, which means that

$$0 = \text{ev}(\sigma_{-p}^d(r)) = \sum_{i=0}^m ((-p)_* \alpha^i)(\mu(a^i)) = \sum_{i=0}^m \alpha^i[\mathbf{z} \mapsto \mu(a^i)(\mathbf{z} - p)], \quad (26)$$

and this is what we need to prove that $M_D^{d,W}(r \otimes e)$ is a relation by using the associativity of \mathcal{V} . We denote the first i coordinates of $D^{i+n} \subseteq \mathbf{C}^{i+n}$ by \mathbf{z} and the last n by \mathbf{w} .

$$\begin{aligned} & \text{ev}_D(M_D^{d,W}(r \otimes e)) \\ &= \sum_{i=0}^m \alpha^i \beta[(\mathbf{z}, \mathbf{w}) \mapsto \mu(a^i \otimes b)(z, w)] \\ &= \sum_{i=0}^m \beta[\mathbf{w} \mapsto \alpha^i[\mathbf{z} \mapsto \mu(a^i \otimes b)(z, w)]] \\ &= \sum_{i=0}^m \beta[\mathbf{w} \mapsto \alpha^i \left(\sum_{k \in \mathbf{Z}} [\mathbf{z} \mapsto \mu(p_k(\mu(a^i)(\mathbf{z} - p)) \otimes \mathbf{b})(p, \mathbf{w})] \right)] \end{aligned}$$

(because of associativity, which holds for tensors like a^i , not just elementary tensors)

$$= \sum_{i=0}^m \beta[\mathbf{w} \mapsto \sum_{k \in \mathbf{Z}} \alpha^i[\mathbf{z} \mapsto \mu(p_k(\mu(a^i)(\mathbf{z} - p)) \otimes \mathbf{b})(p, \mathbf{w})]]$$

(because of the continuity of α^i)

$$\begin{aligned} &= \sum_{i=0}^m \beta[\mathbf{w} \mapsto \sum_{k \in \mathbf{Z}} \mu(p_k (\alpha^i[\mathbf{z} \mapsto \mu(a^i)(\mathbf{z} - p)]) \otimes \mathbf{b})(p, \mathbf{w})] \\ &= \sum_{i=0}^m \beta[\mathbf{w} \mapsto \sum_{k \in \mathbf{Z}} \mu(\alpha^i[\mathbf{z} \mapsto p_k (\mu(a^i)(\mathbf{z} - p)]) \otimes \mathbf{b})(p, \mathbf{w})] \end{aligned}$$

(because evaluating analytic functionals on vector-valued functions is compatible with linear maps)

$$= \beta[\mathbf{w} \mapsto \sum_{k \in \mathbf{Z}} \mu(p_k \left(\sum_{i=0}^m \alpha^i[\mathbf{z} \mapsto \mu(a^i)(\mathbf{z} - p)] \right) \otimes \mathbf{b})(p, \mathbf{w})]$$

(because of the componentwise definition of the evaluation of analytic functionals on functions with values in a \mathbf{Z} -graded vector space)

$$\begin{aligned} &= \beta[\mathbf{w} \mapsto \sum_{k \in \mathbf{Z}} \mu(p_k \left(\sum_{i=0}^m \alpha^i[\mathbf{z} \mapsto \mu(a^i)(\mathbf{z} - p)] \right) \otimes \mathbf{b})(p, \mathbf{w})] \\ &= \beta[\mathbf{w} \mapsto \sum_{k \in \mathbf{Z}} \mu(0 \otimes \mathbf{b})(p, \mathbf{w})] \quad (\text{see Equation (26)}) \\ &= 0 \end{aligned}$$

The case of $U = \mathbf{C}$ follows because every analytic functional on \mathbf{C}^n arises from some bounded subset, so $U = \mathbf{C}$ reduces to the case of U some sufficiently large disc. \square

Corollary 3.5.2. *For every open subset $U \subseteq \mathbf{C}$, the evaluation map ev_U factors through $F(U)$.*

Proof. By Lemma 3.5.1, this is the case for $U = \mathbf{C}$ and $\text{ev}_U = \text{ev}_{\mathbf{C}} \circ E(U \subseteq \mathbf{C})$. \square

We denote the induced map $F(U) \rightarrow \bar{\mathcal{V}}$ by ev_U , too.

Remark 3.5.3. Another consequence of the lemma is that $F(D) = E(D)/R(D)$ for D a disc since we know $R(D)$ to be b-closed; see Remark 3.4.5. By the definition of $R(D)$, this means that we can identify $F(D)$ with a sub vector space of $\bar{\mathcal{V}}$.

The comparison map $I : \mathcal{V} \rightarrow \mathbf{VFV}$ is defined as follows. For every $\varepsilon > 0$, evaluation at zero is an element $\delta_0 \in \mathcal{O}'(B_\varepsilon(0))$, which is fixed under the action of \mathbf{D}^\times . For homogeneous $a \in \mathcal{V}_k$, we define $I(a) \in \mathbf{VFV}$ to be the element corresponding to

$$[\delta_0 \otimes a] \in F(B_\varepsilon(0))_k \cong F(0)_k = (\mathbf{VFV})_k.$$

It is clear that this is a linear map on the summands of \mathcal{V} , and we extend I to \mathcal{V} by linearity. In other words, I is determined by the formula $P_R(I(a)) = [\delta_0 \otimes a]$ where we recall that P_R is the natural map $\mathbf{VFV} \rightarrow \mathbf{FV}(B_R(0))$ for $R > 0$.

Proposition 3.5.4. *The linear map*

$$\begin{aligned} I : \mathcal{V} &\longrightarrow \mathbf{VFV} \\ a &\longmapsto [\delta_0 \otimes a] \quad (\text{for } a \text{ homogeneous}) \end{aligned}$$

is an isomorphism of \mathbf{Z} -graded complete bornological spaces.

Proof. It is clear that I is bounded since \mathcal{V} is discrete. For all $r > 0$, let J_r be the composite of $P_r : \mathbf{VFV} \rightarrow \mathbf{FV}(B_r(0))$ from Section 3.3 (for $F := \mathbf{FV}$) with the evaluation map $\mathbf{FV}(B_r(0)) \rightarrow \overline{\mathcal{V}}$. The evaluation map, which takes values in $\overline{\mathcal{V}}$ in general, maps elements arising from \mathbf{VFV} to \mathcal{V} because the evaluation map is \mathbf{C}^\times -equivariant. We can thus view J_r as a map to \mathcal{V} . The insertion-at-zero axiom implies that

$$J_r(I(a)) = \text{ev}_{B_R(0)}(P_R(I(a))) = \text{ev}_{B_R(0)}([\delta_0 \otimes a]) = \mu(a, 0) = a$$

for homogeneous a . This proves that $J_r \circ I = \text{id}_{\mathcal{V}}$ because every element of \mathcal{V} is a sum of homogeneous elements.

