

Fourth-neighbour two-point functions of the XXZ chain and the Fermionic basis approach

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Abstract

We give a descriptive review of the Fermionic basis approach to the theory of correlation functions of the XXZ quantum spin chain. The emphasis is on explicit formulae for short-range correlation functions which will be presented in a way that allows for their direct implementation on a computer. Within the Fermionic basis approach a huge class of stationary reduced density matrices, compatible with the integrable structure of the model, assumes a factorized form. This means that all expectation values of local operators and all two-point functions, in particular, can be represented as multivariate polynomials in only two functions ρ and ω and their derivatives with coefficients that are rational in the deformation parameter q of the model. These coefficients are of ‘algebraic origin’. They do not depend on the choice of the density matrix, which only impacts the form of ρ and ω . As an example we work out in detail the case of the grand canonical ensemble at temperature T and magnetic field h for q in the critical regime. We compare our exact results for the fourth-neighbour two-point functions with asymptotic formulae for $h, T = 0$ and for finite h and T .

1 Introduction

The study of correlation functions of integrable quantum systems has remained a challenge for many years now. In our quest for a general theory we are still proceeding case by case. One of the most thoroughly studied cases is the spin- $\frac{1}{2}$ XXZ quantum chain. Quantum spin chains are defined on tensor product spaces $\mathcal{H}_{[k,l]}$, where $k < l \in \mathbb{Z}$, $[k, l] = \{k, k+1, \dots, l\}$ and $\mathcal{H}_{[k,l]} = \bigotimes_{j=k}^l V_j$. In the spin-1/2 case the local vector spaces can be taken as $V_j = \mathbb{C}^2$. The XXZ-Hamiltonian

$$H_L = J \sum_{j=-L+1}^L \left\{ \sigma_{j-1}^x \sigma_j^x + \sigma_{j-1}^y \sigma_j^y + \Delta (\sigma_{j-1}^z \sigma_j^z - 1) \right\} - \frac{h}{2} \sum_{j=-L+1}^L \sigma_j^z \quad (1)$$

acts on $\mathcal{H}_{[-L+1,L]}$. The σ_j^α , $\alpha = x, y, z$, are Pauli matrices on V_j and the three real parameters are the anisotropy $\Delta = (q + q^{-1})/2$, the exchange interaction $J > 0$, and the strength $h > 0$ of an external magnetic field.

H_L commutes with the transfer matrix of the 6-vertex model [40]. Its integrability can be traced back [42] to the quantum group $U_q(\widehat{sl_2})$ [13, 26, 49] using the representation theory of the latter. Finite and infinite dimensional representations of the quantum group and its Borel subalgebras are also involved in the ‘Fermionic basis approach’ [5, 10, 11, 29, 31] to the calculation of the correlation functions of the model which we shall briefly review. Although this approach was devised already some time ago, we still have many questions about its interpretation and its connection to more traditional parts of the theory of integrable quantum systems. We are not even sure how to properly characterize this fascinating piece of work in a few words. Perhaps the best we can think of is to say that it introduces a module structure on a space of quasi-local spin operators on the infinite chain in a way compatible with a family of generalized reduced density matrices.

In order to fill these words with meaning and to motivate the basic notions of the Fermionic basis approach let us take a detour and recall some general facts about the statistical mechanics of lattice models. For a many-body system like (1) we typically wish to calculate the correlation functions of local operators in a macro state described by a density matrix ρ_L . This is the state the system relaxes to under the influence of its internal interactions encoded in its Hamiltonian and of an additional weak coupling to its environment, if it was initially prepared in a state given by the experimental setup. Accordingly, ρ_L satisfies the relations

$$\rho_L = \rho_L^+, \quad \rho_L \geq 0, \quad \text{tr}_{[-L+1,L]} \{\rho_L\} = 1, \quad (2)$$

valid for any density matrix by definition, and the stationarity condition

$$[\rho_L, H_L] = 0. \quad (3)$$

The fundamental assumption of statistical mechanics is that the coupling to the environment eventually drives every many-body system to a state that can be described by the canonical ensemble. The corresponding canonical density matrix

$$\rho_L^{(c)}(T) = \frac{e^{-H_L/T}}{\text{tr}_{[-L+1,L]} e^{-H_L/T}} \quad (4)$$

depends on a single parameter, the temperature T . If the interaction with the environment is very weak, which can be the case e.g. in modern cold-atom experiments, the relaxation

to the canonical ensemble may become very slow and certain transient behaviours may be observed. For rather long times then the system behaves as if it were isolated. The question of the relaxation of isolated quantum systems has therefore become relevant. A natural way of asking this question is to consider a small subsystem and inquire whether the rest of the system can act as heat bath for the subsystem. One would say that a system like (1) with density matrix $\rho_L(t)$ thermalizes if

$$\begin{aligned} \lim_{t \rightarrow \infty} \lim_{L \rightarrow \infty} \text{tr}_{[-L+1, k-1] \cup [l+1, L]} \{ \rho_L(t) \} \\ = \lim_{L \rightarrow \infty} \text{tr}_{[-L+1, k-1] \cup [l+1, L]} \{ \rho_L^{(c)}(T) \} = D_{[k, l]}(T) \end{aligned} \quad (5)$$

for all $k, l \in \mathbb{Z}$. The operator on the right hand side of this equation is called the reduced density matrix (of the canonical ensemble) associated with the ‘chain segment’ $\mathcal{H}_{[k, l]}$ or with the ‘interval’ $[k, l]$.

The reduced density matrix defined in (5) is a very useful notion as it allows us to define in a sensible way a space of observables of the infinite chain Hamiltonian associated with (1). The Hamiltonian (1) and the canonical density matrix (4) do not have a naive thermodynamic limit. Such a limit would require to define a limiting space of states, spanned by all eigenstates of H_L which have finite excitation energies for $L \rightarrow \infty$. Since such a construction is not at all obvious and may depend on the details of the interaction, a better way to proceed is to define the space of observables inductively by means of the reduced density matrices of all chain segments. For this purpose consider any $X_{[k, l]} \in \text{End } \mathcal{H}_{[k, l]}$ and let $X_L = \text{id}_{[-L+1, k-1]} \otimes X_{[k, l]} \otimes \text{id}_{[l+1, L]}$ for L large enough, such that $[k, l] \subset [-L+1, L]$. Then

$$\langle X \rangle_T = \lim_{L \rightarrow \infty} \text{tr}_{[-L+1, L]} \{ \rho_L^{(c)}(T) X_L \} = \text{tr}_{[k, l]} \{ D_{[k, l]}(T) X_{[k, l]} \}. \quad (6)$$

This equation may be interpreted as defining the action of $X_{[k, l]}$ on an infinite chain by formally setting

$$X = \text{id}_{(-\infty, k-1]} \otimes X_{[k, l]} \otimes \text{id}_{[l+1, \infty)}. \quad (7)$$

Every operator on the infinite chain, which can be represented like this, will be called a local operator. Let $[k, l]$ be the minimal interval for which X has a representation like (7). Then $X_{[k, l]}$ is called the non-trivial part of X , $[k, l]$ (or $\mathcal{H}_{[k, l]}$) is called its support, $\text{supp}(X)$, and $\ell(X) = \text{card}[k, l]$ its length. $X = \text{id}$ is the unique operator of length zero. Clearly the local operators on the infinite chain span a vector space \mathcal{W} .

If the system is initially represented by an ensemble with density matrix ρ_L that thermalizes in the sense of (5), the expectation value of every local operator X evolves in time to its canonical expectation value determined by a reduced density matrix,

$$\begin{aligned} \lim_{t \rightarrow \infty} \langle X(t) \rangle = \lim_{t \rightarrow \infty} \lim_{L \rightarrow \infty} \text{tr}_{[-L+1, L]} \{ \rho_L X_L(t) \} = \lim_{t \rightarrow \infty} \lim_{L \rightarrow \infty} \text{tr}_{[-L+1, L]} \{ \rho_L(t) X_L \} \\ = \text{tr}_{[k, l]} \{ D_{[k, l]}(T) X_{[k, l]} \}. \end{aligned} \quad (8)$$

We are not aware of a proof of this statement, not even for quantum spin systems on 1d lattices, but this behaviour is quite universally observed in experiments. One should have in mind however, that different equilibrium ensembles are equivalent if they produce the same reduced density matrices for $L \rightarrow \infty$.

The above mentioned cold-atom experiments suggest that thermalization will not happen with the Heisenberg time evolution $X_L \mapsto e^{iH_L t} X_L e^{-iH_L t}$ if H_L is the Hamiltonian of an integrable quantum chain such as (1). In this case a coupling to a bath, $H_L \rightarrow H_L + H_{\text{bath}}$, is required for thermalization. The relaxation of integrable lattice systems has been a subject of intensive debate over the past decade. For a review covering most of the above discussion and extending it in several directions see [17]. Some of the natural questions that arise in this context are: Do isolated integrable systems relax at all? What is the space of all reduced density matrices a given integrable system can relax to? What are reasonable classes of initial density matrices of many-body systems that can be realized in experiments? Experimentalists have provided an answer to the latter question. They can realize so-called quenches. For these the initial macro state is assumed to be represented by the projector onto the ground state sector of the Hamiltonian for a certain set of interaction parameters (like a certain value of Δ in (1)) that are then suddenly changed at initial time $t = 0$. The other questions are still not fully answered, at least not with sufficient rigour. It is believed that integrable many-body systems do relax even if they are isolated [17, 52]. There is also a certain amount of evidence that the reduced density matrices that describe the system after relaxation are related to certain generalized Gibbs ensembles [17, 45, 52].

Consider a system with N local conserved charges $\{H_L^{(n)}\}_{n=1}^N$ that mutually commute among each other, one of which, $H_L^{(1)}$ say, is its Hamiltonian. Then a natural generalization of the canonical density matrix (4) is

$$\rho_L^{(N)}(\beta_1, \dots, \beta_N) = \frac{e^{-\sum_{n=1}^N \beta_n H_L^{(n)}}}{\text{tr}_{[-L+1, L]} \{e^{-\sum_{n=1}^N \beta_n H_L^{(n)}}\}} \quad (9)$$

which satisfies a maximum entropy condition under the constraint that the ensemble averages of the conserved charges $H_L^{(n)}$ are fixed. The density matrices $\rho_L^{(N)}$ generate a sequence of reduced density matrices

$$D_{[k, l]}^{(N)}(\beta_1, \dots, \beta_N) = \lim_{L \rightarrow \infty} \text{tr}_{[-L+1, k-1] \cup [l+1, L]} \{\rho_L^{(N)}(\beta_1, \dots, \beta_N)\} \quad (10)$$

which describe the infinite system in formally the same way as $D_{[k, l]}(T)$ in the canonical case.

We shall call a quantum spin chain integrable, if its Hamiltonian H_L commutes with a commuting family of transfer matrices $t_L(\zeta)$ with spectral parameter ζ , whose local commutativity condition is the Yang-Baxter equation. Typically, for $L \rightarrow \infty$, integrable systems have infinitely many local conserved charges generated by the function $\ln t_L^{-1}(0) t_L(\zeta)$. For this reason N in (10) is often formally sent to infinity in the physics literature. This may be interpreted in the following way. Suppose $((\beta_{n, N})_{n=1}^N)_{N \in \mathbb{N}}$ is a sequence of real numbers such that the limits $\lim_{N \rightarrow \infty} \beta_{n, N} = \beta_n$ and $\lim_{N \rightarrow \infty} D_{[k, l]}^{(N)}(\beta_{1, N}, \dots, \beta_{N, N}) = D_{[k, l]}((\beta_n)_{n \in \mathbb{N}})$ exist for every fixed interval $[k, l]$. This would then define a sequence of reduced density matrices in much the same way as in the canonical case. Still, the existence of such a limit, a clear description of the space of admissible sequences $(\beta_n)_{n \in \mathbb{N}}$ and the ‘completeness of the set of local operators’ [25] are questions that will be hard to answer in full generality.

On the other hand, there is a huge class of reduced density matrices that appeared in studies of the correlation functions of integrable lattice models and is compatible with their

integrable structure. We assume that the reader is familiar with the graphical representation of vertex models (otherwise please see e.g. [21]). The generalized reduced density matrices we are referring to can be graphically represented as

$$D_{[k,l]}^{(N)}(\kappa) = \left\langle \kappa \left| \begin{array}{c} \xrightarrow{\times \kappa} \\ \xrightarrow{\times \kappa} \\ \vdots \\ \xrightarrow{\times \kappa} \\ \xrightarrow{\times \kappa} \end{array} \right. \right\rangle \times \frac{\Lambda^{k-l-1}(0|\kappa)}{\langle \kappa|\kappa \rangle}. \quad (11)$$

Here every line crossing represents a $U(1)$ -symmetric R -matrix, the crosses stand for the corresponding local ‘gauge fields’ (or twists), and $|\kappa\rangle$ is an eigenstate of the column-to-column transfer matrix with eigenvalue $\Lambda(\zeta|\kappa)$. The index ‘ N ’ refers to N horizontal spectral parameters $(\mu_{n,N})_{n=1}^N$. The construction of the generalized reduced density matrix (11) offers considerable freedom as the horizontal spectral parameters as well as their number are arbitrary. This freedom can be used to realize the reduced density matrix of the canonical ensemble [23, 24] or, with little more effort [36], reduced density matrices of the form (9) (see [19, 43] for an application to quenches in the XXZ chain). Another important class of reduced density matrices which can be represented by (11) are those connected with projectors onto eigenstates $|n\rangle$ of $t_L(\zeta)$, i.e. reduced density matrices of the form $D_{[k,l]}(L) = \text{tr}_{[-L+1, k-1] \cup [l+1, L]} \{|n\rangle\langle n|\}$.

At least for the XXZ chain, considered from now on in this work, the algebraic Bethe Ansatz [46] and Slavnov’s scalar product formula [47] can be used in order to rewrite (11) as an $(l - k + 1)$ -fold integral. This technique was pioneered for the ground state of the infinite chain in [33]. In [23, 24] it was shown that similar techniques when combined with the methodology of non-linear integral equations [35] work in the finite temperature case as well. An exploration of the case of excited states at finite length was provided in [44]. In [3, 12] it was shown that the multiple integrals factorize which makes it possible, in principle, to evaluate the short-distance correlation functions of the model. The deeper reason for the factorization was unraveled in a remarkable series of works [7–11] that culminated in the discovery of the Fermionic basis and in the proof of what we like to call the JMS theorem [29].

Note that, in general, equation (11) does not define a density matrix, since the expression on the right hand side is not necessarily Hermitian or positive. This is why we call $D_{[k,l]}^{(N)}(\kappa)$ a generalized reduced density matrix. On the other hand, it still satisfies the normalization condition $\text{tr}_{[k,l]} \{D_{[k,l]}^{(N)}(\kappa)\} = 1$ and, more generally, the reduction relations

$$\text{tr}_k \{D_{[k,l]}^{(N)}(\kappa)\} = D_{[k+1,l]}^{(N)}(\kappa), \quad \text{tr}_l \{D_{[k,l]}^{(N)}(\kappa)\} = D_{[k,l-1]}^{(N)}(\kappa). \quad (12)$$

This is enough to define generalized expectation values of local operators on the infinite chain. An interesting question would be to describe the space of all reduced density matrices of the form (11), which would mean to characterize all admissible sets of horizontal spectral parameters that render $D_{[k,l]}^{(N)}(\kappa)$ Hermitian and positive.

We have included this somewhat lengthy reflection on the equilibration of integrable quantum systems and on reduced density matrices in order to motivate some of the central notions of the Fermionic basis approach to the correlation functions of the XXZ chain that will hopefully appear rather natural now. These are the space of quasi-local operators and a further generalization of the reduced density matrices. Following [11] we define the spin operator

$$S_{[k,l]} = \frac{1}{2} \sum_{j=k}^l \sigma_j^z \in \text{End } \mathcal{H}_{[k,l]} \quad (13)$$

and its formal extension to the infinite chain

$$S(k) = S_{(-\infty,k]} \otimes \text{id}_{[k+1,\infty)} . \quad (14)$$

This allows us to deform the notion of local operators introduced in (7). We fix $\alpha \in \mathbb{C}^\times$. For any local operator X with non-trivial part $X_{[k,l]}$ we set

$$X^{(\alpha)} = q^{2\alpha S(k-1)} X . \quad (15)$$

We call $X^{(\alpha)}$ a quasi-local operator with tail $q^{2\alpha S(k-1)}$. The quasi-local operators span a vector space \mathcal{W}_α . The concepts of the non-trivial part and of the length of operators carry over from \mathcal{W} to \mathcal{W}_α . The generalized density matrix (11) can be deformed in a way that is compatible with the α -deformation of the vector space,

$$D_{[k,l]}^{(N)}(\kappa, \alpha) = \left\langle \begin{array}{c} \kappa + \alpha \\ \vdots \\ \kappa \end{array} \right\rangle \times \left(\begin{array}{c} \xrightarrow{\kappa} \\ \xleftarrow{\kappa} \\ \vdots \\ \xrightarrow{\kappa} \\ \xleftarrow{\kappa} \end{array} \right) \times \frac{\Lambda^{k-l-1}(0|\kappa)}{\langle \kappa + \alpha | \kappa \rangle} . \quad (16)$$

Here arbitrary finite-dimensional representations on the horizontal lines will be admitted [29]. The most important generalization as compared to (11) is, however, that we consider eigenstates $|\kappa\rangle$, $|\kappa + \alpha\rangle$ of twisted transfer matrices with different twists, κ and $\kappa + \alpha$, to the left and to the right. Interestingly, this object can still be represented (see appendix of [2]) as a multiple integral of the same form as for the reduced density matrix pertaining to the ground state [27, 28, 33] or to the canonical ensemble at finite temperature [24].

$D_{[k,l]}^{(N)}(\kappa, \alpha)$ can be used [29] to define generalized ensemble averages $Z^\kappa : \mathcal{W}_\alpha \rightarrow \mathbb{C}$,

$$\begin{aligned} Z^\kappa[X^{(\alpha)}] &= \lim_{L \rightarrow \infty} \frac{\text{tr}_{[-L+1,L]} \{ D_{[-L+1,L]}^{(N)}(\kappa, \alpha) q^{2\alpha S_{[-L+1,k-1]}} \otimes X_{[k,l]} \otimes \text{id}_{[l+1,L]} \}}{\text{tr}_{[-L+1,L]} \{ D_{[-L+1,L]}^{(N)}(\kappa, \alpha) q^{2\alpha S_{[-L+1,k-1]}} \otimes \text{id}_{[k,L]} \}} \\ &= \text{tr}_{[k,l]} \{ D_{[k,l]}^{(N)}(\kappa, \alpha) X_{[k,l]} \} . \quad (17) \end{aligned}$$

This clearly generalizes (6). Our above discussion shows that Z^κ can be employed to realize the canonical ensemble and ground state averages, averages with respect to excited

states of finite chains and averages with respect to generalized Gibbs ensembles including any number of local conserved charges. It is tempting to speculate that all ensembles an isolated XXZ chain can relax to have reduced density matrices that can be represented by means of Z^κ (in general a limit $N \rightarrow \infty$ will have to be considered).

The spin provides a natural grading of the space \mathcal{W}_α . For the adjoint action of the spin operator (13) we shall write $\mathbb{S}_{[k,l]} = [S_{[k,l]}, \cdot]$. By definition $X_{[k,l]} \in \text{End } \mathcal{H}_{[k,l]}$ has spin $s \in \mathbb{Z}$ if $\mathbb{S}_{[k,l]} X_{[k,l]} = s X_{[k,l]}$. This carries over to the space \mathcal{W}_α . Let $X^{(\alpha)} \in \mathcal{W}_\alpha$ with non-trivial part $X_{[k,l]}$. Define $\mathbb{S} : \mathcal{W}_\alpha \rightarrow \mathcal{W}_\alpha$ by $\mathbb{S} X^{(\alpha)} = [S(\infty), X^{(\alpha)}]$. $X^{(\alpha)} \in \mathcal{W}_\alpha$ has spin s if $\mathbb{S} X^{(\alpha)} = s X^{(\alpha)}$. The spin of $X^{(\alpha)}$ is equal to the spin of its non-trivial part. Let $\mathcal{W}_{\alpha,s} \subset \mathcal{W}_\alpha$ be the subspace of all operators with spin s . Clearly $\mathcal{W}_\alpha = \bigoplus_{s \in \mathbb{Z}} \mathcal{W}_{\alpha,s}$ and $Z^\kappa[\mathcal{W}_{\alpha,s}] = 0$ for all $s \neq 0$. Thus, only the subspace $\mathcal{W}_{\alpha,0}$ is relevant for the calculation of correlation functions.

As we shall see, the operators that generate the Fermionic basis change the tail and the spin of a quasi-local operator. They can be constructed as acting on the space

$$\mathcal{W}^{(\alpha)} = \bigoplus_{s \in \mathbb{Z}} \mathcal{W}_{\alpha-s,s} \quad (18)$$

rather than on \mathcal{W}_α . Note that $\mathcal{W}_{\alpha,0} = \mathcal{W}_\alpha \cap \mathcal{W}^{(\alpha)}$. Iterating the above definition we say that an operator $\mathbf{x} \in \text{End } \mathcal{W}^{(\alpha)}$ has spin s if $[\mathbb{S}, \mathbf{x}] = s \mathbf{x}$, and similarly in the finite case, where $\mathbf{x}_{[k,l]} \in \text{End } \text{End } \mathcal{H}_{[k,l]}$ has spin s , if $[\mathbb{S}_{[k,l]}, \mathbf{x}_{[k,l]}] = s \mathbf{x}_{[k,l]}$.

In [10, 11] the authors have constructed a module structure on $\mathcal{W}^{(\alpha)}$ that is generated by the coefficients of the following formal series,

$$\mathbf{t}^*(\zeta) = \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} \mathbf{t}_p^*, \quad (19a)$$

$$\mathbf{x}^*(\zeta) = \zeta^{\alpha s(\mathbf{x}^*)+2} \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} \mathbf{x}_p^*, \quad (19b)$$

$$\mathbf{x}(\zeta) = \zeta^{\alpha s(\mathbf{x})} \sum_{p=0}^{\infty} \frac{\mathbf{x}_p}{(\zeta^2 - 1)^p}, \quad (19c)$$

where $\mathbf{x}^* = \mathbf{b}^*, \mathbf{c}^*$ and $\mathbf{x} = \mathbf{b}, \mathbf{c}$. The coefficients $\mathbf{t}_p^*, \mathbf{x}_p^*, \mathbf{x}_p \in \text{End } \mathcal{W}^{(\alpha)}$ will be called the modes and the formal series the ‘mode expansions’ of the Fermionic operators. The modes have definite spins

$$s(\mathbf{c}_p^*) = s(\mathbf{b}_p) = -1, \quad s(\mathbf{t}_p^*) = 0, \quad s(\mathbf{b}_p^*) = s(\mathbf{c}_p) = 1. \quad (20)$$

Moreover, they exhibit the following block structure,

$$\mathbf{x}_p : \mathcal{W}_{\alpha-s,s} \rightarrow \mathcal{W}_{\alpha-s-s(\mathbf{x}_p),s+s(\mathbf{x}_p)}. \quad (21)$$

This is what we meant when we said that the operators that generate the Fermionic basis change the tail and the spin of a quasi-local operator. The effect of the modes on the length of operators is also well controlled. For any $X^{(\alpha)} \in \mathcal{W}^{(\alpha)}$ we have the inequalities

$$\ell(X^{(\alpha)}) \leq \ell(\mathbf{x}_p^* X^{(\alpha)}) \leq \ell(X^{(\alpha)}) + p, \quad \ell(\mathbf{x}_p X^{(\alpha)}) \leq \ell(X^{(\alpha)}), \quad (22)$$

where $\mathbf{x}^* = \mathbf{t}^*, \mathbf{b}^*, \mathbf{c}^*$ and $\mathbf{x} = \mathbf{b}, \mathbf{c}$. Moreover,

$$\mathbf{x}_p X^{(\alpha)} = 0 \quad (23)$$

for $\mathbf{x} = \mathbf{b}, \mathbf{c}$ if $\ell(X^{(\alpha)}) < p$, whence the name annihilation operators for \mathbf{b} and \mathbf{c} . The operators $\mathbf{x}^* = \mathbf{t}^*, \mathbf{b}^*, \mathbf{c}^*$ will be called creation operators. They owe their name to the following

Theorem 1. *Boos et al. [5]. The set*

$$\mathcal{F} = \left\{ \tau^m \mathbf{t}_{p_1}^* \dots \mathbf{t}_{p_j}^* \mathbf{b}_{q_1^+}^* \dots \mathbf{b}_{q_k^+}^* \mathbf{c}_{q_1^-}^* \dots \mathbf{c}_{q_k^-}^* q^{2\alpha S(0)} \left| \begin{array}{l} m \in \mathbb{Z}; j, k \in \mathbb{Z}_{\geq 0}; \\ p_1 \geq \dots \geq p_j \geq 2, q_1^\pm > \dots > q_k^\pm \geq 1 \end{array} \right. \right\} \quad (24)$$

is a basis of $\mathcal{W}_{\alpha,0}$. Here $\tau = \mathbf{t}_1^*/2$ is the right shift operator.

The interpretation of this theorem is that $\mathcal{W}_{\alpha,0}$ is a module generated by the action of the creation operators on a Fock vacuum $q^{2\alpha S(0)}$. The creation operators \mathbf{t}_p^* are central. They commute among themselves and with all other creation and annihilation operators. The modes $\mathbf{b}_p^*, \mathbf{b}_q$ and $\mathbf{c}_p^*, \mathbf{c}_q$ are two sets of creation and annihilation operators of Fermions. Their mutual anticommutators all vanish except for

$$[\mathbf{b}_p, \mathbf{b}_q^*]_+ = \delta_{p,q}, \quad [\mathbf{c}_p, \mathbf{c}_q^*]_+ = \delta_{p,q}. \quad (25)$$

For this reason \mathcal{F} is called the Fermionic basis.

The commutation and anticommutation rules of the modes easily follow from similar relations among the Fermionic operators.

$$[\mathbf{t}^*(\zeta_1), \mathbf{x}(\zeta_2)] = 0, \quad \mathbf{x} = \mathbf{t}^*, \mathbf{b}^*, \mathbf{c}^*, \mathbf{b}, \mathbf{c}, \quad (26)$$

while all anticommutators among $\mathbf{b}^*(\zeta), \mathbf{c}^*(\zeta), \mathbf{b}(\zeta)$ and $\mathbf{c}(\zeta)$ are vanishing, except for

$$[\mathbf{b}(\zeta_1), \mathbf{b}^*(\zeta_2)]_+ = -\psi(\zeta_2/\zeta_1, \alpha), \quad [\mathbf{c}(\zeta_1), \mathbf{c}^*(\zeta_2)]_+ = \psi(\zeta_1/\zeta_2, \alpha), \quad (27)$$

where

$$\psi(\zeta, \alpha) = \frac{\zeta^\alpha (\zeta^2 + 1)}{2(\zeta^2 - 1)}. \quad (28)$$

The proof of the commutation and anticommutation relations for the mode expansions is probably the most involved part of the construction of the Fermionic basis. It can be found in [11] and [31].

The construction of the creation and annihilation operators in [11] is rather explicit. These operators are constructed from finite building blocks whose action is inductively extended to the infinite chain. This is possible due to certain ‘reduction properties’ of the finite building blocks. We shall review the construction of these finite building blocks and the reduction properties in the following section. We shall write them in a form that is suitable for their construction by means of computer algebra programs. This will allow us to obtain explicit expressions for the fourth-neighbour two-point functions.

The most remarkable property of the creation and annihilation operators is their compatibility with the functional Z^κ described above. It reveals itself in the following fundamental theorem due to Jimbo, Miwa and Smirnov.

Theorem 2. *JMS theorem [29]. For any given realization of the functional Z^κ defined in (17) there are two functions $\rho(\zeta)$ and $\omega(\zeta, \xi; \alpha)$ such that, for every $X^{(\alpha)} \in \mathcal{W}_\alpha$,*

$$Z^\kappa \{ \mathbf{t}^*(\zeta) X^{(\alpha)} \} = 2\rho(\zeta) Z^\kappa \{ X^{(\alpha)} \}, \quad (29a)$$

$$Z^\kappa \{ \mathbf{b}^*(\zeta) X^{(\alpha+1)} \} = \int_\Gamma \frac{d\xi^2}{2\pi i \xi^2} \omega(\zeta, \xi; \alpha) Z^\kappa \{ \mathbf{c}(\xi) X^{(\alpha+1)} \}, \quad (29b)$$

$$Z^\kappa \{ \mathbf{c}^*(\zeta) X^{(\alpha-1)} \} = - \int_\Gamma \frac{d\xi^2}{2\pi i \xi^2} \omega(\xi, \zeta; \alpha) Z^\kappa \{ \mathbf{b}(\xi) X^{(\alpha-1)} \} \quad (29c)$$

where Γ is a simple closed contour around $\xi^2 = 1$.

The functions ρ and ω are described in [29]. While ρ is simply the ratio of the eigenvalues

$$\rho(\zeta) = \frac{\Lambda(\zeta|\kappa + \alpha)}{\Lambda(\zeta|\kappa)} \quad (30)$$

pertaining to the left and right eigenvectors $\langle \kappa + \alpha |$ and $|\kappa \rangle$, the definition of $\omega(\zeta, \xi; \alpha)$ in the general case is rather involved. We do not reproduce it here, but refer to [29] for the details. A characterization of ω in terms of solutions of nonlinear integral equations for the two important special cases of the canonical density matrix and of the reduced density matrix connected with the ground state of a finite system of length $2L$ was obtained in [2]. Such kind of characterization turned out to be useful in applications of the Fermionic basis approach to the description of correlation functions of quantum field theories [6, 30]. We shall recall the finite temperature case below when we consider the finite temperature short-range two-point functions.

The importance of the JMS theorem for the calculation of correlation functions reveals itself in the following two corollaries to Theorem 2.

Corollary 1. *Theorem 2 together with the commutation relations of the creation and annihilation operators implies that*

$$\begin{aligned} Z^\kappa \{ \mathbf{t}^*(\zeta_1^0) \cdots \mathbf{t}^*(\zeta_k^0) \mathbf{b}^*(\zeta_1^+) \cdots \mathbf{b}^*(\zeta_l^+) \mathbf{c}^*(\zeta_1^-) \cdots \mathbf{c}^*(\zeta_1^-) q^{2\alpha S(0)} \} \\ = \left[\prod_{i=1}^k 2\rho(\zeta_i^0) \right] \det_{m,n=1,\dots,l} \{ \omega(\zeta_m^+, \zeta_n^-; \alpha) \}. \quad (31) \end{aligned}$$

Using this corollary and the mode expansions (19) we can evaluate Z^κ on the Fermionic basis (24).

Corollary 2.

$$\begin{aligned} Z^\kappa \{ \mathbf{t}_{p_1}^* \cdots \mathbf{t}_{p_k}^* \mathbf{b}_{q_1^+}^* \cdots \mathbf{b}_{q_l^+}^* \mathbf{c}_{q_1^-}^* \cdots \mathbf{c}_{q_1^-}^* q^{2\alpha S(0)} \} = \left[\prod_{i=1}^k \frac{\partial_{(\zeta_i^0)^2}^{p_i-1} 2\rho(\zeta_i^0) |_{(\zeta_i^0)^2=1}}{(p_i-1)!} \right] \\ \times \det_{m,n=1,\dots,l} \left\{ \frac{\partial_{(\zeta_m^+)^2}^{q_m^+-1}}{(q_m^+-1)!} \frac{\partial_{(\zeta_n^-)^2}^{q_n^--1}}{(q_n^--1)!} \left(\frac{\zeta_n^-}{\zeta_m^+} \right)^\alpha \omega(\zeta_m^+, \zeta_n^-; \alpha) \right\} \Big|_{(\zeta_m^+)^2=(\zeta_n^-)^2=1}. \quad (32) \end{aligned}$$

Since \mathcal{F} is a basis of $\mathcal{W}_{\alpha,0}$ it follows that the generalized ‘ Z^κ expectation value’ of every quasi-local operator is a polynomial in $\rho(\zeta)$, $\omega(\zeta, \xi; \alpha)$ and the derivatives of these functions with respect to their arguments ζ, ξ at $\zeta = \xi = 1$. This is true, in particular, for the finite temperature and finite length ground state expectation values which require to perform the limit $\alpha \rightarrow 0$ in the end. We call this property of the physical expectation values ‘factorization’. The Fermionic basis approach and the above corollary can be understood as providing the algebraic reason for the factorization of multiple integrals observed in [3, 12].

The Fermionic basis \mathcal{F} has a number of specific properties worked out in [5]. First of all, the action of the creation operators extends the support of a given operator X only to the right, not to the left. This is why negative powers of τ are needed in (24). Consider the subspace $\mathcal{W}_{\alpha,0,[1,n]} \subset \mathcal{W}_{\alpha,0}$ of quasi-local operators $X^{(\alpha)}$ with $\text{supp}(X^{(\alpha)}) \subset [1, n]$. In [5] the authors provide an explicit construction of a Fermionic basis of this 4^n -dimensional subspace of $\mathcal{W}_{\alpha,0}$. They show that it is generated by the action of polynomials in the modes in which every term has at most n factors and contains no modes higher than the n th. A basis of $\mathcal{W}_{\alpha,0,[1,1]}$, for instance, is generated by the action of \mathbf{b}_1^* , \mathbf{c}_1^* , \mathbf{t}_1^* and id on the Fock vacuum $q^{2\alpha S(0)}$. In general, the action of finite products of modes can be represented by finite matrices. This makes it possible to calculate short-range correlation functions. The non-trivial part of the image of the Fock vacuum under a product of modes of creation operators can be represented by finite matrices which, in turn, can be written as products of operators each acting on a single ‘lattice site’ V_j . These ‘ultra-local’ operators can be represented using any basis of $\text{End } V_j$, for instance the Pauli matrices together with the 2×2 unit matrix.

Unfortunately, no closed formula for the ‘inverse problem’ of expanding products of ultra-local operators in the Fermionic basis is known. This makes the direct use of Corollary 2 for the calculation of short-range correlation functions rather inefficient. By direct calculation we were only able to proceed up to operators of lengths three [34]. A more efficient algorithm for the calculation of the coefficients of the basis transformation from the Fermionic basis to the standard basis of ultra-local operators was devised in [20]. The basic idea of this work is to exploit the freedom in the definition of the functional Z^κ . The authors consider certain particularly simple realizations of Z^κ for which they use a clever algorithm in order to directly calculate the Z^κ -expectation values of the operators under consideration. For the same realizations they can independently calculate the function ω which, according to (32), determines the expectation values of the elements of the Fermionic basis. Proceeding like this for many different realizations of Z^κ they obtain a linear (and typically overdetermined) system of equations for the expansion coefficients which can be solved on a computer. In [20, 41] this is worked out for the XXX chain in zero external field. In [48] the case of quantum group invariant operators on the XXZ chain is considered.

In this work we are going to explore another possibility to calculate short-range correlation functions by means of the Fermionic basis approach. We will be using the so-called exponential form of the density matrix introduced in [11] for the special case of the ground state correlation functions of the infinite chain. As is implicit in [29] the JMS theorem implies the validity of such a formula in a much more general situation. This had been conjectured at an early stage of the development of the method in [4] and was partially explored in [1, 50]. The difference between these older works and the present work is that the construction of the finite building blocks of the annihilation operators in the old work was based on inhomogeneous monodromy matrices, while we will be using

homogeneous monodromy matrices here. This has the advantage that we have to process smaller expressions allowing us a more efficient use of our computers.

Define the κ -trace $\text{tr}^\kappa : \mathcal{W}_\alpha \rightarrow \mathbb{C}$ by

$$\text{tr}^\kappa \{X^{(\alpha)}\} = \frac{\text{tr}_{[k,l]} \{q^{-\kappa S_{[k,l]}} X_{[k,l]}\}}{\text{tr}_{[k,l]} \{q^{-\kappa S_{[k,l]}}\}}, \quad (33)$$

where $X^{(\alpha)} \in \mathcal{W}_\alpha$ with non-trivial part $X_{[k,l]}$, and a function

$$\omega_0(\zeta; \alpha) = -\left(\frac{1 - q^\alpha}{1 + q^\alpha}\right)^2 \Delta_\zeta \psi(\zeta, \alpha). \quad (34)$$

Here we have introduced the notation $\Delta_\zeta f(\zeta) = f(q\zeta) - f(q^{-1}\zeta)$. Further, let

$$\Omega = \int_\Gamma \frac{d\zeta_1^2}{2\pi i \zeta_1^2} \int_\Gamma \frac{d\zeta_2^2}{2\pi i \zeta_2^2} (\omega_0(\zeta_1/\zeta_2; \alpha) - \omega_0(\zeta_1, \zeta_2; \alpha)) \mathbf{b}(\zeta_1) \mathbf{c}(\zeta_2), \quad (35)$$

where Γ is a simple closed contour around 1. Then it is possible to infer from the JMS theorem that, for any $X^{(\alpha)} \in \mathcal{W}_{\alpha,0}$,

$$Z^{-\alpha/2} \{X^{(\alpha)}\} = \text{tr}^\alpha \{e^\Omega X^{(\alpha)}\}. \quad (36)$$

It follows that for any realisation of the functional Z^0 and any local operator $X \in \mathcal{W}$

$$Z^0 \{X\} = \lim_{\alpha \rightarrow 0} \text{tr}^\alpha \{e^\Omega X^{(\alpha)}\}. \quad (37)$$

This is the formula that we shall use in order to calculate short range correlation functions below.

