AUTOMORPHISM GROUPS OF COMPACT COMPLEX SURFACES

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Abstract. We study automorphism groups and birational automorphism groups of compact complex surfaces. We show that the automorphism group of such a surface X is always Jordan, and the birational automorphism group is Jordan unless X is birational to a product of an elliptic and a rational curve.

Contents

1. Introduction

It often happens that some infinite subgroups exhibit a nice and simple behavior on the level of their finite subgroups. An amazing example of such a situation is given by the following result due to C. Jordan (see [CR62, Theorem 36.13]).

Theorem 1.1. *There is a constant $J = J(n)$ such that for every finite subgroup $G \subset \mathrm{GL}_n(\mathbb{C})$ there exists a normal abelian subgroup $A \subset G$ of index at most J.*

This motivates the following definition.

Definition 1.2 (see [Pop11, Definition 2.1])**.** A group Γ is called *Jordan* (alternatively, we say that Γ *has Jordan property*) if there is a constant J such that for every finite subgroup $G \subset \Gamma$ there exists a normal abelian subgroup $A \subset G$ of index at most J.

In other words, Theorem 1.1 says that the group $\mathrm{GL}_n(\mathbb{C})$ is Jordan. The same applies to any linear algebraic group, since it can be realized as a subgroup of a general linear group.

It was noticed by J.-P. Serre that Jordan property sometimes holds for groups of birational automorphisms.

Theorem 1.3 ([Ser09, Theorem 5.3], [Ser10, Théorème 3.1])**.** *The group of birational automorphisms of $\mathbb{P}^2$ over the field $\mathbb{C}$ (or any other field of characteristic 0) is Jordan.*

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Yu. Zarhin pointed out in [Zar14] that there are projective complex surfaces whose birational automorphism groups are not Jordan; they are birational to products $E \times \mathbb{P}^1$, where E is an elliptic curve. The following result of V. Popov classifies projective surfaces with non-Jordan birational automorphism groups.

Theorem 1.4 ([Pop11, Theorem 2.32])**.** *Let X be a projective surface over $\mathbb{C}$. Then the group of birational automorphisms of X is not Jordan if and only if X is birational to $E \times \mathbb{P}^1$, where E is an elliptic curve.*

Automorphism groups having Jordan property were studied recently in many different contexts. Yu. Prokhorov and C. Shramov in [PS16, Theorem 1.8] and [PS14, Theorem 1.8] proved that this property holds for groups of birational selfmaps of rationally connected algebraic varieties, and some other algebraic varieties of arbitrary dimension. Actually, their results were initially obtained modulo a conjectural boundedness of terminal Fano varieties (see e. g. [PS16, Conjecture 1.7]), which was recently proved by C. Birkar in [Bir16, Theorem 1.1]. Also Yu. Prokhorov and C. Shramov classified Jordan birational automorphism groups of algebraic threefolds in [PS18b]. Some results about birational automorphisms of conic bundles were obtained by T. Bandman and Yu. Zarhin in [BZ17]. For other results on Jordan birational automorphism groups see [PS17], [PS18c], and [Yas17].

S. Meng and D.-Q. Zhang proved in [MZ18] that the automorphism group of any projective variety is Jordan, and J. H. Kim generalized this to automorphism groups of compact Kähler manifolds in [Kim18]. T. Bandman and Yu. Zarhin proved a similar result for automorphism groups of quasi-projective surfaces in [BZ15], and also in some particular cases in arbitrary dimension in [BZ18]. For a survey of some other relevant results see [Pop14].

É. Ghys asked (following a more particular question posed earlier by W. Feit) whether the diffeomorphism group of a smooth compact manifold is always Jordan. Recently B. Csikós, L. Pyber, and E. Szabó in [CPS14] provided a counterexample following the method of [Zar14]; see also [Mun17b] for a further development of this method, and [Pop16, Corollary 2] for a non-compact counterexample. However, Jordan property holds for diffeomorphism groups in many cases; see [Mun16], [Mun14], [MT15], [Mun13], [GZ13], [Zim12], [Zim14a], [Zim14b], [MZ15], and references therein. Also there are results for groups of symplectomorphisms, see [Mun17a] and [Mun18].

The goal of this paper is to generalize Theorem 1.4, and to some extent the results of [MZ18] and [Kim18], to a different setting, namely, to the case of compact complex surfaces (see §3 below for basic definitions and background). There are some particular cases that are already known. For instance, automorphism groups of Inoue surfaces (see [Ino74]) and primary Kodaira surfaces (see [Kod64, §6], [BHPVdV04, § V.5]) were studied in [PS18a].

Theorem 1.5 ([PS18a, Theorem 1.2])**.** *Let X be either an Inoue surface or a primary Kodaira surface. Then the automorphism group of X is Jordan.*

We prove the following.

Theorem 1.6. *Let X be a connected compact complex surface. Then the automorphism group of X is Jordan.*

One can also show (see [Mun13, Theorem 1.3] or Theorem 2.11 below) that the number of generators of any finite subgroup of the automorphism group of a compact complex surface X, and actually of any finite subgroup of the diffeomorphism group of an arbitrary compact manifold, is bounded by a constant that depends only on X.

The main result of this paper is as follows.

Theorem 1.7. *Let X be a connected compact complex surface. Then the group of birational automorphisms of X is not Jordan if and only if X is birational to $E \times \mathbb{P}^1$, where E is an elliptic curve. Moreover, there always exists a constant $R = R(X)$ such that every finite subgroup of the birational automorphism group of X is generated by at most R elements.*

The plan of the paper is as follows. In §2 we collect some elementary facts about Jordan property, and other boundedness properties for subgroups. In §3 we recall the basic facts from the theory of compact complex surfaces, most importantly their Enriques–Kodaira classification. In §4 we recall some important general facts concerning automorphisms of complex spaces. In §5 we study automorphism groups of non-projective surfaces with non-zero topological Euler characteristic; an important subclass of such surfaces is formed by minimal surfaces of class VII with non-zero second Betti number (which are still not completely classified). In §6 we study automorphism groups of Hopf surfaces. In §7 we study automorphism groups of (secondary) Kodaira surfaces. In §8 we study automorphism groups of other minimal surfaces of non-negative Kodaira dimension, and prove Theorems 1.6 and 1.7.

Our general strategy is to consider the compact complex surfaces according to Enriques–Kodaira classification. Note that some of our theorems follow from more general results of I. Mundet i Riera, cf. Theorems 5.1 and 5.12 (and also the discussion in the end of §5). Similarly, some other results are implied by [Kim18]. We also point out that Jordan property always holds for the connected component of the identity in the automorphism group of an arbitrary connected compact complex manifold by [Pop18, Theorem 7].

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2. Jordan property

In this section we collect some group-theoretic properties related to the Jordan property, and prove a couple of auxiliary results about them. We start by recalling a useful result that is very well known (see for instance [Spr77, §4.4]).

Theorem 2.1. *Let $G \subset \mathrm{Aut}(\mathbb{P}^1) \cong \mathrm{PGL}_2(\mathbb{C})$ and $\tilde{G} \subset \mathrm{GL}_2(\mathbb{C})$ be finite subgroups. Then G is either cyclic, or dihedral, or isomorphic to one of the groups $\mathfrak{A}_4$, $\mathfrak{S}_4$, or $\mathfrak{A}_5$. In particular, the group G has a cyclic subgroup of index at most 12, and the group $\tilde{G}$ has an abelian subgroup of index at most 12. Furthermore, if $|G|$ is odd, then G is cyclic, and if $|\tilde{G}|$ is odd, then $\tilde{G}$ is abelian.*

Apart from the Jordan property, one can consider other restrictions formulated in terms of finite subgroups of a given group.

Definition 2.2. We say that a group Γ *has bounded finite subgroups* if there exists a constant $B = B(\Gamma)$ such that for any finite subgroup $G \subset \Gamma$ one has $|G| \leqslant B$.

The following result is due to H. Minkowski, see for instance [Ser07, Theorem 1].

Theorem 2.3. *For every n the group $\mathrm{GL}_n(\mathbb{Q})$ has bounded finite subgroups.*

Definition 2.4. We say that a group Γ is *strongly Jordan* if it is Jordan, and there exists a constant $R = R(\Gamma)$ such that every finite subgroup in Γ is generated by at most R elements.

Note that Definition 2.4 is equivalent to a similar definition in [BZ15]. An example of a strongly Jordan group is given by $\mathrm{GL}_n(\mathbb{C})$. This follows from the fact that every abelian subgroup of $\mathrm{GL}_n(\mathbb{C})$ is conjugate to a group that consists of diagonal matrices. Note however that even the group $\mathbb{C}^*$ contains *infinite* abelian subgroups of arbitrarily large rank.

The following elementary result will be useful to study Jordan property.

Lemma 2.5. *Let*

$$1 \longrightarrow \Gamma' \longrightarrow \Gamma \longrightarrow \Gamma''$$

be an exact sequence of groups. Then the following assertions hold.

(i) *If Γ' is Jordan (respectively, strongly Jordan) and Γ'' has bounded finite subgroups, then Γ is Jordan (respectively, strongly Jordan).*

(ii) *If Γ' has bounded finite subgroups and Γ'' is strongly Jordan, then Γ is strongly Jordan.*

Proof. Assertion (i) is obvious. For assertion (ii) see [PS14, Lemma 2.8] or [BZ15, Lemma 2.2]. □

It is easy to see that if Γ_1 is a subgroup of finite index in Γ_2, then Γ_2 is Jordan (respectively, strongly Jordan) if and only so is Γ_1. At the same time Jordan property, as well as strong Jordan property, does not behave well with respect to quotients by infinite groups. Namely, a quotient of a strongly Jordan group by its subgroup may fail to be Jordan or to have all of its finite subgroups generated by a bounded number of elements. In spite of this we will be able to control the properties of some quotients by infinite groups that will be important for us.

