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ON PILLAI'S PROBLEM WITH X-COORDINATES OF PELL EQUATIONS AND POWERS OF 2 II

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Abstract. In this paper, we show that if (X_n, Y_n) is the nth solution of the Pell equation $X^2 - dY^2 = \pm 1$ for some non–square d, then given any integer c, the equation $c = X_n - 2^m$ has at most 2 integer solutions (n, m) with $n \geq 1$ and $m \geq 0$, except for the only pair $(c, d) = (-1, 2)$. Moreover, we show that this bound is optimal. Additionally, we propose a conjecture about the number of solutions of Pillai's problem in linear recurrent sequences.

1. Introduction

Pillai's problem states that for each fixed non zero integer c, the Diophantine equation

$$a^x - b^y = c \tag{1}$$

has only finitely many positive integer solutions (a, b, x, y) with $x, y \geq 2$. This problem is still unsolved for $|c| > 1$ while the case $|c| = 1$ which is known as Catalan's conjecture was solved in 2004 by Mihăilescu [16].

The work of Pillai was continued by Herschfeld [13, 14] who showed that if c is an integer with sufficiently large absolute value, then the equation (1) in the special case $(a, b) = (3, 2)$ has at most one solution (x, y). For small $|c|$ this is not the case (take for example $c = 1 = 3 - 2 = 3^2 - 2^3$). Pillai [17, 18] extended Herschfeld's result to a more general exponential Diophantine equation (1) with fixed coprime integers a, b, c and $a > b \geq 1$. Specifically, Pillai showed that there exists a positive integer $c_0(a, b)$ such that for $|c| > c_0(a, b)$, equation (1) has at most one integer solution (x, y). That is to say that there are only finitely many integers c such that the equation (1) has more than one positive integer solution (x, y). His method was ineffective. This was made effective by Stroeker and Tijdeman in [20] by using Baker's theory on linear forms in logarithms. In particular, they found all such c when $(a, b) = (3, 2)$ together with their multiple representations of the form $c = 3^x - 2^y$.

In this direction, Bennett [1] proved the following theorem.

Theorem 1 (Bennett, 2001). *Let $a, b \geq 2$ be fixed integers. For any integer $c \neq 0$, the Diophantine equation (1) has at most two solutions (x, y) in non negative integers.*

The equation (1) was revisited recently by replacing the powers of a and b by members of sequences $\{U_n\}_{n\geq 0}$ and $\{V_m\}_{m\geq 0}$ satisfying certain properties. For instance, a consequence of the main result of [3] says that if $\{U_n\}_{n\geq 0}$ and $\{V_m\}_{m\geq 0}$ are linearly recurrent sequences of integers with dominant roots α and α_1 which are multiplicatively independent, then there are only finitely many integers c such that the equation $c = U_n - V_m$ has more than one solution in non-negative integers (n, m). Given $(U, V) := \big(\{U_n\}_{n\geq 0},\ \{V_m\}_{m\geq 0}\big)$, we write $m_{U,V}(c)$ for the "multiplicity" of c as an element of the form $U_n - V_m$; that is, as the number of pairs

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(n, m) of positive integers such that $c = U_n - V_m$. With this notation, the above result says that $m_{U,V}(c) \leq 1$ for all but finitely many c.

Since for positive integers $a, b \geq 2$ the sequences $U = \{a^n\}_{n\geq 0}$ and $V = \{b^m\}_{m\geq 0}$ are linearly recurrence sequences of order 1, we can restate Theorem 1 by saying that $m_{U,V}(c) \leq 2$ for all non zero c when U and V are non constant geometric progressions of positive integers.

Given sequences of integers $U = \{U_n\}_{n\geq 0}$ and $V = \{V_m\}_{m\geq 0}$, we call

$$\mathcal{C} := \{c \in \mathbb{Z} \colon m_{U,V}(c) \geq 2\}$$

the *exceptional set of Pillai's equation* of (U, V). Thus, the main result of [3] says that $\mathcal{C}$ is finite and effectively computable if some conditions on the recurrences are met.

Except for some restrictions on the indices, the following table contains all the particular cases treated so far. In the table, $\{F_n^{(k)}\}_{n\geq 0}$ is sequence of k-generalized Fibonacci numbers of initial values $F_0^{(k)} = 0$, $F_1^{(k)} = 1$, $F_n^{(k)} = 2^{n-1}$ for $n = 2, \ldots, k-1$ and recurrence $F_{n+k}^{(k)} = F_{n+k-1}^{(k)} + \cdots + F_n^{(k)}$ for all $n \geq 0$. When $k = 2$ and $k = 3$ this is the sequence of Fibonacci numbers and Tribonacci numbers, respectively, and we omit the superscripts k, we just write F_n for the nth Fibonacci number and T_n for the nth Tribonacci number. The sequence $\{P_n\}_{n\geq 0}$ is the sequence of Pell numbers of initial values $P_0 = 0$, $P_1 = 1$ and recurrence $P_{n+2} = 2P_{n+1} + P_n$ for all $n \geq 0$. Finally, the sequence $\{X_n\}_{n\geq 1}$ is the sequence of X-coordinates for the Pell equation

$$X^2 - dY^2 = \pm 1 \tag{2}$$

corresponding to some non-square positive integer d, where (X_1, Y_1) is the smallest positive integer solution.

$c =$	$\#\mathcal{C}$	$m_{U,V}(c)$	*Authors*
$a^n - b^m$		2	Bennett [1]
$F_n - 2^m$	8	3	Ddamulira, Luca and Rakotomalala [7]
$T_n - 2^m$	5	4	Bravo, Luca and Yazán [2]
$F_n^{(k)} - 2^m$	≤ 2	$\leq k+2$	Ddamulira, Gómez and Luca [5]
$F_n^{(k)} - 3^m$	≤ 8	2	Ddamulira and Luca [6]
$F_n - T_m$	17	4	Chim, Pink and Ziegler [4]
$F_n - P_m$	10	4	Hernández, Luca and Rivera [11]
$P_n - 2^m$	7	3	Hernane, Luca, Rihane and Togbé [12]
$X_n - 2^m$		3	Erazo, Gómez and Luca [9]

Table 1. Pillai's problem in linear recurrent sequences

We want to point out that the bound for $m_{U,V}(c)$ is optimal for each case. Observing the third column of the above table, we note that this optimal bound is small. So we propose the next problem:

Question 1. *Given two linear recurrences $U = \{U_n\}_{n\geq 0}$ and $V = \{V_n\}_{n\geq 0}$ of integers of orders k and ℓ, respectively, having dominant roots which are multiplicatively independent, show that there exist a bound on $m_{U,V}(c)$ that depends only on* $\max\{k, \ell\}$.

In the table, the example with $U = \{F_n^{(k)}\}_{n\geq 0}$ and $V = \{3^m\}_{m\geq 0}$ shows that $\#\mathcal{C}$ is uniformly bounded although k (the order of the recurrence U) tends to infinity. This is an example where a phenomenon even stronger than the one indicated by our question occurs. One may ask if there are other parametric families of pairs of recurrences for which one can prove that $\mathcal{C}$ in uniformly bounded. This is the question we address in this paper for the case $U = \{X_n\}_{n\geq 1}$ and $V = \{2^m\}_{m\geq 0}$. This example is uniform in the parameter d. We recall that the sequence $\{X_n\}_{n\geq 0}$ is a binary recurrent satisfying the recurrence $X_{n+2} = (2X_1)X_{n+1} - \varepsilon X_n$ for all $n \geq 1$, where we put $\varepsilon = X_1^2 - dY_1^2 \in \{\pm 1\}$. This question was already studied by us in [9]. The main result of that paper was that $m_{U,V}(c) \leq 3$ and we provided an example of d for which $m_{U,V}(c) = 3$ for a particular value of c. In that paper we asked whether there are other instances (c, d) for which $m_{U,V}(c) = 3$. Here, we answer to this question and prove the following result. We make our result slightly more general by also allowing $n = 0$ with the convention that $X_0 := 1$.

Theorem 2. *Let $U := \{X_n\}_{n\geq 0}$ be the sequence of X-coordinates of the positive integer solutions (X, Y) of the Pell equation $X^2 - dY^2 = \pm 1$, where we set $X_0 := 1$, and let $V := \{2^m\}_{m\geq 0}$. Then for all integers c we have*

$$m_{U,V}(c) \leq 2, \tag{3}$$

except for the pair $(c, d) = (-1, 2)$, for which we have the representations:

$$\begin{aligned} c &= 1 - 2^1 = 1 - 2^1 = 3 - 2^2 = 7 - 2^3 \\ &= X_0 - 2^1 = X_1 - 2^1 = X_2 - 2^2 = X_3 - 2^3, \end{aligned}$$

which has $m_{U,V}(c) = 4$. Moreover, the bound (3) is optimal for an infinite number of d's.

Let us show that the result is optimal. Take $\varepsilon = 1$. We then have the identity

$$X_3 - X_1 = 2X_1X_2 - 2X_1 = 2X_1(2X_1^2 - 1) - 2X_1 = 4X_1^3 - 4X_1.$$

So, if we chose $X_1 = 2^k$, we get the identity

$$X_3 - 2^{3k+2} = X_1 - 2^{k+2} = c, \qquad c = c_k := -3 \cdot 2^k.$$

Writing $2^{2k} - 1 = d_k Y_k^2$ with a square-free integer d_k and a positive integer Y_k both depending on k, we get $m_{U,V}(c) \geq 2$ for the pair $(c, d) = (c_k, d_k)$ and for all $k \geq 1$. In virtue of (3), we in fact have $m_{U,V}(c) = 2$. In [9], it was justified that $d_k \to \infty$ as $k \to \infty$, so indeed, the number of d's is infinite.