We shall argue below that (37) remains valid for $\kappa \neq 0$ if we restrict the class of operators to those that are invariant under spin reversal. We define the spin-reversal operator on $\text{End } \mathcal{H}_{[k,l]}$ by $\mathbb{J}_{[k,l]} X_{[k,l]} = J_{[k,l]} X_{[k,l]} J_{[k,l]}$, $J = \prod_{j=k}^l \sigma_j^x$. The corresponding spin-reversal operator on \mathcal{W} is denoted \mathbb{J} . $X \in \mathcal{W}$ is spin-reversal invariant if $\mathbb{J}X = X$. We claim that for such operators

$$Z^\kappa \{X\} = \lim_{\alpha \rightarrow 0} \text{tr}^\alpha \{e^\Omega X^{(\alpha)}\}. \quad (38)$$

Here the κ -dependence on the right hand side is hidden in Ω which depends on κ through the function ω .

The paper is organized as follows. In Section 2 we explain the construction of the modes from finite building blocks. Starting point are two operators $\mathbf{k}_{[k,l]}$ and $\mathbf{t}_{[k,l]}^*$ acting on finite chains and their partial fraction decompositions. From these we obtain finite chain versions of the creation and annihilation operators which we present explicitly in terms of the Laurent coefficients of $\mathbf{k}_{[k,l]}$ and $\mathbf{t}_{[k,l]}^*$ and in terms of coefficients characterizing the behaviour of these operators in the limit when the spectral parameter goes to infinity. We also discuss the reduction properties of the operators which allows us to construct operators for the infinite chain from finite building blocks. In Section 3 we apply the formalism to work out the finite-temperature two-point correlation functions of the XXZ chain on up to five lattice sites. We discuss these functions and compare with known asymptotic results. Section 4 is devoted to a short summary and to our conclusions. In Appendix A we obtain the exponential form starting from the JMS theorem. Appendix B connects two different forms of the physical part ρ, ω with multiplicative and with additive spectral parameters. Appendix C lists the expressions for the fourth-neighbour two-point functions in terms of ρ and ω .

2 Construction of the modes from finite building blocks

2.1 L -matrices and monodromy matrices

The creation and annihilation operators introduced in the previous section are constructed from weighted traces of the elements of certain monodromy matrices related to $U_q(\widehat{\mathfrak{sl}}_2)$. These monodromy matrices are products of two types of L -matrices with two-dimensional or infinite-dimensional auxiliary space.

The L -matrices with two-dimensional auxiliary space are directly related to the R -matrix of the six-vertex model,

$$R(\zeta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta(\zeta) & \gamma(\zeta) & 0 \\ 0 & \gamma(\zeta) & \beta(\zeta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (39)$$

where

$$\beta(\zeta) = \frac{(1 - \zeta^2)q}{1 - q^2\zeta^2}, \quad \gamma(\zeta) = \frac{(1 - q^2)\zeta}{1 - q^2\zeta^2}. \quad (40)$$

Fixing an auxiliary space $V_a \cong \mathbb{C}^2$ we define $L_{a,j}(\zeta) = \varrho(\zeta)R_{a,j}(\zeta) \in \text{End}(V_a \otimes \mathcal{H}_{[k,l]})$, where $\varrho(\zeta)$ is a scalar factor to be specified below. This is the standard L -matrix of the six-vertex model. The corresponding monodromy matrix is

$$T_{a,[k,l]}(\zeta) = L_{a,l}(\zeta) \dots L_{a,k}(\zeta). \quad (41)$$

We are going to construct operators acting on $\text{End } \mathcal{H}_{[k,l]}$ or on $\text{End}(V_a \otimes \mathcal{H}_{[k,l]})$. Our first example is the L -operator $\mathbb{L}_{a,j}(\zeta)$ defined by

$$\mathbb{L}_{a,j}(\zeta) \text{id}_a \otimes X_{[k,l]} = L_{a,j}(\zeta) \text{id}_a \otimes X_{[k,l]} L_{a,j}^{-1}(\zeta) \quad (42)$$

for $j \in \{k, \dots, l\}$. These L -operators generate the first type of monodromy matrices needed for the construction of the Fermionic operators,

$$\mathbb{T}_{a,[k,l]}(\zeta, \alpha) \text{id}_a \otimes X_{[k,l]} = \mathbb{L}_{a,l}(\zeta) \dots \mathbb{L}_{a,k}(\zeta) q^{\alpha\sigma_a^z} \text{id}_a \otimes X_{[k,l]}. \quad (43)$$

A second type of monodromy matrix acts adjointly on an infinite dimensional module of the q -oscillator algebra Osc which is defined in terms of generators \mathbf{a} , \mathbf{a}^* , $q^{\pm D}$ that satisfy the relations

$$q^D \mathbf{a} q^{-D} = q^{-1} \mathbf{a}, \quad q^D \mathbf{a}^* q^{-D} = q \mathbf{a}^*, \quad \mathbf{a} \mathbf{a}^* = 1 - q^{2D+2}, \quad \mathbf{a}^* \mathbf{a} = 1 - q^{2D}. \quad (44)$$

We fix a copy Osc_A of Osc and define

$$L_{A,j}(\zeta) = \sigma(\zeta) \begin{pmatrix} 1 - \zeta^2 q^{2D_A+2} & -\zeta \mathbf{a}_A \\ -\zeta \mathbf{a}_A^* & 1 \end{pmatrix}_j \begin{pmatrix} q^{-D_A} & 0 \\ 0 & q^{D_A} \end{pmatrix}_j \quad (45)$$

which acts on $\text{Osc}_A \otimes \mathcal{H}_{[k,l]}$. The scalar factor $\sigma(\zeta)$ is a convenient normalization and is required to solve the functional equation

$$\sigma(\zeta) \sigma(q^{-1}\zeta) = \frac{1}{1 - \zeta^2}. \quad (46)$$

It also fixes the scalar factor in the definition of $L_{a,j}(\zeta)$,

$$\varrho(\zeta) = \frac{q^{-1/2}\sigma(q^{-1}\zeta)}{\sigma(\zeta)}. \quad (47)$$

For the L -operator (45) we define its adjoint action on $\text{End}(\text{Osc}_A \otimes \mathcal{H}_{[k,l]})$

$$\mathbb{L}_{A,j}(\zeta) \text{id}_A \otimes X_{[k,l]} = L_{A,j}(\zeta) \text{id}_A \otimes X_{[k,l]} L_{A,j}^{-1}(\zeta), \quad (48)$$

$j \in \{k, \dots, l\}$, and the corresponding monodromy matrix

$$\mathbb{T}_{A,[k,l]}(\zeta, \alpha) \text{id}_A \otimes X_{[k,l]} = \mathbb{L}_{A,l}(\zeta) \dots \mathbb{L}_{A,k}(\zeta) q^{2\alpha D_A} \text{id}_A \otimes X_{[k,l]}. \quad (49)$$

The monodromy matrices (43) and (49) are the basic ingredients in the definition of the Fermionic operators.

2.2 The basic operators on finite chain segments

The Fermionic operators and their mode expansions (19) are constructed from a parental operator $\mathbf{k}_{[k,l]}(\zeta, \alpha) \in \text{End End } \mathcal{H}_{[k,l]}$ and from an adjointly acting transfer matrix $\mathbf{t}_{[k,l]}^*(\zeta, \alpha) \in \text{End End } \mathcal{H}_{[k,l]}$ which were both introduced in [11]. Before recalling their definitions we have to fix some notation. Throughout this section we shall fix an operator $X_{[k,l]} \in \text{End } \mathcal{H}_{[k,l]}$ of spin s and length $n \leq k - l + 1$. For such an operator we necessarily have that $|s| \leq n$. If $\mathbf{x}_{[k,l]} \in \text{End End } \mathcal{H}_{[k,l]}$ is acting on $X_{[k,l]}$, we omit the subscript and write $\mathbf{x}X_{[k,l]} = \mathbf{x}_{[k,l]}X_{[k,l]}$ for simplicity.

With this convention the parental operator $\mathbf{k}_{[k,l]}(\zeta, \alpha)$ is defined by

$$\mathbf{k}(\zeta, \alpha) X_{[k,l]} = \text{tr}_{A,a} \left\{ \sigma_a^+ \mathbb{T}_a(\zeta, \alpha) \mathbb{T}_A(\zeta, \alpha) \zeta^{\alpha - S} (\text{id}_{A,a} \otimes q^{-2S_z^z} X_{[k,l]}) \right\}. \quad (50)$$

The trace over the oscillator algebra is to be understood in such a way that it agrees with the trace over the module $W^+ = \bigoplus_{k \geq 0} \mathbb{C}|k\rangle$ for $|q| < 1$ (with $q^D|k\rangle = q^k|k\rangle$, $\mathbf{a}|k\rangle = (1 - q^{2k})|k - 1\rangle$, $\mathbf{a}^*|k\rangle = |k + 1\rangle$). For details see appendix A of [11].

The operator $\mathbf{t}_{[k,l]}^*(\zeta, \alpha)$ is defined by

$$\mathbf{t}^*(\zeta, \alpha) X_{[k,l]} = \text{tr}_a \left\{ \mathbb{T}_a(\zeta, \alpha) (\text{id}_a \otimes X_{[k,l]}) \right\}. \quad (51)$$

2.3 Partial fraction decomposition of $\mathbf{k}_{[k,l]}(\zeta, \alpha)$ and the annihilation operators

In the definition of the monodromy operators (43), (49) we could have included an inhomogeneity parameter on each site if we would have replaced ζ by ζ/ξ_j in every factor on the right hand side of these equations (cf. [11]). This would have affected the analytic properties of the operators $\mathbf{k}_{[k,l]}(\zeta, \alpha)$ and $\mathbf{t}_{[k,l]}^*(\zeta, \alpha)$ which have only simple poles in the inhomogeneous case, if the inhomogeneities are mutually distinct, but exhibit poles of higher order in the homogeneous case. In our previous work [1] we considered the inhomogeneous case. Here we restrict ourselves to the homogeneous case. This means that we have to deal with slightly more complicated expressions which depend, however, on less parameters. The latter fact renders them more efficient in computer algebraic calculations.

We will express the Fermionic annihilation operator $\mathbf{c}_{[k,l]}(\zeta, \alpha)$ and some other operators which are important in the construction of the creation operators in terms of the Laurent coefficients of the operator $\mathbf{k}_{\text{skal}[k,l]}(\zeta, \alpha)$ defined by $\mathbf{k}_{\text{skal}}(\zeta, \alpha)X_{[k,l]} = \zeta^{-\alpha-s-1}\mathbf{k}(\zeta, \alpha)X_{[k,l]}$. As can be seen from section 2.5 of [11], $\mathbf{k}_{\text{skal}}(\zeta, \alpha)X_{[k,l]}$ is a rational function in ζ^2 . As a function of ζ^2 it has three n -fold poles at $q^{2\epsilon}$, $\epsilon = 0, \pm 1$, and an s -fold pole at 0 if $s > 0$. From these properties and the asymptotic behaviour we obtain the partial fraction decomposition

$$\mathbf{k}_{\text{skal}}(\zeta, \alpha)X_{[k,l]} = \left[\sum_{j=1}^n \sum_{\epsilon=-1}^1 \frac{\rho_j^{(\epsilon)}(\alpha)}{(\zeta^2 - q^{2\epsilon})^j} + \sum_{j=1}^s \frac{\kappa_j(\alpha)}{\zeta^{2j}} \right] X_{[k,l]}. \quad (52)$$

The definitions of the operators $\bar{\mathbf{c}}_{[k,l]}(\zeta, \alpha)$, $\mathbf{c}_{[k,l]}(\zeta, \alpha)$ and $\mathbf{f}_{[k,l]}(\zeta, \alpha)$ in [11] remain valid in the homogeneous case. They are introduced in such a way that the decomposition

$$\mathbf{k}(\zeta, \alpha)X_{[k,l]} = (\bar{\mathbf{c}}(\zeta, \alpha) + \mathbf{c}(q\zeta, \alpha) + \mathbf{c}(q^{-1}\zeta, \alpha) + \mathbf{f}(q\zeta, \alpha) - \mathbf{f}(q^{-1}\zeta, \alpha))X_{[k,l]} \quad (53)$$

holds, which becomes unique if we fix

$$\bar{\mathbf{c}}(\zeta, \alpha)X_{[k,l]} = \int_{\Gamma} \frac{d\xi^2}{2\pi i} \psi(\zeta/\xi, \alpha + s + 1) \mathbf{k}(\xi, \alpha) X_{[k,l]}, \quad (54a)$$

$$\mathbf{c}(\zeta, \alpha)X_{[k,l]} = \int_{\Gamma} \frac{d\xi^2}{4\pi i} \psi(\zeta/\xi, \alpha + s + 1) (\mathbf{k}(q\xi, \alpha) + \mathbf{k}(q^{-1}\xi, \alpha)) X_{[k,l]}, \quad (54b)$$

$$\mathbf{f}(\zeta, \alpha)X_{[k,l]} = (\mathbf{f}^{\text{sing}}(\zeta, \alpha) + \mathbf{f}^{\text{reg}}(\zeta, \alpha))X_{[k,l]}, \quad (54c)$$

$$\mathbf{f}^{\text{sing}}(\zeta, \alpha)X_{[k,l]} = \int_{\Gamma} \frac{d\xi^2}{4\pi i} \psi(\zeta/\xi, \alpha + s + 1) (-\mathbf{k}(q\xi, \alpha) + \mathbf{k}(q^{-1}\xi, \alpha)) X_{[k,l]}. \quad (54d)$$

Here Γ is a small circle around $\xi^2 = 1$. It is not difficult to evaluate the above integrals and to obtain the operators $\bar{\mathbf{c}}_{[k,l]}(\zeta, \alpha)$, $\mathbf{c}_{[k,l]}(\zeta, \alpha)$ and $\mathbf{f}_{[k,l]}^{\text{sing}}(\zeta, \alpha)$ explicitly in terms of the Laurent coefficients $\rho_j^{(\epsilon)}(\alpha)$ and $\kappa_j(\alpha)$ occurring in the partial fraction decomposition of $\mathbf{k}_{\text{skal}[k,l]}(\zeta, \alpha)$. We have to deal with integrands of the form

$$\begin{aligned} & \xi^{-2} \psi(\zeta/\xi, \alpha + s + 1) \mathbf{k}(\eta, \alpha) X_{[k,l]} \\ &= \zeta^{\alpha+s+1} \frac{\zeta^2 + \xi^2}{2\xi^2(\zeta^2 - \xi^2)} \left(\frac{\eta}{\xi} \right)^{\alpha+s+1} \mathbf{k}_{\text{skal}}(\eta, \alpha) X_{[k,l]}. \end{aligned} \quad (55)$$

with $\eta = \xi q^\epsilon$, $\epsilon = -1, 0, 1$. Using the decomposition

$$\frac{\zeta^2 + \xi^2}{2\xi^2(\zeta^2 - \xi^2)} = \frac{1}{2} \sum_{k=0}^{\infty} (-1)^k (\xi^2 - 1)^k + \frac{1}{\zeta^2 - 1} \sum_{k=0}^{\infty} \left(\frac{\xi^2 - 1}{\zeta^2 - 1} \right)^k, \quad (56)$$

valid for $|\xi^2 - 1| < |\zeta^2 - 1| < 1$, the integrals are easily calculated. The remaining part $\mathbf{f}_{[k,l]}^{\text{reg}}(\zeta, \alpha)$ is obtained by inverting the difference operator Δ_ζ on the space of Laurent polynomials in ζ^2 .

Lemma 1. *The operators $\bar{\mathbf{c}}_{[k,l]}(\zeta, \alpha)$, $\mathbf{c}_{[k,l]}(\zeta, \alpha)$ and $\mathbf{f}_{[k,l]}(\zeta, \alpha)$ have the partial fraction decompositions*

$$\bar{\mathbf{c}}(\zeta, \alpha)X_{[k,l]} = \zeta^{\alpha+s+1} \sum_{j=0}^n \frac{\bar{c}_j(\alpha)}{(\zeta^2 - 1)^j} X_{[k,l]}, \quad (57a)$$

$$\mathbf{c}(\zeta, \alpha)X_{[k,l]} = \zeta^{\alpha+s+1} \sum_{j=0}^n \frac{c_j(\alpha)}{(\zeta^2 - 1)^j} X_{[k,l]}, \quad (57b)$$

$$\mathbf{f}(\zeta, \alpha)X_{[k,l]} = \zeta^{\alpha+s+1} \left[\sum_{j=0}^n \frac{f_j(\alpha)}{(\zeta^2 - 1)^j} + \sum_{j=0}^s \frac{\kappa_j(\alpha)\zeta^{-2j}}{q^{\alpha+s+1-2j} - q^{2j-\alpha-s-1}} \right] X_{[k,l]}, \quad (57c)$$

where the coefficients are determined in terms of the coefficients of the partial fraction decomposition of $\mathbf{k}_{\text{skal},[k,l]}(\zeta, \alpha)$,

$$\bar{c}_j(\alpha) = \rho_j^{(0)}(\alpha), \quad j = 1, \dots, n, \quad (58a)$$

$$\bar{c}_0(\alpha) = \frac{1}{2} \sum_{j=1}^n (-1)^{j-1} \bar{c}_j(\alpha), \quad (58b)$$

$$c_j(\alpha) = \frac{1}{2} (q^{2j-\alpha-s-1} \rho_j^{(-)}(\alpha) + q^{\alpha+s+1-2j} \rho_j^{(+)}(\alpha)), \quad j = 1, \dots, n, \quad (58c)$$

$$c_0(\alpha) = \frac{1}{2} \sum_{j=1}^n (-1)^{j-1} c_j(\alpha), \quad (58d)$$

$$f_j(\alpha) = \frac{1}{2} (q^{2j-\alpha-s-1} \rho_j^{(-)}(\alpha) - q^{\alpha+s+1-2j} \rho_j^{(+)}(\alpha)), \quad j = 1, \dots, n, \quad (58e)$$

$$f_0(\alpha) = \frac{1}{2} \sum_{j=1}^n (-1)^{j-1} f_j(\alpha), \quad (58f)$$

$$\kappa_0(\alpha) = \frac{1}{2} \sum_{j=1}^n (-1)^j \sum_{\epsilon=-1}^1 q^{-2\epsilon j} \rho_j^{(\epsilon)}(\alpha). \quad (58g)$$

The operator $\mathbf{b}_{[k,l]}(\zeta, \alpha)$ that is needed in the construction of the density matrix is defined as

$$\mathbf{b}_{[k,l]}(\zeta, \alpha) = -q^{-1} (q^{\alpha-\mathbb{S}_{[k,l]}-1} - q^{-\alpha+\mathbb{S}_{[k,l]}+1}) \mathbb{J}_{[k,l]} \mathbf{c}_{[k,l]}(\zeta, -\alpha) \mathbb{J}_{[k,l]}. \quad (59)$$

Setting

$$b_j(\alpha) = -q^{-1} (q^{\alpha-\mathbb{S}_{[k,l]}-1} - q^{-\alpha+\mathbb{S}_{[k,l]}+1}) \mathbb{J}_{[k,l]} c_j(-\alpha) \mathbb{J}_{[k,l]} \quad (60)$$

we obtain it in the form

$$\mathbf{b}(\zeta, \alpha)X_{[k,l]} = \zeta^{-\alpha-s+1} \sum_{j=0}^n \frac{b_j(\alpha)}{(\zeta^2 - 1)^j} X_{[k,l]}. \quad (61)$$

All one has to do in order to obtain explicit representations for these operators is to extract the coefficients $\rho_j^{(\epsilon)}(\alpha)$ and $\kappa_j(\alpha)$ from the expansion (52). They can be obtained as

$$\rho_{n-k}^{(\epsilon)}(\alpha) = \lim_{\zeta^2 \rightarrow q^{2\epsilon}} \frac{1}{k!} \partial_{\zeta^2}^k (\zeta^2 - q^{2\epsilon})^n \mathbf{k}_{\text{skal},[k,l]}(\zeta, \alpha) \quad (62)$$

for $k = 0, \dots, n-1$, and

$$\kappa_{s-k}^{(\epsilon)}(\alpha) = \lim_{\zeta^2 \rightarrow 0} \frac{1}{k!} \partial_{\zeta^2}^k \zeta^{2s} \mathbf{k}_{\text{skal},[k,l]}(\zeta, \alpha), \quad (63)$$

where $k = 0, \dots, s-1$.

2.4 Partial fraction decomposition of the operator $\mathbf{t}^*(\zeta, \alpha)$ in the homogeneous case

In section 2.5 of [11] it is shown that $\mathbf{t}_{[k,l]}^*(\zeta, \alpha)$ is a rational function in ζ^2 . Directly from its definition we further find that for $\zeta^2 \rightarrow \infty$

$$\mathbf{t}^*(\zeta, \alpha)X_{[k,l]} \sim (q^{\alpha+s} + q^{-\alpha-s})X_{[k,l]} \quad (64)$$

and that this operator has precisely two n -fold poles at $\zeta^2 = q^{\pm 2}$. Thus, it has the partial fraction decomposition

$$\mathbf{t}^*(\zeta, \alpha)X_{[k,l]} = \left[\sum_{j=1}^n \sum_{\epsilon=\pm} \frac{\tau_j^{(\epsilon)}(\alpha)}{(\zeta^2 - q^{\epsilon 2})^j} + q^{\alpha+s} + q^{-\alpha-s} \right] X_{[k,l]}. \quad (65)$$

2.5 Construction of the finite generators of creation operators

The construction of creation operators is intimately connected with the Taylor expansion of $\mathbf{t}_{[k,l]}^*(\zeta, \alpha)$ around $\zeta^2 = 1$.

Lemma 2. *Taylor expanding (65) around $\zeta^2 = 1$ we obtain*

$$\mathbf{t}^*(\zeta, \alpha)X_{[k,l]} = \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} t_p^*(\alpha) X_{[k,l]}, \quad (66)$$

where

$$t_1^*(\alpha)X_{[k,l]} = \left[\sum_{j=1}^n \sum_{\epsilon=\pm} \frac{\tau_j^{(\epsilon)}(\alpha)}{(1 - q^{\epsilon 2})^j} + q^{\alpha+s} + q^{-\alpha-s} \right] X_{[k,l]}, \quad (67a)$$

$$t_p^*(\alpha)X_{[k,l]} = \sum_{j=1}^n \sum_{\epsilon=\pm} \binom{-j}{p-1} \frac{\tau_j^{(\epsilon)}(\alpha)X_{[k,l]}}{(1 - q^{\epsilon 2})^{p+j-1}}, \quad p = 2, 3, \dots \quad (67b)$$

We now proceed with the construction of the Fermionic creation operators. In equation (2.29) of [11] their finite generators $\mathbf{b}_{[k,l]}^*(\zeta, \alpha)$ are defined by

$$\mathbf{b}^*(\zeta, \alpha)X_{[k,l]} = (\mathbf{f}(q\zeta, \alpha) + \mathbf{f}(q^{-1}\zeta, \alpha) - \mathbf{t}^*(\zeta, \alpha)\mathbf{f}(\zeta, \alpha))X_{[k,l]}. \quad (68)$$

Here $\mathbf{f}_{[k,l]}(\zeta, \alpha)$ is given by (57c) and $\mathbf{t}_{[k,l]}^*(\zeta, \alpha)$ by (66).

It is shown in [11] (see Lemma 3.8) that the operator $\mathbf{b}_{[k,l]}^*(\zeta, \alpha)$ is regular at $\zeta^2 = 1$. Inserting (57c) and (66) into (68) regularity implies the identities

$$\sum_{j=-p+1}^n t_{p+j}^*(\alpha) f_j(\alpha) = 0 \quad \text{for } p = -n + 1, \dots, 0. \quad (69)$$

Using this identity in the Taylor expansion of the right hand side of (68) around $\zeta^2 = 1$ we obtain the following

Lemma 3.

$$\mathbf{b}^*(\zeta, \alpha)X_{[k,l]} = \zeta^{\alpha+s+1} \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} b_p^*(\alpha) X_{[k,l]}, \quad (70)$$

where

$$\begin{aligned} b_p^*(\alpha)X_{[k,l]} = & \left[\sum_{j=0}^n \binom{-j}{p-1} \frac{q^{\alpha+s+p-j} - (-1)^{p+j} q^{j-p-\alpha-s}}{(q - q^{-1})^{p+j-1}} f_j(\alpha) \right. \\ & + \sum_{j=0}^s \binom{-j}{p-1} \frac{q^{\alpha+s+1-2j} + q^{2j-\alpha-s-1}}{q^{\alpha+s+1-2j} - q^{2j-\alpha-s-1}} \kappa_j(\alpha) \\ & \left. - \sum_{j=0}^n t_{p+j}^*(\alpha) f_j(\alpha) - \sum_{j=0}^s \sum_{k=0}^{p-1} \binom{-j}{k} \frac{t_{p-k}^*(\alpha) \kappa_j(\alpha)}{q^{\alpha+s+1-2j} - q^{2j-\alpha-s-1}} \right] X_{[k,l]}. \quad (71) \end{aligned}$$

The operator $\mathbf{c}_{[k,l]}^*(\zeta, \alpha)$ is defined as

$$\mathbf{c}_{[k,l]}^*(\zeta, \alpha) = q^{-1} (q^{\alpha - \mathbb{S}_{[k,l]} - 1} - q^{-\alpha + \mathbb{S}_{[k,l]} + 1}) \mathbb{J}_{[k,l]} \mathbf{b}_{[k,l]}^*(\zeta, -\alpha) \mathbb{J}_{[k,l]}. \quad (72)$$

Setting

$$c_p^*(\alpha) = q^{-1} (q^{\alpha - \mathbb{S}_{[k,l]} - 1} - q^{-\alpha + \mathbb{S}_{[k,l]} + 1}) \mathbb{J}_{[k,l]} b_p^*(-\alpha) \mathbb{J}_{[k,l]} \quad (73)$$

we obtain the Taylor expansion

$$\mathbf{c}^*(\zeta, \alpha)X_{[k,l]} = \zeta^{-\alpha-s+1} \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} c_p^*(\alpha) X_{[k,l]}. \quad (74)$$

In the following subsection we will use certain reduction properties of the operators acting on finite segments of the infinite spin chain in order to construct operators that act on the whole infinite chain.

2.6 Reduction relations and extension of the action to infinite chains

The family of operators acting on $\text{End } \mathcal{H}_{[j,m]}$ that was introduced above has certain simple so-called ‘reduction properties’ describing how their action on operators of the form $q^{\alpha(\sigma_j^z + \dots + \sigma_{k-1}^z)} \text{id}_{[j,k-1]} \otimes X_{[k,l]} \otimes \text{id}_{[l+1,m]}$, where $[k, l] \subset [j, m]$ is a proper subset, reduces to the action of ‘shorter operators’. The operators $\mathbf{k}_{[k,l]}$ and $\mathbf{t}_{[k,l]}^*$ inherit the following left reduction relations from the ‘gauge invariance’ of the universal R -matrix

$$\mathbf{t}_{[k-1,l]}^*(\zeta, \alpha) (q^{\alpha\sigma^z} \otimes X_{[k,l]}) = q^{\alpha\sigma^z} \otimes (\mathbf{t}_{[k,l]}^*(\zeta, \alpha) X_{[k,l]}), \quad (75)$$

$$\mathbf{k}_{[k-1,l]}(\zeta, \alpha) (q^{(\alpha+1)\sigma^z} \otimes X_{[k,l]}) = q^{\alpha\sigma^z} \otimes (\mathbf{k}_{[k,l]}(\zeta, \alpha) X_{[k,l]}). \quad (76)$$

These properties can be easily understood by means of the graphical representation of the operators. As an immediate consequence we obtain the left reduction relations

$$\mathbf{x}_{[k-1,l]}(\zeta, \alpha) (q^{(\alpha+s(\mathbf{x}))\sigma^z} \otimes X_{[k,l]}) = q^{\alpha\sigma^z} \otimes (\mathbf{x}_{[k,l]}(\zeta, \alpha) X_{[k,l]}), \quad (77)$$

where $\mathbf{x} = \mathbf{b}, \mathbf{b}^*, \mathbf{c}, \mathbf{c}^*, \mathbf{t}^*$, and where $s(\mathbf{x})$ denotes the spin of \mathbf{x} which, as we recall, is -1 for \mathbf{b}, \mathbf{c}^* , $+1$ for \mathbf{c}, \mathbf{b}^* and 0 for \mathbf{t}^* .

The left reduction relations can be used to extend the action of the operators $\mathbf{x}_{[k,l]}(\zeta, \alpha)$ on products of local operators with tail inductively to the semi-infinite interval $(-\infty, l]$. Iterating (77) we obtain

$$\mathbf{x}_{(-\infty, l]}(\zeta, \alpha)(q^{2(\alpha+s(\mathbf{x}))S_{(-\infty, k-1]}} \otimes X_{[k,l]}) = q^{2\alpha S_{(-\infty, k-1]}} \otimes (\mathbf{x}_{[k,l]}(\zeta, \alpha)X_{[k,l]}). \quad (78)$$

An extension of these operators to the right is less obvious and is essentially different for the creation and the annihilation operators.

The annihilation operators have the property that they do not enlarge, in particular not to the right, the support of an operator $X_{[k,l]}$ they are acting on,

$$\mathbf{x}_{[k, l+m]}(\zeta, \alpha)(X_{[k,l]} \otimes \text{id}_{[l+1, l+m]}) = (\mathbf{x}_{[k,l]}(\zeta, \alpha)X_{[k,l]}) \otimes \text{id}_{[l+1, l+m]} \quad (79)$$

for $\mathbf{x} = \mathbf{b}, \mathbf{c}$ and $m \in \mathbb{N}$. The proof is non-trivial and can be found in section 3.5 of [11]. Since (79) is valid for every $m \in \mathbb{N}$ we may use this equation to extend the action of the operators $\mathbf{x} = \mathbf{b}, \mathbf{c}$ to a semi-infinite interval, setting

$$\mathbf{x}_{[k, \infty)}(\zeta, \alpha)(X_{[k,l]} \otimes \text{id}_{[l+1, \infty)}) = (\mathbf{x}_{[k,l]}(\zeta, \alpha)X_{[k,l]}) \otimes \text{id}_{[l+1, \infty)}. \quad (80)$$

By way of contrast the creation operators extend the support of any operator $X_{[k,l]}$ indefinitely to the right. Yet, interestingly, their modes do not. Every mode x_p , $x = b^*, c^*, t^*$, in the expansions (66), (70) and (74) extends the length of an operator $X_{[k,l]}$ at most by p to the right. For $\mathbf{t}_{[k,l]}^*$ a relatively simple and elegant proof of this fact is presented in section 3.4 of [11]. The authors derive an expansion of the form

$$\begin{aligned} \mathbf{t}_{[k, l+m]}^*(\zeta, \alpha)(X_{[k,l]} \otimes \text{id}_{[l+1, l+m]}) \\ = \sum_{p=1}^m (\mathbf{y}_{[k, l+p]}^{(p)}(\zeta, \alpha)(X_{[k,l]} \otimes \text{id}_{[l+1, l+p]})) \otimes \text{id}_{[l+p+1, l+m]} \\ + \mathbf{z}_{[k, l+m]}^{(m)}(\zeta, \alpha)(X_{[k,l]} \otimes \text{id}_{[l+1, l+m]}). \end{aligned} \quad (81)$$

From the explicit form of the operators $\mathbf{y}_{[k, l+p]}^{(p)}$ and $\mathbf{z}_{[k, l+m]}^{(m)}$ we can read off the following features. The $\mathbf{y}_{[k, l+p]}^{(p)}(\zeta, \alpha)(X_{[k,l]} \otimes \text{id}_{[l+1, l+p]})$ have support $[k, l+p]$, and $\mathbf{z}_{[k, l+m]}^{(m)}(\zeta, \alpha)(X_{[k,l]} \otimes \text{id}_{[l+1, l+m]})$ has support $[k, l+m]$. These operators depend rationally on ζ^2 . Close to $\zeta^2 = 1$ they behave as $\mathbf{y}_{[k, l+p]}^{(p)}(\zeta, \alpha) = \mathcal{O}((\zeta^2 - 1)^{p-1})$ and $\mathbf{z}_{[k, l+m]}^{(m)}(\zeta, \alpha) = \mathcal{O}((\zeta^2 - 1)^m)$. This implies that $\mathbf{y}_{[k, l+r]}^{(r)}(\zeta, \alpha)(X_{[k,l]} \otimes \text{id}_{[l+1, l+m]})$ for $r > p$ does not contribute to the first p terms of the Taylor expansion (66) of $\mathbf{t}_{[k, l+r]}^*(\zeta, \alpha)(X_{[k,l]} \otimes \text{id}_{[l+1, l+m]})$ in ζ^2 around 1. In other words the p th coefficient $t_{p, [k, l+m]}^*(\alpha)$ of the Taylor expansion extends the support at most by p to the right,

$$\begin{aligned} t_{p, [k, l+m]}^*(\alpha)(X_{[k,l]} \otimes \text{id}_{[l+1, l+m]}) \\ = (t_{p, [k, l+p]}^*(\alpha)(X_{[k,l]} \otimes \text{id}_{[l+1, l+p]})) \otimes \text{id}_{[l+p+1, l+m]} \end{aligned} \quad (82)$$

for $p = 1, \dots, m$. We may also read this as being valid for fixed p and every $m \geq p$. Since the first factor on the right hand side of this equation is independent of m , we may use (82) to extend the action of the modes infinitely to the right, setting

$$t_{p, [k, \infty)}^*(\alpha)(X_{[k,l]} \otimes \text{id}_{[l+1, \infty)}) = (t_{p, [k, l+p]}^*(\alpha)(X_{[k,l]} \otimes \text{id}_{[l+1, l+p]})) \otimes \text{id}_{[l+p+1, \infty)}. \quad (83)$$

As for the Fermionic creation operators it follows from (73) that the modes of \mathbf{c}^* have the same reduction relations as those of \mathbf{b}^* . The latter can be obtained from a similar argument as above by combining Lemma 3.1 and Lemma 3.7 of [11]. These lemmata imply that

$$\begin{aligned} & \mathbf{b}_{[k,l+m]}^*(\zeta, \alpha) (X_{[k,l]} \otimes \text{id}_{[l+1,l+m]}) \\ &= \zeta^{\alpha+s+1} \left[\sum_{p=1}^m (\mathbf{u}_{[k,l+p]}^{(p)}(\zeta, \alpha) (X_{[k,l]} \otimes \text{id}_{[l+1,l+p]}) \otimes \text{id}_{[l+p+1,l+m]} \right. \\ & \quad \left. + \mathbf{v}_{[k,l+m]}^{(m)}(\zeta, \alpha) (X_{[k,l]} \otimes \text{id}_{[l+1,l+m]}) \right], \end{aligned} \quad (84)$$

where $\mathbf{u}_{[k,l+p]}^{(p)}(\zeta, \alpha)$, $p = 1, \dots, m$, and $\mathbf{v}_{[k,l+m]}^{(m)}(\zeta, \alpha)$ have the same properties as the operators $\mathbf{y}_{[k,l+p]}^{(p)}(\zeta, \alpha)$ and $\mathbf{z}_{[k,l+m]}^{(m)}(\zeta, \alpha)$ introduced above. A comparison with (70) then implies that

$$\begin{aligned} & b_{p,[k,l+m]}^*(\alpha) (X_{[k,l]} \otimes \text{id}_{[l+1,l+m]}) \\ &= (b_{p,[k,l+p]}^*(\alpha) (X_{[k,l]} \otimes \text{id}_{[l+1,l+p]})) \otimes \text{id}_{[l+p+1,l+m]} \end{aligned} \quad (85)$$

for all $m \geq p$, and therefore also

$$\begin{aligned} & c_{p,[k,l+m]}^*(\alpha) (X_{[k,l]} \otimes \text{id}_{[l+1,l+m]}) \\ &= (c_{p,[k,l+p]}^*(\alpha) (X_{[k,l]} \otimes \text{id}_{[l+1,l+p]})) \otimes \text{id}_{[l+p+1,l+m]} \end{aligned} \quad (86)$$

for all $m \geq p$. Eqs. (85) and (86) allow us to extend the action of the modes on local operators infinitely to the right (or, turning it the other way round, to define the action of every mode on a local operator in terms of finite matrices),

$$\begin{aligned} & x_{p,[k,\infty]}^*(\alpha) (X_{[k,l]} \otimes \text{id}_{[l+1,\infty]}) \\ &= (x_{p,[k,l+p]}^*(\alpha) (X_{[k,l]} \otimes \text{id}_{[l+1,l+p]})) \otimes \text{id}_{[l+p+1,\infty]}, \end{aligned} \quad (87)$$

where $x = b^*, c^*$.