Lemma 2.6. *Let A be an abelian group whose torsion subgroup A_{t} is isomorphic to $(\mathbb{Q}/\mathbb{Z})^n$, and let $\Lambda \subset A$ be a subgroup isomorphic to $\mathbb{Z}^m$. Then the quotient group $\Gamma = A/\Lambda$ is strongly Jordan.*

Proof. The group Γ is abelian and thus Jordan. Let $V \subset \Gamma$ be a finite subgroup and let $\tilde{V} \subset A$ be its preimage. Clearly, $\tilde{V}$ is finitely generated and can be decomposed into a direct product $\tilde{V} = \tilde{V}_{\mathrm{t}} \times \tilde{V}_{\mathrm{f}}$ of its torsion and torsion free parts. In particular, $\tilde{V}_{\mathrm{f}}$ is a free abelian group. Since $\tilde{V}_{\mathrm{f}}/(\tilde{V}_{\mathrm{f}} \cap \Lambda)$ is a finite group, one has

$$\operatorname{rk} \tilde{V}_{\mathrm{f}} = \operatorname{rk}(\tilde{V}_{\mathrm{f}} \cap \Lambda) \leqslant \operatorname{rk} \Lambda = m.$$

The group $\tilde{V}_{\mathrm{t}}$ is contained in $A_{\mathrm{t}} \cong (\mathbb{Q}/\mathbb{Z})^n$ and so it can be generated by n elements. Thus $\tilde{V}$ can be generated by $n + m$ elements, and the images of these elements in Γ generate the subgroup V. □

Lemma 2.7. *Let*

$$1 \longrightarrow \Gamma' \longrightarrow \Gamma \longrightarrow \Gamma'' \tag{2.8}$$

be an exact sequence of groups. Suppose that Γ' is central in Γ (so that in particular Γ' is abelian) and there exists a constant R such that every finite subgroup of Γ' is generated by at most R elements. Suppose also that there exists a constant J such that for every *finite subgroup $G \subset \Gamma''$ there is a cyclic subgroup $C \subset G$ of index at most J (so that in particular Γ'' is strongly Jordan). Then the group Γ is strongly Jordan.*

Proof. Let $G \subset \Gamma$ be a finite subgroup. The exact sequence (2.8) induces an exact sequence of groups

$$1 \longrightarrow G' \longrightarrow G \longrightarrow G'',$$

where G' is a subgroup of Γ' (in particular, G' is abelian), while G'' is a subgroup of Γ''. There is a subgroup $\bar{G} \subset G$ of index at most J such that $\bar{G}$ contains G', and the quotient $\bar{G}/G'$ is a cyclic group. To prove that the group Γ is Jordan it is enough to check that $\bar{G}$ is an abelian group. The latter follows from the fact that G' is a central subgroup of $\bar{G}$.

The assertion about the bounded number of generators is obvious. □

Lemma 2.9. *Let Λ be a finitely generated central subgroup of $\mathrm{GL}_2(\mathbb{C})$. Then the quotient group $\Gamma = \mathrm{GL}_2(\mathbb{C})/\Lambda$ is strongly Jordan.*

Proof. We have an exact sequence of groups

$$1 \longrightarrow \mathbb{C}^*/\Lambda \longrightarrow \Gamma \longrightarrow \mathrm{PGL}_2(\mathbb{C}) \longrightarrow 1.$$

The group $\mathbb{C}^*/\Lambda$ is a central subgroup of Γ. Also, the group $\mathbb{C}^*/\Lambda$ is strongly Jordan by Lemma 2.6.

On the other hand, we know from the classification of finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$ that every finite subgroup therein contains a cyclic subgroup of bounded index, see Theorem 2.1. Therefore, the assertion follows from Lemma 2.7. □

We will need the following simple observation in §6.

Lemma 2.10. *Let Γ be a group containing a subgroup $\Lambda \cong \mathbb{Z}$ of finite index. Then there is a subgroup $\Lambda_0 \cong \mathbb{Z}$ in Λ that is characteristic in Γ.*

Proof. The intersection

$$\Lambda_0 = \bigcap_{\theta \in \mathrm{Aut}(\Gamma)} \theta(\Lambda)$$

is a characteristic subgroup in Γ. Therefore, it is enough to check that Λ_0 is not a trivial group.

Denote $r = [\Gamma : \Lambda]$. For any $\theta \in \mathrm{Aut}(\Gamma)$ the group $\theta(\Lambda)$ has index r in Γ. Hence the index of the intersection $\Lambda \cap \theta(\Lambda)$ in Λ is at most r. This means that the intersection of Λ with *all* groups $\theta(\Lambda)$, $\theta \in \mathrm{Aut}(\Gamma)$, contains an intersection of all these subgroups in Λ. Since a subgroup of given index in Λ is unique, we see that the latter intersection is non-trivial. □

Most of the groups we will be working with in the remaining part of the paper will be strongly Jordan. However, we will only need to check Jordan property for them due to the following result.

Theorem 2.11 ([Mun13, Theorem 1.3])**.** *For any compact manifold X there is a constant R such that every finite group acting effectively by diffeomorphisms of X can be generated by at most R elements.*

3. Minimal surfaces

In this section we recall the basic properties of compact complex surfaces. Everything here (as well as in §4 below) is well known to experts, but in some important cases we provide proofs for the reader's convenience.

A complex surface is a complex manifold of (complex) dimension 2. Starting from this point we will always assume that our complex surfaces are connected. Throughout the paper $\mathscr{K}_X$ denotes the canonical line bundle of a compact complex surface X. One has $\mathrm{c}_1(\mathscr{K}_X) = -\mathrm{c}_1(X)$. By $\mathrm{a}(X)$ we denote the algebraic dimension of X, i.e. the transcendence degree of the field of meromorphic functions on X.

Definition 3.1. Let X and Y be compact complex surfaces. A proper holomorphic map $f\colon X \to Y$ is said to be a *proper modification* if there are closed analytic subsets $Z_1 \subsetneq X$ and $Z_2 \subsetneq Y$ such that the restriction $f_{X\setminus Z_1}\colon X\setminus Z_1 \to Y\setminus Z_2$ is biholomorphic. A *birational* (or *bimeromorphic*) *map* $X \dashrightarrow Y$ is an equivalence class of diagrams

$$\begin{array}{ccc} & Z & \\ {}^{f}\swarrow & & \searrow^{g} \\ X & \dashrightarrow & Y \end{array}$$

where f and g are proper modifications, modulo natural equivalence relation.

Birational maps from a given compact complex surface X to itself form a group, which we will denote by $\mathrm{Bir}(X)$. As usual, we say that two complex surfaces are birationally equivalent, or just birational, if there exists a birational map between them.

Remark 3.2. If X and Y are birationally equivalent compact complex surfaces, then the fields of meromorphic functions on X and Y are isomorphic. The converse is not true if the algebraic dimension of X (and thus also of Y) is less than 2.

There are easy ways to find whether a given compact complex surface is projective.

Theorem 3.3 (see [BHPVdV04, Corollary IV.6.5])**.** *A compact complex surface X is projective if and only if* $\mathrm{a}(X) = 2$. *In particular, any compact complex surface birational to a projective one is itself projective.*

Lemma 3.4 (see [BHPVdV04, Theorem IV.6.2])**.** *Let X be a compact complex surface. Suppose that there is a line bundle $\mathcal{L}$ on X such that $\mathcal{L}^2 > 0$. Then X is projective.*

A (-1)-*curve* on a compact complex surface is a smooth rational curve with self-intersection equal to -1. A compact complex surface is *minimal* if it does not contain (-1)-curves. The following fact is well known, see e.g. Corollary III.2.4, Claim on p. 99, and the first paragraph of § VI.7 in [BHPVdV04]). For convenience of the reader we provide its short proof.

Proposition 3.5. *Let X be a minimal surface. Suppose that X is neither rational nor ruled. Then every birational map from an arbitrary compact complex surface X' to X is a proper modification. In particular, X is the unique minimal model in its class of birational equivalence, and* $\mathrm{Bir}(X) = \mathrm{Aut}(X)$.

Proof. Suppose that

$$\begin{array}{ccc} & Z & \\ {}^{g}\swarrow & & \searrow^{f} \\ X' & \dashrightarrow & X \end{array}$$

is a birational map that is not a proper modification. We may assume that there are no (-1)-curves that are simultaneously contracted by f and g. Then there exists a (-1)-curve C in Z contracted by g but not contracted by f. Thus C meets a one-dimensional fiber $f^{-1}(x)$ for some point $x \in X$, since otherwise X would contain a (-1)-curve.

First, we consider the case when the surface X is projective. Since X is minimal and not ruled, the canonical class K_X must be numerically effective [BHPVdV04, Theorem VI.2.1]. Write

$$K_Z \sim f^*K_X + \sum a_i E_i,$$

where E_i are f-exceptional curves and a_i are positive integers. Since $K_Z \cdot C < 0$ and $f^*K_X \cdot C \geqslant 0$, we have $\sum a_i E_i \cdot C < 0$. Thus C is a component of the f-exceptional locus. This contradicts our assumptions.

Now we consider the case when the surface X is not projective. Contracting (-1)-curves in $f^{-1}(x)$ consecutively, we get a surface S with a proper modification $h\colon Z \to S$, and a proper modification $t\colon S \to X$ such that $C_1 = h(C)$ is a (-1)-curve and there exists another (-1)-curve C_2 meeting C_1 and contracted by t. If $C_1 \cdot C_2 > 1$, then $(C_1 + C_2)^2 > 0$ and the surface S is projective by Lemma 3.4. Assume that $C_1 \cdot C_2 = 1$. Then for $n \gg 0$ we have

$$\mathrm{c}_1\left(\mathscr{K}_S \otimes \mathscr{O}_S(-nC_1 - nC_2)\right)^2 = \mathrm{c}_1(S)^2 + 4n > 0,$$

so that the surface S is again projective by Lemma 3.4. The obtained contradiction completes the proof. □

Given a compact complex surface X, we can consider its *pluricanonical linear systems* $|\mathscr{K}_X^{\otimes m}|$. If such a linear system is not empty for $m \gg 0$, it defines a rational *pluricanonical map*. The dimension of its image is called the *Kodaira dimension* of X and is denoted by $\varkappa(X)$. If the linear system $|\mathscr{K}_X^{\otimes m}|$ is empty for all $m > 0$, we put $\varkappa(X) = -\infty$. By $\mathrm{b}_i(X)$ we denote the i-th Betti number of X. By $\mathrm{h}^{p,q}(X)$ we denote the Hodge numbers $\mathrm{h}^{p,q} = \dim H^q(X, \Omega_X^p)$, where Ω_X^p is the sheaf of holomorphic p-forms on X.

