Mathematische Zeitschrift (2022) 300:1455–1507
https://doi.org/10.1007/s00209-021-02810-x

Mathematische Zeitschrift

# Automorphisms and periods of cubic fourfolds

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Received: 26 November 2019 / Accepted: 7 June 2021 / Published online: 17 August 2021


**Abstract**
We classify the symplectic automorphism groups for cubic fourfolds. The main inputs are the global Torelli theorem for cubic fourfolds and the classification of the fixed-point sublattices of the Leech lattice. Among the highlights of our results, we note that there are 34 possible groups of symplectic automorphisms, with 6 maximal cases. The six maximal cases correspond to 8 non-isomorphic, and isolated in moduli, cubic fourfolds; six of them previously identified by other authors. Finally, the Fermat cubic fourfold has the largest possible order (174,960) for the automorphism group (non-necessarily symplectic) among all smooth cubic fourfolds.

## 1 Introduction

Cubic fourfolds are some of the most intensely studied objects in algebraic geometry in connection to rationality questions and to constructing compact hyper-Kähler manifolds. What sets the cubic fourfolds apart is that they are Fano fourfolds whose middle cohomology is of level 2 with $h^{3,1} = 1$ (i.e., of $K3$ type). Consequently, the moduli space of cubic fourfolds behaves very similarly to the moduli space of polarized $K3$ surfaces. Specifically, Voisin [49] proved a global Torelli theorem for cubic fourfolds. Later, Hassett [23] identified some natural Noether–Lefschetz divisors $\mathcal{C}_d$ (for $d \in \mathbb{Z}_+$ with $d \equiv 0, 2 \pmod 6$) in the moduli space of cubic fourfolds, and conjectured that the image of the period map is the complement of $\mathcal{C}_2$ and $\mathcal{C}_6$. This was subsequently verified by the first author [32] and Looijenga [36]. More recently, the second author [55] proved a stronger version of the Torelli theorem: the automorphisms of cubic fourfolds are detected by (polarized) Hodge isometries.

The purpose of this paper is to use the period map to study and classify the possible *symplectic* automorphism groups (Definition 2.7) for cubic fourfolds. The model for our study is the well-known case of $K3$ surfaces. Namely, a consequence of the Torelli theorem for $K3$ surfaces is that there is a close connection between the automorphism group $\mathrm{Aut}(Y)$

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of a $K3$ surface $Y$ and the Hodge isometries on $H^2(Y, \mathbb{Z})$. Nikulin [42] started a systematic investigation of the possible finite automorphism groups for $K3$ surfaces by means of lattice theory [43]. This study culminated with the celebrated result of Mukai [41] relating the classification of the finite groups of symplectic automorphisms acting on $K3$ surfaces with certain subgroups of the Mathieu group $M_{23}$. Kondō [29] simplified Mukai's proof by relating this classification problem to the isometries of the Niemeier lattices. Kondō's approach avoids the Leech lattice (the unique Niemeier lattice containing no roots), but it turns out that a related construction that involves only the Leech lattice $\mathbb{L}$ behaves more uniformly and adapts to higher dimensions [15,27]. In particular, one sees that all the symplectic automorphism groups $G$ occurring are subgroups of the Conway group $\mathrm{Co}_0 (= O(\mathbb{L}))$ satisfying a certain rank condition on the fixed-point sublattice $\mathbb{L}^G$.

The higher dimensional analogue of the $K3$ surfaces are the hyper-Kähler manifolds (simply connected, compact Kähler manifold, carrying a unique holomorphic symplectic 2-form). Due to Verbitsky's Torelli Theorem and recent results on Mori cones of hyper-Kähler manifolds (e.g., [3]), the approach to automorphisms via lattices that works for $K3$ surfaces can be extended to the case of hyper-Kähler manifolds of $K3^{[n]}$ type, leading to a flurry of activity on the subject. In particular, we note the work of Mongardi [38,40] who started a systematic study of the symplectic automorphisms of hyper-Kähler manifolds of $K3^{[n]}$ type. Around the same time, Höhn and Mason [24] have completed the classification of the fixed-point sublattices $\mathbb{L}^G$ of $\mathbb{L}$ with respect to subgroups $G$ of $\mathrm{Co}_0$ (the case $G$ is cyclic was previously settled by Harada–Lang [20]). Using this classification, in subsequent work [25], Höhn and Mason have completed Mongardi's analysis for hyper-Kähler manifolds of $K3^{[2]}$, obtaining an analogue of Mukai's results in the 4-dimensional case. There are 15 maximal groups [25, Table 2] that are listed in Table 1 in our paper.

The cubic fourfolds are intricately related to hyper-Kähler fourfolds of $K3^{[2]}$ type. Specifically, Beauville–Donagi [4] proved that the Fano variety $F(X)$ of lines on a smooth cubic

**Table 1** Maximal rank Leech pairs satisfying Condition 4.8

| Number | Order | Group | Discriminant form $q_K$ |
|---|---|---|---|
| 1 | 29,160 | $3^4 : A_6$ | $3^{+2}9^{+1}$ |
| 2 | 20,160 | $L_3(4)$ | $2_{\mathrm{II}}^{-2}3^{-1}7^{-1}$ |
| 3 | 5760 | $2^4 : A_6$ | $4_5^{-1}8_1^{+1}3^{+1}$ |
| 4 | 2520 | $A_7$ | $3^{+1}5^{+1}7^{+1}$ |
| 5 | 1944 | $3^{1+4} : 2.2^2$ | $2_2^{+2}3^{+3}$ |
| 6 | 1920 | $2^4 : S_5$ | $4_3^{-1}8_1^{+1}5^{-1}$ |
| 7 | 1344 | $2^3 : L_2(7)$ | $4_2^{+2}7^{+1}$ |
| 8 | 1152 | $Q(3^2 : 2)$ | $8_6^{-2}3^{-1}$ |
| 9 | 720 | $S_6$ | $2_{\mathrm{II}}^{-2}3^{+2}5^{+1}$ |
| 10 | 720 | $M_{10}$ | $2_3^{-1}4_7^{+1}3^{-1}5^{+1}$ |
| 11 | 660 | $L_2(11)$ | $11^{+2}$ |
| 12 | 576 | $2^4 : (S_3 \times S_3)$ | $4_7^{+1}8_1^{+1}3^{+2}$ |
| 13 | 360 | $A_{3,5}$ | $3^{-2}5^{-2}$ |
| 14 | 336 | $2 \times L_2(7)$ | $2_{\mathrm{II}}^{+2}7^{+2}$ |
| 15 | 144 | $3^2 : \mathrm{QD}_{16}$ | $2_1^{+1}4_1^{+1}3^{-1}9^{-1}$ |

fourfold is in fact a hyper-Kähler fourfold of $K3^{[2]}$ type. An interesting aspect here is that by varying the cubic fourfold, one obtains a locally complete moduli space for polarized hyper-Kähler manifolds of $K3^{[2]}$ type (i.e., 20 moduli vs. 19 moduli coming from $K3$ surfaces). In the context of automorphism groups, this leads to the construction of exotic automorphisms for hyper-Kähler manifolds of $K3^{[2]}$ type (i.e., not induced from $K3$ surfaces).

Via the Fano variety construction, the classification of automorphisms of cubic fourfolds is closely related (but some differences arise due to the polarization) to the classification for hyper-Kähler manifolds of $K3^{[2]}$ type, and the above mentioned results. In particular, we note that Höhn–Mason [25, Table 11] have shown that 6 of the 15 maximal groups arising in the classification of automorphisms for the $K3^{[2]}$ case are actually realized by some smooth cubic fourfolds. In a different direction, using more geometric arguments, Fu [14] classified all possible symplectic automorphism groups of cubic fourfolds which are cyclic of primary (i.e., a power of a prime number) order. He also gave the corresponding normal forms for the associated cubic fourfolds (for some earlier results and other examples see [17] and [38,39]). Building on these results, we complete (and give a systematic account of) the classification of the possible groups of symplectic automorphisms for cubic fourfolds. Specifically, we classify all possible groups $G = \mathrm{Aut}^s(X)$ of symplectic automorphisms for cubic fourfolds, and for many of them, we give the corresponding normal forms.

**Notation 1.1** We follow the standard notation from group theory for finite groups. For reader's convenience, we recall in Appendix B the relevant notation and definitions. Briefly, we mention that $p^n$ corresponds to $(\mathbb{Z}/p\mathbb{Z})^n$, $L_p(k)$ corresponds to $\mathrm{PGL}(k, \mathbb{F}_p)$, $D_{2k}$ is the dihedral group of order $2k$, $Q_8$ is the quaternion group, $\mathrm{QD}_{16}$ (which is denoted $\Gamma_3 a_2$ in [25]) is the semidihedral group of order 16, $M_9, M_{10}, M_{3,8}$ are the Mathieu groups (see §B.2), $A_{m,n}$ is the subgroup of $S_{m,n} := S_m \times S_n \subset S_{m+n}$ consisting of elements of even signature, and $3^{1+4}$ is one of the extraspecial groups of order $3^5 = 243$ (see B.3). We use $N : Q$ to denote a semidirect product $N \rtimes Q$ that is not a direct product, and $N.Q$ to denote an extension of $Q$ by $N$ for which we are not sure whether it is split or not.

**Theorem 1.2** *Let $X$ be a smooth cubic fourfold with symplectic automorphism group $G = \mathrm{Aut}^s(X)$. Let $S := S_G(X)$ be the covariant lattice (i.e., the orthogonal complement of the invariant sublattice of $H^4(X, \mathbb{Z})$ under the induced action of $G$). Then one of the following situations holds:*

(0) $\mathrm{rank}(S) = 0$, $G \cong 1$.

(1) $\mathrm{rank}(S) = 8$, $G \cong 2$ *and* $S \cong E_8(2)$. *For an appropriate choice of coordinates, $X$ is given by*

$$X = V\left(F_1(x_1, x_2, x_3, x_4) + x_5^2 L_1(x_1, x_2, x_3, x_4) + x_5 x_6 L_2(x_1, x_2, x_3, x_4)\right.$$
$$\left.+ x_6^2 L_3(x_1, x_2, x_3, x_4)\right).$$

*With respect to these coordinates, $G$ is generated by* $g = \frac{1}{2}(0, 0, 0, 0, 1, 1)$.

(2) $\mathrm{rank}(S) = 12$, $G \cong 2^2$ *or* $G \cong 3$.

(a) *If $G \cong 2^2$, for an appropriate choice of coordinates,*

$$X = V\left(F_1(x_1, x_2, x_3) + x_4^2 L_1(x_1, x_2, x_3) + x_5^2 L_2(x_1, x_2, x_3)\right.$$
$$\left.+ x_6^2 L_3(x_1, x_2, x_3) + x_4 x_5 x_6\right),$$

*and $G$ is generated by* $g_1 = \frac{1}{2}(0, 0, 0, 0, 1, 1)$ *and* $g_2 = \frac{1}{2}(0, 0, 0, 1, 1, 0)$.

(b) *If* $G \cong 3$*, then* $X$ *is either*

$$X = V\left(F_1(x_1,x_2,x_3,x_4) + x_5^3 + x_6^3 + x_5x_6L_1(x_1,x_2,x_3,x_4)\right),$$

*in which case* $G$ *is generated by* $g = \frac{1}{3}(0,0,0,0,1,2)$*, or*

$$X = V\left(F_1(x_1,x_2) + F_2(x_3,x_4) + F_3(x_5,x_6) + \Sigma_{i=1,2;j=3,4;k=5,6}(a_{ijk}x_ix_jx_k)\right),$$

*with* $G$ *generated by* $g = \frac{1}{3}(0,0,1,1,2,2)$.

(3) $\mathrm{rank}(S) = 14$*,* $G \cong 4$ *or* $S_3$.

(a) *If* $G \cong 4$*, for an appropriate choice of coordinates, the defining equations of the corresponding cubic fourfolds belong to*

$$\mathrm{Span}\{x_1N_1(x_3,x_4), x_2N_2(x_3,x_4), F_1(x_1,x_2), x_5x_6L_1(x_1,x_2), x_5^2L_2(x_3,x_4), x_6^2L_3(x_3,x_4)\}.$$

*With respect to these coordinates,* $G$ *is generated by* $g = \frac{1}{4}(0,0,2,2,1,3)$.

(b) *If* $G \cong S_3$*, we can choose coordinate* $x_1,\ldots,x_6$ *of* $\mathbb{C}^6$*, such that the action of* $S_3$ *on* $(\mathbb{C}^6)^\vee$ *is by permuting* $(x_1,x_2), (x_3,x_4), (x_5,x_6)$ *simultaneously, and the defining equations of the corresponding cubic fourfolds are invariant under such an action.*

(4) $\mathrm{rank}(S) = 15$*,* $G \cong D_8$.

(5) $\mathrm{rank}(S) = 16$*,* $G \cong A_{3,3}$*,* $D_{12}$*,* $A_4$*, or* $D_{10}$.

(a) *If* $G \cong D_{12}$*, then the defining equations of the corresponding cubic fourfolds either belong to*

$$\mathrm{Span}\{x_1^2x_3, x_1^2x_4, x_1x_2x_3, x_1x_2x_4, x_2^2x_3, x_2^2x_4, x_3^3, x_3^2x_4, x_3x_4^2, x_3x_5x_6, x_4^3, x_4x_5x_6, x_5^3, x_6^3\},$$

*while an order* 6 *element of* $G$ *is* $\frac{1}{6}(3,3,0,0,2,4)$*, or belong to*

$$\mathrm{Span}\{x_1^3, x_1x_2^2, x_1x_3x_5, x_1x_3x_6, x_2x_4x_5, x_2x_4x_6, x_3^3, x_3x_4^2, x_5^3, x_5^2x_6, x_5x_6^2, x_6^3\},$$

*while an order* 6 *element of* $G$ *is* $\frac{1}{6}(0,3,2,5,4,4)$. *Moreover, a generic cubic fourfold admitting such an order* 6 *automorphism has symplectic automorphism group* $D_{12}$.

(b) *If* $G \cong D_{10}$*, then for an appropriate choice of coordinates,*

$$X = V\left(F_1(x_1,x_2) + x_3x_6L_1(x_1,x_2) + x_4x_5L_2(x_1,x_2) + x_3^2x_5 + x_3x_4^2 + x_4x_6^2 + x_5^2x_6\right).$$

*An order* 5 *element in* $G$ *is* $g = \frac{1}{5}(0,0,1,2,3,4)$. *Moreover, any smooth cubic fourfolds with a symplectic automorphism of order* 5 *have this form, and a generic such cubic fourfold has symplectic automorphism group* $D_{10}$.

(6) $\mathrm{rank}(S) = 17$*,* $G \cong S_4$ *or* $Q_8$.

(7) $\mathrm{rank}(S) = 18$*,* $G \cong 3^{1+4}:2$*,* $A_{4,3}$*,* $A_5$*,* $3^2.4$*,* $S_{3,3}$*,* $F_{21}(\cong 7:3)$*,* $\mathrm{Hol}(5)$[1] *or* $\mathrm{QD}_{16}$.

(a) *If* $G \cong 3^{1+4}:2$*, then for an appropriate choice of coordinates, the defining equations of the corresponding cubic fourfolds belong to*

$$\mathrm{Span}\{monomials in x_1, x_2, x_3, monomials in x_4, x_5, x_6\},$$

[1] There is a typo in Höhn–Mason list [24]: they wrote Hol(4), and claimed it has order 20. The correct group is $\mathrm{Hol}(5) \cong \mathrm{AGL}_1(\mathbb{F}_5) \cong 4:5$.

*An element of order* 3 *in G is* $\frac{1}{3}(0, 0, 0, 1, 1, 1)$. *Moreover, any smooth cubic fourfold with a symplectic automorphism which can be diagonalized as* $\frac{1}{3}(0, 0, 0, 1, 1, 1)$ *has this form, and a generic such cubic fourfold has symplectic automorphism group* $3^{1+4} : 2$.

(b) *If* $G \cong F_{21}$*, then for an appropriate choice of coordinates,*

$$X = V\left(x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_6 + x_6^2x_1 + ax_1x_3x_5 + bx_2x_4x_6\right).$$

*The automorphisms* $g_1 = \frac{1}{7}(1, 5, 4, 6, 2, 3)$ *and* $g_2 : x_i \longmapsto x_{i+2}$ *generate* $F_{21}$. *Moreover, any smooth cubic fourfold with symplectic automorphism of order* 7 *has this form, and a generic such cubic fourfold has symplectic automorphism group* $F_{21}$.

(c) *If* $G \cong \mathrm{QD}_{16}$*, for an appropriate choice of coordinates, the defining equations of the corresponding cubic fourfolds belong to*

$$\mathrm{Span}\{x_1^3, x_1x_2^2, x_2x_3^2, x_2x_4^2, x_1x_3x_4, x_4x_5^2, x_3x_6^2, x_2x_5x_6\}.$$

*An element of order* 8 *in G is* $g = \frac{1}{8}(0, 4, 2, 6, 1, 3)$. *Moreover, any smooth cubic fourfold with a symplectic automorphism of order* 8 *has this form, and a generic such cubic fourfold has symplectic automorphism group* $\mathrm{QD}_{16}$.

(8) $\mathrm{rank}(S) = 19$, $G \cong 3^{1+4} : 2.2$, $A_6$, $L_2(7)$, $S_5$, $M_9$, $N_{72}(\cong 3^2 : D_8)$, *or* $T_{48}(= M_{3,8} \cong Q_8 : S_3)$.[2] *Except for the case* $G \cong 3^{1+4} : 2.2$, 1*-parameter families of cubics with automorphism group G can be obtained by smoothing fake cubic fourfolds* (*they correspond to* $K3$ *surfaces of degree* 2 *or* 6) *with maximal symplectic symmetry, see Sect.* 5.5.

(9) $\mathrm{rank}(S) = 20$, $G \cong 3^4 : A_6$, $A_7$, $3^{1+4} : 2.2^2$, $M_{10}$, $L_2(11)$ *or* $A_{3,5}$. *More information on these cases is included in Theorem* 1.8.