Inserting the mode expansions (57b) and (61) for the annihilation operators into the reduction relation (79) we obtain

$$\sum_{p=0}^{n+m} \frac{x_{p,[k,l+m]}(\alpha) (X_{[k,l]} \otimes \text{id}_{[l+1,l+m]})}{(\zeta^2 - 1)^p} = \sum_{p=0}^n \frac{(x_{p,[k,l]}(\alpha) X_{[k,l]}) \otimes \text{id}_{[l+1,m+1]}}{(\zeta^2 - 1)^p}, \quad (88)$$

where $x = b, c$, and $n = l - k + 1$ is the length of $X_{[k,l]}$. Comparing coefficients we conclude that

$$\begin{aligned} & x_{p,[k,l+m]}(\alpha) (X_{[k,l]} \otimes \text{id}_{[l+1,l+m]}) \\ &= \begin{cases} (x_{p,[k,l]}(\alpha) X_{[k,l]}) \otimes \text{id}_{[l+1,l+m]} & p = 1, \dots, n \\ 0 & p = n + 1, \dots, m. \end{cases} \end{aligned} \quad (89)$$

Thus, any operator X of length $\ell(X)$ is annihilated by x_p if $p > \ell(X)$. Since m is arbitrary here we may extend the action of the modes infinitely to the right,

$$\begin{aligned}
& x_{p,[k,\infty)}(\alpha)(X_{[k,l]} \otimes \text{id}_{[l+1,\infty)}) \\
&= \begin{cases} (x_{p,[k,l]}(\alpha)X_{[k,l]}) \otimes \text{id}_{[l+1,\infty)} & p = 1, \dots, n \\ 0 & p > n. \end{cases} \quad (90)
\end{aligned}$$

Equations (83), (87) and (90) comprise the reduction properties of the modes for a reduction to the right. Before considering the left reduction properties we would like to modify our formulation in a way that allows us to define uniform mode expansions for the action on operators that have no definite spin. For this purpose note that the variable α is still at our disposal. We may shift it in such a way that the spin dependence in the mode expansions of the creation and annihilation operators is moved from the spectral parameter to the Taylor and Laurent coefficients. For the annihilation operators we obtain from (57b), (61)

$$\mathbf{x}(\zeta, \alpha - s - s(\mathbf{x}))X_{[k,l]} = \zeta^{\alpha s(\mathbf{x})} \sum_{p=0}^n \frac{x_p(\alpha - s - s(\mathbf{x}))X_{[k,l]}}{(\zeta^2 - 1)^p} \quad (91)$$

for $\mathbf{x} = \mathbf{b}, \mathbf{c}$, $x = b, c$. Recall that $s(\mathbf{b}) = -1$, $s(\mathbf{c}) = 1$. Similarly, the mode expansions for the creation operators can be written as

$$\mathbf{x}(\zeta, \alpha - s - s(\mathbf{x}))X_{[k,l]} = \zeta^{\alpha s(\mathbf{x})} \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} x_p(\alpha - s - s(\mathbf{x}))X_{[k,l]} \quad (92)$$

for $\mathbf{x} = \mathbf{b}^*, \mathbf{c}^*, \mathbf{t}^*$. Here $s(\mathbf{b}^*) = 1$, $s(\mathbf{c}^*) = -1$, and $s(\mathbf{t}^*) = 0$.

Inserting (82), (85) or (86) we obtain

$$\begin{aligned}
& \mathbf{x}_{[k,l+m]}(\zeta, \alpha - s - s(\mathbf{x}))(X_{[k,l]} \otimes \text{id}_{[l+1,l+m]}) \\
&= \zeta^{\alpha s(\mathbf{x})} \sum_{p=1}^m (\zeta^2 - 1)^{p-1} (x_{p,[k,l+p]}(\alpha - s - s(\mathbf{x}))(X_{[k,l]} \otimes \text{id}_{[l+1,l+p]})) \otimes \text{id}_{[l+p+1,l+m]} \\
&\quad + \zeta^{\alpha s(\mathbf{x})} \times \mathcal{O}((\zeta^2 - 1)^m) \quad (93)
\end{aligned}$$

for all $m \in \mathbb{N}$. From this equation we understand how the creation operators can be extended infinitely to the right by the inductive limit $m \rightarrow \infty$,

$$\begin{aligned}
& \mathbf{x}_{[k,\infty)}(\zeta, \alpha - s - s(\mathbf{x}))(X_{[k,l]} \otimes \text{id}_{[l+1,\infty)}) \\
&= \zeta^{\alpha s(\mathbf{x})} \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} (x_{p,[k,l+p]}(\alpha - s - s(\mathbf{x}))) \\
&\quad \times (X_{[k,l]} \otimes \text{id}_{[l+1,l+p]}) \otimes \text{id}_{[l+p+1,\infty)} \quad (94)
\end{aligned}$$

for $\mathbf{x} = \mathbf{b}^*, \mathbf{c}^*, \mathbf{t}^*$. Note that each term under the sum is of finite length.

Remark. Comparing the last two equations we may also state that

$$\begin{aligned}
& \mathbf{x}_{[k,\infty)}(\zeta, \alpha - s - s(\mathbf{x}))(X_{[k,l]} \otimes \text{id}_{[l+1,\infty)}) \\
&= (\mathbf{x}_{[k,l+m]}(\zeta, \alpha - s - s(\mathbf{x}))(X_{[k,l]} \otimes \text{id}_{[l+1,l+m]})) \otimes \text{id}_{[m+1,\infty)} \\
&\quad \text{mod } \zeta^{\alpha s(\mathbf{x})} \times \mathcal{O}((\zeta^2 - 1)^m), \quad (95)
\end{aligned}$$

which is the way the inductive limit for the extension to the right is introduced in [11].

For the extension of the mode expansion of the annihilation operators to $+\infty$ we use (80), (90) and (91) to obtain

$$\begin{aligned} & \mathbf{x}_{[k,\infty)}(\zeta, \alpha - s - s(\mathbf{x})) (X_{[k,l]} \otimes \text{id}_{[l+1,\infty)}) \\ &= \zeta^{\alpha s(\mathbf{x})} \sum_{p=1}^n \frac{(x_{p,[k,l]}(\alpha - s - s(\mathbf{x})) X_{[k,l]}) \otimes \text{id}_{[l+1,\infty)}}{(\zeta^2 - 1)^p} \end{aligned} \quad (96)$$

for $\mathbf{x} = \mathbf{b}, \mathbf{c}$.

Equations (94) and (96) describe how the action of the creation and annihilation operators can be extended infinitely to the right. We may use these equations in (78) in order to obtain operators acting on the entire infinite chain. More precisely, these operators define maps $\mathbf{x} : \mathcal{W}_{\alpha-s,s} \rightarrow \mathcal{W}_{\alpha-s-s(\mathbf{x}),s+s(\mathbf{x})}$. Let $X^{(\alpha-s)} \in \mathcal{W}_{\alpha-s,s}$ with non-trivial part $X_{[k,l]} \in \mathcal{H}_{[k,l]}$. Then

$$\begin{aligned} & \mathbf{x}(\zeta, \alpha - s - s(\mathbf{x})) X^{(\alpha-s)} \\ &= \zeta^{\alpha s(\mathbf{x})} \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} q^{2(\alpha-s-s(\mathbf{x}))S_{(-\infty,k-1]}} \otimes (x_{p,[k,l+p]}(\alpha - s - s(\mathbf{x}))) \\ & \quad \times (X_{[k,l]} \otimes \text{id}_{[l+1,l+p]}) \otimes \text{id}_{[l+p+1,\infty)} \end{aligned} \quad (97)$$

for the creation operators $\mathbf{x} = \mathbf{t}^*, \mathbf{b}^*, \mathbf{c}^*$ and

$$\begin{aligned} & \mathbf{x}(\zeta, \alpha - s - s(\mathbf{x})) X^{(\alpha-s)} \\ &= \zeta^{\alpha s(\mathbf{x})} \sum_{p=1}^n q^{2(\alpha-s-s(\mathbf{x}))S_{(-\infty,k-1]}} \otimes \frac{(x_{p,[k,l]}(\alpha - s - s(\mathbf{x})) X_{[k,l]}) \otimes \text{id}_{[l+1,\infty)}}{(\zeta^2 - 1)^p} \end{aligned} \quad (98)$$

for the annihilation operators $\mathbf{x} = \mathbf{b}, \mathbf{c}$. These formulae show how the action of the finite operators $\mathbf{x}_{[k,l]}$ introduced in the previous subsections can be naturally extended to the infinite chain.

We define the modes $\mathbf{x}_p : \mathcal{W}_{\alpha-s,s} \rightarrow \mathcal{W}_{\alpha-s-s(\mathbf{x}),s+s(\mathbf{x})}$ by their action on $X^{(\alpha-s)} \in \mathcal{W}_{\alpha-s,s}$ with finite part $X_{[k,l]}$. For the creation operators we set

$$\begin{aligned} \mathbf{x}_p(\alpha - s - s(\mathbf{x})) X^{(\alpha-s)} &= q^{2(\alpha-s-s(\mathbf{x}))S_{(-\infty,k-1]}} \\ & \otimes (x_{p,[k,l+p]}(\alpha - s - s(\mathbf{x}))) (X_{[k,l]} \otimes \text{id}_{[l+1,l+p]}) \otimes \text{id}_{[l+p+1,\infty)} \end{aligned} \quad (99)$$

where $\mathbf{x} = \mathbf{t}^*, \tilde{\mathbf{b}}^*, \tilde{\mathbf{c}}^*$ and where the finite parts are $x_p = t_p^*, b_p^*, c_p^*$, whose explicit form are given in (67), (71), (73). For the annihilation operators we define

$$\begin{aligned} \mathbf{x}_p(\alpha - s - s(\mathbf{x})) X^{(\alpha-s)} &= q^{2(\alpha-s-s(\mathbf{x}))S_{(-\infty,k-1]}} \\ & \otimes \begin{cases} (x_{p,[k,l]}(\alpha - s - s(\mathbf{x})) X_{[k,l]}) \otimes \text{id}_{[l+1,\infty)} & p = 1, \dots, n \\ 0 & p > n, \end{cases} \end{aligned} \quad (100)$$

where $\mathbf{x} = \mathbf{b}, \mathbf{c}$, and the finite parts $x_p = c_p, b_p$ have been defined in (58c), (60). Altogether we have obtained fully explicit expressions for the Fermionic operators as maps $\mathcal{W}_{\alpha-s,s} \rightarrow \mathcal{W}_{\alpha-s-s(\mathbf{x}),s+s(\mathbf{x})}$. They are defined by the mode expansions

$$\mathbf{t}^*(\zeta, \alpha - s) = \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} \mathbf{t}_p^*(\alpha - s), \quad (101a)$$

$$\mathbf{x}^*(\zeta, \alpha - s - s(\mathbf{x}^*)) = \zeta^{\alpha s(\mathbf{x}^*)} \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} \tilde{\mathbf{x}}_p^*(\alpha - s - s(\mathbf{x})), \quad (101b)$$

$$\mathbf{x}(\zeta, \alpha - s - s(\mathbf{x})) = \zeta^{\alpha s(\mathbf{x})} \sum_{p=1}^{\infty} (\zeta^2 - 1)^{-p} \mathbf{x}_p(\alpha - s - s(\mathbf{x})), \quad (101c)$$

where $\mathbf{x} = \mathbf{b}, \mathbf{c}$.

We may consider these operators as ‘ $\mathcal{W}_{\alpha-s,s} \rightarrow \mathcal{W}_{\alpha-s-s(\mathbf{x}),s+s(\mathbf{x})}$ ’ blocks’ of operators $\mathbf{x}(\zeta) \in \text{End } \mathcal{W}^{(\alpha)}$ (cf. (18)). If $X \in \mathcal{W}^{(\alpha)}$ with $\ell(X) = n$, then

$$X = \sum_{s=-n}^n X^{(\alpha-s)}, \quad (102)$$

where $X^{(\alpha-s)} \in \mathcal{W}_{\alpha-s,s}$, and

$$\mathbf{x}(\zeta)X = \sum_{s=-n}^n \mathbf{x}(\zeta, \alpha - s - s(\mathbf{x}))X^{(\alpha-s)}. \quad (103)$$

In a similar way we can define the action of the modes in (101) on $\mathcal{W}^{(\alpha)}$,

$$\mathbf{x}_p X = \sum_{s=-n}^n \mathbf{x}_p(\alpha - s - s(\mathbf{x}))X^{(\alpha-s)}. \quad (104)$$

The generating functions $\mathbf{x}(\zeta) = \mathbf{t}^*(\zeta), \mathbf{b}^*(\zeta), \mathbf{c}^*(\zeta), \mathbf{b}(\zeta), \mathbf{c}(\zeta)$ hence have the mode expansions

$$\mathbf{t}^*(\zeta) = \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} \mathbf{t}_p^*, \quad (105a)$$

$$\mathbf{x}^*(\zeta) = \zeta^{\alpha s(\mathbf{x}^*)} \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} \tilde{\mathbf{x}}_p^*, \quad (105b)$$

$$\mathbf{x}(\zeta) = \zeta^{\alpha s(\mathbf{x})} \sum_{p=0}^{\infty} \frac{\mathbf{x}_p}{(\zeta^2 - 1)^p}, \quad (105c)$$

where $\mathbf{x} = \mathbf{b}, \mathbf{c}$. Recall that the action of the modes on the right hand side on $\mathcal{W}_{\alpha-s,s}$ is defined by (99), (100).

We use the geometric series

$$\zeta^2 \sum_{k=0}^{\infty} (-1)^k (\zeta^2 - 1)^k = 1 \quad (106)$$

in order to resum the modes in the mode expansion (105b) for $\mathbf{b}^*(\zeta)$ and $\mathbf{c}^*(\zeta)$. Then

$$\begin{aligned} \mathbf{x}^*(\zeta) &= \zeta^{\alpha s(\mathbf{x}^*)+2} \sum_{p=1}^{\infty} \sum_{k=0}^{\infty} (\zeta^2 - 1)^{p+k-1} (-1)^k \tilde{\mathbf{x}}_p^* \\ &= \zeta^{\alpha s(\mathbf{x}^*)+2} \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} \mathbf{x}_p^*, \end{aligned} \quad (107)$$

where

$$\mathbf{x}_p^* = \sum_{k=1}^p (-1)^{p-k} \tilde{\mathbf{x}}_k^* \quad (108)$$

for $p \in \mathbb{N}$ and $\mathbf{x}^* = \mathbf{b}^*, \mathbf{c}^*$. As opposed to the modes $\tilde{\mathbf{x}}_p^*$ the modes \mathbf{x}_p^* satisfy canonical anticommutation relations with the modes of the corresponding annihilation operators (see below).

To sum up, we have presented a fully explicit description of the operators in the Fermionic basis which is suitable for its implementation in a computer algebra program.

2.7 Remarks on the commutation relations

The commutation relations of the operators acting on finite chains are discussed in section 4 of [11] together with the commutation relations of the operators acting on the space of quasi-local operators on the infinite chain. The proofs are at the same time very technical and rather sketchy. They are certainly the most challenging part of the theory. Here we extract the information about the operators defined on the finite chain which is needed if we wish to verify their commutation relations in special cases on a computer.

Section 4 of [11] provides the commutation relations in a somewhat implicit form involving what the authors call q -exact forms. The commutativity of the Fermionic annihilation operators with \mathbf{t}^* , for instance, is expressed in equation (4.3) of [11] as

$$\begin{aligned} \mathbf{k}(\xi, \alpha) \left((\mathbf{t}^*(\zeta, \alpha + 1) X_{[k,m]}) \otimes \text{id}_{[m+1,l]} \right) - \mathbf{t}^*(\zeta, \alpha) \left((\mathbf{k}(\xi, \alpha) X_{[k,m]}) \otimes \text{id}_{[m+1,l]} \right) \\ \simeq_{\xi} 0 \quad \text{mod } (\zeta^2 - 1)^{l-m} \quad (109) \end{aligned}$$

for $k \leq m < l$. Here ‘ \simeq_{ξ} ’ means equality up to a q -exact form in ξ , a notion that is explained in section 2.6 of [11]. It means that the right hand side of (109) can be written as $\Delta_{\xi} \xi^{\alpha+s} m(\xi^2, \alpha)$, where m is rational in ξ^2 with a pole at most at $\xi^2 = 1$ and where s is the spin of $X_{[k,m]}$. It follows that

$$\begin{aligned} & (\mathbf{k}(q\xi, \alpha) + \mathbf{k}(q^{-1}\xi, \alpha)) \left((\mathbf{t}^*(\zeta, \alpha + 1) X_{[k,m]}) \otimes \text{id}_{[m+1,l]} \right) \\ & \quad - \mathbf{t}^*(\zeta, \alpha) \left((\mathbf{k}(q\xi, \alpha) + \mathbf{k}(q^{-1}\xi, \alpha)) X_{[k,m]} \right) \otimes \text{id}_{[m+1,l]} \\ & = \Delta_{\xi} (q\xi)^{\alpha+s} m((q\xi)^2, \alpha) + \Delta_{\xi} (q^{-1}\xi)^{\alpha+s} m((q^{-1}\xi)^2, \alpha) \quad \text{mod } (\zeta^2 - 1)^{l-m} \\ & = (q^2\xi)^{\alpha+s} m(q^4\xi^2, \alpha) - (q^{-2}\xi)^{\alpha+s} m(q^{-4}\xi^2, \alpha) \quad \text{mod } (\zeta^2 - 1)^{l-m}. \quad (110) \end{aligned}$$

If we multiply by $\psi(\zeta_1/\xi, \alpha + s + 1)/(4\pi i \xi^2)$, then both sides are rational in ξ^2 and the right hand side is obviously regular at $\xi^2 = 1$. Integrating over ξ^2 on a small circle around $\xi^2 = 1$ and using the definition of \mathbf{c} , Eq. (54b), we obtain

$$\begin{aligned} \mathbf{c}(\xi, \alpha) \left((\mathbf{t}^*(\zeta, \alpha + 1) X_{[k,m]}) \otimes \text{id}_{[m+1,l]} \right) - \mathbf{t}^*(\zeta, \alpha) \left((\mathbf{c}(\xi, \alpha) X_{[k,m]}) \otimes \text{id}_{[m+1,l]} \right) \\ = 0 \quad \text{mod } (\zeta^2 - 1)^{l-m}. \quad (111) \end{aligned}$$

In a similar way we can obtain all other commutation relations of the finite generating functions from the relations in section 4 of [11]. If we shift $\alpha \rightarrow \alpha - s - 1$ and perform the inductive limit described in the previous subsection on equation (111), we obtain the commutation relation $[\mathbf{c}(\xi), \mathbf{t}^*(\zeta)] = 0$. The ideology for the derivation of the other

commutation relations (26), (27) is similar. In any case, the hard part of the proof is to derive relations like (109).

We will not touch this subject any deeper. Yet, we would like to comment on how the commutation relations for the modes follow from those for the generating functions. They are obtained by inserting the mode expansions into the commutation relations for the generating functions. Whenever two generating functions commute or anticommute the same is trivially true also for the corresponding modes. The only cases which need extra attention are the anticommutators $[\mathbf{b}(\zeta_1), \mathbf{b}^*(\zeta_2)]_+$ and $[\mathbf{c}(\zeta_1), \mathbf{c}^*(\zeta_2)]_+$ which can be uniformly written as

$$[\mathbf{x}(\zeta_1), \mathbf{x}^*(\zeta_2)]_+ = \psi(\zeta_1/\zeta_2, \alpha s(\mathbf{x})) \quad (112)$$

for $\mathbf{x} = \mathbf{b}, \mathbf{c}$. Inserting the mode expansions (105c), (107) on the left hand side we obtain an equation which is equivalent to

$$\sum_{k=0}^{\infty} \sum_{p=1}^{\infty} [\mathbf{x}_k, \mathbf{x}_p^*]_+ \frac{(\zeta_2^2 - 1)^{p-1}}{(\zeta_1^2 - 1)^k} = \frac{\zeta_1^2 + \zeta_2^2}{2\zeta_2^2(\zeta_1^2 - \zeta_2^2)}. \quad (113)$$

Using (56) and comparing coefficients we obtain the anticommutation relations

$$[\mathbf{x}_0, \mathbf{x}_p^*]_+ = \frac{1}{2}(-1)^{p-1}, \quad p \in \mathbb{N}, \quad (114a)$$

$$[\mathbf{x}_k, \mathbf{x}_p^*]_+ = \delta_{k,p}, \quad k, p \in \mathbb{N}. \quad (114b)$$

Here the first set of relations is consistent with the definitions of the the modes \mathbf{x}_0 (see (58d), (60)) while the second set of relations is just the set of canonical anticommutation relations for Fermions. At this point it is also clear that the coefficients $\tilde{\mathbf{x}}_p^* = \mathbf{x}_p^* + \mathbf{x}_{p-1}^*$ do not satisfy the canonical anticommutation relations.

2.8 Remarks on the implementation on the computer

As explained in the previous subsections we are able to represent operators acting on the infinite chain in terms of finite matrices. Hence, it is possible to construct such operators explicitly on the computer. To do so a system is needed in which symbolic expressions, including non-commutative symbols, can be manipulated. For this purpose we have used FORM [51]. While there exist packages for e.g. Mathematica for this purpose, our experience is, that full blown computer algebra systems like Mathematica are not efficient enough in terms of memory management. In comparison, FORM is a rather primitive language with less features, which allows for a much simpler internal representation of expressions, resulting in a more efficient memory management.

As pointed out before, no closed formula is known for expanding products of ultra-local operators (e.g. $\sigma_1^z \sigma_n^z$) in the Fermionic basis. For this reason as well as for a few others [34], the direct computation of correlation functions by means of the JMS theorem is rather inefficient. Instead we used the so-called exponential form which means that we only need to construct the modes of the annihilation operators \mathbf{b}, \mathbf{c} .

In order to obtain these, in first place, the parental operator \mathbf{k} needs to be constructed. As discussed before, the construction can be done for the case of a finite chain. For our program we changed formula (50) in order to express $\mathbf{k}_{\text{skal}, [k,l]}$ in terms of the fused L -matrices $L_{\{a,A\},j}$ introduced in [11]:

$$L_{\{a,A\},j} = F_{a,A}^{-1} L_{a,j} L_{A,j} F_{a,A}, \quad F_{a,A} = 1 - \mathbf{a}_A \sigma_a^+, \quad (115)$$

which can be written explicitly as

$$L_{\{a,A\},j}(\zeta) = \begin{pmatrix} 1 & 0 \\ \frac{\gamma(\zeta)}{\beta(\zeta)}\sigma_j^+ & 1 \end{pmatrix}_a \begin{pmatrix} L_{A,j}(q\zeta)q^{-\sigma_j^z/2} & 0 \\ 0 & L_{A,j}(q^{-1}\zeta)q^{\sigma_j^z/2} \end{pmatrix}_a. \quad (116)$$

It is possible to express $\mathbf{k}_{\text{skal},[k,l]}$ in terms of the adjoint action of this L -matrix,

$$\mathbb{L}_{\{a,A\},j} \text{id}_a \otimes \text{id}_A \otimes X_{[k,l]} = L_{\{a,A\},j} \text{id}_a \otimes \text{id}_A \otimes X_{[k,l]} L_{\{a,A\},j}^{-1}, \quad (117)$$

since $[F_{a,A}, \sigma_a^+] = [F_{a,A}, q^{\alpha(\sigma_a^z + 2D_A)}] = 0$.

Our program then uses

$$\begin{aligned} & \mathbf{k}_{\text{skal},[k,l]}(\zeta, \alpha) X_{[k,l]} \\ &= \text{tr}_{a,A} \left\{ \sigma_a^+ \mathbb{L}_{\{a,A\},l}(\zeta) \cdots \mathbb{L}_{\{a,A\},k}(\zeta) \left(\zeta^{-2s-1} y^{\sigma_a^z + 2D_A} q^{-2S_{[k,l]}} X_{[k,l]} \right) \right\}, \quad (118) \end{aligned}$$

where again the operator $X_{[k,l]}$ is of spin s . We also set $y = q^\alpha$ in our program because it is only a single symbol. Using the fused operators $L_{\{a,A\},j}$ instead of the simple L -operators effectively means an early simplification of the building blocks of \mathbf{k} .

It is then convenient to compute the action of $\mathbf{k}_{\text{skal},[k,l]}(\zeta, \alpha)$ on the canonical basis of $\text{End } \mathcal{H}_{[k,l]}$ constructed with the single-entry matrices $e_{j\alpha}^\beta$. This will also allow for an easy parallelization later. For each element of the basis the ‘innermost’ part

$$\zeta^{-2s-1} y^{\sigma_a^z + 2D_A} q^{-2S_{[k,l]}} X_{[k,l]} \quad (119)$$

is constructed first. The spin- $\frac{1}{2}$ auxiliary space is explicitly used, whereas $q^{2\alpha D_A}$ is represented by a single non-commuting symbol. Then, in a loop, each step applies a single fused L matrix $\mathbb{L}_{\{a,A\},j}(\zeta)$ after which all symbols are commuted and sorted. When the loop is finished, the elements $e_{a_+}^+$, $e_{a_-}^-$ and $e_{a_+}^-$ can be discarded, because of the operator σ_a^+ and the trace tr_A . The remaining trace tr_A can then be taken by discarding all terms which are not ‘balanced’ in \mathbf{a}_A^* and \mathbf{a}_A and replacing $y^{2D_A} q^{mD_A} \rightarrow \frac{1}{1-y^2 q^m}$.

Since the intermediate expressions can become very large (a few gigabytes for $n = 5$), it is crucial to carefully choose which simplification is done at which point. Too many simplifications can slow down the calculations, but, on the other hand, too few can increase the memory usage dramatically. There is no definite rule in this regard, and the appropriate places in the program, where simplifications are most efficient, need to be determined for each calculation individually. It is also helpful to use simple symbols wherever possible, e.g. representing the functions $\beta(\zeta)$, $\gamma(\zeta)$ by single symbols rather than rational expressions. Such symbols should then be expanded at the latest possible stage.

After constructing the operator $\mathbf{k}_{\text{skal},[k,l]}$, the Laurent-coefficients are obtained. This is done slightly differently from (62) and (63). The reason is that using (62) and (63) directly would trigger the same intermediate calculations to be performed multiple times. Instead, $\mathbf{k}_{\text{skal},[k,l]}(\zeta, \alpha)$ is loaded and a loop counts down $j = n, \dots, 1$. For each j we run over $\epsilon = -1, 0, 1$ and set

$$\rho_{j,[k,l]}^{(\epsilon)}(\alpha) = (\zeta^2 - q^{2\epsilon})^j \mathbf{k}_{\text{skal},[k,l]}^{(j,\epsilon)}(\zeta, \alpha) \Big|_{\zeta^2=q^{2\epsilon}}, \quad (120a)$$

$$\kappa_{j,[k,l]}(\alpha) = \zeta^{2j} \mathbf{k}_{\text{skal},[k,l]}^{(j,2)}(\zeta, \alpha) \Big|_{\zeta^2=0}, \quad (120b)$$

where

$$\mathbf{k}_{\text{skal},[k,l]}^{(j,\epsilon)}(\zeta, \alpha) = \mathbf{k}_{\text{skal},[k,l]}(\zeta, \alpha) - \sum_{m=j}^n \sum_{\delta=-1}^{\epsilon-1} \frac{\rho_{m,[k,l]}^{(\delta)}(\alpha)}{(\zeta^2 - q^{2\delta})^m} - \sum_{m=j+1}^n \frac{\kappa_{m,[k,l]}(\alpha)}{\zeta^{2m}}. \quad (121)$$

The $\mathbf{k}_{\text{skal},[k,l]}^{(j,\epsilon)}(\zeta, \alpha)$ can then be ‘accumulated’ during the loop, instead of building them anew in every step, and therefore become smaller in every iteration of the loop. During traversal of the loop $\kappa_0(\alpha)$ is accumulated as well, which contains all $\rho_j^{(\alpha)}$. Since the $\rho_j^{(0)}$ are only needed for $\kappa_0(\alpha)$, they are discarded as soon as they have entered $\kappa_0(\alpha)$. Each time one of the Laurent coefficients is completed, a sorting is done in order to keep $\mathbf{k}_{\text{skal},[k,l]}^{(j,\epsilon)}(\zeta, \alpha)$ as small as possible. The modes of the annihilation operators are then easily constructed according to (58) and (60).

In order to verify the correctness of the obtained operators we utilize the (anti-)commutation relations. The modes of the transfer matrix \mathbf{t}^* have to commute amongst themselves as well as with all modes of the Fermionic operators. The modes of \mathbf{b}, \mathbf{b}^* and \mathbf{c}, \mathbf{c}^* form two families of Fermions and obey the canonical relations as noted before. For the sake of brevity we do not go into detail regarding the construction of the creation operators, since they are not directly needed when using the exponential form. We did, however, construct all of them explicitly, so we were able to verify all (anti-)commutation relations that are sensibly defined on an interval of a given length n . Additionally, for the annihilation operators, the annihilation relations (23) and reduction relations (77), (79), (85), (86) were verified. In our experience most of these relations depend on the correctness of the involved operators in a very sensitive manner. Typically, even small errors in the program led to objects which obey none of the tested relations. So, if the operators constructed obey all above relations, it is a strong indication of their correctness.

3 Correlation functions

In this section we use the exponential form (38) in order to study short-range correlation functions of spin-reversal invariant operators at finite temperature. In Appendix A we show how (37) follows from the JMS theorem and argue that it continues to hold for $\kappa \neq 0$ if we restrict the action of Z^κ to spin-reversal invariant operators.

3.1 The exponential form

Inserting the mode expansions (19c) for the annihilation operators into (35) and using Cauchy’s theorem we obtain a mode expansion of the operator Ω ,

$$\Omega = \sum_{k,p=1}^{\infty} \frac{\omega_{k-1,p-1}(\alpha) \mathbf{b}_k \mathbf{c}_p}{(k-1)!(p-1)!}, \quad (122)$$

where

$$\omega_{k,p} = \partial_{\zeta_1^2}^k \partial_{\zeta_2^2}^p \zeta_1^{-2} \zeta_2^{-2} (\zeta_1/\zeta_2)^{-\alpha} (\omega_0(\zeta_1/\zeta_2; \alpha) - \omega(\zeta_1, \zeta_2; \alpha)) \Big|_{\zeta_1^2=\zeta_2^2=1}. \quad (123)$$

Recall that we denoted by $\mathcal{W}_{\alpha,0,[1,n]} \subset \mathcal{W}_{\alpha,0}$ the space of quasi-local operators of spin 0 with tail α and with support $\text{supp } X^{(\alpha)} \subset [1, n]$. Due to the annihilation property

(23) of the modes, the restriction of Ω to this subspace is a finite sum,

$$\Omega|_{\mathcal{W}_{\alpha,0,[1,n]}} = \sum_{k,p=1}^n \frac{\omega_{k-1,p-1}(\alpha) \mathbf{b}_k \mathbf{c}_p}{(k-1)!(p-1)!}. \quad (124)$$

In [5] the authors constructed a basis of $\mathcal{W}_{\alpha,0,[1,n]}$ as a submodule of the Fermionic basis. This submodule is generated by the action of monomials of the form

$$\mathbf{t}_{p_1}^* \dots \mathbf{t}_{p_j}^* \mathbf{b}_{q_1^+}^* \dots \mathbf{b}_{q_k^+}^* \mathbf{c}_{q_k^-}^* \dots \mathbf{c}_{q_1^-}^* \quad (125)$$

onto the vacuum $q^{2\alpha S(0)}$, where

$$j + 2k \leq n, \quad p_l, q_m^+, q_n^- \leq n. \quad (126)$$

In particular, the number k of creation operators $\mathbf{b}_{q_m^+}^*, \mathbf{c}_{q_n^-}^*$ of Fermions is restricted by

$$k \leq \left\lfloor \frac{n}{2} \right\rfloor, \quad (127)$$

where the bracket denotes the integer part. It follows that

$$\left(\Omega|_{\mathcal{W}_{\alpha,0,[1,n]}} \right)^{\left\lfloor \frac{n}{2} \right\rfloor + 1} = 0, \quad (128)$$

and therefore

$$e^{\Omega}|_{\mathcal{W}_{\alpha,0,[1,n]}} = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(\Omega|_{\mathcal{W}_{\alpha,0,[1,n]}})^k}{k!}. \quad (129)$$

Using (58c), (60), (100) and (124), this can be realized by means of a finite sum over finite products of finite matrices and is the formula used below for the calculation of short-range correlation functions.

In order to explicitly calculate correlation functions on the computer we use (38), (124) and (129). At this point we have shown that all sums involved are finite and all operators can be represented in terms of finite matrices. We do not evaluate the exponential given in (129) directly for two reasons. On the one hand we argue that our method holds for operators that are invariant under spin reversal, meaning that it is only valid on a subspace of $\mathcal{W}_{\alpha,0,[1,n]}$. On the other hand the involved expressions tend to grow rather big, making it important to save as much memory as possible. For these reasons we only calculate the action of the exponential on a given operator $X^{(\alpha)} \in \mathcal{W}_{\alpha,0,[1,n]}$.

We apply $\Omega|_{\mathcal{W}_{\alpha,0,[1,n]}}$ repeatedly to $X^{(\alpha)}$, filtering out every term that becomes zero due to the nilpotence of the modes before inserting explicit matrices and simplifying as much as possible before the next step. During this process the sum (129) can be accumulated. The treatment of the functions ω and $\hat{\omega}$ is explained in the next section and in Appendix B.

This process then results in expressions rational in q that contain $\hat{\omega}$ and its derivatives as symbols, as shown in [1] for $n = 1 \dots 4$ and in Appendix C for $n = 5$.

3.2 Finite temperature short-range correlation functions

As was explained in the introduction, the Fermionic basis approach applies to very general situations. It holds for any realization of the functional Z^κ which, in turns, can be used

to realize a huge class of reduced density matrices including the cases of the canonical ensemble and of generalized Gibbs ensembles. The characteristics of any functional Z^κ enter the formalism only through two functions ρ and ω . These functions were called the ‘physical part’ of the problem in [2] as they entirely fix ‘the experimental conditions’ under which the correlation functions under consideration are determined. For the actual computation of short-range correlation functions what is still needed is an efficient description of the physical part.

Such a description, valid for the case of finite-temperature correlations in the infinite chain, was obtained in [2]. The core part of this description is a non-linear integral equation for an auxiliary function \mathfrak{a} that had occurred before in the derivation of an efficient thermodynamics of the XXZ chain [35] and in the derivation of multiple-integral representations for static finite-temperature correlation functions of the same model [23]. Let us define the ‘bare energy function’

$$e(\lambda) = \text{cth}(\lambda) - \text{cth}(\lambda - i\gamma) \quad (130)$$

and the kernel function

$$K_\alpha(\lambda) = q^{-\alpha} \text{cth}(\lambda + i\gamma) - q^\alpha \text{cth}(\lambda - i\gamma), \quad (131)$$

where $-i\gamma = \ln q$. Using these functions the nonlinear integral equation for \mathfrak{a} takes the form

$$\ln(\mathfrak{a}(\lambda, \kappa)) = 2i\gamma\kappa + \frac{2iJ \sin(\gamma) e(\lambda)}{T} - \int_{\mathfrak{e}} \frac{d\mu}{2\pi i} K_0(\lambda - \mu) \ln(1 + \mathfrak{a}(\mu, \kappa)). \quad (132)$$

Here the ‘Trotter limit’, corresponding to infinitely many horizontal lines in the definition of Z^κ , is already taken (for more details see [2]). The magnetic field $h > 0$ appearing in the Hamiltonian can be taken into account by setting

$$\kappa = \frac{ih}{2\gamma T}. \quad (133)$$

The precise definition of the integration contour depends on the parameter regime. For $\gamma \in (0, \pi/2)$ (implying that $0 < \Delta < 1$) and $h > 0$, for instance, we may take the contour sketched in Figure 1.

The function ρ , defined as an eigenvalue ratio in (30), has an integral representation involving the auxiliary function and the bare energy,

$$\rho(\zeta) = \hat{\rho}(\lambda) = q^\alpha \exp \left\{ \int_{\mathfrak{e}} \frac{d\mu}{2\pi i} e(\mu - \lambda) \ln \left(\frac{1 + \mathfrak{a}(\mu, \kappa + \alpha)}{1 + \mathfrak{a}(\mu, \kappa)} \right) \right\}, \quad (134)$$

where $\lambda = \ln \zeta$. For the definition of ω in the finite temperature case we first of all need to introduce a function G which is the unique solution of the linear integral equation

$$G(\lambda, \nu) = q^{-\alpha} \text{cth}(\lambda - \nu + i\gamma) - \hat{\rho}(\nu) \text{cth}(\lambda - \nu) + \int_{\mathfrak{e}} \frac{d\mu}{2\pi i} \frac{K_\alpha(\lambda - \mu) G(\mu, \nu)}{\hat{\rho}(\mu)(1 + \mathfrak{a}(\mu, \kappa))}. \quad (135)$$

For $\lambda_j = \ln \zeta_j$, $j = 1, 2$, we set

$$\Psi(\zeta_1, \zeta_2; \alpha) = \int_{\mathfrak{e}} \frac{d\mu}{2\pi i} \frac{(q^\alpha \text{cth}(\mu - \lambda_1 + i\gamma) - \hat{\rho}(\lambda_1) \text{cth}(\mu - \lambda_1)) G(\mu, \lambda_2)}{\hat{\rho}(\mu)(1 + \mathfrak{a}(\mu, \kappa))}. \quad (136)$$

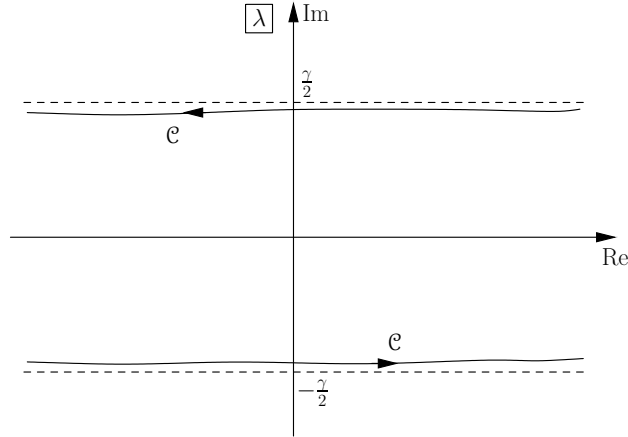


Figure 1: For $0 < \Delta < 1$, $h > 0$ the canonical contour \mathcal{C} surrounds the real axis in a counterclockwise manner inside the strip $-\frac{\gamma}{2} < \text{Im } \lambda < \frac{\gamma}{2}$.