The following is the famous Enriques–Kodaira classification of compact complex surfaces, see e.g. [BHPVdV04, Chapter VI].

Theorem 3.6. *Let X be a minimal compact complex surface. Then X is of one of the following types.*

$\varkappa(X)$	type	$\mathrm{a}(X)$	$\mathrm{b}_1(X)$	$\chi_{\mathrm{top}}(X)$
$-\infty$	rational surfaces	2	0	3, 4
	ruled surfaces of genus $g > 0$	2	$2g$	$4(1-g)$
	surfaces of class VII	0, 1	1	$\geqslant 0$
0	complex tori	0, 1, 2	4	0
	K3 surfaces	0, 1, 2	0	24
	Enriques surfaces	2	0	12
	bielliptic surfaces	2	2	0
	primary Kodaira surfaces	1	3	0
	secondary Kodaira surfaces	1	1	0
1	properly elliptic surfaces	1, 2		$\geqslant 0$
2	surfaces of general type	2	$\equiv 0 \mod 2$	> 0

4. Automorphisms

In this section we recall some important general facts about automorphisms of complex spaces.

Let U be a reduced complex space, see e.g. [Ser56] or [Mal68] for a definition and basic properties. Recall that a complex space is called *irreducible* if it cannot be represented as a union of two proper closed analytic subsets. We denote by $T_{P,U}$ the Zariski tangent space (see [Mal68, §2]) to U at a point $P \in U$. If a group $\Gamma \subset \mathrm{Aut}(X)$ has a fixed point $P \in X$, then Γ naturally acts on the local ring $\mathscr{O}_{P,X}$ and the tangent space $T_{P,X}$ so that the action on $T_{P,X}$ is linear.

The following fact is well-known (see e.g. [Car57] or [Akh95, §2.2]).

Theorem 4.1. *Let X be a Hausdorff (reduced) complex space, and $\Gamma \subset \mathrm{Aut}(X)$ be a finite group. Suppose that Γ has a fixed point P on X. Then there exist Γ-invariant neighborhoods U of P in X and V of 0 in $T_{P,X}$, and a Γ-equivariant closed embedding $U \hookrightarrow V$.*

Corollary 4.2. *Let X be an irreducible Hausdorff reduced complex space, and $\Gamma \subset \mathrm{Aut}(X)$ be a finite group. Suppose that Γ has a fixed point P on X. Then the natural representation*

$$\Gamma \longrightarrow \mathrm{GL}\left(T_{P,X}\right)$$

is faithful.

Proof. Choose an arbitrary transformation f from the kernel of the action of Γ on $T_{P,X}$. By Theorem 4.1, there exists a neighborhood U of P in X such that f restricts to the identity transformation on U.

The fixed point locus $\mathrm{Fix}(f)$ of f is a closed analytic subset in X. Indeed, since X is Hausdorff, it can be covered by f-invariant charts isomorphic to open subsets of $\mathbb{C}^N$ for some positive integer N, and in every such chart the fixed point locus is given by vanishing of certain equations. Since $\mathrm{Fix}(f)$ contains the open subset U and X is irreducible, we conclude that $\mathrm{Fix}(f) = X$. This means that the kernel of the action of Γ on $T_{P,X}$ is trivial. □

Remark 4.3. One cannot drop the assumption that X is irreducible in Corollary 4.2. Indeed, the assertion fails for the variety given by equation $xy = 0$ in $\mathbb{A}^2$ with coordinates x and y, the point P with coordinates $x = 1$ and $y = 0$, and the group $\Gamma \cong \mathbb{Z}/2\mathbb{Z}$ whose generator acts by $(x, y) \mapsto (x, -y)$. Similarly, the assertion fails for the simplest example of a non-Hausdorff reduced complex space, namely, for two copies of $\mathbb{A}^1$ glued along the common open subset $\mathbb{A}^1 \setminus \{0\}$, and the natural involution acting on this space.

Corollary 4.2 easily implies the following result.

Corollary 4.4. *Let X be an irreducible Hausdorff reduced complex space, and $\Delta \subset \mathrm{Aut}(X)$ be a subgroup. Suppose that Δ has a fixed point P on X, and let*

$$\varsigma\colon \Delta \longrightarrow \mathrm{GL}\left(T_{P,X}\right)$$

be the natural representation. Suppose that there is a subgroup $\Gamma \subset \Delta$ of finite index such that the restriction $\varsigma|_\Gamma$ is a group monomorphism. Then ς is an embedding as well.

Proof. Let $\Delta_0 \subset \Delta$ be the kernel of ς. Since $[\Delta : \Gamma] < \infty$, we see that Δ_0 is finite. Thus Δ_0 is trivial by Corollary 4.2. □

Another application of Corollary 4.2 is as follows.

Lemma 4.5. *Let X be a compact complex surface. Suppose that there is a finite non-empty $\mathrm{Aut}(X)$-invariant set $\mathcal{S}$ of curves on X such that $\mathcal{S}$ does not contain smooth elliptic curves. Then the group $\mathrm{Aut}(X)$ is Jordan.*

Proof. Let C be one of the curves from $\mathcal{S}$. Then the group $\mathrm{Aut}_C(X)$ of automorphisms of X that preserve the curve C has finite index in $\mathrm{Aut}(X)$. Since C is not a smooth elliptic curve, there is a constant $B = B(C)$ such that every finite subgroup of $\mathrm{Aut}_C(X)$ contains a subgroup of index at most B that fixes some point on C. Indeed, if C is singular, this is obvious; if C is a smooth rational curve, this follows from Theorem 2.1. If C is a smooth curve of genus $g \geqslant 2$, this follows from the fact that the index of the kernel of the action on C in the group $\mathrm{Aut}_C(X)$ is at most $|\,\mathrm{Aut}(C)|$, which does not exceed the Hurwitz bound $84(g-1)$, see for instance [Har77, Exercise IV.2.5]. Now Corollary 4.2 implies that every finite subgroup of $\mathrm{Aut}_C(X)$ contains a subgroup of index at most B that is embedded into $\mathrm{GL}_2(\mathbb{C})$. Therefore, the group $\mathrm{Aut}_C(X)$ is Jordan by Theorem 1.1, and hence the group $\mathrm{Aut}(X)$ is Jordan as well. □

Using Theorem 4.1, one can also deduce the following facts.

Corollary 4.6. *Let X be a complex manifold, and $\Gamma \subset \mathrm{Aut}(X)$ be a finite group. Then the fixed point locus of Γ is a closed submanifold.*

Proof. Let Y be the fixed point locus of Γ. It is obvious that Y is a closed subset of X.

Choose a point $P \in Y$. We know from Theorem 4.1 that there is a Γ-equivariant closed embedding $U \hookrightarrow V$ for some Γ-invariant neighborhoods U of P in X and V of 0 in $T_{P,X}$. Under this embedding the neighborhood $U_Y = Y \cap U$ of P in Y is isomorphically mapped onto the fixed point locus F of Γ in V. Since the action of Γ on $T_{P,X}$ is linear, we conclude that F is an intersection of V with some linear subspace of $T_{P,X}$. In particular, we see that F is smooth at 0, which implies that Y is smooth at P. □

Note that the fixed point locus discussed in Corollary 4.6 may consist of several connected components of different dimensions.

Corollary 4.7. *Let X be a complex manifold, and $\Gamma \subset \mathrm{Aut}(X)$ be a finite group. Suppose that Γ has a fixed point P on X and let $T \subset T_{P,X}$ be the maximal subspace on which the action of Γ is trivial. Then there exists a Γ-invariant submanifold $Y \subset X$ containing P such that $T = T_{P,Y}$ and the action of Γ on Y is trivial.*

Proof. The fixed point locus Y of Γ is a closed submanifold by Corollary 4.6. Clearly, one has $T_{P,Y} \subset T$. On the other hand, by Theorem 4.1 we have $\dim T_{P,Y} = \dim T$. □

5. Non-projective surfaces with $\chi_{\mathrm{top}}(X) \neq 0$

In this section we will (mostly) work with non-projective compact complex surfaces X with $\chi_{\mathrm{top}}(X) \neq 0$. In this case, by the Enriques–Kodaira classification (see Theorem 3.6) one has $\chi_{\mathrm{top}}(X) > 0$. The main purpose of this section is to prove the following result.

Theorem 5.1. *Let X be a non-projective compact complex surface with $\chi_{\mathrm{top}}(X) \neq 0$. Then the group $\mathrm{Aut}(X)$ is Jordan.*

Recall that an *algebraic reduction* of a compact complex surface X with $\mathrm{a}(X) = 1$ is the morphism $\pi\colon X \to B$ to a curve B obtained as follows. We start with a meromorphic map $X \dashrightarrow \mathbb{P}^1$ defined by a non-constant meromorphic function, regularize it by blow ups, and apply the Stein factorization to the regularization. One can check that the obtained morphism provides a holomorphic elliptic fibration π on X. We refer the reader to [BHPVdV04, Proposition VI.5.1] for details.