Other examples of parametric families for $X_1 = 2^k$ are:

$\varepsilon = 1$	$c = -1$	$X_2 - 2^{2k+1} = X_0 - 2^1 = c,$
$\varepsilon = -1$	$c = 1 - 2^{2k+1}$	$X_2 - 2^{2k+2} = X_0 - 2^{2k+1} = c,$
$\varepsilon = 1$	$c = 1 - 2^{2k+3}$	$X_4 - 2^{4k+3} = X_0 - 2^{2k+3} = c,$
$\varepsilon = 1$	$c = 1 - 2^{k+2}$	$X_2 - 2^{2k+1} = X_0 - 2^{k+2} = c.$

Remark 1. *It is natural to ask what happens when the powers of 2 are replaced by powers of some other prime (or more generally, by powers of any other integer) in the Diophantine equation $X_n - 2^m = c$. The obvious obstruction would be Lemma 3. Work of Sanna [19] might provide a suitable replacement of our Lemma 3 for odd values of p. We leave this for future investigation.*

2. Preliminary results

In this section, we present the tools required to develop our work. For more details, one can consult [9]. In particular, we will use some properties on Pell equations, a result of the Baker theory and a pair of results in Diophantine approximation.

2.1. **Pell equations.**

Let (X_1, Y_1) be the minimal solution in positive integers of the Pell equation (2). It is well–known that all the non–negative integer solutions (X, Y) of the Pell equation (2) have the form

$$X + Y\sqrt{d} = X_n + Y_n\sqrt{d} = (X_1 + Y_1\sqrt{d})^n$$

for some $n \in \mathbb{N}$. For $\alpha := X_1 + Y_1\sqrt{d}$ and $\beta := X_1 - Y_1\sqrt{d}$, we have the usual Binet's formulas

$$X_n = \frac{\alpha^n + \beta^n}{2} \quad \text{and} \quad Y_n = \frac{\alpha^n - \beta^n}{2\sqrt{d}}, \qquad \text{for all} \quad n \geq 0. \tag{4}$$

We need the following lemmas. The following result is Lemma 1 in [9].

Lemma 1. *Let $\alpha > 0$ be the fundamental solution of $X^2 - dY^2 = \pm 1$ for nonsquare $d > 1$. Then*

$$\left(\frac{1}{1+\sqrt{2}}\right)\alpha^\ell \leq X_\ell \leq (2 - \sqrt{2})\alpha^\ell, \quad \textit{for all} \quad \ell \geq 1.$$

The following result is a restatement of Lemma 2 in [9].

Lemma 2. *We have:*

(*i*)

$$\nu_2(X_n) = \begin{cases} 0 & \textit{if } n \equiv 0 \pmod 2, \\ \nu_2(X_1) & \textit{if } n \equiv 1 \pmod 2. \end{cases}$$

(*ii*)

$$\nu_2(Y_n) = \begin{cases} \nu_2(Y_1) + \nu_2(X_1) + \nu_2(n) & \textit{if } n \equiv 0 \pmod 2, \\ \nu_2(Y_1) & \textit{if } n \equiv 1 \pmod 2. \end{cases}$$

The following result is Lemma 4 in [9].

Lemma 3. *Assume that $a > b$ satisfy $a \equiv b \pmod 2$.*

(*i*) *The following inequality holds:*

$$2^{\nu_2(X_a - X_b)} \leq 2(X_1^2 + 1)a^2.$$

(*ii*) *If X_1 is odd, then*

$$2^{\nu_2(X_a - X_b)} \leq 4(X_1 + 1)a^2.$$

Lemma 4. *Assume that $a > b$ satisfy $a \not\equiv b \pmod 2$. If $\varepsilon = 1$, then*

$$2^{\nu_2(X_a - X_b)} \leq X_1^2 - 1.$$

Proof. Let $e = \nu_2(X_a - X_b)$. Since a and b are incongruent modulo 2, it follows that one is even and one is odd. If m is even, then $X_m = 2X_{m/2}^2 - 1$ so X_m is odd, while if m is odd then $X_m \equiv X_1 \pmod 2$. Hence, the only interesting case is when X_1 is odd since otherwise $e = 0$ and the stated inequality is obviously true. Thus, $dY_1^2 \equiv X_1^2 - 1 \equiv 0 \pmod 4$ and

$$X_a^2 - dY_a^2 = X_b^2 - dY_b^2,$$

therefore

$$d(Y_a^2 - Y_b^2) = X_a^2 - X_b^2 = (X_a - X_b)(X_a + X_b).$$

So, $2^e \mid d(Y_a^2 - Y_b^2)$ and we get

$$e \leq \nu_2(d(Y_a^2 - Y_b^2)).$$

Without loss of generality we assume that a is even. Then, by Lemma 2, we have

$$\nu_2(Y_a^2) = 2(\nu_2(Y_1) + \nu_2(X_1) + \nu_2(a)) > 2\nu_2(Y_1) = \nu_2(Y_b^2),$$

where we used again that $a \not\equiv b \pmod 2$. So, $\nu_2(Y_a^2 - Y_b^2) = 2\nu_2(Y_1)$, which implies

$$e \leq \nu_2(dY_1^2) = \nu_2(X_1^2 - 1).$$

□

2.2. **Linear forms in logarithms.** For an algebraic number η of degree d over $\mathbb{Q}$, with conjugate roots $\eta := \eta^{(1)}, \ldots, \eta^{(d)}$, it is defined its *logarithmic height* by

$$h(\eta) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log \max\{|\eta^{(i)}|, 1\}\right),$$

where $a_0 > 0$ is the leading coefficient of the minimal primitive polynomial of η over $\mathbb{Z}$.

In order to find a lower bounds for linear forms in logarithms in two logarithms, we will use the following theorem due to Laurent [15].

Theorem 3 (**Laurent's Theorem**). *Let η_1 and η_2 be multiplicatively independent algebraic numbers. Set $D' = [\mathbb{Q}(\eta_1, \eta_2) : \mathbb{Q}]/[\mathbb{R}(\eta_1, \eta_2) : \mathbb{R}]$. Let $\log A_1$ and $\log A_2$ be real numbers such that*

$$\log A_j \geq \max\left\{h(\eta_j), \frac{1}{D'}, \frac{|\log \eta_1|}{D'}\right\}, \quad j = 1, 2.$$

Let b_1 and b_2 be integers, not both zero, and set

$$\log B' = \max\left\{\log\left(\frac{|b_1|}{D' \log A_2} + \frac{|b_2|}{D' \log A_1}\right) + 0.21, \frac{20}{D'}, 1\right\}.$$

Then, we have the lower bound

$$\log |b_1\eta_1 + b_2\eta_2| \geq -25.2 D'^4 (\log A_1)(\log A_2)(\log B')^2.$$

2.3. **Reduction by continuous fractions.** The following lemma will be used for the treatment of small linear forms homogeneous in two integer variables.

Lemma 5. *Let τ be an irrational number, M be a positive integer and $p_0/q_0, p_1/q_1, \ldots$ be the sequence of convergents of the continued fraction $[a_0, a_1, \ldots]$ of τ. Let N be such that $q_N > M$. Then putting $a(M) := \max\{a_t : t = 0, 1, \ldots, N\}$, the inequality*

$$|m\tau - n| > \frac{1}{(a(M) + 2)m},$$

holds for all pairs (n, m) of integers with $0 < m < M$.

For the treatment of nonhomogeneous linear forms in two integer variables, we will use a slight variation of a result due to Dujella and Pethő [8]. For a real number x, we put $\|x\| := \min\{|x - n| : n \in \mathbb{Z}\}$ for the distance from x to the nearest integer.

Lemma 6. *Let τ be an irrational number, M be a positive integer, and p/q be a convergent of the continued fraction of τ such that $q > 6M$. Let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Put $\epsilon := \|\mu q\| - M\|\tau q\|$. If $\epsilon > 0$, then there is no solution to the inequality*

$$0 < |m\tau - n + \mu| < AB^{-k},$$

in positive integers m, n and k with $m \leq M$ and $k \geq \log(Aq/\epsilon)/\log B$.

3. Other useful results

Before starting the proof of the Theorem 2, we present four additional lemmas for which it is assumed that the equation $c = X_n - 2^m$ has at least two solutions (n_i, m_i), (n_j, m_j) with $n_i > n_j$. Here we assume that $n_j \geq 1$ and we will treat the case $n_j = 0$ at the end of Section 5. Observe that since in this range the sequence $\{X_n\}_{n\geq 1}$ is strictly increasing we also get that $m_i > m_j$. We have

$$X_{n_i} - X_{n_j} = 2^{m_i} - 2^{m_j}. \tag{5}$$

We also have

$$\frac{\alpha^{n_i-1}}{4} < \frac{\alpha^{n_i-1}}{1+\sqrt{2}} < X_{n_i-1} \leq X_{n_i} - X_{n_j} = 2^{m_i} - 2^{m_j} < 2^{m_i}, \tag{6}$$

$$2^{m_i-1} \leq 2^{m_i} - 2^{m_j} = X_{n_i} - X_{n_j} < X_{n_i} < \alpha^{n_i}. \tag{7}$$

From (6) and (7) we get

$$m_i < 2.2 n_i \log\alpha \quad \text{and} \quad n_i < \frac{4}{5} m_i + 2. \tag{8}$$

Lemma 7. *If $c \neq 0$, then $n_i(m_j+1) \neq n_j(m_i+1)$.*

Proof. From (5) and the Binet formula (4), we get

$$\alpha^{n_i} - 2^{m_i+1} = \alpha^{n_j} - 2^{m_j+1} + \beta^{n_j} - \beta^{n_i}.$$

Put $x := \alpha^{n_j}$, $y := 2^{m_j+1}$ and $n_i/n_j = (m_i+1)/(m_j+1) =: 1+\epsilon$. Therefore

$$x^{1+\epsilon} - y^{1+\epsilon} = x - y + \beta^{n_j} - \beta^{n_i}.$$

Thus,

$$\frac{x^{1+\epsilon} - y^{1+\epsilon}}{x-y} = 1 + \frac{\beta^{n_j} - \beta^{n_i}}{x-y}.$$

By the Mean Value Theorem there exists $z \in (x, y)$ such that

$$(1+\epsilon)z^{\epsilon} = 1 + \frac{\beta^{n_j} - \beta^{n_i}}{x-y}.$$

Note that $|x-y| = |2c - \beta^{n_j}| \geq |c| \geq 1$ and that

$$z^{\epsilon} > \min\{x^{\epsilon}, y^{\epsilon}\} = \min\{\alpha^{n_i-n_j}, 2^{m_i-m_j}\} \geq 2.$$

Hence, given that $|\beta| = \alpha^{-1}$, we obtain

$$2 < 2(1+\epsilon) < 1 + \frac{|\beta^{n_j} - \beta^{n_i}|}{|x-y|} < 1 + 2\alpha^{-n_j},$$

so

$$1+\sqrt{2} \leq \alpha^{n_j} < 2,$$

a contradiction. ∎

The next lemma allows us to reduce some computations.