Then

$$\begin{aligned} \omega(\zeta_1, \zeta_2; \alpha) \\ = 2(\zeta_1/\zeta_2)^\alpha \Psi(\zeta_1, \zeta_2; \alpha) - \Delta \psi(\zeta_1/\zeta_2, \alpha) + 2(\rho(\zeta_1) - \rho(\zeta_2)) \psi(\zeta_1/\zeta_2, \alpha). \end{aligned} \quad (137)$$

This function has to be used in (123) for the actual computation of the short-range correlation functions.

As we have to take the limit $\alpha \rightarrow 0$ in (38) and as the Fermi operators have first order poles in α by construction, it suffices to consider the function ω up to first order in α . Let

$$\hat{\omega}(\lambda_1, \lambda_2; \alpha) = (\zeta_1/\zeta_2)^{-\alpha} (\omega_0(\zeta_1/\zeta_2; \alpha) - \omega(\zeta_1, \zeta_2; \alpha)), \quad (138)$$

where $\lambda_j = \ln \zeta_j$, $j = 1, 2$. The functions that occur in the description of the physical correlation functions in the limit $\alpha \rightarrow 0$ can be chosen as

$$\hat{\omega}(\lambda_1, \lambda_2) = \hat{\omega}(\lambda_1, \lambda_2; 0), \quad (139a)$$

$$\hat{\omega}'(\lambda_1, \lambda_2) = \frac{1}{2} \partial_\alpha (\hat{\omega}(\lambda_1, \lambda_2; \alpha) - \hat{\omega}(\lambda_2, \lambda_1; \alpha)) \Big|_{\alpha=0}. \quad (139b)$$

Only the antisymmetric combination (139b) remains in the $\alpha \rightarrow 0$ limit of equation (124). The choice (139) is convenient for the numerical evaluation of the short-range correlation functions. As has been shown in Appendix B of [2] these functions coincide with the functions $\omega(\lambda_1, \lambda_2)$ and $\omega'(\lambda_1, \lambda_2)$ described in Section 4 of [1].

We have calculated the Fermionic basis representation of the two-point functions $\langle \sigma_1^z \sigma_n^z \rangle$ and $\langle \sigma_1^x \sigma_n^x \rangle$ for $n = 2, 3, 4, 5$ from (38), (129). At the last stage we have used Appendix B to replace the $\omega_{k,p}$ by derivatives of $\hat{\omega}$ and $\hat{\omega}'$. The results for $n = 2, 3, 4$ are the same as previously obtained in the inhomogeneous case [1, 4]. The formulae for $n = 5$ are new. They are shown in Appendix C. We have then used the representations of $\hat{\omega}(\lambda_1, \lambda_2)$ and $\hat{\omega}'(\lambda_1, \lambda_2)$ derived in [1] in order to evaluate the two-point functions numerically.

The graphs show a rich, non-monotonous behaviour of the correlation functions, reflecting the interplay of temperature, external magnetic field and the relative strength of

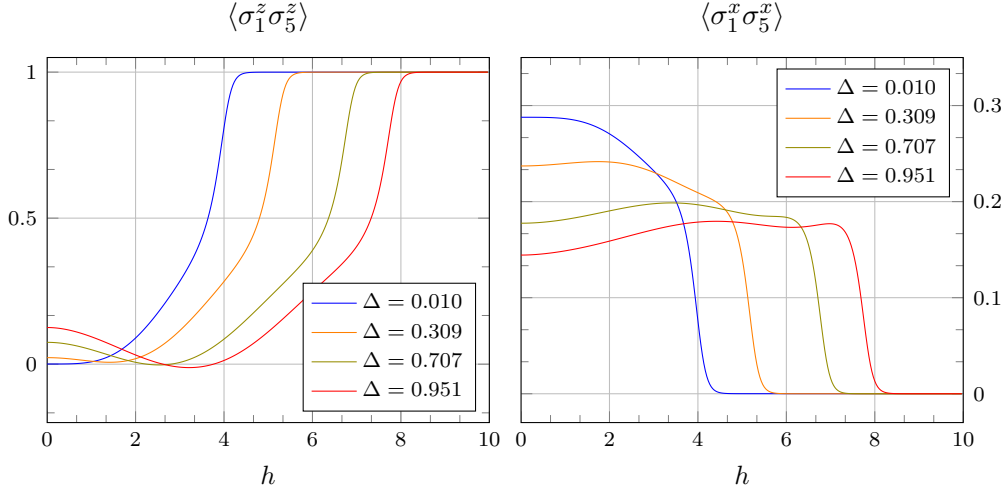


Figure 2: Fourth-neighbour correlation functions as functions of the magnetic field for various values of Δ in the massless regime and $T/J = 0.1$.

the Ising and exchange interactions. Fig. 2 shows how the fourth-neighbour two-point functions depend on the magnetic field at a relatively low temperature of $T/J = 0.1$ and for various values of the anisotropy parameter Δ . The longitudinal correlation functions saturate for h above the upper critical field $h_u = 4J(\Delta + 1)$, where the transverse correlation functions show a complementary behaviour and vanish. At $h = 0$ the longitudinal correlation functions show larger positive correlations for larger Δ , corresponding to an increased Ising interaction, which is the intuitively expected behavior. With increasing magnetic field the correlation functions first diminish before they start growing again in opposite order, such that the correlations are largest for the smallest Ising interaction. This is perhaps somewhat counterintuitive, but is at least in accordance with the fact that smaller Δ corresponds to a smaller saturation field.

Fig. 3 shows the temperature dependence of the two-point functions at $h = 0$ and for various values of Δ between -1 and 1 . For negative values of Δ we observe a characteristic sign change of the longitudinal correlation functions as the temperature increases. This sign change was first discovered in a numerical study [18], where it was interpreted as a ‘quantum to classical crossover’. Fig. 4 shows again the temperature dependence of the fourth-neighbour two-point function, this time at a fixed value $\Delta = 0.707$ of the anisotropy for various magnetic fields. We see that the correlations may change monotonously or non-monotonously depending on the value of the external field.

For the sake of completeness and also for comparison we have included plots of the two-point functions of shorter range, $n = 2, 3, 4$, in Figs. 5-7. We have shown these plots before in [1] with the same choice of parameters. But when we recomputed them for the present work we noticed that the data for anisotropy parameters close to $\Delta = 1$ did not have the accuracy claimed in that earlier work. This concerns mostly the plots for $n = 4$ and $\Delta = 0.995$ for which the numerical error was of an order of magnitude that could be recognized with the naked eye. The reason for the numerical error is that the individual terms in the Fermionic basis representation of the correlation functions of the two-point functions become very large in modulus as Δ goes to 1. The representation becomes a huge sum of large terms that alternate in sign and sum up to a small number. This is

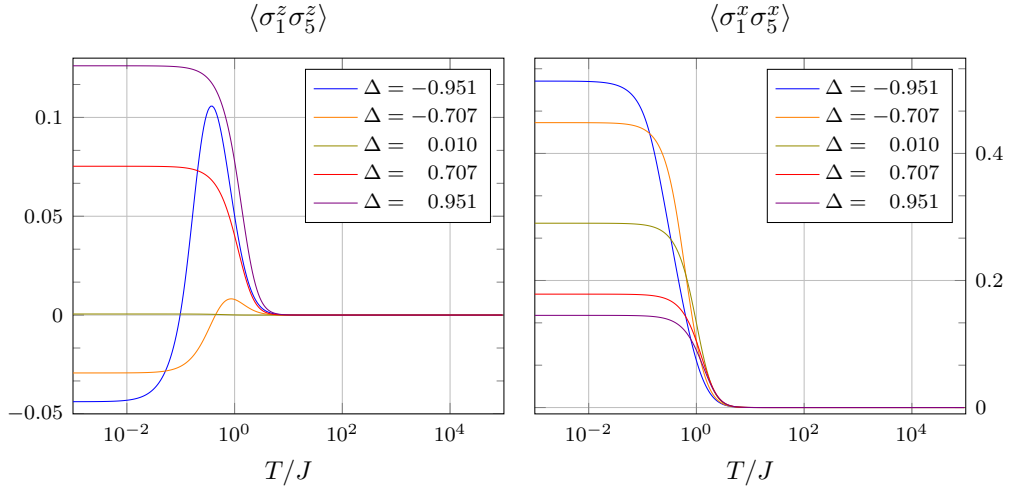


Figure 3: Fourth-neighbour correlation functions as functions of T/J for various values of Δ in the massless regime and $h = 0$.

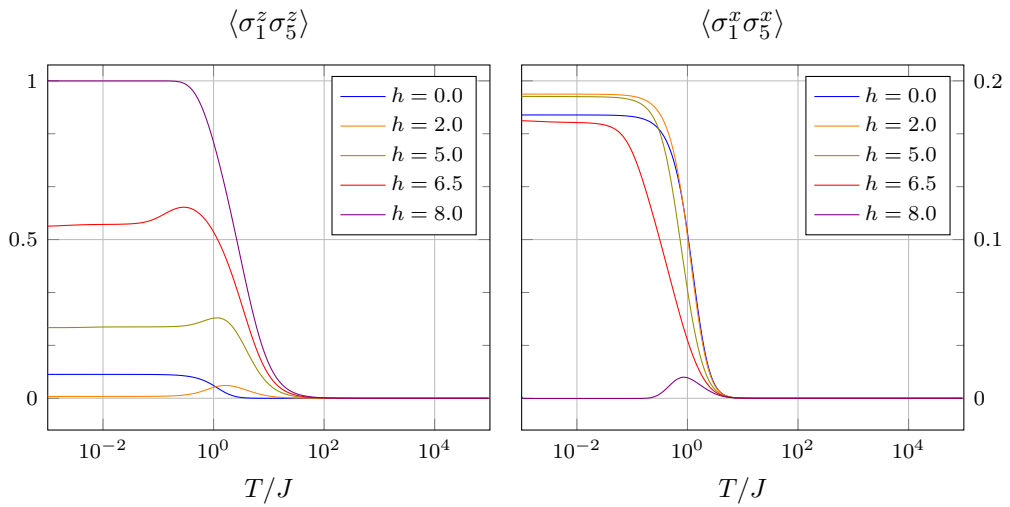


Figure 4: Fourth-neighbour correlation functions as functions of T/J for various magnetic fields and $\Delta = 0.707$.

numerically delicate and requires to have a good accuracy for the individual terms.

3.3 Comparison with asymptotic results

An interesting application of exact results is the test of asymptotic formulae which often do not come with error estimates. In the literature there are at least two different results for the large-distance asymptotics of the static correlation functions of the XXZ chain in the critical regime at zero and small finite temperature, the work of Lukyanov and Terras [38, 39], in which fully explicit formulae for $T = 0$ and $h = 0$ were derived, and the work [14, 16] treating the case of finite h at small T . The question we would like to answer is, how large is large, or, starting from which distance do the asymptotic formulae provide reliable approximations to the correlation functions.

Lukyanov and Terras consider the long-distance asymptotic behaviour of the two-point correlation functions $\langle \sigma_1^x \sigma_n^x \rangle$ and $\langle \sigma_1^z \sigma_n^z \rangle$ combining a Gaussian conformal field theory with input from the q -vertex operator approach applied to the XYZ chain [37]. They show that

$$\begin{aligned} \langle \sigma_1^x \sigma_n^x \rangle &\sim \frac{(-1)^n A}{n^\nu} \left\{ 1 - \frac{B}{n^{4/\nu-4}} + \mathcal{O}\left(n^{-2} \log n, n^{8-8/\nu}\right) \right\} \\ &\quad - \frac{\tilde{A}}{n^{\nu+1/\nu}} \left\{ 1 + \frac{\tilde{B}}{n^{2/\nu-2}} + \mathcal{O}\left(n^{-2} \log n, n^{4-4/\nu}\right) \right\} + \dots \end{aligned} \quad (140)$$

and

$$\begin{aligned} \langle \sigma_1^z \sigma_n^z \rangle &\sim -\frac{1}{\pi^2 \nu n^2} \left\{ 1 + \frac{\tilde{B}_z}{n^{4/\nu-4}} \left(1 + \frac{2-\nu}{2(1-\nu)} \right) + \mathcal{O}\left(n^{-2} \log n, n^{8-8/\nu}\right) \right\} \\ &\quad + \frac{(-1)^n A_z}{n^{1/\nu}} \left\{ 1 - \frac{B_z}{n^{2/\nu-2}} + \mathcal{O}\left(n^{-2} \log n, n^{4-4/\nu}\right) \right\} + \dots \end{aligned} \quad (141)$$

where the functions $A, B, \tilde{A}, \tilde{B}, A_z, B_z, \tilde{B}_z$ depend only on ν and are given explicitly in their work. Note that we have adapted their formulae to our conventions by supplying a factor of $(-1)^n$ to the transverse correlation functions. Lukyanov and Terras are using the Hamiltonian $H_{LT} = -H_L/2$ which is unitarily equivalent to H_L [54]. The unitary transformation is induced by the adjoint action of $U = \prod_{j=-L}^L \sigma_{2j}^z$ accompanied by a reparametrization of Δ which we can take into account with the identification

$$\nu = 1 - \frac{\gamma}{\pi}. \quad (142)$$

For a comparison with our results we take the definition of the function $\hat{\omega}$ from [1]. For $T = h = 0$ the auxiliary functions \mathfrak{b} and $\bar{\mathfrak{b}}$ given in that paper vanish. This reduces the calculation of the functions $\hat{\omega}$ and $\hat{\omega}'$ to the calculation of certain definite integrals which is easily done numerically on a computer.

In (140), (141) it depends on ν which of the terms is asymptotically dominant. For the sake of simplicity, we separate the asymptotically dominant terms only if they can be identified uniformly for all $0 < \nu < 1$. For this reason we consider two levels of approximations for $\langle \sigma_1^x \sigma_n^x \rangle$: one consists of only the first term $\frac{A}{n^\nu}$, which is the leading term for general ν , the other one consists of the whole expression. In the case of $\langle \sigma_1^z \sigma_n^z \rangle$

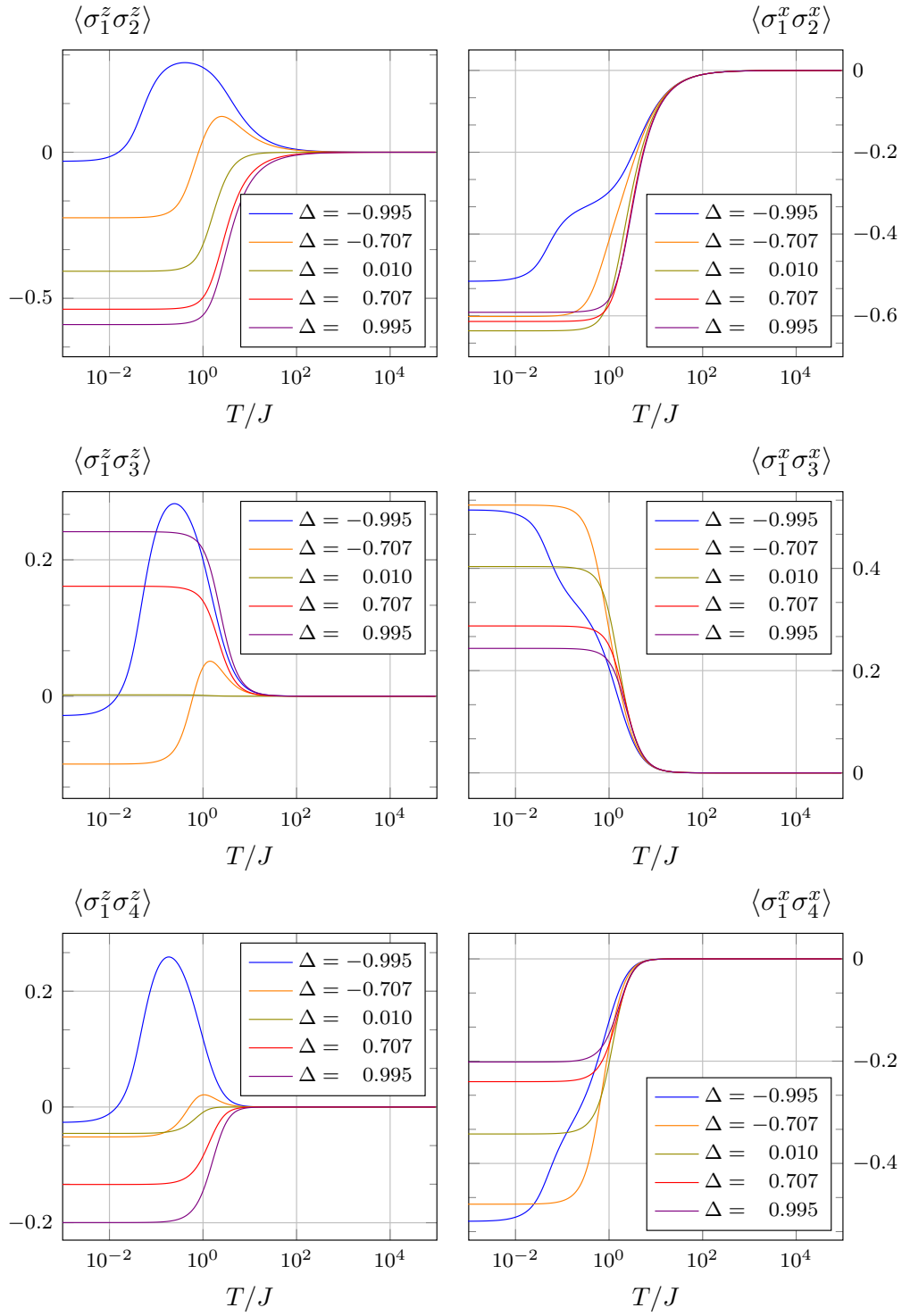


Figure 5: Short-range correlation functions ranging over up to four lattice sites as a function of temperature for various values of Δ in the massless regime and $h = 0$.

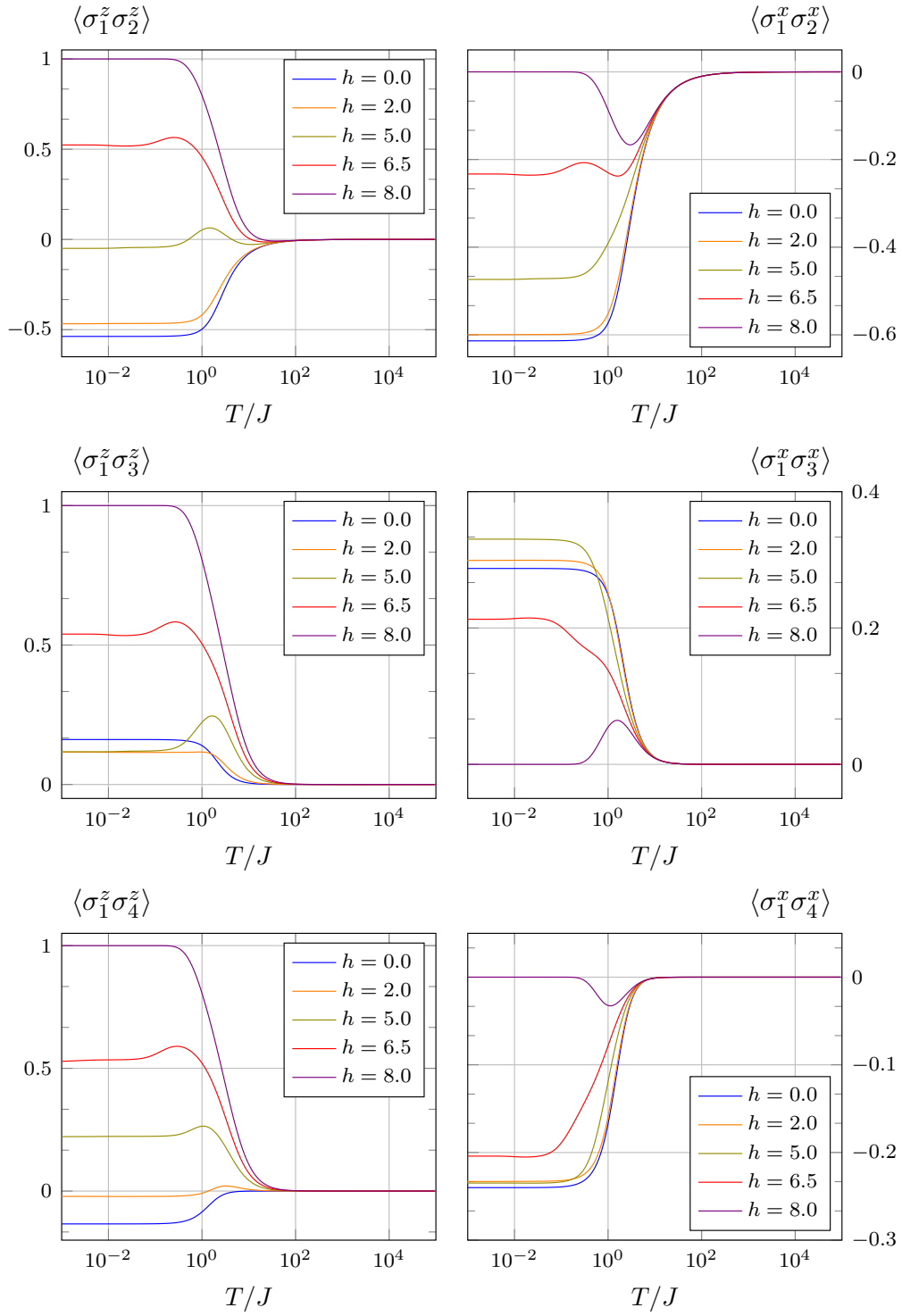


Figure 6: Short-range correlation functions ranging over up to four lattice sites as a function of temperature for various values of the magnetic field and $\Delta = 0.707$.

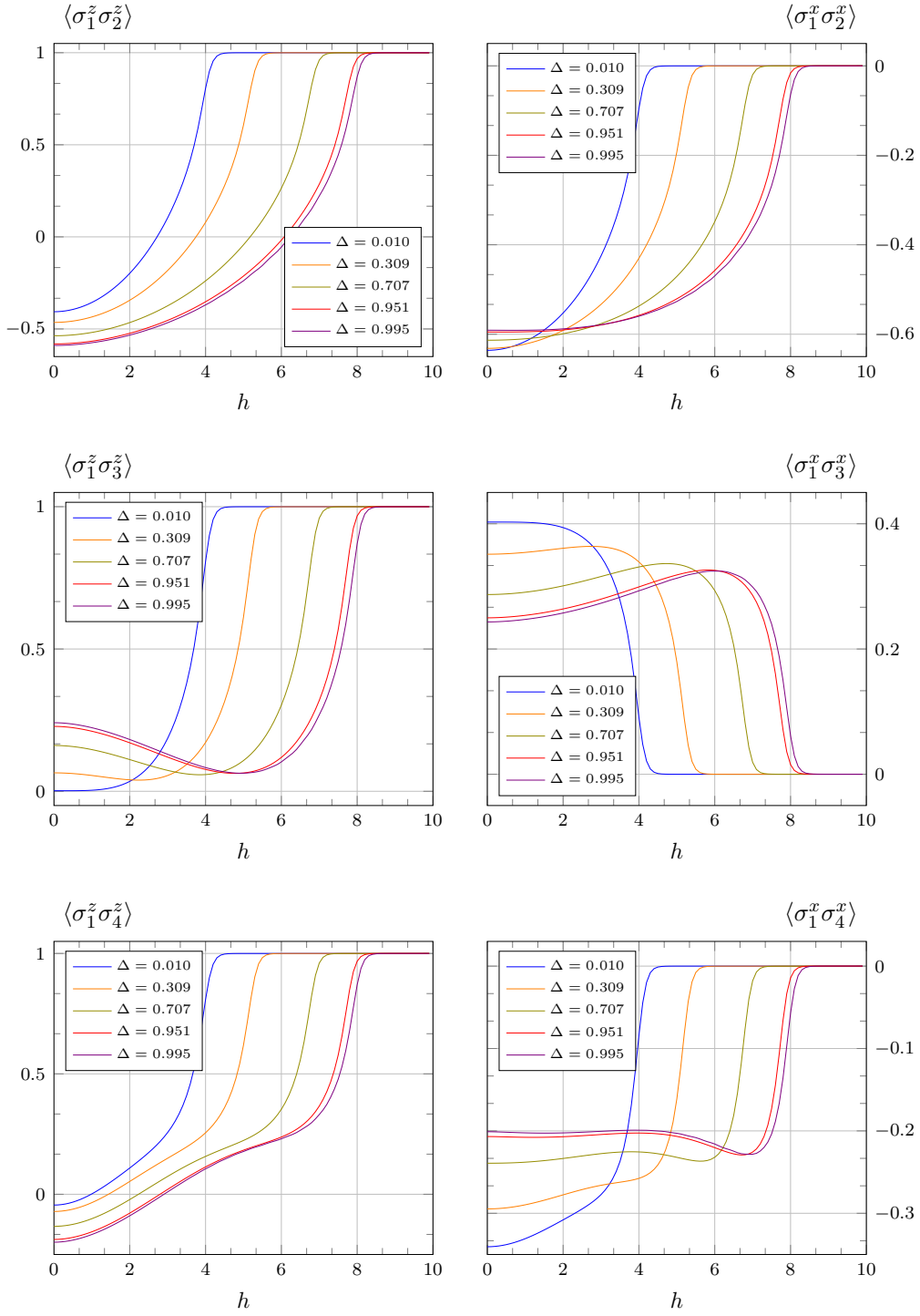


Figure 7: Short-range correlation functions ranging over up to four lattice sites as a function of the magnetic field for various value of Δ and $T/J = 0.1$.

we consider only the whole expression. In this case the term containing \tilde{B}_z can be seen to be of higher order than the rest. Still, it makes no visible difference whether we include it in our plots or not.

Looking at the figures 8 and 9 showing $\langle \sigma_1^z \sigma_n^z \rangle$ and $\langle \sigma_1^x \sigma_n^x \rangle$, respectively, we observe that there are poles in the asymptotic expansion. For $\Delta \rightarrow -1$, corresponding to $\nu \rightarrow 0$, rapid oscillations are visible in all plots, becoming less pronounced with increasing n . These oscillations are not a numerical error but rather a feature of the functions $B, \tilde{B}, B_z, \tilde{B}_z$. The poles visible in $\langle \sigma_1^x \sigma_n^x \rangle$ stem from the function B , which has poles of order 2 at $\nu = \frac{2}{3+2l}$ for $l \in \mathbb{N}$. At these positions q is a root of unity. As can be seen in the plots, the poles become narrower with increasing n .

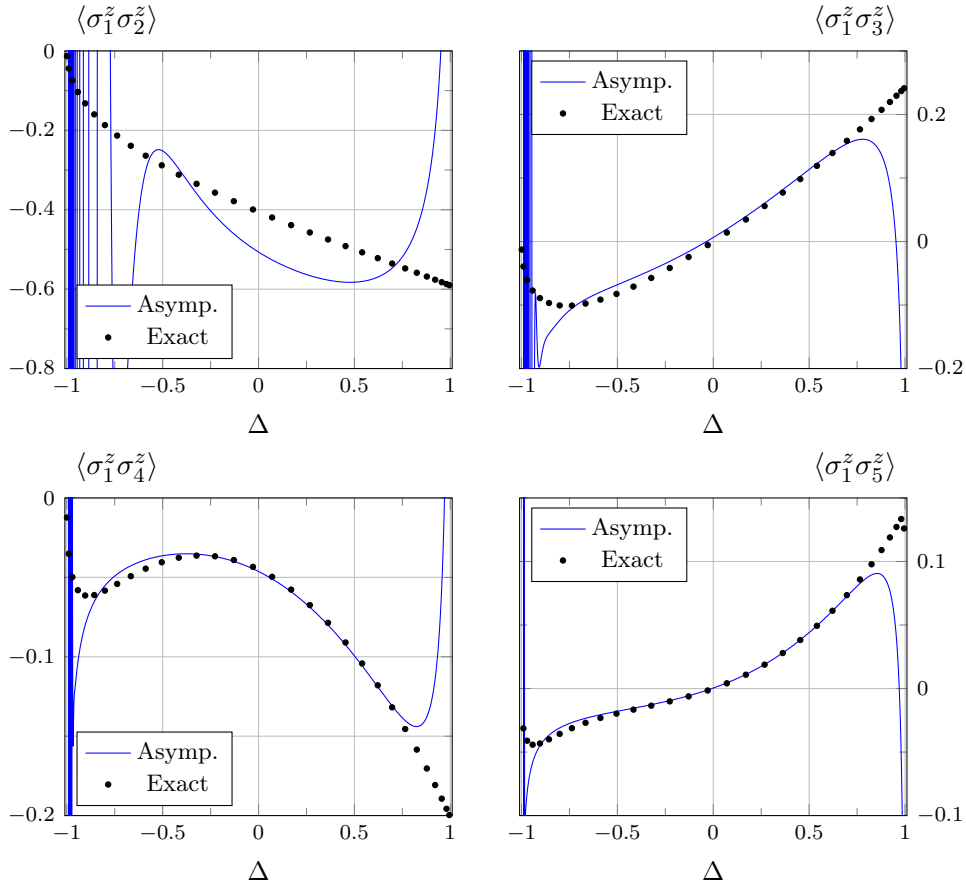


Figure 8: Two-point correlators $\langle \sigma_1^z \sigma_n^z \rangle$ for $n = 2, 3, 4, 5$ in the ground state and at zero magnetic field in the massless regime. Comparison of the asymptotic results of Lukyanov and Terras with our exact results.

Away from roots of unity, we observe the expected behaviour. For the nearest-neighbour functions the asymptotic expansion deviates considerably from the exact results. With increasing n the agreement between the results becomes better. It is noteworthy, that the asymptotics agrees very well with the exact values even for short distances, especially in the transverse case.

In [14, 16] the two-point functions of the XXZ chain in the low-temperature limit were studied within a thermal form factor approach [14]. In the low- T limit the long-distance asymptotics is determined by those terms in the form factor series pertaining to the quantum

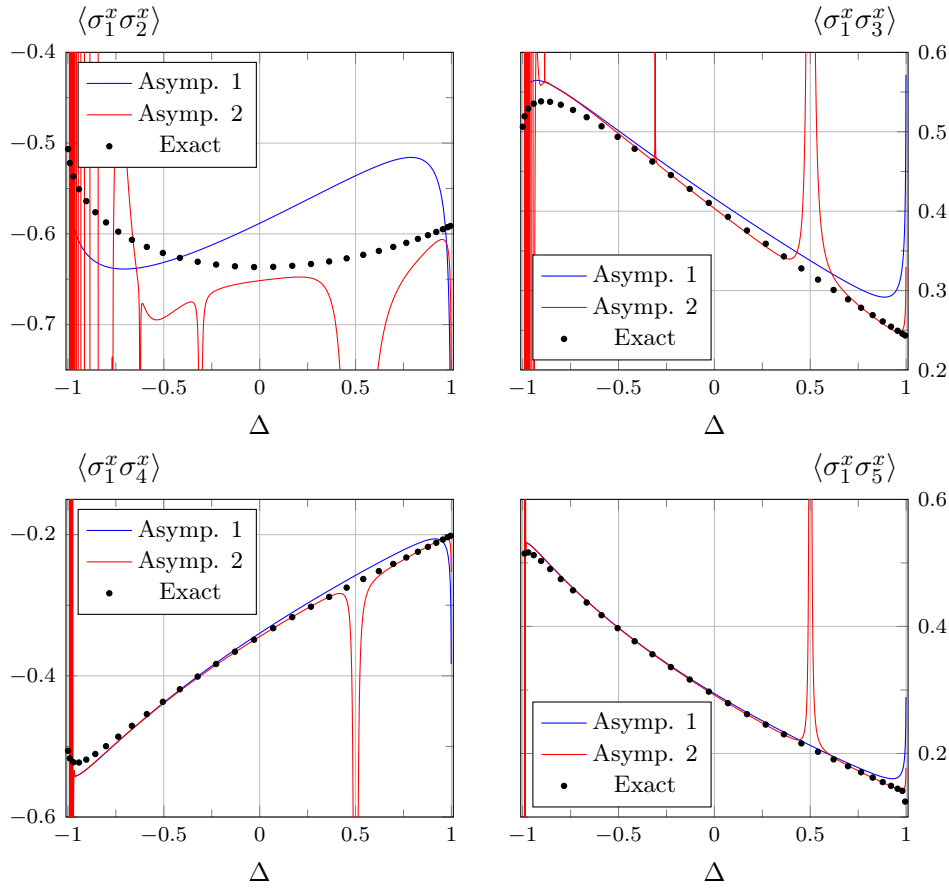


Figure 9: Two-point correlators $\langle \sigma_1^x \sigma_n^x \rangle$ for $n = 2, 3, 4, 5$ in the ground state and at zero magnetic field in the massless regime. Comparison of the asymptotic results of Lukyanov and Terras with our exact results.

transfer matrix of the model for which the correlation lengths diverge for $T \rightarrow 0$. Using a technique developed in [32] the authors of [14, 16] were able to sum these contributions and obtained, from a microscopic calculation, the expected asymptotic behaviour of a conformal field theory on a cylinder. The formulae for the amplitudes as functions of the magnetic field were new and numerically efficient and complemented those for $h = 0$ of Lukyanov and Terras.

Let us briefly recall the main results of [14, 16]. The required low- T data are the dressed charge function Z , the density of Bethe roots ρ , and the dressed energy ε . They are defined as the unique solutions of the Fredholm-type integral equations

$$Z(\lambda) = 1 + \int_{-Q}^Q \frac{d\mu}{2\pi i} K_0(\lambda - \mu) Z(\mu), \quad (143a)$$

$$\rho(\lambda) = -\frac{e(\lambda + i\gamma/2)}{2\pi i} + \int_{-Q}^Q \frac{d\mu}{2\pi i} K_0(\lambda - \mu) \rho(\mu), \quad (143b)$$

$$\varepsilon(\lambda) = \varepsilon_0(\lambda) + \int_{-Q}^Q \frac{d\mu}{2\pi i} K_0(\lambda - \mu) \varepsilon(\mu), \quad \varepsilon_0(\lambda) = h - \frac{4J(1 - \Delta^2)}{\text{ch}(2\lambda) - \Delta}. \quad (143c)$$

The two points $\pm Q$ are called the Fermi points and $Q > 0$ is determined by $\varepsilon(Q) = 0$.

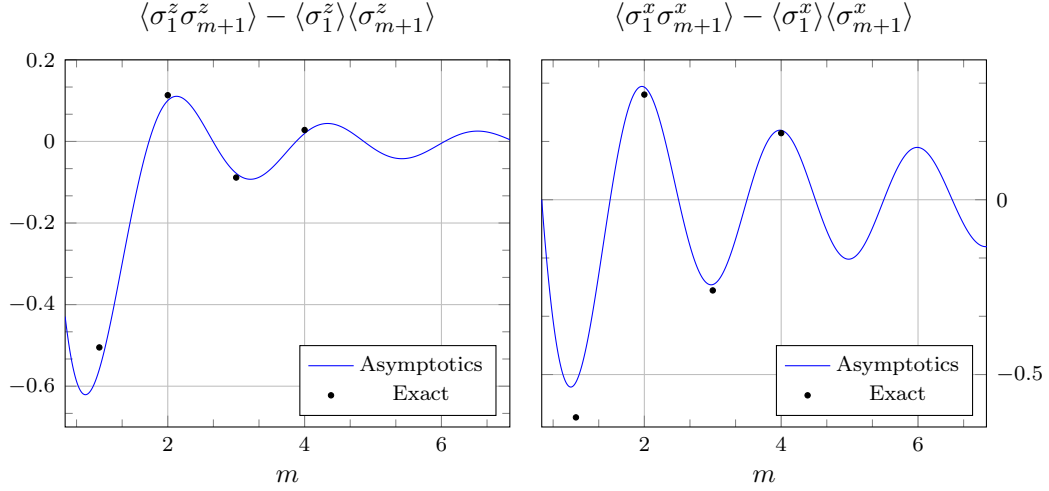


Figure 10: Comparison between the asymptotic expansion obtained in [14, 16] and our exact results for $\Delta = 0.6$, $h = 1$ and $T/J = 0.04$.

