KIRILLOV–RESHETIKHIN CRYSTALS $B^{7,s}$ FOR TYPE $E_7^{(1)}$

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Abstract. We construct a combinatorial crystal structure on the Kirillov–Reshetikhin crystal $B^{7,s}$ in type $E_7^{(1)}$, where 7 is the unique node in the orbit of 0 in the affine Dynkin diagram.

1. Introduction

An important class of finite-dimensional representations for affine Lie algebras are the *Kirillov–Reshetikhin (KR) modules*, which are characterized by their Drinfel'd polynomials [CP95, CP98]. We denote a KR module by $W^{r,s}$, where r is a node of the classical Dynkin diagram and s is a positive integer. KR modules have been well-studied and have many interesting properties. For example, their characters (resp. q-characters) are solutions the Q-system (resp. T-system) [Her10] (see also [KNS11] and references therein). Moreover, graded Demazure characters of tensor products of KR modules are nonsymmetric Macdonald polynomials at $t = 0$ for untwisted affine types [LNS+15, LNS+16, LNS+17].

One significant aspect of a KR module $W^{r,s}$ is that it (conjecturally) admits a crystal base [HKO+99, HKO+02] despite not being a highest weight module. The corresponding crystal of $W^{r,s}$ is called a *Kirillov–Reshetikhin (KR) crystal* and denoted by $B^{r,s}$. KR crystals have been shown to exist in all nonexceptional types in [OS08], types $G_2^{(1)}$ and $D_4^{(3)}$ in [Nao18], and for r being in the orbit of or adjacent to 0 in all affine types from the general theory [KKM+92a, KKM+92b]. An open problem is to determine a uniform model for KR crystals. This has been achieved for $B^{r,1}$ by using Kashiwara's construction of projecting an extremal level-zero module/crystal [Kas02]. This was done explicitly by Naito and Sagaki using Lakshmibai–Seshadri (LS) paths [NS03, NS06, NS08]. The construction of Kashiwara was also recently shown to partially extend to general $B^{r,s}$ in nonexceptional affine types (conjecturally in all affine types) [LS18]. In contrast, the models in [FOS09, JS10, KMOY07, Yam98] are all type-dependent, but are given for $B^{r,s}$ for all s.

KR crystals are connected with mathematical physics. For instance, tensor products of KR modules are used to describe certain vertex models and are related with Heisenberg spin chains by the $X = M$ conjecture of [HKO+99, HKO+02]. The $X = M$ conjecture implies a fermionic formula for the graded characters of a tensor product of KR crystals; see [OSS18, Scr17] for recent progress. Furthermore, tensor products of KR crystals describe the dynamics of soliton cellular automata, a generalization of the Takehashi–Satsuma box-ball system (which is an ultradiscrete version of the Korteweg–de Vries (KdV) equation). We refer the reader to [IKT12, LS17] for more details. Another important (conjectural) property of KR crystals is that they are perfect [FOS10, KKM+92a, KKM+92b, KMOY07, Yam98], a technical condition that allows highest weight crystals to be modeled using a semi-infinite tensor product known as the Kyoto path model [KKM+92b].

In this note, we give a combinatorial model for the KR crystal $B^{7,s}$ in type $E_7^{(1)}$, where 7 is the unique node in the orbit of 0 in the Dynkin diagram (see Figure 1 below). We achieve this by considering the (Levi) decomposition of the classical (type E_7) highest weight crystal $B(s\overline{\varpi}_7)$ into A_6 highest weight crystals, which is multiplicity free. From this, we reconstruct the A_7 decomposition of $B^{7,s}$ since $B^{7,s} \cong B(s\overline{\varpi}_7)$ as E_7 crystals and the decomposition of A_7 highest weight crystals into A_6 crystals is multiplicity free. We note that the KR crystal $B^{7,s}$ exists since 7 is a minuscule node, so $B^{7,s}$ is irreducible as a classical crystal.

As a potential application of our results, we could construct the combinatorial R-matrix $R\colon B^{7,s} \otimes B^{7,1} \to B^{7,1} \otimes B^{7,s}$. Then this can be used to study the soliton cellular automata of $B^{7,1}$ using different techniques from [LS18]. Moreover, our results could potentially be used to show that $B^{7,s}$ is a perfect crystal of level s.

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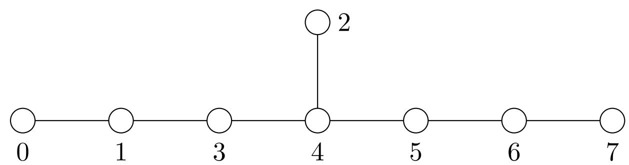


FIGURE 1. The Dynkin diagram of type $E_7^{(1)}$.

This paper is organized as follows. In Section 2, we give the necessary background. In Section 3, we give our main results. In Section 4, we give a conjecture about the decomposition of $B^{1,s}$ into A_7 crystals in an effort to prove [JS10, Conj. 3.26].

2. Background

Let $\mathfrak{g}$ be an affine Kac–Moody Lie algebra with index set I, Cartan matrix $(A_{ij})_{i,j\in I}$, simple roots $(\alpha_i)_{i\in I}$, fundamental weights $(\omega_i)_{i\in I}$, and simple coroots $(\alpha_i^\vee)_{i\in I}$. Let $U_q(\mathfrak{g})$ denote the corresponding (Drinfel'd–Jimbo) quantum group, and we will be using $U_q'(\mathfrak{g}) := U_q([\mathfrak{g},\mathfrak{g}])$, which has weight lattice $P = \sum_{i\in I} \mathbb{Z}\omega_i$. Let P^+ denote the positive weight lattice. Let Q be the root lattice with Q^+ being the positive root lattice. We denote the canonical pairing $\langle\ ,\ \rangle \colon P^\vee \times P \to \mathbb{Z}$, which is given by $\langle \alpha_i^\vee, \alpha_j\rangle = A_{ij}$.

Recall that $\operatorname{lev}(\lambda) := \langle c, \lambda\rangle$ is the level of the weight λ, where c is the canonical central element of $\mathfrak{g}$. In particular, for $\mathfrak{g}$ of type $E_7^{(1)}$, we have

$$\begin{aligned} &\operatorname{lev}(\omega_0) = 1, && \operatorname{lev}(\omega_1) = 2, && \operatorname{lev}(\omega_2) = 2, && \operatorname{lev}(\omega_3) = 3, \\ &\operatorname{lev}(\omega_4) = 4, && \operatorname{lev}(\omega_5) = 3, && \operatorname{lev}(\omega_6) = 2, && \operatorname{lev}(\omega_7) = 1. \end{aligned} \tag{2.1}$$

We denote the dominant weights of level ℓ by P_ℓ^+.

