Physical meaning of the radial index of Laguerre-Gauss beams

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The Laguerre-Gauss modes are a class of fundamental and well-studied optical fields. These stable shape-invariant photons, exhibiting circular-cylindrical symmetry, are familiar from laser optics, micromechanical manipulation, quantum optics, communication, and foundational studies in both classical optics and quantum physics. They are characterized, chiefly, by two mode numbers: the azimuthal index indicating the orbital angular momentum of the beam, which itself has spawned a burgeoning and vibrant subfield, and the radial index, which up until recently has largely been ignored. In this paper we develop a differential operator formalism for dealing with the radial modes in both the position and momentum representations and, more importantly, give the meaning of this quantum number in terms of a well-defined physical parameter: the intrinsic hyperbolic momentum charge.

I. INTRODUCTION

The ability to shape optical beams has increased dramatically in recent years due to a number of newly developed tools and techniques. This progress has been driven by the many promising potential applications of the orbital angular momentum degree of freedom, as well as more generalized beam shaping. Allen *et al.* [1] showed the connection between the azimuthal index of a Laguerre-Gauss (LG) mode (the natural beamlike light modes exhibiting circular-cylindrical symmetry) and the physical quantity of orbital angular momentum (OAM). This physical parameter is useful for doing micromechanical work, is a well-defined quantum number with applications to foundations, and informatics, and also is important in the foundational study of optics (see Ref. [2] and references therein).

However, despite the vast amount of attention OAM has received over the past two decades, very little research has been conducted on the *other* mode number of LG photons: the radial index. Much of the time it is regarded as little more than unwanted noise that unfortunately arises when one is trying to produce pure OAM modes. In another work we referred to this as the forgotten quantum number [3]. In this paper we will summarize the previous work by us and by others [4,5], then significantly extend and simplify these previous analyses, elucidate the physical meaning of this quantum number, and give a brief prospectus for the use of the radial number in future quantum technologies.

Laguerre-Gauss beams are typically found by taking the paraxial wave equation and finding solutions in the circular-cylindrical coordinate system. The paraxial equation is the result of making the assumption that the beam is not highly divergent or focused. Ironically, we find that the *mathematical* analysis of the radial number simplifies significantly when this assumption is removed and exact solutions of Maxwell's equations are considered. This is done by taking the photonic wave function in momentum coordinates as derived through the Riemann-Silberstein vector formalism [6,7]. However,

the physical meaning does remain clearer in the position representation, an oddity we will discuss.

Our conclusion is that the radial index of Laguerre-Gauss photons is a compound physical parameter influenced by, but not influencing, other fundamental parameters of the mode, as well as an additional property intrinsic to itself: the hyperbolic momentum, which is a kind of mathematically well-formed radial-like momentum with subtle and interesting properties. Thus we call the radial index the hyperbolic momentum charge. The hyperbolic momentum is the result of the restriction that the radial coordinate is only defined for values greater than zero, as we will explain.

This paper is organized as follows. In the following section we review the LG beams. In Sec. III we derive the radial mode operator in the paraxial position-space representation and introduce the hyperbolic momentum operator. In Sec. IV we briefly summarize the background for the exact quantum momentum-space wave function of the photon, namely, the Riemann-Silberstein vector, a more detailed derivation being included in the Appendix. In Sec. V we derive the momentum-space formalism for the radial modes. In Sec. VI is the main purpose of this paper: the physical interpretation of the preceding mathematics as well as a discussion of some potential applications of the radial quantum number. In Sec. VII we summarize.

II. LAGUERRE-GAUSS BEAMS AND THE ORBITAL ANGULAR MOMENTUM OF LIGHT

The equation of a LG beam, under the paraxial assumption, in circular-cylindrical coordinates (r,ϕ,z) is

$$\mathrm{LG}_{nl}(r,\phi,z)$$
$$= \sqrt{\frac{2n!}{\pi(n+|l|)!}}\frac{1}{w_z}\left(\frac{\sqrt{2}r}{w_z}\right)^{|l|}L_n^{|l|}\left(\frac{2r^2}{w_z^2}\right)$$
$$\times \exp\left[-\frac{r^2}{w_z^2}+i\left(l\phi+\frac{kr^2}{2R_z}-(2n+|l|+1)\varphi_g\right)\right],$$
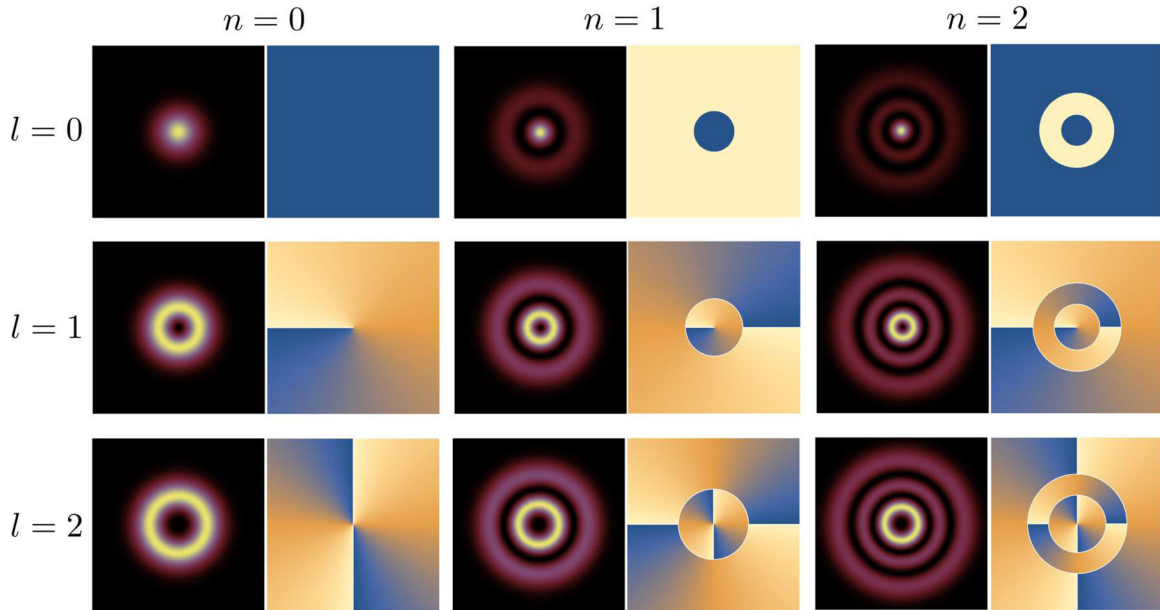
$$(1)$$