*For each group G among the* 34 *groups appeared above, the corresponding lattice S is unique up to isomorphism. Those* 34 *lattices S are primitive sublattices of the Leech lattice. The dimension of the moduli space of cubic fourfolds with associated pair* $(G, S)$ (*see Sect.* 2.2 *and* 4.1.2 *for the precise definition of this moduli space*) *is* $20 - \mathrm{rank}(S)$.
*(Here,* $F_i$, $N_i$, $L_i$ *denote cubic, quadric, and linear polynomials respectively. We denote by* $\frac{1}{n}(k_1, \ldots, k_6)$ *the diagonal matrix* $(\zeta^{k_1}, \ldots, \zeta^{k_6}) \in \mathrm{SL}(6)$*, where* $\zeta$ *is a primitive n-root of unity.)*

***Remark 1.3*** The 6 maximal cases of Theorem 1.2(9) were already identified by Höhn–Mason [25, Table 11] (including explicit realizations for each case), but it is not shown that they are the only possible cases for $\mathrm{rank}(S) = 20$. This is indeed the case, but as we note in Theorem 1.8 below, in two of the six cases there are two non-isomorphic cubic fourfolds realizing the pair $(G, S)$.

***Remark 1.4*** A direct corollary of Theorem 1.2 is that the possible orders $n$ of symplectic automorphisms $g$ for smooth cubic fourfolds are 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 15. Furthermore, we obtain all geometric realizations for cubic fourfolds with a given order $n$ symplectic automorphism (see Sect. 4.3, esp. Theorem 4.17). This is a strengthening of results of Fu [14] (see also [17]) who discussed the primary order case.

Let us briefly review the key ingredients for the proof of Theorem 1.2. Suppose $G$ is a finite group acting symplectically on a smooth cubic fourfold $X$. Then $G$ acts on the middle cohomology $H^4(X, \mathbb{Z})$, which in turn determines the covariant lattice $S_G(X)$. Following

[2] The notation $N_{72}$ and $T_{48}$ was introduced by Mukai [41] in his classification of symplectic automorphism groups for $K3$ surfaces.

Mongardi (with the main ideas going back to Nikulin and Kondō), we see that the pair $(G, S_G(X))$ can be embedded into $(\mathrm{Co}_0, \mathbb{L})$. More precisely, there is a primitive embedding of $S_G(X)$ into the Leech lattice $\mathbb{L}$ such that the action of $G$ on $S_G(X)$ extends to a faithful action on $\mathbb{L}$ with $G$ acting trivially on the orthogonal complement of $S_G(X)$ in $\mathbb{L}$ (see Proposition 3.4 and Lemma 4.2). This leads to the abstract notion of *Leech pair* $(S, G)$ (Definition 3.3). Our classification theorem is essentially equivalent to the classification of Leech pairs that can arise from groups of symplectic automorphisms of cubic fourfolds. In the context of the work of Mongardi and Höhn–Mason, the main difference is that we are dealing with polarized hyper-Kähler manifolds of $K3^{[2]}$ type (specifically, $F(X)$ is of $K3^{[2]}$ type with a degree 6 polarization). Instead of dealing directly with the natural polarization, we are using the so-called *Kondō–Scattone trick*. Namely, we note that the primitive cohomology $\Lambda_0 = H^4(X, \mathbb{Z})_{prim}$ of a cubic fourfold $X$ admits a unique primitive embedding into the Borcherds lattice (i.e., $\mathrm{II}_{26,2}$, the unique even unimodular lattice of signature (26, 2)) with orthogonal complement $E_6$ (Lemma 2.12). This allows us to view $X$ (or equivalently $F(X)$) as being $E_6$ *Borcherds polarized*.[3] Using this perspective, we are able to formulate a lattice theoretic criterion (Theorem 4.5) for a Leech pair $(G, S)$ to arise as $(G, S_G(X))$ for $G = \mathrm{Aut}^s(X)$ for a smooth cubic fourfold $X$. Finally, using this criterion, the classification of fixed-point sublattices in $\mathbb{L}$ [20,24], and a case by case analysis, we are able to complete the proof of Theorem 1.2. One complication that we deal with is the possibility that a Leech pair $(G, S)$ (which is compatible with the $E_6$ Borcherds polarization) might lead to some fake cubic fourfolds, i.e., either singular cubics (with ADE singularities) or degenerations to the secant to the Veronese surface (see [32]). These form the divisors $\mathcal{C}_6$ and $\mathcal{C}_2$ excluded from the image of the period map (see Theorem 2.3). Geometrically and motivically, the divisors $\mathcal{C}_6$ and $\mathcal{C}_2$ are naturally associated with $K3$ surfaces $Y$ (or hyper-Kähler $Y^{[2]}$) of degree 6 and 2 respectively. It turns out (see Sect. 5.5) that any symplectic automorphism of a $K3$ surface of degree 2 or 6 can be lifted to an automorphism of a singular cubic fourfold $X_0$, which can then be smoothed, while preserving the automorphism. In particular, all $\mathrm{rank}(S) = 19$ cases of Theorem 1.2(8), except for the $3^{1+4} : 2.2$ case, can be recovered by starting with a $K3$ surface of degree 2 or 6 with a maximal group of symplectic automorphisms.

***Remark 1.5*** (Automorphisms of low degree $K3$ surfaces) The Kondō–Scattone trick can also be applied to polarized $K3$ surfaces. Namely, the primitive middle cohomology of a degree $d$ $K3$ surface can be embedded (up to a Tate twist) into the Borcherds lattice. The complement of this embedding is a rank 7 positive lattice $M$ with discriminant form $(-\frac{1}{d})$. For the low degree cases, degree 2, 4, and 6, $M$ can be chosen to be $E_7$, $D_7$, and $E_6 \oplus A_1$ respectively. Similarly, the elliptic $K3$ surfaces can be viewed as $E_8$ Borcherds polarized. For these cases, our arguments can be easily adapted. In particular, in Sect. 5, we discuss briefly the case of $K3$ surfaces of degree 2 and 6, as they are closely related to cubic fourfolds.

***Remark 1.6*** (Automorphisms of low dimensional cubics) The possible automorphism groups for cubic surfaces were classified by Segre [46] (see [26] for a modern and corrected account). From our perspective, the salient point is that, for smooth cubic surfaces (and similarly cubic threefolds), the induced action of the automorphism groups on the middle cohomology is faithful. This realizes the automorphism group of a cubic surface as a subgroup of $W(E_6)$. The maximal groups of automorphisms for cubic threefolds were classified recently by Wei–Yu

[3] This should be understood in the context of $M$-polarized $K3$ surfaces in the sense of Dolgachev [11], but here we use the Borcherds lattice, instead of the $K3$ lattice, as the ambient lattice. Regarding the $K3$ surfaces (or hyper-Kähler manifolds) as being Borcherds polarized is a powerful arithmetic trick well-known to experts. The first author learned about it from Kondō long time ago. Presumably, the first use of this construction occurs in the thesis of Scattone [45].

[50] via direct geometric methods (see also [17,18], [1] for some earlier results). More in the spirit of this paper, using the period map of Allcock–Carlson–Toledo [2], the classification of the automorphisms groups for cubic threefolds can be related to the Suzuki sporadic group Suz (N.B. an index 6 extension of Suz is isomorphic to the centralizer of an order 3 element in $Co_0$; see [51]).

We note that once a Leech pair $(G, S)$ as in Theorem 1.2 is specified, one obtains a moduli space $\mathcal{M}_{(G,S)}$ (see Sect. 4.1.2) of dimension $20 - \operatorname{rank}(S)$ parametrizing cubic fourfolds $X$ with $G \subset \operatorname{Aut}^s(X)$. However, it is not necessary that this moduli space is irreducible. This corresponds to $S$ having different primitive embeddings into the primitive lattice $\Lambda_0$ for cubic fourfolds (the existence of the embedding $S \hookrightarrow \Lambda_0$ is essentially the content of Theorem 1.2). It is thus a natural question to study the uniqueness of the embedding $S \hookrightarrow \Lambda_0$ for the pairs $(G, S)$ listed in Theorem 1.2. The analogous question for (unpolarized) $K3$ surfaces was studied by Hashimoto [22] (for polarized symplectic involutions, see [48]). Here, we are restricting ourselves to the maximal cases (i.e., $\operatorname{rank}(S) = 20$), as those are the most interesting cases. For instance, these cases give interesting examples of maximal algebraic cubics (in the sense of maximal possible rank for the group of algebraic cycles $H^4(X, \mathbb{Z}) \cap H^{2,2}(X)$; equivalently the transcendental lattice $T$ is negative definite of rank 2). We obtain a somewhat surprising result: while there are 6 groups that occur (cf. Theorem 1.2(9)), there are 8 cubic fourfolds (automatically isolated in moduli) corresponding to them. Six out of the eight cases are identified in [25, Table 2]; we are not able to give equations for the remaining two special cubics (cases $X^2(A_7)$ and $X^2(M_{10})$ below).

**Notation 1.7** We denote by $a^bc$ the rank 2 quadratic form $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$. We write $-(a^bc) := (-a)^{(-b)}(-c)$.

**Theorem 1.8** *Let* $(G, S)$ *be a Leech pair such that* $\operatorname{rank}(S) = 20$ *and there exists a smooth cubic fourfold X with* $G = \operatorname{Aut}^s(X)$ *and* $(G, S) \cong (G, S_G(X))$. *We denote by T the orthogonal complement of S in* $H^4_0(X, \mathbb{Z})$. *Then we have only the following possibilities:*

(1) $G \cong 3^4 : A_6$, *the corresponding cubic fourfold is the Fermat one*

$$X(3^4 : A_6) = V(x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3)$$

*and* $T = -(6^36) = A_2(-3)$. *Moreover, this is the only smooth cubic fourfold with a symplectic automorphism of order* 9. *In this case* $\operatorname{Aut}(X)/\operatorname{Aut}^s(X) \cong 6$.

(2) $G \cong A_7$, *there are two smooth cubic fourfolds with symplectic action of G. One of them is*

$$X^1(A_7) = V(x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 - (x_1 + x_2 + x_3 + x_4 + x_5 + x_6)^3)$$

*with* $T = -(2^118)$ *and* $\operatorname{Aut}(X^1)/\operatorname{Aut}^s(X^1) \cong 2$. *The other one, denoted* $X^2(A_7)$, *has* $T = -(18^318)$ *and admits no non-symplectic automorphisms.*

(3) $G \cong 3^{1+4} : 2.2^2$, *the cubic fourfold is*

$$X(3^{1+4} : 2.2^2) = V(x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 - 3(\sqrt{3} + 1)(x_1x_2x_3 + x_4x_5x_6))$$

*and* $T = -(6^06) = (A_1 \oplus A_1)(-3)$. *Moreover, this is the only smooth cubic fourfold with a symplectic automorphism of order* 12. *In this case* $\operatorname{Aut}(X)/\operatorname{Aut}^s(X) \cong 4$.

(4) $G \cong M_{10}$, *there are two smooth cubic fourfolds with symplectic action of G, and both of them have* $T = -(12^030)$. *See Equation (4.3) for an explicit description of one such cubic fourfold, which is denoted by* $X^1(M_{10})$. *The other one is denoted by* $X^2(M_{10})$. *Both* $X^1(M_{10})$ *and* $X^2(M_{10})$ *have no non-symplectic automorphisms.*

(5) $G \cong L_2(11)$, *the cubic fourfold is*

$$X(L_2(11)) = V(x_1^3 + x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_6 + x_6^2x_2)$$

*and* $T = -(22^{11}22) = A_2(-11)$. *Moreover, this is the only smooth cubic fourfold with a symplectic automorphism of order* 11. *In this case* $\mathrm{Aut}(X)/\mathrm{Aut}^s(X) \cong 3$.

(6) $G \cong A_{3,5}$, *the cubic fourfold is*

$$X(A_{3,5}) = V(x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 + x_7^3 + x_8^3) \cap V(x_1 + x_2 + x_3) \\ \cap V(x_4 + x_5 + x_6 + x_7 + x_8)$$

*and* $T = -(10^5 10) = A_2(-5)$. *Moreover, this is the only smooth cubic fourfold with a symplectic automorphism of order* 15. *In this case* $\mathrm{Aut}(X)/\mathrm{Aut}^s(X) \cong 6$.

***Remark 1.9*** The transcendental lattice $T$ for cubic fourfolds with nontrivial symplectic automorphisms is relatively small (of rank at most $22 - \mathrm{rank}(S)$). It follows that except the case of symplectic involutions (i.e. Theorem 1.2(1)), $T$ embeds into the $K3$ lattice $(E_8)^2 \oplus U^3$ (cf. [33, Prop. 2.5, Cor. 2.9]). Thus, a priori, the cubics with large group of symplectic automorphisms are not interesting from the perspective of the standard rationality conjectures (we refer to [33] and [44] for further discussion on the subject). Nonetheless, they are Hodge theoretically interesting as the cases of Theorem 1.8 give examples of maximally algebraic cubics (i.e., with maximal rank for $H^{2,2}(X) \cap H^4(X, \mathbb{Z})$) for which the transcendental lattice is explicitly known.

## Structure of the paper

In Sect. 2, we introduce and briefly review the properties of the period map for cubic fourfolds. Additionally, in Sect. 2.4, we review the notion of Borcherds marking for cubic fourfolds. In the following Sect. 3, we review the necessary material on the Leech lattice, Niemeier lattices, and Conway group. These two review sections (specific to our situation) are complemented by two appendix sections, which cover very standard material, but which nonetheless might be helpful to the reader. Specifically, in Appendix A, we collect results in lattice theory (mostly due to Nikulin) which are essential in our arguments. In Appendix B, we review some basic facts and notation for finite groups.

The main content of the proof of Theorem 1.2 is discussed in Sects. 3 and 4. First, following Mongardi's work, we introduce the notion of Leech pair (Definition 3.3), and give a key lemma (Lemma 3.4). We then focus on the polarized case. In particular, we establish a criterion (Theorem 4.5) for a Leech pair $(G, S)$ to arise from a group of symplectic automorphisms for some cubic fourfold $X$. In §4.5 we prove Theorem 1.8 using methods from lattice theory.

The remaining two sections are complementing our main classification result. Namely, in Sect. 5, we partially discuss the completely analogous (and somewhat easier) situation for $K3$ surfaces of degree 2 and 6. Finally, while the focus of this paper is on symplectic automorphisms, we make some comments on the non-symplectic case in Sect. 6. In particular, we determine the full automorphism groups of the 8 maximal cases of Theorem 1.8 (see Proposition 6.12). This allows us to distinguish geometrically the two cases of Theorem 1.8(2) with $\mathrm{Aut}^s(X) \cong A_7$ (i.e., one has an anti-symplectic involution, while the other does not). As a consequence of this classification, we also conclude that the maximal possible order of automorphism group for a cubic fourfold is 174,960, which is reached only by the Fermat cubic fourfold (an analogous result for $K3$ surfaces was obtained by Kondō [30]).

After the posting of our manuscript, we have learned of the work of Ouchi [44], who explores the interplay between automorphisms of cubic fourfolds and the automorphisms of

the associated K3 category (the Kuznetsov component). We thank Ouchi for sharing an early version of his work, and for some comments on our paper. We are also grateful to S. Mukai for sharing with us some of his partial work on the classification of automorphisms of cubic fourfolds (from late eighties). As a consequence, we have updated some of our notation (and added some remarks) to be aligned with Mukai's work. While preparing a final revision of our paper, Höhn explained to us a strategy for obtaining the equations for all the maximal cases of Theorem 1.8. We expect details to appear elsewhere. Finally, we are very grateful to the anonymous referee for numerous detailed comments which helped us improve the paper.

## 2 Automorphisms and periods

In this section we review some well-known facts, which are the starting point of our classification of the automorphism groups for cubic fourfolds. First, the Global Torelli Theorem (Theorem 2.3 and Proposition 2.4) allows one to reduce the classification of automorphisms for cubic fourfolds to the classification of automorphisms of Hodge structures, which in turn is essentially a lattice theoretic question. Classically, this approach was successfully applied to the case of $K3$ surfaces (Nikulin, Mukai, Kondō and others). More recently, it was (partially) adapted to the case of hyper-Kähler manifolds of $K3^{[n]}$ type. The Fano variety $F(X)$ of a cubic fourfold $X$ is a hyper-Kähler of $K3^{[2]}$ type. Thus, the classification of automorphisms of $X$ is closely related to the classification of automorphisms of $F(X)$. We review this in Sect. 2.3 below. Finally, the difference to most of related work that we cite is that we need to keep track of the polarization. It turns out that it is better to keep track of a "Borcherds polarization" instead of the natural polarization of $X$ (or equivalently $F(X)$). We introduce this notion in Sect. 2.4.

### 2.1 Periods for cubics

Let $X$ be a smooth cubic fourfold. The middle cohomology group $H^4(X,\mathbb{Z})$, with the natural intersection pairing, is a unimodular odd lattice $\Lambda$ of signature $(21,2)$ (uniquely specified by these conditions). Let $\eta_X \in H^4(X,\mathbb{Z})$ be the square of the hyperplane class of $X$. The primitive cohomology $H^4(X,\mathbb{Z})_{prim} = \langle \eta_X\rangle^{\perp}$ carries a polarized Hodge structure of $K3$ type (i.e., Hodge numbers $(0,1,20,1,0)$). As lattice, $H^4(X,\mathbb{Z})_{prim} \cong \Lambda_0$ where $\Lambda_0 := (E_8)^2 \oplus U^2 \oplus A_2$ (with $A_2$ and $E_8$ the standard root lattices, and $U$ the hyperbolic plane). Similarly to the well-known case of $K3$ surfaces, the period domain for Hodge structures on $H^4(X,\mathbb{Z})_{prim}$ is the 20-dimensional type IV period domain

$$\mathcal{D} = \{x \in \mathbb{P}((\Lambda_0)_{\mathbb{C}}) \big| (x,x)=0, (x,\overline{x})<0\}^{+}$$

(where the superscript $+$ indicates a choice of one of the two connected components).

Associated to the lattice $\Lambda_0$, there are several natural groups:

(1) $\mathscr{O}(\Lambda_0)$ the automorphism group of lattice $\Lambda_0$;
(2) $\widetilde{\mathscr{O}}(\Lambda_0)$ the subgroup of $\mathscr{O}(\Lambda_0)$ which acts trivially on the discriminant group $A_{\Lambda_0}(=(\Lambda_0)^{\vee}/\Lambda_0 \cong 3)$;
(3) $\mathscr{O}^{+}(\Lambda_0)$ the subgroup of $\mathscr{O}(\Lambda_0)$ which preserves the spinor norm on $\Lambda_0$ (or equivalently preserves $\mathcal{D}$);
(4) $\mathscr{O}^{*}(\Lambda_0) := \mathscr{O}^{+}(\Lambda_0) \cap \widetilde{\mathscr{O}}(\Lambda_0)$.

The global monodromy group $\Gamma$ for cubic fourfold is $\mathscr{O}^{*}(\Lambda_0)$ (cf. Beauville [5]). Since $\Gamma = \mathscr{O}^{*}(\Lambda_0)$ is an arithmetic group, $\Gamma$ acts properly discontinuously on $\mathcal{D}$. The resulting analytic variety $\mathcal{D}/\Gamma$ is in fact a quasi-projective variety; we refer to it as the global period domain for cubic fourfolds.

