ON PETERSSON'S PARTITION LIMIT FORMULA

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Abstract. For each prime $p \equiv 1 \pmod 4$ consider the Legendre character $\chi = (\frac{\cdot}{p})$. Let $p_{\pm}(n)$ be the number of partitions of n into parts $\lambda > 0$ such that $\chi(\lambda) = \pm 1$. Petersson proved a beautiful limit formula for the ratio of $p_+(n)$ to $p_-(n)$ as $n \to \infty$ expressed in terms of important invariants of the real quadratic field $K = \mathbb{Q}(\sqrt{p})$. But his proof is not illuminating and Grosswald conjectured a more natural proof using a Tauberian converse of the Stolz-Cesàro theorem. In this paper we suggest an approach to address Grosswald's conjecture. We discuss a monotonicity conjecture which looks quite natural in the context of the monotonicity theorems of Bateman-Erdős.

1. Introduction

Let K be a real quadratic field, h_K its class number, and $\varepsilon_K > 1$ its fundamental unit. Let us assume that the discriminant of K is a prime number p, so in particular $p \equiv 1 \pmod 4$. Consider the Nebentypus cover $X_\chi(p)$ of degree two of the modular curve $X_0(p)$ introduced by Shimura [16, p. 174] in his work towards a theory of "real multiplication".[1] The Fricke involution w_p of $X_\chi(p)$ is defined over K and the curve $X_\chi(p)$ corresponds to the congruence subgroup

$$\Gamma_\chi(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p) : \chi(a) = 1 \right\},$$

where χ denotes the Legendre character $\chi = (\frac{\cdot}{p})$ of conductor p. To simplify the discussion here, we will assume that $p > 5$. Let f be the modular unit on the curve $X_\chi(p)$ introduced by Ogg an Ligozat, as described by Mazur [9, pp. 107–108]. In this paper we define a certain normalization u of f and use its Fourier expansion and that of the composite $\breve{u} = u \circ w_p$ to generalize a limit formula due Schur. (See Proposition 2.) We use this limit formula together with a monotonicity theorem of Bateman and Erdős [2], a consequence of the work of Meinardus [10] (described in the appendix), and a ratio Tauberian theorem due to Sato [14], to prove the following.

[1] Shimura determines division points of certain one-dimensional factors of the Jacobian $J_\chi(p)$ of $X_\chi(p)$ that generate abelian extensions of K. These one-dimensional factors are cut out by the action of the Hecke algebra and the involution w_p over K on $J_\chi(p)$.

Theorem 1. *With the above assumptions, for each $n \in \mathbb{Z}_{\geq 0}$ let $p_{\pm}(n)$ denote the number of partitions*

$$n = \lambda_1 + \cdots + \lambda_r$$

with parts $\lambda_i \in \mathbb{Z}_{>0}$ such that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_r$ and $\chi(\lambda_i) = \pm 1$, for $i = 1, 2, \ldots, r$. Then

$$\lim_{\nu \to \infty} \frac{\sum_{n=0}^{\nu} p_+(n)}{\sum_{n=0}^{\nu} p_-(n)} = \varepsilon_K^{h_K}.$$

Of course, the above theorem follows directly from the classical Stolz-Cesàro theorem [12, p. 14] applied to a celebrated partition limit formula due to Petersson [11],

$$\lim_{n \to \infty} \frac{p_+(n)}{p_-(n)} = \varepsilon_K^{h_K}. \tag{1}$$

But our proof does not use Eq. (1). In fact, Petersson's partition limit formula follows directly from Theorem 1 and a converse of the Stolz-Cesàro theorem[2] due to Păltănea [13], provided we assume a special case of Conjecture 1, discussed in Section 4. This is a monotonicity conjecture which looks quite natural in the context of the monotonicity theorems of Bateman and Erdős [2].