Let $\mathfrak{g}_0$ denote the canonical simple Lie algebra given by the index set $I_0 = I \setminus \{0\}$, and $U_q(\mathfrak{g}_0)$ the corresponding quantum group. Let P_0 and Q_0 be the weight and root lattice of $\mathfrak{g}_0$, and let $\overline{\omega}_i$ be the natural projection of the fundamental weight ω_i onto P_0. Let W_0 be the Weyl group of $\mathfrak{g}_0$.

2.1. **Crystals.** An *abstract $U_q(\mathfrak{g})$-crystal* is a set B endowed with *crystal operators* $e_i, f_i \colon B \to B \sqcup \{0\}$, for $i \in I$, and *weight function* $\operatorname{wt}\colon B \to P$ that satisfy the following conditions:

(1) $\varphi_i(b) = \varepsilon_i(b) + \langle \alpha_i^\vee, \operatorname{wt}(b)\rangle$, for all $b \in B$ and $i \in I$,
(2) $f_i b = b'$ if and only if $b = e_i b'$, for $b, b' \in B$ and $i \in I$,
(3) $\operatorname{wt}(f_i b) = \operatorname{wt}(b) - \alpha_i$ if $f_i b \neq 0$;

where the statistics $\varepsilon_i, \varphi_i \colon B \to \mathbb{Z}_{\geq 0}$ are defined by

$$\varepsilon_i(b) := \max\{k \mid e_i^k b \neq 0\}, \qquad \varphi_i(b) := \max\{k \mid f_i^k b \neq 0\}.$$

Remark 2.1. The definition of an abstract crystal given in this paper is sometimes called a *regular* or *seminormal* abstract crystal in the literature. See, *e.g.*, [BS17] for the more general definition.

Using the axioms, we identify B with an I-edge colored weighted directed graph whose vertices are B and having an i-colored edge $b \xrightarrow{i} b'$ if and only if $f_i b = b'$. Therefore, we can depict an entire i-string through an element $b \in B$ diagrammatically by

$$e_i^{\varepsilon_i(b)} b \xrightarrow{i} \cdots \xrightarrow{i} e_i^2 b \xrightarrow{i} e_i b \xrightarrow{i} b \xrightarrow{i} f_i b \xrightarrow{i} f_i^2 b \xrightarrow{i} \cdots \xrightarrow{i} f_i^{\varphi_i(b)} b.$$

Let $J \subseteq I$. An element $b \in B$ is *J-highest (resp. lowest) weight* if $e_i b = 0$ (resp. $f_i b = 0$) for all $i \in J$. When $J = I$, we simply say b is *highest (resp. lowest) weight*.

For abstract $U_q(\mathfrak{g})$-crystals $B_1, B_2, \ldots, B_L$, the action of the crystal operators on the *tensor product* $B_L \otimes \cdots \otimes B_2 \otimes B_1$, which equals the Cartesian product $B_L \times \cdots \times B_2 \times B_1$ as sets, can be defined by the *signature rule*. Let $b := b_L \otimes \cdots \otimes b_2 \otimes b_1 \in B$, and for $i \in I$, we write

$$\underbrace{-\cdots-}_{\varphi_i(b_L)}\ \underbrace{+\cdots+}_{\varepsilon_i(b_L)}\ \cdots\ \underbrace{-\cdots-}_{\varphi_i(b_1)}\ \underbrace{+\cdots+}_{\varepsilon_i(b_1)}.$$

Then by successively deleting consecutive $+-$-pairs (in that order), we obtain a sequence

$$\operatorname{sgn}_i(b) := \underbrace{-\cdots-}_{\varphi_i(b)}\ \underbrace{+\cdots+}_{\varepsilon_i(b)},$$

called the *reduced signature*. If there does not exist a $+$ (resp. $-$) in $\operatorname{sgn}_i(b)$, then $e_i b = 0$ (resp. $f_i b = 0$). Otherwise, suppose $1 \leq j_- \leq j_+ \leq L$ are such that b_{j_-} contributes the rightmost $-$ in $\operatorname{sgn}_i(b)$ and b_{j_+} contributes the leftmost $+$ in $\operatorname{sgn}_i(b)$. Then, we have

$$e_i b := b_L \otimes \cdots \otimes b_{j_+ +1} \otimes e_i b_{j_+} \otimes b_{j_+ -1} \otimes \cdots \otimes b_1\,,$$
$$f_i b := b_L \otimes \cdots \otimes b_{j_- +1} \otimes f_i b_{j_-} \otimes b_{j_- -1} \otimes \cdots \otimes b_1\,.$$

Remark 2.2. Our tensor product convention follows [BS17], which is opposite to that of Kashiwara [Kas91].

Let B_1 and B_2 be two abstract $U_q(\mathfrak{g})$-crystals. A *crystal morphism* $\psi\colon B_1 \to B_2$ is a map $B_1 \sqcup \{0\} \to B_2 \sqcup \{0\}$ with $\psi(0) = 0$, such that the following properties hold for all $b \in B_1$ and $i \in I$:

(1) if $\psi(b) \in B_2$, then $\operatorname{wt}\big(\psi(b)\big) = \operatorname{wt}(b)$, $\varepsilon_i\big(\psi(b)\big) = \varepsilon_i(b)$, and $\varphi_i\big(\psi(b)\big) = \varphi_i(b)$;
(2) we have $\psi(e_i b) = e_i \psi(b)$ if $\psi(e_i b) \neq 0$ and $e_i\psi(b) \neq 0$;
(3) we have $\psi(f_i b) = f_i \psi(b)$ if $\psi(f_i b) \neq 0$ and $f_i\psi(b) \neq 0$.

An *embedding* (resp. *isomorphism*) is a crystal morphism such that the induced map $B_1 \sqcup \{0\} \to B_2 \sqcup \{0\}$ is an embedding (resp. bijection).

An abstract crystal B is a *$U_q(\mathfrak{g})$-crystal* if B is the crystal basis of some $U_q(\mathfrak{g})$-module. Kashiwara [Kas91] has shown that the irreducible highest weight module $V(\lambda)$, for $\lambda \in P^+$, admits a crystal basis denoted $B(\lambda)$. The highest weight crystal $B(\lambda)$ is generated by a unique highest weight element u_λ that satisfies $\operatorname{wt}(u_\lambda) = \lambda$.