**Definition 2.1** (i) A norm 2 vector $v$ in $\Lambda_0$ is called a *short root*. The set of short roots in $\Lambda_0$ determines a $\Gamma$-invariant hyperplane arrangement $\mathcal{H}_6$ in $\mathcal{D}$. Let $\mathcal{C}_6 := \mathcal{H}_6/\Gamma \subset \mathcal{D}/\Gamma$ be the associated Heegner divisor.
(ii) A norm 6 vector $v$ in $\Lambda_0$ with divisibility 3 is called a *long root*. The set of long roots in $\Lambda_0$ determines a $\Gamma$-invariant hyperplane arrangement $\mathcal{H}_2$ in $\mathcal{D}$. Let $\mathcal{C}_2 := \mathcal{H}_2/\Gamma \subset \mathcal{D}/\Gamma$.

***Remark 2.2*** It is well known that there exists a single $\{\pm 1\} \times \Gamma$-orbit of short and long roots respectively, and thus $\mathcal{C}_6$ and $\mathcal{C}_2$ are irreducible divisors. Furthermore, $\Gamma (= \mathscr{O}^*(\Lambda_0))$ is generated by reflections in short roots [5], and $\Gamma$ has index 2 in $\widetilde{\mathscr{O}}(\Lambda_0)$ with $\widetilde{\mathscr{O}}(\Lambda_0)/\Gamma$ generated by the class of a reflection in a long root.

Let $\mathcal{M}$ be the moduli space of smooth cubic fourfolds. It is a quasi-projective 20-dimensional variety, which can be constructed by GIT (see [31] for a full GIT analysis). By associating with a cubic fourfold $X$, the Hodge structure on its middle cohomology, one obtains a period map

$$\mathscr{P} : \mathcal{M} \longrightarrow \mathcal{D}/\Gamma.$$

Voisin [49] proved that the Global Torelli Theorem is valid for cubic fourfolds. It follows that $\mathscr{P}$ is an open embedding. For the purpose of this paper, it is important to understand also the image of the period map $\mathscr{P}(\mathcal{M}) \subset \mathcal{D}/\Gamma$. This type of question was first investigated by Hassett [23]. In particular, he defined certain Heegner divisors $\mathcal{C}_d$ in $\mathcal{D}/\Gamma$ (indexed by $d \in \mathbb{Z}_+$ with $d \equiv 0, 2 \pmod 6$) corresponding to cubic fourfolds containing additional Hodge classes. The relevant divisors here are $\mathcal{C}_2 = \mathcal{H}_2/\Gamma$ and $\mathcal{C}_6 = \mathcal{H}_6/\Gamma$ as defined above. Geometrically, $\mathcal{C}_6$ corresponds to singular cubic fourfolds, while $\mathcal{C}_2$ correspond to degenerations of cubics to the secant variety of the Veronese surface in $\mathbb{P}^5$. The image of the period map misses the divisors $\mathcal{C}_2$ and $\mathcal{C}_6$. Conversely, as shown by Laza [32] and Looijenga [36], any period outside these two divisors is realized for some smooth cubic fourfold.

**Theorem 2.3** (Voisin, Hassett, Laza, Looijenga) *The period map for cubic fourfolds gives an isomorphism of quasi-projective varieties*

$$\mathcal{P} : \mathcal{M} \xrightarrow{\sim} (\mathcal{D} \setminus (\mathcal{H}_2 \cup \mathcal{H}_6))/\Gamma. \tag{2.1}$$

We note that both sides of (2.1) have natural orbifold structures. For instance, since any smooth cubic fourfold is GIT stable [31], the moduli space of smooth cubic fourfolds is a smooth Deligne–Mumford stack $\mathfrak{M}$ with quasi-projective coarse moduli space $\mathcal{M}$. A natural question is whether the period map $\mathscr{P}$ identifies the two sides of (2.1) as orbifolds. This is equivalent to the strong global Torelli theorem, i.e., the statement that any isomorphism between the polarized Hodge structures of two smooth cubic fourfolds is induced by a unique isomorphism between the two cubic fourfolds. Using the fact that automorphisms of cubic fourfolds $X$ are induced by linear transformations of the ambient projective space $\mathbb{P}^5$, and that $\mathrm{Aut}(X)$ acts faithfully on the middle cohomology $H^4(X, \mathbb{Z})$ (e.g., [28, Proposition 2.16]), the second author [55] has verified the Strong Global Torelli Theorem.

**Proposition 2.4** [55] *Let $X_1$ and $X_2$ be two smooth cubic fourfolds. Assume that there is an isomorphism*

$$\varphi : H^4(X_2, \mathbb{Z}) \cong H^4(X_1, \mathbb{Z})$$

*of polarized Hodge structures (in particular $\varphi(\eta_{X_2}) = \eta_{X_1}$). Then, there exists a unique isomorphism $f\colon X_1 \cong X_2$ such that $\varphi = f^*$. In particular, for any smooth cubic fourfold $X$,*

$$\mathrm{Aut}(X) \cong \mathrm{Aut}_{HS}(H^4(X,\mathbb{Z}), \eta_X), \tag{2.2}$$

*where $\mathrm{Aut}_{HS}$ stands for group of Hodge isometries.*

***Remark 2.5*** We note that while the period map extends to an isomorphism of quasi-projective varieties

$$\mathcal{M}^{\mathrm{ADE}} \cong (\mathcal{D} \setminus \mathcal{H}_2)/\,\Gamma$$

where $\mathcal{M}^{\mathrm{ADE}}$ is the moduli space of cubics with ADE singularities (see [31,32]), the orbifold structure along the discriminant divisor is different. Simply, a general cubic fourfold with a node (i.e., $A_1$ singularity) has no automorphism, while on the periods side, there is a $\mathbb{Z}/2$ stabilizer corresponding to the reflection in a short root.

## 2.2 Moduli space of lattice-polarized cubic fourfolds

Let $M$ be a positive definite lattice with a fixed primitive embedding into the primitive cubic lattice $\Lambda_0$. Assume that $M$ does not contain short or long roots. Inspired by Dogachev's theory of $M$-polarized $K3$ surfaces (cf. [11]), we define a moduli space $\mathcal{M}_M$ of cubic fourfolds containing a specified lattice $M$ as a sublattice of the lattice of primitive algebraic cycle. We note however that there is a slight difference to [11], namely for our purposes it is better to view $\mathcal{M}_M$ as a subspace of the moduli space of cubic fourfolds $\mathcal{M}$ (in particular, it can be non-normal). In contrast, Dolgachev's moduli space is a locally symmetric variety of type $\mathcal{D}_M/\,\Gamma_M$, and in particular normal. As explained below, $\mathcal{M}_M$ is essentially the image in $\mathcal{M}$ of $\mathcal{D}_M/\,\Gamma_M$.

As a set, $\mathcal{M}_M$ is the subspace of $\mathcal{M}$ consisting of smooth cubic fourfolds $X$ with an embedding $M \subseteq H^{2,2}(X) \cap H^4(X,\mathbb{Z})_{prim} \subset H^4(X,\mathbb{Z})_{prim} \cong \Lambda_0$ such that the composition $M \subset \Lambda_0$ is equivalent to the fixed embedding. The image $\mathcal{P}(\mathcal{M}_M)$ of $\mathcal{M}_M$ under the period map is a closed subset of $(\mathcal{D} \setminus (\mathcal{H}_2 \cup \mathcal{H}_6))\,/\,\Gamma$ (in fact, it is the Noether-Lefschetz cycle associated with $M \hookrightarrow \Lambda_0$). In particular, the space $\mathcal{M}_M$ admits a natural structure of a quasi-projective variety.

The variety $\mathcal{M}_M$ may be not normal. In any case, the normalization of $\mathcal{M}_M$ is (the complement of some Hegneer divisors in) a locally symmetric variety $\mathcal{D}_M/\,\Gamma_M$, where $\mathcal{D}_M$ is the type IV domain associated with the transcendental lattice $T = M^{\perp}_{\Lambda_0}$, and $\Gamma_M$ an arithmetic group acting on $\mathcal{D}_M$. We refer to [54, Proposition A.5] for further details. In particular, $\dim \mathcal{M}_M = 20 - \mathrm{rank}(M)$. Furthermore, if $M \subset M' \subset \Lambda_0$ (primitive embeddings) then $\mathcal{M}_{M'} \subset \mathcal{M}_M$ (i.e., the more algebraic cycles, the smaller the moduli). The moduli of cubic fourfolds $\mathcal{M}$ corresponds to $M = \emptyset$, and the Hassett divisors $\mathcal{C}_d$ correspond to $\mathrm{rank}(M) = 1$. (Equivalently, as in Hassett's work, one can consider the full lattice of algebraic cycles $\widetilde{M} = \mathrm{Sat}(M \oplus \langle\eta\rangle)_\Lambda \subset \Lambda \cong H^4(X,\mathbb{Z})$. Here, it is more convenient to work with the primitive lattices $M$ and $\Lambda_0$.)

## 2.3 The hyper-Kähler fourfold associated with a cubic fourfold *X*

For a smooth cubic fourfold $X$, the Fano variety $F(X)$ of lines on $X$ is a smooth hyper-Kähler fourfold, deformation equivalent to $K3^{[2]}$ (cf. [4]). There is a natural polarization on

$F(X)$ induced from the Plücker embedding $F(X) \hookrightarrow \mathrm{Gr}(1, \mathbb{P}^5) \subset \mathbb{P}(\wedge^2(\mathbb{C}^6))$. Since any automorphism of $X$ is linear, there is a natural group homomorphism

$$\mathrm{Aut}(X) \longrightarrow \mathrm{Aut}(F(X)).$$

Conversely, the following holds (e.g., [14, Corollary 2.3]):

**Proposition 2.6** *The homomorphism* $\mathrm{Aut}(X) \longrightarrow \mathrm{Aut}(F(X))$ *is injective with image the subgroup preserving the Plücker polarization on* $F(X)$.

An automorphism of a hyper-Kähler manifold sends $H^{2,0}$ to $H^{2,0}$, hence induces a scalar action on $H^{2,0}$. If the scalar is the identify, the automorphism is called *symplectic*. Otherwise, it is called *non-symplectic*. Adapting this to the case of cubic fourfolds, we make the following definition:

**Definition 2.7** An automorphism of a smooth cubic fourfold $X$ is called *symplectic*, iff the induced automorphism on $F(X)$ is symplectic. Equivalently, an automorphism of $X$ is symplectic iff the induced action on $H^{3,1}(X)$ is the identity. We denote the group of symplectic automorphisms of $X$ by $\mathrm{Aut}^s(X)$.

***Remark 2.8*** In view of Theorem 2.3 and Proposition 2.4, it is clear that essential arithmetic input in the classification of automorphisms of cubic fourfolds is the primitive cohomology lattice $\Lambda_0 = H^4(X, \mathbb{Z})_{prim} \cong A_2 \oplus (E_8)^2 \oplus U^2$. Let us note that the associated hyper-Kähler $F(X)$ has the same primitive lattice. More precisely, $H^2(F(X), \mathbb{Z})$ carries a natural quadratic form, the so-called Beauville–Bogomolov quadratic form. With respect to this form, there is a natural lattice isometry $H_0^2(F(X), \mathbb{Z})(-1) \cong H_0^4(X, \mathbb{Z})$, which is also an isomorphism of Hodge structures (see [4, Proposition 6]). In particular, via this isomorphism $H^{2,0}(F(X))$ maps to $H^{3,1}(X)$, justifying our definition above. In summary, the discussion of this subsection says that the classification of the automorphisms of cubic fourfolds is essentially equivalent to the classification of automorphisms of degree 6 (the degree of the Plücker polarization) polarized hyper-Kähler manifolds of $K3^{[2]}$ type.

***Remark 2.9*** One should note that there is a subtle difference to the case of $K3$ surfaces. While for $K3$ surfaces the full cohomology lattice $H^2(S, \mathbb{Z})$ is even unimodular, the full cohomology lattice for cubic fourfolds $H^4(X, \mathbb{Z})$ is odd unimodular. If one prefers to work with hyper-Kähler manifolds of $K3^{[2]}$ type, we note that the full cohomology lattice (w.r.t. the Beauville–Bogomolov form) is even, but not unimodular (it is (up to sign) $A_1 \oplus (E_8)^2 \oplus U^3$).

## 2.4 Borcherds polarizations

In view of Nikulin's theory [43], it is preferable to work with even unimodular lattices (compare Remark 2.9). The smallest (with definite orthogonal complement) even unimodular lattice that contains the primitive cubic lattice $\Lambda_0$ is the *Borcherds lattice* $\mathbb{B}$, i.e., the unique even unimodular lattice $\mathrm{II}_{26,2} \cong (E_8)^3 \oplus U^2$ of signature $(26, 2)$. (Here, we prefer to denote it $\mathbb{B}$ and call it the Borcherds lattice in honor of Borcherds, who studied the automorphic forms on the associated type IV symmetric domain.)

***Remark 2.10*** Even in the $K3$ case, the embedding of the primitive cohomology lattice for a polarized $K3$ surface into the Borcherds lattice $\mathbb{B}$ turns out to be a powerful arithmetic trick (the geometric reason why it works is not yet completely understood). As examples of applications of this artifice (that we baptized *Kondō–Scattone trick*), we mention Scattone's work [45] on the Baily–Borel compactification for polarized $K3$ surfaces, Kondō's work

[29] on symplectic automorphisms, and the Gristsenko–Hulek–Sankaran work [19] on the Kodaira dimensions on the moduli spaces of $K3$ surfaces.

***Remark 2.11*** We recall that there exist 24 even unimodular lattices of rank 24, called the *Niemeier lattices* (see Sect. 3.1 below). What is relevant to note here is that these lattices are intricately related to the Borcherds lattice $\mathbb{B}$. Namely, for any Niemeier lattice $N$, we have $\mathbb{B} \cong N \oplus U^2$. Conversely, the classification of the Niemeier lattices follows from the classification of isotropic vectors in the hyperbolic lattice $\mathrm{II}_{25,1}$ (see [8]), or equivalently the type II boundary components (i.e., rank 2 totally isotropic subspaces in $\mathbb{B}$) of the Baily–Borel compactification for the Borcherds period domain.

Returning to cubic fourfolds, in analogy with the work of $M$-polarized $K3$ surfaces of Dolgachev [11], we can view a cubic fourfold as being *Borcherds $E_6$-polarized* (i.e., $\Lambda_0$ admits a primitive embedding into $\mathbb{B}$ with orthogonal complement $E_6$). More interestingly, the periods missing from the image of the period map for cubic fourfolds (see Theorem 2.3), i.e., the divisors $\mathcal{C}_2$ and $\mathcal{C}_6$, correspond to $E_7$ and $E_6 + A_1$ Borcherds polarizations respectively. This allows a more uniform view on "singular" cubic fourfolds (i.e., singular cubics, or degenerations to the Veronese surface) – simply $X$ is singular if it acquires an additional root (i.e., the existing "algebraic" lattice $E_6$ is enlarged to either $E_7$ or $E_6 + A_1$ by adding a root). This is of course equivalent to the more classical view of Hassett [23] where $H^4_{alg,prim} = H^4(X,\mathbb{Z})_{prim} \cap H^{2,2}$ acquires a short root (equivalently, in terms of Borcherds polarizations $E_6 \subset E_6 + A_1$) or long root (case $E_7$). From either perspective, the transcendental lattices for the two cases $\mathcal{C}_2$ and $\mathcal{C}_6$ are

$$\Lambda_2 := \langle 2\rangle \oplus (E_8)^2 \oplus U^2, \text{ and}$$
$$\Lambda_6 := \langle 6\rangle \oplus (E_8)^2 \oplus U^2$$

respectively, where the transcendental lattice for Borcherds $R$-polarized objects is defined as $R^{\perp}_{\mathbb{B}}$. One recognizes $\Lambda_2$ and $\Lambda_6$ (up to a sign) as the primitive lattices for $K3$ surfaces of degree 2 and 6 respectively. There is indeed a close geometric relationship between degree 6 (and respectively degree 2) $K3$ surfaces and singular cubic fourfolds (respectively degenerations to the Veronese surface); see [23,32].

From the perspective of this paper, the relevant fact is the following easy proposition (see [32, §6]).

**Proposition 2.12** (i) *There is a unique primitive embedding of $\Lambda_0$ into $\mathbb{B}$, with orthogonal complement $E_6$; in another words, $\Lambda_0 \oplus E_6$ can be saturated as $\mathbb{B}$ in a unique way.*

(ii) *There is a unique primitive embedding of $\Lambda_6$ into $\mathbb{B}$, with orthogonal complement $A_1 \oplus E_6$; in another words, $\Lambda_0 \oplus A_1 \oplus E_6$ can be saturated as $\mathbb{B}$ in a unique way.*

(iii) *There is a unique primitive embedding of $\Lambda_2$ into $\mathbb{B}$, with orthogonal complement $E_7$; in another words, $\Lambda_0 \oplus E_7$ can be saturated as $\mathbb{B}$ in a unique way.*

## 3 Automorphisms and the Conway group

Via the Global Torelli Theorem, we have reduced the study of automorphisms for cubic fourfolds to the study of automorphisms of Hodge structures. This is in turn a question about the symmetries (satisfying certain properties) of the underlying cohomology lattice $L$.

**Notation 3.1** For a lattice $L$ with action by a group $G \subset O(L)$, we call $L^G := \{x \in L \big| gx = x, \forall g \in G\}$ *the invariant sublattice*, and $S_G(L) := (L^G)^{\perp}_L$ *the covariant lattice*.[4]

In the case of a finite group of *symplectic* automorphisms $G$ acting on the cohomology lattice $L$ of a $K3$ surface, Nikulin made two key observations:

(i) *the covariant lattice* $S_G(L)$ *is a definite lattice* (this is equivalent to the symplectic condition), and
(ii) $S_G(L)$ *does not contain any* $\underline{effective}$ *algebraic cycle* (in fact, the symplectic condition implies that the algebraicity is automatic). In particular, for $K3$ surfaces, by Riemann–Roch, $S_G(L)$ (which is negative definite in this case) should not contain any $-2$ classes (or equivalently roots).