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FIG. 1. (Color online) Transverse spatial profiles of nine different Laguerre-Gauss beams in both intensity (left) and phase (right) at $z = 0$, for three different values of the radial index and three different values of the azimuthal index. The diameter of all the plots is 6 mm.

where n and l are the radial and orbital angular momentum quantum numbers, respectively, and $L_n^{|l|}$ is the generalized Laguerre polynomial of order n and degree $|l|$. The functions w_z, R_z, and φ_g are the beam waist, radius of curvature, and Gouy phase of the fundamental beam, respectively, and are given by

$$w_z = w_0 \sqrt{1 + \frac{4z^2}{k^2 w_0^4}}, \tag{2}$$

$$R_z = z + \frac{k^2 w_0^4}{4z}, \tag{3}$$

$$\varphi_g = \arctan\left[\frac{2z}{k w_0^2}\right], \tag{4}$$

where $k = 2\pi/\lambda$ is the overall wave number of the beam and w_0 is the beam waist at $z = 0$ (defined where the beam is narrowest). The beam or photon is then completely defined by four numbers n, l, k, and w_0. In Fig. 1 we plot LG beams for various parameters. The LG beams are solutions to the paraxial wave equation

$$\nabla_t^2 E - 2ik\frac{\partial}{\partial z}E = 0, \tag{5}$$

where ∇_t^2 is the transverse Laplacian and E is a complex electric scalar field (i.e., we assume that the electric field vector points in the same direction at every point in the transverse plane). The paraxial wave equation is a version of the full wavelike Maxwell equations under the small-angle approximation for propagation. In this paper we consider both the cases where this approximations is met and when it may not be.

In the paraxial limit the angular momentum of light separates out into a spin and orbital part, both of which are well defined [8]. The form of the differential OAM operator in

the paraxial limit, about the direction of beam propagation, is well known and straightforward

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}. \tag{6}$$

The OAM is also sometimes called the winding number as it incrementally changes the number of helical equal-phase surfaces of the photon (see Fig. 1). If a particle absorbs a photon with both spin (arising from the polarization of light) and orbital angular momentum the spin will cause the particle to rotate about its own axis, whereas the OAM will cause it to rotate about the optical axis of the beam [9].

III. RADIAL MODES: OPERATOR FORMALISM IN THE PARAXIAL REGIME

The orbital angular momentum of Laguerre-Gauss beams has received an extensive amount of attention, however the radial index (n in our notation) has been the subject of only a handful of papers. The radial index is so called because the intensity pattern of LG beams exhibits $n + 1$ concentric rings if $l \neq 0$ (there are n rings around a central dot for $l = 0$). Also, the phase structure of the beam displays n concentric radial discontinuities with no smooth transitions at $z = 0$ (see Fig. 1). This is in contrast to the OAM azimuthal coordinate where the phase does have smooth transitions at $z = 0$. Unlike the OAM operator (6), the differential operator for the radial number has not been studied until very recently. The operator formalism for the radial modes of LG beams was first derived by Karimi and Santamato [4] via group-theoretic techniques and then independently by Plick *et al.* [3] via the method described below.

We start with the Laguerre polynomial. There exists a series of relations between Laguerre polynomials of varying order and degree (the so-called three-point relations). One such

relation is

$$nL_n^l(x) = (l + 1 - x)L_{n-1}^{l+1}(x) - xL_{n-2}^{l+2}(x), \quad (7)$$

which, when combined with the rule for differentiation of the polynomials

$$\frac{\partial}{\partial x}L_n^l(x) = -L_{n-1}^{l+1}(x), \quad (8)$$

yields

$$\left[(x - l - 1)\frac{\partial}{\partial x} - x\frac{\partial^2}{\partial x^2}\right]L_n^l(x) = nL_n^l(x). \quad (9)$$

Given this differential relation, it is possible to arrive at a relationship between the full LG modes by left multiplying the other factors (the nonpolynomial terms) in the LG function (1) onto Eq. (9) and commuting those factors past the differentials on the left. Doing this we obtain

$$\left[-\frac{1}{4x}\frac{\partial^2}{\partial\phi^2} - x\frac{\partial^2}{\partial x^2} + \frac{i}{2}\frac{\partial}{\partial\phi} - \frac{\partial}{\partial x} + \frac{x}{4} - \frac{1}{2}\right]$$
$$\times \mathrm{LG}_{nl}(x,\phi) = n\mathrm{LG}_{nl}(x,\phi). \quad (10)$$

Now if we make the identification $x = 2r^2/w_0^2$ and perform a change of coordinates, we obtain

$$\hat{N}_0 \equiv \left[-\frac{w_0^2}{8}\left(\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}\right)\right.$$
$$\left. + \frac{i}{2}\frac{\partial}{\partial\phi} + \frac{1}{2}\left(\frac{r^2}{w_0^2} - 1\right)\right], \quad (11)$$

where $\hat{N}_0\mathrm{LG}_{nl}(r,\phi,0) = n\mathrm{LG}_{nl}(r,\phi,0)$ and we have defined the differential n-mode operator for $z = 0$. A trivial transformation shows that this is indeed the operator derived in [4].

Further simplification is possible by identification with other, better known, operators yielding the opportunity for some physical insight

$$\hat{N}_o = -\frac{w_0^2}{8}\nabla_t^2 - \frac{\hat{L}_z}{2} + \frac{1}{2}\left(\frac{r^2}{w_0^2} - 1\right). \quad (12)$$

The operator is composed of the transverse Laplacian in cylindrical coordinates and thus has a direct relationship to the transverse momentum squared (up to some constants). For completeness this Laplacian is defined as

$$\nabla_t^2 = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}. \quad (13)$$

Equation (12) also contains the OAM operator (with $\hbar = 1$), as well as a term reminiscent of a harmonic potential. It commutes with the OAM operator, but not with the Hamiltonian for free propagation due to the final potential-like term. This indicates that n is *not* a conserved quantity of the beam with respect to evolution through free space. This point will be discussed further in Sec. VI.

It is also useful to derive a general operator for any value of z. Continuing with the same methods as above we have

$$\hat{N}_z = -\frac{w_z^2}{8}\nabla_t^2 + \frac{iz}{kw_0^2}\frac{\partial}{\partial r}r - \frac{\hat{L}_z}{2} + \frac{1}{2}\left(\frac{r^2}{w_0^2} - 1\right), \quad (14)$$

where we have, similar to before, $\hat{N}_z\mathrm{LG}_{nl}(r,\phi,z) = n\mathrm{LG}_{nl}(r,\phi,z)$, the differences being that the prefactor of the Laplacian is now the z-dependent beam waist and an additional term resultant from the phase imparted from the radius of curvature R_z (this can be seen from the fact that if the LG beam is written without the radius of curvature phase factor then this term does not appear). Interestingly, the final term is unchanged and remains z independent.