In [15] it was proven that such a Q exists and is unique. With these quantities we then define the Fermi momentum k_F , the Fermi sound velocity v_0 and the dressed charge \mathcal{Z} at the Fermi point,

$$k_F = 2\pi \int_0^Q d\lambda \rho(\lambda), \quad v_0 = \frac{\varepsilon'(Q)}{2\pi\rho(Q)}, \quad \mathcal{Z} = Z(Q). \quad (144)$$

The asymptotic expressions consist of products of amplitudes times terms that oscillate and decay with distance. The amplitudes $A_{0,n}^{zz}$ and $A_{0,0}^{-+}$ are slightly complicated expressions given in equations (90) and (97b) of [16]. We refrain from reproducing them here. The leading oscillating and decaying contribution can be expressed in terms of the above defined functions. For the longitudinal case the asymptotic behaviour takes the form

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle - \langle \sigma_1^z \rangle \langle \sigma_{m+1}^z \rangle \sim A_{0,0}^{zz} \left(\frac{\pi T/v_0}{\text{sh}(m\pi T/v_0)} \right)^2 + A_{0,1}^{zz} \cos(2mk_F) \left(\frac{\pi T/v_0}{\text{sh}(m\pi T/v_0)} \right)^{2\mathcal{Z}^2}. \quad (145)$$

Here the term containing $A_{0,0}^{zz}$ is the leading term for $\Delta < 0$, whereas the term with $A_{0,1}^{zz}$ is dominant for $\Delta > 0$. In the transverse case the asymptotic behaviour is described by

$$\langle \sigma_1^- \sigma_{m+1}^+ \rangle \sim A_{0,0}^{-+} (-1)^m \left(\frac{\pi T/v_0}{\text{sh}(m\pi T/v_0)} \right)^{\frac{1}{2\mathcal{Z}^2}}. \quad (146)$$

It should be noted that these expressions are numerically efficient and can be evaluated on a laptop computer in rather short time. Most of the numerical cost goes into the calculation of the amplitudes which are independent of the distance and of the temperature and have to be computed only once for given values of the anisotropy parameter and of the magnetic field.

Fig. 10 shows the comparison between the asymptotic and exact results as functions of the distance m . It can be seen that the asymptotics come very close to the exact results for

surprisingly small distances, starting with $m = 3, 4$. This is of course dependent on the chosen parameters. For example, close to the isotropic point the agreement becomes worse. Fig. 11 shows both results as a function of the external field h . Again, for distance $m = 4$ the agreement is remarkable even for a non-trivial structure as shown for the longitudinal case.

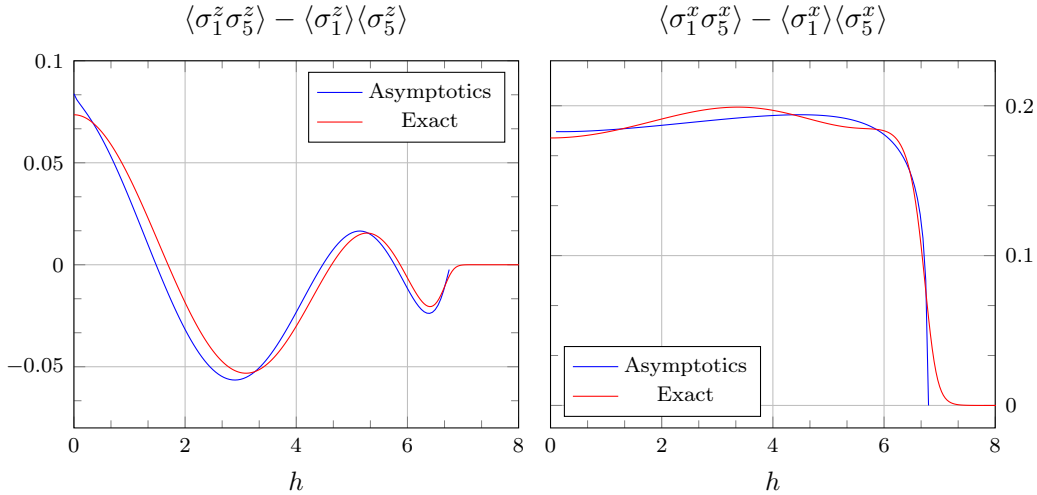


Figure 11: Comparison of exact and asymptotic behaviour as a function of the magnetic field for $n = 5$, $\Delta = 0.7$ and $T/J = 0.1$.

We can conclude that the asymptotic formulae derived in [14, 16] are very close to the exact results for surprisingly small values of the distance m . A rough estimation for the temperatures for which the asymptotics are valid would be $T/J < 0.1$. In addition, this comparison provides another test to see that our results are consistent with previous works.

4 Conclusions

We have given a descriptive review of the Fermionic basis approach to the theory of correlation functions of the XXZ chain. In the course of the review we worked out a few details that were omitted in the original literature. Our main interest in this work was to explore the efficiency of the exponential form in the homogenous case for the actual computation of short-range correlation functions. For this purpose we wrote out the mode expansions in explicit form and worked out the explicit formulae for the action of the modes on operators of finite length.

We further worked out explicitly the expressions for $\langle \sigma_1^x \sigma_n^x \rangle$ and $\langle \sigma_1^z \sigma_n^z \rangle$ for $n = 2, 3, 4, 5$ in terms of $\hat{\omega}$, $\hat{\omega}'$ and the derivatives of these functions. The lengthy result for $n = 5$ was heretofore unknown and is listed in Appendix C. We believe that the corresponding formulae for n up to 9 or 10 could be worked out in a similar way. However, the length of the final answer will be rapidly growing. It would fill many pages, and printing it out would make little sense. What would rather be needed would be a better understanding of the structure of such formulae. It was part of the motivation of the work [20] to guess this structure, but so far this attempt was only partially successful.

Using the explicit Fermionic basis expansions we have discussed in some details the two-point functions for the system coupled to a heat bath of temperature T . For this case we have also compared the exact correlation functions in the critical regime with asymptotic formulae for their large-distance behaviour. These asymptotic formulae, valid in the low- T limit, turned out to be very good approximations even for distances as short as three or four lattice sites.

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Appendix A From the JMS theorem to the exponential form

The JMS theorem [29] is central to the Fermionic basis approach. Here we show how the exponential form (38) of the reduced density matrix that had appeared earlier in the literature [4, 8, 11] is a natural consequence of the JMS theorem.

We define

$$\hat{\Omega} = \int_{\Gamma} \frac{d\zeta_1^2}{2\pi i \zeta_1^2} \int_{\Gamma} \frac{d\zeta_2^2}{2\pi i \zeta_2^2} \omega(\zeta_1, \zeta_2; \alpha) \mathbf{b}(\zeta_1) \mathbf{c}(\zeta_2), \quad (\text{A.1})$$

and

$$B(\zeta) = \int_{\Gamma} \frac{d\xi^2}{2\pi i \xi^2} \omega(\xi, \zeta; \alpha) \mathbf{b}(\xi), \quad C(\zeta) = \int_{\Gamma} \frac{d\xi^2}{2\pi i \xi^2} \omega(\zeta, \xi; \alpha) \mathbf{c}(\xi), \quad (\text{A.2})$$

where Γ is small circle around 1.

Lemma 4.

$$[e^{\hat{\Omega}}, \mathbf{b}^*(\zeta)] = -C(\zeta) e^{\hat{\Omega}}, \quad [e^{\hat{\Omega}}, \mathbf{c}^*(\zeta)] = B(\zeta) e^{\hat{\Omega}}. \quad (\text{A.3})$$

Proof. Using (27) we see that $[\hat{\Omega}, \mathbf{b}^*(\zeta)] = -C(\zeta)$, implying $[\hat{\Omega}^k, \mathbf{b}^*(\zeta)] = -C(\zeta) k \hat{\Omega}^{k-1}$. Hence,

$$[e^{\hat{\Omega}}, \mathbf{b}^*(\zeta)] = [e^{\hat{\Omega}} - 1, \mathbf{b}^*(\zeta)] = -C(\zeta) \sum_{k=1}^{\infty} \frac{\hat{\Omega}^{k-1}}{(k-1)!} = -C(\zeta) e^{\hat{\Omega}} \quad (\text{A.4})$$

which is the first identity. The second one follows in a similar way. \square

Lemma 5.

$$Z^{\kappa} \{ e^{\hat{\Omega}} \mathbf{t}^*(\zeta) X^{(\alpha)} \} = 2\rho(\zeta) Z^{\kappa} \{ e^{\hat{\Omega}} X^{(\alpha)} \}, \quad (\text{A.5a})$$

$$Z^{\kappa} \{ e^{\hat{\Omega}} \mathbf{b}^*(\zeta) X^{(\alpha+1)} \} = 0, \quad (\text{A.5b})$$

$$Z^{\kappa} \{ e^{\hat{\Omega}} \mathbf{c}^*(\zeta) X^{(\alpha-1)} \} = 0. \quad (\text{A.5c})$$

Proof. The first equation follows from the JMS theorem, since $[e^{\hat{\Omega}}, \mathbf{t}^*(\zeta)] = 0$. The JMS also implies that $Z^{\kappa} \{ (\mathbf{b}^*(\zeta) - C(\zeta)) e^{\hat{\Omega}} X^{(\alpha+1)} \} = Z^{\kappa} \{ (\mathbf{c}^*(\zeta) + B(\zeta)) e^{\hat{\Omega}} X^{(\alpha-1)} \} = 0$. Combining these two identities with Lemma 4 we obtain (A.5b), (A.5c). \square

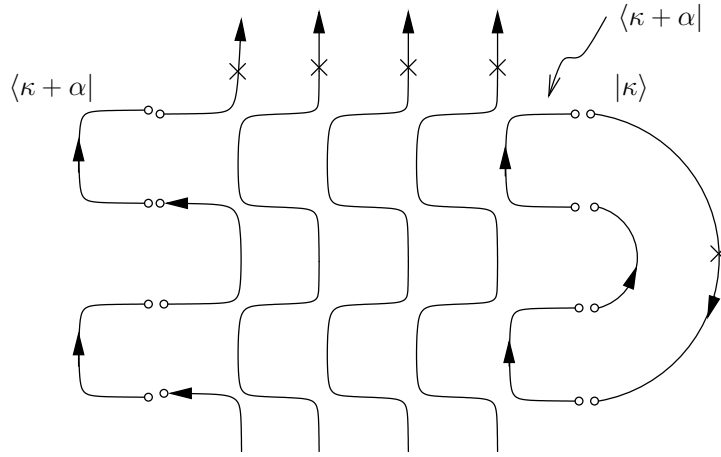


Figure 12: Graphical representation of the (unnormalized) reduced density matrix $D_{[k,l]}^N(T, \kappa, \alpha)$ in the limit $T \rightarrow \infty$. Crosses denote operators $q^{\kappa \sigma^z}$.

Lemma 5 means that the functional $Z^\kappa \{e^{\hat{\Omega}} \cdot\}$ acts as a left vacuum for the creation operators $\mathbf{b}^*(\zeta)$, $\mathbf{c}^*(\zeta)$. In general, this functional is hard to evaluate. A very special realization is obtained if we use the finite Trotter number approximant to the reduced density matrix $D_{[k,l]}^N(T, \kappa, \alpha)$ of the canonical ensemble for the definition of Z^κ . This means to put spin- $\frac{1}{2}$ representations on the horizontal lines in (16) and, in an alternating manner, spectral parameters $\pm \frac{c}{NT}$, where c is an appropriate constant (for more details see e.g. [21]). Sending then $T \rightarrow \infty$ at fixed κ all R -matrices in (16) degenerate into permutation matrices and the left and right eigenvectors become independent of α and κ (for a graphical representation of the limit see Fig. 12). All in all we see that Z^κ has the limit

$$\lim_{T \rightarrow \infty} Z^\kappa \{X(\alpha)\} = \frac{\text{tr}_{[k,l]} \{q^{2\kappa S_{[k,l]}} X_{[k,l]}\}}{\text{tr}_{[k,l]} \{q^{2\kappa S_{[k,l]}}\}} = \text{tr}^{-2\kappa} \{X(\alpha)\}, \quad (\text{A.6})$$

where we used the definition (33) of the κ -trace.

The limits $T \rightarrow \infty$ and $N \rightarrow \infty$ commute for $D_{[k,l]}^N(T, \kappa, \alpha)$ [22]. The unique solution of (132) for $T \rightarrow \infty$ is $\mathbf{a}_0(\lambda, \kappa) = q^{-2\kappa}$. Inserting it into (134) we obtain

$$\lim_{T \rightarrow \infty} \rho(\zeta) = \frac{q^{\kappa+\alpha} + q^{-\kappa-\alpha}}{q^\kappa + q^{-\kappa}}. \quad (\text{A.7})$$

Then, using the latter in (135)-(137), we see that the high- T limit of ω is

$$\lim_{T \rightarrow \infty} \omega(\zeta_1, \zeta_2; \alpha) = - \left(\frac{q^\kappa - q^{-\kappa}}{q^\kappa + q^{-\kappa}} \right)^2 \Delta\psi(\zeta_1/\zeta_2, \alpha). \quad (\text{A.8})$$

For the special value $\kappa = -\alpha/2$ we have

$$\lim_{T \rightarrow \infty} \rho(\zeta) \Big|_{\kappa=-\alpha/2} = 1, \quad \lim_{T \rightarrow \infty} \omega(\zeta_1, \zeta_2; \alpha) \Big|_{\kappa=-\alpha/2} = \omega_0(\zeta_1/\zeta_2; \alpha), \quad (\text{A.9})$$

where ω_0 was defined in (34). Setting

$$\hat{\Omega}_0 = \int_\Gamma \frac{d\zeta_1^2}{2\pi i \zeta_1^2} \int_\Gamma \frac{d\zeta_2^2}{2\pi i \zeta_2^2} \omega_0(\zeta_1/\zeta_2; \alpha) \mathbf{b}(\zeta_1) \mathbf{c}(\zeta_2) \quad (\text{A.10})$$

and taking the limit $T \rightarrow \infty$ at $\kappa = -\alpha/2$ in (A.5) we conclude that the following lemma holds true.

Lemma 6.

$$\mathrm{tr}^\alpha \{ e^{\hat{\Omega}_0} \mathbf{t}^*(\zeta) X^{(\alpha)} \} = 2 \mathrm{tr}^\alpha \{ e^{\hat{\Omega}_0} X^{(\alpha)} \}, \quad (\text{A.11a})$$

$$\mathrm{tr}^\alpha \{ e^{\hat{\Omega}_0} \mathbf{b}^*(\zeta) X^{(\alpha+1)} \} = 0, \quad (\text{A.11b})$$

$$\mathrm{tr}^\alpha \{ e^{\hat{\Omega}_0} \mathbf{c}^*(\zeta) X^{(\alpha-1)} \} = 0. \quad (\text{A.11c})$$

This lemma means that the functional $\mathrm{tr}^\alpha \{ e^{\hat{\Omega}_0} \cdot \}$ can be interpreted as a left vacuum for the creation operators $\mathbf{b}^*(\zeta)$ and $\mathbf{c}^*(\zeta)$. The lemma can be used to prove the validity of the exponential form, equation (37) of the main text, for $\kappa = 0$. For this purpose we introduce some additional notation. Let

$$\mathcal{B}^{(l)} = \mathbf{b}^*(\zeta_1^+) \dots \mathbf{b}^*(\zeta_l^+), \quad \mathcal{C}^{(l)} = \mathbf{c}^*(\zeta_l^-) \dots \mathbf{c}^*(\zeta_1^-). \quad (\text{A.12})$$

By $\mathcal{B}_j^{(l)}$ we denote $\mathcal{B}^{(l)}$ with the j th factor from the left omitted, whereas $\mathcal{C}_k^{(l)}$ will stand for $\mathcal{C}^{(l)}$ with the k th factor from the right omitted. We would further like to recall the definition (35) of the operator Ω . With (A.1) and (A.10) it can be expressed as $\Omega = \hat{\Omega}_0 - \hat{\Omega}$.

Lemma 7.

$$\mathrm{tr}^\alpha \{ e^{\Omega} \mathcal{B}^{(l)} \mathcal{C}^{(l)} q^{2\alpha S(0)} \} = \sum_{k=1}^l (-1)^{j+k} \omega(\zeta_j^+, \zeta_k^-; \alpha) \mathrm{tr}^\alpha \{ e^{\Omega} \mathcal{B}_j^{(l)} \mathcal{C}_k^{(l)} q^{2\alpha S(0)} \}. \quad (\text{A.13})$$

Proof.

$$\begin{aligned} & \mathrm{tr}^\alpha \{ e^{\Omega} \mathcal{B}^{(l)} \mathcal{C}^{(l)} q^{2\alpha S(0)} \} \\ &= (-1)^{j-1} \mathrm{tr}^\alpha \{ e^{\hat{\Omega}_0} e^{-\hat{\Omega}} \mathbf{b}^*(\zeta_j^+) \mathcal{B}_j^{(l)} \mathcal{C}^{(l)} q^{2\alpha S(0)} \} \\ &= (-1)^{j-1} \mathrm{tr}^\alpha \{ e^{\hat{\Omega}_0} (\mathbf{b}^*(\zeta_j^+) + C(\zeta_j^+)) e^{-\hat{\Omega}} \mathcal{B}_j^{(l)} \mathcal{C}^{(l)} q^{2\alpha S(0)} \} \\ &= (-1)^{j+l} \mathrm{tr}^\alpha \{ e^{\Omega} \mathcal{B}_j^{(l)} C(\zeta_j^+) \mathcal{C}^{(l)} q^{2\alpha S(0)} \} \\ &= \sum_{k=1}^l (-1)^{j+k} \mathrm{tr}^\alpha \{ e^{\Omega} \mathcal{B}_j^{(l)} [C(\zeta_j^+), \mathbf{c}^*(\zeta_k^-)]_+ \mathcal{C}_k^{(l)} q^{2\alpha S(0)} \} \\ &= \sum_{k=1}^l (-1)^{j+k} \omega(\zeta_j^+, \zeta_k^-; \alpha) \mathrm{tr}^\alpha \{ e^{\Omega} \mathcal{B}_j^{(l)} \mathcal{C}_k^{(l)} q^{2\alpha S(0)} \}. \quad (\text{A.14}) \end{aligned}$$

Here we used Lemma 4 in the second equation and Lemma 6 in the third equation. \square

Let

$$\mathcal{T}^{(k)} = \mathbf{t}^*(\zeta_1^0) \dots \mathbf{t}^*(\zeta_k^0). \quad (\text{A.15})$$

Lemma 8.

$$\mathrm{tr}^\alpha \{ e^{\Omega} \mathcal{T}^{(k)} \mathcal{B}^{(l)} \mathcal{C}^{(l)} q^{2\alpha S(0)} \} = 2^k \det_{m,n=1,\dots,l} \{ \omega(\zeta_m^+, \zeta_n^-; \alpha) \}. \quad (\text{A.16})$$

Proof. The operators $\mathcal{T}^{(k)}$ and $e^{\hat{\Omega}}$ commute. It follows from Lemma 6 that

$$\mathrm{tr}^\alpha \{ e^{\hat{\Omega}} \mathcal{T}^{(k)} \mathcal{B}^{(l)} \mathcal{C}^{(l)} q^{2\alpha S(0)} \} = 2^k \mathrm{tr}^\alpha \{ e^{\hat{\Omega}} \mathcal{B}^{(l)} \mathcal{C}^{(l)} q^{2\alpha S(0)} \}. \quad (\text{A.17})$$

Furthermore, since $e^{\hat{\Omega}} q^{2\alpha S(0)} = \mathrm{id}$, Lemma 7 in conjunction with the Laplace expansion formula for determinants implies that

$$\mathrm{tr}^\alpha \{ e^{\hat{\Omega}} \mathcal{B}^{(l)} \mathcal{C}^{(l)} q^{2\alpha S(0)} \} = \det_{m,n=1,\dots,l} \{ \omega(\zeta_m^+, \zeta_n^-; \alpha) \}. \quad (\text{A.18})$$

□

Now recall that the function ρ was originally defined as an eigenvalue ratio in (30). If the eigenvalue in the definition of ρ is non-degenerate, ρ is an even function of its second argument κ . This is, for instance, the case with the dominant eigenvalue of the quantum transfer matrix that occurs in the description of the finite temperature reduced density matrix. Then (30) implies that $\rho|_{\kappa=-\alpha/2} = 1$. Combining Corollary 1 and Lemma 8 we conclude that

$$Z^{-\alpha/2} \{ \mathcal{T}^{(k)} \mathcal{B}^{(l)} \mathcal{C}^{(l)} q^{2\alpha S(0)} \} = \mathrm{tr}^\alpha \{ e^{\hat{\Omega}} \mathcal{T}^{(k)} \mathcal{B}^{(l)} \mathcal{C}^{(l)} q^{2\alpha S(0)} \}. \quad (\text{A.19})$$

Inserting the mode expansions and using Theorem 1 we arrive at equation (36) and (37) of the main text.

We claim that (37) remains valid for non-zero values of κ if we restrict the action of Z^κ to spin-reversal invariant operators such as $\sigma_1^z \sigma_n^z$ or $\sigma_1^x \sigma_n^x$. We have checked this by direct use of the Fermionic basis for $n = 1, 2, 3$ [34]. It also follows if we assume the existence of an operator $\mathbf{t}(\zeta)$ conjugate to $\mathbf{t}^*(\zeta)$ [34]. Such an operator has been defined by its properties in [2] in the inhomogeneous case and in [34] in the homogeneous case. In the inhomogeneous case we have checked that the postulated properties are sufficient to fix the operator $\mathbf{t}(\zeta)$ for $n = 1, 2$ in [2] and for $n = 3$ in [53].

Appendix B Multiplicative and additive spectral parameters

The following formula can be used to switch from the definition of the function ω with multiplicative spectral parameters to the corresponding function $\hat{\omega}$ with additive spectral parameters favoured in [1]. Let $u = \ln(\zeta)$. Then

$$\begin{aligned} \partial_{\zeta^2}^k \zeta^{2\ell} f(\ln(\zeta)) &= \left(\frac{1}{2} e^{-2u} \partial_u \right)^k e^{2u\ell} f(u) = \\ &= e^{2u(\ell-k)} (-1)^k \left(-\ell - \frac{1}{2} \partial_u \right) \dots \left(-\ell - \frac{1}{2} \partial_u + k - 1 \right) f(u) \end{aligned} \quad (\text{B.1})$$

for all $\ell \in \mathbb{Z}$. Introducing the Pochhammer symbol

$$(x)_k = x(x+1) \dots (x+k-1) \quad (\text{B.2})$$

we obtain

$$\partial_{\zeta^2}^k \zeta^{2\ell} f(\ln(\zeta)) \Big|_{\zeta^2=1} = (-1)^k \left(-\ell - \frac{1}{2} \partial_u \right)_k f(u) \Big|_{u=0}, \quad (\text{B.3})$$

which can be nicely implemented on a computer.

Appendix C Two-point functions for $n = 5$

This appendix contains the explicit expressions for the two independent two-point functions $\langle \sigma_1^x \sigma_5^x \rangle$ and $\langle \sigma_1^z \sigma_5^z \rangle$. In the following we shall employ the shorthand notation $\hat{\omega}_{ij} = \partial_\lambda^i \partial_\mu^j \hat{\omega}(\lambda, \mu)_{\lambda=\mu=0}$ and $\hat{\omega}'_{ij} = \partial_\lambda^i \partial_\mu^j \hat{\omega}'(\lambda, \mu)_{\lambda=\mu=0}$.

$$\begin{aligned}
\langle \sigma_1^z \sigma_5^z \rangle = & \\
& \{ -8\eta^2 q^2 \{ 112\hat{\omega}_{11}\hat{\omega}_{20} - 112\hat{\omega}_{10}\hat{\omega}_{21} - 120\hat{\omega}_{21}^2 + 120\hat{\omega}_{11}\hat{\omega}_{22} + 40\hat{\omega}_{21}\hat{\omega}_{30} + 12\hat{\omega}_{22}\hat{\omega}_{31} \\
& - 8\hat{\omega}_{31}^2 - 40\hat{\omega}_{10}\hat{\omega}_{32} - 12\hat{\omega}_{21}\hat{\omega}_{32} + 4\hat{\omega}_{30}\hat{\omega}_{32} + 8\hat{\omega}_{11}\hat{\omega}_{33} - 4\hat{\omega}_{20}\hat{\omega}_{33} \\
& - 20\hat{\omega}_{11}\hat{\omega}_{40} - 3\hat{\omega}_{22}\hat{\omega}_{40} + 2\hat{\omega}_{31}\hat{\omega}_{40} + 20\hat{\omega}_{10}\hat{\omega}_{41} + 6\hat{\omega}_{21}\hat{\omega}_{41} - 2\hat{\omega}_{30}\hat{\omega}_{41} \\
& - 32q^{14} (9216\hat{\omega} + 18112\hat{\omega}_{11} - 12864\hat{\omega}_{20} - 3084\hat{\omega}_{22} + 3016\hat{\omega}_{31} + 148\hat{\omega}_{33} - 240\hat{\omega}_{40} - 111\hat{\omega}_{42}) \\
& - 6\hat{\omega}_{11}\hat{\omega}_{42} + 3\hat{\omega}_{20}\hat{\omega}_{42} \\
& + q^{18} (-51200\hat{\omega}_{10}^2 + 25600\hat{\omega}_{11} - 9216\hat{\omega}_{20} - 233488\hat{\omega}_{11}\hat{\omega}_{20} + 155136\hat{\omega}_{20}^2 - 15288\hat{\omega}_{21}^2 \\
& + 14016\hat{\omega}_{22} + 15288\hat{\omega}_{11}\hat{\omega}_{22} + 11240\hat{\omega}_{21}\hat{\omega}_{30} - 2048\hat{\omega}_{30}^2 - 18560\hat{\omega}_{31} - 6144\hat{\omega}_{20}\hat{\omega}_{31} \\
& - 1236\hat{\omega}_{22}\hat{\omega}_{31} + 824\hat{\omega}_{31}^2 + 1236\hat{\omega}_{21}\hat{\omega}_{32} - 412\hat{\omega}_{30}\hat{\omega}_{32} - 1856\hat{\omega}_{33} - 824\hat{\omega}_{11}\hat{\omega}_{33} \\
& + 412\hat{\omega}_{20}\hat{\omega}_{33} + 2304\hat{\omega}_{40} - 2548\hat{\omega}_{11}\hat{\omega}_{40} + 1536\hat{\omega}_{20}\hat{\omega}_{40} + 309\hat{\omega}_{22}\hat{\omega}_{40} - 206\hat{\omega}_{31}\hat{\omega}_{40} \\
& - 618\hat{\omega}_{21}\hat{\omega}_{41} + 206\hat{\omega}_{30}\hat{\omega}_{41} + 4\hat{\omega}_{10} (58372\hat{\omega}_{21} - 25856\hat{\omega}_{30} + 637 (-2\hat{\omega}_{32} + \hat{\omega}_{41})) \\
& + 512\hat{\omega} (-864 + 100\hat{\omega}_{11} - 303\hat{\omega}_{22} + 202\hat{\omega}_{31} + 4\hat{\omega}_{33} - 3\hat{\omega}_{42}) \\
& + 1392\hat{\omega}_{42} + 618\hat{\omega}_{11}\hat{\omega}_{42} - 309\hat{\omega}_{20}\hat{\omega}_{42}) \\
& + q^{12} (-456704\hat{\omega}_{10}^2 + 151552\hat{\omega}_{11} - 142848\hat{\omega}_{20} - 77136\hat{\omega}_{11}\hat{\omega}_{20} + 81408\hat{\omega}_{20}^2 \\
& - 16920\hat{\omega}_{21}^2 - 69504\hat{\omega}_{22} + 16920\hat{\omega}_{11}\hat{\omega}_{22} + 21000\hat{\omega}_{21}\hat{\omega}_{30} - 5120\hat{\omega}_{30}^2 + 69376\hat{\omega}_{31} \\
& - 15360\hat{\omega}_{20}\hat{\omega}_{31} - 900\hat{\omega}_{22}\hat{\omega}_{31} + 600\hat{\omega}_{31}^2 + 900\hat{\omega}_{21}\hat{\omega}_{32} - 300\hat{\omega}_{30}\hat{\omega}_{32} + 3712\hat{\omega}_{33} \\
& - 600\hat{\omega}_{11}\hat{\omega}_{33} + 300\hat{\omega}_{20}\hat{\omega}_{33} - 5760\hat{\omega}_{40} - 2820\hat{\omega}_{11}\hat{\omega}_{40} + 3840\hat{\omega}_{20}\hat{\omega}_{40} + 225\hat{\omega}_{22}\hat{\omega}_{40} \\
& - 150\hat{\omega}_{31}\hat{\omega}_{40} - 450\hat{\omega}_{21}\hat{\omega}_{41} + 150\hat{\omega}_{30}\hat{\omega}_{41} + 4\hat{\omega}_{10} (19284\hat{\omega}_{21} - 13568\hat{\omega}_{30} + 705 (-2\hat{\omega}_{32} + \hat{\omega}_{41})) \\
& + 256\hat{\omega} (-2880 + 1784\hat{\omega}_{11} - 318\hat{\omega}_{22} + 212\hat{\omega}_{31} + 20\hat{\omega}_{33} - 15\hat{\omega}_{42}) \\
& - 2784\hat{\omega}_{42} + 450\hat{\omega}_{11}\hat{\omega}_{42} - 225\hat{\omega}_{20}\hat{\omega}_{42}) \\
& + q^4 (7168\hat{\omega}_{10}^2 - 7296\hat{\omega}_{20}^2 + 840\hat{\omega}_{21}^2 - 576\hat{\omega}_{22} + 7296\hat{\omega}\hat{\omega}_{22} - 2200\hat{\omega}_{21}\hat{\omega}_{30} \\
& + 640\hat{\omega}_{30}^2 + 384\hat{\omega}_{31} - 4864\hat{\omega}\hat{\omega}_{31} + 1920\hat{\omega}_{20}\hat{\omega}_{31} - 84\hat{\omega}_{22}\hat{\omega}_{31} + 56\hat{\omega}_{31}^2 \\
& + 84\hat{\omega}_{21}\hat{\omega}_{32} - 28\hat{\omega}_{30}\hat{\omega}_{32} - 192\hat{\omega}_{33} - 640\hat{\omega}\hat{\omega}_{33} + 28\hat{\omega}_{20}\hat{\omega}_{33} - 480\hat{\omega}_{20}\hat{\omega}_{40} \\
& + 21\hat{\omega}_{22}\hat{\omega}_{40} - 14\hat{\omega}_{31}\hat{\omega}_{40} - 42\hat{\omega}_{21}\hat{\omega}_{41} + 14\hat{\omega}_{30}\hat{\omega}_{41} \\
& - 4\hat{\omega}_{10} (3932\hat{\omega}_{21} - 1216\hat{\omega}_{30} - 70\hat{\omega}_{32} + 35\hat{\omega}_{41}) \\
& - 2\hat{\omega}_{11} (-768 + 3584\hat{\omega} - 7864\hat{\omega}_{20} + 420\hat{\omega}_{22} + 28\hat{\omega}_{33} - 70\hat{\omega}_{40} - 21\hat{\omega}_{42}) \\
& + 144\hat{\omega}_{42} + 480\hat{\omega}\hat{\omega}_{42} - 21\hat{\omega}_{20}\hat{\omega}_{42}) \\
& + q^{28} (120\hat{\omega}_{21}^2 - 40\hat{\omega}_{21}\hat{\omega}_{30} - 12\hat{\omega}_{22}\hat{\omega}_{31} + 8\hat{\omega}_{31}^2 + 12\hat{\omega}_{21}\hat{\omega}_{32} - 4\hat{\omega}_{30}\hat{\omega}_{32} \\
& + 4\hat{\omega}_{20}\hat{\omega}_{33} + 3\hat{\omega}_{22}\hat{\omega}_{40} - 2\hat{\omega}_{31}\hat{\omega}_{40} + 4\hat{\omega}_{10} (28\hat{\omega}_{21} + 10\hat{\omega}_{32} - 5\hat{\omega}_{41}) \\
& - 6\hat{\omega}_{21}\hat{\omega}_{41} + 2\hat{\omega}_{30}\hat{\omega}_{41} - 2\hat{\omega}_{11} (56\hat{\omega}_{20} + 60\hat{\omega}_{22} + 4\hat{\omega}_{33} - 10\hat{\omega}_{40} - 3\hat{\omega}_{42}) \\
& - 3\hat{\omega}_{20}\hat{\omega}_{42}) \\
& + 8q^6 (2176\hat{\omega}_{10}^2 - 1968\hat{\omega}_{20}^2 - 228\hat{\omega}_{21}^2 - 120\hat{\omega}_{22} + 1968\hat{\omega}\hat{\omega}_{22} - 260\hat{\omega}_{21}\hat{\omega}_{30} \\
& + 112\hat{\omega}_{30}^2 + 80\hat{\omega}_{31} - 1312\hat{\omega}\hat{\omega}_{31} + 336\hat{\omega}_{20}\hat{\omega}_{31} + 12\hat{\omega}_{22}\hat{\omega}_{31} - 8\hat{\omega}_{31}^2 - 12\hat{\omega}_{21}\hat{\omega}_{32}
\end{aligned}$$