Lemma 8. *If $n_i = 2$ and X_1 is even, then $X_1 = 2$.*

Proof. If $n_i = 2$, then $n_j = 1$. If X_1 is even, then

$$X_2 - X_1 = 2X_1^2 - X_1 \mp 1 = 2^{m_i} - 2^{m_j},$$

and since X_1 is even we get $m_j = 0$. Thus, either

$$2X_1^2 - X_1 - 1 = 2^{m_i} - 1 \qquad \text{so} \qquad X_1(2X_1 - 1) = 2^{m_i},$$

or

$$2X_1^2 - X_1 + 1 = 2^{m_i} - 1 \qquad \text{so} \qquad (4X_1 - 1)^2 + 15 = 2^{m_i+3}.$$

The first situation gives no solutions since $2X_1 - 1 > 1$ is an odd number. Rewrite the second one as

$$2^a - u^2 = 15. \tag{9}$$

For a odd, we obtain the equation $u^2 - 2v^2 = -15$ which has no solutions (this can be seen by reducing it modulo 3, for example). Now, if a is even, then rewriting the equation (9) as $v^2 - u^2 = 15$ gives the solutions $v \in \{4, 8\}$, so $m_i \in \{1, 3\}$. The case $m_i = 1$ gives $2X_1 = 1$ while the case $m_i = 3$ leads to $X_1 = 2$. □

4. The proof of Theorem 2

Assume that $m_{U,V}(c) \geq 3$ and let (n_1, m_1), (n_2, m_2), (n_3, m_3) be such that $X_{n_i} - 2^{m_i} = c$ for $i = 1, 2, 3$, with $n_1 > n_2 > n_3 \geq 1$. In particular, $m_1 > m_2 > m_3$. Let us recall some useful statements from [9] with some slightly differences. Let n and n' both in $\{n_1, n_2, n_3\}$.

Claim 1. *If either $n \equiv n' \pmod 2$ and $n' \leq n/1.5$, or there are $n_i \equiv n_j \pmod 2$ with $n_i < n_j < n$ and $n_i \leq n/1.5$, then $n \leq 60381$.*

Claim 2. *Let $n \equiv n' \pmod 2$. If $n' > n/1.5$ and there is n_i such that $n_i < n'$, then $n \leq 16$.*

Both these claims hold even when $n_3 = 0$. Let us add another claim.

Claim 3. *If $n_1 \not\equiv n_2 \pmod 2$ and $n_1 > 15$, then either the inequality $\alpha^{n_2 - n_3} < 13n_1$ or the inequality $\alpha^{n_2 - 1} < 104n_1^3$ holds.*

Proof. Since $n_1 \not\equiv n_2 \pmod 2$, it follows that X_1 is odd. Indeed, if X_1 is even, then by Lemma 2 we must have $m_2 = \nu_2(X_{n_1} - X_{n_2}) = \min\{0, \nu_2(X_1)\} = 0$, a contradiction since $m_2 > m_3$. Note that $n_3 \equiv n_1 \pmod 2$ or $n_3 \equiv n_2 \pmod 2$, so by Lemma 3 in either case

$$2^{m_3} \leq 4(X_1 + 1)n_1^2. \tag{10}$$

Using Binet's formula (4) in

$$X_{n_i} - X_{n_3} = 2^{m_i} - 2^{m_3}, \qquad i = 1, 2, \tag{11}$$

we get

$$\left|1 - 2^{m_i+1}\alpha^{-n_i}\right| \leq \frac{\alpha^{n_3} + 2^{m_3+1} + 2}{\alpha^{n_i}} \leq \frac{\alpha^{n_3} + 8(X_1 + 1)n_1^2 + 2}{\alpha^{n_i}} \leq \frac{\alpha^{n_3} + 8.01\alpha n_1^2}{\alpha^{n_2}} \tag{12}$$

for $i = 1, 2$. Define

$$\Gamma_i := (m_i + 1)\log 2 - n_i \log \alpha, \quad \text{for } i = 1, 2. \tag{13}$$

Using (11) for $i=1$ and (6), we obtain

$$\begin{aligned}\left|2^{-m_1-1}\alpha^{n_1}-1\right| &\le \frac{\alpha^{n_3}+2^{m_3+2}}{2^{m_1+1}} \le \frac{\alpha^{n_3}}{2^{m_1+1}}+\frac{1}{2^{m_1-m_3-1}}\\ &\le \frac{\alpha^{n_1-2}}{2^{m_1+1}}+\frac{1}{2^{m_1-m_3-1}}\\ &< \frac{(1+\sqrt{2})2^{m_1}}{\alpha 2^{m_1+1}}+\frac{1}{2^{m_1-m_3-1}}\\ &\le \frac{1}{2}+\frac{1}{2^{m_1-m_3-1}}. \qquad (14)\end{aligned}$$

If $m_1-m_3\le 5$, then we use (6), (10) and (14) to conclude

$$\alpha^{n_1-1}<2^{m_1+2}\le 2^{m_3+7}\le 512(X_1+1)n_1^2<512\alpha n_1^2,$$

so $\alpha^{n_1-2}<512n_1^2$. Since $\alpha>2(X_1^2-1)^{1/2}$ we get

$$\left(2\sqrt{X_1^2-1}\right)^{n_1-2}<512n_1^2. \qquad (15)$$

For $X_1\ge 2$, we get

$$\left(2\sqrt{3}\right)^{n_1-2}\le\left(2\sqrt{X_1^2-1}\right)^{n_1-2}<512n_1^2,$$

which implies $n_1\le 10$. If $X_1=1$, we have $\alpha=1+\sqrt{2}$ and so

$$(1+\sqrt{2})^{n_1-2}<512n_1^2,$$

and this gives $n_1\le 15$.

So assuming $n_1>15$, we obtain from (14) that $e^{|\Gamma_1|}<2.1$. Therefore

$$|\Gamma_i|<e^{|\Gamma_i|}\left|e^{\Gamma_i}-1\right|<\frac{18n_1^2}{\alpha^{n_i-n_3}},\quad \text{for } i=1,2. \qquad (16)$$

We perform the linear combination

$$\Gamma:=n_1\Gamma_2-n_2\Gamma_1=(n_1(m_2+1)-n_2(m_1+1))\log 2. \qquad (17)$$

Thus, we conclude that

$$|\Gamma|<\frac{2.1n_1}{\alpha^{n_2-n_3}}+\frac{18n_1^3}{\alpha^{n_2-1}}+\frac{2.1n_2}{\alpha^{n_1-n_3}}+\frac{18n_1^2n_2}{\alpha^{n_1-1}}<\frac{4.2n_1}{\alpha^{n_2-n_3}}+\frac{36n_1^3}{\alpha^{n_2-1}}.$$

Hence, if both terms in the right–hand side are $<\log 2/2$, then $|\Gamma|<\log 2$. But this last inequality implies $n_1(m_2+1)=n_2(m_1+1)$, which is contrary to Lemma 7.

□

4.1. **Bounding n_1.** We continue with the proof of Theorem 2. We begin by ruling out the case $c=0$. If $X_{n_i}=2^{m_i}$ then by Lemma 2 we have $m_i\in\{0,\nu_2(X_1)\}$, so $m_{U,V}(0)\le 2$.

Consider the following situations.

Case 1. $n_1\equiv n_2 \pmod 2$.

If $n_2>n_1/1.5$, it follows from Claim 2 that $n_1\le 16$. If on the contrary $n_2\le n_1/1.5$, it follows immediately from Claim 1 that $n_1\le 60381$.

Case 2. $n_1\not\equiv n_2 \pmod 2$.

Subcase $n_3\le n_1/1.5$.

If $n_1 \not\equiv n_2 \equiv n_3 \pmod 2$, from $n_3 \le n_1/1.5$ and the second case of Claim 1 we get that $n_1 \le 60381$. If the other possibility happens, namely $n_1 \equiv n_3 \not\equiv n_2 \pmod 2$, we get from Claim 1 that $n_1 \le 60381$.

Subcase $n_3 > n_1/1.5$.

This subcase is more delicate. Recall (from the arguments in Claim 3) that X_1 must be odd. Furthermore,

$$2^{m_3} \le 4(X_1+1)n_1^2, \tag{18}$$

and by hypothesis

$$n_2 < n_1 < 1.5n_3.$$

We apply Laurent's Theorem to the linear form in logarithms Γ_1 (corresponding to $i=1$ in (13)), with $(\eta_1, b_1) := (2, m_1+1)$, $(\eta_2, b_2) = (\alpha, n_1)$, $\mathbb{K} := \mathbb{Q}(\alpha)$, $D' := 2$, $\log A_1 = \log 2$, $\log A_2 = (\log\alpha)/2$. Using (8), we have

$$\begin{aligned}\log B' &= \max\left\{\log\left(\frac{m_1+1}{\log\alpha} + \frac{n_1}{2\log 2}\right) + 0.21, 10\right\}\\ &\le \max\left\{\log\left(\frac{n_1}{\log 2} + \frac{2}{\log\alpha} + \frac{n_1}{2\log 2}\right) + 0.21, 10\right\}\\ &\le \max\left\{\log(4n_1) + 0.21, 10\right\} = \log B''. \end{aligned}\tag{19}$$

By Theorem 3, we get

$$\begin{aligned}\log|\Gamma_1| &\ge -25.2\cdot 2^4\cdot(\log 2)\cdot((\log\alpha)/2)\cdot(\log B'')^2\\ &\ge -140(\log\alpha)(\log B'')^2.\end{aligned}\tag{20}$$

Assume $n_1 > 15$, so that (16) holds. Comparing the inequality (16) with (20), we get

$$n_1 - n_3 < 140(\log B'')^2 + 2\log(69n_1^2). \tag{21}$$

We highlight that we have not used so far the condition $n_1/1.5 < n_3$. Therefore (14) and (21) hold for all $n_1 \not\equiv n_2 \pmod 2$ if $n_1 > 12$.