It remains to prove that J_r is injective and bounded. It is a bounded linear map as the composite of two bounded linear maps. The inclusion $\mathbf{VF} \hookrightarrow \overline{\mathbf{VF}}$ is the composite of P_R with the map $L^R : F(B_R(0)) \rightarrow \overline{\mathbf{VF}}$, so P_R is injective. Lemma 3.5.1 says that $R^{\text{disc}}(B_r(0)) = R(B_R(0))$, so the evaluation map $F(B_R(0)) \rightarrow \overline{\mathcal{V}}$ is injective. Thus $J_r = \text{ev}_{B_R(0)} \circ P_R$ is injective. \square

The previous proposition implies that \mathbf{VFV} is discrete. The comparison map I induces a map $\overline{I} : \overline{\mathcal{V}} \rightarrow \overline{\mathbf{VFV}}$ on the products of the weight spaces. Let $\tilde{\mu}$ denote the multiplication maps of \mathbf{VFV} .

Proposition 3.5.5. *I is a homomorphism w. r. t. the geometric vertex algebra structures, that is, for $m \geq 0$, and $z \in \mathbf{C}^m \setminus \Delta$, the square*

$$\begin{array}{ccc} \mathcal{V}^{\otimes m} & \xrightarrow{I^{\otimes m}} & \mathbf{VFV}^{\otimes m} \\ \downarrow \mu^z & & \downarrow \tilde{\mu}^z \\ \overline{\mathcal{V}} & \xrightarrow{\overline{I}} & \overline{\mathbf{VFV}} \end{array}$$

commutes.

It follows that \mathbf{VFV} satisfies the meromorphicity axiom; hence \mathbf{FV} has meromorphic OPE, \mathbf{VFV} is a geometric vertex algebra, and I is an isomorphism of geometric vertex algebras

$$\mathcal{V} \cong \mathbf{VFV}.$$

Proof. Let $k \in \mathbf{Z}$ and $R > 0$ be big enough so that $z_1, \dots, z_m \in B_R(0)$. The image in $F(B_R(0))$ of the k -th component of the lower composition evaluated at $a \in \mathcal{V}^{\otimes m}$ is

$$P_R(p_k(\overline{I}(\mu(a)(z)))) = P_R(I(p_k(\mu(a)(z)))) = [\delta_0 \otimes p_k(\mu(a)(z))],$$

and this evaluates to $p_k(\mu(a)(z))$. For the upper composition and a an elementary tensor $a_1 \otimes \dots \otimes a_m \in \mathcal{V}^{\otimes m}$ with $a_i \in V$ homogeneous, we get

$$\begin{aligned}
P_R(p_k(\tilde{\mu}(I^{\otimes m}(a))(z))) &= P_R(p_k(\tilde{\mu}([\delta_0 \otimes a_1] \otimes \dots \otimes [\delta_0 \otimes a_m])(z))) \\
&= P_R(p_k(\tilde{\mu}([\delta_{(z_1, \dots, z_m)} \otimes a_1 \otimes \dots \otimes a_m]))) \\
&= P_R(p_k(\tilde{\mu}([\delta_z \otimes a]))) \\
&= l_k^R([\delta_z \otimes a]) \\
&= \frac{1}{2\pi i} \oint w^{-k-1} w \cdot [\delta_z \otimes a] dw
\end{aligned}$$

and this evaluates to

$$\frac{1}{2\pi i} \oint w^{-k-1} w \cdot \text{ev}_{B_R(0)}([\delta_z \otimes a]) dw$$

because evaluation is equivariant. This equals

$$\frac{1}{2\pi i} \oint w^{-k-1} w \cdot \mu(a, z) dw = p_k(\mu(a, z))$$

in \mathcal{V}_k . Two elements of \mathbf{VFV}_k which evaluate to the same element of \mathcal{V}_k must be equal, so the square commutes and I is a homomorphism.

The meromorphicity axiom for (\mathcal{V}, μ) implies the meromorphicity axiom for $(\mathbf{VFV}, \tilde{\mu})$ since I and hence $I^{\otimes m}$ and \bar{I} are isomorphisms of vector spaces by Proposition 3.5.4. \square

Remark 3.5.6. It can be shown that the image of \mathcal{V} in $\mathbf{FV}(B_r(0))$ is dense for all $0 < r \leq \infty$. Furthermore, for $0 < r \leq R \leq \infty$, the maps

$$\mathcal{V} \hookrightarrow \mathbf{FV}(B_r(0)) \hookrightarrow \mathbf{FV}(B_R(0)) \hookrightarrow \bar{\mathcal{V}}$$

are embeddings of vector spaces, all of which are complete bornological spaces.

4 Factorization Algebras

In this section, we prove that \mathbf{FV} is a factorization algebra with values in the symmetric monoidal category of complete bornological spaces after recalling the definition of factorization algebras.

Definition 4.0.1. A prefactorization algebra F on a topological space X with values in a symmetric monoidal category \mathcal{C} is called *multiplicative* if the map

$$M_{U \sqcup V}^{U,V} : F(U) \otimes F(V) \rightarrow F(U \sqcup V)$$

is an isomorphism for all disjoint opens $U, V \subseteq X$ and the unit map $\mathbf{1}_{\mathcal{C}} \rightarrow F(\emptyset)$ is an isomorphism.

In our context, \otimes is the completed tensor product $\bar{\otimes}$ of complete bornological vector spaces defined in 3.1.12.

Recall that a presheaf G on a topological space X is called a *sheaf* if it satisfies descent along open covers, that is, if

$$G(Y) \rightarrow \prod_{U \in \mathcal{U}} G(U) \rightrightarrows \prod_{U, V \in \mathcal{U}} G(U \cap V)$$

is a limit diagram for every open cover \mathcal{U} of every open subset Y of X . Similarly, a precosheaf F on X is called a *cosheaf* if it satisfies codescent along open covers, that is, if the diagram

$$\bigoplus_{U, V \in \mathcal{U}} F(U \cap V) \rightrightarrows \bigoplus_{U \in \mathcal{U}} F(U) \rightarrow F(Y) \quad (27)$$

is a colimit diagram. In the definition of a factorization algebra, we demand codescent for special open covers called Weiss covers.

Definition 4.0.2. A family \mathcal{U} of open subsets of a topological space X is called a *Weiss cover* if, for every finite subset $F \subseteq X$, there is a $U \in \mathcal{U}$ with $F \subseteq U$. A precosheaf on a topological space X is a *Weiss cosheaf* if diagram (27) is a colimit diagram for all open subsets $Y \subseteq X$ and Weiss covers \mathcal{U} of Y .

Every Weiss cover is an open cover. A family \mathcal{U} of subsets of a space X is a Weiss cover if and only if $\{U^n \mid U \in \mathcal{U}\}$ is an open cover of X^n for all $n \geq 0$. Another way of phrasing the (Weiss) cosheaf condition is the following, if the category in which the precosheaf F takes values has all colimits: For $Y \subseteq X$ open and all (Weiss) open covers \mathcal{U} of Y closed under taking pairwise intersections, the natural map

$$\operatorname{colim}_{U \in \mathcal{U}} F(U) \longrightarrow F(X)$$

is an isomorphism, where \mathcal{U} considered as a poset under inclusions of subsets of Y , and the colimit is taken over the extension maps of F .

Definition 4.0.3. A *factorization algebra* is a prefactorization algebra which is multiplicative and whose underlying precosheaf is a Weiss cosheaf.