Petersson obtained Eq. (1) by first establishing the asymptotic expression for $p_+(n)$ and for $p_-(n)$ separately, after a rather laborious calculation. So given the simplicity of Eq. (1), it seems desirable to have a simpler proof. In fact, Grosswald [5] conjectured that a monotonicity theorem of Bateman and Erdős [2] together with a suitable Tauberian converse to the Stolz-Cesàro theorem, would furnish a nicer proof of Eq. (1). It is hoped that our approach can shed new light on Grosswald's conjecture. Moreover, the key role played here by the modular unit u on $X_\chi(p)$ and the Fricke involution w_p of $X_\chi(p)$ may help pave the way towards an explanation why h_K and ε_K appear in Eq. (1), a question which was raised by Petersson [11].

The rest of the paper is organized as follows. In Section 2 we use Klein forms to define the modular unit u on $X_\chi(p)$. Then we use the class number formula for real quadratic fields to obtain the constant term of the Fourier expansion of u. We express the Fourier expansion of $\breve{u}$ as an infinite product and conclude this section with a discussion of the $p = 5$ case, where we express the Rogers-Ramanujan continued fraction in terms of $\breve{u}$. In Section 3 we use the Fourier expansions of u and of $\breve{u}$ to obtain a generalization of a limit formula due to Schur, which we use to prove Theorem 1. In Section 4 we discuss Conjecture 1, including the numerical evidence that supports it, and also suggest an open question. In the appendix Luca shows how Eq. (1) follows from

[2] Păltănea stated his theorem as a converse of L'Hôpital rule for locally integrable functions. But applying his theorem to suitable step functions yields a converse of the Stolz-Cesàro theorem.

the work of Meinardus. The appendix also includes a discussion of the inequality (4), which is used in our proof of Theorem 1.

2. Two Fourier expansions

Following Kubert and Lang [7, p. 27], for each $z \in \mathbb{C}$ and each lattice $L \subset \mathbb{C}$ we may define the *Klein form*

$$\mathfrak{k}(z, L) = e^{-\frac{1}{2}\eta(z,L)z}\sigma(z, L),$$

where $\sigma(z, L)$ is the Weierstraß sigma-function and $z \mapsto \eta(z, L)$ is the $\mathbb{R}$-linear function that gives the quasi-periods of the Weierstraß zeta-function with respect to the lattice L. Put $\mathfrak{k}_a(\tau) = \mathfrak{k}(z, L_\tau)$, where the point $a = (a_1, a_2) \in \mathbb{R}^2$ is uniquely determined by $z = a_1\tau + a_2$ and $L_\tau = \mathbb{Z}\tau \oplus \mathbb{Z}$, with τ lying in the Poincaré upper-half plane

$$\mathcal{H} = \{z \in \mathbb{C} : \Im(z) > 0\}.$$

As before, consider a prime number $p > 5$ such that $p \equiv 1 \pmod 4$ and define

$$u(\tau) = \prod_{r=1}^{\frac{p-1}{2}} \mathfrak{k}_{(0,r/p)}(\tau)^{\chi(r)}.$$

As we shall see, up to a multiplicative constant this is Ogg and Ligozat's modular unit f on the Nebentypus cover $X_\chi(p)$ described by Mazur [9, pp. 107–108]. The cover $X_\chi(p)$ has 4 cusps, namely ∞ and $\overline{\infty}$ conjugate over K, above the cusp ∞ of $X_0(p)$ and cusps o and $\overline{o}$ defined over $\mathbb{Q}$, above the cusp o of $X_0(p)$. Mazur also showed that

$$(f) = \frac{1}{2}B_{2,\chi}((o) - (\overline{o})),$$

where $B_{n,\chi}$ is the generalized n-th Bernoulli number attached to χ defined by

$$\sum_{n=0}^{\infty} B_{n,\chi}\frac{X^n}{n!} = \sum_{r=1}^{p} \chi(r)\frac{Xe^{rX}}{e^{pX} - 1}.$$

The Fricke involution w_p of $X_\chi(p)$ interchanges the cusps o and ∞ (resp. $\overline{o}$ and $\overline{\infty}$). So the composite $\breve{u} = u \circ w_p$ has a zero of order $\frac{1}{2}B_{2,\chi}$ at the cusp ∞ of $X_\chi(p)$. The following proposition provides further details.