2.2. **Minuscule crystals.** We say a highest weight $U_q(\mathfrak{g}_0)$-crystal $B(\lambda)$ is *minuscule* if W_0 acts transitively on $B(\lambda)$. In other words, there exists a bijection between $B(\lambda)$ and W_0^λ, the set of minimal length coset representatives of $W_0/\operatorname{stab}_{W_0}(\lambda)$ (recall $\operatorname{stab}_{W_0}(\lambda) := \{w \in W_0 \mid w\lambda = w\}$ is the stabilizer of λ and a parabolic subgroup of W_0). Indeed, consider a minimal length coset representative $w \in W_0^\lambda$ with a reduced expression $s_{i_1} s_{i_2} \cdots s_{i_k}$, then the corresponding element is $u_{w\lambda} := f_{i_1} f_{i_2} \cdots f_{i_k} u_\lambda$. We note that the element $u_{u\lambda}$ is independent of the choice of reduced expression.

For a minuscule representation $B(\lambda)$, we can characterize the elements in $B(s\lambda)$ as follows. Recall that since $B(\lambda)$ is a highest weight crystal, it can be considered as the Hasse diagram of a poset with u_λ being the smallest element.

Proposition 2.3 ([Scr17, Prop. 7.11])**.** *Let $B(\lambda)$ be a minuscule representation. The crystal $B(s\lambda)$ is isomorphic to the set of* semistandard tableaux *whose shape is a single row of length s*

$$T = \boxed{x_1}\boxed{x_2}\boxed{\cdots}\boxed{x_s},$$

with $x_1 \leq x_2 \leq \cdots \leq x_s$ in $B(\lambda)$, and the crystal structure is given by considering T as the element $x_1 \otimes x_2 \otimes \cdots \otimes x_s \in B(\lambda)^{\otimes s}$.

2.3. **Type A_n crystals.** In this section, we consider the Lie algebra of type A_n, which is $\mathfrak{sl}_{n+1}$. We denote the fundamental weights of type A_n by $\{\eta_i \mid 1 \leq i \leq n\}$. Recall that we have a natural bijection between P^+ and partitions of length at most n by η_i corresponding to a column of height i. Let $B(\eta_1)$ denote the crystal of the vector representation of $\mathfrak{sl}_{n+1}$:

$$\boxed{1} \xrightarrow{\ 1\ } \boxed{2} \xrightarrow{\ 2\ } \boxed{3} \xrightarrow{\ 3\ } \cdots \xrightarrow{\ n-1\ } \boxed{n} \xrightarrow{\ n\ } \boxed{n+1}\,.$$

Furthermore, the crystal $B(\eta)$ can be described by *semistandard Young tableaux (SSYT)*, written in English convention, whose entries are at most $n+1$. The crystal structure is given by embedding a SSYT $T \in B(\eta)$ into $B(\eta_1)^{\otimes|\eta|}$ by the reverse Far-Eastern reading word: reading bottom-to-top and left-to-right.

Next, we recall the Levi branching rule $\mathfrak{sl}_{n+1} \searrow \mathfrak{sl}_n$ given at the level of crystals. In terms of the crystal graph, we simply remove all n-colored edges. For the SSYT, this amounts to fixing all $n+1$'s that appear. Since any $n+1$ must be the bottom entry of every column and the largest entry in a given row, we obtain the following statement (which is well-known to experts).

Proposition 2.4. *As $U_q(\mathfrak{sl}_n)$-crystals, we have*

$$B(\eta) \cong \bigoplus_{\mu} B(\mu)$$

where the sum is taken over all μ such that η/μ is a horizontal strip (i.e. a skew partition that does not contain a vertical domino).

Note that the decomposition of Proposition 2.4 is *multiplicity-free*.

We recall that there exists a natural order 2 diagram automorphism σ on type A_n crystals given by $i^\sigma = n+1-i$. Indeed, we define an automorphism of the weight lattice, which we abuse notation and also denote by $\sigma\colon P \to P$, by $\eta_i \mapsto \eta_{i^\sigma} = \eta_{n+1-i}$; in particular, we note that $\eta \mapsto -w_0\eta$. Next, we define a map also denoted by $\sigma\colon B(\eta) \mapsto B(-w_0\eta)$ given by

$$\sigma(f_{i_1}\cdots f_{i_k} u_\eta) = f_{i_1^\sigma}\cdots f_{i_k^\sigma} u_{-w_0\eta}.$$

We recall from [SS06] that $\sigma = \vee \circ *$, where $\vee$ denotes the contragredient dual map and $*$ is the the Lusztig involution (which equals the Schützenberger involution [Len00]).

In the sequel, we require the A_7 Levi subalgebra $\mathfrak{g}_2$ of the type $E_7^{(1)}$ affine Lie algebra $\mathfrak{g}$ given by the index set $I_2 := I \setminus \{2\}$. Therefore, the fundamental weights correspond by

$$\eta_1 = \omega_7, \qquad \eta_2 = \omega_6, \qquad \eta_3 = \omega_5, \qquad \eta_4 = \omega_4, \qquad \eta_5 = \omega_3, \qquad \eta_6 = \omega_1, \qquad \eta_7 = \omega_0.$$

We let $I_{0,2} := J \setminus \{0,2\}$ be index set of the the A_6 Levi subalgebra of $\mathfrak{g}_0$ (of type E_7).

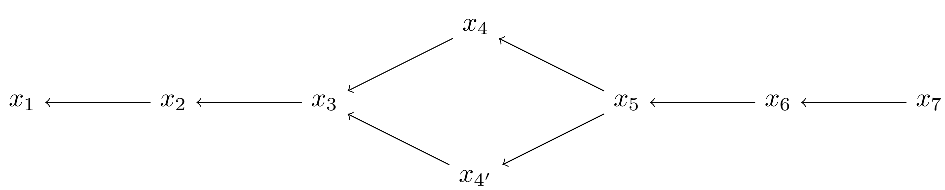
3. Results

3.1. **Multiplicity freeness of $B(s\overline{\omega}_7)$.** We first prove that when we decompose the E_7 crystals $B(s\overline{\omega}_7)$ into the Levi subalgebra of type A_6, we obtain a multiplicity free decomposition.

The $I_{0,2}$-highest weight elements of $B(s\overline{\omega}_7)$ are single row tableaux of size k whose entries consist of

$$x_1 = 7, \qquad x_2 = \overline{6}5, \qquad x_3 = \overline{4}23, \qquad \begin{matrix} x_4 = \overline{1}2, \\ x_{4'} = \overline{2}3, \end{matrix} \qquad x_5 = \overline{12}4 \qquad x_6 = \overline{2}6, \qquad x_7 = \overline{2}1.$$

This can be seen by a direct computation using Proposition 2.3, the crystal graph $B(\overline{\omega}_7)$, and the signature rule. Note that in the crystal graph of $B(\overline{\omega}_7)$, we have



(3.1)

and so, nearly all of the elements x_i are comparable except x_4 with $x_{4'}$.