The same holds for hyper-Kähler manifolds of $K3^{[n]}$ type (e.g., by involving Markman's theory of prime exceptional divisors) and for cubic fourfolds (i.e., there is no norm 2 vector in $S_G(L)$; e.g., as a consequence of Theorem 2.3). Normally, one would try to classify $S_G(L)$ and its embeddings into the cohomology lattice $L$. However, using Nikulin's theory, Kondō made the observation that (in the geometric situations considered here: $K3$s, $K3^{[n]}$, or cubics) $S_G(L)$ *embeds into one of the Niemeier lattices* $N$, and furthermore $G$ *extends to an isometry of* $N$ (thus $G \subset O(N)$). Niemeier lattices $N$ show up here since they are the smallest even unimodular definite lattices $N$ containing $S_G(L)$ for any $G$. The lattice $N$ being definite is important as the associated orthogonal group $O(N)$ is finite. Kondō [29] successfully applied this approach to the classification of symplectic automorphisms for $K3$ surfaces. Kondō avoids the Leech lattice $\mathbb{L}$ (namely, he noted that $A_1 \oplus S_G(L)$ embeds into $N$ for $K3$ surfaces, and thus $N \neq \mathbb{L}$), but in fact, since $S_G(L)$ contains no roots, it is possible to embed it into the Leech lattice $\mathbb{L}$ (cf. [15,27]). Considering embeddings into the Leech lattice $\mathbb{L}$ leads to a more uniform behavior. Note however that there is a trade-off here: we deal with a single larger group $\mathrm{Co}_0 := O(\mathbb{L})$ versus 23 smaller groups $O(N)$ for $N \neq \mathbb{L}$. With the advent of more powerful computational tools, and a better understanding of the Leech lattice (esp. relevant here is [24]), we can work throughout with the Leech lattice.

In this section, we briefly review the Leech lattice, the Conway group, and introduce the key concept (due to Mongardi, but with origins going back to Nikulin) of *Leech pair*. We then close with the Höhn–Mason [24] classification of the fixed-point lattices for the Leech lattice $\mathbb{L}$. The material here is standard (and applies equally to $K3$s and $K3^{[n]}$s); we will apply it in the following section to the actual classification of the automorphisms of cubic fourfolds.

### 3.1 The Leech lattice and the Conway group

We recall the following classification result of Niemeier.

**Theorem 3.2** (Niemeier) *Up to isometry, there exist* 24 *even unimodular positive definite lattices* $N$ *of rank* 24. *Let* $R \subset N$ *be the sublattice spanned by the roots (i.e., norm* 2 *vectors) of* $N$. *Then* $R$ *is of one of the following* 24 *types:* $\varnothing, 24A_1, 12A_2, 8A_3, 6A_4, 6D_4, 4A_5 \oplus D_4, 4A_6, 2A_7 \oplus 2D_5, 3A_8, 4D_6, 2A_9 \oplus D_6, 4E_6, A_{11} \oplus D_7 \oplus E_6, 2A_{12}, 3D_8, A_{15} \oplus D_9, D_{10} \oplus 2E_7, A_{17} \oplus E_7, 2D_{12}, A_{24}, 3E_8, D_{16} \oplus E_8, D_{24}$. *In particular,* $R$ *uniquely determines* $N$.

A lattice $N$ as in the theorem is called a *Niemeier lattice*. In all but one of the cases $N$ is spanned (over $\mathbb{Q}$) by roots. The remaining case, i.e., the Niemeier lattice containing no roots,

[4] Some authors call the quotient $M/S_G(M)$ the covariant lattice, since it is the maximal quotient such that the induced action of $G$ on it is trivial.

is called the *Leech lattice*, and we denote it by $\mathbb{L}$. The automorphism group of the Leech lattice is the *Conway group*

$$\mathrm{Co}_0 := O(\mathbb{L}).$$

The center of $\mathrm{Co}_0$ is just $\mu_2 = \{\pm \mathrm{id}\}$, and the quotient

$$\mathrm{Co}_1 := \mathrm{Co}_0 / Z(\mathrm{Co}_0)$$

is one of the largest sporadic simple groups. In fact,

$$|\mathrm{Co}_0| = 2^{22} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23 (\sim 8 \cdot 10^{18}).$$

As we will see below, a group $G$ of symplectic automorphisms for $K3$ surfaces, hyper-Kähler manifolds of type $K3^{[n]}$, or cubic fourfolds can be realized as a subgroup of the Conway group $\mathrm{Co}_0$. Thus, only the prime factors 2, 3, 5, 7, 11, 13, and 23 can occur in $\mathrm{ord}(G)$. For $K3$ surfaces, only the primes $p \leq 7$ can occur, while for cubics all primes $p \leq 11$ occur (compare Theorem 4.15). In particular, the Fano variety $F(X)$ of a cubic fourfold $X$ admitting an order 11 symplectic automorphism will give an example of an exotic automorphism (i.e., not induced from $K3$ surfaces) on a hyper-Kähler of $K3^{[2]}$ type (see [38, §4.5]).

### 3.2 Leech pairs

As already mentioned, the study of symplectic automorphisms on $K3$s and $K3^{[n]}$'s leads to the following notion (first formalized in the thesis of Mongardi [38]):

**Definition 3.3** A pair $(G, S)$ consisting of a finite group $G$ acting faithfully on an even lattice $S$ is called a *Leech pair*, if it satisfies the following conditions:

(i) $S$ is positive definite,
(ii) $S$ does not contain any 2-vector,
(iii) $G$ fixes no nontrivial vector in $S$,
(iv) the induced action of $G$ on the discriminant group $A_S$ is trivial.

The condition (iv) of the Definition 3.3 should be understood as saying that given a primitive embedding $S \hookrightarrow L$ into a unimodular lattice $L$, the action of $G$ on $S$ extends to $L$ in such a way that, the restriction of the extended action to $S_L^{\perp}$ is trivial. The condition (iii) complements this by saying that $S$ is the covariant lattice for the action of $G$ on $L$. Note then that the smallest unimodular lattice satisfying the first 2 conditions of the definition above is the Leech lattice $\mathbb{L}$. Obviously, any sublattice of the Leech lattice will also satisfy (i) and (ii) of the definition. Thus choosing a subgroup $G \subset \mathrm{Co}_0 (= O(\mathbb{L}))$, the associated covariant lattice $S_G(\mathbb{L})$ in $\mathbb{L}$ will give an example of Leech pair $(G, S_G(\mathbb{L}))$. The next proposition says the converse: under a mild condition on $S$ (satisfied in the geometric context relevant to this paper), any Leech pair $(S, G)$ is obtained as a covariant lattice in $\mathbb{L}$. This argument seems to occur first in [15, Appendix B] (see also [27, Prop. 2.2]; related arguments go back to Scattone [45] and Kondō [29]). For completeness, we sketch the proof. Recall that for a lattice $S$, we denote by $A_S$ the associated discriminant group, and by $l(A_S)$ the minimal number of generators in $A_S$, see Sect. A.3 for more details.

**Proposition 3.4** *For a Leech pair $(G, S)$ the following two statements are equivalent:*

(i) $\mathrm{rank}(S) + l(A_S) \leq 24$,
(ii) *There exists a primitive embedding of $S$ into the Leech lattice $\mathbb{L}$.*

*Once these two condition are fulfilled, there is an action of G on $\mathbb{L}$ with $(G, S) \cong (G, S_G(\mathbb{L}))$.*

***Proof*** Assume (ii), and denote by $K$ the orthogonal complement of the given primitive embedding of $S$ into $\mathbb{L}$. Then $l(A_S) = l(A_K) \leq \mathrm{rank}(K) = 24 - \mathrm{rank}(S)$. Thus (ii) implies (i).

Now assume (i). Since $l(A_S) \leq 24 - \mathrm{rank}(S) < \mathrm{rank}(\mathbb{L} \oplus U) - \mathrm{rank}(S)$, by Nikulin's existence Theorem A.8, there exists a primitive embedding $S \hookrightarrow \mathbb{L} \oplus U$. Denote by $N$ the orthogonal complement of $S$ in $\mathbb{L} \oplus U$. Then $N$ has signature $(25 - \mathrm{rank}(S), 1)$. Thus $N_{\mathbb{R}}$ intersects with the positive cone of $\mathbb{L} \oplus U$. Since $S$ contains no 2-vector, $N_{\mathbb{R}}$ intersects with one of the chambers of the positive cone of $\mathbb{L} \oplus U$.

Let $w \in U$ be primitive and isotropic. The vector $w \in \mathbb{L} \oplus U$ is called a Weyl vector.[5] We call a vector $v \in \mathbb{L} \oplus U$ with $(v, v) = 2$ and $(v, w) = -1$ a Leech root. By [8, Chap. 27], the automorphism group of $\mathbb{L} \oplus U$ is generated by reflections with respect to Leech roots. Therefore, there exists a chamber $\mathcal{C}_0$ given by $\mathcal{C}_0 = \{x \in (\mathbb{L} \oplus U) \otimes \mathbb{R} \big| (x, v) > 0$, for any Leech root v$\}$. By adjusting the embedding $S \hookrightarrow \mathbb{L} \oplus U$ via an automorphism of $\mathbb{L} \oplus U$, we may assume that $N_{\mathbb{R}}$ intersects with $\mathcal{C}_0$, hence $G$ leaves the chamber $\mathcal{C}_0$ stable. By [7], $G$ fixes the Weyl vector $w$. Equivalently, $w \in N$. Then we have:

$$S \hookrightarrow w^{\perp} \longrightarrow w^{\perp}/\langle w \rangle \cong \mathbb{L}$$

which gives rise to a primitive embedding of $S$ into the Leech lattice $\mathbb{L}$. The group action of $G$ on $S$ extends to an action on $\mathbb{L}$ with $(G, S) \cong (G, S_G(\mathbb{L}))$. □

**Corollary 3.5** *For a Leech pair $(G, S)$ satisfying the statements in Lemma 3.4, there is an embedding $G \hookrightarrow \mathrm{Co}_0$, with image avoiding $-id$ unless* $\mathrm{rank}(S) = 24$.

### 3.3 Höhn–Mason classification of saturated Leech pairs

In view of the discussion above, to classify the Leech pairs relevant to the classification of automorphisms, one can proceed by considering subgroups $G \subset \mathrm{Co}_0$ and the associated covariant lattices $S_G(\mathbb{L})$. The only issue is that there might be several groups $G$ leading to the same covariant lattice. For the classification of automorphism groups, we are interested in the maximal cases (i.e., in $G = \mathrm{Aut}^s(X)$ and not subgroups $G' \subset G$ that happen to have the same invariant/covariant lattice). The following two definitions formalize this idea.

**Definition 3.6** A Leech pair (G, S) is called *saturated*, if $G$ is the maximal group acting faithfully on $S$ and trivially on the discriminant group $A_S$.

Let $G$ be a finite group acting on the Leech lattice $\mathbb{L}$. One can consider the (point-wise) stabilizer $G'$ of $\mathbb{L}^G$. Obviously, $G \subseteq G'$, $\mathbb{L}^G = \mathbb{L}^{G'}$, and $G'$ is the largest group stabilizing $\mathbb{L}^G$. The induced action of $G'$ on $A_{S_G(\mathbb{L})} \cong A_{\mathbb{L}^G}$ is trivial. Conversely, every automorphism of $S_G(\mathbb{L})$ which trivializes $A_{S_G(\mathbb{L})}$ can be extended to an automorphism of $\mathbb{L}$ which stabilizes $\mathbb{L}^G$. Thus $G'$ is equal to the automorphism group of $S_G(\mathbb{L})$ trivializing the discriminant. The Leech pair $(G, S_G(\mathbb{L}))$ is saturated if and only if $G = G'$.

**Definition 3.7** Let $(G_1, S_1)$ and $(G_2, S_2)$ be two Leech pairs. We say $(G_1, S_1) \leq (G_2, S_2)$ if $G_1$ is a subgroup of $G_2$ and $S_1 = S_2^{G_1}$. We call $(G_1, S_1)$ a sub-pair of $(G_2, S_2)$. Two

[5] Up to conjugacy by $O(\mathrm{I}_{25,1})$, the choice of a primitive isotropic vector $w$ in $\mathrm{I}_{25,1} \cong \mathbb{L} \oplus U$ is equivalent to the choice of the isometry type of a Niemeier lattice $N \cong \langle w \rangle^{\perp}/\langle w \rangle$ (N.B. $N \oplus U \cong \mathrm{I}_{25,1}$). Intrinsically a Weyl vector in $\mathrm{I}_{25,1}$ is the primitive isotropic vector with the associated Niemeier lattice $N$ equal to the Leech lattice.

sub-pairs $(G_1, S_1)$, $(G_2, S_2)$ of a Leech pair $(G, S)$ are conjugate if there exists $g \in G$ such that $gG_1g^{-1} = G_2$ and $gS_1 = S_2$.

We denote by $\mathscr{A}$ the set of conjugacy classes of sub-pairs of $(\mathrm{Co}_0, \mathbb{L})$. There is a natural poset structure on $\mathscr{A}$. Denote by $\mathscr{A}_{sat}$ the sub-poset of $\mathscr{A}$ consisting of saturated Leech pairs. *A fixed-point sublattice* of $\mathbb{L}$ is the invariant sublattice $\mathbb{L}^G$ for some $G \subset \mathrm{Co}_0$. It is clear that associating with $(G, S) \in \mathscr{A}$ the fixed-point sublattice $\mathbb{L}^G$ gives rise to a one-to-one correspondence between $\mathscr{A}_{sat}$ and the set of ($\mathrm{Co}_0$-)orbits in the set of fixed-point sublattices of the Leech lattice $\mathbb{L}$. The fixed-point sublattices of $\mathbb{L}$ were classified by Höhn and Mason [24]. This classification will play a key role for us. For further reference, we mention:

**Theorem 3.8** (Höhn–Mason) *Under the action of* $\mathrm{Co}_0$, *there are exactly* 290 *orbits on the set of fixed-point sublattices of* $\mathbb{L}$. *In another word,* $|\mathscr{A}_{sat}| = 290$.

***Remark 3.9*** Harada and Lang [20] classified all fixed-point sublattices $K$ which are induced by actions of cyclic groups $G \cong n$ on the Leech lattice. The information contained in [20] is sometimes richer and more handy than that in [24].

## 4 The case of cubic fourfolds

In this section, we are classifying the symplectic automorphism groups of smooth cubic fourfolds. First, following the standard argument for $K3$ surfaces and hyper-Kähler manifolds, we establish that a group $G$ acting symplectically on a cubic $X$, determines a Leech pair $(G, S = S_G(X))$, which further can be embedded into the Leech lattice $\mathbb{L}$ (Corollary 4.3). Since $S$ arises from a cubic fourfold $X$, it is clear that $S$ embeds into the primitive lattice $\Lambda_0$. By Theorem 2.3 (we use the surjectivity part), this condition is essentially a sufficient one. We state this in terms of the Borcherds polarization (see Sect. 2.4) as an iff criterion in Theorem 4.5. Using this criterion, the actual classification (Sect. 4.4) is accomplished by using the Höhn–Masson [24] (see also [20]) classification of the fixed-point sublattices in the Leech lattice, and Fu's classification [14] of automorphism groups of primary orders. The uniqueness of embeddings in the maximal cases (Theorem 1.8) is discussed in Sect. 4.5.

### 4.1 Leech pairs associated to symplectic automorphisms on cubic fourfolds and *K*3 surfaces

A finite group of symplectic automorphisms on a $K3$ surface, on a hyper-Kähler manifold of $K3^{[n]}$ type, or on a cubic fourfold leads to a Leech pair. The argument essentially goes back to Nikulin [42], and was refined recently in the context of groups of symplectic automorphisms for hyper-Kähler manifolds (see esp. [27] and [38]). We review the situation for the cases relevant to us: cubic fourfolds and polarized $K3$ surfaces.

**Notation 4.1** Let $X$ be a smooth cubic fourfold, and $G \subset \mathrm{Aut}^s(X)$. We denote by $S_G(X)$ the covariant lattice for the induced action of $G$ on $H^4(X, \mathbb{Z})$. Similarly, if $Y$ is a smooth algebraic $K3$ surface, and $G \subset \mathrm{Aut}^s(Y)$ a finite group, we denote by $S_G(Y)$ the covariant lattice for the induced action of $G$ on $H^2(Y, \mathbb{Z})(-1)$.

**Lemma 4.2** *Let* $X$ *be either a smooth cubic fourfold or an algebraic* $K3$ *surface with an action of a finite group* $G \subset \mathrm{Aut}^s(X)$. *Then* $(G, S_G(X))$ *is a Leech pair.*

***Proof*** The assumption that the action of $G$ is symplectic implies that $S_G(X) \subset H^{2,2}(X) \cap H^4(X, \mathbb{Z})_{prim}$. By Hodge index Theorem, $S_G(X)$ is positive definite, and by Theorem 2.3, $S_G(X)$ contains no short roots (i.e., the period point avoids $\mathcal{C}_6$). Since $G$ acts trivially on the invariant cohomology $H^4(X, \mathbb{Z})^G$ and $S_G(X) = (H^4(X, \mathbb{Z})^G)^\perp$, it follows that $G$ acts trivially on $A_{S_G(X)}$. Finally, since $\mathrm{Aut}(X)$ acts faithfully on $H^4(X)$, it is clear that $G$ acts faithfully on $S_G(X)$. We conclude that $(G, S_G(X))$ is a Leech pair (cf. Definition 3.3).

The argument for $K3$ surfaces is similar (and due to Nikulin), except for invoking Riemann–Roch to prove that there is no norm 2 vector (corresponding, via our scaling, to a $-2$ class) in $S_G(X)$. □

**Corollary 4.3** *Let $X$ be either a smooth cubic fourfold or an algebraic $K3$ surface with a faithful action of a finite group $G \subset \mathrm{Aut}^s(X)$. There exists a primitive embedding of $S_G(X)$ into $\mathbb{L}$, and hence an embedding of $G$ into $\mathrm{Co}_0$ with image avoiding $-id$.*

***Proof*** By Lemma 4.2, $(G, S_G(X))$ is a Leech pair. Since $S_G(X)$ has a primitive embedding into a unimodular lattice of rank 23 (or 22) for cubic fourfolds (or $K3$ surfaces respectively), the rank condition of Proposition 3.4 is satisfied; the claim follows. □

Let us now discuss the role of the polarization. If $X$ is a cubic fourfold, any automorphism $f$ is induced from a linear automorphism of the ambient projective space, and thus $\varphi = f^*$ preserves the class $\eta \in H^4(X, \mathbb{Z})$ (recall $\eta$ is the square of a hyperplane class). It follows that there is a primitive embedding

$$S_G(X) \hookrightarrow \Lambda_0, \tag{4.1}$$

where $\Lambda_0$ is the primitive cohomology (recall $\Lambda_0 \cong A_2 \oplus (E_8)^2 \oplus U^2$).