The z-dependent form in Eq. (14) offers another insight. The second term (radius-of-curvature term) can be identified as the operator for a well-formed radial momentum known as the hyperbolic momentum.

It has been known for a long time [10] that the operator for a strictly radial momentum cannot be well defined. To see this consider the most direct attempt $\hat{p}_r = -i\hbar\partial_r$ and its action on the radial coordinate operator

$$e^{i\gamma\hat{p}_r/\hbar}\hat{r}e^{-i\gamma\hat{p}_r/\hbar} = \hat{r} + \gamma. \quad (15)$$

However, the domain of $\hat{r}$ is only the non-negative real numbers. The operator $\hat{p}_r$ can take $\hat{r}$ out of this domain and is thus not well formed. An operator that *can* be well defined in the circular-cylindrical coordinate system is the hyperbolic momentum

$$\widehat{P_H} = -i\hbar\left(r\frac{\partial}{\partial r} + 1\right), \quad (16)$$

which, up to constants (and the z coordinate), is the second term in Eq. (14). The action of this on the radial coordinate is

$$e^{i\gamma\widehat{P_H}/\hbar}\hat{r}e^{-i\gamma\widehat{P_H}/\hbar} = \hat{r}e^{\gamma}. \quad (17)$$

Thus the hyperbolic momentum generates dilations, not linear translations, and cannot cause the radial coordinate to be negative no matter the value of γ. As linear momentum is associated with invariance under translation, hyperbolic momentum is associated with invariance under scale transformations. However, since $[\nabla_t^2, \widehat{P_H}] = -2i\hbar\nabla_t^2$ hyperbolic momentum is not a conserved property of paraxial photon propagation (unlike, obviously, the linear-z momentum and the OAM). It is clear that in our case the position r cannot be negative, that is, $r \in \mathrm{Re}^+$. A convenient method of dealing with this is to make the transformation $\eta = \ln(r)$, which is only defined for positive positions. The conjugate variable to η is the hyperbolic momentum. With this construction it is unsurprising that the hyperbolic momentum should appear in our formalism. The hyperbolic momentum has many other interesting properties; for a detailed investigation the interested reader is referred to Refs. [11,12].

For the case of a LG beam the expectation value of the hyperbolic momentum increases linearly as a function of the propagation distance z. Thus the expectation value of the second term in Eq. (14) as a whole has a quadratic scaling in z. The hyperbolic momentum is always zero where the beam waist is narrowest, since at this point the beam is neither dilating nor contracting (see Fig. 2). If we look at the same quantity as a function of beam waist w_0 we see an exponential decay. This latter effect can be understood from the fact that as the beam waist increases so does the degree of collimation of the beam, thus the beam dilates less (see Fig. 3). These
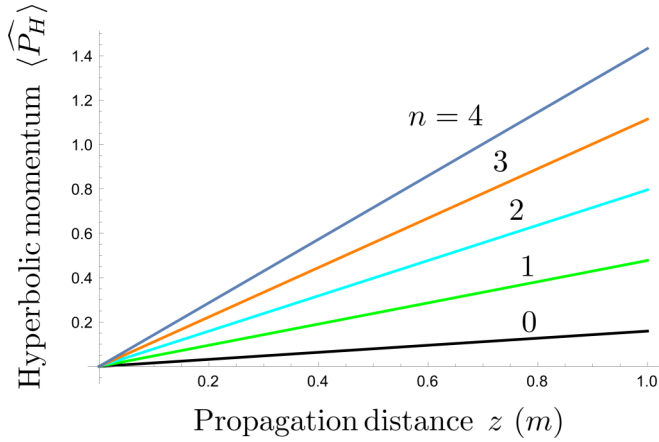
FIG. 2. (Color online) Expectation value of the hyperbolic momentum as a function of propagation distance for five different values of the radial index. In all cases the scaling is linear, passing through the origin where the beam waist is smallest. Note that this does not include the prefactor of $-z/kw_o^2$ in Eq. (14). For that whole term together the z scaling is quadratic, which, in the far field, is equivalent to the scaling of the Laplacian term. Note that the hyperbolic momentum is scale invariant and thus unitless.

functions can be directly calculated as

$$\langle \widehat{P_H} \rangle = \int d\vec{x}\, \mathrm{LG}_{nl}^*(r,\phi,z)\, \widehat{P_H}\, \mathrm{LG}_{nl}(r,\phi,z). \tag{18}$$

See Figs. 2 and 3 for an illustration of the numerical results calculated from Eq. (18). We will examine in more detail the physical interpretation of the operator in Eq. (14) in Sec. VI.

It was previously noted by Karimi and Santamato in Ref. [4] that the OAM and the radial index are inextricably linked. From the referenced paper we know that "OAM and radial intensity distribution are strictly correlated, and different OAM generators produce specific (and different) distributions of
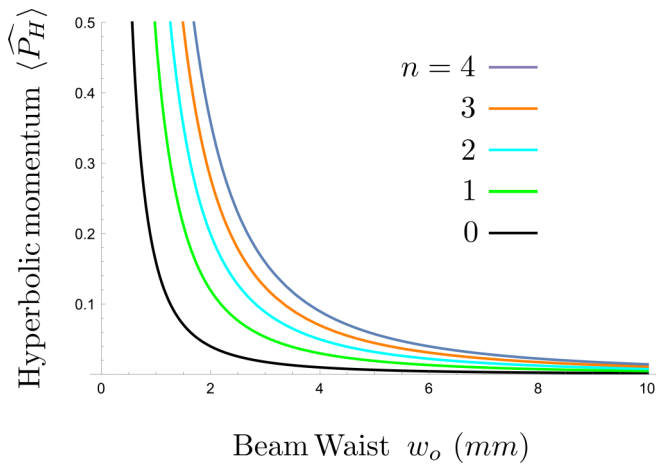

FIG. 3. (Color online) Expectation value of the hyperbolic momentum as a function of beam waist at the origin for five different values of the radial index. The hyperbolic momentum has a finite value at w_o (not pictured due to scaling in order to show contrast between modes). Note that the hyperbolic momentum is scale invariant and thus unitless.

radial modes." (We will comment again on this fact in Sec. VI.) In that paper radial coherent (displaced vacuum) and intelligent (minimum-uncertainty) beams are also derived, the latter of which can be generalized to squeezed states, which was done by Karimi *et al.* in Ref. [5] via a sophisticated algebraic technique employing the raising and lowering operators on the radial number, which form a lie algebra. It is also possible to alternatively define the coherent state as the eigenstate of the lowering operator, which was also done in Ref. [4]. Unlike light in the number basis, where all three definitions are connected via the familiar coherent state $|\alpha\rangle$ and its generalization, the single-mode squeezed state, in the radial representation, all three concepts result in distinct beams.