We recall from [JS10, Def. 3.10] that a *(reduced) composition graph* $G_k(\overline{\omega})$ is essentially the smallest acyclic digraph with loops whose vertices are elements of $B(\overline{\omega})$ such that for any $s > 0$ and every $(I_0 \setminus \{k\})$-highest weight element $b_1 \otimes \cdots \otimes b_s \in B(s\overline{\omega}) \subseteq B(\overline{\omega})^{\otimes s}$, the elements $(b_1, \ldots, b_s)$ occurs as a subsequence of a directed path in $G_k(\overline{\omega})$. We remark by reversing the arrows and adding loops to every vertex in (3.1), we obtain the composition graph $G_1(\overline{\omega}_7)$.

Note that for a semistandard tableau $T \in B(s\overline{\omega}_7)$, we have

$$\mathrm{wt}(T) = \sum_{i \in I} (a_i - a_{\overline{\imath}})\overline{\omega}_i,$$

where a_i (resp. $a_{\overline{\imath}}$) equals the number if i's (resp. $\overline{\imath}$'s) that appear in T. Therefore, from (3.1) and the signature rule, we have the following.

Lemma 3.1. *Let m_i denote the number of occurrences of x_i in $T \in B(s\overline{\omega}_7)$. Then T is a $I_{0,2}$-highest weight element if and only if entries in T consist of $\{x_1, x_2, x_3, x_4, x_{4'}, x_5, x_6, x_7\}$ and*

$$m_2 \le m_6, \qquad m_3 \le m_5, \qquad m_4 + m_5 \le m_7, \qquad \min(m_4, m_{4'}) = 0,$$

with $\sum_i m_i = s$. *Moreover, if* $T \in B(s\overline{\varpi}_7)$ *is a* $I_{0,2}$*-highest weight element, then*

$$\begin{aligned}\mathrm{wt}(T) = (m_7 - m_4 - m_5)\overline{\varpi}_1 + (m_3 + m_4 - m_{4'} - m_5 - m_6 - m_7)\overline{\varpi}_2 + (m_3 + m_{4'})\overline{\varpi}_3 \\ + (m_5 - m_3)\overline{\varpi}_4 + m_2\overline{\varpi}_5 + (m_6 - m_2)\overline{\varpi}_6 + m_1\overline{\varpi}_7.\end{aligned}$$

We note that the condition $\min(m_4, m_{4'}) = 0$ is precisely the fact that x_4 and $x_{4'}$ cannot simultaneously appear in an element of $B(s\overline{\varpi}_7)$.

Proposition 3.2. *The decomposition of* $B(s\overline{\varpi}_7)$ *into type* A_6 *crystals is given by*

$$B(s\overline{\varpi}_7) \cong \bigoplus_{\mu} B(\mu),$$

where

$$\mu = (m_7 - m_4 - m_5)\eta_6 + (m_3 + m_{4'})\eta_5 + (m_5 - m_3)\eta_4 + m_2\eta_3 + (m_6 - m_2)\eta_2 + m_1\eta_1$$

such that $m_1, \dots, m_7$ *satisfy*

$$\begin{gathered} m_2 \le m_6, \qquad m_3 \le m_5, \qquad m_4 + m_5 \le m_7, \\ \min(m_4, m_{4'}) = 0, \qquad s = m_1 + m_2 + m_3 + m_4 + m_{4'} + m_5 + m_6 + m_7. \end{gathered}$$

Moreover, this decomposition is multiplicity free.

Proof. The first claim follows immediately from Lemma 3.1 and relabeling the fundamental weights. For the second claim, consider a weight $\mu = \sum_{i=1}^{6} a_i\eta_i$ such that $B(\mu)$ appears in the A_6 decomposition of $B(s\overline{\varpi}_7)$. Thus, we have

$$\begin{aligned} m_1 &= a_1, & m_2 &= a_3, \\ m_6 &= a_2 + m_2 = a_2 + a_3, & m_3 &= a_5 - m_{4'}, \\ m_5 &= a_4 + m_3 = a_4 + a_5 - m_{4'}, & m_7 &= a_6 + m_4 + m_5 = a_4 + a_5 + a_6 + m_4 - m_{4'}. \end{aligned}$$

Since $\min(m_4, m_{4'}) = 0$ with $m_4, m_{4'} \ge 0$, there exists a unique m_4 and $m_{4'}$ such that $m_4 - m_{4'} = C$ for any constant C. Next, we have

$$\begin{aligned} s &= m_1 + m_2 + m_3 + m_4 + m_{4'} + m_5 + m_6 + m_7 \\ &= a_1 + a_3 + (a_5 - m_{4'}) + m_4 + m_{4'} + (a_4 + a_5 - m_{4'}) + (a_2 + a_3) + (a_4 + a_5 + a_6 + m_4 - m_{4'}) \\ &= a_1 + a_2 + 2a_3 + 2a_4 + 3a_5 + a_6 + 2(m_4 - m_{4'}). \end{aligned}$$

Hence, we have

$$m_4 - m_{4'} = \frac{1}{2}(s - a_1 - a_2 - 2a_3 - 2a_4 - 3a_5 - a_6).$$

Therefore, there is a unique $m_1, \dots, m_7$ that yields the weight μ. ☐

3.2. **Reconstructing the A_7 crystals.** In this section, we continue to use the notation of Proposition 3.2.

Lemma 3.3. *Let* $T \in B^{7,s}$ *be a* $I_{0,2}$*-highest weight element. Then*

$$-\langle \alpha_0^\vee, \mathrm{wt}(T) \rangle = m_1 + m_2 + m_3 + m_{4'}$$

Proof. This follows from Proposition 3.2, that all elements in $B^{7,s}$ are of level 0, and Equation (2.1). ☐

Since $\langle \alpha_0^\vee, \mathrm{wt}(T) \rangle \le 0$ for all $I_{0,2}$-highest weight elements $T \in B^{7,s}$, we know that any potential I_2-highest weight element must have $m_1 = m_2 = m_3 = m_{4'} = 0$. Hence, the possible I_2-dominant weights are

$$\mu = (m_7 - m_4 - m_5)\eta_6 + m_5\eta_4 + m_6\eta_2$$

(*i.e.*, $\langle \alpha_i^\vee, \mu \rangle \ge 0$ for all $i \in I_2$) such that

$$m_4 + m_5 \le m_7, \qquad m_4 + m_5 + m_6 + m_7 = s.$$

Because we can only remove horizontal strips for the branching rule from $A_7 \searrow A_6$ (Proposition 2.4), each of such I_2-dominant weights μ must correspond to a I_2-highest weight component $B(\mu)$. Hence, we obtain the following.