For $K3$ surfaces $Y$, the situation is similar, but there is a subtle difference. Namely, under the assumption that $Y$ is algebraic (i.e., $\mathrm{NS}(Y)$ contains an ample class $h$), and $G$ is finite, *any automorphism $\varphi \in G$ will preserve <u>some</u> ample class $h'$* (e.g., obtained by "averaging" $h$). This is the set-up of the classical results of Nikulin and Mukai. However, when talking about <u>polarized</u> $K3$ surfaces, we will fix an ample class $h$ on $Y$ and insist that the automorphism $f$ preserves $h$ (i.e., $f^*h = h$ in cohomology). With this assumption, we have again a primitive embedding

$$S_G(X) \hookrightarrow \Lambda_d$$

where $d = h^2 \in 2\mathbb{Z}_+$, and $\Lambda_d = (\langle h\rangle^\perp_{H^2(Y,\mathbb{Z})})(-1)$ is the primitive cohomology (we twist the form by $-1$ to get consistency with the cubic fourfold case).

***Remark 4.4*** We are not aware of a systematic study of the symplectic automorphisms in the polarized case for any degree (in Sect. 5 below, we will partially discuss the degree 2 and 6 cases as they are tightly connected to the cubic fourfold case). One situation where the polarized case was studied is the symplectic involutions. We recall that Nikulin proved that there is a single class of symplectic involutions for algebraic $K3$ surfaces (with notation as above, $S_G(X) \cong E_8(2)$). The polarized symplectic involutions were classified by van Geemen and Sarti [48]; a richer picture emerges (as one needs to keep track of the embedding of $E_8(2)$ into $\Lambda_d$, versus the unimodular $K3$ lattice).

#### 4.1.1 A criterion for Leech pairs to arise from symplectic automorphisms

So far we have discussed how a finite group of symplectic automorphisms $G \subset \mathrm{Aut}^s(X)$ leads to a Leech pair $(G, S_G(X))$, which in turn can be classified by Höhn–Mason [24] results. Now we are interested in the converse, given a Leech pair $(G, S)$, *when does it come from a symplectic automorphism group $G$ acting on $X$?* By Global Torelli Theorem (and surjectivity of the period map), this becomes a question about embeddings of lattices. For instance, note that (4.1) is a necessary condition if $X$ is a cubic fourfold. In fact, by Theorem 2.3 (and Proposition 2.6), (4.1) is essentially also sufficient, but some care is needed as $S$ needs to avoid both short roots (automatic since $(G, S)$ is a Leech pair) and long roots. To deal with both cases uniformly, it is better to view a smooth cubic fourfold $X$ as being $E_6$ Borcherds polarized (see Sect. 2.4). Based on these considerations, we obtain the following key result which allows us to go back and forth between geometry (automorphisms of $X$) and arithmetic (fixed-point sublattices of the Leech lattice $\mathbb{L}$).

**Theorem 4.5** (Criterion for Leech pairs associated with cubic fourfolds) *Let $(G, S)$ be a Leech pair. The following are equivalent:*

(i) *There exists a smooth cubic fourfold $X$ with a faithful and symplectic action of $G$ such that $(G, S) \cong (G, S_G(X))$,*
(ii) *There exists a faithful action of $G$ on the Leech lattice $\mathbb{L}$ with $(G, S) \cong (G, S_G(\mathbb{L}))$ and $K = \mathbb{L}^G$, such that there exists a primitive embedding of $E_6$ into $K \oplus U^2$,*
(iii) *There exists an embedding of $S \oplus E_6$ into the Borcherds lattice $\mathbb{B}$, such that the image of $S$ is primitive.*

***Proof*** $(i) \Longrightarrow (ii)$: From Corollary 4.3, there exists a primitive embedding $S \hookrightarrow \mathbb{L}$ with an extension of the $G$-action on $\mathbb{L}$ such that $\mathbb{L}^G$ is the orthogonal complement of $S$ in $\mathbb{L}$. We have now two ways to embed $S$ into $\mathbb{B}$, explicitly:

$$S \hookrightarrow \Lambda_0 \hookrightarrow \Lambda_0 \oplus E_6 \subset \mathbb{B}$$

and

$$S \hookrightarrow \mathbb{L} \hookrightarrow \mathbb{L} \oplus U^2 \cong \mathbb{B}.$$

Clearly, both embeddings are primitive (e.g., $\Lambda_0 \subset \mathbb{B}$ is primitive by Proposition 2.12, and $S$ is primitive in $\Lambda_0$ by (4.1)). By Nikulin's results (see Theorem A.9), we know that there is a single conjugacy class of primitive embeddings $S \hookrightarrow \mathbb{B}$. Therefore, we can choose the isomorphism $\mathbb{L} \oplus U^2 \cong \mathbb{B}$, such that the following diagram commutes:

$$\begin{array}{ccccc} S & \hookrightarrow & \mathbb{L} & \hookrightarrow & \mathbb{L} \oplus U^2 \\ \downarrow & & & & \downarrow \cong \\ \Lambda_0 & \hookrightarrow & \Lambda_0 \oplus E_6 & \hookrightarrow & \mathbb{B} \end{array}$$

We have $K = S_{\mathbb{L}}^{\perp}$, giving $S_{\mathbb{B}}^{\perp} \cong K \oplus U^2$. On the other hand, $E_6 \cong (\Lambda_0)_{\mathbb{B}}^{\perp}$, thus $E_6 \subset S_{\mathbb{B}}^{\perp} \cong K \oplus U^2$. Since $E_6$ does not admit any overlattice, $E_6$ embeds primitively into $K \oplus U^2$.

$(ii) \Longrightarrow (iii)$: There is the embedding:

$$S \oplus E_6 \hookrightarrow S \oplus K \oplus U^2 \subset \mathbb{L} \oplus U^2 \cong \mathbb{B}$$

Notice that $S$ has primitive image in $\mathbb{L}$, hence also has primitive image in $\mathbb{B}$.

$(iii) \Longrightarrow (i)$: The action of $G$ on $S$ induces trivial action on $(A_S, q_S)$, hence extends to an action on $\mathbb{B}$ such that its restriction to the orthogonal complement of $S$ is trivial. Since $S \subset \mathbb{B}$ is primitive (by assumption), we get $S = S_G(\mathbb{B})$ (recall $S_G(\mathbb{B}) = (\mathbb{B}^G)^{\perp} = (S^{\perp})^{\perp}$). On the other hand, we note that $G$ acts trivially on $E_6 \subset \mathbb{B}$ (since by construction $E_6 \subset$

$S_{\mathbb{B}}^{\perp}$). We view $\Lambda_0$ as the orthogonal complement of $E_6$ in $\mathbb{B}$ (cf. Proposition 2.12). Via this identification, the $G$ action on $\mathbb{B}$ induces a $G$ action on $\Lambda_0$. By construction $S \hookrightarrow \Lambda_0$ (primitive, as $S$ is primitive in $\mathbb{B}$), and clearly $(G, S) \cong (G, S_G(\Lambda_0))$. We can choose a Hodge structure $H$ on $\Lambda_0$ of type $(0, 1, 20, 1, 0)$ (i.e., $H$ is a decomposition of $\Lambda_{0,\mathbb{C}}$ with the obvious properties) such that $H^{2,2} \cap \Lambda_0 = S$ (i.e., $S$ is the algebraic lattice). Assuming that $S$ contains neither short nor long roots, the Global Torelli Theorem (Theorem 2.3) says that there exists a smooth cubic fourfold with $H^4(X, \mathbb{Z})_{prim} \cong H$ (as Hodge structures). Finally, by Proposition 2.4, we conclude that $X$ has a faithful and symplectic action of $G$ such that $(G, S_G(X)) \cong (G, S)$.

It remains to prove that $S \subset \Lambda_0$ contains no short or long roots of $\Lambda_0$ (see Definition 2.1). By assumption $S$ is a sublattice of the Leech lattice $\mathbb{L}$, so it contains no short roots (i.e., norm 2 vectors). Assume now $S$ contains a long root $\delta$, i.e., $(\delta, \delta) = 6$ and $\mathrm{div}_{\Lambda_0}(\delta) = 3$. Since $\mathbb{B}$ is obtained by gluing $E_6$ and $\Lambda_0$, we conclude that $\delta$ and $E_6$ span a $E_7$ lattice in $\mathbb{B}$. More precisely, there exists $\epsilon \in E_6$ (with $\epsilon^2 = 12$ and $\mathrm{div}_{E_6}(\epsilon) = 3$) such that $(\delta + \epsilon)/3 \in \mathbb{B}$. Since $G$ acts on $S$ without fixed nonzero vector, there exists $g \in G$ such that $g\delta \neq \delta$. We distinguish two cases, either $g\delta = -\delta$ or not. Assume first $g\delta = -\delta$; then $g((\delta+\epsilon)/3) = (-\delta+\epsilon)/3 \in \mathbb{B}$. We conclude $v = 2\delta/3 \in \mathbb{B}$, but this is a contradiction due to the fact that $(\delta, \delta) = 6$ ($v$ will not have integral norm). Thus, we can assume that $\delta' = g\delta$ is a long root non-proportional to $\delta$. Consider the lattice $M = \mathrm{Sat}_{\mathbb{B}}(\langle \delta, \delta', E_6\rangle) \subset \mathrm{Sat}_{\mathbb{B}}(S \oplus E_6)$. Then $M$ is a positive definite rank 8 lattice containing two sublattices $\mathrm{Sat}_{\mathbb{B}}(\langle \delta, E_6\rangle)$ and $\mathrm{Sat}_{\mathbb{B}}(\langle \delta', E_6\rangle)$ of type $E_7$. Clearly, $M \cong E_8$ (first, the root sublattice of $M$ is of type $E_8$ as it is strictly larger than $E_7$, then $E_8 \subset M$ forces equality for reasons of rank and determinant). It is well known that $E_6$ admits a unique embedding in $E_8$ with orthogonal complement $A_2$. We get $A_2 \subset S = (E_6)^{\perp}_{\mathrm{Sat}_{\mathbb{B}}(S \oplus E_6)}$ (using the primitivity of $S$ in $\mathbb{B}$). In particular, $S$ contains some short roots, contradicting the fact that $(G, S)$ is a Leech pair. □

#### 4.1.2 Moduli of cubics associated with a Leech pair $(G, S)$

We denote by $\mathscr{A}_{cub}$ the sub-poset of $\mathscr{A}$ consisting of Leech pairs isomorphic to $(G, S_G(X))$ for some smooth cubic fourfold $X$ with $G = \mathrm{Aut}^s(X)$. It is clear that such a Leech pair $(G, S_G(X))$ is saturated. Therefore we have $\mathscr{A}_{cub} \subset \mathscr{A}_{sat}$. Our purpose is to determine the poset $\mathscr{A}_{cub}$. We now discuss the geometric loci ("moduli") associated with the elements of this poset. By studying the minimal and maximal loci, in Sects. 4.2 and 4.3 respectively, we will be able to complete the proof of our main Theorem 1.2.

Suppose we have a faithful action of a finite group $G$ on $\mathbb{P}^5_{x_1,\ldots,x_6}$ which preserves a smooth cubic fourfold $V(F_0(x_1, \ldots, x_6))$. The action is equivalent to a group embedding $G \hookrightarrow \mathrm{PGL}(6, \mathbb{C})$. Denote by $\widetilde{G}$ the preimage of $G$ in $\mathrm{SL}(6, \mathbb{C})$. Let $\lambda\colon \widetilde{G} \to \mathbb{C}^\times$ be the character of $\widetilde{G}$ satisfying that $g(F_0)(:= F_0 \circ g^{-1}) = \lambda(g)F$ for any $g \in \widetilde{G}$. Let $\mathcal{V}_\lambda$ be the vector space of all cubic polynomials $F(x_1, \ldots, x_6)$ such that $g(F) = \lambda(g)F$ for any $g \in \widetilde{G}$. Denote by $\mathcal{V}_\lambda^{sm}$ the subset of $\mathcal{V}_\lambda$ consisting of polynomials which defines smooth cubic fourfolds. Define a group $N_\lambda := \{a \in \mathrm{SL}(6, \mathbb{C}) \big| aGa^{-1} = G \text{ and } \lambda(aga^{-1}) = \lambda(g), \forall g \in \widetilde{G}\}$. Then $N_\lambda$ is a reductive group acting on $\mathcal{V}_\lambda$ and $\mathcal{V}_\lambda^{sm}$. With respect to this action, all points in $\mathcal{V}_\lambda^{sm}$ are stable (namely, have closed orbits and finite stabilizer groups). We define $\mathcal{F}_G$ to be the GIT quotient of $\mathbb{P}\mathcal{V}_\lambda^{sm}$ by $N_\lambda$. We regard $\mathcal{F}_G$ as the moduli space of smooth cubic fourfolds with the specified group action of $G$ on $\mathbb{P}^5$.

In Sect. 2.2 we have defined the moduli space $\mathcal{M}_M$ of cubic fourfolds associated with a specified lattice embedding $M \hookrightarrow \Lambda_0$. The normalization of $\mathcal{M}_M$ is an arithmetic quotient

$\mathcal{D}_M/\Gamma_M$ (minus some Heegner divisors). From [54, Theorem 1.1], we have the following result which illustrates the relation between $\mathcal{F}_G$ and $\mathcal{M}_M$:

**Proposition 4.6** *Suppose $G$ is a finite group acting faithfully and symplectically on a smooth cubic fourfold $X \subset \mathbb{P}^5$, and $S = S_G(X)$. Let $\mathcal{F}_G$ be the moduli space of smooth cubic fourfolds preserved by the group action of $G$ on $\mathbb{P}^5$. Let $\mathcal{M}_S$ be the moduli space of smooth cubic fourfolds associated with the embedding $S = S_G(X) \subset H^4(X, \mathbb{Z})_0 \cong \Lambda_0$. Let $(\mathbb{D}_S \setminus \mathcal{H})/\Gamma_S$ be the normalization of $\mathcal{M}_S$, where $\mathcal{H}$ is a $\Gamma_S$-invariant hyperplane arrangement in $\mathbb{D}_S$ induced by restricting the $\mathcal{H}_2$ and $\mathcal{H}_6$ hyperplane arrangements (see Definition 2.1) to $\mathbb{D}_S$. Then the period map for cubic fourfolds gives rise to a natural isomorphism*

$$\mathcal{F}_G \cong (\mathbb{D}_S \setminus \mathcal{H})/\Gamma_G,$$

*where $\Gamma_G$ is a subgroup of $\Gamma_S$ of finite index. In particular,* $\dim \mathcal{V}_\lambda - \dim N_\lambda = \dim \mathcal{F} = \dim \mathcal{M}_S = 20 - \operatorname{rank}(S)$.

Let $(G, S)$ be a Leech pair with saturation $(G', S) \in \mathscr{A}_{cub}$. For any primitive embedding of $S$ into $\Lambda_0$ with the orthogonal complement containing neither short roots nor long roots, we have a moduli space $\mathcal{M}_S$ of smooth cubic fourfolds with this specified $S$-polarization. There may be more than one such primitive embeddings of $S$ up to conjugacy of $\Lambda_0$, hence there may be more than one moduli spaces $\mathcal{M}_S$. We define $\mathcal{M}_{(G,S)}$ to be the union of all possible $\mathcal{M}_S$. This is a closed subvariety of $\mathcal{M}$ (the moduli of cubic fourfolds) which only depends on the saturation of $(G, S)$. For a Leech pair $(G, S)$ whose saturation does not belong to $\mathscr{A}_{cub}$, we simply define $\mathcal{M}_{(G,S)}$ to be the empty set. While $\mathcal{M}_{(G,S)}$ is nonempty, its dimension is equal to $20 - \operatorname{rank}(S)$. Proposition 4.6 allows us to calculate $\dim \mathcal{M}_{(G,S)}$ once a corresponding group action of $G$ on $\mathbb{P}^5$ is known. This idea for the calculation of $\dim \mathcal{M}_{(G,S)}$ is used in proof of Theorem 1.2.

***Remark 4.7*** In Theorem 1.2, our classification is about saturated pairs, but in the arguments below it is convenient not to require $(G, S)$ to be saturated.

It is clear that the moduli spaces $\mathcal{M}_{(G,S)}$ have a natural poset structure that matches with the poset structure on $\mathscr{A}_{cub}$. Theorem 1.2 is organized by the dimensions of $\mathcal{M}_{(G,S)}(\neq \emptyset)$ (or equivalently $\operatorname{rank}(S)$).

### 4.2 The maximal Leech pairs for cubic fourfolds

We now note that the Leech pairs arising from automorphisms of cubic fourfolds satisfy an easy necessary condition (in terms of the rank of covariant lattice and the rank of the discriminant group). In order to simplify the relation to the Höhn–Mason classification [24], we state the condition in terms of the fixed-point sublattice $K$ in the Leech lattice $\mathbb{L}$.

**Condition 4.8** Let $(G, S)$ be a Leech sub-pair of $(\mathrm{Co}_0, \mathbb{L})$, and $K = S^{\perp}_{\mathbb{L}}$. We require $K$ to satisfy the following conditions:

(i) $\operatorname{rank}(K) \geq 4$ (or equivalently $\operatorname{rank}(S) \leq 20$);
(ii) for every prime number $p \neq 3$, $\alpha_p(K)(:= \operatorname{rank}(K) - l_p(A_K)) \geq 2$, and $\alpha_3(K) \geq 1$ (in particular $\alpha(K) = \operatorname{rank}(K) - l(A_K) \geq 1$).

**Proposition 4.9** *The equivalent conditions in Theorem 4.5 imply Condition 4.8.*

***Proof*** Since $S$ embeds into $\Lambda_0$ which has signature $(20, 2)$, the rank condition is clear. Assume now that $(G, S)$ is a Leech pair with a primitive embedding of $S$ into $\mathbb{L}$, and $K$ is the orthogonal complement of $S$ in $\mathbb{L}$. By Theorem 4.5, there exists a primitive embedding of $E_6$ into $K \oplus U^2$. Denote by $M$ the orthogonal complement of $E_6$ in $K \oplus U^2$. We have a saturation $E_6 \oplus M \hookrightarrow K \oplus U^2$. By Nikulin's glueing theory, there exists an isotropic subspace $H$ of $A_{E_6} \oplus A_M$, such that $A_K \cong H^\perp/H$. Since $M$ is primitive in $K \oplus U^2$, there is no nontrivial element in $H \cap A_M$. Therefore, we have either $H = 0$ or $H = \{(x, f(x)) \big| x \in A_{E_6}\}$, where $f : A_{E_6} \longrightarrow A_M$ is an isometry onto $f(A_{E_6})$ equipped with $-q_M$.