Proposition 1. *We have Fourier expansions*

$$\breve{u}(\tau) = q^{\frac{1}{2}B_{2,\chi}} \prod_{n=1}^{\infty}(1 - q^n)^{\chi(n)},$$

and

$$u(\tau) = \varepsilon_K^{-h_K}(1 - \sqrt{p}\, q_\tau + \dots),$$

where $q_\tau = e^{2\pi i\tau}$ and τ lies in the Poincaré upper-half plane $\mathcal{H}$. Actually,

$$u = \varepsilon_K^{-h_K} f,$$

where f is the modular unit of Ogg and Ligozat.

Proof. Let $\eta(\tau)$ denote Dedekind's eta-function

$$\eta(\tau) = e^{\pi i\tau/12}\prod_{n=1}^{\infty}(1 - e^{2\pi in\tau}).$$

The *Siegel function* $g_a(\tau) = \mathfrak{k}_a(\tau)\eta(\tau)^2$ has a product expansion

$$g_a(\tau) = -q_\tau^{\frac{1}{2}B_2(a_1)}e^{2\pi ia_2(a_1-1)/2}(1 - q_z)\prod_{n=1}^{\infty}(1 - q_\tau^n q_z)(1 - q_\tau^n q_z^{-1}),$$

where $B_2(X) = X^2 - X + \frac{1}{6}$ is the second Bernoulli polynomial, and $q_z = e^{2\pi iz}$ with $z \in \mathbb{C}$. So if we let $\zeta_p = e^{2\pi i/p}$, then

$$\begin{aligned}
u(\tau) &= \prod_{r=1}^{\frac{p-1}{2}} g_{(0,r/p)}(\tau)^{\chi(r)} \\
&= \prod_{r=1}^{\frac{p-1}{2}}\left(\zeta_p^{-\frac{r}{2}}(1 - \zeta_p^r)\prod_{n=1}^{\infty}(1 - q_\tau^n\zeta_p^r)(1 - q_\tau^n\zeta_p^{-r})\right)^{\chi(r)} \\
&= \left(\prod_{r=1}^{\frac{p-1}{2}}\zeta_p^{-\chi(r)\frac{r}{2}}(1 - \zeta_p^r)^{\chi(r)}\right)\left(\prod_{n=1}^{\infty}\prod_{r=1}^{\frac{p-1}{2}}(1 - q_\tau^n\zeta_p^r)^{\chi(r)}(1 - q_\tau^n\zeta_p^{-r})^{\chi(r)}\right) \\
&= \kappa f(\tau),
\end{aligned}$$

where

$$\kappa = \prod_{r=1}^{\frac{p-1}{2}}(\zeta_p^{-\frac{r}{2}} - \zeta_p^{\frac{r}{2}})^{\chi(r)} = \varepsilon_K^{-h_K}.$$

The last equality follows from the first equation in Théorème 1 of Borevič and Šafarevič [3, p. 385], namely

$$h_K = -\frac{1}{\log\varepsilon_K}\sum_{\substack{(r,D)=1\\ 0<r<\frac{D}{2}}}\chi(r)\log\sin\frac{\pi r}{D},$$

specialized to the positive fundamental discriminant $D = p$, which is a well-known consequence of the formula

$$L(1,\chi) = \frac{2h_K}{\sqrt{p}}\log\varepsilon_K.$$

Here $L(s,\chi)$ is the Dirichlet L-series attached to χ

$$L(s,\chi) = \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s} \quad \Re(s) > 0.$$

Therefore $u = \varepsilon_K^{-h_K} f$, which is the third assertion in our proposition. To prove the second assertion note that

$$f(\tau) = \prod_{n=1}^{\infty} \Psi(q^n),$$

where

$$\begin{aligned}\Psi(X) &= \prod_{r=1}^{\frac{p-1}{2}} (1 - X\zeta_p^r)^{\chi(r)} (1 - X\zeta_p^{-r})^{\chi(r)} \\ &= \prod_{r=1}^{\frac{p-1}{2}} (1 - X\zeta_p^r)^{\chi(r)} (1 - X\zeta_p^{-r})^{\chi(-r)} \\ &= \prod_{r=1}^{p-1} (1 - X\zeta_p^r)^{\chi(r)} \\ &\equiv 1 - S_p X \pmod{X^2}.\end{aligned}$$

where $S_p = \sum_{r=1}^{p} \chi(r)\zeta_p^r$ is the Gauß sum attached to χ. But we assumed that $p \equiv 1 \pmod 4$, so $S_p = \sqrt{p}$. Hence

$$f(\tau) = 1 - \sqrt{p}\, q_\tau + \dots .$$

and the third assertion of our proposition implies that

$$u(\tau) = \varepsilon_K^{-h_K} f(\tau) = \varepsilon_K^{-h_K} (1 - \sqrt{p}\, q_\tau + \dots).$$