Lemma 5.2. *Let X be a non-projective compact complex surface. If X contains an irreducible curve C which is not a smooth elliptic curve, then the group $\mathrm{Aut}(X)$ is Jordan.*

Proof. We claim that the surface X contains at most a finite number of such curves. Indeed, if $\mathrm{a}(X) = 0$, then X contains at most a finite number of curves at all, see [BHPVdV04, Theorem IV.8.2]. If $\mathrm{a}(X) = 1$, then all curves on X are contained in the fibers of the algebraic reduction by Lemma 3.4. The latter fibration is elliptic, so every non-elliptic curve is contained in one of its degenerate fibers. Now the assertion follows from Lemma 4.5. □

Lemma 5.3. *Let X be a compact complex surface with $\chi_{\mathrm{top}}(X) \neq 0$. If $\mathrm{a}(X) = 1$, then the group $\mathrm{Aut}(X)$ is Jordan.*

Proof. Let $\pi\colon X \to B$ the algebraic reduction, so that B is a smooth curve and π is an elliptic fibration. Since $\chi_{\mathrm{top}}(X) \neq 0$, the fibration π has at least one fiber X_b such that $F = (X_b)_{\mathrm{red}}$ is not a smooth elliptic curve. So the group $\mathrm{Aut}(X)$ is Jordan by Lemma 5.2. □

For every compact complex surface X, we denote by $\overline{\mathrm{Aut}}(X)$ the subgroup of $\mathrm{Aut}(X)$ that consists of all elements acting trivially on $H^*(X, \mathbb{Q})$. This is a normal subgroup of $\mathrm{Aut}(X)$, and the quotient group $\mathrm{Aut}(X)/\overline{\mathrm{Aut}}(X)$ has bounded finite subgroups by Theorem 2.3. Thus Lemma 2.5(i) implies that the group $\mathrm{Aut}(X)$ is Jordan if and only if $\overline{\mathrm{Aut}}(X)$ is Jordan.

Lemma 5.4. *Let X be a compact complex surface. Suppose that every irreducible curve contained in X is a smooth elliptic curve. Let $g \in \overline{\mathrm{Aut}}(X)$ be a non-trivial element of finite order, and $\Xi_0(g)$ be the set of all isolated fixed points of g. Then*

$$|\Xi_0(g)| = \chi_{\mathrm{top}}(X).$$

Proof. The fixed locus $\Xi(g)$ of g is a disjoint union $\Xi_0(g) \sqcup \Xi_1(g)$, where $\Xi_1(g)$ is of pure dimension 1. Note that the curve $\Xi_1(g)$ is smooth by Corollary 4.6, so that every irreducible component of $\Xi_1(g)$ is its connected component.

We see that every connected component of $\Xi_1(g)$ is a smooth elliptic curve, so that $\chi_{\mathrm{top}}(\Xi_1(g)) = 0$. On the other hand, one has

$$\chi_{\mathrm{top}}(\Xi(g)) = \chi_{\mathrm{top}}(X)$$

by the topological Lefschetz fixed point formula, see [Die79, Proposition 5.3.11]. Therefore, we have

$$\chi_{\mathrm{top}}(X) = \chi_{\mathrm{top}}(\Xi(g)) = \chi_{\mathrm{top}}(\Xi_0(g)) + \chi_{\mathrm{top}}(\Xi_1(g)) = \chi_{\mathrm{top}}(\Xi_0(g)) = |\Xi_0(g)|. \quad \square$$

Lemma 5.5. *Let X be a compact complex surface with $\chi_{\mathrm{top}}(X) \neq 0$. Suppose that every irreducible curve contained in X is a smooth elliptic curve. Let $G \subset \overline{\mathrm{Aut}}(X)$ be a finite subgroup. If G contains a non-trivial normal cyclic subgroup, then G contains an abelian subgroup of index at most $12\chi_{\mathrm{top}}(X)$.*

Proof. Let $N \subset G$ be a non-trivial normal cyclic subgroup. By Lemma 5.4 the group N has exactly $\chi_{\mathrm{top}}(X) > 0$ isolated fixed points on X (and maybe also several curves that consist of fixed points). Since N is normal in G, the group G permutes these points. Thus there exists a subgroup of index at most $\chi_{\mathrm{top}}(X)$ in G acting on X with a fixed point. Now the assertion follows from Corollary 4.2 and Theorem 2.1 (cf. [PS17, Corollary 2.2.2]). $\square$

Lemma 5.6. *Let X be a compact complex surface with $\mathrm{a}(X) = 0$ and $\chi_{\mathrm{top}}(X) \neq 0$. If X contains at least one curve, then $\mathrm{Aut}(X)$ is Jordan.*

Proof. It is enough to prove that the group $\overline{\mathrm{Aut}}(X)$ is Jordan. The surface X contains at most a finite number of curves by [BHPVdV04, Theorem IV.8.2]. By Lemma 5.2 we may assume that all these curves are smooth and elliptic. Let $C_1, \ldots, C_m$ be all curves on X, and let $\mathrm{Aut}^\sharp(X) \subset \overline{\mathrm{Aut}}(X)$ be the stabilizer of C_1. Clearly, the subgroup $\mathrm{Aut}^\sharp(X)$ has index at most m in $\overline{\mathrm{Aut}}(X)$, so it is sufficient to prove that $\mathrm{Aut}^\sharp(X)$ is Jordan. For any finite subgroup $G \subset \mathrm{Aut}^\sharp(X)$ we have an exact sequence

$$1 \longrightarrow \Gamma \longrightarrow G \longrightarrow \mathrm{Aut}(C_1),$$

where Γ is the kernel of the action of G on C_1.

Let P be a point on C_1. Then $\Gamma \subset \mathrm{GL}(T_{P,X})$ by Corollary 4.2, and Γ has a trivial one-dimensional subrepresentation in $T_{P,X}$ corresponding to the tangent space T_{P,C_1}. This implies that Γ is a cyclic group. If $\Gamma = \{1\}$, then G is contained in $\mathrm{Aut}(C_1)$. Since C_1 is an elliptic curve, the group G has an abelian subgroup of index at most 6. If $\Gamma \neq \{1\}$, then G has an abelian subgroup of index at most $12\chi_{\mathrm{top}}(X)$ by Lemma 5.5. Therefore, in both cases G also has a normal abelian subgroup of bounded index. $\square$

In the following lemmas we will deal with compact complex surfaces X that contain no curves. In particular, this implies that $\mathrm{a}(X) = 0$. Furthermore, we conclude that the action of any finite subgroup G of $\mathrm{Aut}(X)$ is free in codimension one, that is, there exists a finite subset $\Xi \subset X$ such that the action of G on $X \setminus \Xi$ is free.

Lemma 5.7. *Let X be a compact complex surface with $\chi_{\mathrm{top}}(X) \neq 0$. Suppose that X contains no curves. Then the group $\overline{\mathrm{Aut}}(X)$ has no elements of even order.*

Proof. Let $g \in \overline{\mathrm{Aut}}(X)$ be an element of order 2 (such elements always exist provided that there are elements of even order).

First assume that $\varkappa(X) = -\infty$. We have $\mathrm{b}_1(X) = 1$ and $\mathrm{b}_2(X) = \chi_{\mathrm{top}}(X) > 0$ (see Theorem 3.6). Moreover, we know that $\mathrm{h}^{2,0}(X) = 0$ because $\varkappa(X) = -\infty$. Hodge relations (see e.g. [BHPVdV04, § IV.2]) give us

$$\mathrm{h}^{0,1}(X) = 1, \quad \mathrm{h}^{1,0}(X) = 0, \quad \text{and} \quad \mathrm{h}^{2,0}(X) = \mathrm{h}^{0,2}(X) = 0.$$

Thus, one has $\chi(\mathscr{O}_X) = 0$, and the canonical embedding $H^1(X, \mathscr{O}_X) \hookrightarrow H^1(X, \mathbb{C})$ is an isomorphism. In particular, the element g acts trivially on $H^1(X, \mathscr{O}_X)$. We also know that there are no curves consisting of g-fixed points. Therefore, the holomorphic Lefschetz fixed point formula (see e. g. [GH78, §3.4]) can be written as follows:

$$\sum_{P \in \mathrm{Fix}(g)} \frac{1}{\det(\mathrm{Id} - g_P)} = \sum_{q=0}^{2} (-1)^q \operatorname{tr} g^*|_{H^q(X, \mathscr{O}_X)} = \mathrm{b}_0(X) - \mathrm{h}^{0,1}(X) + \mathrm{h}^{0,2}(X) = 0,$$

where $\mathrm{Fix}(g)$ is the fixed point locus of g, and $g_P \colon T_{P,X} \to T_{P,X}$ is the differential of g at a fixed point P. Since the order of g equals 2, one has $g_p = -\mathrm{Id}$, because otherwise there exists an analytic germ of a curve in a neighborhood of P in X that consists of fixed points of g by Corollary 4.7. Hence

$$\frac{|\,\mathrm{Fix}(g)|}{4} = \sum_{P \in \mathrm{Fix}(g)} \frac{1}{\det(\mathrm{Id} - g_P)} = 0.$$

Thus, we conclude that g has no fixed points at all. The latter contradicts Lemma 5.4.

Now assume that $\varkappa(X) \geqslant 0$. Since $\mathrm{a}(X) = 0$, this implies that $\varkappa(X) = 0$ and X is a K3 surface (see Theorem 3.6). Therefore, one has $\chi_{\mathrm{top}}(X) = 24$ and $\chi(\mathscr{O}_X) = 2$. As above the holomorphic Lefschetz fixed point formula shows that g has exactly 8 fixed points. This again contradicts Lemma 5.4. ☐

Lemma 5.8. *Let X be a compact complex surface with $\chi_{\mathrm{top}}(X) \neq 0$. Suppose that X contains no curves. Let $G \subset \overline{\mathrm{Aut}}(X)$ be a finite subgroup. Suppose that G has a fixed point on X. Then G is cyclic.*

Proof. Let $P \in X$ be a fixed point of G. By Corollary 4.2 we have an embedding

$$G \subset \mathrm{GL}(T_{P,X}) \cong \mathrm{GL}_2(\mathbb{C}).$$

Since the group G does not contain elements of order 2 by Lemma 5.7, the order of G is odd. Hence G is abelian by Theorem 2.1. Recall that the action of G is free in codimension one. By Corollary 4.2, the action of G on $T_{P,X} \cong \mathbb{C}^2$ is faithful.