Assume first that the glueing group $H$ is trivial, then $A_K = A_{E_6} \oplus A_M$. Since $A_{E_6} \cong \mathbb{Z}/3$, we conclude $l_p(A_M) = l_p(A_K)$ for $p \neq 3$ and $l_3(A_M) = l_3(A_K) - 1$. Otherwise, we have $H = \{(x, f(x)) \big| x \in A_{E_6}, f(x) \in A_M, q_{E_6}(x) = -q_M(f(x))\} \cong \mathbb{Z}/3$. We have an isometry

$$A_K \cong H^\perp/H \cong A_M/f(A_{E_6}).$$

Thus, $l_p(A_M) = l_p(A_K)$ for $p \neq 3$ and $l_3(A_M) = l_3(A_K) + 1$.

In any case, we get

$$l_p(A_K) = l_p(A_M) \leq \mathrm{rank}(M) = \mathrm{rank}(K) - 2$$

for $p \neq 3$, and

$$l_3(A_K) \leq l_3(A_M) + 1 \leq \mathrm{rank}(M) + 1 = \mathrm{rank}(K) - 1$$

hence Condition 4.8. □

***Remark 4.10*** To understand the restriction imposed by Condition 4.8 on Leech pairs, let us consider the case $G \cong 2$ (i.e., symplectic involutions). According to [20] (also [24]), there are three nontrivial conjugacy classes of involutions in $\mathrm{Co}_0 = O(\mathbb{L})$. The fixed-point sublattices $K$ in the three cases are $E_8(2)$, $D_{12}^+(2)$, and $BW_{16}$ (the Barnes–Wall lattice), while the covariant lattices $S_G(\mathbb{L}) = K_{\mathbb{L}}^\perp$ are $BW_{16}$, $D_{12}^+(2)$, and $E_8(2)$ respectively. For $E_8(2)$ and $D_{12}^+(2)$, we have $\mathrm{rank}(K) = l(K)$ (this holds true whenever $K = K'(n)$ for some integral lattice $K'$, $n \in \mathbb{Z}_{>1}$), while $BW_{16}$ obviously satisfies Condition 4.8. We conclude that the only possible Leech pair arising from symplectic involutions on cubic fourfolds is $(2, E_8(2))$. To conclude that there is a unique class of symplectic involutions, we would need to prove that there exists a unique primitive embedding of $E_8(2)$ in $\Lambda_0$. In this particular case, a direct geometric argument (via a diagonalization of the involution) is easier. This concludes item (1) of Theorem 1.2.

In Sect. 3.3, we have defined a natural poset $\mathscr{A}$ on the set of Leech pairs in $(\mathrm{Co}_0, \mathbb{L})$. We are now interested in identifying the maximal Leech pairs $(G, S)$ arising from cubic fourfolds. As noted above, these pairs satisfy Condition 4.8. Focusing on the maximal rank cases, by inspecting [24], we note that there are 15 Leech pairs $(G, S) \in A_{sat}$ with $\mathrm{rank}(S) = 20$ (or equivalently $\mathrm{rank}(K) = 4$) and satisfying Condition 4.8. In fact, these cases precisely coincide with those of [25, Table 9]. For reader's convenience, we list them (sometimes corrected[6]) in Table 1 below.

***Remark 4.11*** In Table 1, the items in the last column represent discriminant forms of the invariant sublattices of the actions of $G$ on the Leech lattice. See [8, Page 379–380] and also our Appendix A for an explanation of the notation of discriminant forms.

6 There are some typos in the listing of the discriminant forms in [24]. For example, the discriminant form corresponds to case of $M_{10}$ is listed as $2_5^{+1}4_1^{+1}3^{-1}5^{+1}$ in [24], but this is not allowed in the Conway–Sloane [8] notation. See also Sect. A.2.

***Remark 4.12*** The group $Q$ appearing in item 8 is a group of order 128, see [25, Theorem 5.1, Case 5(b)]. We expect that the semi-direct product $3^2 : \mathrm{QD}_{16}$ appearing in item 15 is in fact isomorphic to $M_{2,9}$ (see §B.2).

It turns out that the 15 groups listed in Table 1 occur as maximal groups of symplectic automorphisms for some hyper-Kähler manifold of $K3^{[2]}$ type (algebraic, but not polarized). Specifically, the following result holds.

**Theorem 4.13** (Höhn–Mason [25, Theorem 8.7]) *A finite group acts symplectically on a hyper-Kähler manifold of type $K3^{[2]}$ if and only if it is a subgroup of a group in Table 1.*

We are interested in the maximal rank cases that can occur for cubic fourfolds, or equivalently the saturated Leech pairs $(G, S)$ for which $\mathcal{M}_{(G,S)} \neq \emptyset$ and $\dim \mathcal{M}_{(G,S)} = 0$. Höhn and Mason [25, Table 11] have identified six cases that do occur for cubic fourfolds, and in fact they gave explicit equations of cubic fourfolds realizing these groups of automorphisms. Using our Criterion 4.5, we prove the converse: these six cases are all the maximal rank possibilities for cubic fourfolds. Note however (see Sect. 4.5 below) that in two of the cases, there are two distinct embeddings of $S$ into $\Lambda_0$, leading to two more isolated cubic fourfolds with large symmetry in addition to the six cubics found by Höhn and Mason.

**Theorem 4.14** *Let $X$ be a smooth cubic fourfold, and $G = \mathrm{Aut}^s(X)$. Assume that $\mathrm{rank}(S_G(X)) = 20$, then the Leech pair $(G, S_G(X))$ corresponds to one of the entries 1, 4, 5, 10, 11 and 13 in Table 1.*

***Proof*** By Theorem 4.5, we need to determine for which $(G, S)$ among the 15 candidates, there exists an embedding of $S \oplus E_6$ into the Borcherds lattice $\mathbb{B}$ for which the image of $S$ is primitive. There are two possibilities for such an embedding $S \oplus E_6 \subset \mathbb{B}$. Either $S \oplus E_6 \subset \mathbb{B}$ is primitive or not. If $S \oplus E_6 \subset \mathbb{B}$ is not primitive, there exists a coindex 3 saturation $\widetilde{S}$ of $S \oplus E_6$, in which $S$ is primitive. Then $\widetilde{S}$ embeds primitively into $\mathbb{B}$. Since $S \oplus E_6$ (or $\widetilde{S}$) has rank 26, and $\mathbb{B}$ is the unique even unimodular lattice of signature $(26, 2)$, by Nikulin's theory, we conclude that $S \oplus E_6$ (or $\widetilde{S}$ respectively) embeds primitively into $\mathbb{B}$ iff there exists a negative definite rank 2 even lattice $T$ with discriminant form $q_T = -q_{S \oplus E_6}$ (or $q_T = -q_{\widetilde{S}}$ respectively). By Theorem A.8, such a lattice $T$ exists iff four conditions are satisfied. The first condition on the signature is automatically satisfied here. The remaining conditions are on the discriminant form $q_T$ (that is determined by $S \oplus E_6$ or the index 3 overlattice $\widetilde{S}$ of $S \oplus E_6$). We do a case by case analysis of the 15 possibilities from Table 1. The computations are standard manipulations with finite groups, and finite quadratic forms, we list only the essential details. (For a prime $p$, $\mathbb{Z}_p$ denotes the ring of $p$-adic integers.)

(1) The discriminant form of $S \oplus E_6$ is $3^{+2}9^{-1} \oplus 3^{+1}$. There is a nontrivial saturation $\widetilde{S}$ of $S \oplus E_6$ with discriminant form $3^{-1}9^{-1}$. There exists a negative rank 2 even lattice $T$ with discriminant form $3^{+1}9^{+1}$. Thus there exists a primitive embedding of $\widetilde{S}$ into $\mathbb{B}$ with orthogonal complement $T$.

(2) The discriminant form of $S \oplus E_6$ is $2_{\mathrm{II}}^{-2}3^{+2}7^{+1}$. There is no nontrivial saturation of $S \oplus E_6$. Since $3^2 \times 7$ is a square in $\mathbb{Z}_2$, there does not exist a negative rank 2 even lattice with discriminant form $2_{\mathrm{II}}^{-2}3^{+2}7^{-1}$. Thus there does not exist embedding of $S \oplus E_6$ into $\mathbb{B}$.

(3) The discriminant form of $S \oplus E_6$ is $4_3^{+1}8_7^{-1}3^{-1} \oplus 3^{+1}$. There is a nontrivial saturation $\widetilde{S}$ of $S \oplus E_6$ with discriminant form $4_3^{+1}8_7^{-1}$. By the third condition in Nikulin's criterion, there does not exist a negative rank 2 even lattice with discriminant form $4_5^{-1}8_1^{+1}$ nor $4_5^{-1}8_1^{+1}3^{-1} \oplus 3^{+1}$. Thus there does not exist embedding of $S \oplus E_6$ into $\mathbb{B}$.

(4) The discriminant form of $S \oplus E_6$ is $3^{-1}5^{+1}7^{-1} \oplus 3^{+1}$. There is a nontrivial saturation $\widetilde{S}$ of $S \oplus E_6$ with discriminant form $5^{+1}7^{-1}$. There exists a negative rank 2 even lattice $T$ with discriminant form $5^{+1}7^{+1}$. Thus there exists a primitive embedding of $\widetilde{S}$ into $\mathbb{B}$ with orthogonal complement $T$. Since $5 \times 7 = 35$ is not a square in $\mathbb{Z}_3$, there exists a negative rank 2 even lattice $T'$ with discriminant form $3^{-2}5^{+1}7^{+1}$. Thus there exists a primitive embedding of $S \oplus E_6$ into $\mathbb{B}$ with orthogonal complement $T'$.

(5) The discriminant form of $S \oplus E_6$ is $2_6^{+2}3^{-3} \oplus 3^{+1}$. There is a nontrivial saturation $\widetilde{S}$ of $S \oplus E_6$ with discriminant form $2_2^{+2}3^{+2}$. Since $2 \times 2 = 4$ is obviously a square in $\mathbb{Z}_3$, there exists a negative rank 2 even lattice $T$ with discriminant form $2_6^{+2}3^{+2}$. Thus there exists a primitive embedding of $\widetilde{S}$ into $\mathbb{B}$ with orthogonal complement $T$.

(6) The discriminant form of $S \oplus E_6$ is $4_5^{+1}8_7^{-1}5^{-1} \oplus 3^{+1}$, and there is no nontrivial saturation of $S \oplus E_6$. Since $5 \times 3 = 15 \equiv -1 \pmod 8$, there is no negative rank 2 even lattice with discriminant form $4_3^{-1}8_1^{+1}3^{-1}5^{-1}$. Thus there does not exist embedding of $S \oplus E_6$ into $\mathbb{B}$.

(7) The discriminant form of $S \oplus E_6$ is $4_6^{+2}7^{-1} \oplus 3^{+1}$ and there is no nontrivial saturation of $S \oplus E_6$. Since $3 \times 7 = 21 \equiv -3 \pmod 8$, there is no negative rank 2 even lattice with discriminant form $4_2^{+2}3^{-1}7^{+1}$. Thus there does not exist embedding of $S \oplus E_6$ into $\mathbb{B}$.

(8) The discriminant form of $S \oplus E_6$ is $8_2^{-2}3^{+1} \oplus 3^{+1}$ and there is no nontrivial saturation of $S \oplus E_6$ with $S$ primitive. Since $3 \times 3 = 9 \equiv 1 \pmod 8$, there is no negative rank 2 even lattice with discriminant form $8_6^{-2}3^{+2}$. Thus there is no embedding of $S \oplus E_6$ into $\mathbb{B}$ with primitive image of $S$.

(9) The discriminant form of $S \oplus E_6$ is $2_{\mathrm{II}}^{-2}3^{+2}5^{-1} \oplus 3^{+1}$. There is a nontrivial saturation $\widetilde{S}$ of $S \oplus E_6$ with discriminant form $2_{\mathrm{II}}^{-2}3^{-1}5^{-1}$. Since $3 \times 5 = 15 \equiv -1 \pmod 8$, there is no negative rank 2 even lattice with discriminant form $2_{\mathrm{II}}^{-2}3^{+1}5^{+1}$. Thus there is no primitive embedding of $\widetilde{S}$ into $\mathbb{B}$.

(10) The discriminant form of $S \oplus E_6$ is $2_5^{-1}4_1^{+1}3^{+1}5^{+1} \oplus 3^{+1}$ and there is no nontrivial saturation of $S \oplus E_6$. Since $2 \times 4 \times 5 = 40 \equiv 1$ is a square in $\mathbb{Z}_3$, there exists a negative rank 2 even lattice $T$ with discriminant form $2_3^{-1}4_7^{+1}3^{+2}5^{+1}$. Thus there exists a primitive embedding of $S \oplus E_6$ into $\mathbb{B}$ with orthogonal complement $T$.

(11) The discriminant form of $S \oplus E_6$ is $11^{+2} \oplus 3^{+1}$, and there is no nontrivial saturation of $S \oplus E_6$. Since 3 is a square in $\mathbb{Z}_{11}$ (notice that $5^2 \equiv 3 \pmod{11}$), there exists a negative rank 2 even lattice $T$ with discriminant form $3^{-1}11^{+2}$. Thus there exists a primitive embedding of $S \oplus E_6$ into $\mathbb{B}$ with orthogonal complement $T$.

(12) The discriminant form of $S \oplus E_6$ is $4_1^{-1}8_7^{-1}3^{+2} \oplus 3^{+1}$. There is a nontrivial saturation $\widetilde{S}$ of $S \oplus E_6$ with discriminant form $4_1^{-1}8_7^{-1}3^{-1}$. Since 3 is not congruent to $\pm 1$ modulo 8, there is no negative rank 2 even lattice with discriminant form $4_7^{+1}8_1^{+1}3^{+1}$. Thus there is no embedding of $\widetilde{S}$ into $\mathbb{B}$.

(13) The discriminant form of $S \oplus E_6$ is $3^{-2}5^{-2} \oplus 3^{+1}$. There is a unique nontrivial saturation $\widetilde{S}$ of $S \oplus E_6$ with discriminant form $3^{+1}5^{-2}$. Since 3 is not a square in $\mathbb{Z}_5$, there exists a negative rank 2 even lattice $T$ with discriminant form $3^{-1}5^{-2}$. Thus there exists a primitive embedding of $\widetilde{S}$ into $\mathbb{B}$ with orthogonal complement $T$.

(14) The discriminant form of $S \oplus E_6$ is $2_{\mathrm{II}}^{+2}7^{+2} \oplus 3^{+1}$, and there is no nontrivial saturation of $S \oplus E_6$. Since $2 \times 2 \times 3 = 12$ is not a square in $\mathbb{Z}_7$, there is no negative rank 2 even lattice with discriminant form $2_{\mathrm{II}}^{+2}3^{-1}7^{+2}$. Thus there is no embedding of $S \oplus E_6$ into $\mathbb{B}$.

(15) The discriminant form of $S \oplus E_6$ is $2_7^{-1}4_7^{-1}3^{+1}9^{+1} \oplus 3^{+1}$, and there is no nontrivial saturation of $S \oplus E_6$. Since $l_3(2_7^{-1}4_7^{-1}3^{+1}9^{+1} \oplus 3^{+1}) = 3$, there is no negative rank 2

even lattice with discriminant form the opposite of $2_7^{-1}4_7^{-1}3^{+1}9^{+1} \oplus 3^{+1}$. Thus there is no embedding of $S \oplus E_6$ into $\mathbb{B}$.

The proposition follows. □

### 4.3 Cubics with special groups of automorphisms (cyclic, Klein, and $S_3$)

Theorem 4.14 classifies the 0-dimensional moduli spaces $\mathcal{M}_{(G,S)}$. The top dimensional moduli spaces $\mathcal{M}_{(G,S)}$ will correspond to small groups $G$. In particular, the minimal elements in the poset $\mathscr{A}_{cub}$ can be determined by considering cyclic groups $G$ of prime orders. The cubics with a symplectic automorphism of prime order were studied previously, especially by Fu [14] (see also [17]), who classified all the possibilities for symplectic automorphisms of primary order.