To prove the first assertion of our proposition recall that the Fricke involution w_p of $X_\chi(p)$ is induced by the Möbius transformation

$$\tau \mapsto -\frac{1}{p\tau}$$

acting on the extended upper-half plane $\mathcal{H}^* = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$, which is the composition of the Möbius transformation attached to

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

with the map $\tau \mapsto p\tau$. But the basic properties **K0** and **K1** of Kubert and Lang [7, p. 27] imply that for each $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ and each $\tau \in \mathcal{H}$ we have

$$\mathfrak{k}_{a\alpha}(\tau) = (c\tau + d)\mathfrak{k}_a(\alpha\tau).$$

Therefore

$$\begin{aligned}
\breve{u}(\tau) &= \prod_{r=1}^{\frac{p-1}{2}} \mathfrak{k}_{(-r/p,0)}(p\tau)^{\chi(r)} \\
&= \prod_{r=1}^{\frac{p-1}{2}} \left(q_\tau^{\frac{1}{2}B_2(\frac{r}{p})p} (1-q_\tau^r) \prod_{n=1}^{\infty} (1-q_\tau^{pn+r})(1-q_\tau^{pn-r}) \right)^{\chi(r)} \\
&= q_\tau^{\frac{1}{2}B_{2,\chi}} \prod_{r=1}^{\frac{p-1}{2}} (1-q_\tau^r)^{\chi(r)} \prod_{r=1}^{\frac{p-1}{2}} \prod_{n=1}^{\infty} (1-q_\tau^{pn+r})^{\chi(r)} (1-q_\tau^{pn-r})^{\chi(-r)} \\
&= q_\tau^{\frac{1}{2}B_{2,\chi}} \prod_{n=1}^{\infty} (1-q_\tau^n)^{\chi(n)},
\end{aligned}$$

which is the first assertion of our proposition. □

For $p = 5$ we have $\frac{1}{2}B_{2,\chi} = \frac{1}{5}$ and we may see from the above proposition that in this case u is not an element of the function field of the curve $X_\chi(p)$, but u^5 is in fact a Hauptmodul for $X_\chi(p)$. Moreover, in this notation the Rogers-Ramanujan continued fraction becomes

$$\breve{u}(\tau) = \cfrac{q^{1/5}}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \ddots}}}}. \tag{2}$$

3. A limit formula

For each $n \in \mathbb{Z}_{\geq 0}$ let $p_A(n)$ denote the number of partitions

$$n = \lambda_1 + \cdots + \lambda_r$$

with parts $\lambda_i \in A$ such that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_r$, for $i = 1, 2, \ldots, r$. In particular, consider the set of quadratic residues S_+ and the set of quadratic non-residues S_- modulo p,

$$S_\pm = \{m \in \mathbb{Z}_{>0} : \chi(m) = \pm 1\},$$

so that $p_\pm(n) = p_{S_\pm}(n)$. Here as before, $\chi = (\frac{\cdot}{p})$ is the Legendre character attached to p.

Proposition 2. *As before consider a prime $p \equiv 1 \pmod 4$. We have the limit*

$$\lim_{t \to 0^+} \frac{\sum_{n=0}^{\infty} p_+(n) e^{-2\pi n t}}{\sum_{n=0}^{\infty} p_-(n) e^{-2\pi n t}} = \varepsilon_K^{h_K}.$$

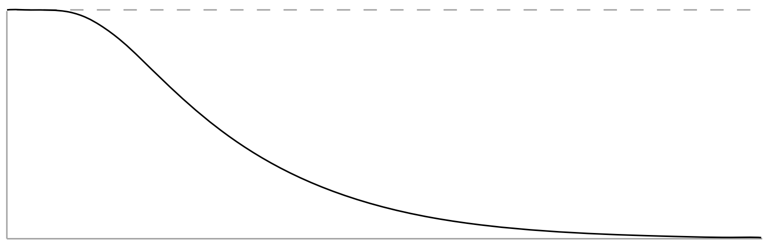

FIGURE 1. Graph of $t \mapsto \breve{u}(it)$ near $t = 0$ for $p = 13$.