Suppose that G is not a cyclic group. Since G is abelian, its action on $\mathbb{C}^2$ is diagonalizable and so there exists a non-trivial element $g \in G$ such that g has an eigen-vector with eigen-value 1 in $T_{P,X}$. By Corollary 4.7 there exists an analytic germ of a curve in a neighborhood of P in X that consists of fixed points of g. The latter is impossible since the action of g is free in codimension one. The obtained contradiction shows that the group G is cyclic. ☐

Lemma 5.9. *Let X be a compact complex surface with $\chi_{\mathrm{top}}(X) \neq 0$. Suppose that X contains no curves. Let $G \subset \overline{\mathrm{Aut}}(X)$ be a finite cyclic subgroup, and $g \in G$ be a non-trivial element. Then g has the same set of fixed points as G.*

Proof. For an arbitrary element $h \in G$ denote by $\mathrm{Fix}(h)$ the fixed locus of h. By Lemma 5.4 one has

$$|\mathrm{Fix}(h)| = \chi_{\mathrm{top}}(X)$$

for every non-trivial element h.

Let f be a generator of G. Then for some positive integer n one has $f^n = g$, so that

$$\mathrm{Fix}(f) \subset \mathrm{Fix}(g).$$

Therefore, one has $\mathrm{Fix}(f) = \mathrm{Fix}(g)$, which means that every non-trivial element of G has one and the same set of fixed points. □

Lemma 5.10. *Let X be a compact complex surface with $\chi_{\mathrm{top}}(X) \neq 0$. Suppose that X contains no curves. Then every finite subgroup $G \subset \overline{\mathrm{Aut}}(X)$ is a union $G = \bigcup_{i=1}^m G_i$ of cyclic subgroups such that $G_i \cap G_j = \{1\}$ for $i \neq j$.*

Proof. Choose some representation of G as a union $G = \bigcup_{i=1}^m G_i$, where G_i are cyclic groups that possibly have non-trivial intersections. Let G_1 and G_2 be subgroups such that $G_1 \cap G_2 \neq \{1\}$. Let $g \in G_1 \cap G_2$ be a non-trivial element. By Lemma 5.4 it has a fixed point, say x. By Lemma 5.8 the stabilizer G_x is a cyclic group. By Lemma 5.9 the groups G_1 and G_2 fix the point x, so that $G_1, G_2 \subset G_x$. Replacing G_1 and G_2 by G_x, we proceed to construct the required system of subgroups by induction. □

Lemma 5.11. *Let X be a compact complex surface with $\chi_{\mathrm{top}}(X) \neq 0$. Suppose that X contains no curves. Then there exists a constant $J = J(X)$ such that any finite subgroup $G \subset \mathrm{Aut}(X)$ contains a normal cyclic subgroup of index at most J. In particular, the group $\mathrm{Aut}(X)$ is Jordan.*

Proof. It is enough to prove that any finite subgroup of $\overline{\mathrm{Aut}}(X)$ contains a normal cyclic subgroup of index at most J. Let $G \subset \overline{\mathrm{Aut}}(X)$ be a finite subgroup. Let $\Xi \subset X$ be the set of points with non-trivial stabilizers in G.

By Lemma 5.10 the group G is a union $G = \bigcup_{i=1}^m G_i$ of cyclic subgroups such that $G_i \cap G_j = \{1\}$ for $i \neq j$. We claim that the stabilizer of any point $x \in \Xi$ is one of the groups $G_1, \ldots, G_m$. Indeed, choose a point $x \in \Xi$, and let G_x be its stabilizer. Then G_x is a cyclic group by Lemma 5.8. Let g_x be a generator of G_x, and let $1 \leqslant r \leqslant m$ be the index such that the group G_r contains g_x. Then $G_x \subset G_r$. Now Lemma 5.9 implies that $G_x = G_r$.

By Lemma 5.4 every element of G has exactly $\chi_{\mathrm{top}}(X)$ fixed points. The set Ξ is a disjoint union of orbits of the group G. Therefore, for some positive integers k_i one has

$$|\Xi| = m\chi_{\mathrm{top}}(X) = \sum_{i=1}^m k_i [G : G_i].$$

Hence, for some i we have $[G : G_i] \leqslant \chi_{\mathrm{top}}(X)$, i.e. G contains a cyclic subgroup G_i of index at most $\chi_{\mathrm{top}}(X)$. This implies that G contains a normal cyclic subgroup of bounded index. □

Now we are ready to prove Theorem 5.1.

Proof of Theorem 5.1. If $\mathrm{a}(X) = 1$, then the assertion follows from Lemma 5.3. If $\mathrm{a}(X) = 0$ and X contains at least one curve, then the assertion follows from Lemma 5.6. Finally, if X contains no curves, then the assertion follows from Lemma 5.11. □

An alternative way to prove Theorem 5.1 is provided by the following more general result due to I. Mundet i Riera. Our proof of Theorem 5.1 is a simplified version of the proof of this result given in [Mun16].

Theorem 5.12 ([Mun16, Theorem 1.1])**.** *Let X be a compact, orientable, connected four-dimensional smooth manifold with $\chi_{\mathrm{top}}(X) \neq 0$. Then the group of diffeomorphisms of X is Jordan. In particular, if X is a compact complex surface with non-vanishing topological Euler characteristic, then the group* $\mathrm{Aut}(X)$ *is Jordan.*

Note however that our proof of Theorem 5.1 implies that for a compact complex surface X with $\chi_{\mathrm{top}}(X) \neq 0$ and containing no curves, there exists a constant J such that for every finite subgroup $G \subset \mathrm{Aut}(X)$ there exists a normal *cyclic* subgroup of index at most J (see Lemma 5.11), while the results of [Mun16] provide only a normal abelian subgroup of bounded index generated by at most 2 elements.

6. Hopf surfaces

In this section we study automorphism groups of Hopf surfaces, and make some general conclusions about automorphisms of surfaces of class VII.

Recall that a *Hopf surface* X is a compact complex surface whose universal cover is (analytically) isomorphic to $\mathbb{C}^2 \setminus \{0\}$. Thus $X \cong (\mathbb{C}^2 \setminus \{0\})/\Gamma$, where $\Gamma \cong \pi_1(X)$ is a group acting freely on $\mathbb{C}^2 \setminus \{0\}$. A Hopf surface X is said to be *primary* if $\pi_1(X) \cong \mathbb{Z}$. One can show that a primary Hopf surface is isomorphic to a quotient

$$X(\alpha, \beta, \lambda, n) = \left(\mathbb{C}^2 \setminus \{0\}\right)/\Lambda,$$

where $\Lambda \cong \mathbb{Z}$ is a group generated by the transformation

$$(x, y) \mapsto (\alpha x + \lambda y^n, \beta y). \tag{6.1}$$

Here n is a positive integer, and α and β are complex numbers satisfying

$$0 < |\alpha| \leqslant |\beta| < 1;$$

moreover, one has $\lambda = 0$, or $\alpha = \beta^n$. A *secondary* Hopf surface is a quotient of a primary Hopf surface by a free action of a finite group. Every Hopf surface is either primary or secondary. We refer the reader to [Kod66, §10] for details.

The following result shows the significance of Hopf and Inoue surfaces (see [Ino74]) from the point of view of Enriques–Kodaira classification.

Theorem 6.2 (see [Bog77] and [Tel94])**.** *Every minimal surface of class VII with vanishing second Betti number is either a Hopf surface or an Inoue surface.*

Automorphisms of Hopf surfaces were studied in detail in [Kat75], [Kat89], [Nam74], [Weh81], and [MN00]. Our approach does not use these results.

We will need the following easy observation.

Lemma 6.3. *Let*

$$M = \begin{pmatrix} \alpha & \lambda \\ 0 & \beta \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$$

be an upper triangular matrix, and $Z \subset \mathrm{GL}_2(\mathbb{C})$ be the centralizer of M. The following assertions hold.

(i) *If $\alpha = \beta$ and $\lambda = 0$, then $Z = \mathrm{GL}_2(\mathbb{C})$.*

(ii) *If* $\alpha \neq \beta$ *and* $\lambda = 0$*, then* $Z \cong (\mathbb{C}^*)^2$.
(iii) *If* $\alpha = \beta$ *and* $\lambda \neq 0$*, then* $Z \cong \mathbb{C}^* \times \mathbb{C}^+$.

Proof. Simple linear algebra. □

Lemma 6.4. *Let* X *be a Hopf surface. Then the group* $\mathrm{Aut}(X)$ *is Jordan.*

Proof. The non-compact surface $\mathbb{C}^2 \setminus \{0\}$ is the universal cover of X. Moreover, X is obtained from $\mathbb{C}^2 \setminus \{0\}$ as a quotient by a free action of some group Γ that contains a subgroup $\Lambda \cong \mathbb{Z}$ of finite index such that a generator of Λ acts as in (6.1); if X is a primary Hopf surface, then $\Gamma = \Lambda$, and otherwise Λ is identified with the fundamental group of (some) primary Hopf surface covering X. By Lemma 2.10 we can replace Λ by its subgroup $\Lambda_0 \cong \mathbb{Z}$ such that Λ_0 is characteristic in Γ. Since the generator of Λ_0 is a power of a generator of Λ, it also acts on $\mathbb{C}^2 \setminus \{0\}$ by a transformation of type (6.1) (but possibly with different parameters α, β, and λ). Therefore, without loss of generality we may assume that Λ itself was a characteristic subgroup of Γ.

There is an exact sequence of groups

$$1 \longrightarrow \Gamma \longrightarrow \widetilde{\mathrm{Aut}}(X) \longrightarrow \mathrm{Aut}(X) \longrightarrow 1,$$

where $\widetilde{\mathrm{Aut}}(X)$ acts by automorphisms of $\mathbb{C}^2 \setminus \{0\}$. By Hartogs theorem the action of $\widetilde{\mathrm{Aut}}(X)$ extends to $\mathbb{C}^2$ so that $\widetilde{\mathrm{Aut}}(X)$ fixes the origin $0 \in \mathbb{C}^2$. The image of the generator of Λ is mapped by the natural homomorphism

$$\varsigma \colon \widetilde{\mathrm{Aut}}(X) \longrightarrow \mathrm{GL}\left(T_{0,\mathbb{C}^2}\right) \cong \mathrm{GL}_2(\mathbb{C})$$

to the matrix

$$M = \begin{pmatrix} \alpha & \lambda\delta_1^n \\ 0 & \beta \end{pmatrix}$$

where δ is the Kronecker symbol.