On the other hand, returning to (5) with $j=3$ and $i=1,2$, and using Binet's formula we get

$$\left|\alpha^{n_3}(\alpha^{n_i-n_3}-1) - 2^{m_i+1}\right| \le 2^{m_3+2} \le 16\alpha n_1^2,$$

so

$$\left|2^{m_i+1}\alpha^{-n_3}(\alpha^{n_i-n_3}-1)^{-1} - 1\right| \le \frac{16\alpha n_1^2}{\alpha^{n_i}-\alpha^{n_3}} \le \frac{16n_1^2}{\alpha^{n_i-2}}. \tag{22}$$

If $\alpha^{n_i-2} \le 32n_1^2$, since $n_i > n_3 > n_1/1.5$, we arrive to $\alpha^{n_1/1.5-2} < 32n_1^2$. Using the same argument in (15) we obtain $n_1 \le 18$.

Assuming $n_1 > 18$, we get $\alpha^{n_i-2} > 32n_1^2$ for $i=1,2$. Hence,

$$|\Gamma_{i+2}| < e^{|\Gamma_{i+2}|}\left|e^{\Gamma_{i+2}}-1\right| < \frac{32n_1^2}{\alpha^{n_i-2}},$$

where

$$\Gamma_{i+2} := (m_i+1)\log 2 - n_3\log\alpha - \log(\alpha^{n_i-n_3}-1).$$

We perform the linear combination

$$\tilde{\Gamma} := \Gamma_3 - \Gamma_4 = (m_1-m_2)\log 2 - \log\left(\frac{\alpha^{n_1-n_3}-1}{\alpha^{n_2-n_3}-1}\right).$$

Thus,

$$|\tilde{\Gamma}| < \frac{32n_1^2}{\alpha^{n_1-2}} + \frac{32n_1^2}{\alpha^{n_2-2}} < \frac{64n_1^2}{\alpha^{n_2-2}}. \tag{23}$$

We use Laurent's Theorem on $\tilde{\Gamma}$ with $\eta_1 = 2$ and $\eta_2 = (\alpha^{n_1-n_3}-1)/(\alpha^{n_2-n_3}-1)$, $b_1 = m_1-m_2$ and $b_2 = 1$. First

$$\begin{aligned} h(\eta_2) &\leq h(\alpha^{n_1-n_3}-1) + h(\alpha^{n_2-n_3}-1) \\ &\leq (n_1-n_3)\log\alpha + (n_2-n_3)\log\alpha + 2\log 2 \\ &\leq 2(n_1-n_3)\log\alpha, \end{aligned}$$

so we can take $\log A_2 = 2(n_1-n_3)\log\alpha$. Therefore,

$$\log \tilde{B}' = \max\left\{\log\left(\frac{m_1-m_2}{4(n_1-n_3)\log\alpha} + \frac{1}{2\log 2}\right) + 0.21, 10\right\}.$$

Using (6) and (7), we get

$$\begin{aligned} \frac{m_1-m_2}{4(n_1-n_3)\log\alpha} + \frac{1}{2\log 2} &< \frac{(n_1-n_2+1)\log\alpha + \log 4}{4(n_1-n_3)\log\alpha} + \frac{1}{2\log 2} \\ &< \frac{(n_1-n_3)\log\alpha}{4(n_1-n_3)\log\alpha} + 1 + \frac{1}{2\log 2} < 3, \end{aligned}$$

so, we actually have $\log \tilde{B}' = 10$.

On the other hand, if in (19) it holds that $\log B'' = 10$, then $n_i \leq e^{9.79}/3 < 5970$. From now on suppose $n_1 > 5970$. Then (21) reads

$$n_1 - n_3 < 140(\log(4n_1) + 0.21)^2 + 2\log(69n_1^2). \tag{24}$$

Thus, by Theorem 3, we get

$$\begin{aligned} \log|\tilde{\Gamma}| &\geq -25.2 \cdot 2^4 \cdot (\log 2) \cdot (2(n_1-n_3)\log\alpha) \cdot 10^2 \\ &\geq -55900(n_1-n_3)(\log\alpha). \end{aligned}$$

Comparing this last inequality with (23) and using (24), we get

$$\begin{aligned} n_1/1.5 - 2 < n_2 - 2 &< 55900(n_1-n_3) + \log(64n_1^2) \\ &< 55900\left(140(\log(4n_1)+0.21)^2 + 2\log(69n_1^2)\right) + \log(64n_1^2) \\ &< 7.83 \cdot 10^6((\log(4n_1)+0.21)^2 + \log(69n_1^2)), \end{aligned}$$

so $n_1 < 7.55 \cdot 10^9$.

In summary, we have proved the following result.

Lemma 9. *Let $\{X_n\}_{n\geq 1}$ be the sequence of X–coordinates of the positive integer solutions (X,Y) of the Pell equation $X^2 - dY^2 = \pm 1$ and let (n_i, m_i) be pairs of nonegative integers with $n_i \geq 1$ and $X_{n_i} - 2^{m_i} = c$ for $i = 1,2,3$. Let us put $n_1 := \max\{n_i : i = 1,2,3\}$. If $n_1 \equiv n_2 \pmod 2$, or $n_1 \not\equiv n_2 \pmod 2$ with $n_1 > 1.5n_2$, then*

$$n_1 \leq 60381.$$

Otherwise

$$n_1 < 7.55 \cdot 10^9.$$

4.2. **Reducing the bounds.** In this reduction step, we work again in the same cases division as above.

Case 1. $n_1 \equiv n_2 \pmod 2$.

From (5) and Lemma 3(i)

$$\begin{aligned} X_{n_2} - X_{n_2-1} &\le X_{n_2} - X_{n_3} = 2^{m_2} - 2^{m_3} < 2^{m_2} \\ &\le 2(X_1^2+1)n_1^2 \le 2(X_1^2+1)(60381)^2. \end{aligned}$$

It is useful at this point to recall the *Chebyshev polynomials of the first kind* which are useful in order to represent X_ℓ in terms of X_1:

$$\begin{aligned} X_\ell =& \frac{1}{2}\left(\alpha^\ell + \beta^\ell\right) = \frac{1}{2}\left(\left(X_1 + \sqrt{d}Y_1\right)^\ell + \left(X_1 - \sqrt{d}Y_1\right)^\ell\right) \\ =& \frac{1}{2}\left(\left(X_1 + \sqrt{X_1^2 \mp 1}\right)^\ell + \left(X_1 - \sqrt{X_1^2 \mp 1}\right)^\ell\right) := P_\ell^{\pm}(X_1). \end{aligned}$$

From (25), if $n_2 \ge 20$ then

$$P_{20}^{+}(X_1) - P_{19}^{+}(X_1) \le X_{20} - X_{19} \le X_{n_2} - X_{n_2-1} \le 2(X_1^2+1)\cdot 60381^2, \tag{25}$$

and we obtain that $X_1 < 2$, so $n_2 \le 19$ or $X_1 = 1$. If $X_1 = 1$, then $d = 2$, so we can compute explicitly the terms X_n to obtain that $X_{28} - X_{27} > 4 \cdot 60381^2$. Thus, $n_2 \le 27$ for all X_1. In case that $n_2 = 2$, if X_1 odd then, according to Lemma 3(ii), we can replace $2(X_1^2+1)$ by $4(X_1+1)$ in (25), so

$$P_2^{+}(X_1) - P_1^{+}(X_1) \le X_2 - X_1 \le 4(X_1+1)\cdot 60381^2,$$

and we obtain $X_1 < 7.3 \cdot 10^9$. It is easy to see that for $n_2 \in [3,19]$ one gets a bound for X_1 which is smaller than the one given above. In case that X_1 is even, by Lemma 8, we get $X_1 = 2$. Note that from Lemma 3(i), we have

$$2^{m_2} \le 2(X_1^2+1)n_1^2 \le 2((7.3\cdot 10^9)^2+1)\cdot 60381^2,$$

which implies that $m_2 \le 98$. Thus, returning to (5), we consider the equations:

$$P_{n_2}^{+}(X_1) - P_{n_3}^{+}(X_1) = 2^{m_2} - 2^{m_3}, \tag{26}$$

$$P_{n_2}^{-}(X_1) - P_{n_3}^{-}(X_1) = 2^{m_2} - 2^{m_3}, \tag{27}$$

both with $n_2 \in [2,27],\ n_3 \in [1, n_2-1],\ m_2 \in [1,109],\ m_3 \in [0, m_2-1]$. We consider a larger range in m_2 for later purposes. The output from this search is recorded in Table 2.

X_1	(n_2,n_3)	(m_2,m_3)	ε	X_1	(n_2,n_3)	(m_2,m_3)	ε
3	$(2,1)$	$(4,1)$	$+$	1	$(3,2)$	$(3,2)$	$-$
1	$(2,1)$	$(2,1)$	$-$	1	$(4,1)$	$(5,4)$	$-$
2	$(2,1)$	$(3,0)$	$-$	1	$(4,2)$	$(4,1)$	$+$
3	$(2,1)$	$(5,4)$	$-$	7	$(5,1)$	$(18,6)$	$+$
3	$(3,1)$	$(7,5)$	$+$	1	$(5,4)$	$(5,3)$	$-$
5	$(3,1)$	$(9,5)$	$+$	1	$(6,2)$	$(7,5)$	$-$
1	$(3,1)$	$(3,1)$	$-$	2	$(10,2)$	$(18,6)$	$+$
5	$(3,1)$	$(3,1)$	$-$	$2^k,\ 0 \le k \le 41$	$(3,1)$	$(2^{3k+2}, 2^{k+2})$	$+$

TABLE 2. The solutions of equations (26) and (27)

Case 2. $n_1 \not\equiv n_2 \pmod 2$.