Proof. From the first part of Proposition 1 we have

$$\frac{1}{\breve{u}} = q^{-\frac{1}{2}B_{2,\chi}} \frac{\prod_{m \in S_+} \frac{1}{1-q^m}}{\prod_{m \in S_-} \frac{1}{1-q^m}} = q^{-\frac{1}{2}B_{2,\chi}} \frac{\sum_{n=0}^{\infty} p_+(n) q^n}{\sum_{n=0}^{\infty} p_-(n) q^n}.$$

So the second part of Proposition 1 yields

$$\lim_{t \to 0^+} \breve{u}(it) = \lim_{t \to 0^+} u(w_p(it)) = \lim_{t \to \infty} u(it) = \varepsilon_K^{-h_K}$$

and the proposition follows. □

The above proposition is a generalization of a limit formula due to Schur[15, p. 321] for $p = 5$. It may be regarded as a consequence of the Rogers-Ramanujan continued fraction, since the right-hand side of Eq. (2) tends to

$$\cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}} = \frac{-1 + \sqrt{5}}{2}$$

as $t \to 0$, and we know that $\varepsilon_K^{-1} = \frac{-1+\sqrt{5}}{2}$ and that $h_K = 1$ for the real quadratic field $K = \mathbb{Q}(\sqrt{5})$.

Remark 1. *From the first two terms of the Fourier expansion of* $u(\tau)$ *we may see that the real-analytic function* $h : \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{>0}$ *defined by* $t \mapsto \breve{u}(it)$ *is a monotone concave function in a neighbourhood of* $t = 0$*, as depicted in Figure 1 for* $p = 13$*. Moreover, the function* h *is log-concave on* $\mathbb{R}_{>0}$*. This is an easy consequence of the fact that the logarithmic derivative of* $u(\tau)$ *is, up to a positive scalar multiple, the well-known Eisenstein series* $G_{2,\chi}(\tau)$ *of weight* 2 *attached to the character* χ*. (Cf. Lang* [8, p. 250]*.)*

Fix $k \in \mathbb{Z}$ and define $p^{(k)}(n) = p_A^{(k)}(n)$ by the formal power series equality

$$\sum_{n=0}^{\infty} p^{(k)}(n) X^n = (1 - X)^k \prod_{a \in A} \frac{1}{1 - X^a}$$

Following Bateman and Erdős [2], we say that a subset $A \subset \mathbb{Z}_{>0}$ satisfies property P_k if $|A| > k$ and if $\gcd(A \setminus S) = 1$, for each $S \subset A$ such that $|S| = k$. Note that $p^{(k)}(n)$ is the k-th difference of $p(n)$ if $k > 0$, the $-k$-th order summatory function of $p(n)$, and $p^{(0)}(n) = p(n)$.

Lemma 1. *If A is an infinite subset of $\mathbb{Z}_{>0}$ such that $gcd(A) = 1$, then for each positive integer h we have the limit*

$$\lim_{n\to\infty} \frac{p(0) + \cdots + p(n+h)}{p(0) + \cdots + p(n)} = 1.$$

Proof. From Bateman and Erdős [2, p. 10], the corollary after Theorem 6 says that for each positive integer h we have

$$\frac{p^{(k-1)}(n+h) - p^{(k-1)}(n)}{h} = (1 + o(1))\, p^{(k)}(n).$$

For $k = 0$ the assumption $\gcd(A) = 1$ yields

$$\frac{p(n+1) + \cdots + p(n+h)}{p(n)} = (1 + o(1))\, h, \tag{3}$$

for each positive integer h. But from Theorem 5 of Bateman and Erdős [2, p. 7] we know that

$$\lim_{n\to\infty} \frac{p^{(k+1)}(n)}{p^{(k)}(n)} = 0,$$

provided A is infinite and such that property P_k holds. In particular, for $k = -1$ we see that property P_k is trivially satisfied and we thus have the limit