Let $G \subset \mathrm{Aut}(X)$ be a finite subgroup, and $\tilde{G}$ be its preimage in $\widetilde{\mathrm{Aut}}(X)$. Thus, one has $G \cong \tilde{G}/\Gamma$. By Corollary 4.4 we know that $\varsigma|_{\tilde{G}}$ is an embedding. Let Ω be the normalizer of $\varsigma(\Lambda)$ in $\mathrm{GL}_2(\mathbb{C})$. By construction $\varsigma(\tilde{G})$ is contained in the normalizer of $\varsigma(\Gamma)$ in $\mathrm{GL}_2(\mathbb{C})$, which in turn is contained in Ω because Λ is a characteristic subgroup of Γ. We see that every finite subgroup of $\mathrm{Aut}(X)$ is contained in the group $\Omega/\varsigma(\Gamma)$. On the other hand, $\Omega/\varsigma(\Gamma)$ is a quotient of $\Omega/\varsigma(\Gamma)$ is a quotient of $\Omega/\varsigma(\Lambda)$ by a finite subgroup isomorphic to $\varsigma(\Gamma)/\varsigma(\Lambda)$. Thus, by Lemma 2.5(ii) it is sufficient to show that the group $\Omega/\varsigma(\Lambda)$ is strongly Jordan.

Since $\varsigma(\Lambda) \cong \mathbb{Z}$, the group Ω has a (normal) subgroup Ω' of index at most 2 that coincides with the centralizer of the matrix M. It remains to check that the group $\Omega'/\varsigma(\Lambda)$ is strongly Jordan. If $\lambda = 0$ and $\alpha = \beta$, then this follows from Lemmas 6.3(i) and 2.9. If either $\lambda = 0$ and $\alpha \neq \beta$, or $\lambda \neq 0$ and $n \geqslant 2$, then this follows from Lemmas 6.3(ii) and 2.6. If $\lambda \neq 0$ and $n = 1$, then this follows from Lemmas 6.3(iii) and 2.6. □

Remark 6.5. Suppose that for a primary Hopf surface $X \cong X(\alpha, \beta, \lambda, n)$ one has $\lambda = 0$ and $\alpha^k = \beta^l$ for some positive integers k and l. Then there is an elliptic fibration

$$X \cong \left(\mathbb{C}^2 \setminus \{0\}\right)/\Lambda \to \mathbb{P}^1 \cong \left(\mathbb{C}^2 \setminus \{0\}\right)/\mathbb{C}^*,$$

and one has an exact sequence of groups

$$1 \longrightarrow E \longrightarrow \mathrm{Aut}(X) \longrightarrow \mathrm{PGL}_2(\mathbb{C}),$$

where E is the group of points of the elliptic curve $\mathbb{C}^*/\mathbb{Z}$.

Finally, we put together the information about automorphisms of surfaces of class VII.

Corollary 6.6. *Let X be a minimal surface of class VII. Then the group* $\mathrm{Aut}(X)$ *is Jordan.*

Proof. If the second Betti number $\mathrm{b}_2(X)$ vanishes, then X is either a Hopf surface or an Inoue surface by Theorem 6.2. Thus the assertion follows from Lemma 6.4 and Theorem 1.5 in this case. If $\mathrm{b}_2(X)$ does not vanish, then the assertion follows from Theorem 5.1. □

Remark 6.7. Except for Hopf surfaces, there are some other types of minimal compact complex surfaces of class VII whose automorphism groups have been studied in detail. For instance, this is the case for so called hyperbolic and parabolic Inoue surfaces, see [Pin84] and [Fuj09], respectively. Note that surfaces of both of these types have positive second Betti numbers (and thus they are not to be confused with Inoue surfaces introduced in [Ino74]). Also, automorphism groups of Enoki surfaces are known due to [DK98, Theorem 3.1] and [DK98, Proposition 3.2(2)].

7. Kodaira surfaces

In this section we study automorphism groups of Kodaira surfaces.

Recall (see e.g. [BHPVdV04, § V.5]) that a Kodaira surface is a compact complex surface of Kodaira dimension 0 with odd first Betti number. There are two types of Kodaira surfaces: primary and secondary ones. A primary Kodaira surface is a compact complex surface with the following invariants [Kod64, Theorem 19]:

$$\mathscr{K}_X \sim 0,\ \mathrm{a}(X)=1,\ \mathrm{b}_1(X)=3,\ \mathrm{b}_2(X)=4,\ \chi_{\mathrm{top}}(X)=0,\ \mathrm{h}^{0,1}(X)=2,\ \mathrm{h}^{0,2}(X)=1.$$

A secondary Kodaira surface is a quotient of a primary Kodaira surface by a free action of a finite cyclic group. The invariants of a secondary Kodaira surface are [Kod66, §9]:

$$\mathrm{a}(X)=1,\ \ \mathrm{b}_1(X)=1,\ \ \mathrm{b}_2(X)=0,\ \ \chi_{\mathrm{top}}(X)=0,\ \ \mathrm{h}^{0,1}(X)=1,\ \ \mathrm{h}^{0,2}(X)=0.$$

Due to Theorem 1.5, we know that automorphism groups of primary Kodaira surfaces are Jordan. Thus, we are left with the case of secondary Kodaira surfaces.

Lemma 7.1. *Let X be a secondary Kodaira surface. Then the group* $\mathrm{Aut}(X)$ *is Jordan.*

Proof. Since $\mathrm{a}(X)=1$, the algebraic reduction is an $\mathrm{Aut}(X)$-equivariant elliptic fibration $\pi\colon X\to B$. Thus there is an exact sequence of groups

$$1\longrightarrow \mathrm{Aut}(X)_\pi \longrightarrow \mathrm{Aut}(X)\longrightarrow \Gamma\longrightarrow 1,$$

where the action of $\mathrm{Aut}(X)_\pi$ is fiberwise with respect to π, and Γ is a subgroup of $\mathrm{Aut}(B)$.

We claim that the group $\mathrm{Aut}(X)_\pi$ is Jordan. Indeed, if H is a finite subgroup of $\mathrm{Aut}(X)_\pi$, then H acts faithfully on a typical fiber of π, which is a smooth elliptic curve. This implies that H has a normal abelian subgroup of index at most 6.

Since

$$\mathrm{h}^{1,0}(X)=\mathrm{b}_1(X)-\mathrm{h}^{0,1}(X)=0,$$

the base curve B is rational. Furthermore, one has

$$\chi(\mathscr{O}_X)=\mathrm{h}^{0,0}(X)-\mathrm{h}^{0,1}(X)+\mathrm{h}^{0,2}(X)=1-1+0=0.$$

By the canonical bundle formula (see e.g. [BHPVdV04, Theorem V.12.1]) we have

$$\mathscr{K}_X \sim \pi^*\left(\mathscr{K}_B \otimes \mathcal{L}\right) \otimes \mathscr{O}_X\left(\sum (m_i - 1)F_i\right),$$

where F_i are all (reduced) multiple fibers of π, the fiber F_i has multiplicity m_i, and $\mathcal{L}$ is a line bundle of degree $\chi(\mathscr{O}_X) = 0$. Since X has Kodaira dimension 0, we see that

$$\sum (1 - 1/m_i) = 2.$$

In particular, the number of multiple fibers equals either 3 or 4. This means that Γ has a finite non-empty invariant subset in $B \cong \mathbb{P}^1$ that consists of 3 or 4 points. Hence Γ is finite, so that the assertion follows by Lemma 2.5(i). □

An alternative way to prove the Jordan property for the automorphism group of a secondary Kodaira surface is to use the fact that its canonical cover is a primary Kodaira surface together with Lemma 2.5(ii) and Theorem 2.11.

8. Non-negative Kodaira dimension

In this section we study automorphism groups of surfaces of non-negative Kodaira dimension, and prove Theorems 1.6 and 1.7.

The case of Kodaira dimension 2 is well known.

Theorem 8.1. *Let X be a (minimal) surface of general type. Then the group* $\mathrm{Aut}(X)$ *is finite.*

Proof. The surface X is projective, see Theorem 3.6. Thus the group $\mathrm{Aut}(X)$ is finite, see for instance [HMX13] where a much more general result is established for varieties of general type of arbitrary dimension. □

Now we consider the case of Kodaira dimension 1.

Lemma 8.2 (cf. [PS18b, Lemma 3.3])**.** *Let X be a minimal surface of Kodaira dimension* 1. *Then the group* $\mathrm{Aut}(X)$ *is Jordan.*

Proof. Let $\phi\colon X \to B$ be the pluricanonical fibration, where B is some (smooth) curve. It is equivariant with respect to the action of $\mathrm{Aut}(X)$. Thus there is an exact sequence of groups

$$1 \longrightarrow \mathrm{Aut}(X)_\phi \longrightarrow \mathrm{Aut}(X) \longrightarrow \Gamma \longrightarrow 1,$$

where the action of $\mathrm{Aut}(X)_\phi$ is fiberwise with respect to ϕ, and Γ is a subgroup of $\mathrm{Aut}(B)$. As in the proof of Lemma 7.1, we see that the group $\mathrm{Aut}(X)_\phi$ is Jordan. Hence by Lemma 2.5(i) it is enough to check that Γ has bounded finite subgroups. In particular, this holds if the genus of B is at least 2, since the group $\mathrm{Aut}(B)$ is finite in this case. Thus we will assume that the genus of B is at most 1.

Suppose that ϕ has a fiber F such that F_{red} is not a smooth elliptic curve. Then every irreducible component of F is a rational curve, see e.g. [BHPVdV04, § V.7]. Hence Lemma 4.5 applied to the set of irreducible components of fibers of the morphism ϕ shows that the group $\mathrm{Aut}(X)$ is Jordan.