Subcase $\varepsilon = 1$. According to Lemma 4 we have that

$$2^{m_2} \le X_1^2 - 1 \le \alpha^2.$$

By comparing the above inequality with (6), we obtain

$$\alpha^{n_2-1} \le (1+\sqrt{2}) \cdot 2^{m_2} < \alpha^3,$$

therefore $n_2 \le 3$. Furthermore, from (7), we have

$$m_2 \le \frac{2\log(104 n_1^3)}{\log 2} + 1 < 110.$$

Hence, we are led to consider the equation:

$$P_{n_2}^{+}(X_1) - P_{n_2-1}^{+}(X_1) = 2^{m_2} - 2^{m_3} \tag{28}$$

with $n_2 \in [2,3]$, $m_2 \in [1,109]$, $m_3 \in [0, m_2-1]$, which was already solved in equations (26) and (27).

Subcase $\varepsilon = -1$.

Assume $n_1 > 15$. First, we reduce the bound on n_2. According to Claim 3, we have two possibilities namely $\alpha^{n_2-n_3} < 13n_1$, or $\alpha^{n_2-1} < 104n_1^3$. Since the former implies $n_2 \le 83$, suppose the contrary, that is, that we have $\alpha^{n_2-n_3} < 13n_1$. Thus, $\alpha < (9.82 \cdot 10^{10})^{1/(n_2-n_3)}$. Using (7), we conclude that $m_1 < 2.76 \cdot 10^{11}$.

Subcase $n_2 - n_3 \ge 2$ or $n_2 - n_3 = 1$ with $X_1 \le 10^7$.

This case is treatable with continued fractions. From (22) for $i = 2$, we get

$$\left|2^{m_2+1}\alpha^{-n_3}(\alpha^{n_2-n_3}-1)^{-1} - 1\right| \le \frac{16\alpha n_1^2}{\alpha^{n_2} - \alpha^{n_3}} \le \frac{16 n_1^2}{\alpha^{n_2-2}}.$$

If $\alpha^{n_2-2} < 32n_1^2 < 1.83 \cdot 10^{21}$, then $n_2 \le 57$. Consider

$$\Gamma_4 := (m_2+1)\log 2 - n_3 \log\alpha - \log(\alpha^{n_2-n_3} - 1).$$

Thus,

$$\left|(m_2+1)\frac{\log 2}{\log\alpha} - n_3 - \frac{\log(\alpha^{n_2-n_3}-1)}{\log\alpha}\right| < \frac{32n_1^2}{\alpha^{n_2-2}\log\alpha} < \frac{1.83\cdot 10^{21}}{\alpha^{n_2-2}\log\alpha}. \tag{29}$$

We use Lemma 6 with

$$\tau := \log 2/\log\alpha,\ \mu := \log(\alpha^{n_2-n_3}-1)/\log\alpha,\ A := 1.83\cdot 10^{21}\alpha^2/\log\alpha,\ B := \alpha,$$

and with $m := m_2 + 1 \le M := 2.76 \cdot 10^{12}$ and $k = n_2$. To do the computation, we recover α by the formula

$$\alpha = X_1 + \sqrt{X_1^2 + 1}$$

and loop over positive integers X_1 in the range

$$X_1 < \min\{(9.82 \cdot 10^{10})^{1/(n_2-n_3)}/2, 10^7\}.$$

In the above, we used the fact that $\alpha > 2X_1$. Finally, since $\alpha^{n_2-n_3} < 9.82\cdot 10^{10}$, we get that $n_2 - n_3 \le 28$. So, for each $1 \le n_2 - n_3 \le 28$, we ran the program implementing Lemma 6. The output of this computation was $n_2 \le 101$ for $n_2 - n_3 = 1$ and $n_2 \le 104$ for $2 \le n_2 - n_3 \le 28$.

Actually for the situation $n_2 - n_3 = 2$ some delicate analysis is needed. Indeed, if $X_1 = 2^k$ then

$$\frac{\log(\alpha^2-1)}{\log\alpha} = 1 + \frac{\log(\alpha-\alpha^{-1})}{\log\alpha} = 1 + \frac{\log(2X_1)}{\log\alpha} = 1 + (k+1)\frac{\log 2}{\log\alpha}.$$

Thus, in this particular case, we rewrite (29) as

$$\left|(n_3+1)\frac{\log\alpha}{\log 2} - (m_2 - k)\right| < \frac{32n_1^2}{\alpha^{n_2-2}\log 2},$$

and use instead Lemma 5 with $\tau := \log\alpha/\log 2$ and $0 < n_3 + 1 < 7.56 \cdot 10^9 := M$, for each $0 \le k \le 16$. Hence, Lemma 5 gives the inequality

$$\alpha^{n_2-2} < 32n_1^3(a(M)+2) < 1.4 \cdot 10^{31}(a(M)+2).$$

We concluded that $n_2 \le 117$ after obtaining computationally that

$$\max\{a(M)+2 : 0 \le k \le 16\} < 10^{13}.$$

If $n_1 > 176 > 1.5 \cdot 117 > 1.5n_2$, then by Claim 1 necessarily $n_1 \le 60381$. So, to summarise, we have proved that $n_2 \le 117$ and $n_1 \le 60381$ unconditionally, so we can drop the assumptions on n_2.

We need to improve further our bound on n_1 for this situation. We assume

$$n_1 > 176 > 1.5 \cdot 117 > 1.5n_2.$$

According to Claim 3, we have two situations to consider namely when

$$\alpha \le \alpha^{n_2-n_3} < 13n_1 \le 13 \cdot 60381 < 8 \cdot 10^5,$$

or when

$$\alpha^{n_2-1} < 104n_1^3 < 104 \cdot 60381^3 < 2.29 \cdot 10^{16}.$$

The first situation gives $X_1 < 4 \times 10^5$. Let us look at the second situation. Together with inequality (7), this implies

$$m_2 \le \frac{2(n_2-1)\log\alpha}{\log 2} \le \frac{2\log(2.29 \cdot 10^{16})}{\log 2} + 1 < 110. \tag{30}$$

Thus, for $n_2 \le 27$, we are led to the same equations (26) and (27), whose all solutions are listed in Table 2. For $n_2 \ge 28$, this second situation implies $\alpha < 5$. So, we can assume $X_1 < 4 \cdot 10^5$. We can go back to (16) which we rewrite as

$$\left|n_1\frac{\log\alpha}{\log 2} - (m_1+1)\right| < \frac{34n_1^2}{\alpha^{n_1/3}}, \tag{31}$$

and use Lemma 5 with $\tau := \log\alpha/\log 2$ and $176 < n_1 < 60381 := M$ (notice that we used the condition $n_3 \le n_1/1.5$). Therefore, Lemma 5 gives the inequality

$$\alpha^{n_1/3} < 34n_1^3(a(M)+2).$$

Since $q_n \ge F_n$ for all $n \ge 1$ where F_n denotes the nth Fibonacci number, we can use q_{26} for all X_1. Recall that we can recover α by the formula $\alpha = X_1 + \sqrt{X_1^2+1}$, where $X_1 < 4 \cdot 10^5$. For each X_1, the above relation allows us to dramatically reduce the bound for n_1. The output of this computation was $n_1 \le 64$. Thus, $n_1 \le 176$.

Now we find all possible candidates for X_1. If $\alpha^{n_2-1} < 104n_1^3 < 104 \cdot 176^3 < 5.7 \cdot 10^8$ then $n_2 \le 23$ and together with (30) yields again equations (26) and (27). Now suppose $\alpha^{n_2-n_3} < 13n_1 \le 2288$. This implies that $n_2 - n_3 \le 8$. We return to (5) which is

$$X_{n_2} - X_{n_3} = 2^{m_2} - 2^{m_3} \tag{32}$$

with $n_2 \in [2, 117]$ and $n_2 - n_3 \in [1, 8]$. Observe that given an integer N, if it can be written as a number of the form $N = 2^a - 2^b$, then this representation is unique. Thus, we only have to check if $X_{n_2} - X_{n_3}$ is of the form $2^{m_2} - 2^{m_3}$. Therefore, for each $1 \le n_2 - n_3 \le 8$, we fix $X_1 < \alpha/2 < 2288^{1/(n_2-n_3)}/2$ and we generate the sequence $\{X_n\}_{n\ge 1}$ up to $n \le 117$ and we check if equation (32) has a solution.

Finally, note that all the solutions given in Table 3 are included in Table 2.

X_1	n_3	n_2	ε	X_1	n_2	n_3	ε
1	1	2	–	1	1	3	–
1	2	3	–	1	2	4	–
1	4	5	–	5	1	3	–
2	1	2	–	1	1	4	–
3	1	2	–	1	2	6	–

TABLE 3. Solutions of equation (32) with $1 \le n_2 - n_3 \le 8$

Subcase $n_2 - n_3 = 1$ *and* $X_1 > 10^7$.

Then

$$2(X_1^2 + 1)n_1^2 < 2(X_1 + 1)^2 n_1^2 < 10^{-10}\alpha^6. \tag{33}$$

The above holds because $X_1 + 1 < \alpha$, so

$$2(X_1 + 1)^2 n_1^2 < 2\alpha^2(7.6 \cdot 10^9)^2 < 2 \cdot 10^{20}\alpha^2 < 10^{21}\alpha^2 < 10^{-7}\alpha^6,$$

where the last inequality holds since $\alpha > X_1 > 10^7$.