We fix a geometric vertex algebra \mathcal{V} and use the abbreviations $E = \mathbf{E}\mathcal{V}$ and $F = \mathbf{F}\mathcal{V}$.

Related work. Factorization algebras, as considered here, were introduced by Costello and Gwilliam and the main reference is [2] (see also [7]). We include multiplicativity in the definition of factorization algebra; in [2] multiplicativity is a property of some factorization algebras. Earlier, Beilinson and Drinfeld introduced factorization algebras in algebraic geometry [1].

4.1 More Functional-Analytic Preliminaries

There are two more topics in functional analysis which we did not treat in our summary of bornological spaces in Section 3.1 and reference in the proofs of multiplicativity of F and Weiss codescent. The first is nuclearity, the bornological vector space $\mathcal{O}'(U)$ of analytic functionals on an open $U \subseteq \mathbf{C}^n$ is nuclear. References for nuclear spaces include [18] and [10], the latter includes the bornological setting. The second is the generalization of the identity theorem for holomorphic functions with values in a Banach space to the case of a complete bornological space.

We call a completely normable bornological space *Hilbertable* if one of the complete norms inducing its bornology is a Hilbert space norm. The notion of Hilbert-Schmidt map between Hilbert spaces only depends on the bornologies of source and target. A complete bornological space is *nuclear* if every bounded subset B is bounded in a Hilbertable subobject H_1 so that H_1 is contained in a Hilbertable subobject H_2 such that the inclusion $H_1 \subseteq H_2$ is Hilbert-Schmidt. Let Hilb denote the category of Hilbert spaces and bounded linear maps. The functor from Hilb to CBVS sending a Hilbert space to its underlying complete bornological space is a full embedding with image the Hilbertable spaces. In [10, Chapter III], a locally convex space X is defined to be *nuclear* if X' is a nuclear bornological space, and this definition is shown to equivalent to some of the usual ones, e. g., in terms of Hilbert-Schmidt maps. Since $\mathcal{O}(X)$ is nuclear for $X \subseteq \mathbf{C}^n$ open, it follows that $\mathcal{O}'(X)$ is a nuclear bornological space. Our next goal is to prove that \mathcal{O}' is a cosheaf for Stein open covers of Stein subsets of \mathbf{C}^n . A Stein open cover is an open cover in which every open set is Stein. The intersection of two Stein open subsets is again Stein. For $X \subseteq \mathbf{C}^n$ open, let $\mathcal{O}_{L^2}(X)$ denote the Hilbert space of holomorphic L^2 -functions on X .

Proposition 4.1.1. *The presheaf \mathcal{O}_{L^2} of Hilbert spaces satisfies the sheaf condition for finite covers.*

Proof. Finite products and kernels exist in the category of Hilbert spaces and bounded maps, and their underlying vector spaces are products resp. kernels in the category of vector spaces. Let $V \subseteq \mathbf{C}^m$ be open and V_1, \dots, V_n cover V . If $f \in \mathcal{O}(V)$ denotes the result of gluing L^2 -functions on the V_i , then

$$\|f\|_{L^2(V)}^2 \leq \|f|_{V_1}\|_{L^2(V_1)}^2 + \dots + \|f|_{V_n}\|_{L^2(V_n)}^2,$$

so \mathcal{O}_{L^2} is a sheaf of vector spaces. This inequality also proves that the gluing map is bounded. \square

Corollary 4.1.2. *The precosheaf \mathcal{O}'_{L^2} of Hilbert spaces satisfies the cosheaf condition for finite covers.*

Proof. Taking the dual is a functor $\text{Hilb} \rightarrow \text{Hilb}^{\text{op}}$ which preserves finite limits because it is an equivalence of categories. \square

If U and X are open subsets of \mathbf{C}^n , we use the notation $U \subset\subset X$ to mean that $\bar{U} \subseteq X$ and \bar{U} is compact.

Proposition 4.1.3. *If an open $X \subseteq \mathbf{C}^n$ is Stein, then*

$$\mathcal{O}'(X) \cong \text{colim}_{U \subset\subset X} \mathcal{O}'_{L^2}(U)$$

in the category of bornological spaces.

Proof. The comparison map is

$$\begin{aligned} \Phi : \operatorname{colim}_{U \subset\subset X} \mathcal{O}'_{L^2}(U) &\longrightarrow \mathcal{O}'(X) \\ \alpha &\longmapsto [f \mapsto \alpha(f|_U)] \end{aligned}$$

whose well-definedness uses the fact that supremum norm $\|\cdot\|_{\infty, \bar{U}}$ of continuous functions on \bar{U} bounds $\|\cdot\|_{L^2(U)}$ from above. We prove that Φ is surjective. Let $\alpha \in \mathcal{O}'(X)$. There is a compact set $K \subset X$ s.t. α is continuous as a map from $(\mathcal{O}(X), \|\cdot\|_{\infty, K})$ to \mathbf{C} . Let $U \subset\subset X$ be a neighborhood of K . Let $\operatorname{res} : (\mathcal{O}(X), \|\cdot\|_{\infty, K}) \rightarrow (\mathcal{O}_{L^2}(U), \|\cdot\|_{\infty, K})$ denote the restriction map. It is continuous. Then

$$\tilde{\alpha} = \alpha \circ \operatorname{res}^{-1} : \operatorname{im} \operatorname{res} \rightarrow \mathbf{C}$$

is continuous, in particular well-defined. By the Hahn-Banach theorem, $\tilde{\alpha}$ extends to a continuous linear functional β defined on $(\mathcal{O}_{L^2}(U), \|\cdot\|_{\infty, K})$ with operator norm $\|\beta\| = \|\tilde{\alpha}\| = \|\alpha\|$. The map

$$i : (\mathcal{O}_{L^2}(U), \|\cdot\|_{L^2(U)}) \rightarrow (\mathcal{O}_{L^2}(U), \|\cdot\|_{\infty, K})$$

given by the identity map on elements is continuous as a consequence of the Cauchy integral formula. Then the class of $\beta \circ i$ is a preimage of α under Φ .