There has also been experimental work done on the value of the radial index as a quantum number. Again this was carried out by Karimi *et al.* in Ref. [13]. They showed via Hong-Ou-Mandel interference that the radial index is indeed a quantum number and could in principle be used in quantum-information tasks. However, a small caveat is that care must be taken to ensure that the beam waists are equivalent, as stated in that paper: "The chosen basis is beam-waist dependent; an eigenstate for a specific beam waist turns into a superposition of radial modes for any other beam waist."

Several experiments have taken advantage of both the OAM and radial indices, for example, Ref. [14]. It has also been shown [15] that via the use of both quantum numbers, entangled states of very high dimension may be produced, even as high as 103 dimensions. Research has also been done on efficiently producing radial modes [16]. It has also recently been shown that LG beams of high radial indices exhibit self-healing properties in the same way that Bessel beams do [17].

IV. RIEMANN-SILBERSTEIN VECTOR, EXACT SOLUTIONS TO MAXWELL'S EQUATIONS, AND THE PHOTONIC WAVE FUNCTION

Here we briefly outline some previously existing mathematical formalism we will need. A more complete version is included in the Appendix. The Riemann-Silberstein (RS) vector is a representation of the electromagnetic field given by

$$\vec{F} = \sqrt{\frac{\epsilon_0}{2}}(\vec{E} + ic\vec{B}), \tag{19}$$

where c is the speed of light in vacuum, ϵ_0 is the permittivity of vacuum, and $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields, respectively. The vector consists of three complex numbers and many useful quantities can be calculated directly with simple equations. For a full review of the RS vectors the interested reader is directed to Refs. [6,7]. In the Appendix we review this derivation in the interest of completeness and notational consistency. The previously acquainted reader may skip it.

It is convenient to work with the scalar function χ (which contains the full information of $\vec{F}$, up to choice of gauge, as explained in the Appendix) as opposed to the full RS vector $\vec{F}$. Given the formalism from the Appendix, we can directly write a general beam in the Bessel-Gauss basis as

$$\chi_m^\sigma(r,\phi,z,t) = \int d\vec{k}\, \psi^\sigma(\vec{k})e^{-i\sigma(ck-k_z z)} J_m(k_t r) \tag{20}$$

using Eqs. (A13) and (A14). Note that we take a different definition than is used in Ref. [18] as we pull the azimuthal phase and Jacobian into ψ.

Now the derivation of the Laguerre-Gauss modes is facilitated by a change of momentum coordinates $k_\pm = (k \pm k_z)/2$ and also time $t_\pm = t \pm z/c$, yielding

$$\chi_m^\sigma(r,\phi,z,t) = \int_0^\infty dk_+ \int_0^\infty dk_- \int_0^{2\pi} dk_\phi \psi^\sigma(k_+,k_-)$$
$$\times e^{-i\sigma c(k_+ t_- + k_- t_+)} J_m(2r\sqrt{k_+ k_-}). \quad (21)$$

It is of key importance to note that since χ is effectively the position-space wave function (given the caveats described in the Appendix) that $\psi^\sigma(k_+,k_-)$ is the momentum-space wave function in the beamlike (i.e., Bessel) basis. Now if we make the choice

$$\psi^\sigma(k_+,k_-,k_\phi) = e^{i\sigma m k_\phi}\delta\left(k_+ - \frac{\Omega}{c}\right)k_-^{n+|m|/2}e^{-(w^2\Omega/c)k_-}k, \quad (22)$$

where n is some integer and w is some real number, we can solve Eq. (21) exactly:

$$\chi_m^\sigma(r,\phi,z,t) = \frac{\mathcal{N}r^{|m|}}{a(t_+)^{n+|m|+1}}e^{-i\sigma\Omega(t-z/c-m\phi)}$$
$$\times e^{-r^2/a(t_+)}L_n^{|m|}\left(\frac{r^2}{a(t_+)}\right), \quad (23)$$

which is clearly a form of the Laguerre-Gauss beams, with $\mathcal{N}$ a normalization constant, m an angular momentum number (we avoid calling this the orbital angular momentum since the OAM is only completely well defined in the paraxial limit), n the radial number, and $a(t_+) = w^2 + i\sigma c^2 t_+/\Omega$, a form of the complex beam parameter. Thus we can identify Eq. (22) as the exact momentum-space wave function of a Laguerre-Gauss beam (up to normalization constants).

V. MOMENTUM-SPACE FORMALISM FOR THE RADIAL MODES

In the previous section (and in the Appendix) we summarized the mathematical tools from Ref. [18] that we will need for the investigation of the radial number as it exists for the exact photon wave function. From the momentum representation of Laguerre-Gauss beams in Eq. (22) we can easily derive a radial momentum operator

$$k_-\frac{\partial}{\partial k_-}\psi^\sigma = \left(\frac{k_-}{k} + n + \frac{m}{2} - \frac{w^2\Omega}{c}k_-\right)\psi^\sigma, \quad (24)$$

$$\left(k_-\frac{\partial}{\partial k_-} + \frac{i}{2\sigma}\frac{\partial}{\partial k_\phi} - \frac{k_-}{k} + \frac{w^2\Omega}{c}k_-\right)\psi^\sigma = n\psi^\sigma. \quad (25)$$

Therefore, one finds the (general, nonparaxial) radial momentum operator as

$$\hat{N}_k = \left(k_-\frac{\partial}{\partial k_-} + \frac{i}{2\sigma}\frac{\partial}{\partial k_\phi} - \frac{k_-}{k} + \frac{w^2\Omega}{c}k_-\right). \quad (26)$$

In order to interpret the operator, one can transform it to polar coordinates. By using the relations

$$k = \sqrt{k_z^2 + k_t^2}, \quad (27)$$

$$k_\pm = \frac{\sqrt{k_z^2 + k_t^2} \pm k_z}{2}, \quad (28)$$

we find directly the polar-coordinate representation of the radial-momentum operator in momentum space

$$\hat{N}_k = \frac{1}{2}\left[k_t\frac{\partial}{\partial k_t} + \frac{i}{\sigma}\frac{\partial}{\partial k_\phi} - (k - k_z)\left(\frac{\partial}{\partial k_z} + \frac{1}{k} - \frac{w^2\Omega}{c}\right)\right]. \quad (29)$$