Proposition 3.4. *The decomposition of $B(s\overline{\varpi}_7)$ into type A_7 crystals is given by*

$$B(s\overline{\varpi}_7) \cong \bigoplus_{\mu} B(\mu),$$

where

$$\mu = (m_7 - m_4 - m_5)\eta_6 + m_5\eta_4 + m_6\eta_2$$

such that m_4, m_5, m_6, m_7 satisfy

$$m_4 + m_5 \leq m_7, \qquad s = m_4 + m_5 + m_6 + m_7.$$

Moreover, Proposition 2.4 states that for any element T expressed as an A_6 tableau, we add in a horizontal strip to T with every entry an 8 such that every column has even height to obtain the representation as an A_7 tableau. In particular, for a $I_{0,2}$-highest weight element b, we have

$$\varepsilon_0(b) = m_1 + m_2 + m_3 + m_{4'}, \qquad \varphi_0(b) = 0.$$

Now we combine this to form a combinatorial crystal structure on $B^{7,s}$ by extending the E_7 crystal structure on $B(s\overline{\varpi}_7)$ as follows. Let $\overline{\psi}\colon B(s\overline{\varpi}_7) \to \bigoplus_{\overline{\mu}} B(\overline{\mu})$ be the $I_{0,2}$-crystal (*i.e.*, type A_6) isomorphism given by Proposition 3.2. From Proposition 3.4 and Proposition 2.4, we can uniquely extend the image of $\overline{\psi}$ to highest weight crystals of type A_7. Therefore, we define

$$e_0 := \overline{\psi}^{-1} \circ e_7^A \circ \overline{\psi}, \qquad f_0 := \overline{\psi}^{-1} \circ f_7^A \circ \overline{\psi},$$

where e_7^A and f_7^A are the crystal operators from this extended type A_7 crystal. Let $\mathcal{B}^{7,s}$ denote the corresponding crystal.

In order to show this is the combinatorial structure of KR crystal, we need the following uniqueness theorem. The proof is similar to [JS10, Thm. 3.15] with $K = I_{0,2}$ and using Proposition 3.2 instead of [JS10, Lemma 3.12].

Theorem 3.5. *Let $\mathcal{B}$ and $\mathcal{B}'$ be two affine type $E_7^{(1)}$ crystals such that there exists a I_2-crystal (i.e., type A_7) isomorphism and I_0-crystal (i.e., type E_7) isomorphism*

$$\Psi_{I_2}\colon \mathcal{B}|_{I_2} \to \mathcal{B}'|_{I_2} \cong \bigoplus_{\mu} B(\mu) \qquad \Psi_{I_0}\colon \mathcal{B}|_{I_0} \to \mathcal{B}'|_{I_0} \cong B(s\overline{\varpi}_7),$$

where the sum is over μ as given in Proposition 3.4. Then, we have $\Psi_J(b) = \Psi_{I_0}(b)$ for all $b \in \mathcal{B}$. Moreover, there exists an I-crystal isomorphism $\Psi\colon \mathcal{B} \to \mathcal{B}'$.

Corollary 3.6. *We have $\mathcal{B}^{7,s} \cong B^{7,s}$.*

Proof. We have that $B^{7,s} \cong B(s\overline{\varpi}_7)$ as I_0-crystals since the corresponding KR module is irreducible as a $U_q(\mathfrak{g}_0)$-module [Cha01]. Hence, the KR crystal $B^{7,s}$ exists by [KKM+92a, KKM+92b]. We can then decompose the crystal $B(s\overline{\varpi}_7)$ into $I_{0,2}$-crystals according to Proposition 3.2, and so the I_2-crystal decomposition of $B^{7,s}$ is given by Proposition 3.4. Hence, we have isomorphisms between $\mathcal{B}^{7,s}$ and $B^{7,s}$ as I_0-crystals and I_2-crystals because these decompositions are multiplicity free. ∎

We have verified that $B^{7,s}$ is a perfect crystal of level s for all $s \leq 4$ using the implementation of $B^{7,s}$ in SageMath [Dev18] done by the second author. From [FSS07], we have that the tensor product $(B^{7,s})^{\otimes m}$ is connected and isomorphic to a Demazure subcrystal of $B(s\omega_0)$.

3.3. **Combinatorial isomorphism.** Let $\langle b \rangle$ denote the $I_{0,2}$-subcrystal generated by an element b. Consider the subcrystals

$$\mathcal{C}_1 = \langle x_1 \rangle, \qquad \mathcal{C}_2 = \langle x_6 \rangle, \qquad \mathcal{C}_3 = \langle x_2 \otimes x_6 \rangle,$$
$$\mathcal{C}_4 = \langle x_5 \otimes x_7 \rangle, \qquad \mathcal{C}_5 = \langle x_3 \otimes x_5 \otimes x_7 \rangle, \qquad \mathcal{C}_6 = \langle x_7 \rangle.$$

It is clear that we have $I_{0,2}$-crystal isomorphisms $\psi_i\colon B(\eta_i) \to \mathcal{C}_i$ for all i. (Note that we could have $\mathcal{C}_7 = \langle x_4 \otimes x_7 \rangle$, which would correspond to m_4.) Let $\mathcal{C}_{5'} = \langle x_{4'} \rangle$, and we also have a $I_{0,2}$-crystal isomorphism $\psi_{5'}\colon B(\eta_5) \to \mathcal{C}_{5'}$.

Therefore, we can construct a bijection between SSYT for $B(\mu)$ with the A_6 component

$$\mathcal{C}_6^{\otimes a_6} \otimes \mathcal{C}_{5'}^{\otimes m_{4'}} \otimes \mathcal{C}_5^{\otimes a_5 - m_{4'}} \otimes \mathcal{C}_4^{\otimes a_4} \otimes \mathcal{C}_3^{\otimes a_3} \otimes \mathcal{C}_2^{\otimes a_2} \otimes \mathcal{C}_1^{\otimes a_1}$$

by applying ψ_i on each column of height i, with possibly $\psi_{5'}$ on columns of height 5, of a SSYT (of type A_6). Note that the result is "out-of-order" in that it does not result is $x_{i_1} \otimes \cdots \otimes x_{i_\ell}$ with $i_1 \leq \cdots \leq i_\ell$ (*i.e.*, a semistandard tableaux given by Proposition 2.3). Thus it would remain to "sort" these elements to obtain an honest component of $B(s\overline{\varpi}_7)$. However, we will instead do the reverse process: we refine the isomorphisms ψ_i^{-1} to apply them to elements of $B(\overline{\varpi}_7)$, which we combine to apply to an element of $B(s\overline{\varpi}_7)$.