We prove that Φ is injective. Let $\alpha \in \mathcal{O}'_{L^2}(U)$, $U \subset\subset X$, represent a class in $\ker \Phi$. This means that $\alpha(f|_U) = 0$ for all $f \in \mathcal{O}(X)$. Without loss of generality, we may assume that \bar{U} is $\mathcal{O}(X)$ -convex, since the $\mathcal{O}(X)$ -convex hull of a compact set in a Stein manifold is again compact. Let $V \subset\subset X$ be a neighborhood of \bar{U} . It suffices to see that $\alpha(g|_U) = 0$ for all $g \in \mathcal{O}_{L^2}(V)$. Let $A(g) = \alpha(g|_U)$ for $g \in \mathcal{O}_{L^2}(V)$. We now argue that the map $A : \mathcal{O}_{L^2}(V) \rightarrow \mathbf{C}$ is continuous w.r.t. the semi-norm $\|\cdot\|_{\infty, \bar{U}}$ on the source. First, the semi-norm $\|\cdot\|_{\infty, \bar{U}}$ is at least as large as the L^2 -norm on U with respect to which α is continuous by assumption. Second, recall that the inclusion $\mathcal{O}_{L^2}(V) \hookrightarrow \mathcal{O}(V)$ is continuous. In particular, up to a constant factor, the semi-norm $\|\cdot\|_{L^2(V)}$ is at least as large as the continuous semi-norm $\|\cdot\|_{\infty, \bar{U}}$ on $\mathcal{O}(V)$. This concludes the argument that A is continuous for the semi-norm $\|\cdot\|_{\infty, \bar{U}}$. This is the semi-norm of uniform convergence on \bar{U} . Since \bar{U} is $\mathcal{O}(X)$ -convex, every holomorphic function on a neighborhood of \bar{U} can be uniformly approximated on \bar{U} by holomorphic functions on X ; see [6, Chapter VII, A, Theorem 6]. The restriction to V of a holomorphic function on X is a holomorphic L^2 -function on V because \bar{V} is compact. By assumption, α and thus A vanish on functions which extend holomorphically to X . The set of such functions is $\|\cdot\|_{\infty, \bar{U}}$ -dense in $\mathcal{O}_{L^2}(V)$, the source of A , so $A = 0$. Hence the class of α is zero in the colimit.

It remains to prove that Φ^{-1} is bounded. Let $B \subseteq \mathcal{O}'(X)$ be bounded. There is a compact $K \subseteq X$ and a $C > 0$ with $\|\alpha\| \leq C$ for all $\alpha \in B$ where the operator norm is taken w.r.t. $\|\cdot\|_{\infty, K}$ on $\mathcal{O}(X)$. Let $U \subset\subset X$ be a neighborhood of K . Then $\Phi^{-1}(\alpha)$ is represented by $\beta \circ i \in \mathcal{O}'_{L^2}(U)$, with

$$\|\beta \circ i\| \leq \|\beta\| \|i\| = \|\alpha\| \|i\| \leq C \|r\|.$$

Since $C \|r\|$ is independent of α , so $\Phi^{-1}(\alpha)$ contained in the image of a bounded subset in $\mathcal{O}'_{L^2}(U)$, namely the closed ball of radius $C \|i\|$. \square

Proposition 4.1.4. *The precosheaf \mathcal{O}' of complete bornological spaces satisfies the cosheaf condition for Stein open covers of Stein subsets of \mathbf{C}^n .*

Proof. Let Y be a Stein subset of \mathbf{C}^n and \mathcal{U} be a Stein open cover of Y closed under pairwise intersections. Corollary 4.1.2 implies that \mathcal{O}'_{L^2} satisfies the cosheaf property in the category of complete bornological spaces because the inclusion $\text{Hilb} \hookrightarrow \text{CBVS}$ preserves finite colimits. In the following, \mathcal{F} ranges over all finite sets of open subsets of Y closed under intersection such that, for every $V \in \mathcal{F}$, there is a $U \in \mathcal{U}$ with $V \subset\subset U$. The collection of such \mathcal{F} is a partially ordered set w.r.t. inclusions. It is filtered because \mathcal{F} and \mathcal{F}' are both contained in the closure of \mathcal{F} and \mathcal{F}' under pairwise intersections. The map $\text{colim}_{U \in \mathcal{U}} \mathcal{O}'(U) \rightarrow \mathcal{O}'(Y)$ factors as

$$\text{colim}_{U \in \mathcal{U}} \mathcal{O}'(U) \cong \text{colim}_{U \in \mathcal{U}} \text{colim}_{V \subset\subset U} \mathcal{O}'_{L^2}(V) \quad (\text{Proposition 4.1.3}) \quad (28)$$

$$\cong \text{colim}_{\mathcal{F}} \text{colim}_{V \in \mathcal{F}} \mathcal{O}'_{L^2}(V) \quad (29)$$

$$\cong \text{colim}_{\mathcal{F}} \mathcal{O}'_{L^2}(\cup \mathcal{F}) \quad (\text{Corollary 4.1.2}) \quad (30)$$

$$\cong \text{colim}_{W \subset\subset Y} \mathcal{O}'_{L^2}(W) \quad (31)$$

$$\cong \mathcal{O}'(Y), \quad (\text{Proposition 4.1.3}) \quad (32)$$

where it remains to justify (29) and (31). Given $U \in \mathcal{U}$ and $V \subset\subset U$, we use $\mathcal{F} = \{V\}$ to define the comparison map in (29). For its inverse, we pick a $U \in \mathcal{U}$ with $V \subset\subset U$ given \mathcal{F} and $V \in \mathcal{F}$; the resulting map from $\mathcal{O}'_{L^2}(V)$ to the r.h.s. of (29) is independent of the choice of U because \mathcal{U} is closed under pairwise intersections. For the map in (31), we use $W = \cup \mathcal{F}$. For its inverse, let $W \subset\subset \mathcal{F}$. Pick a finite subcover of \mathcal{U} of \overline{W} and shrink it to get a choice of \mathcal{F} . This is possible because \overline{W} is compact. \square

We let $X \otimes Y$ resp. $X \bar{\otimes} Y$ denote the tensor product of nuclear locally convex spaces X and Y resp. the completed tensor product of nuclear locally convex spaces. We include a proof that dualizing is monoidal, i. e., dualizing takes the completed tensor products of nuclear locally convex spaces to the completed tensor product of complete bornological spaces, see Definition 3.1.12.

Proposition 4.1.5. *Let X and Y be nuclear locally convex spaces. Then*

$$\begin{aligned} X' \bar{\otimes} Y' &\longrightarrow (X \bar{\otimes} Y)' \\ \alpha \otimes \beta &\longmapsto [x \otimes y \mapsto \alpha(x)\beta(y)] \end{aligned}$$

is an isomorphism of nuclear convex bornological spaces.

If X and Y are Hilbert spaces, then

$$X' \bar{\otimes}_{HS} Y' \cong (X \bar{\otimes}_{HS} Y)'$$

as Hilbert spaces where $\bar{\otimes}_{HS}$ denotes the completed Hilbert-Schmidt tensor product. If X and Y are Banach spaces, then the bornology on the projective tensor product $X \otimes_{\pi} Y$ agrees with the bornology on $X \otimes Y$ from Definition 3.1.11, so the underlying bornological space of the completed projective tensor product of X and Y is the completed tensor product of the underlying complete bornological spaces of X and Y . The completed tensor product of nuclear bornological spaces can be computed in terms of the completed Hilbert-Schmidt tensor product, similar to how the projective tensor product of nuclear locally convex topological vector spaces can be computed in terms of the Hilbert-Schmidt tensor product.