This is the most exact form of the operator. Now if we take the case of paraxial photons (which is in most cases the situation of interest), the momentum along the direction of propagation will be many orders of magnitude larger than the transverse momentum. So if we Taylor-series expand the square root in k_- and k_+ about k_z we have

$$k_- = \frac{1}{2}\left(\sqrt{k_t^2 + k_z^2} - k_z\right)$$
$$= \frac{1}{2}\left(k_z + \frac{k_t^2}{2k_z} - \frac{k_t^4}{8k_z^3} - \cdots - k_z\right)$$
$$\approx \frac{k_t^2}{4k_z}, \quad (30)$$
$$k_+ = \frac{1}{2}\left(\sqrt{k_t^2 + k_z^2} + k_z\right)$$
$$= \frac{1}{2}\left(k_z + \frac{k_t^2}{2k_z} - \frac{k_t^4}{8k_z^3} - \cdots + k_z\right)$$
$$\approx k_z \quad (31)$$

as well as $k = \sqrt{k_t^2 + k_z^2} \approx k_z$. Now, again taking Eq. (22), or using directly Eq. (29), rewriting it in terms of these more familiar momentum coordinates

$$\psi^\sigma \approx e^{i\sigma m k_\phi}\delta\left(k_z - \frac{\Omega}{c}\right)\left(\frac{k_t^2}{2k_z}\right)^{n+|m|/2}e^{-(w^2\Omega/c)(k_t^2/2k_z)}k_z, \quad (32)$$

and making use of the δ function, we have up to constants

$$\psi^\sigma(k_t,k_z,k_\phi) \propto e^{i\sigma m k_\phi}k_t^{2n+|m|}e^{-(w^2/2)k_t^2}. \quad (33)$$

From this, unlike in the position-space paraxial-regime case, it is quite straightforward to find the operator that returns the value n. By simple inspection we can write

$$\hat{N}_k' = \frac{1}{2}\left(k_t\frac{\partial}{\partial k_t} + \frac{i}{\sigma}\frac{\partial}{\partial k_\phi} + w^2 k_t^2\right), \quad (34)$$

which has the most straightforward form. The first term is similar to the hyperbolic momentum operator, however it lacks the i in front. This is nontrivial as without the i its action on the radial coordinate is given by

$$e^{i\gamma k_t(\partial/\partial k_t)/\hbar}\hat{r}e^{-i\gamma k_t(\partial/\partial k_t)/\hbar} = \hat{r}e^{i\gamma}, \quad (35)$$

where the radial position coordinate is defined in the momentum representation as

$$\hat{r} \equiv -i\hbar \frac{\partial}{\partial k_t}. \tag{36}$$

Obviously this is problematic as the radial coordinate must remain a real number. Also, without the i the first term (by itself) is not in general Hermitian. Its potentially interesting to note that if the action γ is a multiple of π then it generates a real (periodic) scaling. Whether this can have some physical meaning is unknown to us as of yet.

The second term is again the OAM operator, scaled by the helicity. The final term is the transverse Laplacian in momentum coordinates scaled by the beam waist at the origin. There are also a couple other advantages of writing the operator in this way. It is propagation invariant, that is, this is the operator for all time (as the momenta are conserved quantities). Also, the transverse momentum coordinate may be rescaled (as $k'_t = w k_t$) to make the operator independent of the beam waist, should this be desirable.

We want to test is its Hermiticity (in total). Does $\langle \hat{N}'_k \psi^\sigma | \psi^\sigma \rangle = \langle \psi^\sigma | \hat{N}'_k \psi^\sigma \rangle$? A straightforward computation shows that if we desire real eigenvalues, then $\hat{N}'_k$ cannot, in general, be Hermitian. It is Hermitian instead only on a restricted class of wave functions where $\psi(k_t, k_\phi) = \Theta(k_t)\Phi(k_\phi)$, where $\Theta(k_t)$ is a real function. Only in such beams can $\hat{N}_k$ be considered a valid observable. Some examples of such beams are the Bessel beams and the Laguerre-Gauss beams, which are our main interest.

VI. PHYSICAL INTERPRETATION AND TECHNOLOGICAL PROSPECTS

Now that we have laid down the mathematical foundations for the radial index operator we can investigate the meaning of the radial index. Though the RS vector momentum-representation version of the operator (34) is accurate and simple, its interpretation is more difficult to parse than the paraxial coordinate-representation operator (14). This is indicative of the fact that the radial-index operator is *not* representative of a general quantum observable like its partner the OAM operator. Instead n can be considered a proper observable in a restricted class of scenarios, more specifically, for beams whose momentum-space wave function is separable into radial and angular portions and for which the radial part is always real. The radial-index operator also requires additional information to write down, specifically, the beam waist at origin, the propagation distance, and the orbital angular momentum.

This latter point has implications for future potential uses of the radial index as a tool in quantum technologies. Suppose we wish to measure the (unknown) radial index of a single photon. A common technique for doing this would be to shine the photon on a phase pattern that takes a photon of the radial index in question and turns it into the fundamental Gaussian mode; this photon then either couples, or does not couple, to a single mode fiber. This constitutes a projective measurement onto the chosen radial number. However, in order to choose the correct phase pattern, the OAM index, the beam waist, and the distance of propagation must be known. Note that

this is not merely a problem with a particular experimental implementation of radial-index sorting, but a general problem. If we examine the paraxial version of the operator (14) we can see that it contains all three of the other parameters (z, w_0, and l), meaning that we must know these values *a priori* (or measure them simultaneously) in order to get a meaningful result from a radial-index measurement. It may also be possible to perform a measurement if the other parameters can be measured *passively*, that is, nondestructively for the same photon (for example, OAM may be passively measured with a mode sorter [19]).

To further illuminate this problem it should be noted that Laguerre-Gauss beams of a different radial index do not have zero overlap if they are at different distances in their propagation. Mathematically, that is,

$$\int r\, dr \int d\phi \, \mathrm{LG}^*_{nl}(r,\phi,z)\mathrm{LG}_{n'l}(r,\phi,z') \neq 0. \tag{37}$$

In Fig. 4 we plot the overlap of a beam of a particular n with several others for increasing propagation-distance mismatches.

However, if these problems can be overcome (which can be achieved by careful calibration), then indeed the radial index can be used as a carrier of quantum information in realistic scenarios. This was demonstrated recently by two separate experiments, as discussed earlier. These experiments show that the radial index can be a valuable resource in quantum experiments, but one needs to be careful to satisfy the subtle requirements in the measurements.