$$\lim_{n\to\infty} \frac{p(n)}{p(0) + \cdots + p(n)} = 0$$

This limit together with Eq. (3) yield

$$\lim_{n\to\infty} \frac{p(n+1) + \cdots + p(n+h)}{p(0) + \cdots + p(n)} = 0,$$

which gives

$$\lim_{n\to\infty} \frac{p(0) + \cdots + p(n+h)}{p(0) + \cdots + p(n)} = 1 + \lim_{n\to\infty} \frac{p(n+1) + \cdots + p(n+h)}{p(0) + \cdots + p(n)} = 1$$

and the proposition follows. □

Now we shall prove Theorem 1. From the appendix, for all large enough n we have

$$p_-(n) < p_+(n). \tag{4}$$

But Lemma 1 yields

$$\frac{\sum_{m=0}^{\mu} p_+(m)}{\sum_{n=0}^{\nu} p_+(n)} \to 1 \quad \text{as} \quad \mu, \nu \to \infty \quad \text{with} \quad \frac{\mu}{\nu} \to 1.$$

Hence the limit formula of Proposition 2 satisfies all the hypothesis of Theorem 3.2 of Sato [14, p. 85] and Theorem 1 follows.

Remark 2. *If we replace Inequality 4 by the weaker condition*

$$p_-(n) = O(p_+(n)),$$

then Sato's ratio Tauberian theorem still applies here.

4. A conjecture and some open questions

Given $k \in \mathbb{Z}$, let $p^{(k)}(n)$ be as in Section 3 and for each $n \in \mathbb{Z}_{\geq 0}$ define

$$\rho^{(k)}(n) = \frac{p^{(k+1)}(n)}{p^{(k)}(n)} = \frac{p^{(k)}(n) - p^{(k)}(n-1)}{p^{(k)}(n)}.$$

Note that the corollary of Theorem 5 of Bateman and Erdős [2, p. 9] says that if A has property P_k then

$$p^{(k)}(n) \to \infty$$

and

$$\rho^{(k)}(n) \to 0,$$

as $n \to \infty$. They also show that if A satisfies property P_{k+1}, then $p^{(k)}(n)$ is eventually strictly increasing. But the question of the monotonicity of $\rho^{(k)}(n)$ has not been raised before. This is an interesting question, as the eventual monotonicity of $\rho^{(k)}(n)$ is a natural generalization of the log-concavity of $p(n)$ for all large enough n. Indeed, we have

$$\rho^{(k)}(n) > \rho^{(k)}(n+1)$$

if and only if

$$0 < p^{(k)}(n)^2 - p^{(k)}(n-1)p^{(k)}(n+1).$$

This monotonicity question has been settled for the case $A = \mathbb{Z}_{>0}$ and $k = 0$ by DeSalvo and Pak [4], as they proved that the classical partition function $p(n)$ is log-concave for all $n > 25$. Moreover, we may also see that the monotonicity of $\rho^{(-1)}(n)$ for all large enough n is equivalent to having the sequence

$$\left\{ \frac{1}{p(\nu)} \sum_{n=0}^{\nu} p(n) \right\}_{\nu=0}^{\infty} \tag{5}$$

eventually strictly increasing. Eq. (5) is the key condition of the converse of Stolz-Cesàro theorem due to Păltănea [13] which (together with Theorem 1) yields Petersson's partition limit formula. Considering the Tauberian condition T_2 of the conjecture due to Grosswald [5, pp. 55-56], we propose the following.

Conjecture 1. *If $A \subset \mathbb{Z}_{>0}$ is such that $P_{k+1}, P_{k+2}, \ldots$, then $\rho^{(k)}(n)$ is eventually strictly decreasing.*

With the help of PARI/GP [17] we obtained strong numerical evidence supporting Conjecture 1 for $p(n) = p_{\pm}(n)$ in the range $p \leq 1987$, with $|k| \leq 5$ and $n \leq 10000$ and also for the classical partition function $p(n)$, with $|k| \leq 10$ and $n \leq 100000$.