Using the tensor product rule, we obtain the following defining crystal isomorphism with a tensor product of A_6 columns:

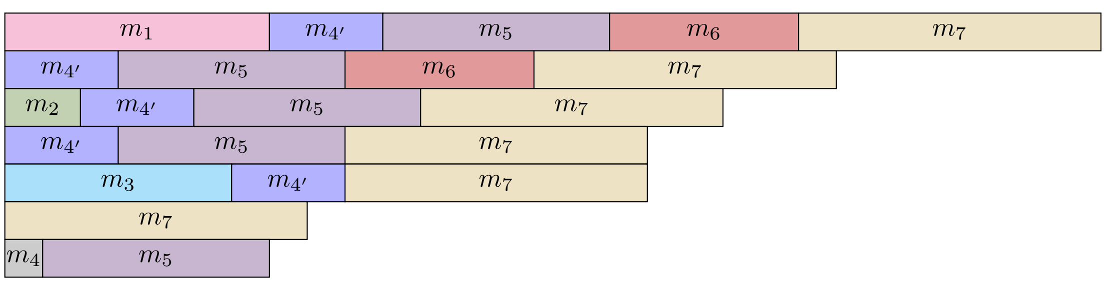
$$x_1 \mapsto \boxed{1}, \quad x_2 \mapsto \boxed{3}, \quad x_3 \mapsto \boxed{5}, \quad x_4 \mapsto \boxed{7}, \quad x_{4'} \mapsto \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline 5 \\ \hline \end{array}, \quad x_5 \otimes x_7 \mapsto \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array}, \quad x_6 \mapsto \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}, \quad x_7 \mapsto \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline 5 \\ \hline 6 \\ \hline \end{array}.$$

We need one additional step to decouple $x_5 \otimes x_7$, which we do by

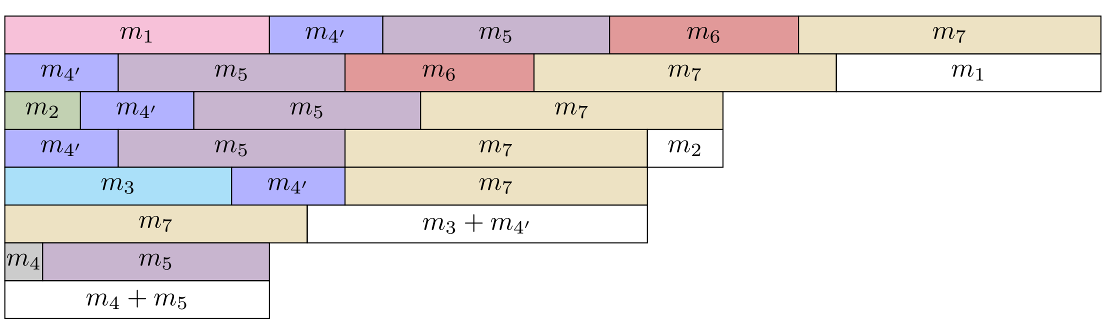
$$x_5 \mapsto \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline 7 \\ \hline \end{array}.$$

We could have arrived at this refined isomorphism by using the A_6 decomposition of $B(\overline{\varpi}_7)$. Now using our refined isomorphism and jeu-de-taquin, we obtain a crystal isomorphism Ψ from elements of $B(s\overline{\varpi}_7)$ to a direct sum of A_6 SSYT given by Proposition 3.2. Recall that jeu-de-taquin is a crystal isomorphism.

Indeed, performing jeu-de-taquin (see, *e.g.*, [Ful97]) on a highest weight element of the tensor product of single column SSYT, we obtain the SSYT



where row i is filled with only i. Note that we have included an entry from both m_4 and $m_{4'}$ for illustrative purposes, but both cannot appear. Augmenting the result with a horizontal strip of 8's to obtain all columns of even height (which agrees with Proposition 3.2 after ignoring the columns of height 8), we have



Note that the 8's in the even rows can overlap any of the blocks depending on m_1, m_2, and $m_3 + m_{4'}$ compares with $m_{4'} + m_5 + m_6 + m_7$, $m_{4'} + m_5 + m_7$, and m_7 respectively.

3.4. **Diagram automorphisms.** We can construct the type E_6 crystal decomposition as follows. Let $I_{0,7} := I_0 \setminus \{7\}$. We construct the $I_{0,7}$-highest weight elements by considering semistandard tableaux in $B(s\overline{\varpi}_7)$ consisting of the elements of

$$7 \longleftarrow \overline{7}6 \longleftarrow \overline{7}1 \longleftarrow \overline{7}.$$

The computation is similar to (3.1). Similarly, by reversing the arrows and adding loops at every vertex, we obtain the composition graph $G_7(\overline{\varpi}_7)$. Thus, the $I_{0,7}$-highest weight elements in $B(s\overline{\varpi}_7)$ are given by

$$7^{\otimes m_7} \otimes \overline{7}6^{\otimes m_{\overline{7}6}} \otimes \overline{7}1^{\otimes m_{\overline{7}1}} \otimes \overline{7}^{\otimes m_{\overline{7}}}$$

(recall that we naturally identify this with a semistandard tableau) with $m_7 + m_{\overline{7}6} + m_{\overline{7}1} + m_{\overline{7}} = s$.

Proposition 3.7. *Let m_b be the number of occurrences of b in a single row tableau $R \in B(s\overline{\varpi}_7)$. Then R is an $I_{0,7}$-highest weight element. Moreover, as E_6 crystals, we have*

$$B(s\overline{\varpi}_7) \cong \bigoplus_{m_{\overline{7}6}+m_{\overline{7}1}+m_{\overline{7}} \leq s} B(m_{\overline{7}6}\overline{\varpi}_6 + m_{\overline{7}1}\overline{\varpi}_1).$$

Note that the decomposition into E_6 crystals from Proposition 3.7 is not multiplicity free; indeed, the multiplicity of $B(m_{\overline{7}6}\overline{\varpi}_6 + m_{\overline{7}1}\overline{\varpi}_1)$ is $s - m_{\overline{7}6} - m_{\overline{7}1}$ (the number of distinct values $m_{\overline{7}}$ can take). However, we can distinguish each of the $I_{0,7}$-highest weight elements by using the extra information (in addition to the weight) of $\langle \mathrm{wt}(b), \alpha_7^\vee \rangle = m_7 - m_{\overline{7}1} - m_{\overline{7}6} - m_{\overline{7}}$. Furthermore, by the level-zero condition, we have $\langle \mathrm{wt}(b), \alpha_0^\vee \rangle = m_7 - m_{\overline{7}6} - m_{\overline{7}1} - m_{\overline{7}}$. By also using $m_7 + m_{\overline{7}6} + m_{\overline{7}1} + m_{\overline{7}} = s$, we thus obtain the following.