The maxim "One man's noise is another man's data" may also find some application for the radial modes. To wildly speculate, it could be the case that the index's sensitivity to distance and beam-width mismatch could find some application and that by (for example) measuring one n mode's projection onto $n \pm 1$ some information about propagation distance or dilation could be obtained.

Returning to the question of physical meaning, it is more illuminating to consider the paraxial version of the operator (14), which we rewrite below for the convenience of the reader:

$$\hat{N}_z = -\frac{\hbar w_z^2}{8}\nabla_t^2 - \frac{z}{kw_0^2}\widehat{P_H} - \frac{\hat{L}_z}{2} + \frac{\hbar}{2}\left(\frac{r^2}{w_0^2} - 1\right), \tag{38}$$

where we have multiplied through by $\hbar$ and substituted the second term with the hyperbolic momentum operator (16). There are four terms, three of which are related to fundamental properties of the beam that can be identified with quantities other than, and independent of, the radial index. That is to say, these three parameters influence, but are not influenced by, the radial index. This leaves a single term that represents the degree of freedom that the radial index arises from. The first term in Eq. (38) is the transverse Laplacian scaled by the z-dependent beam waist and a numerical constant. In the free-space paraxial regime the transverse Laplacian alone is the Hamiltonian (and, of course, a constant of motion). The third term is the OAM. The final term produces, up to constants, the second moment of the radial position at $z = 0$, which is clearly related to the transverse spatial variance, i.e., the spatial confinement at origin. These are three independent parameters of the beam (and of individual photons that occupy these modes).
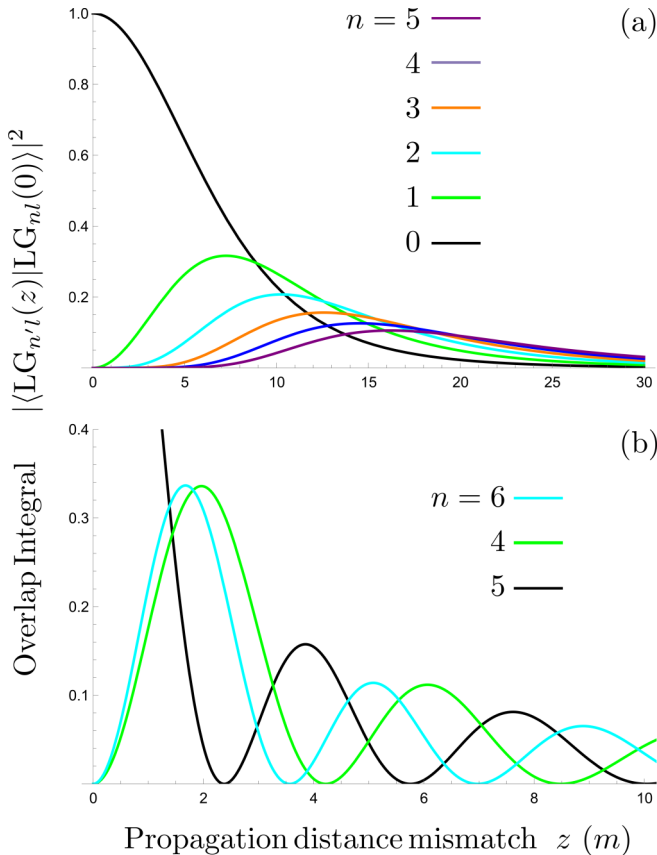
FIG. 4. (Color online) Graphs of the overlap function of two Laguerre-Gauss modes with unequal radial indices as a function of the mismatch in the distance the modes have propagated. In both graphs the black line represents the mode's overlap with itself. In (a) we show the $n = 0$ mode's overlap with the five next higher indexed modes. It is clear from the graph that very quickly the depicted modes no longer add to one. This is due to the fact that as the beam propagates it acquires nontrivial overlaps with more and more modes. After only about 15 m, 20 modes are necessary to describe the beam to good accuracy (not shown). In (b) we show the overlap of an $n = 5$ mode with a mode of higher index and a mode of lower index. Scaling is different than in (a) to highlight detail. The oscillatory behavior is due to the fact that after moving from one mode to an adjacent one, the photon may then move back as it continues to spread in the radial-index space.

It is the second term, however, that is most important for understanding the physical meaning of the radial index. This is the hyperbolic momentum operator, which generates dilations. As discussed previously, it is a mathematically-well-formed kind of radial momentum operator. It is, however, *not* a constant of motion (as is demonstrated in Fig. 2) and thus does not have one of the quantities we would traditionally expect from a momentum. Nonetheless, if the propagation distance is accounted for in the operator itself (as is the case in $\hat{N}_z$), then it is sensible to speak of it as an intrinsic property; here we see why the operator must be z dependent in the coordinate representation and becomes somewhat unmanageable in the momentum representation. In other words, the radial index is best understood in the coordinate picture, as in that representation the fact that it is not conserved under time

evolution is easy to account for, however the momentum representation deals best with quantities that are preserved for all time.

The radially transverse momentum of a photon must be dependent on the total transverse momentum, as well as the orbital angular momentum (as the orbital part of the total transverse momentum needs to be compensated for), and on the degree of spatial confinement of the beam (because greater confinement must lead to higher divergence and greater radial momentum). However, this does not account for intrinsic freedom of the hyperbolic momentum. The radial index can then be seen as representing the promotion of the radial-like hyperbolic momentum *itself* to a quantum observable. We therefore believe that the radial index could best be described as the intrinsic hyperbolic momentum charge.

We might have initially expected the radial index to be related to the radial momentum, as the azimuthal index is related to a rotational momentum, completing the set of two momenta necessary to move about in the transverse two-dimensional plane. We have found that, while $\hat{N}_z$ does not correspond to a full radial momentum operator (as no such well-formed operator exists), the hyperbolic momentum operator, which is up to scalings (z/kw_0^2) and other observables (the paraxial Hamiltonian $-\nabla_t^2$, the OAM $\hat{L}_z$, and the radial position squared r^2) of the operator we find, has a direct connection to the momentum along the radial coordinate.

It has been shown [14] that the radial index is conserved in the process of spontaneous parametric down-conversion. More specifically, in the ideal case, the two daughter photons will have the same radial index as each other. In the nonideal case (finite crystal length and width) there is a spread in the radial index of the down-converted photons. As we have seen in the research in this paper, the radial index can only be said to be well defined given a precise propagation distance, OAM, and beam width; therefore, we gain insight into why the effect of crystal length (and thus uncertainty in the point of origin of the down-conversion and walkoff) should blur the resultant radial indices. This is in addition to a similar effect from a limited transverse spatial extent of the crystal, investigated in detail in Ref. [20].