$$\begin{aligned}
& + 4\hat{\omega}_{30}\hat{\omega}_{32} + 8\hat{\omega}_{33} - 112\hat{\omega}\hat{\omega}_{33} - 4\hat{\omega}_{20}\hat{\omega}_{33} - 84\hat{\omega}_{20}\hat{\omega}_{40} - 3\hat{\omega}_{22}\hat{\omega}_{40} \\
& + 2\hat{\omega}_{31}\hat{\omega}_{40} + 6\hat{\omega}_{21}\hat{\omega}_{41} - 2\hat{\omega}_{30}\hat{\omega}_{41} \\
& + \hat{\omega}_{10} (-4168\hat{\omega}_{21} + 1312\hat{\omega}_{30} - 76\hat{\omega}_{32} + 38\hat{\omega}_{41}) \\
& + \hat{\omega}_{11} (128 - 2176\hat{\omega} + 4168\hat{\omega}_{20} + 228\hat{\omega}_{22} + 8\hat{\omega}_{33} - 38\hat{\omega}_{40} - 6\hat{\omega}_{42}) \\
& - 6\hat{\omega}_{42} + 84\hat{\omega}\hat{\omega}_{42} + 3\hat{\omega}_{20}\hat{\omega}_{42}) \\
& + q^2 (4096\hat{\omega}_{10}^2 - 3840\hat{\omega}_{20}^2 + 216\hat{\omega}_{21}^2 - 960\hat{\omega}_{22} + 3840\hat{\omega}\hat{\omega}_{22} - 840\hat{\omega}_{21}\hat{\omega}_{30} + 256\hat{\omega}_{30}^2 \\
& + 640\hat{\omega}_{31} - 2560\hat{\omega}\hat{\omega}_{31} + 768\hat{\omega}_{20}\hat{\omega}_{31} + 36\hat{\omega}_{22}\hat{\omega}_{31} - 24\hat{\omega}_{31}^2 - 36\hat{\omega}_{21}\hat{\omega}_{32} + 12\hat{\omega}_{30}\hat{\omega}_{32} \\
& + 64\hat{\omega}_{33} - 256\hat{\omega}\hat{\omega}_{33} - 12\hat{\omega}_{20}\hat{\omega}_{33} - 192\hat{\omega}_{20}\hat{\omega}_{40} - 9\hat{\omega}_{22}\hat{\omega}_{40} + 6\hat{\omega}_{31}\hat{\omega}_{40} \\
& + 18\hat{\omega}_{21}\hat{\omega}_{41} - 6\hat{\omega}_{30}\hat{\omega}_{41} - 4\hat{\omega}_{10} (1524\hat{\omega}_{21} - 640\hat{\omega}_{30} - 18\hat{\omega}_{32} + 9\hat{\omega}_{41}) \\
& + \hat{\omega}_{11} (1024 - 4096\hat{\omega} + 6096\hat{\omega}_{20} - 216\hat{\omega}_{22} + 24\hat{\omega}_{33} + 36\hat{\omega}_{40} - 18\hat{\omega}_{42}) \\
& - 48\hat{\omega}_{42} + 192\hat{\omega}\hat{\omega}_{42} + 9\hat{\omega}_{20}\hat{\omega}_{42}) \\
& + q^{24} (-7168\hat{\omega}_{10}^2 + 7296\hat{\omega}_{20}^2 - 840\hat{\omega}_{21}^2 - 576\hat{\omega}_{22} - 7296\hat{\omega}\hat{\omega}_{22} + 2200\hat{\omega}_{21}\hat{\omega}_{30} \\
& - 640\hat{\omega}_{30}^2 + 384\hat{\omega}_{31} + 4864\hat{\omega}\hat{\omega}_{31} - 1920\hat{\omega}_{20}\hat{\omega}_{31} + 84\hat{\omega}_{22}\hat{\omega}_{31} - 56\hat{\omega}_{31}^2 - 84\hat{\omega}_{21}\hat{\omega}_{32} \\
& + 28\hat{\omega}_{30}\hat{\omega}_{32} - 192\hat{\omega}_{33} + 640\hat{\omega}\hat{\omega}_{33} - 28\hat{\omega}_{20}\hat{\omega}_{33} + 480\hat{\omega}_{20}\hat{\omega}_{40} - 21\hat{\omega}_{22}\hat{\omega}_{40} \\
& + 14\hat{\omega}_{31}\hat{\omega}_{40} + 42\hat{\omega}_{21}\hat{\omega}_{41} - 14\hat{\omega}_{30}\hat{\omega}_{41} + 4\hat{\omega}_{10} (3932\hat{\omega}_{21} - 1216\hat{\omega}_{30} - 70\hat{\omega}_{32} + 35\hat{\omega}_{41}) \\
& + 2\hat{\omega}_{11} (768 + 3584\hat{\omega} - 7864\hat{\omega}_{20} + 420\hat{\omega}_{22} + 28\hat{\omega}_{33} - 70\hat{\omega}_{40} - 21\hat{\omega}_{42}) \\
& + 144\hat{\omega}_{42} - 480\hat{\omega}\hat{\omega}_{42} + 21\hat{\omega}_{20}\hat{\omega}_{42}) \\
& + q^{16} (456704\hat{\omega}_{10}^2 + 151552\hat{\omega}_{11} - 142848\hat{\omega}_{20} + 77136\hat{\omega}_{11}\hat{\omega}_{20} - 81408\hat{\omega}_{20}^2 + 16920\hat{\omega}_{21}^2 \\
& - 69504\hat{\omega}_{22} - 16920\hat{\omega}_{11}\hat{\omega}_{22} - 21000\hat{\omega}_{21}\hat{\omega}_{30} + 5120\hat{\omega}_{30}^2 + 69376\hat{\omega}_{31} + 15360\hat{\omega}_{20}\hat{\omega}_{31} \\
& + 900\hat{\omega}_{22}\hat{\omega}_{31} - 600\hat{\omega}_{31}^2 - 900\hat{\omega}_{21}\hat{\omega}_{32} + 300\hat{\omega}_{30}\hat{\omega}_{32} + 3712\hat{\omega}_{33} + 600\hat{\omega}_{11}\hat{\omega}_{33} \\
& - 300\hat{\omega}_{20}\hat{\omega}_{33} - 5760\hat{\omega}_{40} + 2820\hat{\omega}_{11}\hat{\omega}_{40} - 3840\hat{\omega}_{20}\hat{\omega}_{40} - 225\hat{\omega}_{22}\hat{\omega}_{40} + 150\hat{\omega}_{31}\hat{\omega}_{40} \\
& + \hat{\omega}_{10} (-77136\hat{\omega}_{21} + 54272\hat{\omega}_{30} + 5640\hat{\omega}_{32} - 2820\hat{\omega}_{41}) \\
& - 256\hat{\omega} (2880 + 1784\hat{\omega}_{11} - 318\hat{\omega}_{22} + 212\hat{\omega}_{31} + 20\hat{\omega}_{33} - 15\hat{\omega}_{42}) \\
& + 450\hat{\omega}_{21}\hat{\omega}_{41} - 150\hat{\omega}_{30}\hat{\omega}_{41} - 2784\hat{\omega}_{42} - 450\hat{\omega}_{11}\hat{\omega}_{42} + 225\hat{\omega}_{20}\hat{\omega}_{42}) \\
& + q^{10} (51200\hat{\omega}_{10}^2 + 25600\hat{\omega}_{11} - 9216\hat{\omega}_{20} + 233488\hat{\omega}_{11}\hat{\omega}_{20} - 155136\hat{\omega}_{20}^2 + 15288\hat{\omega}_{21}^2 \\
& + 14016\hat{\omega}_{22} - 15288\hat{\omega}_{11}\hat{\omega}_{22} - 11240\hat{\omega}_{21}\hat{\omega}_{30} + 2048\hat{\omega}_{30}^2 - 18560\hat{\omega}_{31} + 6144\hat{\omega}_{20}\hat{\omega}_{31} \\
& + 1236\hat{\omega}_{22}\hat{\omega}_{31} - 824\hat{\omega}_{31}^2 - 1236\hat{\omega}_{21}\hat{\omega}_{32} + 412\hat{\omega}_{30}\hat{\omega}_{32} - 1856\hat{\omega}_{33} + 824\hat{\omega}_{11}\hat{\omega}_{33} \\
& - 412\hat{\omega}_{20}\hat{\omega}_{33} + 2304\hat{\omega}_{40} + 2548\hat{\omega}_{11}\hat{\omega}_{40} - 1536\hat{\omega}_{20}\hat{\omega}_{40} - 309\hat{\omega}_{22}\hat{\omega}_{40} + 206\hat{\omega}_{31}\hat{\omega}_{40} \\
& + 618\hat{\omega}_{21}\hat{\omega}_{41} - 206\hat{\omega}_{30}\hat{\omega}_{41} - 4\hat{\omega}_{10} (58372\hat{\omega}_{21} - 25856\hat{\omega}_{30} + 637(-2\hat{\omega}_{32} + \hat{\omega}_{41})) \\
& - 512\hat{\omega} (864 + 100\hat{\omega}_{11} - 303\hat{\omega}_{22} + 202\hat{\omega}_{31} + 4\hat{\omega}_{33} - 3\hat{\omega}_{42}) \\
& + 1392\hat{\omega}_{42} - 618\hat{\omega}_{11}\hat{\omega}_{42} + 309\hat{\omega}_{20}\hat{\omega}_{42}) \\
& - 8q^{22} (2176\hat{\omega}_{10}^2 - 1968\hat{\omega}_{20}^2 - 228\hat{\omega}_{21}^2 + 120\hat{\omega}_{22} + 1968\hat{\omega}\hat{\omega}_{22} - 260\hat{\omega}_{21}\hat{\omega}_{30} \\
& + 112\hat{\omega}_{30}^2 - 80\hat{\omega}_{31} - 1312\hat{\omega}\hat{\omega}_{31} + 336\hat{\omega}_{20}\hat{\omega}_{31} + 12\hat{\omega}_{22}\hat{\omega}_{31} - 8\hat{\omega}_{31}^2 - 12\hat{\omega}_{21}\hat{\omega}_{32} \\
& + 4\hat{\omega}_{30}\hat{\omega}_{32} - 8\hat{\omega}_{33} - 112\hat{\omega}\hat{\omega}_{33} - 4\hat{\omega}_{20}\hat{\omega}_{33} - 84\hat{\omega}_{20}\hat{\omega}_{40} - 3\hat{\omega}_{22}\hat{\omega}_{40} \\
& + 2\hat{\omega}_{31}\hat{\omega}_{40} + 6\hat{\omega}_{21}\hat{\omega}_{41} - 2\hat{\omega}_{30}\hat{\omega}_{41} + \hat{\omega}_{10} (-4168\hat{\omega}_{21} + 1312\hat{\omega}_{30} - 76\hat{\omega}_{32} + 38\hat{\omega}_{41}) \\
& + 6\hat{\omega}_{42} + 84\hat{\omega}\hat{\omega}_{42} + 3\hat{\omega}_{20}\hat{\omega}_{42} - 2\hat{\omega}_{11} (64 + 1088\hat{\omega} - 2084\hat{\omega}_{20} - 114\hat{\omega}_{22} - 4\hat{\omega}_{33} + 19\hat{\omega}_{40} + 3\hat{\omega}_{42})) \\
& + q^{26} (-4096\hat{\omega}_{10}^2 + 3840\hat{\omega}_{20}^2 - 216\hat{\omega}_{21}^2 - 960\hat{\omega}_{22} - 3840\hat{\omega}\hat{\omega}_{22} + 840\hat{\omega}_{21}\hat{\omega}_{30} \\
& - 256\hat{\omega}_{30}^2 + 640\hat{\omega}_{31} + 2560\hat{\omega}\hat{\omega}_{31} - 768\hat{\omega}_{20}\hat{\omega}_{31} - 36\hat{\omega}_{22}\hat{\omega}_{31} + 24\hat{\omega}_{31}^2 + 36\hat{\omega}_{21}\hat{\omega}_{32}
\end{aligned}$$

$$\begin{aligned}
& -12\hat{\omega}_{30}\hat{\omega}_{32} + 64\hat{\omega}_{33} + 256\hat{\omega}\hat{\omega}_{33} + 12\hat{\omega}_{20}\hat{\omega}_{33} + 192\hat{\omega}_{20}\hat{\omega}_{40} + 9\hat{\omega}_{22}\hat{\omega}_{40} \\
& -6\hat{\omega}_{31}\hat{\omega}_{40} - 18\hat{\omega}_{21}\hat{\omega}_{41} + 6\hat{\omega}_{30}\hat{\omega}_{41} + 4\hat{\omega}_{10} (1524\hat{\omega}_{21} - 640\hat{\omega}_{30} - 18\hat{\omega}_{32} + 9\hat{\omega}_{41}) \\
& -48\hat{\omega}_{42} - 192\hat{\omega}\hat{\omega}_{42} - 9\hat{\omega}_{20}\hat{\omega}_{42} + 2\hat{\omega}_{11} (512 + 2048\hat{\omega} - 3048\hat{\omega}_{20} + 108\hat{\omega}_{22} - 12\hat{\omega}_{33} - 18\hat{\omega}_{40} + 9\hat{\omega}_{42})) \\
& + q^{20} (-74752\hat{\omega}_{10}^2 + 109056\hat{\omega}_{11} - 53760\hat{\omega}_{20} - 118704\hat{\omega}_{11}\hat{\omega}_{20} + 48768\hat{\omega}_{20}^2 + 3672\hat{\omega}_{21}^2 \\
& + 8640\hat{\omega}_{22} - 3672\hat{\omega}_{11}\hat{\omega}_{22} - 8520\hat{\omega}_{21}\hat{\omega}_{30} + 2432\hat{\omega}_{30}^2 - 4224\hat{\omega}_{31} + 7296\hat{\omega}_{20}\hat{\omega}_{31} \\
& + 612\hat{\omega}_{22}\hat{\omega}_{31} - 408\hat{\omega}_{31}^2 - 612\hat{\omega}_{21}\hat{\omega}_{32} + 204\hat{\omega}_{30}\hat{\omega}_{32} + 576\hat{\omega}_{33} + 408\hat{\omega}_{11}\hat{\omega}_{33} \\
& - 204\hat{\omega}_{20}\hat{\omega}_{33} - 384\hat{\omega}_{40} + 612\hat{\omega}_{11}\hat{\omega}_{40} - 1824\hat{\omega}_{20}\hat{\omega}_{40} - 153\hat{\omega}_{22}\hat{\omega}_{40} + 102\hat{\omega}_{31}\hat{\omega}_{40} \\
& + 4\hat{\omega}_{10} (29676\hat{\omega}_{21} - 8128\hat{\omega}_{30} + 306\hat{\omega}_{32} - 153\hat{\omega}_{41}) \\
& + 306\hat{\omega}_{21}\hat{\omega}_{41} - 102\hat{\omega}_{30}\hat{\omega}_{41} - 432\hat{\omega}_{42} - 306\hat{\omega}_{11}\hat{\omega}_{42} + 153\hat{\omega}_{20}\hat{\omega}_{42} \\
& + 32\hat{\omega} (-13824 + 2336\hat{\omega}_{11} - 1524\hat{\omega}_{22} + 1016\hat{\omega}_{31} - 76\hat{\omega}_{33} + 57\hat{\omega}_{42})) \\
& - q^8 (-74752\hat{\omega}_{10}^2 - 109056\hat{\omega}_{11} + 53760\hat{\omega}_{20} - 118704\hat{\omega}_{11}\hat{\omega}_{20} + 48768\hat{\omega}_{20}^2 + 3672\hat{\omega}_{21}^2 \\
& - 8640\hat{\omega}_{22} - 3672\hat{\omega}_{11}\hat{\omega}_{22} - 8520\hat{\omega}_{21}\hat{\omega}_{30} + 2432\hat{\omega}_{30}^2 + 4224\hat{\omega}_{31} + 7296\hat{\omega}_{20}\hat{\omega}_{31} \\
& + 612\hat{\omega}_{22}\hat{\omega}_{31} - 408\hat{\omega}_{31}^2 - 612\hat{\omega}_{21}\hat{\omega}_{32} + 204\hat{\omega}_{30}\hat{\omega}_{32} - 576\hat{\omega}_{33} + 408\hat{\omega}_{11}\hat{\omega}_{33} \\
& - 204\hat{\omega}_{20}\hat{\omega}_{33} + 384\hat{\omega}_{40} + 612\hat{\omega}_{11}\hat{\omega}_{40} - 1824\hat{\omega}_{20}\hat{\omega}_{40} - 153\hat{\omega}_{22}\hat{\omega}_{40} + 102\hat{\omega}_{31}\hat{\omega}_{40} \\
& + 4\hat{\omega}_{10} (29676\hat{\omega}_{21} - 8128\hat{\omega}_{30} + 306\hat{\omega}_{32} - 153\hat{\omega}_{41}) \\
& + 306\hat{\omega}_{21}\hat{\omega}_{41} - 102\hat{\omega}_{30}\hat{\omega}_{41} + 432\hat{\omega}_{42} - 306\hat{\omega}_{11}\hat{\omega}_{42} + 153\hat{\omega}_{20}\hat{\omega}_{42} \\
& + 32\hat{\omega} (13824 + 2336\hat{\omega}_{11} - 1524\hat{\omega}_{22} + 1016\hat{\omega}_{31} - 76\hat{\omega}_{33} + 57\hat{\omega}_{42})) \} \\
& - 4\eta q^2 (-1 - q^2 - q^4 + q^8 + q^{10} + q^{12}) \\
& \{ -256\hat{\omega}_{20}\hat{\omega}'_{10} - 288\hat{\omega}_{22}\hat{\omega}'_{10} - 64\hat{\omega}_{31}\hat{\omega}'_{10} - 32\hat{\omega}_{33}\hat{\omega}'_{10} + 64\hat{\omega}_{40}\hat{\omega}'_{10} + 24\hat{\omega}_{42}\hat{\omega}'_{10} \\
& + 256\hat{\omega}_{10}\hat{\omega}'_{20} + 288\hat{\omega}_{21}\hat{\omega}'_{20} - 64\hat{\omega}_{30}\hat{\omega}'_{20} - 48\hat{\omega}_{32}\hat{\omega}'_{20} + 24\hat{\omega}_{41}\hat{\omega}'_{20} + 64\hat{\omega}'_{21} \\
& - 256\hat{\omega}\hat{\omega}'_{21} - 288\hat{\omega}_{20}\hat{\omega}'_{21} + 48\hat{\omega}_{31}\hat{\omega}'_{21} + 12\hat{\omega}_{33}\hat{\omega}'_{21} - 12\hat{\omega}_{40}\hat{\omega}'_{21} - 9\hat{\omega}_{42}\hat{\omega}'_{21} \\
& + 64\hat{\omega}_{11}\hat{\omega}'_{30} + 64\hat{\omega}_{20}\hat{\omega}'_{30} + 48\hat{\omega}_{22}\hat{\omega}'_{30} + 32\hat{\omega}_{31}\hat{\omega}'_{30} - 16\hat{\omega}_{40}\hat{\omega}'_{30} - 64\hat{\omega}_{10}\hat{\omega}'_{31} \\
& - 48\hat{\omega}_{21}\hat{\omega}'_{31} - 32\hat{\omega}_{30}\hat{\omega}'_{31} - 12\hat{\omega}_{32}\hat{\omega}'_{31} + 6\hat{\omega}_{41}\hat{\omega}'_{31} + 16\hat{\omega}'_{32} - 64\hat{\omega}\hat{\omega}'_{32} \\
& + 48\hat{\omega}_{11}\hat{\omega}'_{32} - 48\hat{\omega}_{20}\hat{\omega}'_{32} + 12\hat{\omega}_{31}\hat{\omega}'_{32} - 3\hat{\omega}_{40}\hat{\omega}'_{32} - 64\hat{\omega}_{10}\hat{\omega}'_{40} - 48\hat{\omega}_{21}\hat{\omega}'_{40} \\
& + 16\hat{\omega}_{30}\hat{\omega}'_{40} - 16\hat{\omega}'_{41} + 64\hat{\omega}\hat{\omega}'_{41} + 36\hat{\omega}_{20}\hat{\omega}'_{41} + 9\hat{\omega}_{22}\hat{\omega}'_{41} - 6\hat{\omega}_{31}\hat{\omega}'_{41} \\
& + 12\hat{\omega}_{10}\hat{\omega}'_{42} - 9\hat{\omega}_{21}\hat{\omega}'_{42} + 3\hat{\omega}_{30}\hat{\omega}'_{42} + 4\hat{\omega}'_{43} - 16\hat{\omega}\hat{\omega}'_{43} + 6\hat{\omega}_{11}\hat{\omega}'_{43} - 3\hat{\omega}_{20}\hat{\omega}'_{43} \\
& - 24q^8 (12288\hat{\omega}'_{10} + 1840\hat{\omega}'_{21} + 156\hat{\omega}'_{32} + 20\hat{\omega}'_{41} + 15\hat{\omega}'_{43}) \\
& - 2q^2 (1216\hat{\omega}_{31}\hat{\omega}'_{10} + 224\hat{\omega}_{33}\hat{\omega}'_{10} + 64\hat{\omega}_{40}\hat{\omega}'_{10} - 168\hat{\omega}_{42}\hat{\omega}'_{10} \\
& - 4352\hat{\omega}_{10}\hat{\omega}'_{20} + 2208\hat{\omega}_{21}\hat{\omega}'_{20} - 1024\hat{\omega}_{30}\hat{\omega}'_{20} - 144\hat{\omega}_{32}\hat{\omega}'_{20} + 72\hat{\omega}_{41}\hat{\omega}'_{20} \\
& - 960\hat{\omega}'_{21} + 4352\hat{\omega}\hat{\omega}'_{21} + 144\hat{\omega}_{31}\hat{\omega}'_{21} + 12\hat{\omega}_{33}\hat{\omega}'_{21} - 36\hat{\omega}_{40}\hat{\omega}'_{21} - 9\hat{\omega}_{42}\hat{\omega}'_{21} \\
& - 1216\hat{\omega}_{11}\hat{\omega}'_{30} - 224\hat{\omega}_{31}\hat{\omega}'_{30} + 32\hat{\omega}_{40}\hat{\omega}'_{30} + 1216\hat{\omega}_{10}\hat{\omega}'_{31} - 144\hat{\omega}_{21}\hat{\omega}'_{31} \\
& + 224\hat{\omega}_{30}\hat{\omega}'_{31} - 12\hat{\omega}_{32}\hat{\omega}'_{31} + 6\hat{\omega}_{41}\hat{\omega}'_{31} + 288\hat{\omega}'_{32} - 1024\hat{\omega}\hat{\omega}'_{32} \\
& + 144\hat{\omega}_{11}\hat{\omega}'_{32} + 12\hat{\omega}_{31}\hat{\omega}'_{32} - 3\hat{\omega}_{40}\hat{\omega}'_{32} - 64\hat{\omega}_{10}\hat{\omega}'_{40} + 96\hat{\omega}_{21}\hat{\omega}'_{40} \\
& - 32\hat{\omega}_{30}\hat{\omega}'_{40} - 48\hat{\omega}'_{41} + 64\hat{\omega}\hat{\omega}'_{41} - 6\hat{\omega}_{31}\hat{\omega}'_{41} \\
& + \hat{\omega}_{22} (-2208\hat{\omega}'_{10} + 144\hat{\omega}'_{30} + 9\hat{\omega}'_{41}) + 36\hat{\omega}_{10}\hat{\omega}'_{42} - 9\hat{\omega}_{21}\hat{\omega}'_{42} + 3\hat{\omega}_{30}\hat{\omega}'_{42} \\
& + \hat{\omega}_{20} (4352\hat{\omega}'_{10} - 2208\hat{\omega}'_{21} + 1024\hat{\omega}'_{30} - 144\hat{\omega}'_{32} - 132\hat{\omega}'_{41} - 3\hat{\omega}'_{43}) \\
& + 32\hat{\omega}\hat{\omega}'_{43} + 6\hat{\omega}_{11}\hat{\omega}'_{43}) \\
& + 2q^{14} (1216\hat{\omega}_{31}\hat{\omega}'_{10} + 224\hat{\omega}_{33}\hat{\omega}'_{10} + 64\hat{\omega}_{40}\hat{\omega}'_{10} - 168\hat{\omega}_{42}\hat{\omega}'_{10}
\end{aligned}$$

$$\begin{aligned}
& -4352\hat{\omega}_{10}\hat{\omega}'_{20} + 2208\hat{\omega}_{21}\hat{\omega}'_{20} - 1024\hat{\omega}_{30}\hat{\omega}'_{20} - 144\hat{\omega}_{32}\hat{\omega}'_{20} + 72\hat{\omega}_{41}\hat{\omega}'_{20} \\
& + 960\hat{\omega}'_{21} + 4352\hat{\omega}_{21}\hat{\omega}'_{21} + 144\hat{\omega}_{31}\hat{\omega}'_{21} + 12\hat{\omega}_{33}\hat{\omega}'_{21} - 36\hat{\omega}_{40}\hat{\omega}'_{21} \\
& - 9\hat{\omega}_{42}\hat{\omega}'_{21} - 1216\hat{\omega}_{11}\hat{\omega}'_{30} - 224\hat{\omega}_{31}\hat{\omega}'_{30} + 32\hat{\omega}_{40}\hat{\omega}'_{30} + 1216\hat{\omega}_{10}\hat{\omega}'_{31} \\
& - 144\hat{\omega}_{21}\hat{\omega}'_{31} + 224\hat{\omega}_{30}\hat{\omega}'_{31} - 12\hat{\omega}_{32}\hat{\omega}'_{31} + 6\hat{\omega}_{41}\hat{\omega}'_{31} - 288\hat{\omega}'_{32} \\
& - 1024\hat{\omega}_{11}\hat{\omega}'_{32} + 144\hat{\omega}_{11}\hat{\omega}'_{32} + 12\hat{\omega}_{31}\hat{\omega}'_{32} - 3\hat{\omega}_{40}\hat{\omega}'_{32} - 64\hat{\omega}_{10}\hat{\omega}'_{40} \\
& + 96\hat{\omega}_{21}\hat{\omega}'_{40} - 32\hat{\omega}_{30}\hat{\omega}'_{40} + 48\hat{\omega}'_{41} + 64\hat{\omega}_{41}\hat{\omega}'_{41} - 6\hat{\omega}_{31}\hat{\omega}'_{41} \\
& + \hat{\omega}_{22}(-2208\hat{\omega}'_{10} + 144\hat{\omega}'_{30} + 9\hat{\omega}'_{41}) + 36\hat{\omega}_{10}\hat{\omega}'_{42} - 9\hat{\omega}_{21}\hat{\omega}'_{42} + 3\hat{\omega}_{30}\hat{\omega}'_{42} \\
& + \hat{\omega}_{20}(4352\hat{\omega}'_{10} - 2208\hat{\omega}'_{21} + 1024\hat{\omega}'_{30} - 144\hat{\omega}'_{32} - 132\hat{\omega}'_{41} - 3\hat{\omega}'_{43}) \\
& + 32\hat{\omega}_{11}\hat{\omega}'_{43} + 6\hat{\omega}_{11}\hat{\omega}'_{43}) \\
& - 2q^{10}(-4288\hat{\omega}_{31}\hat{\omega}'_{10} + 544\hat{\omega}_{33}\hat{\omega}'_{10} + 192\hat{\omega}_{40}\hat{\omega}'_{10} - 408\hat{\omega}_{42}\hat{\omega}'_{10} \\
& - 9984\hat{\omega}_{10}\hat{\omega}'_{20} - 5280\hat{\omega}_{21}\hat{\omega}'_{20} + 3456\hat{\omega}_{30}\hat{\omega}'_{20} + 336\hat{\omega}_{32}\hat{\omega}'_{20} - 168\hat{\omega}_{41}\hat{\omega}'_{20} \\
& - 7232\hat{\omega}'_{21} + 9984\hat{\omega}_{21}\hat{\omega}'_{21} - 336\hat{\omega}_{31}\hat{\omega}'_{21} + 36\hat{\omega}_{33}\hat{\omega}'_{21} + 84\hat{\omega}_{40}\hat{\omega}'_{21} - 27\hat{\omega}_{42}\hat{\omega}'_{21} \\
& + 4288\hat{\omega}_{11}\hat{\omega}'_{30} - 544\hat{\omega}_{31}\hat{\omega}'_{30} + 192\hat{\omega}_{40}\hat{\omega}'_{30} - 4288\hat{\omega}_{10}\hat{\omega}'_{31} + 336\hat{\omega}_{21}\hat{\omega}'_{31} \\
& + 544\hat{\omega}_{30}\hat{\omega}'_{31} - 36\hat{\omega}_{32}\hat{\omega}'_{31} + 18\hat{\omega}_{41}\hat{\omega}'_{31} - 800\hat{\omega}'_{32} + 3456\hat{\omega}_{11}\hat{\omega}'_{32} \\
& - 336\hat{\omega}_{11}\hat{\omega}'_{32} + 36\hat{\omega}_{31}\hat{\omega}'_{32} - 9\hat{\omega}_{40}\hat{\omega}'_{32} - 192\hat{\omega}_{10}\hat{\omega}'_{40} + 576\hat{\omega}_{21}\hat{\omega}'_{40} \\
& - 192\hat{\omega}_{30}\hat{\omega}'_{40} - 208\hat{\omega}'_{41} + 192\hat{\omega}_{41}\hat{\omega}'_{41} - 18\hat{\omega}_{31}\hat{\omega}'_{41} \\
& + 3\hat{\omega}_{22}(1760\hat{\omega}'_{10} - 112\hat{\omega}'_{30} + 9\hat{\omega}'_{41}) - 84\hat{\omega}_{10}\hat{\omega}'_{42} - 27\hat{\omega}_{21}\hat{\omega}'_{42} + 9\hat{\omega}_{30}\hat{\omega}'_{42} \\
& + \hat{\omega}_{20}(9984\hat{\omega}'_{10} + 5280\hat{\omega}'_{21} - 3456\hat{\omega}'_{30} + 336\hat{\omega}'_{32} - 492\hat{\omega}'_{41} - 9\hat{\omega}'_{43}) \\
& - 128\hat{\omega}'_{43} + 192\hat{\omega}_{11}\hat{\omega}'_{43} + 18\hat{\omega}_{11}\hat{\omega}'_{43}) \\
& + 2q^6(-4288\hat{\omega}_{31}\hat{\omega}'_{10} + 544\hat{\omega}_{33}\hat{\omega}'_{10} + 192\hat{\omega}_{40}\hat{\omega}'_{10} - 408\hat{\omega}_{42}\hat{\omega}'_{10} \\
& - 9984\hat{\omega}_{10}\hat{\omega}'_{20} - 5280\hat{\omega}_{21}\hat{\omega}'_{20} + 3456\hat{\omega}_{30}\hat{\omega}'_{20} + 336\hat{\omega}_{32}\hat{\omega}'_{20} - 168\hat{\omega}_{41}\hat{\omega}'_{20} \\
& + 7232\hat{\omega}'_{21} + 9984\hat{\omega}_{21}\hat{\omega}'_{21} - 336\hat{\omega}_{31}\hat{\omega}'_{21} + 36\hat{\omega}_{33}\hat{\omega}'_{21} + 84\hat{\omega}_{40}\hat{\omega}'_{21} - 27\hat{\omega}_{42}\hat{\omega}'_{21} \\
& + 4288\hat{\omega}_{11}\hat{\omega}'_{30} - 544\hat{\omega}_{31}\hat{\omega}'_{30} + 192\hat{\omega}_{40}\hat{\omega}'_{30} - 4288\hat{\omega}_{10}\hat{\omega}'_{31} + 336\hat{\omega}_{21}\hat{\omega}'_{31} \\
& + 544\hat{\omega}_{30}\hat{\omega}'_{31} - 36\hat{\omega}_{32}\hat{\omega}'_{31} + 18\hat{\omega}_{41}\hat{\omega}'_{31} + 800\hat{\omega}'_{32} + 3456\hat{\omega}_{11}\hat{\omega}'_{32} \\
& - 336\hat{\omega}_{11}\hat{\omega}'_{32} + 36\hat{\omega}_{31}\hat{\omega}'_{32} - 9\hat{\omega}_{40}\hat{\omega}'_{32} - 192\hat{\omega}_{10}\hat{\omega}'_{40} + 576\hat{\omega}_{21}\hat{\omega}'_{40} \\
& - 192\hat{\omega}_{30}\hat{\omega}'_{40} + 208\hat{\omega}'_{41} + 192\hat{\omega}_{41}\hat{\omega}'_{41} - 18\hat{\omega}_{31}\hat{\omega}'_{41} \\
& + 3\hat{\omega}_{22}(1760\hat{\omega}'_{10} - 112\hat{\omega}'_{30} + 9\hat{\omega}'_{41}) - 84\hat{\omega}_{10}\hat{\omega}'_{42} - 27\hat{\omega}_{21}\hat{\omega}'_{42} + 9\hat{\omega}_{30}\hat{\omega}'_{42} \\
& + \hat{\omega}_{20}(9984\hat{\omega}'_{10} + 5280\hat{\omega}'_{21} - 3456\hat{\omega}'_{30} + 336\hat{\omega}'_{32} - 492\hat{\omega}'_{41} - 9\hat{\omega}'_{43}) \\
& + 128\hat{\omega}'_{43} + 192\hat{\omega}_{11}\hat{\omega}'_{43} + 18\hat{\omega}_{11}\hat{\omega}'_{43}) \\
& + q^{16}(64\hat{\omega}_{31}\hat{\omega}'_{10} + 32\hat{\omega}_{33}\hat{\omega}'_{10} - 64\hat{\omega}_{40}\hat{\omega}'_{10} - 24\hat{\omega}_{42}\hat{\omega}'_{10} \\
& - 256\hat{\omega}_{10}\hat{\omega}'_{20} - 288\hat{\omega}_{21}\hat{\omega}'_{20} + 64\hat{\omega}_{30}\hat{\omega}'_{20} + 48\hat{\omega}_{32}\hat{\omega}'_{20} - 24\hat{\omega}_{41}\hat{\omega}'_{20} \\
& + 64\hat{\omega}'_{21} + 256\hat{\omega}_{21}\hat{\omega}'_{21} - 48\hat{\omega}_{31}\hat{\omega}'_{21} - 12\hat{\omega}_{33}\hat{\omega}'_{21} + 12\hat{\omega}_{40}\hat{\omega}'_{21} + 9\hat{\omega}_{42}\hat{\omega}'_{21} \\
& - 64\hat{\omega}_{11}\hat{\omega}'_{30} - 32\hat{\omega}_{31}\hat{\omega}'_{30} + 16\hat{\omega}_{40}\hat{\omega}'_{30} + 64\hat{\omega}_{10}\hat{\omega}'_{31} + 48\hat{\omega}_{21}\hat{\omega}'_{31} \\
& + 32\hat{\omega}_{30}\hat{\omega}'_{31} + 12\hat{\omega}_{32}\hat{\omega}'_{31} - 6\hat{\omega}_{41}\hat{\omega}'_{31} + 16\hat{\omega}'_{32} + 64\hat{\omega}_{11}\hat{\omega}'_{32} - 48\hat{\omega}_{11}\hat{\omega}'_{32} \\
& - 12\hat{\omega}_{31}\hat{\omega}'_{32} + 3\hat{\omega}_{40}\hat{\omega}'_{32} + 64\hat{\omega}_{10}\hat{\omega}'_{40} + 48\hat{\omega}_{21}\hat{\omega}'_{40} - 16\hat{\omega}_{30}\hat{\omega}'_{40} \\
& + 3\hat{\omega}_{22}(96\hat{\omega}'_{10} - 16\hat{\omega}'_{30} - 3\hat{\omega}'_{41}) - 16\hat{\omega}'_{41} - 64\hat{\omega}_{41}\hat{\omega}'_{41} + 6\hat{\omega}_{31}\hat{\omega}'_{41} \\
& - 12\hat{\omega}_{10}\hat{\omega}'_{42} + 9\hat{\omega}_{21}\hat{\omega}'_{42} - 3\hat{\omega}_{30}\hat{\omega}'_{42} + 4\hat{\omega}'_{43} + 16\hat{\omega}_{11}\hat{\omega}'_{43} \\
& - 6\hat{\omega}_{11}\hat{\omega}'_{43} + \hat{\omega}_{20}(256\hat{\omega}'_{10} + 288\hat{\omega}'_{21} - 64\hat{\omega}'_{30} + 48\hat{\omega}'_{32} - 36\hat{\omega}'_{41} + 3\hat{\omega}'_{43}))
\end{aligned}$$

$$\begin{aligned}
& + 2q^4 (-2880\hat{\omega}_{31}\hat{\omega}'_{10} + 96\hat{\omega}_{33}\hat{\omega}'_{10} - 64\hat{\omega}_{40}\hat{\omega}'_{10} - 72\hat{\omega}_{42}\hat{\omega}'_{10} \\
& - 1792\hat{\omega}_{10}\hat{\omega}'_{20} - 4704\hat{\omega}_{21}\hat{\omega}'_{20} + 256\hat{\omega}_{30}\hat{\omega}'_{20} - 336\hat{\omega}_{32}\hat{\omega}'_{20} + 168\hat{\omega}_{41}\hat{\omega}'_{20} \\
& + 2816\hat{\omega}'_{21} + 1792\hat{\omega}\hat{\omega}'_{21} + 336\hat{\omega}_{31}\hat{\omega}'_{21} - 12\hat{\omega}_{33}\hat{\omega}'_{21} - 84\hat{\omega}_{40}\hat{\omega}'_{21} + 9\hat{\omega}_{42}\hat{\omega}'_{21} \\
& + 2880\hat{\omega}_{11}\hat{\omega}'_{30} - 96\hat{\omega}_{31}\hat{\omega}'_{30} - 32\hat{\omega}_{40}\hat{\omega}'_{30} - 2880\hat{\omega}_{10}\hat{\omega}'_{31} - 336\hat{\omega}_{21}\hat{\omega}'_{31} \\
& + 96\hat{\omega}_{30}\hat{\omega}'_{31} + 12\hat{\omega}_{32}\hat{\omega}'_{31} - 6\hat{\omega}_{41}\hat{\omega}'_{31} + 416\hat{\omega}'_{32} + 256\hat{\omega}\hat{\omega}'_{32} \\
& + 336\hat{\omega}_{11}\hat{\omega}'_{32} - 12\hat{\omega}_{31}\hat{\omega}'_{32} + 3\hat{\omega}_{40}\hat{\omega}'_{32} + 64\hat{\omega}_{10}\hat{\omega}'_{40} - 96\hat{\omega}_{21}\hat{\omega}'_{40} + 32\hat{\omega}_{30}\hat{\omega}'_{40} \\
& + \hat{\omega}_{22} (4704\hat{\omega}'_{10} + 336\hat{\omega}'_{30} - 9\hat{\omega}'_{41}) - 128\hat{\omega}'_{41} - 64\hat{\omega}\hat{\omega}'_{41} + 6\hat{\omega}_{31}\hat{\omega}'_{41} \\
& + 84\hat{\omega}_{10}\hat{\omega}'_{42} + 9\hat{\omega}_{21}\hat{\omega}'_{42} - 3\hat{\omega}_{30}\hat{\omega}'_{42} - 40\hat{\omega}'_{43} - 32\hat{\omega}\hat{\omega}'_{43} - 6\hat{\omega}_{11}\hat{\omega}'_{43} \\
& + \hat{\omega}_{20} (1792\hat{\omega}'_{10} + 4704\hat{\omega}'_{21} - 256\hat{\omega}'_{30} - 336\hat{\omega}'_{32} + 12\hat{\omega}'_{41} + 3\hat{\omega}'_{43})) \\
& - 2q^{12} (-2880\hat{\omega}_{31}\hat{\omega}'_{10} + 96\hat{\omega}_{33}\hat{\omega}'_{10} - 64\hat{\omega}_{40}\hat{\omega}'_{10} - 72\hat{\omega}_{42}\hat{\omega}'_{10} \\
& - 1792\hat{\omega}_{10}\hat{\omega}'_{20} - 4704\hat{\omega}_{21}\hat{\omega}'_{20} + 256\hat{\omega}_{30}\hat{\omega}'_{20} - 336\hat{\omega}_{32}\hat{\omega}'_{20} + 168\hat{\omega}_{41}\hat{\omega}'_{20} \\
& - 2816\hat{\omega}'_{21} + 1792\hat{\omega}\hat{\omega}'_{21} + 336\hat{\omega}_{31}\hat{\omega}'_{21} - 12\hat{\omega}_{33}\hat{\omega}'_{21} - 84\hat{\omega}_{40}\hat{\omega}'_{21} + 9\hat{\omega}_{42}\hat{\omega}'_{21} \\
& + 2880\hat{\omega}_{11}\hat{\omega}'_{30} - 96\hat{\omega}_{31}\hat{\omega}'_{30} - 32\hat{\omega}_{40}\hat{\omega}'_{30} - 2880\hat{\omega}_{10}\hat{\omega}'_{31} - 336\hat{\omega}_{21}\hat{\omega}'_{31} \\
& + 96\hat{\omega}_{30}\hat{\omega}'_{31} + 12\hat{\omega}_{32}\hat{\omega}'_{31} - 6\hat{\omega}_{41}\hat{\omega}'_{31} - 416\hat{\omega}'_{32} + 256\hat{\omega}\hat{\omega}'_{32} \\
& + 336\hat{\omega}_{11}\hat{\omega}'_{32} - 12\hat{\omega}_{31}\hat{\omega}'_{32} + 3\hat{\omega}_{40}\hat{\omega}'_{32} + 64\hat{\omega}_{10}\hat{\omega}'_{40} - 96\hat{\omega}_{21}\hat{\omega}'_{40} + 32\hat{\omega}_{30}\hat{\omega}'_{40} \\
& + \hat{\omega}_{22} (4704\hat{\omega}'_{10} + 336\hat{\omega}'_{30} - 9\hat{\omega}'_{41}) + 128\hat{\omega}'_{41} - 64\hat{\omega}\hat{\omega}'_{41} + 6\hat{\omega}_{31}\hat{\omega}'_{41} \\
& + 84\hat{\omega}_{10}\hat{\omega}'_{42} + 9\hat{\omega}_{21}\hat{\omega}'_{42} - 3\hat{\omega}_{30}\hat{\omega}'_{42} + 40\hat{\omega}'_{43} - 32\hat{\omega}\hat{\omega}'_{43} \\
& - 6\hat{\omega}_{11}\hat{\omega}'_{43} + \hat{\omega}_{20} (1792\hat{\omega}'_{10} + 4704\hat{\omega}'_{21} - 256\hat{\omega}'_{30} - 336\hat{\omega}'_{32} + 12\hat{\omega}'_{41} + 3\hat{\omega}'_{43})) \} \\
& + (-1 + q^4)^3 (1 + q^2 + 2q^4 + q^6 + q^8) \\
& (\hat{\omega}'_{32}\hat{\omega}'_{41} - \hat{\omega}'_{31}\hat{\omega}'_{42} + \hat{\omega}'_{21}\hat{\omega}'_{43} \\
& + q^4 (-7040\hat{\omega}'_{20}\hat{\omega}'_{31} + 7040\hat{\omega}'_{10}\hat{\omega}'_{32} - 160\hat{\omega}'_{31}\hat{\omega}'_{40} + 160\hat{\omega}'_{30}\hat{\omega}'_{41} - 33\hat{\omega}'_{32}\hat{\omega}'_{41} \\
& + 33\hat{\omega}'_{31}\hat{\omega}'_{42} + 11\hat{\omega}'_{21} (640\hat{\omega}'_{30} - 3\hat{\omega}'_{43}) - 160\hat{\omega}'_{10}\hat{\omega}'_{43}) \\
& + q^8 (-7040\hat{\omega}'_{20}\hat{\omega}'_{31} + 7040\hat{\omega}'_{10}\hat{\omega}'_{32} - 160\hat{\omega}'_{31}\hat{\omega}'_{40} + 160\hat{\omega}'_{30}\hat{\omega}'_{41} - 33\hat{\omega}'_{32}\hat{\omega}'_{41} \\
& + 33\hat{\omega}'_{31}\hat{\omega}'_{42} + 11\hat{\omega}'_{21} (640\hat{\omega}'_{30} - 3\hat{\omega}'_{43}) - 160\hat{\omega}'_{10}\hat{\omega}'_{43}) \\
& + q^{12} (\hat{\omega}'_{32}\hat{\omega}'_{41} - \hat{\omega}'_{31}\hat{\omega}'_{42} + \hat{\omega}'_{21}\hat{\omega}'_{43}) \\
& + q^2 (320\hat{\omega}'_{20}\hat{\omega}'_{31} - 320\hat{\omega}'_{10}\hat{\omega}'_{32} - 80\hat{\omega}'_{31}\hat{\omega}'_{40} + 80\hat{\omega}'_{30}\hat{\omega}'_{41} + 6\hat{\omega}'_{32}\hat{\omega}'_{41} - 6\hat{\omega}'_{31}\hat{\omega}'_{42} \\
& - 80\hat{\omega}'_{10}\hat{\omega}'_{43} + \hat{\omega}'_{21} (-320\hat{\omega}'_{30} + 6\hat{\omega}'_{43})) \\
& + q^{10} (320\hat{\omega}'_{20}\hat{\omega}'_{31} - 320\hat{\omega}'_{10}\hat{\omega}'_{32} - 80\hat{\omega}'_{31}\hat{\omega}'_{40} + 80\hat{\omega}'_{30}\hat{\omega}'_{41} + 6\hat{\omega}'_{32}\hat{\omega}'_{41} - 6\hat{\omega}'_{31}\hat{\omega}'_{42} \\
& - 80\hat{\omega}'_{10}\hat{\omega}'_{43} + \hat{\omega}'_{21} (-320\hat{\omega}'_{30} + 6\hat{\omega}'_{43})) \\
& + 4q^6 (-96\hat{\omega}'_{20}\hat{\omega}'_{31} + 96\hat{\omega}'_{10}\hat{\omega}'_{32} + 120\hat{\omega}'_{31}\hat{\omega}'_{40} - 120\hat{\omega}'_{30}\hat{\omega}'_{41} + 13\hat{\omega}'_{32}\hat{\omega}'_{41} \\
& - 13\hat{\omega}'_{31}\hat{\omega}'_{42} + 120\hat{\omega}'_{10}\hat{\omega}'_{43} + \hat{\omega}'_{21} (96\hat{\omega}'_{30} + 13\hat{\omega}'_{43})) \} \\
& / (1179648\eta^2 q^{10} (-1 - q^2 - q^4 + q^8 + q^{10} + q^{12}))
\end{aligned}$$