Therefore, we will assume that all (set-theoretic) fibers of ϕ are smooth elliptic curves. Then the topological Euler characteristic $\chi_{\mathrm{top}}(X)$ equals 0. By the Noether's formula one has

$$\chi(\mathscr{O}_X) = \frac{1}{12}\left(\mathrm{c}_1(X)^2 + \chi_{\mathrm{top}}(X)\right) = 0.$$

By the canonical bundle formula we have

$$\mathscr{K}_X \sim \phi^* \left(\mathscr{K}_B \otimes \mathcal{L}\right) \otimes \mathscr{O}_X \left(\sum (m_i - 1) F_i\right),$$

where F_i are all (reduced) multiple fibers of ϕ, the fiber F_i has multiplicity m_i, and $\mathcal{L}$ is a line bundle of degree $\chi(\mathscr{O}_X) = 0$. Since X has Kodaira dimension 1, we see that

$$2g(B) - 2 + \sum (1 - 1/m_i) = \deg\left(\mathscr{K}_B \otimes \mathcal{L}\right) + \sum (1 - 1/m_i) > 0. \tag{8.3}$$

Suppose that B is an elliptic curve, so that $g(B) = 1$. Then (8.3) implies that ϕ has at least one multiple fiber. This means that Γ has a finite non-empty invariant subset in B, so that Γ is finite.

Now suppose that B is a rational curve, so that $g(B) = 0$. Then (8.3) implies that ϕ has at least three multiple fibers, cf. the proof of Lemma 7.1. This means that Γ has a finite non-empty invariant subset in B that consists of at least three points. Therefore, Γ is finite in this case as well. $\square$

Finally, we consider the case of Kodaira dimension 0. The following result is well known.

Theorem 8.4. *Let $X = \mathbb{C}^n/\Lambda$ be a complex torus. Then*

$$\mathrm{Aut}(X) \cong (\mathbb{C}^n/\Lambda) \rtimes \Gamma, \tag{8.5}$$

where Γ is isomorphic to the stabilizer of the lattice Λ in $\mathrm{GL}_n(\mathbb{C})$.

Proof. The proof is standard, but we include it for the reader's convenience. Let Γ be the stabilizer of the point $0 \in X$. Then the decomposition (8.5) holds, and it remains to prove that Γ is isomorphic to the stabilizer of the lattice Λ in $\mathrm{GL}_n(\mathbb{C})$.

Since $\mathbb{C}^n$ is the universal cover of X, there is an embedding $\Gamma \hookrightarrow \mathrm{Aut}(\mathbb{C}^n)$, and there is a point in Λ that is invariant with respect to Γ. We may assume that this is the origin in $\mathbb{C}^n$.

Let g be an element of Γ. One has $g(\Lambda) = \Lambda$. We claim that $g \in \mathrm{GL}_n(\mathbb{C})$. Indeed, let λ be an arbitrary element of the lattice Λ. Consider a holomorphic map

$$f_\lambda \colon \mathbb{C}^n \to \mathbb{C}^n, \quad f_\lambda(z) = g(z + \lambda) - g(z).$$

One has $f_\lambda(z) \in \Lambda$ for every $z \in \mathbb{C}^n$. This means that $f_\lambda(z)$ is constant, so that all partial derivatives of f_λ vanish. Hence the partial derivatives of $g(z)$ are periodic with respect to the lattice Λ. This in turn means that these partial derivatives are bounded and thus constant, so that $g(z)$ is a linear function in z. $\square$

Remark 8.6. A complete classification of finite groups that can act by automorphisms of a two-dimensional complex torus (preserving a point therein) was obtained in [Fuj88].

Theorem 8.4 immediately implies the following result (which is already known in a more general setup, see [Mun10, Theorem 1.4], [Ye17, Corollary 1.7]).

Corollary 8.7. *Let X be a complex torus. Then the group $\mathrm{Aut}(X)$ is Jordan.*

Proof. By Lemma 2.5(i) it is enough to check that in the notation of Theorem 8.4 the group Γ has bounded finite subgroups. Since Γ is a subgroup in the automorphism group of Λ, the latter follows from Theorem 2.3. $\square$

Lemma 8.8. *Let X be either a $K3$ surface, or an Enriques surface. Then the group $\mathrm{Aut}(X)$ has bounded finite subgroups.*

Proof. Suppose that X is a $K3$ surface. If X is projective, then the assertion follows from [PS14, Theorem 1.8(i)]. If X is non-projective, then the assertion follows from a stronger result of [Ogu08, Theorem 1.5].

Now suppose that X is an Enriques surface. Then it is projective (see Theorem 3.6), so that the assertion again follows from [PS14, Theorem 1.8(i)]. □

Note that in the assumptions of Lemma 8.8, the (weaker) assertion that the group $\mathrm{Aut}(X)$ is Jordan follows directly from Theorem 5.1 or Theorem 5.12.

Lemma 8.9. *Let X be a bielliptic surface. Then the group $\mathrm{Aut}(X)$ is Jordan.*

Proof. The surface X is projective (see Theorem 3.6). Thus the assertion follows from Theorem 1.4 (or [BZ15], or [MZ18], or [PS14, Theorem 1.8(ii)]). □

For a more precise description of automorphism groups of bielliptic surfaces, we refer the reader to [BM90].

Corollary 8.10. *Let X be a minimal surface of Kodaira dimension 0. Then the group $\mathrm{Aut}(X)$ is Jordan.*

Proof. We know from Theorem 3.6 that X is either a complex torus, or a $K3$ surface, or an Enriques surface, or a bielliptic surface, or a Kodaira surface.

If X is a complex torus, then the assertion holds by Corollary 8.7. If X is a $K3$ surface or an Enriques surface, then the assertion is implied by Lemma 8.8. If X is a bielliptic surface, then the assertion holds by Lemma 8.9. If X is a Kodaira surface, then the assertion holds by Theorem 1.5 and Lemma 7.1. □

Proposition 8.11. *Let X be a minimal surface. Then the group $\mathrm{Aut}(X)$ is Jordan.*

Proof. We check the possibilities for the birational type of X listed in Theorem 3.6 case by case. If X is rational or ruled, then X is projective (see Theorem 3.6), and thus the group $\mathrm{Aut}(X)$ is Jordan by [Zar15, Corollary 1.6] or [MZ18]. If X is a surface of class VII, then the group $\mathrm{Aut}(X)$ is Jordan by Corollary 6.6. If the Kodaira dimension of X is 0, then the group $\mathrm{Aut}(X)$ is Jordan by Corollary 8.10. If the Kodaira dimension of X is 1, then the group $\mathrm{Aut}(X)$ is Jordan by Lemma 8.2. Finally, if the Kodaira dimension of X is 2, then the group $\mathrm{Aut}(X)$ is finite by Theorem 8.1. □

Now we are ready to prove Theorem 1.6.

Proof of Theorem 1.6. If X is rational or ruled, then X is projective (see Theorem 3.6), and thus the group $\mathrm{Aut}(X)$ is Jordan by [BZ15] or [MZ18]. Otherwise Proposition 3.5 implies that there is a unique minimal surface X' birational to X, and

$$\mathrm{Aut}(X) \subset \mathrm{Bir}(X) \cong \mathrm{Bir}(X') = \mathrm{Aut}(X').$$

Now the assertion follows from Proposition 8.11. □

Finally, we are going to prove Theorem 1.7.

Proof of Theorem 1.7. There always exists a minimal surface birational to a given one, so we may assume that X is a minimal surface itself.

If X is rational, then the group $\mathrm{Bir}(X)$ is Jordan by Theorem 1.3. Also, by [PS16, Theorem 4.2] and Corollary 4.2 every finite subgroup of $\mathrm{Bir}(X)$ contains a subgroup of bounded index that can be embedded into $\mathrm{GL}_2(\mathbb{C})$. Hence every finite subgroup of $\mathrm{Bir}(X)$ can be generated by a bounded number of elements.

If X is ruled and non-rational, let $\phi\colon X \to B$ be the $\mathbb{P}^1$-fibration over a (smooth) curve. Since X is projective (see Theorem 3.6), the group $\mathrm{Bir}(X)$ is Jordan if and only if B is not an elliptic curve by Theorem 1.4. Moreover, we always have an exact sequence of groups

$$1 \to \mathrm{Bir}(X)_\phi \to \mathrm{Bir}(X) \to \mathrm{Aut}(B),$$

where the action of the subgroup $\mathrm{Bir}(X)_\phi$ is fiberwise with respect to ϕ. In particular, the group $\mathrm{Bir}(X)_\phi$ acts faithfully on the schematic general fiber of ϕ, which is a conic over the field $\mathbb{C}(B)$. This implies that finite subgroups of $\mathrm{Bir}(X)_\phi$ are generated by a bounded number of elements. Also, finite subgroups of $\mathrm{Aut}(B)$ are generated by a bounded number of elements. Therefore, the same holds for finite subgroups of $\mathrm{Bir}(X)$ as well.

In the remaining cases we have $\mathrm{Bir}(X) = \mathrm{Aut}(X)$ by Proposition 3.5, so the assertion follows from Proposition 8.11 and Theorem 2.11. □

References

[Akh95] Dmitri N. Akhiezer. *Lie group actions in complex analysis*. Aspects of Mathematics, E27. Friedr. Vieweg & Sohn, Braunschweig, 1995.

[BHPVdV04] Wolf P. Barth, Klaus Hulek, Chris A. M. Peters, and Antonius Van de Ven. *Compact complex surfaces*, volume 4 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 2004.

[Bir16] Caucher Birkar. Singularities of linear systems and boundedness of Fano varieties. *ArXiv e-print*, 1609.05543, 2016.

[BM90] Curtis Bennett and Rick Miranda. The automorphism groups of the hyperelliptic surfaces. *Rocky Mountain J. Math.*, 20(1):31–37, 1990.

[Bog77] F.A. Bogomolov. Classification of surfaces of class VII_0 with $b_2 = 0$. *Math. USSR, Izv.*, 10:255–269, 1977.

[BZ15] Tatiana Bandman and Yuri G. Zarhin. Jordan groups and algebraic surfaces. *Transform. Groups*, 20(2):327–334, 2015.

[BZ17] Tatiana Bandman and Yuri G. Zarhin. Jordan groups, conic bundles and abelian varieties. *Algebraic Geometry*, 4(2):229–246, 2017.

[BZ18] Tatiana Bandman and Yuri G. Zarhin. Jordan properties of automorphism groups of certain open algebraic varieties. *Transform. Groups*, 2018.