Since this paper and previous work of ours and others have shown that both the orbital angular momentum and the intrinsic hyperbolic momentum charge are well-formed quantum numbers with a physical meaning (in the proper circumstances), it follows that that the mode indices of beams with different symmetries (noncircular cylindrical, e.g., elliptic or Cartesian) are also well-formed quantum numbers. These other mode indices are just algebraic combinations of l and n. Therefore, it should follow that these other numbers should have interesting physical meanings as well. For the case of paraxial photons with elliptic symmetry, the so-called Ince-Gauss modes [21], it has been shown that their mode indices can be be entangled [22] and also that their OAM properties are nontrivial and show interesting complexity [23]. It may be the case that these effects could be better understood by combining the concepts of orbital and hyperbolic momentum.

There are still some unanswered questions. Most significantly, it is not clear what should be considered the conjugate variable. Also, we do not yet understand why the intrinsic hyperbolic momentum charge (radial index) should

take discreet values. Furthermore, it is unknown why the hyperbolic momentum should be tied to the number of rings in the transverse spatial pattern of the beam.

We would also like to briefly sketch some potential technological applications and concerns tied to the radial index. As we have mentioned already, it has been shown experimentally that the radial index is indeed a valid quantum number [13] and could be used for quantum communication. However, the fact that different n modes can have nonzero overlap if there is mismatch in the beam waist w_0, or propagation distance z, adds some additional complication, especially in realistic scenarios where complete control or knowledge of the distances and amount of focusing involved may not be available. However, if we take the $z = 0$ form of the radial-index operator (12), we find that this is exactly the Hamiltonian of a graded-index fiber [24]. Graded-index fibers are commercially available fibers that have a linear radial variation of their refractive indices. It is possible that a hyperbolic-momentum-carrying photon could be matched with standard optics to such a fiber (i.e., beam waist matched to the correct steepness of index variation) and then the radial index would be intrinsically preserved under propagation in the fiber. This would allow for quantum or classical communication multiplexing using the radial index in a widely available kind of fiber. This is an exciting prospect that we believe merits further investigation.

VII. CONCLUSION

In this paper we have developed the differential-operator formalism for the radial index of Laguerre-Gauss modes in both the paraxial coordinate representation and in the exact momentum-space representation. We identified the various parts of these operators with certain physical parameters, most of which are tied to beam characteristics that are not influenced by the radial index but are influential on it. Put another way, for each value of these parameters (beam waist, orbital angular momentum, and propagation distance) there is a different representation of the radial-index operator. There is one remaining part of the operator that is not tied to other properties of the photon that we have identified: the hyperbolic momentum that generates dilations. It is this term that corresponds to the true degree of freedom that is not represented by other mode numbers (l, k, and w_0) in our operator and thus we call the radial index the intrinsic hyperbolic momentum charge. We have shown that the radial index is not tied exclusively to the transverse spatial profile of the beam but also has a physical meaning. We hope that this opens up a new area of investigation and inspires some new potential technological prospects.

We briefly outlined one such potential application, which is that the radial index may be naturally preserved under propagation in a graded-index fiber. This may have application to fiber-optic multiplexing in the classical and quantum domains.

In conclusion, we conjecture that, despite the resources devoted to the study of the orbital angular momentum mode number, the *radial* mode number is vastly richer due to its mathematical complexity and the plethora of questions that remains unanswered.

APPENDIX: DERIVATION OF THE PHOTON WAVE FUNCTION

This section is for the most part a summary of the formalism collected or derived in Ref. [18]. Note, however, that some definitions and notation will differ.

In the RS vector formalism, Maxwell's equations in free space reduce to

$$\frac{\partial}{\partial t}\vec{F}(\vec{x},t) = -ic\nabla \times \vec{F}(\vec{x},t), \tag{A1}$$

$$\nabla \cdot \vec{F}(\vec{x},t) = 0. \tag{A2}$$

The RS vector can also be expressed as a complex vector field $\vec{Z}$:

$$\vec{F}(\vec{x},t) = \nabla \times \left(\frac{i}{c}\vec{Z}(\vec{x},t) + \nabla \times \vec{Z}(\vec{x},t)\right). \tag{A3}$$

The function $\vec{Z}$ satisfies the wave equation

$$\left(\frac{1}{c^2}\frac{\partial}{\partial t} - \nabla^2\right)\vec{Z}(\vec{x},t) = 0, \tag{A4}$$

where ∇^2 is the full Laplacian. The potential $\vec{Z}$ can itself be recast in a more tractable form $\vec{Z} = (e_1,e_2,e_3)\chi(\vec{x},t)$, where χ is a field and the e's may be chosen to suit the situation at hand (i.e., the desired symmetries of the system to be studied), for beams propagating in a straight line $\vec{Z} = (0,0,1)\chi(\vec{x},t)$ is common.

All solutions to the wave equation (A4) may be expressed as a superposition of plane waves, so

$$\chi(\vec{x},t) = \int d\vec{k}\, N(\vec{k})[\psi^+(\vec{k})e^{-i\omega_k t+i\vec{k}\cdot\vec{x}} + \psi^-(\vec{k})e^{i\omega_k t-i\vec{x}\cdot\vec{r}}], \tag{A5}$$

where $N(\vec{k})$ represents a normalization factor. Then the RS vector can be written as

$$\vec{F}(\vec{x},t) = \int d\vec{k}\vec{e}(\vec{k})[\psi^+(\vec{k})e^{-i\omega_k t+i\vec{k}\cdot\vec{x}} + \psi^-(\vec{k})e^{i\omega_k t-i\vec{x}\cdot\vec{r}}], \tag{A6}$$

where $\vec{e}$ is a k-dependent complex polarization vector that is determined by the choice of the vector part of the potential $\vec{Z}$ and includes the normalization factor. This represents a gauge freedom. Note that the polarization vector factors completely from the rest of the expression. It can thus be considered a part of the transformation from momentum space to position space. This is a special feature of the RS formulation. The weight

functions $\psi^-(\vec{k})$ and $\psi^+(\vec{k})$ are the positive and negative frequency components of the RS vector in momentum space, the choice of which completely defines the physical degrees of freedom of the EM field in momentum space.