**Theorem 4.15** (Fu [14, Theorem 1.1]) *Let $X = V(F) \subset \mathbb{P}^5$ be a smooth cubic fourfold with a symplectic action by a cyclic group $G = \langle g \rangle$ of primary order. We can choose coordinates $(x_1, x_2, \dots, x_6)$ on $\mathbb{P}^5$, and a generator $g \in G$, such that $(g, F)$ belongs to one of the following cases:*

(0) $\mathrm{ord}(g) = 1$, $g = id$, $\dim(\mathcal{F}) = 20$, *and $F$ any smooth cubic.*
(1) $\mathrm{ord}(g) = 2$, $g = \frac{1}{2}(0,0,0,0,1,1)$, $\dim(\mathcal{F}) = 12$, *and*

$$F = F_1(x_1,x_2,x_3,x_4) + x_5^2 L_1(x_1,x_2,x_3,x_4) + x_5x_6L_2(x_1,x_2,x_3,x_4) \\ + x_6^2 L_3(x_1,x_2,x_3,x_4);$$

(2) $\mathrm{ord}(g) = 4$, $g = \frac{1}{4}(0,0,2,2,1,3)$, $\dim(\mathcal{F}) = 6$, *and*

$$F \in \mathrm{Span}\{x_1N_1(x_3,x_4), x_2N_2(x_3,x_4), F_1(x_1,x_2), x_5x_6L_1(x_1,x_2), \\ x_5^2L_2(x_3,x_4), x_6^2L_3(x_3,x_4)\};$$

(3) $\mathrm{ord}(g) = 8$, $g = \frac{1}{8}(0,4,2,6,1,3)$, $\dim(\mathcal{F}) = 2$, *and*

$$F \in \mathrm{Span}\{x_1^3, x_1x_2^2, x_2x_3^2, x_2x_4^2, x_1x_3x_4, x_4x_5^2, x_3x_6^2, x_2x_5x_6\};$$

(4) $\mathrm{ord}(g) = 3$, $g = \frac{1}{3}(0,0,0,0,1,2)$, $\dim(\mathcal{F}) = 8$, *and*

$$F = F_1(x_1,x_2,x_3,x_4) + x_5^3 + x_6^3 + x_5x_6L_1(x_1,x_2,x_3,x_4);$$

(5) $\mathrm{ord}(g) = 3$, $g = \frac{1}{3}(0,0,1,1,2,2)$, $\dim(\mathcal{F}) = 8$, *and*

$$F = F_1(x_1,x_2) + F_2(x_3,x_4) + F_3(x_5,x_6) + \Sigma_{i=1,2;\,j=3,4;\,k=5,6}(a_{ijk}x_ix_jx_k);$$

(6) $\mathrm{ord}(g) = 3$, $g = \frac{1}{3}(0,0,0,1,1,1)$, $\dim(\mathcal{F}) = 2$, *and*

$$F \in \mathrm{Span}\{monomials in x_1, x_2, x_3, monomials in x_4, x_5, x_6\};$$

(7) $\mathrm{ord}(g) = 9$, $g = \frac{1}{9}(0,6,3,1,4,7)$, $\dim(\mathcal{F}) = 0$, *and*

$$F \in \mathrm{Span}\{x_1^2x_2, x_2^2x_3, x_3^2x_1, x_4^2x_5, x_5^2x_6, x_6^2x_4\};$$

(8) $\mathrm{ord}(g) = 9$, $g = \frac{1}{9}(0,3,6,1,1,4)$, $\dim(\mathcal{F}) = 0$, *and*

$$F \in \mathrm{Span}\{x_1^2x_2, x_2^2x_3, x_3^2x_1, x_4^2x_5, x_4x_5^2, x_4^3, x_5^3, x_6^3\};$$

(9) $\mathrm{ord}(g) = 5$, $g = \frac{1}{5}(0, 0, 1, 2, 3, 4)$, $\dim(\mathcal{F}) = 4$, *and*

$$F = F_1(x_1, x_2) + x_3x_6L_1(x_1, x_2) + x_4x_5L_2(x_1, x_2) + x_3^2x_5 + x_3x_4^2 + x_4x_6^2 + x_5^2x_6;$$

(10) $\mathrm{ord}(g) = 7$, $g = \frac{1}{7}(1, 5, 4, 6, 2, 3)$, $\dim(\mathcal{F}) = 2$, *and*

$$F = x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_6 + x_6^2x_1 + ax_1x_3x_5 + bx_2x_4x_6;$$

(11) $\mathrm{ord}(g) = 11$, $g = \frac{1}{11}(1, 9, 4, 3, 5, 0)$, $\dim(\mathcal{F}) = 0$, *and*

$$F \in \mathrm{Span}\{x_1^2x_2, x_2^2x_3, x_3^2x_4, x_4^2x_5, x_5^2x_1, x_6^3\}.$$

*Moreover, in all situations, the generic members of the defined families of cubic fourfolds are smooth.*

***Remark 4.16*** For further reference, we give the condition for a diagonal matrix $g = \frac{1}{n}(w_1, \ldots, w_6) \in \mathrm{GL}(6)$ to act symplectically on a cubic $X = V(F)$. Denote by $\underline{w} = (w_1, \ldots, w_6) \in (\mathbb{Z}/n)^6$ the set of weights. Then a simple application of Griffiths' residue calculus (see [14, Lemma 3.2]) gives that $g$ acts symplectically on $X$ iff

$$|\underline{w}| \equiv 2\deg_{\underline{w}}(F) \pmod n \tag{4.2}$$

where $|\underline{w}| = \sum_{i=1}^6 w_i$ and $\deg_{\underline{w}}(F) = \sum_{i=1}^6 w_i\alpha_i$ for some monomial $x_1^{\alpha_1}\ldots x_6^{\alpha_6}$ occurring with non-zero coefficient in $F$ (N.B. since $V(F)$ is stabilized by $g$, $\deg_{\underline{w}}(F)$ is well defined in $\mathbb{Z}/n$). For most of the cases above, $|\underline{w}| = \deg_{\underline{w}}(F) = 0$ (equivalently $g \in \mathrm{SL}(6)$), but this does not always hold (e.g., in the case $\mathrm{ord}(g) = 9$ above).

From the lattice theoretic approach (our main approach in this paper), Fu's classification is closely related to Harada–Lang classification [20] of fixed-point sublattices in the Leech lattice with respect to cyclic groups (see Remark 4.10 for the case of involutions). In fact, using the lattice theoretic approach and [20], we can improve Fu's result. Specifically, the following holds:

**Theorem 4.17** *Let $G$ be a cyclic group acting symplectically on some smooth cubic fourfold $X$ (i.e., $G \subset \mathrm{Aut}^s(X)$). Then, the order of $|G|$ is one of the following:*

$$|G| \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 15\}$$

*Furthermore, the following holds:*

(1) *(Primary Cases). For the cases $|G| = p^k$, we have the following correspondences among Fu's classification, Harada-Lang classification and Höhn–Mason classification.*

| Case in Theorem 4.15 | (0) | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Case in [20] | $1_A$ | $2_A$ | $4_C$ | $8_E$ | $3_B$ | $3_B$ | $3_C$ | $9_C$ | $9_C$ | $5_B$ | $7_B$ | $11_A$ |
| Case in [24] | 1 | 2 | 9 | 55 | 4 | 4 | 35 | 101 | 101 | 20 | 52 | 120 |
| The saturated group | 1 | 2 | 4 | $QD_{16}$ | 3 | 3 | $3^{1+4}:2$ | $3^4:A_6$ | $3^4:A_6$ | $D_{10}$ | $F_{21}$ | $L_2(11)$ |

(2) *(Composite Cases). There are 4 Leech pairs $(G, S)$ occurring for cubic fourfolds with $G$ cyclic of order $n$ divisible by two distinct primes.*

(3) *(Maximal Cases). A cubic fourfold with a symplectic automorphism of order 9, 11, 12, or 15 is isolated in moduli (i.e., $\dim \mathcal{M}_{(G,S)} = 0$).*

| | | | | |
|---|---|---|---|---|
| Case in [20] | $-6_D$ | $6_E$ | $-12_H$ | $15_D$ |
| Case in [24] | 35 | 18 | 109 | 128 |
| The saturated group | $3^{1+4}:2$ | $D_{12}$ | $3^{1+4}:2.2^2$ | $A_{3,5}$ |

***Remark 4.18*** The maximal cases in item (3) above are in fact unique. This is proved in Sect. 4.5 below. Thus, considering cubic fourfolds with a symplectic action by a cyclic group of order $\geq 9$ gives four of the maximal cases listed in Theorem 1.8.

***Proof*** Harada–Lang [20] classified the conjugacy classes of cyclic subgroups in the Conway group $\mathrm{Co}_0$ and their associated fixed lattices $K$ (recall $S = K_{\mathbb{L}}^{\perp}$). The necessary Condition 4.8 says (in particular) that $\mathrm{rank}(K) \geq 4$ and that $K$ is not divisible as a lattice (otherwise we would have $K = K'(n)$ for some integral, not necessarily even, lattice $K'$ and integer $n \geq 2$, which is impossible as in this situation $\mathrm{rank}(K) = l(A_K)$). Inspecting the list of [20] in the primary order case gives an easy match with the list of Theorem 4.15 (essentially, there is only one possibility for $(G, S)$ once the order of $G$ and the rank of $S$ are specified). The pairs $(G, S)$ are not saturated, but the knowledge of $K$ (essentially, rank and discriminant) suffices to identify the relevant case in Höhn–Mason [24] list, and to find the saturated pair $(G', S)$ (with $G \subset G'$).

Assuming that $n = |G|$ has at least 2 prime divisors, and that $K$ is a non-divisible lattice of rank at least 4, leaves only the following cases in [20]: $-6_D, 6_E, -10_E, -12_H, 14_B$ and $15_D$. As before, for each case we can associate a unique saturated Leech pair from [24]. Using Theorem 4.5 (our main criterion), cases $-10_E$ and $14_B$ cannot arise from cubic fourfolds, while the others can occur. Finally, the cases $-12_H$ and $15_D$ correspond to maximal cases (i.e., $\mathrm{rank}(K) = 4$, or equivalently $\mathrm{rank}(S) = 20$). Considering also the cases of order 9 and 11 identified in Theorem 4.15, we obtain item (3) (compare also with Theorem 4.14). □

***Remark 4.19*** Let us comment on the two apparent repetitions in the matching of the cases in Theorem 4.17. First, the two order 9 cases (case (8) and (9)) correspond to a unique cubic fourfold, in fact the Fermat cubic fourfold

$$X = V(x_1^3 + \cdots + x_6^3) \subset \mathbb{P}^5,$$

which has $\mathrm{Aut}^s(X) = 3^4 : A_6$. The fact that we list two cases of order 9 in Theorem 4.15 corresponds to the existence of two non-conjugate cyclic subgroups of order 9 in $3^4 : A_6$ (induced from the two conjugacy classes of order 3 elements in $A_6$). For reference, we note (cf. [20, Case $9_C$]) that the fixed-point lattice $K$ is

$$\begin{pmatrix} 4 & 1 & 1 & 2 \\ 1 & 4 & 1 & 2 \\ 1 & 1 & 4 & -1 \\ 2 & 2 & -1 & 4 \end{pmatrix}$$

which has discriminant form $3^{+2}9^{+1}$. The cases (4) and (5) of order 3 lead to the same Leech pair $(G, S)$ (with $K = S_{\Lambda}^{\perp}$ being the Coxeter–Todd lattice), but in this case the two (8-dimensional) families of cubics are different, due to the fact that $S$ has two different primitive embeddings into the lattice $\Lambda_0 (= A_2 \oplus (E_8)^2 \oplus U^2)$. The other order 3 case (namely (6)) is easily distinguished; it corresponds to $K$ being $E_6^*(3)$ which has discriminant form $3^{+5}$.

***Remark 4.20*** Let us also note that the order 6 case $-6_D$ in fact coincides with the case $3_C$. This is clear by noticing that they both correspond to case 35 in [24] (with saturated group

$3^{1+4}:2$). This also follows by inspecting [20]; in both cases $K = E_6^*(3)$ (N.B. $E_6^*$ is not an integral lattice, thus scaling by 3 does not contradict our non-divisibility assumption on $K$).

***Remark 4.21*** The order 11 case is very interesting, as 11 cannot occur as a prime order for symplectic automorphisms of $K3$ surfaces (and thus this example can be used to construct exotic automorphisms for hyper-Kähler's of $K3^{[2]}$ type; e.g. [38, §4.5]). The equation of the unique cubic with an order 11 symplectic automorphism is well known, namely

$$X = V(x_1^3 + x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_6 + x_6^2x_2).$$

From our perspective, this corresponds to case $(11_A)$ in [20]. The saturated Leech pair is $(\mathrm{PSL}(2, \mathbb{F}_{11}), S)$ and the fixed-point lattice $K$ is

$$\begin{pmatrix} 4 & 0 & 2 & -1 \\ 0 & 4 & -1 & 2 \\ 2 & -1 & 4 & -1 \\ -1 & 2 & -1 & 4 \end{pmatrix}$$

which has discriminant form $11^{+2}$.

In view of Theorem 4.17, we note that the only cyclic case that needs further investigation is $G \cong 6$ (the primary cases are covered by Theorem 4.15, while the maximal cases are discussed later in Sect. 4.5). According to Theorem 4.17, there are two order 6 cases relevant for us ($6_E$ and $-6_D$). However, the case $-6_D$ was already covered by Theorem 4.15 (cf. Remark 4.20). The last cyclic group case is handled by the following result.

**Lemma 4.22** *Let $X$ be a smooth cubic fourfold with a symplectic automorphism of order 6. Suppose the moduli of cubic fourfolds with such an automorphism has dimension greater than 2 (i.e., $\dim \mathcal{M}_{(G,S)} > 2$). Then for an appropriate choice of coordinates, the defining equation for $X$ either belongs to*

$$\mathrm{Span}\{x_1^2x_3, x_1^2x_4, x_1x_2x_3, x_1x_2x_4, x_2^2x_3, x_2^2x_4, x_3^3, x_3^2x_4, x_3x_4^2, x_3x_5x_6, x_4^3, x_4x_5x_6, x_5^3, x_6^3\},$$

*while the order 6 automorphism is $\frac{1}{6}(3, 3, 0, 0, 2, 4)$, or belongs to*

$$\mathrm{Span}\{x_1^3, x_1x_2^2, x_1x_3x_5, x_1x_3x_6, x_2x_4x_5, x_2x_4x_6, x_3^3, x_3x_4^2, x_5^3, x_5^2x_6, x_5x_6^2, x_6^3\},$$

*while the order 6 automorphism is $\frac{1}{6}(0, 3, 2, 5, 4, 4)$. In both cases, the corresponding moduli spaces $\mathcal{F}$ have dimension 4. They both correspond to the case $6_E$ in [20], and the associated saturated group is $D_{12}$.*

***Proof*** Denote by $\rho$ the order 6 automorphism. Since the moduli space of cubic fourfolds with such an automorphism has dimension geater than 2, the order 3 automorphism $\rho^2$ belongs to cases (4) or (5) in Theorem 4.15. Thus, we can choose coordinates $(x_1, x_2, \ldots, x_6)$ such that $\rho^2 = \frac{1}{3}(0, 0, 0, 0, 1, 2)$ or $\frac{1}{3}(0, 0, 1, 1, 2, 2)$, meanwhile $\rho^3$ has two $-1$ on the diagonal. Denote by $F = F(x_1, \ldots, x_6)$ a defining equation for $X$. Then $F$ is a linear combination of $\rho$-invariant monomials in $x_1, \ldots, x_6$. Since $X$ is smooth, there exists a $\rho$-invariant monomial divisible by $x_i^2$ for any $i = 1, 2, \ldots, 6$ .

If $\rho^2 = \frac{1}{3}(0, 0, 0, 0, 1, 2)$, then a $\rho^2$-invariant monomial divisible by $x_5^2$ must be $x_5^3$. Therefore $x_5^3$ is $\rho$-invariant, hence also $\rho^3$-invariant. So does $x_6^3$. We may then take $\rho^3 = \frac{1}{2}(1, 1, 0, 0, 0, 0)$. Then $\rho = \frac{1}{6}(3, 3, 0, 0, 2, 4)$ and

$$F \in \mathrm{Span}\{x_1^2x_3, x_1^2x_4, x_1x_2x_3, x_1x_2x_4, x_2^2x_3, x_2^2x_4, x_3^3, x_3^2x_4, x_3x_4^2, x_3x_5x_6, x_4^3, x_4x_5x_6, x_5^3, x_6^3\}.$$

This 14-dimensional vector space contains $x_1^2x_3+x_2^2x_4+x_3^3+x_4^3+x_5^3+x_6^3$ which determines a smooth cubic fourfold. Therefore, a generic cubic fourfold with this automorphism $\rho$ is smooth. The dimension of the centralizer of $\rho$ in $\mathrm{GL}(6,\mathbb{C})$ is $4+4+1+1=10$, hence the dimension of the moduli space $\mathcal{F}$ is $14-10=4$.

If $\rho^2=\frac{1}{3}(0,0,1,1,2,2)$, then a $\rho^2$-invariant monomial divisible by $x_1^2$ must be $x_1^3$ or $x_1^2x_2$. Therefore, the two $-1$ on the diagonal of $\rho^3$ cannot occupy the first two positions simultaneously. So do the third and fourth positions, the fifth and sixth positions. By symmetry, we may take $\rho=\frac{1}{6}(0,3,2,5,4,4)$ and

$$F\in \mathrm{Span}\{x_1^3, x_1x_2^2, x_1x_3x_5, x_1x_3x_6, x_2x_4x_5, x_2x_4x_6, x_3^3, x_3x_4^2, x_5^3, x_5^2x_6, x_5x_6^2, x_6^3\}.$$

This 12 dimensional vector space contains $x_1^3+x_1x_2^2+x_3^3+x_3x_4^2+x_5^3+x_6^3$ which determines a smooth cubic fourfold, hence a generic element also determines a smooth cubic fourfold. Moreover, the dimension of the centralizer of $\rho$ in $\mathrm{GL}(6,\mathbb{C})$ is $1+1+1+1+4=8$, hence the dimension of the moduli space $\mathcal{F}$ is $12-8=4$. □

### 4.3.1 Small non-cyclic groups

In addition to the cyclic groups identified above, we discuss also the cases of cubics with symplectic action by the simplest non-cyclic groups, Klein group and respectively $S_3$. First, for the Klein group $2^2$, relevant to item $(c1)$ in Theorem 1.2, the following holds.

**Lemma 4.23** *Suppose $X=V(F)$ is a smooth cubic fourfold with symplectic action of $G\cong 2^2$. Then we can choose coordinates $(x_1,x_2,x_3,x_4,x_5,x_6)$ for $V$ such that $G=\langle \mathrm{diag}(1,1,1,1,-1,-1), \mathrm{diag}(1,1,1,-1,-1,1)\rangle$, and $F$ can be written as $F_1(x_1,x_2,x_3)+x_4^2L_1(x_1,x_2,x_3)+x_5^2L_2(x_1,x_2,x_3)+x_6^2L_3(x_1,x_2,x_3)+x_4x_5x_6$. The dimension of the associated moduli space $\mathcal{F}$ is* 8.

***Proof*** Since $G$ is a finite abelian subgroup of $\mathrm{PSL}(V)$, we can choose coordinates $(x_1,x_2,x_3,x_4,x_5,x_6)$ for $V$, such that all element in $G$ are diagonal matrices. For any $g\in G$, since $g^2=id$ and $g$ acts symplectically on the smooth cubic fourfold $X$, there are four eigenvalues 1 and two eigenvalues $-1$ (see Theorem 4.15(1)). We now choose generators $g_1,g_2$ of $G$. Up to coordinate choices, we may assume $g_1=\mathrm{diag}(1,1,1,1,-1,-1)$, and $g_2$ is either $\mathrm{diag}(1,1,1,-1,-1,1)$ or $\mathrm{diag}(1,1,-1,-1,1,1)$. Suppose $g_2=\mathrm{diag}(1,1,-1,-1,1,1)$, then there is no smooth cubic fourfold preserved by the action of $G$. Thus $g_2=\mathrm{diag}(1,1,1,1,-1,-1)$. The defining polynomial $F$ can then be written as

$$F_1(x_1,x_2,x_3)+x_4^2L_1(x_1,x_2,x_3)+x_5^2L_2(x_1,x_2,x_3)+x_6^2L_3(x_1,x_2,x_3)+x_4x_5x_6.$$

A generic cubic of this type is smooth. Moreover, the dimension of the vector space of such cubic polynomials is $10+3+3+3+1=20$, and the dimension of the reductive group $\mathrm{GL}(3)\times\mathbb{C}^\times\times\mathbb{C}^\times\times\mathbb{C}^\times\subset\mathrm{GL}(6)$ preserving the normal form is $9+1+1+1=12$. Thus $\dim(\mathcal{F})=20-12=8$. □

We now consider the symmetric group $S_3$, relevant to item $(d2)$ in Theorem 1.2.