Proof. It suffices to prove that the comparison map $c : X' \bar{\otimes} Y' \rightarrow (X \otimes Y)'$ is an isomorphism of bornological spaces, it then follows that the map $X' \bar{\otimes} Y' \rightarrow (X \bar{\otimes} Y)'$ is an isomorphism because $(X \bar{\otimes} Y)' \rightarrow (X \otimes Y)'$ is an isomorphism by the universal property of the completion $X \bar{\otimes} Y$. Letting p, q run over a family of semi-norms exhibiting X resp. Y as an inverse limit of Hilbert-Schmidt maps, the map c factors as

$$\begin{aligned} X' \bar{\otimes} Y' &\cong \operatorname{colim}_p X'_p \bar{\otimes} \operatorname{colim}_q Y'_q \\ &\cong \operatorname{colim}_{p,q} (X'_p \bar{\otimes}_\pi Y'_q) \\ &\cong \operatorname{colim}_{p,q} (X'_p \bar{\otimes}_{HS} Y'_q) \\ &\cong \operatorname{colim}_{p,q} (X_p \bar{\otimes}_{HS} Y_q)' \\ &\cong \operatorname{colim}_{p,q} (X_p \otimes_{HS} Y_q)' \\ &\cong \left(\lim_{p,q} (X_p \otimes_{HS} Y_q) \right)' \end{aligned}$$

(since the underlying vector space of $X_p \otimes_{HS} Y_q$ is independent of p and q)

$$\cong (X \otimes Y)' .$$

□

Therefore, if $U \subseteq \mathbf{C}^m$ and $V \subseteq \mathbf{C}^n$, there is an isomorphism

$$\mathcal{O}'(U) \bar{\otimes} \mathcal{O}'(V) \cong (\mathcal{O}(U) \bar{\otimes} \mathcal{O}(V))' \cong \mathcal{O}(U \times V)' = \mathcal{O}'(U \times V)$$

of complete bornological spaces. This map sends $\alpha \otimes \beta$ to a functional called $\alpha \times \beta$, which on product functions $f(z)g(w)$ of $(z, w) \in U \times V$ is $\alpha(f)\beta(g)$, and hence agrees with the map defined by Equation 19. A version of $\mathcal{O}'(U) \bar{\otimes} \mathcal{O}'(V) \cong \mathcal{O}'(U \times V)$ in the context of locally convex spaces is stated in the Corollary to Theorem 51.6 of [18]. The author has found [8, Chapter 4] helpful for the proof of $\mathcal{O}(U) \bar{\otimes} \mathcal{O}(V) \cong \mathcal{O}(U \times V)$, even though this book does not seem to cleanly state this isomorphism in the generality we use it.

We now formulate the identity theorem for holomorphic functions with values in a complete bornological space and deduce it from the identity theorem for holomorphic functions with values in a Banach space.

Proposition 4.1.6. *Let f be a holomorphic function on a domain $D \subseteq \mathbf{C}$ with values in a complete bornological space. If D contains an accumulation point of $f^{-1}(0)$, then f is zero.*

Proof. Let S be the set of points $z \in D$ which have an open neighborhood $U \subseteq D$ s. t. $f|_U = 0$. By definition S is open. Locally, f admits a power series expansion in a Banachable subobject of X and thus

$$S = \bigcap_{n \geq 0} \{z \in D \mid f^{(n)}(z) = 0\} .$$

Thus S is closed as the intersection of sets which are closed by the continuity of f and its derivatives w. r. t. the topology of b-closed sets. It remains to

show that S is not empty because the claim then follows since D is connected. Let $z_0 \in f^{-1}(0)$ be an accumulation point of $f^{-1}(0)$. After restricting f to an open neighborhood of z_0 , we may apply the identity theorem for holomorphic functions with values in a Banach space and conclude that $z \in S$. \square

Corollary 4.1.7. *Let f be a holomorphic function on a domain $D \subseteq \mathbf{C}$ with values in a complete bornological vector space X . Let Y be a sub vector space of X . If $f^{-1}(Y)$ contains an accumulation point, then $\text{im } f \subseteq \overline{Y}$.*

Proof. Let $q : X \rightarrow X/\overline{Y}$ denote the quotient map. Its target X/\overline{Y} is the quotient of a complete bornological vector space by a b-closed subspace and hence complete. Proposition 4.1.6 applies to $q \circ f$, so $q \circ f = 0$ and thus $\text{im } f \subseteq \overline{Y}$. \square

4.2 Multiplicativity

Proposition 4.2.1. *The prefactorization algebra E is multiplicative.*

Proof. Let $U, V \subseteq \mathbf{C}$ be open and disjoint. The multiplication map

$$M_{U \sqcup V}^{U, V} : E(U) \bar{\otimes} E(V) \rightarrow E(U \sqcup V)$$

factors as

$$\begin{aligned} & E(U) \bar{\otimes} E(V) \\ &= \left(\bigoplus_{m \geq 0} (\mathcal{O}'(U^m \setminus \Delta_m) \otimes \mathcal{V}^{\otimes m})_{\Sigma_m} \right) \bar{\otimes} \left(\bigoplus_{n \geq 0} (\mathcal{O}'(V^n \setminus \Delta_n) \otimes \mathcal{V}^{\otimes n})_{\Sigma_n} \right) \\ &\cong \bigoplus_{m \geq 0} \bigoplus_{0 \leq i \leq m} (\mathcal{O}'(U^i \setminus \Delta_i) \otimes \mathcal{V}^{\otimes i})_{\Sigma_i} \bar{\otimes} \left(\mathcal{O}'(V^{m-i} \setminus \Delta_{m-i}) \otimes \mathcal{V}^{\otimes(m-i)} \right)_{\Sigma_{m-i}} \\ &\cong \bigoplus_{m \geq 0} (\mathcal{O}'((U \sqcup V)^m \setminus \Delta) \otimes \mathcal{V}^{\otimes m})_{\Sigma_m}, \end{aligned}$$

where the first isomorphism holds because the completed tensor product commutes with direct sums separately in each variable. To deduce that the second map is an isomorphism, we fix $m \geq 0$ and consider the map between the m -th summands. It factors as

$$\begin{aligned} & \bigoplus_{0 \leq i \leq m} (\mathcal{O}'(U^i \setminus \Delta_i) \otimes \mathcal{V}^{\otimes i})_{\Sigma_i} \bar{\otimes} \left(\mathcal{O}'(V^{m-i} \setminus \Delta_{m-i}) \otimes \mathcal{V}^{\otimes(m-i)} \right)_{\Sigma_{m-i}} \\ &\cong \bigoplus_{0 \leq i \leq m} \left(\mathcal{O}'(U^i \setminus \Delta_i) \otimes \mathcal{V}^{\otimes i} \bar{\otimes} \mathcal{O}'(V^{m-i} \setminus \Delta_{m-i}) \otimes \mathcal{V}^{\otimes(m-i)} \right)_{\Sigma_i \times \Sigma_{m-i}} \\ &\cong \bigoplus_{0 \leq i \leq m} \left(\mathcal{O}'(U^i \setminus \Delta_i) \bar{\otimes} \mathcal{O}'(V^{m-i} \setminus \Delta_{m-i}) \otimes \mathcal{V}^{\otimes i} \otimes \mathcal{V}^{\otimes(m-i)} \right)_{\Sigma_i \times \Sigma_{m-i}} \\ &\cong \bigoplus_{0 \leq i \leq m} \left(\mathcal{O}'((U^i \setminus \Delta_i) \times (V^{m-i} \setminus \Delta_{m-i})) \otimes \mathcal{V}^{\otimes i} \otimes \mathcal{V}^{\otimes(m-i)} \right)_{\Sigma_i \times \Sigma_{m-i}} \\ &\cong (\mathcal{O}'(U^m \setminus \Delta) \otimes \mathcal{V}^{\otimes m})_{\Sigma_m}, \end{aligned}$$

where the last isomorphism requires further justification. Let $\mathbf{m} = \{1, \dots, m\}$. For finite sets I , in particular $I \subseteq \mathbf{m}$, we have the finite product \mathbf{C}^I and its subset

$$\Delta_I := \{z \in \mathbf{C}^I \mid z_i = z_j \text{ for some } i, j \in I \text{ with } i \neq j\}.$$