$$\begin{aligned}
\langle \sigma_1^x \sigma_5^x \rangle = & \\
& \{ -8\eta^2 q^2 \{ 512\hat{\omega}_{10}^2 + 128\hat{\omega}_{11} - 512\hat{\omega}\hat{\omega}_{11} + 272\hat{\omega}_{11}\hat{\omega}_{20} - 192\hat{\omega}_{20}^2 - 272\hat{\omega}_{10}\hat{\omega}_{21} \\
& - 168\hat{\omega}_{21}^2 - 48\hat{\omega}_{22} + 192\hat{\omega}\hat{\omega}_{22} + 168\hat{\omega}_{11}\hat{\omega}_{22} + 128\hat{\omega}_{10}\hat{\omega}_{30} + 248\hat{\omega}_{21}\hat{\omega}_{30} \\
& - 64\hat{\omega}_{30}^2 + 32\hat{\omega}_{31} - 128\hat{\omega}\hat{\omega}_{31} - 192\hat{\omega}_{20}\hat{\omega}_{31} - 12\hat{\omega}_{22}\hat{\omega}_{31} + 8\hat{\omega}_{31}^2 \}
\end{aligned}$$

$$\begin{aligned}
& -56\hat{\omega}_{10}\hat{\omega}_{32} + 12\hat{\omega}_{21}\hat{\omega}_{32} - 4\hat{\omega}_{30}\hat{\omega}_{32} - 16\hat{\omega}_{33} + 64\hat{\omega}_{33} - 8\hat{\omega}_{11}\hat{\omega}_{33} \\
& + 4\hat{\omega}_{20}\hat{\omega}_{33} - 28\hat{\omega}_{11}\hat{\omega}_{40} + 48\hat{\omega}_{20}\hat{\omega}_{40} + 3\hat{\omega}_{22}\hat{\omega}_{40} - 2\hat{\omega}_{31}\hat{\omega}_{40} + 28\hat{\omega}_{10}\hat{\omega}_{41} \\
& - 6\hat{\omega}_{21}\hat{\omega}_{41} + 2\hat{\omega}_{30}\hat{\omega}_{41} + 12\hat{\omega}_{42} - 48\hat{\omega}_{42} + 6\hat{\omega}_{11}\hat{\omega}_{42} - 3\hat{\omega}_{20}\hat{\omega}_{42} \\
& + q^{14} \left(-306688\hat{\omega}_{10}^2 - 159872\hat{\omega}_{11} + 133632\hat{\omega}_{20} - 266928\hat{\omega}_{11}\hat{\omega}_{20} + 170304\hat{\omega}_{20}^2 \right. \\
& - 26568\hat{\omega}_{21}^2 + 31152\hat{\omega}_{22} + 26568\hat{\omega}_{11}\hat{\omega}_{22} + 18264\hat{\omega}_{21}\hat{\omega}_{30} - 3136\hat{\omega}_{30}^2 - 25376\hat{\omega}_{31} \\
& - 9408\hat{\omega}_{20}\hat{\omega}_{31} - 2268\hat{\omega}_{22}\hat{\omega}_{31} + 1512\hat{\omega}_{31}^2 + 2268\hat{\omega}_{21}\hat{\omega}_{32} - 756\hat{\omega}_{30}\hat{\omega}_{32} - 368\hat{\omega}_{33} \\
& - 1512\hat{\omega}_{11}\hat{\omega}_{33} + 756\hat{\omega}_{20}\hat{\omega}_{33} + 1152\hat{\omega}_{40} - 4428\hat{\omega}_{11}\hat{\omega}_{40} + 2352\hat{\omega}_{20}\hat{\omega}_{40} \\
& + 567\hat{\omega}_{22}\hat{\omega}_{40} - 378\hat{\omega}_{31}\hat{\omega}_{40} - 1134\hat{\omega}_{21}\hat{\omega}_{41} + 378\hat{\omega}_{30}\hat{\omega}_{41} \\
& + 4\hat{\omega}_{10}(66732\hat{\omega}_{21} - 28384\hat{\omega}_{30} + 1107(-2\hat{\omega}_{32} + \hat{\omega}_{41})) \\
& + 16\hat{\omega}(129024 + 19168\hat{\omega}_{11} - 10644\hat{\omega}_{22} + 7096\hat{\omega}_{31} + 196\hat{\omega}_{33} - 147\hat{\omega}_{42}) \\
& + 276\hat{\omega}_{42} + 1134\hat{\omega}_{11}\hat{\omega}_{42} - 567\hat{\omega}_{20}\hat{\omega}_{42}) \\
& + q^{10} \left(218112\hat{\omega}_{10}^2 + 166144\hat{\omega}_{11} - 110592\hat{\omega}_{20} - 180432\hat{\omega}_{11}\hat{\omega}_{20} + 79488\hat{\omega}_{20}^2 \right. \\
& - 2808\hat{\omega}_{21}^2 - 9888\hat{\omega}_{22} + 2808\hat{\omega}_{11}\hat{\omega}_{22} - 4824\hat{\omega}_{21}\hat{\omega}_{30} + 1920\hat{\omega}_{30}^2 + 6592\hat{\omega}_{31} \\
& + 5760\hat{\omega}_{20}\hat{\omega}_{31} - 972\hat{\omega}_{22}\hat{\omega}_{31} + 648\hat{\omega}_{31}^2 + 972\hat{\omega}_{21}\hat{\omega}_{32} - 324\hat{\omega}_{30}\hat{\omega}_{32} - 224\hat{\omega}_{33} \\
& - 648\hat{\omega}_{11}\hat{\omega}_{33} + 324\hat{\omega}_{20}\hat{\omega}_{33} - 468\hat{\omega}_{11}\hat{\omega}_{40} - 1440\hat{\omega}_{20}\hat{\omega}_{40} + 243\hat{\omega}_{22}\hat{\omega}_{40} \\
& - 486\hat{\omega}_{21}\hat{\omega}_{41} + 162\hat{\omega}_{30}\hat{\omega}_{41} + 36\hat{\omega}_{10}(5012\hat{\omega}_{21} - 1472\hat{\omega}_{30} - 26\hat{\omega}_{32} + 13\hat{\omega}_{41}) \\
& - 96\hat{\omega}(-3072 + 2272\hat{\omega}_{11} + 828\hat{\omega}_{22} - 552\hat{\omega}_{31} + 20\hat{\omega}_{33} - 15\hat{\omega}_{42}) \\
& - 162\hat{\omega}_{31}\hat{\omega}_{40} + 168\hat{\omega}_{42} + 486\hat{\omega}_{11}\hat{\omega}_{42} - 243\hat{\omega}_{20}\hat{\omega}_{42}) \\
& + q^{26} \left(80896\hat{\omega}_{10}^2 + 27648\hat{\omega}_{20} - 21120\hat{\omega}_{20}^2 - 360\hat{\omega}_{21}^2 + 2400\hat{\omega}_{22} + 21120\hat{\omega}_{22} \right. \\
& - 1800\hat{\omega}_{21}\hat{\omega}_{30} + 640\hat{\omega}_{30}^2 - 1600\hat{\omega}_{31} - 14080\hat{\omega}_{21}\hat{\omega}_{31} + 1920\hat{\omega}_{20}\hat{\omega}_{31} - 324\hat{\omega}_{22}\hat{\omega}_{31} \\
& + 216\hat{\omega}_{31}^2 + 324\hat{\omega}_{21}\hat{\omega}_{32} - 108\hat{\omega}_{30}\hat{\omega}_{32} + 32\hat{\omega}_{33} - 640\hat{\omega}_{33} + 108\hat{\omega}_{20}\hat{\omega}_{33} \\
& - 480\hat{\omega}_{20}\hat{\omega}_{40} + 81\hat{\omega}_{22}\hat{\omega}_{40} - 54\hat{\omega}_{31}\hat{\omega}_{40} - 162\hat{\omega}_{21}\hat{\omega}_{41} + 54\hat{\omega}_{30}\hat{\omega}_{41} \\
& - 4\hat{\omega}_{10}(10596\hat{\omega}_{21} - 5(704\hat{\omega}_{30} - 6\hat{\omega}_{32} + 3\hat{\omega}_{41})) \\
& - 2\hat{\omega}_{11}(22400 + 40448\hat{\omega} - 21192\hat{\omega}_{20} - 180\hat{\omega}_{22} + 108\hat{\omega}_{33} + 30\hat{\omega}_{40} - 81\hat{\omega}_{42}) \\
& - 24\hat{\omega}_{42} + 480\hat{\omega}\hat{\omega}_{42} - 81\hat{\omega}_{20}\hat{\omega}_{42}) \\
& + q^{30} \left(-512\hat{\omega}_{10}^2 + 192\hat{\omega}_{20}^2 + 168\hat{\omega}_{21}^2 - 48\hat{\omega}_{22} - 192\hat{\omega}\hat{\omega}_{22} - 248\hat{\omega}_{21}\hat{\omega}_{30} \right. \\
& + 64\hat{\omega}_{30}^2 + 32\hat{\omega}_{31} + 128\hat{\omega}\hat{\omega}_{31} + 192\hat{\omega}_{20}\hat{\omega}_{31} + 12\hat{\omega}_{22}\hat{\omega}_{31} - 8\hat{\omega}_{31}^2 \\
& - 12\hat{\omega}_{21}\hat{\omega}_{32} + 4\hat{\omega}_{30}\hat{\omega}_{32} - 16\hat{\omega}_{33} - 64\hat{\omega}\hat{\omega}_{33} - 4\hat{\omega}_{20}\hat{\omega}_{33} - 48\hat{\omega}_{20}\hat{\omega}_{40} \\
& - 3\hat{\omega}_{22}\hat{\omega}_{40} + 2\hat{\omega}_{31}\hat{\omega}_{40} + 4\hat{\omega}_{10}(68\hat{\omega}_{21} - 32\hat{\omega}_{30} + 14\hat{\omega}_{32} - 7\hat{\omega}_{41}) \\
& + 6\hat{\omega}_{21}\hat{\omega}_{41} - 2\hat{\omega}_{30}\hat{\omega}_{41} + 2\hat{\omega}_{11}(64 + 256\hat{\omega} - 136\hat{\omega}_{20} - 84\hat{\omega}_{22} \\
& + 4\hat{\omega}_{33} + 14\hat{\omega}_{40} - 3\hat{\omega}_{42}) \\
& + 12\hat{\omega}_{42} + 48\hat{\omega}\hat{\omega}_{42} + 3\hat{\omega}_{20}\hat{\omega}_{42}) \\
& + q^4 \left(-80896\hat{\omega}_{10}^2 + 27648\hat{\omega}_{20} + 21120\hat{\omega}_{20}^2 + 360\hat{\omega}_{21}^2 + 2400\hat{\omega}_{22} \right. \\
& - 21120\hat{\omega}\hat{\omega}_{22} + 1800\hat{\omega}_{21}\hat{\omega}_{30} - 640\hat{\omega}_{30}^2 - 1600\hat{\omega}_{31} + 14080\hat{\omega}\hat{\omega}_{31} \\
& - 1920\hat{\omega}_{20}\hat{\omega}_{31} + 324\hat{\omega}_{22}\hat{\omega}_{31} - 216\hat{\omega}_{31}^2 - 324\hat{\omega}_{21}\hat{\omega}_{32} + 108\hat{\omega}_{30}\hat{\omega}_{32} \\
& + 32\hat{\omega}_{33} + 640\hat{\omega}\hat{\omega}_{33} - 108\hat{\omega}_{20}\hat{\omega}_{33} + 480\hat{\omega}_{20}\hat{\omega}_{40} - 81\hat{\omega}_{22}\hat{\omega}_{40} \\
& + 54\hat{\omega}_{31}\hat{\omega}_{40} + 162\hat{\omega}_{21}\hat{\omega}_{41} - 54\hat{\omega}_{30}\hat{\omega}_{41} \\
& + 4\hat{\omega}_{10}(10596\hat{\omega}_{21} - 5(704\hat{\omega}_{30} - 6\hat{\omega}_{32} + 3\hat{\omega}_{41}))
\end{aligned}$$

$$\begin{aligned}
& + 2\hat{\omega}_{11}(-22400 + 40448\hat{\omega} - 21192\hat{\omega}_{20} - 180\hat{\omega}_{22} + 108\hat{\omega}_{33} + 30\hat{\omega}_{40} - 81\hat{\omega}_{42}) \\
& - 24\hat{\omega}_{42} - 480\hat{\omega}\hat{\omega}_{42} + 81\hat{\omega}_{20}\hat{\omega}_{42}) \\
& - q^6(89600\hat{\omega}_{10}^2 - 56064\hat{\omega}_{20} - 58944\hat{\omega}_{20}^2 + 4488\hat{\omega}_{21}^2 + 624\hat{\omega}_{22} + 58944\hat{\omega}\hat{\omega}_{22} \\
& - 7064\hat{\omega}_{21}\hat{\omega}_{30} + 1856\hat{\omega}_{30}^2 - 1184\hat{\omega}_{31} - 39296\hat{\omega}\hat{\omega}_{31} + 5568\hat{\omega}_{20}\hat{\omega}_{31} \\
& + 444\hat{\omega}_{22}\hat{\omega}_{31} - 296\hat{\omega}_{31}^2 - 444\hat{\omega}_{21}\hat{\omega}_{32} + 148\hat{\omega}_{30}\hat{\omega}_{32} - 176\hat{\omega}_{33} - 1856\hat{\omega}\hat{\omega}_{33} \\
& - 148\hat{\omega}_{20}\hat{\omega}_{33} + 192\hat{\omega}_{40} - 1392\hat{\omega}_{20}\hat{\omega}_{40} - 111\hat{\omega}_{22}\hat{\omega}_{40} + 74\hat{\omega}_{31}\hat{\omega}_{40} + 222\hat{\omega}_{21}\hat{\omega}_{41} \\
& - 74\hat{\omega}_{30}\hat{\omega}_{41} - 4\hat{\omega}_{10}(26668\hat{\omega}_{21} - 9824\hat{\omega}_{30} + 187(-2\hat{\omega}_{32} + \hat{\omega}_{41})) \\
& + \hat{\omega}_{11}(104320 - 89600\hat{\omega} + 106672\hat{\omega}_{20} - 4488\hat{\omega}_{22} + 296\hat{\omega}_{33} + 748\hat{\omega}_{40} - 222\hat{\omega}_{42}) \\
& + 132\hat{\omega}_{42} + 1392\hat{\omega}\hat{\omega}_{42} + 111\hat{\omega}_{20}\hat{\omega}_{42}) \\
& + q^{20}(-218112\hat{\omega}_{10}^2 + 166144\hat{\omega}_{11} - 110592\hat{\omega}_{20} + 180432\hat{\omega}_{11}\hat{\omega}_{20} - 79488\hat{\omega}_{20}^2 \\
& + 2808\hat{\omega}_{21}^2 - 9888\hat{\omega}_{22} - 2808\hat{\omega}_{11}\hat{\omega}_{22} + 4824\hat{\omega}_{21}\hat{\omega}_{30} - 1920\hat{\omega}_{30}^2 + 6592\hat{\omega}_{31} \\
& - 5760\hat{\omega}_{20}\hat{\omega}_{31} + 972\hat{\omega}_{22}\hat{\omega}_{31} - 648\hat{\omega}_{31}^2 - 972\hat{\omega}_{21}\hat{\omega}_{32} + 324\hat{\omega}_{30}\hat{\omega}_{32} - 224\hat{\omega}_{33} \\
& + 648\hat{\omega}_{11}\hat{\omega}_{33} - 324\hat{\omega}_{20}\hat{\omega}_{33} + 468\hat{\omega}_{11}\hat{\omega}_{40} + 1440\hat{\omega}_{20}\hat{\omega}_{40} - 243\hat{\omega}_{22}\hat{\omega}_{40} \\
& + 162\hat{\omega}_{31}\hat{\omega}_{40} + 486\hat{\omega}_{21}\hat{\omega}_{41} - 162\hat{\omega}_{30}\hat{\omega}_{41} \\
& - 36\hat{\omega}_{10}(5012\hat{\omega}_{21} - 1472\hat{\omega}_{30} - 26\hat{\omega}_{32} + 13\hat{\omega}_{41}) \\
& + 96\hat{\omega}(3072 + 2272\hat{\omega}_{11} + 828\hat{\omega}_{22} - 552\hat{\omega}_{31} + 20\hat{\omega}_{33} - 15\hat{\omega}_{42}) \\
& + 168\hat{\omega}_{42} - 486\hat{\omega}_{11}\hat{\omega}_{42} + 243\hat{\omega}_{20}\hat{\omega}_{42}) \\
& + q^{16}(306688\hat{\omega}_{10}^2 - 159872\hat{\omega}_{11} + 133632\hat{\omega}_{20} + 266928\hat{\omega}_{11}\hat{\omega}_{20} - 170304\hat{\omega}_{20}^2 \\
& + 26568\hat{\omega}_{21}^2 + 31152\hat{\omega}_{22} - 26568\hat{\omega}_{11}\hat{\omega}_{22} - 18264\hat{\omega}_{21}\hat{\omega}_{30} + 3136\hat{\omega}_{30}^2 - 25376\hat{\omega}_{31} \\
& + 9408\hat{\omega}_{20}\hat{\omega}_{31} + 2268\hat{\omega}_{22}\hat{\omega}_{31} - 1512\hat{\omega}_{31}^2 - 2268\hat{\omega}_{21}\hat{\omega}_{32} + 756\hat{\omega}_{30}\hat{\omega}_{32} - 368\hat{\omega}_{33} \\
& + 1512\hat{\omega}_{11}\hat{\omega}_{33} - 756\hat{\omega}_{20}\hat{\omega}_{33} + 1152\hat{\omega}_{40} + 4428\hat{\omega}_{11}\hat{\omega}_{40} - 2352\hat{\omega}_{20}\hat{\omega}_{40} - 567\hat{\omega}_{22}\hat{\omega}_{40} \\
& + 378\hat{\omega}_{31}\hat{\omega}_{40} + 1134\hat{\omega}_{21}\hat{\omega}_{41} - 378\hat{\omega}_{30}\hat{\omega}_{41} \\
& - 4\hat{\omega}_{10}(66732\hat{\omega}_{21} - 28384\hat{\omega}_{30} + 1107(-2\hat{\omega}_{32} + \hat{\omega}_{41})) \\
& - 16\hat{\omega}(-129024 + 19168\hat{\omega}_{11} - 10644\hat{\omega}_{22} + 7096\hat{\omega}_{31} + 196\hat{\omega}_{33} - 147\hat{\omega}_{42}) \\
& + 276\hat{\omega}_{42} - 1134\hat{\omega}_{11}\hat{\omega}_{42} + 567\hat{\omega}_{20}\hat{\omega}_{42}) \\
& + q^2(2048\hat{\omega}_{10}^2 + 1200\hat{\omega}_{11}\hat{\omega}_{20} - 768\hat{\omega}_{20}^2 + 648\hat{\omega}_{21}^2 - 648\hat{\omega}_{11}\hat{\omega}_{22} + 552\hat{\omega}_{21}\hat{\omega}_{30} \\
& - 256\hat{\omega}_{30}^2 - 768\hat{\omega}_{20}\hat{\omega}_{31} - 108\hat{\omega}_{22}\hat{\omega}_{31} + 72\hat{\omega}_{31}^2 + 108\hat{\omega}_{21}\hat{\omega}_{32} - 36\hat{\omega}_{30}\hat{\omega}_{32} \\
& - 72\hat{\omega}_{11}\hat{\omega}_{33} + 36\hat{\omega}_{20}\hat{\omega}_{33} + 108\hat{\omega}_{11}\hat{\omega}_{40} + 192\hat{\omega}_{20}\hat{\omega}_{40} + 27\hat{\omega}_{22}\hat{\omega}_{40} - 18\hat{\omega}_{31}\hat{\omega}_{40} \\
& - 54\hat{\omega}_{21}\hat{\omega}_{41} + 18\hat{\omega}_{30}\hat{\omega}_{41} - 4\hat{\omega}_{10}(300\hat{\omega}_{21} - 128\hat{\omega}_{30} - 54\hat{\omega}_{32} + 27\hat{\omega}_{41}) \\
& + 54\hat{\omega}_{11}\hat{\omega}_{42} - 27\hat{\omega}_{20}\hat{\omega}_{42} - 64\hat{\omega}(32\hat{\omega}_{11} - 12\hat{\omega}_{22} + 8\hat{\omega}_{31} - 4\hat{\omega}_{33} + 3\hat{\omega}_{42})) \\
& + q^{28}(-2048\hat{\omega}_{10}^2 - 1200\hat{\omega}_{11}\hat{\omega}_{20} + 768\hat{\omega}_{20}^2 - 648\hat{\omega}_{21}^2 + 648\hat{\omega}_{11}\hat{\omega}_{22} - 552\hat{\omega}_{21}\hat{\omega}_{30} \\
& + 256\hat{\omega}_{30}^2 + 768\hat{\omega}_{20}\hat{\omega}_{31} + 108\hat{\omega}_{22}\hat{\omega}_{31} - 72\hat{\omega}_{31}^2 - 108\hat{\omega}_{21}\hat{\omega}_{32} + 36\hat{\omega}_{30}\hat{\omega}_{32} \\
& + 72\hat{\omega}_{11}\hat{\omega}_{33} - 36\hat{\omega}_{20}\hat{\omega}_{33} - 108\hat{\omega}_{11}\hat{\omega}_{40} - 192\hat{\omega}_{20}\hat{\omega}_{40} - 27\hat{\omega}_{22}\hat{\omega}_{40} + 18\hat{\omega}_{31}\hat{\omega}_{40} \\
& + 54\hat{\omega}_{21}\hat{\omega}_{41} - 18\hat{\omega}_{30}\hat{\omega}_{41} + 4\hat{\omega}_{10}(300\hat{\omega}_{21} - 128\hat{\omega}_{30} - 54\hat{\omega}_{32} + 27\hat{\omega}_{41}) \\
& - 54\hat{\omega}_{11}\hat{\omega}_{42} + 27\hat{\omega}_{20}\hat{\omega}_{42} + 64\hat{\omega}(32\hat{\omega}_{11} - 12\hat{\omega}_{22} + 8\hat{\omega}_{31} - 4\hat{\omega}_{33} + 3\hat{\omega}_{42})) \\
& + q^{18}(157696\hat{\omega}_{10}^2 + 194048\hat{\omega}_{11} - 132096\hat{\omega}_{20} + 226960\hat{\omega}_{11}\hat{\omega}_{20} - 119040\hat{\omega}_{20}^2 \\
& - 9768\hat{\omega}_{21}^2 - 27840\hat{\omega}_{22} + 9768\hat{\omega}_{11}\hat{\omega}_{22} + 16312\hat{\omega}_{21}\hat{\omega}_{30} - 4352\hat{\omega}_{30}^2 + 24704\hat{\omega}_{31} \\
& - 13056\hat{\omega}_{20}\hat{\omega}_{31} - 1284\hat{\omega}_{22}\hat{\omega}_{31} + 856\hat{\omega}_{31}^2 + 1284\hat{\omega}_{21}\hat{\omega}_{32} - 428\hat{\omega}_{30}\hat{\omega}_{32} + 704\hat{\omega}_{33}
\end{aligned}$$

$$\begin{aligned}
& - 856\hat{\omega}_{11}\hat{\omega}_{33} + 428\hat{\omega}_{20}\hat{\omega}_{33} - 1536\hat{\omega}_{40} - 1628\hat{\omega}_{11}\hat{\omega}_{40} + 3264\hat{\omega}_{20}\hat{\omega}_{40} \\
& + 321\hat{\omega}_{22}\hat{\omega}_{40} - 214\hat{\omega}_{31}\hat{\omega}_{40} - 4\hat{\omega}_{10}(56740\hat{\omega}_{21} - 19840\hat{\omega}_{30} + 814\hat{\omega}_{32} - 407\hat{\omega}_{41}) \\
& - 642\hat{\omega}_{21}\hat{\omega}_{41} + 214\hat{\omega}_{30}\hat{\omega}_{41} - 528\hat{\omega}_{42} + 642\hat{\omega}_{11}\hat{\omega}_{42} - 321\hat{\omega}_{20}\hat{\omega}_{42} \\
& - 64\hat{\omega}(-18432 + 2464\hat{\omega}_{11} - 1860\hat{\omega}_{22} + 1240\hat{\omega}_{31} - 68\hat{\omega}_{33} + 51\hat{\omega}_{42})) \\
& + q^{12}(-157696\hat{\omega}_{10}^2 + 194048\hat{\omega}_{11} - 132096\hat{\omega}_{20} - 226960\hat{\omega}_{11}\hat{\omega}_{20} + 119040\hat{\omega}_{20}^2 \\
& + 9768\hat{\omega}_{21}^2 - 27840\hat{\omega}_{22} - 9768\hat{\omega}_{11}\hat{\omega}_{22} - 16312\hat{\omega}_{21}\hat{\omega}_{30} + 4352\hat{\omega}_{30}^2 + 24704\hat{\omega}_{31} \\
& + 13056\hat{\omega}_{20}\hat{\omega}_{31} + 1284\hat{\omega}_{22}\hat{\omega}_{31} - 856\hat{\omega}_{31}^2 - 1284\hat{\omega}_{21}\hat{\omega}_{32} + 428\hat{\omega}_{30}\hat{\omega}_{32} \\
& + 704\hat{\omega}_{33} + 856\hat{\omega}_{11}\hat{\omega}_{33} - 428\hat{\omega}_{20}\hat{\omega}_{33} - 1536\hat{\omega}_{40} + 1628\hat{\omega}_{11}\hat{\omega}_{40} - 3264\hat{\omega}_{20}\hat{\omega}_{40} \\
& - 321\hat{\omega}_{22}\hat{\omega}_{40} + 214\hat{\omega}_{31}\hat{\omega}_{40} + 4\hat{\omega}_{10}(56740\hat{\omega}_{21} - 19840\hat{\omega}_{30} + 814\hat{\omega}_{32} - 407\hat{\omega}_{41}) \\
& + 642\hat{\omega}_{21}\hat{\omega}_{41} - 214\hat{\omega}_{30}\hat{\omega}_{41} - 528\hat{\omega}_{42} - 642\hat{\omega}_{11}\hat{\omega}_{42} + 321\hat{\omega}_{20}\hat{\omega}_{42} \\
& + 64\hat{\omega}(18432 + 2464\hat{\omega}_{11} - 1860\hat{\omega}_{22} + 1240\hat{\omega}_{31} - 68\hat{\omega}_{33} + 51\hat{\omega}_{42})) \\
& + q^{24}(89600\hat{\omega}_{10}^2 + 56064\hat{\omega}_{20} - 58944\hat{\omega}_{20}^2 + 4488\hat{\omega}_{21}^2 - 624\hat{\omega}_{22} + 58944\hat{\omega}_{22} \\
& - 7064\hat{\omega}_{21}\hat{\omega}_{30} + 1856\hat{\omega}_{30}^2 + 1184\hat{\omega}_{31} - 39296\hat{\omega}_{31} + 5568\hat{\omega}_{20}\hat{\omega}_{31} + 444\hat{\omega}_{22}\hat{\omega}_{31} \\
& - 296\hat{\omega}_{31}^2 - 444\hat{\omega}_{21}\hat{\omega}_{32} + 148\hat{\omega}_{30}\hat{\omega}_{32} + 176\hat{\omega}_{33} - 1856\hat{\omega}_{33} - 148\hat{\omega}_{20}\hat{\omega}_{33} - 192\hat{\omega}_{40} \\
& - 1392\hat{\omega}_{20}\hat{\omega}_{40} - 111\hat{\omega}_{22}\hat{\omega}_{40} + 74\hat{\omega}_{31}\hat{\omega}_{40} + 222\hat{\omega}_{21}\hat{\omega}_{41} - 74\hat{\omega}_{30}\hat{\omega}_{41} \\
& - 4\hat{\omega}_{10}(26668\hat{\omega}_{21} - 9824\hat{\omega}_{30} + 187(-2\hat{\omega}_{32} + \hat{\omega}_{41})) \\
& - 132\hat{\omega}_{42} + 1392\hat{\omega}_{42} + 111\hat{\omega}_{20}\hat{\omega}_{42} - 2\hat{\omega}_{11}(52160 + 44800\hat{\omega} - 53336\hat{\omega}_{20} \\
& + 2244\hat{\omega}_{22} - 148\hat{\omega}_{33} - 374\hat{\omega}_{40} + 111\hat{\omega}_{42})) \\
& + q^{22}(-193024\hat{\omega}_{10}^2 + 25344\hat{\omega}_{20} - 45120\hat{\omega}_{20}^2 - 5976\hat{\omega}_{21}^2 + 4848\hat{\omega}_{22} \\
& + 45120\hat{\omega}_{22} + 5640\hat{\omega}_{21}\hat{\omega}_{30} - 1216\hat{\omega}_{30}^2 - 5536\hat{\omega}_{31} - 30080\hat{\omega}_{31} \\
& - 3648\hat{\omega}_{20}\hat{\omega}_{31} - 756\hat{\omega}_{22}\hat{\omega}_{31} + 504\hat{\omega}_{31}^2 + 756\hat{\omega}_{21}\hat{\omega}_{32} - 252\hat{\omega}_{30}\hat{\omega}_{32} \\
& - 304\hat{\omega}_{33} + 1216\hat{\omega}_{33} + 252\hat{\omega}_{20}\hat{\omega}_{33} + 576\hat{\omega}_{40} + 912\hat{\omega}_{20}\hat{\omega}_{40} + 189\hat{\omega}_{22}\hat{\omega}_{40} \\
& - 126\hat{\omega}_{31}\hat{\omega}_{40} - 4\hat{\omega}_{10}(25884\hat{\omega}_{21} - 7520\hat{\omega}_{30} + 498\hat{\omega}_{32} - 249\hat{\omega}_{41}) \\
& - 378\hat{\omega}_{21}\hat{\omega}_{41} + 126\hat{\omega}_{30}\hat{\omega}_{41} + 228\hat{\omega}_{42} - 912\hat{\omega}_{42} - 189\hat{\omega}_{20}\hat{\omega}_{42} \\
& + 2\hat{\omega}_{11}(-25664 + 96512\hat{\omega} + 51768\hat{\omega}_{20} + 2988\hat{\omega}_{22} - 252\hat{\omega}_{33} - 498\hat{\omega}_{40} + 189\hat{\omega}_{42})) \\
& + q^8(193024\hat{\omega}_{10}^2 + 25344\hat{\omega}_{20} + 45120\hat{\omega}_{20}^2 + 5976\hat{\omega}_{21}^2 + 4848\hat{\omega}_{22} - 45120\hat{\omega}_{22} \\
& - 5640\hat{\omega}_{21}\hat{\omega}_{30} + 1216\hat{\omega}_{30}^2 - 5536\hat{\omega}_{31} + 30080\hat{\omega}_{31} + 3648\hat{\omega}_{20}\hat{\omega}_{31} + 756\hat{\omega}_{22}\hat{\omega}_{31} \\
& - 504\hat{\omega}_{31}^2 - 756\hat{\omega}_{21}\hat{\omega}_{32} + 252\hat{\omega}_{30}\hat{\omega}_{32} - 304\hat{\omega}_{33} - 1216\hat{\omega}_{33} - 252\hat{\omega}_{20}\hat{\omega}_{33} \\
& + 576\hat{\omega}_{40} - 912\hat{\omega}_{20}\hat{\omega}_{40} - 189\hat{\omega}_{22}\hat{\omega}_{40} + 126\hat{\omega}_{31}\hat{\omega}_{40} \\
& + 4\hat{\omega}_{10}(25884\hat{\omega}_{21} - 7520\hat{\omega}_{30} + 498\hat{\omega}_{32} - 249\hat{\omega}_{41}) \\
& + 378\hat{\omega}_{21}\hat{\omega}_{41} - 126\hat{\omega}_{30}\hat{\omega}_{41} + 228\hat{\omega}_{42} + 912\hat{\omega}_{42} + 189\hat{\omega}_{20}\hat{\omega}_{42} \\
& - 2\hat{\omega}_{11}(25664 + 96512\hat{\omega} + 51768\hat{\omega}_{20} + 2988\hat{\omega}_{22} - 252\hat{\omega}_{33} - 498\hat{\omega}_{40} + 189\hat{\omega}_{42})) \\
& + (-1 + q^2)^3(1 + q^2)^2(1 + q^2 + 2q^4 + q^6 + q^8) \\
& \{-32(1 - 10q^2 - 256q^4 - 342q^6 - 514q^8 - 342q^{10} - 256q^{12} - 10q^{14} + q^{16})\hat{\omega}'_{20}\hat{\omega}'_{31} \\
& + 32\hat{\omega}'_{10}\hat{\omega}'_{32} - 320q^2\hat{\omega}'_{10}\hat{\omega}'_{32} - 8192q^4\hat{\omega}'_{10}\hat{\omega}'_{32} - 10944q^6\hat{\omega}'_{10}\hat{\omega}'_{32} - 16448q^8\hat{\omega}'_{10}\hat{\omega}'_{32} \\
& - 10944q^{10}\hat{\omega}'_{10}\hat{\omega}'_{32} - 8192q^{12}\hat{\omega}'_{10}\hat{\omega}'_{32} - 320q^{14}\hat{\omega}'_{10}\hat{\omega}'_{32} + 32q^{16}\hat{\omega}'_{10}\hat{\omega}'_{32} + 8\hat{\omega}'_{31}\hat{\omega}'_{40} \\
& + 112q^2\hat{\omega}'_{31}\hat{\omega}'_{40} + 256q^4\hat{\omega}'_{31}\hat{\omega}'_{40} - 240q^6\hat{\omega}'_{31}\hat{\omega}'_{40} - 272q^8\hat{\omega}'_{31}\hat{\omega}'_{40} - 240q^{10}\hat{\omega}'_{31}\hat{\omega}'_{40}
\end{aligned}$$