As described by Iwasawa [6, p. 61], there is a remarkable non-archimedean analogue of

$$L(1,\chi) = -\frac{S_p}{p} \sum_{r=1}^{p-1} \chi(r) \log |1 - \zeta_p^r|$$

known as Leopoldt's formula. Moreover, Siegel functions have natural rigid-analytic avatars. It seems to be an interesting open question whether there are analogues of Proposition 2 (which is a generalization of a limit formula due to Schur) and of Petersson's limit partition formula within this realm.

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Appendix A. By Florian Luca

Here, we show how equation (1) follows from Meinardus' scheme [10] (see also [1]).

Let us recall Meinardus' scheme. Let $\mathcal{A} \subseteq \mathbb{N}$ be a set of positive integers. Put

$$\begin{aligned} p_{\mathcal{A}}(n) &= \#\{(\lambda_1, \ldots, \lambda_k) : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1,\ \lambda_1 + \cdots + \lambda_k = n, \\ &\quad \lambda_i \in \mathcal{A}, i = 1, \ldots, k\} \end{aligned}$$

for the number of partitions of n with parts from $\mathcal{A}$. Writing $\{a_n\}_{n\geq 1}$ for the characteristic function of $\mathcal{A}$; that is, $a_n = 1$ if $n \in \mathcal{A}$ and $a_n = 0$ otherwise, the generating function of $p_{\mathcal{A}}$ is

$$\prod_{n\geq 1} (1 - e^{-n\tau})^{-a_n} = 1 + \sum_{n\geq 1} p_{\mathcal{A}}(n) e^{-n\tau}, \quad \text{with} \quad \mathrm{Re}(\tau) > 0.$$

Meinardus, in his 1954 paper [10], makes the following assumptions:

(i) Let

$$D(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \quad \text{where} \quad s = \sigma + it.$$

Assume that $D(s)$ is convergent for $\sigma > \alpha > 0$. Assume further that $D(s)$ can be analytically continued up to $\sigma = -c_0$, where $0 < c_0 < 1$. Assume that for $\sigma \geq -c_0$, $D(s)$ is holomorphic except for $s = \alpha$ where it has a pole of order 1 with residue A. Assume further that in this region, we have

$$D(s) = O(|t|^{c_1})$$

as $t \to \infty$ for some $c_1 > 0$.

(ii) For $\tau = y + 2\pi i x$ with $y > 0$ put

$$g(\tau) = \sum_{n \geq 0} a_n e^{-n\tau}.$$

Assume that for $|\arg(\tau)| > \pi/4$, $|x| \leq 1/2$, one has

$$\mathrm{Re}(g(\tau) - g(y)) \leq -c_2 y^{-\varepsilon}$$

for y sufficiently small, where $c_2 > 0$ and $\epsilon > 0$ are some positive real numbers.

Under (i) and (ii), Meinardus proves that

(6)

$$p_{\mathcal{A}}(n) = C n^{\chi} e^{n^{\frac{\alpha}{\alpha+1}}\left(1+\frac{1}{\alpha}\right)(A\Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1}{\alpha+1}}} (1 + O(n^{-\chi_1})) \quad \text{as} \quad n \to \infty,$$

where

$$\begin{aligned} C &= e^{D'(0)}(2\pi(1+\alpha))^{-1/2}(A\Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1-2D(0)}{2(1+\alpha)}}; \\ \chi &= \frac{2D(0) - 2 - \alpha}{2(1+\alpha)}. \end{aligned}$$

He also gives some estimates for χ_1 which we don't need. Well, let us apply it to our case. For us, $a_n = \chi(n)$ in the case of p_+ and $a_n = -\chi(n)$ in the case of p_-, where $\chi(n) = \left(\frac{n}{p}\right)$ is the Legendre character modulo p. Putting again

$$L(s, \chi) = \sum_{n \geq 1} \frac{\chi(n)}{n^s},$$

one sees easily that

$$D_+(s) = \sum_{\substack{n \geq 1 \\ \chi(n) = 1}} n^{-s} = \frac{1}{2} \sum_{\substack{n \geq 1 \\ p \nmid n}} \frac{1 + \chi(n)}{n^s} = \frac{1}{2}\left(\zeta(s)(1 - p^{-s}) + L(s, \chi)\right),$$