There are Σ_m -equivariant complex-analytic isomorphisms

$$\begin{aligned} (U \sqcup V)^m \setminus \Delta_m &= (U \sqcup V)^{\mathbf{m}} \setminus \Delta_{\mathbf{m}} \\ &\cong \bigsqcup_{I \subseteq \mathbf{m}} (U^I \times V^{\mathbf{m} \setminus I}) \setminus \Delta_I \\ &\cong \bigsqcup_{I \subseteq \mathbf{m}} (U^I \setminus \Delta_I) \times (V^{\mathbf{m} \setminus I} \setminus \Delta_{\mathbf{m} \setminus I}) \quad (U \cap V = \emptyset) \end{aligned}$$

given by the inclusions and associativity isomorphism. Combining the induced isomorphism on \mathcal{O}' with the isomorphism $\mathcal{O}'(X) \bar{\otimes} \mathcal{O}'(Y) \cong \mathcal{O}'(X \times Y)$ for X, Y

open subsets of finite-dimensional complex vector spaces, we get

$$\begin{aligned} \bigoplus_{I \subseteq \mathbf{m}} \mathcal{O}'(U^I \setminus \Delta_I) \bar{\otimes} \mathcal{O}'(V^{\mathbf{m} \setminus I} \setminus \Delta_{\mathbf{m} \setminus I}) &\cong \bigoplus_{I \subseteq \mathbf{m}} \mathcal{O}'((U^I \setminus \Delta_I) \times (V^{\mathbf{m} \setminus I} \setminus \Delta_{\mathbf{m} \setminus I})) \\ &\cong \mathcal{O}'((U \sqcup V)^{\mathbf{m}} \setminus \Delta_{\mathbf{m}}), \end{aligned}$$

again Σ_m -equivariantly. Tensoring with the Σ_m -representation $\mathcal{V}^{\mathbf{m}}$ gives a Σ_m -isomorphism

$$\begin{aligned} \bigoplus_{I \subseteq \mathbf{m}} (\mathcal{O}'(U^I \setminus \Delta_I) \otimes \mathcal{V}^I) \bar{\otimes} (\mathcal{O}'(V^{\mathbf{m} \setminus I} \setminus \Delta_{\mathbf{m} \setminus I}) \otimes \mathcal{V}^{\mathbf{m} \setminus I}) \\ \cong \mathcal{O}'((U \sqcup V)^{\mathbf{m}} \setminus \Delta_{\mathbf{m}}) \otimes \mathcal{V}^{\mathbf{m}}. \end{aligned}$$

This yields the second isomorphism in

$$\begin{aligned} \bigoplus_{0 \leq i \leq m} (\mathcal{O}'(U^i \setminus \Delta_i) \otimes \mathcal{V}^{\otimes i})_{\Sigma_i} \bar{\otimes} (\mathcal{O}'(V^{m-i} \setminus \Delta_{m-i}) \otimes \mathcal{V}^{\otimes(m-i)})_{\Sigma_{m-i}} \\ \cong \bigoplus_{0 \leq i \leq m} ((\mathcal{O}'(U^i \setminus \Delta_i) \otimes \mathcal{V}^i) \bar{\otimes} (\mathcal{O}'(V^{m-i} \setminus \Delta_{m-i}) \otimes \mathcal{V}^{m-i}))_{\Sigma_i \times \Sigma_{m-i}} \\ \cong (\mathcal{O}'((U \sqcup V)^m \setminus \Delta_m) \otimes \mathcal{V}^m)_{\Sigma_m}, \end{aligned}$$

and the composite is the map on the m -th summand. \square

Proposition 4.2.2. *The prefactorization algebra F is multiplicative.*

Proof. Let $U, V \subseteq \mathbf{C}$ be open and disjoint. Recall from the proof of Proposition 3.4.9 that the map $E(U) \bar{\otimes} E(V) \rightarrow F(U) \bar{\otimes} F(V)$ is the cokernel of the multiplication map

$$T = S^{disc}(U) \bar{\otimes} E(V) \oplus E(U) \bar{\otimes} S^{disc}(V) \rightarrow E(U) \bar{\otimes} E(V).$$

We already know from Proposition 4.2.1 that $E(U) \bar{\otimes} E(V) \cong E(U \sqcup V)$ via the same map that induces the multiplication map $F(U) \bar{\otimes} F(V) \rightarrow F(U \sqcup V)$ given by Proposition 3.4.9. Therefore, it remains to see that the cokernel of $E(U \sqcup V)$ by $S^{disc}(U \sqcup V)$ agrees with the cokernel of

$$T \rightarrow E(U) \bar{\otimes} E(V) \rightarrow E(U \sqcup V).$$

For this, we use that there is an isomorphism

$$S^{disc}(U \sqcup V) \cong S^{disc}(U) \bar{\otimes} \left(\bigoplus_{W' \subseteq V} E(W') \right) \oplus \left(\bigoplus_{W' \subseteq U} E(W') \right) \bar{\otimes} S^{disc}(V) \quad (33)$$

compatible with the multiplication maps to $E(U \sqcup V)$. Using (33), we see that $S^{disc}(U \sqcup V)$ has maps to and from T compatible with the maps to $E(U \sqcup V)$. Hence the images of $S^{disc}(U \sqcup V)$ and T in $E(U \sqcup V)$ agree. We obtain the isomorphism in (33) from the fact that any disc $d \subseteq U \sqcup V$ is contained in

exactly one of U and V and from the multiplicativity of E ,

$$\begin{aligned}
S^{disc}(U \sqcup V) &\cong \bigoplus_{\substack{d \subseteq U \\ W \subseteq (\bar{U} \sqcup V) \setminus d}} E(W) \bar{\otimes} R(d) \oplus \bigoplus_{\substack{d \subseteq V \\ W \subseteq (\bar{U} \sqcup V) \setminus d}} E(W) \bar{\otimes} R(d) \\
&\cong \bigoplus_{\substack{d \subseteq U \\ W \subseteq (\bar{U} \sqcup V) \setminus d}} E(W \cap V) \bar{\otimes} E(W \cap U) \bar{\otimes} R(d) \\
&\quad \oplus \bigoplus_{\substack{d \subseteq V \\ W \subseteq (\bar{U} \sqcup V) \setminus d}} E(W \cap U) \bar{\otimes} E(W \cap V) \bar{\otimes} R(d) \\
&\cong \bigoplus_{\substack{d \subseteq U \\ W \subseteq U \setminus d, W' \subseteq V}} E(W') \bar{\otimes} E(W) \bar{\otimes} R(d) \\
&\quad \oplus \bigoplus_{\substack{d \subseteq V \\ W \subseteq V \setminus d, W' \subseteq U}} E(W') \bar{\otimes} E(W) \bar{\otimes} R(d) \\
&\cong \left(\bigoplus_{\substack{d \subseteq U \\ W \subseteq U \setminus d}} E(W) \bar{\otimes} R(d) \right) \bar{\otimes} \left(\bigoplus_{W' \subseteq V} E(W') \right) \\
&\quad \oplus \left(\bigoplus_{W' \subseteq U} E(W') \right) \otimes \left(\bigoplus_{\substack{d \subseteq V \\ W \subseteq V \setminus d}} E(W) \bar{\otimes} R(d) \right) \\
&\cong S^{disc}(U) \bar{\otimes} \left(\bigoplus_{W' \subseteq V} E(W') \right) \oplus \left(\bigoplus_{W' \subseteq U} E(W') \right) \bar{\otimes} S^{disc}(V).
\end{aligned}$$