Proposition 3.8. *The map $\Phi \colon B^{7,s} \to B^{7,s}$ given by*

$$\Phi\bigl(7^{\otimes a} \otimes \overline{7}6^{\otimes b} \otimes \overline{7}1^{\otimes c} \otimes \overline{7}^{\otimes d}\bigr) = 7^{\otimes b+c+d} \otimes \overline{7}6^{\otimes c} \otimes \overline{7}1^{\otimes b} \otimes \overline{7}^{\otimes a+b+c}$$

and extended as a twisted $I_{0,7}$-crystal morphism is a twisted $I_{0,7}$-crystal isomorphism. Moreover Φ is an involution that is induced from the order 2 diagram automorphism of $E_7^{(1)}$.

By restricting to $I_{0,2}$-highest weight elements, we have that $\Phi = \psi^{-1} \circ \sigma \circ \psi$, where ψ is the isomorphism from Proposition 3.4. This can be seen by equating the weight and $\langle \mathrm{wt}(b), \alpha_7^\vee \rangle$. Thus, Φ when restricted to the A_7 crystal of $B^{7,s}$ is the order 2 diagram automorphism of A_7.

4. Conjectures for $B^{1,s}$

We conclude with a conjectural decomposition of $B^{1,s}$ into A_7 crystals. Recall that A_7 has a natural diagram symmetry σ that respects the $E_7^{(1)}$ diagram symmetry. Thus, proving this conjecture using E_7 crystal decomposition of $B^{1,s} \cong \bigoplus_{k=0}^{s} B(k\overline{\varpi}_1)$ could possibly lead to a proof of [JS10, Conj. 3.26].

Conjecture 4.1. *Let $a, b, c, d \in \mathbb{Z}_{\geq 0}$ such that $a + 2b + 3c + d \leq s$. Then we have*

$$B^{1,s} \cong \bigoplus B(a(\eta_1 + \eta_7) + b(\eta_2 + \eta_6) + c(\eta_3 + \eta_5) + d\eta_4)^{\oplus m_{a,b,c,d}},$$

as A_7 crystals, where the multiplicities are $m_{a,b,c,d} = m_{d,s-a-2b-3c}$, where

$$m_{d,s'} = \sum_{i=M}^{d+1} \left\lceil \frac{i}{2} \right\rceil$$

with $M = \max(d + 1 - (s' - d), 0)$.

We compute the multiplicity of η_4 in $B^{1,s}$ using SageMath [Dev18]:

```
sage: for s in range(10):
....:     [sum(ceil(i/2) for i in range(max(0,2*d+1-s),d+1+1)) for d in range(s+1)]
[1]
[1, 1]
[1, 2, 2]
[1, 2, 3, 2]
[1, 2, 4, 4, 3]
[1, 2, 4, 5, 5, 3]
[1, 2, 4, 6, 7, 6, 4]
```

```
[1, 2, 4, 6, 8, 8, 7, 4]
[1, 2, 4, 6, 9, 10, 10, 8, 5]
[1, 2, 4, 6, 9, 11, 12, 11, 9, 5]
```

We compute the decomposition of $B^{7,s}$ into A_7 crystals using SageMath:

```
sage: def compute_branching(s):
....:     A7 = WeylCharacterRing(['A',7], style="coroots")
....:     E7 = WeylCharacterRing(['E',7], style="coroots")
....:     La = E7.fundamental_weights()
....:     chi = sum(E7(k*La[1]) for k in range(s+1))
....:     return chi.branch(A7, rule="extended")
sage: compute_branching(1)
A7(0,0,0,0,0,0,0) + A7(0,0,0,1,0,0,0) + A7(1,0,0,0,0,0,1)
sage: compute_branching(2)
2*A7(0,0,0,0,0,0,0) + 2*A7(0,0,0,1,0,0,0) + A7(0,0,0,2,0,0,0)
 + A7(0,1,0,0,0,1,0) + A7(1,0,0,0,0,0,1) + A7(1,0,0,1,0,0,1)
 + A7(2,0,0,0,0,0,2)
sage: compute_branching(3)
2*A7(0,0,0,0,0,0,0) + 3*A7(0,0,0,1,0,0,0) + 2*A7(0,0,0,2,0,0,0)
 + A7(0,0,1,0,1,0,0) + A7(0,1,0,0,0,1,0) + 2*A7(1,0,0,0,0,0,1)
 + 2*A7(1,0,0,1,0,0,1) + A7(0,1,0,1,0,1,0) + A7(0,0,0,3,0,0,0)
 + A7(1,0,0,2,0,0,1) + A7(1,1,0,0,0,1,1) + A7(2,0,0,0,0,0,2)
 + A7(2,0,0,1,0,0,2) + A7(3,0,0,0,0,0,3)
```

We have verified Conjecture 4.1 for $s \leq 9$ by using a heavily optimized version of the branching rule code in SageMath [Dev18].

One way to construct the A_7 highest weight elements would be to use the A_6 highest weight elements, which we can compute from the composition graph in Figure 2 (equivalently Figure 3). From there, we will want the elements invariant (as a set) under the diagram automorphism from [JS10] as highest weight elements of weight η must map to a highest weight element of weight $-w_0\eta$ under the A_7 diagram automorphism σ.

As a step towards proving Conjecture 4.1, we show the analog of [JS10, Prop. 2.13] that characterizes the elements in $B(2\overline{\varpi}_1) \subseteq B(\overline{\varpi}_1)^{\otimes 2}$.

Proposition 4.2. *We have*

$$(b_1 \otimes c_1) \otimes (b_2 \otimes c_2) \in B(2\overline{\varpi}_1) \subseteq B(\overline{\varpi}_1)^{\otimes 2} \subseteq B(\overline{\varpi}_7)^{\otimes 4}$$

if and only if $b_1 \leq b_2$ *and* $c_1 \leq c_2$ *(the comparisons are in* $B(\overline{\varpi}_7)$*) and* $(b_1 \otimes c_1) < (b_2 \otimes c_2)$ *(the comparison is in* $B(\overline{\varpi}_1)$*).*