From now on, we assume $n_3 \ge 40000$ since otherwise $n_1 \le 1.5n_3 < 60000$. We exploit the two equations

$$X_{n_2} - X_{n_3} = 2^{m_2} - 2^{m_3} \qquad \text{and} \qquad X_{n_1} - X_{n_3} = 2^{m_1} - 2^{m_3}. \tag{34}$$

From the first one, we have

$$|\alpha^{n_3}(\alpha - 1) - 2^{m_2+1}| = | - 2^{m_3+1} - (\beta^{n_2} - \beta^{n_3})| < 2^{m_3+1} + 1.$$

Thus,

$$|1 - \alpha^{-n_3}(\alpha - 1)^{-1}2^{m_2+1}| < \frac{2^{m_3+1} + 1}{\alpha^{n_3}(\alpha - 1)}.$$

Since $2^{m_3} < 10^{-7} \cdot \alpha^6$ (see (33)), we get that

$$|1 - \alpha^{-n_3}(\alpha - 1)^{-1}2^{m_2+1}| < \frac{2 \cdot 10^{-7}\alpha^6 + 1}{\alpha^{n_3}(\alpha - 1)} < \frac{3 \cdot 10^{-7}\alpha^6}{\alpha^{n_3+1}(3/4)} < \frac{4}{10^7\alpha^{n_3-5}} \le \frac{4}{10^7\alpha^{n_2-6}}.$$

It follows that

$$|n_3 \log \alpha + \log(\alpha - 1) - (m_2 + 1)\log 2| < \frac{8}{10^7\alpha^3}. \tag{35}$$

From the second equation in (34), we have

$$|\alpha^{n_1} - 2^{m_1+1}| = | - 2^{m_3+1} - \beta^{n_1}| \le 2^{m_3+1} + 1,$$

so

$$|1 - \alpha^{-n_1}2^{m_1+1}| < \frac{2^{m_3+1} + 1}{\alpha^{n_1}} < \frac{2 \cdot 10^{-7}\alpha^6 + 1}{\alpha^{n_1}} < \frac{3}{10^7\alpha^{n_1-6}},$$

therefore

$$|n_1 \log \alpha - (m_1+1)\log 2| < \frac{6}{10^7 \alpha^{n_1-6}}. \tag{36}$$

In particular,

$$|n_1 \log \alpha - (m_1+1)\log 2| < \frac{6}{10^7 \alpha^{3}}. \tag{37}$$

Eliminating $\log \alpha$ from (35) and (37), we get

$$|n_1 \log(\alpha-1) - (n_1(m_2+1) - n_3(m_1+1))\log 2| < \frac{8n_1+6n_3}{10^7\alpha^3} < \frac{14n_1}{10^7\alpha^3}.$$

We now write

$$\log(\alpha-1) = \log\alpha + \log(1-1/\alpha) = \log\alpha - 1/\alpha - 1/(2\alpha^2) + \zeta,$$

where

$$\zeta = -\sum_{k\ge 3} \frac{1}{k\alpha^k}.$$

So,

$$|\zeta| < \frac{1}{3\alpha^3}\left(1 + \frac{1}{\alpha} + \frac{1}{\alpha^2} + \cdots\right) = \frac{1}{3\alpha^3(1-\alpha^{-1})} < \frac{1+2\cdot 10^{-7}}{3\alpha^3}.$$

Thus,

$$\begin{aligned}\left|n_1\log\alpha - \frac{n_1}{\alpha} - \frac{n_1}{2\alpha^2} - (n_1(m_2+1) - n_3(m_1+1))\log 2\right| &< \frac{14n_1}{10^7\alpha^3} + n_1|\zeta| \\ &< \frac{14n_1}{10^7\alpha^3} + \frac{(1+2\cdot 10^{-7})n_1}{3\alpha^3} \\ &< \frac{(1+5\cdot 10^{-6})n_1}{3\alpha^3}.\end{aligned}$$

We next replace n/α by n/X_1 and keep track of the resulting error. We write

$$\alpha = 2X_1 + \zeta, \qquad 0 < \zeta < 1/(2X_1).$$

Thus,

$$\frac{1}{\alpha} = \frac{1}{2X_1(1+\zeta/(2X_1))} = \frac{1}{2X_1}\left(1 - \frac{\zeta}{2X_1} + \left(\frac{\zeta}{2X_1}\right)^2 - \cdots\right) := \frac{1}{2X_1}(1+w),$$

where

$$w := \sum_{i\ge 1}(-1)^i\left(\frac{\zeta}{2X_1}\right)^i.$$

Clearly,

$$|w| = \frac{\zeta}{2X_1(1-\zeta/(2X_1))} < \frac{(1+10^{-10})}{4X_1^2}.$$

Thus,

$$\frac{1}{\alpha} = \frac{1}{2X_1} + \zeta', \qquad |\zeta'| < \frac{1+10^{-10}}{8X_1^3}.$$

Squaring it we get

$$\frac{1}{\alpha^2} = \frac{1}{4X_1^2} + \left(\frac{\zeta'}{X_1} + \zeta'^2\right) := \frac{1}{4X_1^2} + \zeta'', \quad |\zeta''| < \frac{1+10^{-10}}{8X_1^4} + \frac{(1+10^{-10})^2}{64X_1^6} < \frac{1}{10^7X_1^3}.$$

Thus,

$$\begin{aligned} & \left|n_1 \log\alpha - \frac{n_1}{2X_1} - \frac{n_1}{8X_1^2} - (n_1(m_2+1) - n_3(m_1+1))\log 2\right| \\ < & \left|n_1\log\alpha - \frac{n_1}{\alpha} - \frac{n_1}{2\alpha^2} - (n_1(m_2+1) - n_3(m_1+1))\log 2\right| \\ + & \; n_1|1/\alpha - 1/(2X_1)| + (n_1/2)|1/\alpha^2 - 1/(4X_1^2)| \\ < & \frac{(1+5\cdot 10^{-6})n_1}{3\alpha^3} + n_1|\zeta'| + (n_1/2)|\zeta''| \\ < & \frac{(1+5\cdot 10^{-6})n_1}{24X_1^3} + \frac{(1+10^{-10})n_1}{8X_1^3} + \frac{n_1}{2\cdot 10^7 X_1^3} < \left(\frac{1}{6} + 10^{-6}\right)\frac{n_1}{X_1^3}. \end{aligned}$$

Hence,

$$\left|n_1\log\alpha - \frac{n_1}{2X_1} - \frac{n_1}{8X_1^2} - (n_1(m_2+1) - n_3(m_1+1))\log 2\right| < \frac{(1/6+10^{-6})n_1}{X_1^3}. \tag{38}$$

Eliminating $n_1 \log\alpha$ between (37) and (38), we get

$$\begin{aligned} \left|\frac{n_1}{2X_1} + \frac{n_1}{8X_1^2} - ((m_1+1)(n_3+1) - n_1(m_2+1))\log 2\right| & < \frac{(1/6+10^{-6})n_1}{X_1^3} + \frac{(3/4\cdot 10^{-7})}{X_1^3} \\ & < \frac{(1/6+2\cdot 10^{-6})n_1}{X_1^3}. \end{aligned}$$

We recognise the coefficient of $\log 2$ as $k := (m_1+1)n_2 - n_1(m_1+1) \neq 0$ by Lemma 7. Thus,

$$\left|\frac{n_1}{2X_1} + \frac{n_1}{8X_1^2} - k\log 2\right| < \frac{(1/6+2\cdot 10^{-6})n_1}{X_1^3}. \tag{39}$$

We work on this last estimate. First of all it gives

$$\left|\frac{n_1}{2X_1} - k\log 2\right| < (1/8 + 1/X_1)\frac{n_1}{X_1^2} < (1/8 + 10^{-7})\frac{n_1}{X_1^2}. \tag{40}$$

Next after multiplying both sides of (39) by $4X_1$ is also gives

$$\left|2n_1 - 4X_1 k\log 2 + \frac{n_1}{2X_1}\right| < (2/3 + 10^{-5})\frac{n_1}{X_1^2}. \tag{41}$$

Eliminating $n_1/(2X_1)$ from (40) and (41), it gives

$$|2n_1 - (4X_1-1)k\log 2| < (2/3 + 1/8 + 2\cdot 10^{-5})\frac{n_1}{X_1^2}. \tag{42}$$

Since $n_1 < 7.6\cdot 10^9$ and $X_1 > 10^7$, it follows that

$$|2n_1 - (4X_1-1)k\log 2| < 6.1\cdot 10^{-5}.$$

Thus,

$$2n_1 - 6.1\times 10^{-5} < (4X_1-1)k\log 2 < 2n_1 + 6.1\times 10^{-5}. \tag{43}$$

In particular,

$$k < \frac{2n_1 + 6.1\cdot 10^{-5}}{4X_1 - 1} < 381,$$

since $n_1 \leq 7.6\cdot 10^9$ and $X_1 > 10^7$. Hence, $k \leq 380$. Next dividing both sides of (42) by $2n_1(\log 2)$, we get

$$\left|\frac{1}{\log 2} - \frac{(4X_1-1)k}{2n_1}\right| < \frac{19/24 + 2\cdot 10^{-5}}{(2\log 2)X_1^2}.$$

We work on the right–hand side. We have

$$\begin{aligned} \frac{19/24+2\cdot 10^{-5}}{(2\log 2)X_1^2} &= \frac{8(19/24+2\cdot 10^{-7})(\log 2)k^2}{(4X_1k\log 2)^2} \\ &< \frac{8(19/24+2\cdot 10^{-5})(\log 2)k^2}{((4X_1-1)k\log 2)^2} \\ &< \frac{8(19/24+2\cdot 10^{-5})(\log 2)k}{(2n_1-6.1\cdot 10^{-5})^2} \\ &< \frac{8(19/24+2\cdot 10^{-5})(\log 2)k^2}{(2n_1)^2(1-10^{-9})^2} \\ &< \frac{8(19/24+3\cdot 10^{-5})(\log 2)k^2}{(2n_1)^2} \\ &< \frac{4.39k^2}{(2n_1)^2}. \end{aligned}$$