□

4.3 Weiss Codescent

Proposition 4.3.1. *The precosheaf E is a Weiss cosheaf of complete bornological vector spaces.*

Proof. Let X be an open subset of \mathbf{C} and \mathcal{U} be a Weiss cover of X . There is a commutative diagram

$$\begin{array}{ccc} \operatorname{colim}_{U \in \mathcal{U}} E(U) & \xrightarrow{\hspace{10em}} & E(X) \\ \cong \uparrow & & \parallel \\ \bigoplus_{n \geq 0} \left(\operatorname{colim}_{U \in \mathcal{U}} (\mathcal{O}'(U^n \setminus \Delta)) \otimes V^{\otimes n} \right)_{\Sigma_n} & \longrightarrow & \bigoplus_{n \geq 0} (\mathcal{O}'(X^n \setminus \Delta) \otimes V^{\otimes n})_{\Sigma_n} \end{array}$$

in BVS. The left hand map is an isomorphism because colimits commute with each other and with the completed tensor product. The bottom map is an isomorphism because \mathcal{O}' is a cosheaf for Stein open covers of Stein subsets of \mathbf{C}^n for each $n \geq 0$ by Proposition 4.1.4 and $U^n \setminus \Delta$ is Stein for all open subsets $U \subseteq \mathbf{C}$. Since \mathcal{U} is a Weiss cover, $\{U^n\}_{U \in \mathcal{U}}$ is an open cover of X^n for each $n \geq 0$. The cosheaf property applied to the open cover $\{U^n \setminus \Delta\}_{U \in \mathcal{U}}$ of $X^n \setminus \Delta$ implies that the bottom map is an isomorphism. \square

Theorem 4.3.2. *The precosheaf F is a Weiss cosheaf of complete bornological vector spaces.*

Lemma 4.3.3. *Let d be a disc with center $z_0 \in \mathbf{C}$ and let $\delta > 0$ s. t. $B_\delta(z_0) \subseteq d$. Then*

$$R(d) = \overline{i(R(B_\delta(z_0)))} \subseteq E(d),$$

where $i = E(B_\delta(z_0) \hookrightarrow d)$ and the b-closure is taken in $E(d)$.

Roughly speaking, the proof of the lemma involves viewing an element of $R(d)$ as the endpoint of a path $[\varepsilon, 1] \rightarrow R(d)$ which extends to a holomorphic function taking values in $i(R(B_\delta(z_0)))$ near its beginning. The path is given by scaling down a given relation, and this preserves the property of being a relation, i. e., evaluating to zero, because the evaluation map is equivariant. It then follows that the holomorphic function has image in the b-closure of the image of $R(B_\delta(z_0))$ by Corollary 4.1.7 to the identity theorem for holomorphic functions.

Proof of Lemma 4.3.3. It suffices to consider the case of $z_0 = 0$ by the translation invariance of E . The inclusion “ \supseteq ” holds because $R(d)$ is b-closed in $E(d)$ by Remark 3.4.5. To prove “ \subseteq ”, let $x \in R(d)$. There exists a strictly smaller concentric disc $d' \subset d$ and an $x' \in E(d')$ which is mapped to x by the map induced by the inclusion of d' into d because E is a Weiss cosheaf and the set of concentric discs strictly smaller than d forms a Weiss cover of d . We have $x' \in R(d')$ because the evaluation map is compatible with inclusions. Let $r > r' > 0$ denote the radius of d resp. d' . We can restrict attention to the case that d' strictly contains $B_\delta(0)$, that is, that $r' > \delta$, because otherwise $d' \subseteq B_\delta(0)$ and thus $x \in i(R(B_\delta(0)))$. Let $R = \frac{r}{r'}$ and $A = B_R(0) - 0 \subseteq \mathbf{C}^\times$. From now on, we only consider rotations and dilations among the affine-linear isomorphisms

of \mathbf{C} , and we identify A with a subset of the interior of $\mathcal{D}_{d',d}$ via the group homomorphism $\mathbf{C}^\times \hookrightarrow \mathbf{C}^\times \times \mathbf{C}$. The map

$$\begin{aligned} f : A &\longrightarrow E(d) \\ q &\longmapsto \rho_{d',d}(q)(x') \end{aligned}$$

is holomorphic because E is holomorphic. Note that $1 \in A$ since $R > 1$ because d' is strictly smaller than d . Also note that $f(1)$ is the image of x' under the inclusion of d' into d and hence $f(1) = x$. Let N be the set of $q \in A$ with $|q| < \frac{\delta}{r'}$. If $q \in N$, then $q \cdot d' \subseteq B_\delta(0)$, so that $q \in \mathcal{D}_{d',B_\delta(0)}$ and

$$\rho_{d',d}(q) = \rho_{B_\delta(0),d}(1) \circ \rho_{d',B_\delta(0)}(q) = i \circ \rho_{d',B_\delta(0)}(q).$$

The evaluation map is equivariant by Proposition 3.4.10, so

$$f(q) = \rho_{d',d}(q)(x') \in i(R(B_\delta(0)))$$

for $q \in N$. Since $N \subseteq A$ is non-empty and open, it has an accumulation point inside A . Corollary 4.1.7 applies to the holomorphic function f defined on the domain A , the complete bornological vector space $E(d)$, and its subspace $i(R(B_\delta(0)))$ to which N maps. Therefore, the image of f is contained in $\overline{i(R(B_\delta(0)))}$. In particular, $x = f(1)$ is an element of this set. \square

Definition 4.3.4. Let X be an open subset of \mathbf{C} and let \mathcal{U} be a Weiss cover of X . The *relation on discs subordinate to \mathcal{U}* are defined as

$$R_{\mathcal{U}}^{disc}(X) = \text{im} \left(\bigoplus_{d,W} R(d) \otimes E(W) \longrightarrow E(X) \right)$$

where the direct sum is taken over pairs of a disc $d \subseteq X$ and $W \subseteq X$ open and disjoint from d such that there exists a $U \in \mathcal{U}$ s. t. d and W are both subsets of U .

The last condition in the definition of $R_{\mathcal{U}}^{disc}(X)$, namely that d and W are contained in a single $U \in \mathcal{U}$, is the only difference from the definition of $R^{disc}(X)$.