$$\begin{aligned}
& + 256q^{12}\hat{\omega}'_{31}\hat{\omega}'_{40} + 112q^{14}\hat{\omega}'_{31}\hat{\omega}'_{40} + 8q^{16}\hat{\omega}'_{31}\hat{\omega}'_{40} - 8\hat{\omega}'_{30}\hat{\omega}'_{41} - 112q^2\hat{\omega}'_{30}\hat{\omega}'_{41} \\
& - 256q^4\hat{\omega}'_{30}\hat{\omega}'_{41} + 240q^6\hat{\omega}'_{30}\hat{\omega}'_{41} + 272q^8\hat{\omega}'_{30}\hat{\omega}'_{41} + 240q^{10}\hat{\omega}'_{30}\hat{\omega}'_{41} - 256q^{12}\hat{\omega}'_{30}\hat{\omega}'_{41} \\
& - 112q^{14}\hat{\omega}'_{30}\hat{\omega}'_{41} - 8q^{16}\hat{\omega}'_{30}\hat{\omega}'_{41} - 3\hat{\omega}'_{32}\hat{\omega}'_{41} + 2q^2\hat{\omega}'_{32}\hat{\omega}'_{41} + 30q^6\hat{\omega}'_{32}\hat{\omega}'_{41} - 58q^8\hat{\omega}'_{32}\hat{\omega}'_{41} \\
& + 30q^{10}\hat{\omega}'_{32}\hat{\omega}'_{41} + 2q^{14}\hat{\omega}'_{32}\hat{\omega}'_{41} - 3q^{16}\hat{\omega}'_{32}\hat{\omega}'_{41} + 3\hat{\omega}'_{31}\hat{\omega}'_{42} - 2q^2\hat{\omega}'_{31}\hat{\omega}'_{42} - 30q^6\hat{\omega}'_{31}\hat{\omega}'_{42} \\
& + 58q^8\hat{\omega}'_{31}\hat{\omega}'_{42} - 30q^{10}\hat{\omega}'_{31}\hat{\omega}'_{42} - 2q^{14}\hat{\omega}'_{31}\hat{\omega}'_{42} + 3q^{16}\hat{\omega}'_{31}\hat{\omega}'_{42} + 8\hat{\omega}'_{10}\hat{\omega}'_{43} + 112q^2\hat{\omega}'_{10}\hat{\omega}'_{43} \\
& + 256q^4\hat{\omega}'_{10}\hat{\omega}'_{43} - 240q^6\hat{\omega}'_{10}\hat{\omega}'_{43} - 272q^8\hat{\omega}'_{10}\hat{\omega}'_{43} - 240q^{10}\hat{\omega}'_{10}\hat{\omega}'_{43} + 256q^{12}\hat{\omega}'_{10}\hat{\omega}'_{43} \\
& + 112q^{14}\hat{\omega}'_{10}\hat{\omega}'_{43} + 8q^{16}\hat{\omega}'_{10}\hat{\omega}'_{43} + \hat{\omega}'_{21} (\\
& 32(1 - 10q^2 - 256q^4 - 342q^6 - 514q^8 - 342q^{10} - 256q^{12} - 10q^{14} + q^{16})\hat{\omega}'_{30} \\
& - (-1 + q^2)^4(3 + 10q^2 + 22q^4 + 10q^6 + 3q^8)\hat{\omega}'_{43}) \} \\
& - 2\eta(-1 - q^2 - q^4 + q^8 + q^{10} + q^{12}) \\
& \{ -16\hat{\omega}_{31}\hat{\omega}'_{21} - 4\hat{\omega}_{33}\hat{\omega}'_{21} + 4\hat{\omega}_{40}\hat{\omega}'_{21} + 3\hat{\omega}_{42}\hat{\omega}'_{21} + 16\hat{\omega}_{21}\hat{\omega}'_{31} + 4\hat{\omega}_{32}\hat{\omega}'_{31} - 2\hat{\omega}_{41}\hat{\omega}'_{31} \\
& - 16\hat{\omega}_{11}\hat{\omega}'_{32} - 4\hat{\omega}_{31}\hat{\omega}'_{32} + \hat{\omega}_{40}\hat{\omega}'_{32} - 4\hat{\omega}_{20}\hat{\omega}'_{41} - 3\hat{\omega}_{22}\hat{\omega}'_{41} + 2\hat{\omega}_{31}\hat{\omega}'_{41} + 4\hat{\omega}_{10}\hat{\omega}'_{42} \\
& + 3\hat{\omega}_{21}\hat{\omega}'_{42} - \hat{\omega}_{30}\hat{\omega}'_{42} - 2\hat{\omega}_{11}\hat{\omega}'_{43} + \hat{\omega}_{20}\hat{\omega}'_{43} \\
& + q^4(8064\hat{\omega}_{31}\hat{\omega}'_{10} + 960\hat{\omega}_{33}\hat{\omega}'_{10} - 1024\hat{\omega}_{40}\hat{\omega}'_{10} - 720\hat{\omega}_{42}\hat{\omega}'_{10} + 11264\hat{\omega}_{10}\hat{\omega}'_{20} \\
& + 5952\hat{\omega}_{21}\hat{\omega}'_{20} - 3200\hat{\omega}_{30}\hat{\omega}'_{20} - 480\hat{\omega}_{32}\hat{\omega}'_{20} + 240\hat{\omega}_{41}\hat{\omega}'_{20} + 1920\hat{\omega}'_{21} - 11264\hat{\omega}\hat{\omega}'_{21} \\
& + 464\hat{\omega}_{31}\hat{\omega}'_{21} - 28\hat{\omega}_{33}\hat{\omega}'_{21} - 116\hat{\omega}_{40}\hat{\omega}'_{21} + 21\hat{\omega}_{42}\hat{\omega}'_{21} - 8064\hat{\omega}_{11}\hat{\omega}'_{30} - 960\hat{\omega}_{31}\hat{\omega}'_{30} \\
& + 160\hat{\omega}_{40}\hat{\omega}'_{30} + 8064\hat{\omega}_{10}\hat{\omega}'_{31} - 464\hat{\omega}_{21}\hat{\omega}'_{31} + 960\hat{\omega}_{30}\hat{\omega}'_{31} + 28\hat{\omega}_{32}\hat{\omega}'_{31} - 14\hat{\omega}_{41}\hat{\omega}'_{31} \\
& + 768\hat{\omega}'_{32} - 3200\hat{\omega}\hat{\omega}'_{32} + 464\hat{\omega}_{11}\hat{\omega}'_{32} - 28\hat{\omega}_{31}\hat{\omega}'_{32} + 7\hat{\omega}_{40}\hat{\omega}'_{32} + 1024\hat{\omega}_{10}\hat{\omega}'_{40} \\
& + 480\hat{\omega}_{21}\hat{\omega}'_{40} - 160\hat{\omega}_{30}\hat{\omega}'_{40} + 96\hat{\omega}'_{41} - 1024\hat{\omega}\hat{\omega}'_{41} + 14\hat{\omega}_{31}\hat{\omega}'_{41} \\
& - 3\hat{\omega}_{22}(1984\hat{\omega}'_{10} - 160\hat{\omega}'_{30} + 7\hat{\omega}'_{41}) + 124\hat{\omega}_{10}\hat{\omega}'_{42} + 21\hat{\omega}_{21}\hat{\omega}'_{42} - 7\hat{\omega}_{30}\hat{\omega}'_{42} \\
& - \hat{\omega}_{20}(11264\hat{\omega}'_{10} + 5952\hat{\omega}'_{21} - 3200\hat{\omega}'_{30} + 480\hat{\omega}'_{32} + 604\hat{\omega}'_{41} - 7\hat{\omega}'_{43}) \\
& + 48\hat{\omega}'_{43} + 160\hat{\omega}\hat{\omega}'_{43} - 14\hat{\omega}_{11}\hat{\omega}'_{43}) \\
& + q^{20}(-384\hat{\omega}_{31}\hat{\omega}'_{10} - 192\hat{\omega}_{33}\hat{\omega}'_{10} - 128\hat{\omega}_{40}\hat{\omega}'_{10} + 144\hat{\omega}_{42}\hat{\omega}'_{10} - 3584\hat{\omega}_{10}\hat{\omega}'_{20} \\
& - 1344\hat{\omega}_{21}\hat{\omega}'_{20} + 512\hat{\omega}_{30}\hat{\omega}'_{20} - 96\hat{\omega}_{32}\hat{\omega}'_{20} + 48\hat{\omega}_{41}\hat{\omega}'_{20} + 896\hat{\omega}'_{21} \\
& + 3584\hat{\omega}\hat{\omega}'_{21} + 16\hat{\omega}_{31}\hat{\omega}'_{21} + 4\hat{\omega}_{33}\hat{\omega}'_{21} - 4\hat{\omega}_{40}\hat{\omega}'_{21} - 3\hat{\omega}_{42}\hat{\omega}'_{21} + 384\hat{\omega}_{11}\hat{\omega}'_{30} \\
& + 192\hat{\omega}_{31}\hat{\omega}'_{30} - 64\hat{\omega}_{40}\hat{\omega}'_{30} - 384\hat{\omega}_{10}\hat{\omega}'_{31} - 16\hat{\omega}_{21}\hat{\omega}'_{31} - 192\hat{\omega}_{30}\hat{\omega}'_{31} - 4\hat{\omega}_{32}\hat{\omega}'_{31} \\
& + 2\hat{\omega}_{41}\hat{\omega}'_{31} + 128\hat{\omega}'_{32} + 512\hat{\omega}\hat{\omega}'_{32} + 16\hat{\omega}_{11}\hat{\omega}'_{32} + 4\hat{\omega}_{31}\hat{\omega}'_{32} - \hat{\omega}_{40}\hat{\omega}'_{32} + 128\hat{\omega}_{10}\hat{\omega}'_{40} \\
& - 192\hat{\omega}_{21}\hat{\omega}'_{40} + 64\hat{\omega}_{30}\hat{\omega}'_{40} - 32\hat{\omega}'_{41} - 128\hat{\omega}\hat{\omega}'_{41} - 2\hat{\omega}_{31}\hat{\omega}'_{41} \\
& + 3\hat{\omega}_{22}(448\hat{\omega}'_{10} + 32\hat{\omega}'_{30} + \hat{\omega}'_{41}) + 44\hat{\omega}_{10}\hat{\omega}'_{42} - 3\hat{\omega}_{21}\hat{\omega}'_{42} + \hat{\omega}_{30}\hat{\omega}'_{42} \\
& + \hat{\omega}_{20}(3584\hat{\omega}'_{10} + 1344\hat{\omega}'_{21} - 512\hat{\omega}'_{30} - 96\hat{\omega}'_{32} + 148\hat{\omega}'_{41} - \hat{\omega}'_{43}) \\
& - 16\hat{\omega}'_{43} - 64\hat{\omega}\hat{\omega}'_{43} + 2\hat{\omega}_{11}\hat{\omega}'_{43}) \\
& + q^{18}(-8064\hat{\omega}_{31}\hat{\omega}'_{10} - 960\hat{\omega}_{33}\hat{\omega}'_{10} + 1024\hat{\omega}_{40}\hat{\omega}'_{10} + 720\hat{\omega}_{42}\hat{\omega}'_{10} - 11264\hat{\omega}_{10}\hat{\omega}'_{20} \\
& - 5952\hat{\omega}_{21}\hat{\omega}'_{20} + 3200\hat{\omega}_{30}\hat{\omega}'_{20} + 480\hat{\omega}_{32}\hat{\omega}'_{20} - 240\hat{\omega}_{41}\hat{\omega}'_{20} + 1920\hat{\omega}'_{21} + 11264\hat{\omega}\hat{\omega}'_{21} \\
& - 464\hat{\omega}_{31}\hat{\omega}'_{21} + 28\hat{\omega}_{33}\hat{\omega}'_{21} + 116\hat{\omega}_{40}\hat{\omega}'_{21} - 21\hat{\omega}_{42}\hat{\omega}'_{21} + 8064\hat{\omega}_{11}\hat{\omega}'_{30} + 960\hat{\omega}_{31}\hat{\omega}'_{30} \\
& - 160\hat{\omega}_{40}\hat{\omega}'_{30} - 8064\hat{\omega}_{10}\hat{\omega}'_{31} + 464\hat{\omega}_{21}\hat{\omega}'_{31} - 960\hat{\omega}_{30}\hat{\omega}'_{31} - 28\hat{\omega}_{32}\hat{\omega}'_{31} + 14\hat{\omega}_{41}\hat{\omega}'_{31} \\
& + 768\hat{\omega}'_{32} + 3200\hat{\omega}\hat{\omega}'_{32} - 464\hat{\omega}_{11}\hat{\omega}'_{32} + 28\hat{\omega}_{31}\hat{\omega}'_{32} - 7\hat{\omega}_{40}\hat{\omega}'_{32} - 1024\hat{\omega}_{10}\hat{\omega}'_{40} \\
& - 480\hat{\omega}_{21}\hat{\omega}'_{40} + 160\hat{\omega}_{30}\hat{\omega}'_{40} + 96\hat{\omega}'_{41} + 1024\hat{\omega}\hat{\omega}'_{41} - 14\hat{\omega}_{31}\hat{\omega}'_{41} \\
& + 3\hat{\omega}_{22}(1984\hat{\omega}'_{10} - 160\hat{\omega}'_{30} + 7\hat{\omega}'_{41}) - 124\hat{\omega}_{10}\hat{\omega}'_{42} - 21\hat{\omega}_{21}\hat{\omega}'_{42} + 7\hat{\omega}_{30}\hat{\omega}'_{42}
\end{aligned}$$

$$\begin{aligned}
& + \hat{\omega}_{20}(11264\hat{\omega}'_{10} + 5952\hat{\omega}'_{21} - 3200\hat{\omega}'_{30} + 480\hat{\omega}'_{32} + 604\hat{\omega}'_{41} - 7\hat{\omega}'_{43}) \\
& + 48\hat{\omega}'_{43} - 160\hat{\omega}'_{43} + 14\hat{\omega}_{11}\hat{\omega}'_{43}) \\
& + q^6(32(4608 + 3632\hat{\omega}_{20} - 972\hat{\omega}_{22} + 440\hat{\omega}_{31} - 20\hat{\omega}_{33} + 52\hat{\omega}_{40} + 15\hat{\omega}_{42})\hat{\omega}'_{10} \\
& + 31104\hat{\omega}_{21}\hat{\omega}'_{20} - 6272\hat{\omega}_{30}\hat{\omega}'_{20} + 768\hat{\omega}_{32}\hat{\omega}'_{20} - 384\hat{\omega}_{41}\hat{\omega}'_{20} - 32256\hat{\omega}'_{21} \\
& + 116224\hat{\omega}'_{21} - 31104\hat{\omega}_{20}\hat{\omega}'_{21} - 1584\hat{\omega}_{31}\hat{\omega}'_{21} + 228\hat{\omega}_{33}\hat{\omega}'_{21} + 396\hat{\omega}_{40}\hat{\omega}'_{21} \\
& - 171\hat{\omega}_{42}\hat{\omega}'_{21} - 14080\hat{\omega}_{11}\hat{\omega}'_{30} + 6272\hat{\omega}_{20}\hat{\omega}'_{30} - 768\hat{\omega}_{22}\hat{\omega}'_{30} + 640\hat{\omega}_{31}\hat{\omega}'_{30} \\
& - 32\hat{\omega}_{40}\hat{\omega}'_{30} + 1584\hat{\omega}_{21}\hat{\omega}'_{31} - 640\hat{\omega}_{30}\hat{\omega}'_{31} - 228\hat{\omega}_{32}\hat{\omega}'_{31} + 114\hat{\omega}_{41}\hat{\omega}'_{31} - 1152\hat{\omega}'_{32} \\
& - 6272\hat{\omega}'_{32} - 1584\hat{\omega}_{11}\hat{\omega}'_{32} + 768\hat{\omega}_{20}\hat{\omega}'_{32} + 228\hat{\omega}_{31}\hat{\omega}'_{32} - 57\hat{\omega}_{40}\hat{\omega}'_{32} - 96\hat{\omega}_{21}\hat{\omega}'_{40} \\
& + 32\hat{\omega}_{30}\hat{\omega}'_{40} + 1664\hat{\omega}'_{41} + 84\hat{\omega}_{20}\hat{\omega}'_{41} + 171\hat{\omega}_{22}\hat{\omega}'_{41} - 114\hat{\omega}_{31}\hat{\omega}'_{41} \\
& - 4\hat{\omega}_{10}(29056\hat{\omega}'_{20} - 3520\hat{\omega}'_{31} + 416\hat{\omega}'_{40} - 3\hat{\omega}'_{42}) - 171\hat{\omega}_{21}\hat{\omega}'_{42} + 57\hat{\omega}_{30}\hat{\omega}'_{42} \\
& - 32\hat{\omega}'_{43} + 114\hat{\omega}_{11}\hat{\omega}'_{43} - 57\hat{\omega}_{20}\hat{\omega}'_{43}) \\
& - q^{16}(32(-4608 + 3632\hat{\omega}_{20} - 972\hat{\omega}_{22} + 440\hat{\omega}_{31} - 20\hat{\omega}_{33} + 52\hat{\omega}_{40} + 15\hat{\omega}_{42})\hat{\omega}'_{10} \\
& + 31104\hat{\omega}_{21}\hat{\omega}'_{20} - 6272\hat{\omega}_{30}\hat{\omega}'_{20} + 768\hat{\omega}_{32}\hat{\omega}'_{20} - 384\hat{\omega}_{41}\hat{\omega}'_{20} + 32256\hat{\omega}'_{21} \\
& + 116224\hat{\omega}'_{21} - 31104\hat{\omega}_{20}\hat{\omega}'_{21} - 1584\hat{\omega}_{31}\hat{\omega}'_{21} + 228\hat{\omega}_{33}\hat{\omega}'_{21} + 396\hat{\omega}_{40}\hat{\omega}'_{21} \\
& - 171\hat{\omega}_{42}\hat{\omega}'_{21} - 14080\hat{\omega}_{11}\hat{\omega}'_{30} + 6272\hat{\omega}_{20}\hat{\omega}'_{30} - 768\hat{\omega}_{22}\hat{\omega}'_{30} + 640\hat{\omega}_{31}\hat{\omega}'_{30} \\
& - 32\hat{\omega}_{40}\hat{\omega}'_{30} + 1584\hat{\omega}_{21}\hat{\omega}'_{31} - 640\hat{\omega}_{30}\hat{\omega}'_{31} - 228\hat{\omega}_{32}\hat{\omega}'_{31} + 114\hat{\omega}_{41}\hat{\omega}'_{31} + 1152\hat{\omega}'_{32} \\
& - 6272\hat{\omega}'_{32} - 1584\hat{\omega}_{11}\hat{\omega}'_{32} + 768\hat{\omega}_{20}\hat{\omega}'_{32} + 228\hat{\omega}_{31}\hat{\omega}'_{32} - 57\hat{\omega}_{40}\hat{\omega}'_{32} - 96\hat{\omega}_{21}\hat{\omega}'_{40} \\
& + 32\hat{\omega}_{30}\hat{\omega}'_{40} + 1664\hat{\omega}'_{41} + 84\hat{\omega}_{20}\hat{\omega}'_{41} + 171\hat{\omega}_{22}\hat{\omega}'_{41} - 114\hat{\omega}_{31}\hat{\omega}'_{41} \\
& - 4\hat{\omega}_{10}(29056\hat{\omega}'_{20} - 3520\hat{\omega}'_{31} + 416\hat{\omega}'_{40} - 3\hat{\omega}'_{42}) - 171\hat{\omega}_{21}\hat{\omega}'_{42} + 57\hat{\omega}_{30}\hat{\omega}'_{42} \\
& - 32\hat{\omega}'_{43} + 114\hat{\omega}_{11}\hat{\omega}'_{43} - 57\hat{\omega}_{20}\hat{\omega}'_{43}) \\
& + 2q^{14}(16(9216 + 1424\hat{\omega}_{20} + 1644\hat{\omega}_{22} - 824\hat{\omega}_{31} + 20\hat{\omega}_{33} - 68\hat{\omega}_{40} - 15\hat{\omega}_{42})\hat{\omega}'_{10} \\
& - 26304\hat{\omega}_{21}\hat{\omega}'_{20} + 9536\hat{\omega}_{30}\hat{\omega}'_{20} - 17152\hat{\omega}'_{21} + 22784\hat{\omega}'_{21} + 26304\hat{\omega}_{20}\hat{\omega}'_{21} \\
& - 1392\hat{\omega}_{31}\hat{\omega}'_{21} + 204\hat{\omega}_{33}\hat{\omega}'_{21} + 348\hat{\omega}_{40}\hat{\omega}'_{21} - 153\hat{\omega}_{42}\hat{\omega}'_{21} + 13184\hat{\omega}_{11}\hat{\omega}'_{30} \\
& - 9536\hat{\omega}_{20}\hat{\omega}'_{30} - 320\hat{\omega}_{31}\hat{\omega}'_{30} + 80\hat{\omega}_{40}\hat{\omega}'_{30} + 1392\hat{\omega}_{21}\hat{\omega}'_{31} + 320\hat{\omega}_{30}\hat{\omega}'_{31} \\
& - 204\hat{\omega}_{32}\hat{\omega}'_{31} + 102\hat{\omega}_{41}\hat{\omega}'_{31} - 1216\hat{\omega}'_{32} + 9536\hat{\omega}'_{32} - 1392\hat{\omega}_{11}\hat{\omega}'_{32} + 204\hat{\omega}_{31}\hat{\omega}'_{32} \\
& - 51\hat{\omega}_{40}\hat{\omega}'_{32} + 240\hat{\omega}_{21}\hat{\omega}'_{40} - 80\hat{\omega}_{30}\hat{\omega}'_{40} - 128\hat{\omega}'_{41} - 1088\hat{\omega}'_{41} - 588\hat{\omega}_{20}\hat{\omega}'_{41} \\
& + 153\hat{\omega}_{22}\hat{\omega}'_{41} - 102\hat{\omega}_{31}\hat{\omega}'_{41} - 4\hat{\omega}_{10}(5696\hat{\omega}'_{20} + 3296\hat{\omega}'_{31} - 272\hat{\omega}'_{40} - 87\hat{\omega}'_{42}) \\
& - 153\hat{\omega}_{21}\hat{\omega}'_{42} + 51\hat{\omega}_{30}\hat{\omega}'_{42} - 64\hat{\omega}'_{43} + 80\hat{\omega}'_{43} + 102\hat{\omega}_{11}\hat{\omega}'_{43} - 51\hat{\omega}_{20}\hat{\omega}'_{43}) \\
& - 2q^8(16(-9216 + 1424\hat{\omega}_{20} + 1644\hat{\omega}_{22} - 824\hat{\omega}_{31} + 20\hat{\omega}_{33} - 68\hat{\omega}_{40} - 15\hat{\omega}_{42})\hat{\omega}'_{10} \\
& - 26304\hat{\omega}_{21}\hat{\omega}'_{20} + 9536\hat{\omega}_{30}\hat{\omega}'_{20} + 17152\hat{\omega}'_{21} + 22784\hat{\omega}'_{21} + 26304\hat{\omega}_{20}\hat{\omega}'_{21} \\
& - 1392\hat{\omega}_{31}\hat{\omega}'_{21} + 204\hat{\omega}_{33}\hat{\omega}'_{21} + 348\hat{\omega}_{40}\hat{\omega}'_{21} - 153\hat{\omega}_{42}\hat{\omega}'_{21} + 13184\hat{\omega}_{11}\hat{\omega}'_{30} \\
& - 9536\hat{\omega}_{20}\hat{\omega}'_{30} - 320\hat{\omega}_{31}\hat{\omega}'_{30} + 80\hat{\omega}_{40}\hat{\omega}'_{30} + 1392\hat{\omega}_{21}\hat{\omega}'_{31} + 320\hat{\omega}_{30}\hat{\omega}'_{31} \\
& - 204\hat{\omega}_{32}\hat{\omega}'_{31} + 102\hat{\omega}_{41}\hat{\omega}'_{31} + 1216\hat{\omega}'_{32} + 9536\hat{\omega}'_{32} - 1392\hat{\omega}_{11}\hat{\omega}'_{32} + 204\hat{\omega}_{31}\hat{\omega}'_{32} \\
& - 51\hat{\omega}_{40}\hat{\omega}'_{32} + 240\hat{\omega}_{21}\hat{\omega}'_{40} - 80\hat{\omega}_{30}\hat{\omega}'_{40} + 128\hat{\omega}'_{41} - 1088\hat{\omega}'_{41} - 588\hat{\omega}_{20}\hat{\omega}'_{41} \\
& + 153\hat{\omega}_{22}\hat{\omega}'_{41} - 102\hat{\omega}_{31}\hat{\omega}'_{41} - 4\hat{\omega}_{10}(5696\hat{\omega}'_{20} + 3296\hat{\omega}'_{31} - 272\hat{\omega}'_{40} - 87\hat{\omega}'_{42}) \\
& - 153\hat{\omega}_{21}\hat{\omega}'_{42} + 51\hat{\omega}_{30}\hat{\omega}'_{42} + 64\hat{\omega}'_{43} + 80\hat{\omega}'_{43} + 102\hat{\omega}_{11}\hat{\omega}'_{43} - 51\hat{\omega}_{20}\hat{\omega}'_{43}) \\
& + q^{22}(4\hat{\omega}_{33}\hat{\omega}'_{21} - 4\hat{\omega}_{40}\hat{\omega}'_{21} - 3\hat{\omega}_{42}\hat{\omega}'_{21} - 16\hat{\omega}_{21}\hat{\omega}'_{31} - 4\hat{\omega}_{32}\hat{\omega}'_{31} + 2\hat{\omega}_{41}\hat{\omega}'_{31} \\
& + 16\hat{\omega}_{11}\hat{\omega}'_{32} - \hat{\omega}_{40}\hat{\omega}'_{32} + 2\hat{\omega}_{31}(8\hat{\omega}'_{21} + 2\hat{\omega}'_{32} - \hat{\omega}'_{41}) + 4\hat{\omega}_{20}\hat{\omega}'_{41} + 3\hat{\omega}_{22}\hat{\omega}'_{41}
\end{aligned}$$

$$\begin{aligned}
& -4\hat{\omega}_{10}\hat{\omega}'_{42} - 3\hat{\omega}_{21}\hat{\omega}'_{42} + \hat{\omega}_{30}\hat{\omega}'_{42} + 2\hat{\omega}_{11}\hat{\omega}'_{43} - \hat{\omega}_{20}\hat{\omega}'_{43}) \\
& -2q^{10}(32(-2304 + 5208\hat{\omega}_{20} - 552\hat{\omega}_{22} - 184\hat{\omega}_{31} + 52\hat{\omega}_{33} + 138\hat{\omega}_{40} - 39\hat{\omega}_{42})\hat{\omega}'_{10} \\
& + 17664\hat{\omega}_{21}\hat{\omega}'_{20} - 2496\hat{\omega}_{30}\hat{\omega}'_{20} + 672\hat{\omega}_{32}\hat{\omega}'_{20} - 336\hat{\omega}_{41}\hat{\omega}'_{20} - 31872\hat{\omega}'_{21} \\
& + 166656\hat{\omega}\hat{\omega}'_{21} - 17664\hat{\omega}_{20}\hat{\omega}'_{21} + 1680\hat{\omega}_{31}\hat{\omega}'_{21} - 180\hat{\omega}_{33}\hat{\omega}'_{21} - 420\hat{\omega}_{40}\hat{\omega}'_{21} \\
& + 135\hat{\omega}_{42}\hat{\omega}'_{21} + 5888\hat{\omega}_{11}\hat{\omega}'_{30} + 2496\hat{\omega}_{20}\hat{\omega}'_{30} - 672\hat{\omega}_{22}\hat{\omega}'_{30} - 1664\hat{\omega}_{31}\hat{\omega}'_{30} + 528\hat{\omega}_{40}\hat{\omega}'_{30} \\
& - 1680\hat{\omega}_{21}\hat{\omega}'_{31} + 1664\hat{\omega}_{30}\hat{\omega}'_{31} + 180\hat{\omega}_{32}\hat{\omega}'_{31} - 90\hat{\omega}_{41}\hat{\omega}'_{31} - 1344\hat{\omega}'_{32} - 2496\hat{\omega}\hat{\omega}'_{32} \\
& + 1680\hat{\omega}_{11}\hat{\omega}'_{32} + 672\hat{\omega}_{20}\hat{\omega}'_{32} - 180\hat{\omega}_{31}\hat{\omega}'_{32} + 45\hat{\omega}_{40}\hat{\omega}'_{32} + 1584\hat{\omega}_{21}\hat{\omega}'_{40} - 528\hat{\omega}_{30}\hat{\omega}'_{40} \\
& - 96\hat{\omega}'_{41} + 4416\hat{\omega}\hat{\omega}'_{41} - 828\hat{\omega}_{20}\hat{\omega}'_{41} - 135\hat{\omega}_{22}\hat{\omega}'_{41} + 90\hat{\omega}_{31}\hat{\omega}'_{41} + 135\hat{\omega}_{21}\hat{\omega}'_{42} \\
& - 45\hat{\omega}_{30}\hat{\omega}'_{42} - 4\hat{\omega}_{10}(41664\hat{\omega}'_{20} + 1472\hat{\omega}'_{31} + 1104\hat{\omega}'_{40} + 189\hat{\omega}'_{42}) \\
& - 48\hat{\omega}'_{43} + 528\hat{\omega}\hat{\omega}'_{43} - 90\hat{\omega}_{11}\hat{\omega}'_{43} + 45\hat{\omega}_{20}\hat{\omega}'_{43}) \\
& + 2q^{12}(32(2304 + 5208\hat{\omega}_{20} - 552\hat{\omega}_{22} - 184\hat{\omega}_{31} + 52\hat{\omega}_{33} + 138\hat{\omega}_{40} - 39\hat{\omega}_{42})\hat{\omega}'_{10} \\
& + 17664\hat{\omega}_{21}\hat{\omega}'_{20} - 2496\hat{\omega}_{30}\hat{\omega}'_{20} + 672\hat{\omega}_{32}\hat{\omega}'_{20} - 336\hat{\omega}_{41}\hat{\omega}'_{20} + 31872\hat{\omega}'_{21} \\
& + 166656\hat{\omega}\hat{\omega}'_{21} - 17664\hat{\omega}_{20}\hat{\omega}'_{21} + 1680\hat{\omega}_{31}\hat{\omega}'_{21} - 180\hat{\omega}_{33}\hat{\omega}'_{21} - 420\hat{\omega}_{40}\hat{\omega}'_{21} \\
& + 135\hat{\omega}_{42}\hat{\omega}'_{21} + 5888\hat{\omega}_{11}\hat{\omega}'_{30} + 2496\hat{\omega}_{20}\hat{\omega}'_{30} - 672\hat{\omega}_{22}\hat{\omega}'_{30} - 1664\hat{\omega}_{31}\hat{\omega}'_{30} \\
& + 528\hat{\omega}_{40}\hat{\omega}'_{30} - 1680\hat{\omega}_{21}\hat{\omega}'_{31} + 1664\hat{\omega}_{30}\hat{\omega}'_{31} + 180\hat{\omega}_{32}\hat{\omega}'_{31} - 90\hat{\omega}_{41}\hat{\omega}'_{31} \\
& + 1344\hat{\omega}'_{32} - 2496\hat{\omega}\hat{\omega}'_{32} + 1680\hat{\omega}_{11}\hat{\omega}'_{32} + 672\hat{\omega}_{20}\hat{\omega}'_{32} - 180\hat{\omega}_{31}\hat{\omega}'_{32} + 45\hat{\omega}_{40}\hat{\omega}'_{32} \\
& + 1584\hat{\omega}_{21}\hat{\omega}'_{40} - 528\hat{\omega}_{30}\hat{\omega}'_{40} + 96\hat{\omega}'_{41} + 4416\hat{\omega}\hat{\omega}'_{41} - 828\hat{\omega}_{20}\hat{\omega}'_{41} - 135\hat{\omega}_{22}\hat{\omega}'_{41} \\
& + 90\hat{\omega}_{31}\hat{\omega}'_{41} + 135\hat{\omega}_{21}\hat{\omega}'_{42} - 45\hat{\omega}_{30}\hat{\omega}'_{42} \\
& - 4\hat{\omega}_{10}(41664\hat{\omega}'_{20} + 1472\hat{\omega}'_{31} + 1104\hat{\omega}'_{40} + 189\hat{\omega}'_{42}) + 48\hat{\omega}'_{43} + 528\hat{\omega}\hat{\omega}'_{43} \\
& - 90\hat{\omega}_{11}\hat{\omega}'_{43} + 45\hat{\omega}_{20}\hat{\omega}'_{43}) \\
& + q^2(384\hat{\omega}_{31}\hat{\omega}'_{10} + 192\hat{\omega}_{33}\hat{\omega}'_{10} + 128\hat{\omega}_{40}\hat{\omega}'_{10} - 144\hat{\omega}_{42}\hat{\omega}'_{10} + 3584\hat{\omega}_{10}\hat{\omega}'_{20} \\
& + 1344\hat{\omega}_{21}\hat{\omega}'_{20} - 512\hat{\omega}_{30}\hat{\omega}'_{20} + 96\hat{\omega}_{32}\hat{\omega}'_{20} - 48\hat{\omega}_{41}\hat{\omega}'_{20} + 896\hat{\omega}'_{21} - 3584\hat{\omega}\hat{\omega}'_{21} \\
& - 16\hat{\omega}_{31}\hat{\omega}'_{21} - 4\hat{\omega}_{33}\hat{\omega}'_{21} + 4\hat{\omega}_{40}\hat{\omega}'_{21} + 3\hat{\omega}_{42}\hat{\omega}'_{21} - 384\hat{\omega}_{11}\hat{\omega}'_{30} - 192\hat{\omega}_{31}\hat{\omega}'_{30} \\
& + 64\hat{\omega}_{40}\hat{\omega}'_{30} + 384\hat{\omega}_{10}\hat{\omega}'_{31} + 16\hat{\omega}_{21}\hat{\omega}'_{31} + 192\hat{\omega}_{30}\hat{\omega}'_{31} + 4\hat{\omega}_{32}\hat{\omega}'_{31} - 2\hat{\omega}_{41}\hat{\omega}'_{31} \\
& + 128\hat{\omega}'_{32} - 512\hat{\omega}\hat{\omega}'_{32} - 16\hat{\omega}_{11}\hat{\omega}'_{32} - 4\hat{\omega}_{31}\hat{\omega}'_{32} + \hat{\omega}_{40}\hat{\omega}'_{32} - 128\hat{\omega}_{10}\hat{\omega}'_{40} \\
& + 192\hat{\omega}_{21}\hat{\omega}'_{40} - 64\hat{\omega}_{30}\hat{\omega}'_{40} - 32\hat{\omega}'_{41} + 128\hat{\omega}\hat{\omega}'_{41} + 2\hat{\omega}_{31}\hat{\omega}'_{41} \\
& - 3\hat{\omega}_{22}(448\hat{\omega}'_{10} + 32\hat{\omega}'_{30} + \hat{\omega}'_{41}) - 44\hat{\omega}_{10}\hat{\omega}'_{42} \\
& + 3\hat{\omega}_{21}\hat{\omega}'_{42} - \hat{\omega}_{30}\hat{\omega}'_{42} - 16\hat{\omega}'_{43} + 64\hat{\omega}\hat{\omega}'_{43} - 2\hat{\omega}_{11}\hat{\omega}'_{43} \\
& + \hat{\omega}_{20}(-3584\hat{\omega}'_{10} - 1344\hat{\omega}'_{21} + 512\hat{\omega}'_{30} + 96\hat{\omega}'_{32} - 148\hat{\omega}'_{41} + \hat{\omega}'_{43}))\}} \\
& / (2359296\eta^2 q^{10}(1 + q^2)^2(-1 - q^4 + q^6 + q^{10}))
\end{aligned}$$

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