Proof. This is a finite computation that can be done, *e.g.*, by using the following SageMath [Dev18] code:

```
sage: L = crystals.Letters(['E',7])
sage: x = L.highest_weight_vector().f_string([7,6,5,4,2,3,4,5,6,7])
sage: A = tensor([x,L.highest_weight_vector()]).subcrystal()
sage: S = tensor([A.module_generators[0].value,
....:             A.module_generators[0].value]).subcrystal()
sage: all(P.le(x.value[0][1], x.value[1][1]) for x in S) # Check the b_i condition
True
sage: all(P.le(x.value[0][0], x.value[1][0]) for x in S) # Check the c_i condition
True
sage: P = Poset(L.digraph())
sage: PA = Poset(A.digraph())
sage: data = [[x.value[0], x.value[1]] for x in S]
sage: T = tensor([A, A])
sage: all((not PA.le(A(x[0].value), A(x[1].value))) or x[0].value == x[1].value
....:     for x in T if P.le(x[0].value[0], x[1].value[0])
....:                and P.le(x[0].value[1], x[1].value[1])
....:                and [x[0].value, x[1].value] not in data)
True
```

□

Appendix A. SageMath code for composition graphs

To compute the composition graph in Figure 2, we first do some setup:

```
sage: L = crystals.Letters(['E',7])
sage: x = L.highest_weight_vector().f_string([7,6,5,4,2,3,4,5,6,7])
sage: A = tensor([x,L.highest_weight_vector()]).subcrystal()
sage: TA = tensor([A.module_generators[0], A.module_generators[0]]).subcrystal()
sage: _ = A.list()
```

To compute the composition graph $G_2(\overline{\omega}_1)$, we run the following functions:

```
def check_le(x, y):
    return tensor([x,y]) in TA

def composition_graph(J):
    I = A.index_set()
    ImJ = sorted(set(I) - set(J))
    G = DiGraph([[b for b in A if b.is_highest_weight(ImJ)], check_le])
    num_verts = 0
    while num_verts != G.num_verts():
        num_verts = G.num_verts()
        verts = set(G.vertices())
        for b in A:
            ep = set([i for i in I if b.epsilon(i) > 0])
            Jplus = set(list(J) + [i for i in ImJ
                                   if any(bp.phi(i) > 0 and check_le(b,bp)
                                          for bp in verts)
                                  ]
                       )
            if ep.issubset(Jplus):
                G.add_vertex(b)
                for bp in verts:
                    if check_le(b, bp):
                        G.add_edge(b, bp)
    loops = G.loops()
    G = G.transitive_reduction()
    for l in loops:
        G.add_edge(*l)
    G.set_latex_options(format='dot2tex')
    return G
```

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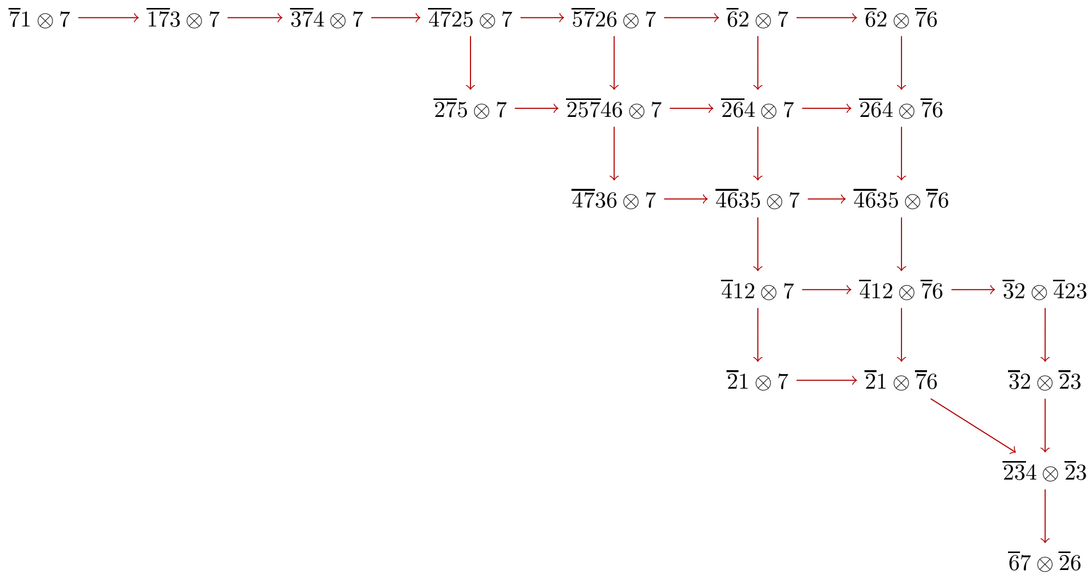

FIGURE 2. The composition graph $G_2(\varpi_1)$ to compute the $(I_0 \setminus \{2\})$-highest weight elements. We have suppressed the loops that occur at every node except $\overline{3}2 \otimes \overline{2}3$.

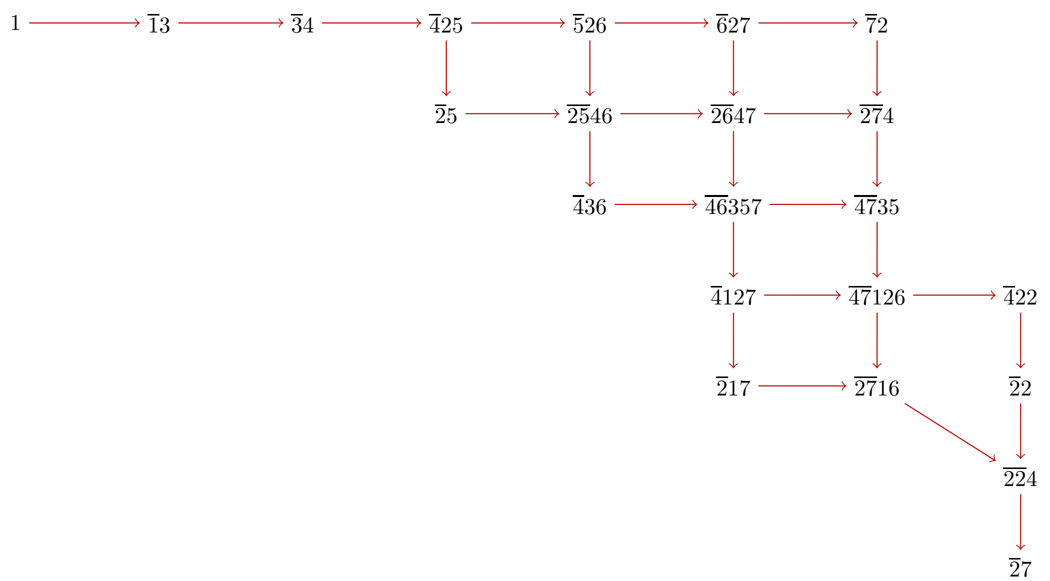

FIGURE 3. The composition graphs of Figure 2 with every node written in "compact form," where a k adds 1 to $\varphi_k(b)$ and $\overline{k}$ adds 1 to $\varepsilon_k(b)$. Recall that the only vertex that does not have a loop is $\overline{2}2$.

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