Lemma 4.3.5. *If X is an open subset of \mathbf{C} and \mathcal{U} a Weiss cover of X , then*

$$R^{disc}(X) \subseteq \overline{R_{\mathcal{U}}^{disc}(X)},$$

where the *b-closure* is taken in $E(X)$.

Proof. By definition, the vector space $R^{disc}(X)$ is generated by elements of the form $M_X^{d,W}(f \otimes g)$ for $f \in R(d)$ and $g \in E(W)$ for some disc $d \subseteq X$ and $W \subseteq X$ open and disjoint from d . Let z_0 be the center of d . The set

$$\mathcal{W} = \{V \subseteq W \text{ open} \mid \exists U \in \mathcal{U} : U \cap W = V, z_0 \in U\}$$

is a Weiss cover of W as we now check. Let $S \subseteq W$ be finite. Since $S \cup \{z_0\}$ is a finite subset of X , there is a $U \in \mathcal{U}$ s. t. $S \cup \{z_0\} \subseteq U$. Thus

$$V := U \cap W \supseteq S.$$

By the definition of the subspace topology, V is an open subset of W . This concludes the proof that \mathcal{W} is a Weiss cover of W .

Let $f \otimes g \in R(d) \otimes E(W)$. Since \mathcal{W} is a Weiss cover of W and E is a Weiss cosheaf, there are $W_1, \dots, W_k \in \mathcal{W}$ and $g_i \in \text{im } E(W_i \hookrightarrow W)$ for $i = 1, \dots, k$ s. t.

$$g = \sum_{i=1}^k g_i,$$

and thus

$$f \otimes g = \sum_{i=1}^k f \otimes g_i.$$

It suffices to consider each summand individually because $\overline{R_{\mathcal{U}}^{disc}(X)}$ is closed under addition because it is a sub vector space as the b-closure of a sub vector space, see [9, 2:12 Proposition 1]. We consider each of these summands individually and may therefore restrict attention to the case that there is a $U \in \mathcal{U}$ s. t. $g \in \text{im } E(U \hookrightarrow W)$ and $z_0 \in U$. Our goal is to prove that $M_X^{d,W}(f \otimes g) \in \overline{R_{\mathcal{U}}^{disc}(X)}$. There is a $\delta > 0$ s. t. $B_\delta(z_0) \subseteq d \cap U$ because both U and d are open and contain z_0 . Lemma 4.3.3 says that $R(d) = \overline{i(R(B_\delta(z_0)))}$ where the b-closure is taken in $E(d)$ and i is the map from $E(B_\delta(z_0))$ to $E(d)$ induced by the inclusion of $B_\delta(z_0)$ into d . Thus $f \otimes g \in \overline{i(R(B_\delta(z_0)))} \otimes E(W)$. The claim follows because the map $M_X^{d,W}(- \otimes g) : E(d) \rightarrow E(X)$ is bounded and hence continuous, so

$$\begin{aligned} M_X^{d,W}(f \otimes g) &\in M_X^{d,W}(\overline{i(R(B_\delta(z_0)))} \otimes g) \\ &\subseteq \overline{M_X^{d,W}(i(R(B_\delta(z_0))) \otimes g)} \subseteq \overline{R_{\mathcal{U}}^{disc}(X)}. \end{aligned}$$

□

Proof of Theorem 4.3.2. Let X be an open subset of \mathbf{C} and let \mathcal{U} be a Weiss cover of X closed under taking pairwise intersections. Our goal is to show that the right vertical map in the following commutative diagram, whose maps are explained below, is an isomorphism.

$$\begin{array}{ccccc} \text{colim}_{U \in \mathcal{U}} R^{disc}(U) & \xrightarrow{i_{\mathcal{U}}} & \text{colim}_{U \in \mathcal{U}} E(U) & \xrightarrow{q_{\mathcal{U}}} & \text{colim}_{U \in \mathcal{U}} F(U) \\ \downarrow \gamma_R & & \downarrow \gamma_E & & \downarrow \gamma_F \\ R^{disc}(X) & \xrightarrow{i_X} & E(X) & \xrightarrow{q_X} & F(X) \end{array}$$

The colimits are taken over \mathcal{U} . The vertical maps are induced by inclusions between subsets of \mathbf{C} . Let

$$q_X : E(X) \rightarrow E(X) / \overline{R^{disc}(X)} = F(X)$$

be the quotient map and let

$$q_{\mathcal{U}} : \text{colim}_{U \in \mathcal{U}} E(U) \longrightarrow \text{colim}_{U \in \mathcal{U}} F(U)$$

be the map induced by the quotient maps q_U for $U \in \mathcal{U}$ instead of X . In the lower row, the map i_X is the inclusion, and the second map q_X is its cokernel

in the category of complete bornological spaces. The upper row is the colimit of the analogue of the lower row for $U \in \mathcal{U}$ instead of X . In the upper row, it also holds that the second map is the cokernel of the first, because colimits commute with each other.

The map γ_E is an isomorphism because E is a cosheaf by Proposition 4.3.1. Hence γ_F is an isomorphism if $\overline{\text{im } i_X} = \overline{\gamma_E(\text{im}(i_U))}$ where the b-closure is taken in $E(X)$, because cokernels in CBVS are computed by modding out the b-closure of the image. Note that $\text{im } i_X = R^{disc}(X)$ by definition. On the other side,

$$\gamma_E(\text{im}(i_U)) = \text{im} \left(\bigoplus_{d,W} R(d) \otimes E(W) \longrightarrow E(X) \right) = R_{\mathcal{U}}^{disc}(X),$$

where $d \subseteq X$ is a disc and $W \subseteq X$ is open and disjoint from d such that there exists a $U \in \mathcal{U}$ s. t. d and W are both subsets of U , and the last equality is the definition of $R_{\mathcal{U}}^{disc}(X)$. Therefore, it remains to show $\overline{R^{disc}(X)} = \overline{R_{\mathcal{U}}^{disc}(X)}$. The inclusion “ \supseteq ” follows from $R^{disc}(X) \supseteq R_{\mathcal{U}}^{disc}(X)$ which is consequence of the definitions. The inclusion “ \subseteq ” follows from Lemma 4.3.5. \square

We now state a more detailed version of the theorem from the introduction.

Theorem 4.3.6. *If \mathcal{V} is a geometric vertex algebra, then \mathbf{FV} is a holomorphic factorization algebra with meromorphic OPE and discrete weight spaces whose associated geometric vertex algebra \mathbf{VFV} is isomorphic to \mathcal{V} .*

Proof. Proposition 3.4.9 says that $F = \mathbf{FV}$ is a prefactorization algebra. Proposition 3.4.18 says that F is holomorphic. Proposition 3.5.4 says that $\mathbf{VFV} \cong \mathcal{V}$ as graded bornological vector spaces. In particular, F has discrete weight spaces. Proposition 3.5.5 says that this isomorphism respects the multiplication maps, and \mathbf{FV} having meromorphic OPE by definition means that \mathbf{VFV} satisfies the meromorphicity axiom of a geometric vertex algebra. Proposition 4.2.2 says that F is multiplicative, and Theorem 4.3.2 says that F is a Weiss cosheaf, so F is a factorization algebra. \square

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