[Car57] Henri Cartan. Quotient d'un espace analytique par un groupe d'automorphismes. In *Algebraic geometry and topology*, pages 90–102. Princeton University Press, Princeton, N. J., 1957. A symposium in honor of S. Lefschetz,.

[CPS14] Balázs Csikós, László Pyber, and Endre Szabó. Diffeomorphism groups of compact 4-manifolds are not always Jordan. *ArXiv e-print*, 1411.7524, 2014.

[CR62] Charles W. Curtis and Irving Reiner. *Representation theory of finite groups and associative algebras*. Pure and Applied Mathematics, Vol. XI. Interscience Publishers, a division of John Wiley & Sons, New York-London, 1962.

[Die79] Tammo tom Dieck. *Transformation groups and representation theory*, volume 766. Springer, Cham, 1979.

[DK98] G. Dloussky and F. Kohler. Classification of singular germs of mappings and deformations of compact surfaces of class VII_0. *Ann. Polon. Math.*, 70:49–83, 1998. Complex analysis and applications (Warsaw, 1997).

[Fuj88] Akira Fujiki. Finite automorphism groups of complex tori of dimension two. *Publ. Res. Inst. Math. Sci.*, 24(1):1–97, 1988.

[Fuj09] A. Fujiki. Automorphisms of parabolic Inoue surfaces. *ArXiv e-print*, 0903.5374, 2009.

[GH78] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. Wiley-Interscience [John Wiley & Sons], New York, 1978. Pure and Applied Mathematics.

[GZ13] Alessandra Guazzi and Bruno Zimmermann. On finite simple groups acting on homology spheres. *Monatsh. Math.*, 169(3-4):371–381, 2013.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[HMX13] Christopher D. Hacon, James McKernan, and Chenyang Xu. On the birational automorphisms of varieties of general type. *Ann. Math. (2)*, 177(3):1077–1111, 2013.

[Ino74] Masahisa Inoue. On surfaces of Class VII_0. *Invent. Math.*, 24:269–310, 1974.

[Kat75] Masahide Kato. Topology of Hopf surfaces. *J. Math. Soc. Japan*, 27:222–238, 1975.

[Kat89] Masahide Kato. Erratum to "Topology of Hopf surfaces". *J. Math. Soc. Japan*, 41(1):173–174, 1989.

[Kim18] Jin Hong Kim. Jordan property and automorphism groups of normal compact Kähler varieties. *Commun. Contemp. Math.*, 20(3):1750024, 9, 2018.

[Kod64] K. Kodaira. On the structure of compact complex analytic surfaces. I. *Amer. J. Math.*, 86:751–798, 1964.

[Kod66] K. Kodaira. On the structure of compact complex analytic surfaces. II. *Amer. J. Math.*, 88:682–721, 1966.

[Mal68] Bernard Malgrange. Analytic spaces. *Enseign. Math. (2)*, 14:1–28, 1968.

[MN00] Takao Matumoto and Noriaki Nakagawa. Explicit description of Hopf surfaces and their automorphism groups. *Osaka J. Math.*, 37(2):417–424, 2000.

[MT15] Ignasi Mundet i Riera and Alexandre Turull. Boosting an analogue of Jordan's theorem for finite groups. *Adv. Math.*, 272:820–836, 2015.

[Mun10] Ignasi Mundet i Riera. Jordan's theorem for the diffeomorphism group of some manifolds. *Proc. Amer. Math. Soc.*, 138(6):2253–2262, 2010.

[Mun13] Ignasi Mundet i Riera. Finite group actions on manifolds without odd cohomology. *ArXiv e-print*, 1310.6565, 2013.

[Mun14] Ignasi Mundet i Riera. Finite group actions on homology spheres and manifolds with nonzero Euler characteristic. *ArXiv e-print*, 1403.0383, 2014.

[Mun16] Ignasi Mundet i Riera. Finite group actions on 4-manifolds with nonzero Euler characteristic. *Math. Z.*, 282(1-2):25–42, 2016.

[Mun17a] Ignasi Mundet i Riera. Finite groups acting symplectically on $T^2 \times S^2$. *Trans. Am. Math. Soc.*, 369(6):4457–4483, 2017.

[Mun17b] Ignasi Mundet i Riera. Non Jordan groups of diffeomorphisms and actions of compact Lie groups on manifolds. *Transform. Groups*, 22(2):487–501, 2017.

[Mun18] Ignasi Mundet i Riera. Finite subgroups of Ham and Symp. *Math. Ann.*, 370(1-2):331–380, 2018.

[MZ15] Mattia Mecchia and Bruno P. Zimmermann. On finite groups of isometries of handlebodies in arbitrary dimensions and finite extensions of Schottky groups. *Fundam. Math.*, 230(3):237–249, 2015.

[MZ18] Sheng Meng and De-Qi Zhang. Jordan property for non-linear algebraic groups and projective varieties. *Amer. J. Math.*, 140(4):1133–1145, 2018.

[Nam74] Makoto Namba. Automorphism groups of Hopf surfaces. *Tôhoku Math. J. (2)*, 26:133–157, 1974.

[Ogu08] Keiji Oguiso. Bimeromorphic automorphism groups of non-projective hyperkähler manifolds – a note inspired by C.T. McMullen. *J. Differential Geom.*, 78(1):163–191, 2008.

[Pin84] H. C. Pinkham. Automorphisms of cusps and Inoue–Hirzebruch surfaces. *Compositio Math.*, 52(3):299–313, 1984.

[Pop11] Vladimir L. Popov. On the Makar-Limanov, Derksen invariants, and finite automorphism groups of algebraic varieties. In *Peter Russell's Festschrift, Proceedings of the conference on Affine Algebraic Geometry held in Professor Russell's honour, 1–5 June 2009, McGill Univ., Montreal.*, volume 54 of *Centre de Recherches Mathématiques CRM Proc. and Lect. Notes*, pages 289–311, 2011.

[Pop14] Vladimir L. Popov. Jordan groups and automorphism groups of algebraic varieties. In I. Cheltsov, C. Ciliberto, H. Flenner, J. McKernan, Y. G. Prokhorov, and M. Zaidenberg, editors, *Automorphisms in birational and affine geometry. Levico Terme, Italy, October 2012*, volume 79 of *Springer Proceedings in Mathematics & Statistics*, pages 185–213. 2014.

[Pop16] Vladimir L. Popov. Finite subgroups of diffeomorphism groups. *Proc. Steklov Inst. Math.*, 289(1):221–226, 2016.

[Pop18] V. L. Popov. The Jordan property for Lie groups and automorphism groups of complex spaces. *Math. Notes*, 103(5-6):811–819, 2018.

[PS14] Yu. Prokhorov and C. Shramov. Jordan property for groups of birational selfmaps. *Compositio Math.*, 150(12):2054–2072, 2014.

[PS16] Yu. Prokhorov and C. Shramov. Jordan property for Cremona groups. *Amer. J. Math.*, 138(2):403–418, 2016.

[PS17] Yu. Prokhorov and C. Shramov. Jordan constant for Cremona group of rank 3. *Moscow Math. J.*, 17(3):457–509, 2017.

[PS18a] Yu. Prokhorov and C. Shramov. Automorphism groups of Inoue and Kodaira surfaces. *ArXiv e-print*, 1812.02400, 2018.

[PS18b] Yu. Prokhorov and C. Shramov. Finite groups of birational selfmaps of threefolds. *Math. Res. Lett.*, 25(3):957–972, 2018.

[PS18c] Yu. Prokhorov and C. Shramov. p-subgroups in the space Cremona group. *Math. Nachrichten*, 291(8–9):1374–1389, 2018.

[Ser56] Jean-Pierre Serre. Géométrie algébrique et géométrie analytique. *Ann. Inst. Fourier*, 6:1–42, 1956.

[Ser07] Jean-Pierre Serre. Bounds for the orders of the finite subgroups of $G(k)$. In *Group representation theory*, pages 405–450. EPFL Press, Lausanne, 2007.

[Ser09] Jean-Pierre Serre. A Minkowski-style bound for the orders of the finite subgroups of the Cremona group of rank 2 over an arbitrary field. *Mosc. Math. J.*, 9(1):193–208, 2009.

[Ser10] Jean-Pierre Serre. Le groupe de Cremona et ses sous-groupes finis. In *Séminaire Bourbaki. Volume 2008/2009. Exposés 997–1011*, pages 75–100, ex. Paris: Société Mathématique de France (SMF), 2010.

[Spr77] T. A. Springer. *Invariant theory.* Lecture Notes in Mathematics, Vol. 585. Springer-Verlag, Berlin, 1977.

[Tel94] Andrei-Dumitru Teleman. Projectively flat surfaces and Bogomolov's theorem on class VII_0 surfaces. *Int. J. Math.*, 5(2):253–264, 1994.

[Weh81] J. Wehler. Versal deformation of Hopf surfaces. *J. Reine Angew. Math.*, 328:22–32, 1981.

[Yas17] Egor Yasinsky. The Jordan constant for Cremona group of rank 2. *Bull. Korean Math. Soc.*, 54(5):1859–1871, 2017.

[Ye17] Shengkui Ye. Symmetries of flat manifolds, Jordan property and the general Zimmer program. *ArXiv e-print*, 1704.03580, 2017.

[Zar14] Yuri G. Zarhin. Theta groups and products of abelian and rational varieties. *Proc. Edinburgh Math. Soc.*, 57(1):299–304, 2014.

[Zar15] Yuri G. Zarhin. Jordan groups and elliptic ruled surfaces. *Transform. Groups*, 20(2):557–572, 2015.

[Zim12] Bruno P. Zimmermann. On finite groups acting on spheres and finite subgroups of orthogonal groups. *Sib. Èlektron. Mat. Izv.*, 9:1–12, 2012.

[Zim14a] Bruno P. Zimmermann. On finite groups acting on a connected sum of 3-manifolds $S^2 \times S^1$. *Fundam. Math.*, 226(2):131–142, 2014.

[Zim14b] Bruno P. Zimmermann. On Jordan type bounds for finite groups acting on compact 3-manifolds. *Arch. Math.*, 103(2):195–200, 2014.

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