In the above, we used in addition to (43) also the fact that $n_1 > 6\cdot 10^4$. Thus,

$$\left|\frac{1}{\log 2} - \frac{(4X_1-1)k}{2n_1}\right| < \frac{4.39k^2}{(2n_1)^2}.$$

We put $w := \gcd(2n_1, (4X_1-1)k)$. Then $(4X_1-1)k/(2n_1) = a/b$, where $2n_1 = wb$ and $(4X_1-1)k = wa$. Thus,

$$\left|\frac{1}{\log 2} - \frac{a}{b}\right| < \frac{4.39(k^2/w^2)}{b^2}.$$

We need a bound on w. We have $b \le 2n_1 < 1.52\cdot 10^{10}$. If p_ℓ/q_ℓ denotes the ℓ convergent of $1/\log 2$, then $q_{28} > 10^{11} > 2n_1$ and $\max\{a_\ell : 0 \le \ell \le 28\} = 13$. Hence, we get

$$\frac{1}{15b^2} < \left|\frac{1}{\log 2} - \frac{a}{b}\right| < \frac{4.39(k^2/w^2)}{b^2},$$

so $w^2 < 4.39\times 15k^2 < 66k^2$. In particular, $w < \sqrt{66\cdot 380^2}$, so $w \le 3100$. Fix w. Put $K := 4.39k^2/w^2$. Then

$$\left|\frac{1}{\log 2} - \frac{a}{b}\right| < \frac{K}{b^2}. \tag{44}$$

At this stage we use the following theorem of Worley [22] for the irrational $1/\log 2$, which generalises Legendre's result.

Theorem 4. *Assume* (44) *holds. There exist r, s with $r > 0$, $s \ge 0$, $rs < 2K$ and $m \ge 1$ such that*

$$a = rp_m \pm sp_{m-1} \qquad \textit{and} \qquad b = rq_m + sq_{m-1},$$

or $1 \le rs < K$, m is such that $a_{m+1} = 1$ and

$$a = rp_{m+1} + sp_{m-1}, \qquad \textit{and} \qquad b = rp_{m+1} + sp_{m-1}.$$

In case $s = 0$, we can take $r = 1$ and then $a/b = p_m/q_m$. This is the only case possible when $K \le 1/2$ since then $2K \le 1$ so $rs < 1$ giving $s = 0$. This is Legendre's result.

For us, $m \le 28$. So, we fix $w \le 3100$ and then $k \le 380$ and calculate $K := 4.39k^2/w^2$. Then we take non–negative numbers $\lambda = rs \le 2K$. For $\lambda = 0$, we take $r = 1, s = 0$. Then we take any $w \le 3100$, we factor $wp_m = (4X_1-1)k$ with some divisor $k \le 380$ and we find X_1. If this has a solution for the given m, w, then we know X_1 and $2n_1 = q_m w$. Next, $\lambda \ge 1$. The largest value of $2K$ is at most 634000 (at $w = 1$ and $k = 380$). Then we take

$w < \sqrt{634000/\lambda} < 800/\sqrt{\lambda}$. We factor $\lambda = rs$ with positive r, s and look at $(rq_m + sq_{m-1})w$. This must be $2n_1 > 10^6$. Thus, if $rq_m + sq_{m-1}$ is even and larger than 10^6, then we take $n_1 = (rp_m + sp_{m-1})w/2$ and look at the possible candidate $(4X_1 - 1)k = (rp_m \pm sp_{m-1})w$. Here, k must be a divisor of $rp_m \pm sp_{m-1}$ in the interval $\sqrt{\lambda/8.78}w \le k \le 380$ and the quotient $4X_1 - 1 = (rp_m \pm sp_{m-1})/k$ must be congruent to 3 modulo 4. If all these happen, then we solve for X_1.

There is another similar test when $a_{m+1} = 1$, where here we take $(rq_{m+1} + sq_{m-1})w$ as a candidate for $2n_1$ and $(rp_{m+1} \pm sp_{m-1})w$ as candidates for $(4X_1 - 1)k$.

At any rate, this procedure creates some candidates (n_1, X_1). To check whether they are convenient we go back to (36). At this stage, X_1 is known, (therefore so is α), and n_1 is also known and (36) shows that $m_1 + 1$ is the closest integer to $n_1(\log \alpha)/\log 2$. We compute it and then find a lower bound on the left–hand side of (36), and then this lower bound together with the right–hand side of (36) puts a bound on n_1.

We wrote a code which ran the entire calculation for $\lambda = 0$ in a few minutes. It also ran the entire calculation for the branch with $rq_m + sq_{m-1}$ of Worley's theorem and all $\lambda \le 634000$ in about 170 hours. Then we ran the calculation for the branch with $rq_{m+1} + sq_{m-1}$ when $a_{m+1} = 1$ of Worley's theorem and all $\lambda \le 634000$ in about 50 hours.

The output of all this computation was empty (the lower bound obtained from the left–hand side of (36) tends to be very good, usually 10^{-3}), so there is no solution when $X_1 > 10^7$. This finishes the proof of the last subclass.

4.2.1. *Final verification.* In conclusion, if Pillai's equation $c = X_n - 2^m$ has three solutions (n, m) for a fixed c all with positive n's, then all possible candidates for (X_1, c) are listed in Table 2. Now we proceed line per line. We recover the value of c from the table through the formula $c = X_{n_3} - 2^{m_3}$. Next, we reduce the bound $n_1 \le 60381$ using the same procedure done in equation (31). This gives us a significantly smaller bound $n_1 \le M(X_1)$. Next, we check if $X_{n_1} - c$ is a power of 2.

Doing the search for each line in the table, we could only find the same pairs (n_3, m_3) and (n_2, m_2) that already appeared in the table except when $X_1 = 1$ and $c = -1$, where we found the three pairs given in the Theorem 2, namely $(1, 1)$, $(2, 2)$, $(3, 3)$. Since none of these equations, except for the last one mentioned above, for the given values of c had 3 pairs of solutions, we conclude that indeed $m_{U,V}(c) \le 2$ for all (c, d) except in the case $(c, d) = (-1, 2)$, where $m_{U,V}(c) = 3$.

5. Case $n = 0$

Here, we treat the missing case when $n_3 = 0$. Then $c = 1 - 2^{m_3}$. If $m_3 = 0$ then $c = 0$ and consequently $X_{n_i} = 2^{m_i}$, so using that $\nu_2(X_{n_i}) \in \{0, \nu_2(X_1)\}$ by Lemma 2 and the fact that $\{X_n\}_{n \ge 1}$ is strictly increasing, we must have $n_i = 1$. Hence, this only gives us $m_{U,V}(0) \le 2$.

Thus, we may assume that $m_3 > 0$, so $c < 0$. If $c = X_{n_i} - 2^{m_i}$ with n_i even, then $c = 2X_{n_i/2}^2 - \varepsilon^{n_i/2} - 2^{m_i}$. Thus

$$2X_{n_i/2}^2 = 2^{m_i} - 2^{m_3} + 1 + \varepsilon^{n_i/2}.$$

This reduces either to $2X_{n_i/2}^2 = 2^{m_i} - 2^{m_3}$ or $2X_{n_i/2}^2 = 2^{m_i} - 2^{m_3} + 2$.

In the first case, the equation $X_{n_i/2}^2 = 2^{m_i - 1} - 2^{m_3 - 1}$ implies first that $m_3 - 1$ is even, and that $x := X_{n_i/2}/2^{(m_3-1)/2}$ satisfies $x^2 = 2^{m_i - m_3} - 1$. The only possibility is $m_i = m_3 + 1$, so $X_{n_i/2} = 2^{(m_3-1)/2}$, whose only solution is $n_i = 2$.

In the second case, we have

$$X_{n_i/2}^2 = 2^{m_i-1} - 2^{m_3-1} + 1. \tag{45}$$

According to [21], this possibility leads to either the parametric family $X_{n_i/2} = 2^{(m-1)/2} - 1$ with $m_3 = (m+3)/2$ and $m_i = m$, or to the sporadic solutions $X_{n_i/2} \in \{5, 11, 181\}$. Since $n_i \equiv n_3 \pmod 2$, by Lemma 3, we have

$$\alpha^{n_i/2-1} \leq X_{n_i/2} < 2^{m_3-2} < (X_1^2+1)n_i^2/2, \tag{46}$$

and by the same method used in (15) we obtain $n_i \leq 16$. Moreover,

$$P_{n_i/2}^{+}(X_1) \leq X_{n_i/2} < 128(X_1^2+1).$$

The previous equation gives us $X_1 \leq 32$ when $n_i \geq 6$.

If both n_2, n_1 are even, then taking $n_i = n_1$ we have $n_i \geq 4$. In case that $n_1 = 4$, then $n_2 = 2$ and so $X_1 = X_{n_2/2} = 2^{(m-1)/2} - 1$ is odd. If $n' \in \{n_2, n_1\}$ is odd, then since $X_{n'} = 1 + 2^{m'} - 2^{m_3}$, we have that $X_{n'}$ is odd and by Lemma 2 we have that $\nu_2(X_1) = \nu_2(X_n')$, and so X_1 is odd.

Hence, if $n_i = 4$, then X_1 is odd, and we can replace the right–hand side of (46) by $(X_1+1)n_i^2$. Thus,

$$P_{n_i/2}^{+}(X_1) \leq X_{n_i/2} < 16(X_1+1),$$

which implies $X_1 \leq 8$.

Note that if $n_i = 2$, then (45) holds only when $\varepsilon = 1$. Furthermore, $n_i = 2$ implies that n_1 must be odd, so by Lemma 4 we have that

$$2(X_1+1)^2 = 2^{m_2} \leq X_1^2 - 1,$$

a contradiction. Thus, $n_i \neq 2$.