In our treatment we wish for a fully-well-formed construction, so we will have as our objective to end up in momentum space, since it has been long known that the definition of the photon wave function in position space is problematic. This is due to the fact that a position operator does not exist for photons [eigenstates of momentum for photons (plane waves), though nonphysical, can be mathematically tractable, but the idea of a position eigenstate for photons is not nonsensical even at the conceptual level as photons are inherently delocalized]. A *probabilistic* position representation can be generated for photons, but we will not take this route. For more information on this and connected topics please see Refs. [6,7].

However, for reasons of clarity we will start in position space and then switch to momentum. The time-dependent Schrödinger equation is

$$i\hbar\frac{\partial}{\partial t}\Psi(\vec{x},t)=\widehat{H}\Psi(\vec{x},t), \tag{A7}$$

where $\widehat{H}$ is the Hamiltonian. One immediately recognizes the similarity between this and Eq. (A1), the first Maxwell equation for the RS vector. We can thus equate the Hamiltonian to the appropriate part of that equation $\widehat{H}\Psi=-ic\nabla\times\Psi$ with the RS vector becoming the photonic wave function.

In order to proceed from here we will rewrite the curl operator. Take a vector $\hat{\vec{s}}$, the components of which are the spin-1 matrices

$$\hat{s}_x=\begin{bmatrix}0&0&0\\0&0&-i\\0&i&0\end{bmatrix},\quad\hat{s}_y=\begin{bmatrix}0&0&i\\0&0&0\\-i&0&0\end{bmatrix},$$

$$\hat{s}_z=\begin{bmatrix}0&-i&0\\i&0&0\\0&0&0\end{bmatrix}.$$

Written in index summation notation, the tensor $\hat{\vec{s}}=s_{ijk}$ is equivalent to both the Levi-Città symbol ϵ_{ijk} times a factor of $-i$ and the spin-1 matrices in quantum mechanics (when these matrices are acting on the Cartesian vector components of the wave function and not the eigenstates of $\hat{s}_z$). Since it is the case that $\vec{a}\times\vec{b}=\epsilon_{ijk}a_jb_k$ and $\vec{a}\cdot\vec{b}=a_ib_i$, it is also the case that

$$\nabla\times\Psi=-i\hat{\vec{s}}\cdot\nabla\Psi, \tag{A8}$$

where $\hat{p}=-i\hbar\nabla$ is the very-well-known momentum operator. Slightly less well known is the helicity operator in optics and particle physics, which is the sign of the projection of the angular momentum on the momentum

$$\hat{\Lambda}=\text{sgn}[\widehat{M}\cdot\hat{p}] \tag{A9}$$

$$=\text{sgn}[(-i\hbar\vec{r}\times\nabla+\hbar\hat{\vec{s}})\cdot(-i\hbar\nabla)] \tag{A10}$$

$$=\frac{1}{p_T}\hat{\vec{s}}\cdot\hat{p}. \tag{A11}$$

where the momentum and angular momentum operators are substituted for their definitions on the second line (the latter is

composed of an orbital and spin part) $p_T=\sqrt{p_x^2+p_y^2+p_z^2}$ and the simplification on the last line is due to the fact that $(\vec{r}\times\nabla)\cdot\nabla=0$. So we can see directly by comparing the above to Eq. (A8) that the helicity operator (up to constants) is the Hamiltonian. The helicity operator generates the duality rotation and is associated with that symmetry in free space. Other than the fact that it serves as the Hamiltonian for a Schrödinger equation with the RS vectors as wave functions, the properties of $\hat{\Lambda}$ are not strictly relevant to our work here; nonetheless, the interested reader is redirected to Refs. [25,26].

It will be convenient to find the eigenfunctions of the helicity (and thus the energy eigenstates) in a beamlike basis. We return to Eq. (A6) and utilize the following expansion in cylindrical coordinates:

$$e^{i\vec{k}\cdot\vec{x}}=e^{ik_zz}\sum_{m=-\infty}^{\infty}i^m e^{im(\phi-k_\phi)}J_m(k_tr), \tag{A12}$$

where k_ϕ and k_t are the polar and radial coordinates in momentum space, respectively. Note that the radial coordinate in momentum space is also the transverse momentum. The functions J_m are Bessel functions of the first kind. This affects a change of basis from plane waves to Bessel waves, which also forms a complete set. Now Eq. (A5) becomes

$$\chi(\vec{x},t)=\sum_{m,\sigma}\int d\vec{k}\,\chi_m^\sigma(\vec{k})\psi^\sigma(\vec{k}), \tag{A13}$$

where σ can take values ±1 and the $\chi_m^\pm$ are defined as

$$\chi_m^\pm(\vec{x},\vec{k})=\frac{(\pm i)^m}{kk_t\sqrt{2}}e^{\pm i(\omega_kt-k_zz-m(\phi-k_\phi))}J_m(k_tr). \tag{A14}$$

We then obtain, via choosing the beamlike form of the potential ($\vec{Z}=(1,0,0)\chi$) and by direct calculation for a particular m and $\vec{k}$,

$$\vec{F}_{k_tk_zm}^\sigma=\begin{bmatrix}F_r\\F_\phi\\F_z\end{bmatrix}=\frac{(i\sigma)^m}{k\sqrt{2}}e^{-i\sigma(\omega_kt-k_zz-m\phi)}$$

$$\times\begin{bmatrix}i\sigma k_z\frac{\partial}{\partial(k_tr)}+i\frac{km}{k_tr}\\-\sigma k\frac{\partial}{\partial(k_tr)}-\frac{k_zm}{k_tr}\\k_t\end{bmatrix}J_m(k_tr). \tag{A15}$$

The number σ represents whether the photon is right or left circularly polarized. It is the case that $\hat{\Lambda}\vec{F}_{k_tk_zm}^\sigma=\sigma\vec{F}_{k_tk_zm}^\sigma$. The wave functions in Eq. (A15) form a basis of solutions to the photonic Schrödinger equation. It is important to note that, since the helicity takes the place of the Hamiltonian, there are solutions to the Schrödinger equation that at first glance might seem to have negative energy. These solutions (those of left-circularly-polarized photons) do not represent antiparticles as photons have no antiparticles. They should be interpreted merely as opposite-helicity photons. These beamlike solutions are in fact the Bessel beams, which have come to be objects of interest due to their diffraction-free properties [27]. However, much like plane waves, these beams are not physical as they turn out to have infinite energy. Typically these beams are made physical without breaking the assumption of monochromicity

by convolving them with a Gaussian function (the Bessel-Gauss beams). However, these beams are no longer exact solutions to the Maxwell equations. However, by dropping the assumption of a monochromatic beam (also usually implicit in the paraxial wave equation) we can use the Bessel-beam basis defined in Eq. (A15) and continue to have exact solutions to the photon wave function that are valid under all circumstances.

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