and similarly

$$D_-(s) = \frac{1}{2}\left(\zeta(s)(1 - p^{-s}) - L(s, \chi)\right).$$

So, we see that hypothesis (i) of Meinardus' scheme is fulfilled for both $D_+(s)$ and $D_-(s)$ with $\alpha = 1$, $A = (1 - p^{-1})$ since for $\sigma > -1/2$, $\zeta(s)$ is holomorphic except for a single pole at $s = 1$ with residue 1 and $L(s, \chi)$ is holomorphic. Condition (ii) is also fulfilled by standard results about vertical growth of $\zeta(s)$ and $L(s,\chi)$. Furthermore, since $\zeta(0) = -1/12$, and $L(0,\chi) = 0$ (because $p \equiv 1 \pmod 4$), we get that

$$D_+(0) = \frac{1}{2}\left((-1/12)(1-p^{-0}) + L(0,\chi)\right) = 0,$$

and similarly $D_-(0) = 0$. So, the "main" terms of $p_+(n)$ and $p_-(n)$ in (6) coincide up to the constants C_+ and C_-, that is

(7)
$$\frac{p_+(n)}{p_-(n)} = (1+o(1))\frac{C_+}{C_-} = (1+o(1))e^{D_+(0)'-D_-(0)'} \qquad \text{as} \qquad n \to \infty,$$

where $C_+ = D'_+(0)$ and $C_- = D_-(0)'$. Now

$$D_+(s)' = \frac{1}{2}\left(\zeta'(s)(1-p^{-s}) + \zeta(s)(\log p)p^{-s} + L'(s,\chi)\right).$$

Evaluating in $s = 0$, we get

$$D_+(0)' = \frac{1}{2}\zeta(0)\log p + \frac{1}{2}L'(0,\chi) = -\frac{\log p}{24} + \frac{1}{2}L(0,\chi)'.$$

A similar argument shows that

$$D_-(0)' = -\frac{\log p}{24} - \frac{1}{2}L(0,\chi)',$$

so

$$D_+(0)' - D_-(0)' = L(0,\chi)'.$$

Since $L(0,\chi)' = (\sqrt{p}/2)L(1,\chi)$, it follows that

$$D_+(0)' - D_-(0)' = (\sqrt{p}/2)L(1,\chi)$$

which is positive (by the proof of Dirichlet's theorem on primes in progressions). In fact, by the class number formula the above difference is $h_k \log \varepsilon_K$ and we recover Petersson's limit from (7).

In particular, the inequality

$$p_+(n) > p_-(n) \quad \text{holds for all} \quad n > n_0(p).$$

One may wonder if the fact that the inequality $p_+(n) > p_-(n)$ holds might be due to the fact that 1 is a quadratic residue and being the smallest positive integer it likely contributes to a lot of elements counted by $p_+(n)$. Well, let us test it. Let $p_{1,+}(n)$ be the number of partitions of n with parts that are > 1 but quadratic residues modulo p. Then

$$D_{1,+}(s) = D_+(s) - 1,$$

so $D_{+,1}(0) = D_+(0) - 1 = -1$. It thus follows that

$$\chi_{1,+} = \frac{2D_{1,+}(0) - 3}{4} = -\frac{5}{4} \qquad \text{while} \qquad \chi_- = \frac{2D_-(0) - 3}{4} = -\frac{3}{4},$$

therefore

$$\frac{p_{1,+}(n)}{p_-(n)} = (1+o(1))c_3 n^{-1/2} \qquad \text{as} \qquad n \to \infty,$$

where $c_3 := C_{1,+}/C_-$. The above asymptotic shows that the inequality $p_{1,+}(n) < p_-(n)$ holds for large n. So, indeed, if we eliminate the 1's from the partitions of $p_+(n)$ we get a number of partitions much smaller (in fact, of a smaller order of magnitude asymptotically) than $p_-(n)$, whereas $p_+(n)$ and $p_-(n)$ are of the same order or magnitude, which can be indeed interpreted by saying that the fact that the inequality $p_+(n) > p_-(n)$ holds for large n is driven by the contribution of the 1's in the $p_+(n)$ side.

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