In conclusion n_2 and X_1 are bounded if one of n_1, n_2 is even. If both are odd, then by Lemma 3 we have

$$2^{m_2} \leq 2(X_1^2+1)n_1^2.$$

As $c < 0$, from the previous inequality we get $X_{n_2} \leq 2(X_1^2+1)n_1^2$. If $n_2 \leq n_1/1.5$ then by Claim 1 we get $n_1 \leq 60381$. Otherwise, we get that $\alpha^{n_1/1.5-1} < 2(X_1^2+1)n_1^2$ and by the same method used in (15) we obtain $n_1 \leq 12$. Thus,

$$P_{n_2}^{+}(X_1) - 1 \leq X_{n_2} - 1 = X_{n_2} - X_{n_3} = 2^{m_2} - 2^{m_3} \leq 2^{m_2} \leq 2(X_1^2+1)(60381)^2.$$

Following the argument used at (25), this implies that $n_2 \leq 27$. Furthermore, it also implies that $X_1 < 1.83 \cdot 10^9$ when $n_2 \geq 3$. So,

$$2^{m_2} \leq 2(X_1^2+1)n_1^2 \leq 2((1.83 \cdot 10^9)^2+1)(60381)^2,$$

which leads to $m_2 \leq 94$.

Following the argument used to deal with (26) and (27), we consider the equations:

$$P_{n_2}^{+}(X_1) = 2^{m_2} - 2^{m_3} + 1,$$
$$P_{n_2}^{-}(X_1) = 2^{m_2} - 2^{m_3} + 1,$$

both with $n_2 \in [2, 27]$, $m_2 \in [1, 94]$, $m_3 \in [1, m_2 - 1]$. Since n_2 is odd, these yield the only solution $X_1 = 1$, $n_2 = 3$, $m_2 = 3$, $m_3 = 1$.

When $n_2 = 1$, we have that $X_1 = 2^{m_2} - 2^{m_3} + 1$ is odd. If $m_3 = 1$, we have that $X_1 = 2^{m_2} - 1$ and $X_{n_1} = 2^{m_1} - 1$ are both Mersenne numbers, which we can think as repunits in base 2. Since n_1 is odd, Lemma 5.2 of [10] implies that $2^{m_2} \mid n_1^2$, so $m_2 = 1$ and $X_1 = 1$.

Suppose $m_3 > 1$. Taking a closer look to the proof of Lemma 3 (which is Lemma 4 in [9]), we have the inequality

$$m_2 \le 1 + \nu_2(dY_1^2) + 2\log_2(n_1). \tag{47}$$

Since $dY_1^2 = X_1^2 - \varepsilon = (2^{m_2} - 2^{m_3})^2 + 2(2^{m_2} - 2^{m_3}) + 1 - \varepsilon$, we have that $\nu_2(dY_1^2) \le m_3 + 1$. In particular, this last inequality implies that

$$m_2 - m_3 \le 2 + 2\log_2(60381) < 34.$$

When $\varepsilon = -1$ we have instead $\nu_2(dY_1^2) = 1$, so from inequality (47) we deduce that $m_2 \le 32$.

Suppose $\varepsilon = 1$. Note that $P_n^+(1) = 1$ for any n. Taking the derivative to $Q(z) := P_n^+(z+1)$ we get

$$Q'(z) = nP_n^+(z+1) + n\left(\frac{\left(z+1+\sqrt{z^2+z}\right)^{n-1} - \left(z+1-\sqrt{z^2+z}\right)^{n-1}}{2\sqrt{z^2+z}}\right),$$

so $Q'(0) = n^2$. This implies that there is $h(z) \in \mathbb{Z}[z]$ such that

$$P_n^+(z+1) = 1 + n^2 z + z^2 h(z).$$

Write $X_1 = b \cdot 2^{m_3} + 1$ where b is a positive odd integer. Therefore

$$2^{m_1} = X_{n_1} - 1 + 2^{m_3} = P_{n_1}^+(X_1) - 1 + 2^{m_3} = n_1^2 b \cdot 2^{m_3} + b^2 2^{2m_3} h(b \cdot 2^{m_3}) + 2^{m_3}.$$

Hence,

$$2^{m_1 - m_3} = n_1^2 b + 1 + b^2 2^{m_3} h(b \cdot 2^{m_3}).$$

If $2^{m_3} > n_1^2 b + 1$ then the above expression cannot be a power of 2. This implies that

$$m_3 \le \frac{\log(n_1^2 b + 1)}{\log 2} \le \frac{\log((60381)^2(2^{33}-1)+1)}{\log 2} < 64. \tag{48}$$

So, $m_2 \le 96$ when $\varepsilon = 1$.

In conclusion, we have boundedly many possibilities for X_1 and n_2. We perform again a reduction using continued fractions via Lemma 5 exactly as we did in (31). More specifically, assuming $n_1 > 10$, from $X_{n_1} - 2^{m_1} = 1 - 2^{m_3}$ we obtain the linear form

$$\left| n_1 \frac{\log \alpha}{\log 2} - (m_1 + 1) \right| < \frac{6n_1^2}{\alpha^{n_1 - 2}}.$$

We got the uniform bound $n_1 \le 17$, for all cases including the cases where some n_i is even.

When $n_2 = 1$ and n_1 is odd, putting this again in the previous inequalities (47) and (48), we get $m_3 \le 17$ and $m_2 \le 26$. In case n_2 or n_1 were even, we have

$$m_3 \le \log_2(2(181^2+1)(17)^2) < 25.$$

Finally, we consider the Diophantine equation

$$X_{n_1} - 1 + 2^{m_3} = 2^{m_1} \tag{49}$$

with $n_1 \in [1, 17]$, $m_3 \in [1, 24]$ with $X_1 \in [1, 181]$ or $X_1 = 2^{m_2} - 2^{m_3} + 1$ with $m_2 \in [1, 26]$, $m_3 \in [1, m_2 - 1]$, for both choices of sign $\varepsilon \in \{\pm 1\}$.

In the second situation the output was empty, while in the first one we obtained several triples of (X_1, m_3, n_1). However, each one of these triples differ in (X_1, m_3) except in the case $(X_1, m_3) = (1, 1)$ with $\varepsilon = -1$, which corresponds to the exceptional case of Theorem 2. Hence, equation (49) has at most one solution, because if (n_2, m_2) exists, it must be a solution of (49). Actually, from what we have already said, we found several pairs (X_1, m_3) with same X_1 but different m_3, so these pairs belong to different c corresponding to solutions

(n, m) of the equation $c = X_n - 2^m$. This finishes the proof of Theorem 2 in the remaining case $n_3 = 0$.

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References

1. M. A. Bennett, *On some exponential equations of s. s. pillai*, Canad. J. Math. **53** (2001), no. 5, 897–922.
2. J. J. Bravo, F. Luca, and K. Yazán, *On pillai's problem with tribonacci numbers and powers of 2*, Bull. Korean Math. Soc. **54** (2017), no. 3, 1069–1080.
3. K. C. Chim, I. Pink, and V. Ziegler, *On a variant of Pillai's problem*, Int. J. Number Theory **13** (2017), no. 7, 1711–1727.
4. ______, *On a variant of Pillai's problem II*, J. Number Theory **183** (2018), 269–290.
5. M. Ddamulira, C. A. Gómez, and F. Luca, *On a problem of pillai with k–generalized fibonacci numbers and powers of 2*, Monatsh. Math. **187** (2018), no. 4, 635–664.
6. M. Ddamulira and F. Luca, *On a problem of pillai with k–generalized fibonacci numbers and powers of 3*, arXiv:1905.01015.
7. M. Ddamulira, F. Luca, and M. Rakotomalala, *On a problem of pillai with fibonacci numbers and powers of 2*, Proc. Indian Acad. Sci. Math. Sci. **127** (2017), no. 3, 411–421.
8. A. Dujella and A. Pethő, *A generalization of a theorem of baker and davenport*, Quart. J. Math. Oxford Ser. (2) **49** (1998), no. 195, 291–306.
9. H. S. Erazo, C. A. Gómez, and F. Luca, *On pillai's problem with x–coordinates of pell equations and powers of 2*, Journal of Number Theory **203** (2019), 294–309.
10. B. Faye and F. Luca, *On X-coordinates of Pell equations that are repdigits*, Fibona cci Quart. **56** (2018), no. 1, 52–62. MR 3782967
11. S. Hernández, F. Luca, and L. M. Rivera, *On pillai's problem with the fibonacci and pell sequences*, Boletín de la Sociedad Matemática Mexicana (2018).
12. M. O. Hernane, F. Luca, S. E. Rihane, and A. Togbé, *On pillai's problem with pell numbers and powers of 2*, Hardy-Ramanujan J. **41** (2018), 22–31.
13. A. Herschfeld, *The equation* $2^x - 3^y = d$, Bull. Amer. Math. Soc. **41** (1935), no. 9, 631.
14. ______, *The equation* $2^x - 3^y = d$, Bull. Amer. Math. Soc. **42** (1936), no. 4, 231–234.
15. M. Laurent, *Linear forms in two logarithms and interpolation determinants. II*, Acta Arith. **133** (2008), no. 4, 325–348. MR 2457264
16. P. Mihăilescu, *Primary cyclotomic units and a proof of catalan's conjecture*, J. Reine Angew. Math. **572** (2004), 167–195.
17. S. S. Pillai, *A correction to the paper on* $a^x - b^y = c$, J. Indian Math. Soc. **2** (1937), 215.
18. ______, *On* $a^x - b^y = c$, J. Indian Math. Soc. **2** (1937), 119–122.
19. C. Sanna, *The p-adic valuation of Lucas sequences*, Fibonacci Quart. **54** (2016), no. 2, 118–124. MR 3512829
20. R. J. Stroeker and R. Tijdeman, *Diophantine equations*, Computational methods in number theory, Part II, Math. Centre Tracts, vol. 155, Math. Centrum, Amsterdam, 1982, pp. 321–369.
21. László Szalay, *The equations* $2^n \pm 2^m \pm 2^l = z^2$, Indag. Math. (N.S.) **13** (2002), no. 1, 131–142. MR 2014980
22. R. T. Worley, *Estinating* $|\alpha - p/q|$, J. Austral. Math. Soc. (Serie A) **31** (1981), 202–206.

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