**Lemma 4.24** *Let $X=V(F)\subset\mathbb{P}(V)$ be a smooth cubic fourfold with symplectic action of $G\cong S_3$. Then the action of $G$ on $\mathbb{P}^5$ can be lifted to a representation of $G$ on $V\cong\mathbb{C}^6$, and one of the following holds:*

(1) *The representation of $G$ on $V$ is the direct sum of two standard representations of $S_3$. The dimension of the moduli space of cubic fourfolds $\mathcal{F}$ with such an action is* 6.

(2) *The representation of G on V is the direct sum of a standard representation, an alternating character, and two trivial characters of* $S_3$. *The dimension of the moduli space of cubic fourfolds* $\mathcal{F}$ *with such an action is* 4.

***Proof*** A projective representation of $S_3$ can be lifted as a linear representation. Suppose we have an action of $S_3$ on $V$ with an invariant smooth cubic form $F \in \mathrm{Sym}^3(V^*)$, such that the induced action of $S_3$ on $V(F)$ is faithful and symplectic. There are three involutions in $S_3$, and their actions on $V$ must have two-dimensional $(-1)$-eigenspace.

There are three linear irreducible representations of $S_3$, namely, the trivial character, the alternating character, and the standard representation on $\mathbb{C}^3$. Since the action of an order 3 element in $G$ is faithful on $V$, the representation of $G$ on $V$ has the standard representation of $S_3$ as an irreducible component. It is then clear that the two cases mentioned in the lemma are the only possible ones.

Suppose $V$ is a direct sum of two standard representations. We can choose coordinates $(x_1, x_2, x_3, x_4, x_5, x_6)$ of $V^*$, such that $G \cong S_3$ is acting via permutating $(x_1, x_2)$, $(x_3, x_4)$, $(x_5, x_6)$ simultaneously. A cubic form which is invariant under this action can be written uniquely as a linear combinations of 14 cubic forms which are also invariant. The centralizer group of $S_3$ in $\mathrm{GL}(V)$ can be written as $\begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix}$ where $A, B$ are two by two matrices. This group has dimension 8. Hence the dimension of the moduli of cubic fourfolds with this action is $14 - 8 = 6$.

Suppose $V$ is a direct sum of a standard representation, an alternating character, and two trivial characters. We can choose coordinate $(x_1, x_2, x_3, x_4, x_5, x_6)$ of $V^*$, such that $G \cong S_3$ is acting via permutating $(x_1, x_2, x_3)$, identically on $x_5, x_6$, and alternatively on $x_4$. A cubic form which is invariant under this action can be written uniquely as a linear combinations of 15 cubic forms which are also invariant. The centralizer group of $S_3$ in $\mathrm{GL}(V)$ has dimension 11. Thus, the dimension of the moduli space of cubic fourfolds with this action is $15-11 = 4$. □

### 4.4 Proof of Theorem 1.2

At this point, we can complete the proof of our classification theorem (Theorem 1.2). The main ingredients of our proof are the criterion given by Theorem 4.5, the Höhn–Mason classification [24] of the fixed-point sublattices in the Leech lattice $\mathbb{L}$, and Fu's classification discussed above (Theorem 4.15). Nikulin's criterion for the existence of even lattices with specified discriminant form (Theorem A.8) is a well-known tool that we use repeatedly.

Höhn and Mason [24] list all possibilities (290 in total) for saturated Leech pairs $(G, S)$. Condition 4.8 allows us to rapidly remove a large number of cases (e.g., about half of the cases have $\mathrm{rank}(S) \geq 21$). We analyze the remaining cases one by one using Theorem 4.5 (our main criterion) and Nikulin's theory. The most delicate case, $\mathrm{rank}(K) = 4$, was analyzed in detail in Theorem 4.14. The cases when $\mathrm{rank}(K) \geq 5$ are similar and in fact easier. Namely, as $K$ becomes larger, it is easier to embed $E_6$ into $K \oplus U^2$ (in particular, note that except $\mathrm{rank}(K) = 4$, $(E_6)^{\perp}_{K \oplus U^2}$ is indefinite, i.e., the "easy" case of Nikulin's theory). By a routine inspection (we only need to compare the rank of $K$ and $l_p(A_K)$) of the list of Höhn–Mason, we see that there are 43 cases (among them, there are 12, 12, 5, 5, 2, 3, 2, 1, 1 cases with $\mathrm{rank}(K) = 5, 6, 7, 8, 9, 10, 12, 16, 24$ respectively) in Höhn–Mason list with $\mathrm{rank}(K) \geq 5$ and satisfying Condition 4.8. Out of these 43 potential cases with $\mathrm{rank}(K) \geq 5$, only 28 satisfy the equivalent conditions in our main criterion Theorem 4.5. We omit the details. Including the 6 cases of maximal rank, we obtain the list of 34 possibilities for $(G, S) \in \mathscr{A}_{cub}$. We list them in Theorem 1.2 in the order of decreasing dimension of moduli $\mathcal{M}_{(G,S)}$ (or equivalently by $\mathrm{rank}(S)$). (Note however that $\mathcal{M}_{(G,S)}$ is not necessarily irreducible. When possible, we list also the irreducible components of $\mathcal{M}_{(G,S)}$.)

The second part of Theorem 1.2 is to give explicit equations for some of the cases. As discussed above, Theorem 4.15, Lemma 4.22, Lemma 4.23 and Lemma 4.24 give normal equations for cubic fourfolds $X$ which admit faithful actions by some special group $G$ (either cyclic of primary order, $\mathbb{Z}/6$, Klein group or $S_3$ respectively). Starting with this classification, we proceed in two ways. First, we have the saturation procedure: given a normal form $F$ stabilized by such a $G$, we obtain a Leech pair $(G, S = S_G(V(F)))$, with saturation $(G', S) \in \mathscr{A}_{cub}$ (i.e., in the list of the previous paragraph). For a generic cubic fourfold $X$ in $\mathcal{M}_{(G,S)}$, we have $G \subset G' = \mathrm{Aut}^s(X)$. By Proposition 4.6, we can calculate $\mathrm{rank}(S) = 20 - \dim \mathcal{M}_{(G,S)} = 20 - \dim \mathcal{F}_G$. Typically, using the information on the order of $G$ (note that $\mathrm{ord}(G')$ is a multiple of $\mathrm{ord}(G)$) and $\mathrm{rank}(S)$ suffices to identify the pair $(G', S)$. As an illustration of this saturation procedure see item (5) case $D_{10}$ in Theorem 1.2.

A second way to proceed is to start with $(G, S) \in \mathscr{A}_{cub}$, and consider elements of primary order $g \in G$ (say $\mathrm{ord}(g) = p^k$). By Theorem 4.15, we know the possible normal form(s) $F$ of $X$ with an action by $g$ (similar arguments apply to $2^2 \subset G$ or $S_3 \subset G$). We then try to specialize $F$ so that it admits an action by $G \supset \langle g \rangle$ (e.g., see proof of Lemma 4.24). Again, the knowledge of the dimension of $\mathcal{F}_G$ (from the normal form) and that of $\mathcal{M}_{(G,S)}$ proved very handy in practice.

Concretely, for $G \cong 1, 2, 3$ or 4, we can directly apply the second method ($G = \langle g \rangle$) and (0), (1), (2$b$), (3$a$) are clear. For (2$a$), we can apply the second method for $G \cong 2^2$ and use Lemma 4.23. For (3$b$), we can apply the second method for $G \cong S_3$ and use Lemma 4.24. For (5$a$), we can apply the second method for $G \cong D_{12}$ and use Lemma 4.22. Then applying the first method we see that a generic cubic fourfold described in Lemma 4.22 has symplectic automorphism group $D_{12}$. For items (5$b$), (7$b$) and (7$c$) of Theorem 1.2, we apply a combination of the two methods.

The last case left is (7$a$). From Harada-Lang [20] (case ($3_C$)), there is a Leech pair $(G, S)$ with $G \cong 3$ and $K = E_6^*(3)$. From Höhn–Mason classification, the only saturated Leech subpair of $(\mathrm{Co}_0, \mathbb{L})$ with discriminant $3^5$ is $(3^{1+4} : 2, S)$. Thus this is the saturation of $(G, S)$. By Theorem 4.5, there exist cubic fourfolds with certain order 3 automorphism such that the induced Leech pair is $(G, S)$. The moduli space of such cubic fourfolds has dimension 2. These cubic fourfolds must be given by case (6) in Theorem 4.15. Using the first method described above, any cubic fourfold with such an order 3 automorphism has automatically symplectic automorphism group $3^{1+4} : 2$. We conclude case (7$a$). □

### 4.5 Uniqueness in maximal case

As discussed in the previous subsection, we are able to identify explicit equations for a number of cases in Theorem 1.2. The cases that are more difficult are those with large, non-abelian group. One further complication that can arise is the fact that $\mathcal{M}_{(G,S)}$ might not be irreducible. We discuss in detail this situation for the maximal rank case, $\mathrm{rank}(S) = 20$ or equivalently $\dim \mathcal{M}_{(G,S)} = 0$. In Theorem 4.14 we have identified six cases for such pairs $(G, S)$. On the other hand, Höhn–Mason [25, Table 11] have listed for each of these cases a cubic fourfold in $\mathcal{M}_{(G,S)}$. It turns out, that in two of the six cases, there is an additional point in $\mathcal{M}_{(G,S)}$. This is the new content of our Theorem 1.8. Our arguments are lattice theoretic;

we do not have explicit equations for these cubic fourfolds with large automorphism groups. We start with two lemmas:

**Lemma 4.25** *For Leech pairs* $(G, S)$ *with numbers* 1, 4, 5, 10, 11, *or* 13 *in Table 1, the natural group homomorphisms*

$$\mathrm{Aut}(S) \longrightarrow \mathrm{Aut}(q_S)$$

*are surjective.*

***Proof*** Direct inspection of Table 9 in [25]. □

From the reduction theory of lattices (e.g., see [8, Chap. 15, §3.2]), we have:

**Lemma 4.26** *Every positive rank* 2 *lattice admits a basis, such that the corresponding intersection matrix is* $a^b c = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ *with* $-a < 2b \leq a \leq c$, *and* $b \geq 0$ *if* $a = c$. *In particular, we have* $3b^2 \leq d = ac - b^2$.

***Proof of Theorem 1.8*** The issue that we need to investigate is the uniqueness of the primitive embedding $S \hookrightarrow \Lambda_0$ (where $\Lambda_0 \cong A_2 \oplus (E_8)^2 \oplus U^2$ is the primitive cohomology of the cubic fourfold). We let $T = S^{\perp}_{\Lambda_0}$ be the transcendental lattice. The maximal rank case is very special, as $T$ is in fact a negative definite lattice of rank 24 (in all other cases, $T$ is indefinite, the easy case of Nikulin's theory). We now analyze case by case, the six cases of the Theorem 1.8, corresponding to items 1, 4, 5, 10, 11, or 13 in Table 1.

(1) For $3^4 : A_6$, the lattice $T$ has discriminant form $3^{+1}9^{+1}$. By Lemma 4.26, we see that the negative rank 2 even lattices with discriminant 27 are $-(2^1 14)$ and $-(6^3 6)$. Only $-(6^3 6)$ has discriminant form $3^{+1}9^{+1}$. Hence $T = -(6^3 6)$ is unique. A saturation $S \oplus T \hookrightarrow \Lambda_0$ is given by an injective morphism $-q_T \hookrightarrow q_S$. Every two such morphisms differ by an automorphism of $q_S$, which is induced by an automorphism of $S$ (from Lemma 4.25). Thus all primitive embeddings of $S$ into $\Lambda_0$ with orthogonal complement $T$ give the same primitive sublattice (up to automorphisms of $\Lambda_0$). Therefore, this case recovers a unique smooth cubic fourfold, which must be the Fermat cubic fourfold $V(x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3)$.

(2) For $G \cong A_7$ and $q_T = 5^{+1}7^{+1}$. All negative rank 2 even lattices with discriminant 35 are $-(6^1 6)$ and $-(2^1 18)$. After calculating their discriminant forms, we conclude $T = -(2^1 18)$. Similarly to the previous case, all primitive embeddings of $S$ into $\Lambda_0$ with orthogonal complement $T$ give the same primitive sublattice. Therefore, this case recovers a unique smooth cubic fourfold which is the diagonal cubic fourfold $V(x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 - (x_1 + x_2 + x_3 + x_4 + x_5 + x_6)^3)$ as we will show in Sect. 6 using the existence of certain anti-symplectic involutions (equivalently, Eckardt points). See Corollary 6.9.

(2′) For $G \cong A_7$ and $q_T = 3^{-2}5^{+1}7^{+1}$. All negative rank 2 even lattices with discriminant 315 are $-(2^1 158)$, $-(6^3 54)$, $-(18^3 18)$, $-(10^5 34)$, $-(14^7 26)$ and $-(18^9 22)$. After calculating their discriminant forms, we must have $T = -(18^3 18)$. A saturation $S \oplus T \hookrightarrow \Lambda_0$ is given by an injective morphism $q_S \hookrightarrow -q_T$. Given two such morphisms $\tau_1$ and $\tau_2$, denote by $e_1, e_2$ the generators of $T$ with intersecting matrix $-(18^3 18)$. One element in the automorphism group of $T$ sends $(e_1, e_2)$ to $(e_1, e_2)$, $(e_2, e_1)$, $(-e_1, -e_2)$ or $(-e_2, -e_1)$. By simple calculation, we can choose an automorphism $\iota$ of $T$, such that $\tau_1$ and $\iota \circ \tau_2$ have the same image. Then $\tau_1$ and $\iota \circ \tau_2$ only differ by an automorphism of $q_S$. By Lemma 4.25, this is induced by an automorphism of $S$. Thus the two primitive embeddings corresponding to $\tau_1$ and $\tau_2$ have the same image in $\Lambda_0$. Therefore, this case recovers a unique smooth cubic fourfold. As we will show in Sect. 6, this cubic fourfold does not admit any Eckardt points, hence is distinguished from case (2) above.

(3) For $3^{1+4} : 2.2^2$, the lattice $T$ has discriminant form $2_2^{+2}3^{+2}$. All negative rank 2 even lattices with discriminant 36 are $-(2^0 18)$, $-(6^0 6)$ and $-(4^2 10)$. After calculating their discriminant forms, we must have $T = -(6^0 6)$. As in case (1), all primitive embeddings of $S$ into $\Lambda_0$ with orthogonal complement $T$ give the same primitive sublattice. Therefore, this case gives rise to a unique smooth cubic fourfold. By [25], this cubic fourfold is $X(3^{1+4} : 2.2^2) = V(x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 - 3(\sqrt{3}+1)(x_1x_2x_3 + x_4x_5x_6))$.

(4) For $M_{10}$, the discriminant form of $T$ is $2_3^{-1}4_7^{+1}3^{+2}5^{+1}$. All negative rank 2 even lattices with discriminant 360 are $-(2^0 180)$, $-(4^0 90)$, $-(6^0 60)$, $-(10^0 36)$, $-(12^0 30)$, $-(18^0 20)$, $-(14^2 26)$, $-(18^6 22)$ and $-(18^{-6} 22)$. After calculating their discriminant forms, we must have $T = -(12^0 30)$. There are two discriminant subforms of $-q_T = 2_5^{-1}4_1^{+1}3^{+2}5^{+1}$ that are isomorphic to $q_S = 2_5^{-1}4_1^{+1}3^{+1}5^{+1}$. Moreover, these two are not identified via an automorphism of $T$. Therefore, there are two non-conjugate embeddings of $S$ into $\Lambda_0$, both with orthogonal complement isomorphic to $T$. From [25, Table 11] there is an explicit description for one smooth cubic fourfold with symplectic automorphism group $M_{10}$:

$$X^1(M_{10}) = x_1^3 + \cdots + x_6^3 + \frac{1}{5}(-3\zeta^7 - 3\zeta^5 + 3\zeta^4 - 3\zeta^3 + 6\zeta - 3) \times F \quad (4.3)$$

where $\zeta = e^{2\pi\sqrt{-1}/24}$ and $F = x_1x_2x_3 + x_1x_2x_4 + (\zeta^4 - 1)x_1x_2x_5 + x_1x_2x_6 + (\zeta^4 - 1)x_1x_3x_4 + x_1x_3x_5 + x_1x_3x_6 + (\zeta^4 - 1)x_1x_4x_5 - \zeta^4 x_1x_4x_6 - \zeta^4 x_1x_5x_6 + (\zeta^4 - 1)x_2x_3x_4 + (\zeta^4 - 1)x_2x_3x_5 - \zeta^4 x_2x_3x_6 + x_2x_4x_5 + x_2x_4x_6 - \zeta^4 x_2x_5x_6 + x_3x_4x_5 - \zeta^4 x_3x_4x_6 + x_3x_5x_6 + x_4x_5x_6$.

(5) For $L_2(11)$, the lattice $T$ has discriminant form $11^{+2}3^{-1}$. All negative rank 2 even lattices with discriminant 363 are $-(2^1 182)$, $-(14^1 26)$, $-(14^{-1} 26)$, $-(6^3 62)$ and $-(22^{11} 22)$. After calculating their discriminant forms, we must have $T = -(22^{11} 22)$. Similarly to the case (3), the image of $S$ in $\Lambda_0$ is unique up to automorphisms of $\Lambda_0$. Thus this recovers a unique cubic fourfold, which must be $V(x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_1 + x_6^3)$. Namely, Adler [1] showed that $L_2(11)$ is the automorphism group of the Klein cubic threefold $V(x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_1)$. It is then easy to see that $L_2(11)$ acts symplectically on the fourfold $V(x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_1 + x_6^3)$ (obtained as a cyclic cover of $\mathbb{P}^4$ branched over the Klein cubic threefold).

(6) For $A_{3,5}$, the lattice $T$ has discriminant form $3^{-1}5^{-2}$. All negative rank 2 even lattices with discriminant 75 are $-(2^1 38)$, $-(6^3 14)$ and $-(10^5 10)$. After calculating their discriminant forms, we must have $T = -(10^5 10)$. Similarly to the case (1), the image of $S$ in $\Lambda_0$ is unique up to automorphisms of $\Lambda_0$. Thus, this recovers a unique cubic fourfold, which must be

$$X(A_{3,5}) = V(x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 + x_7^3 + x_8^3) \cap V(x_1 + x_2 + x_3) \\ \cap V(x_4 + x_5 + x_6 + x_7 + x_8).$$

as $A_{3,5}$ is acting symplectically on this cubic fourfold by permuting the two tuples $(x_1, x_2, x_3)$ and $(x_4, \ldots, x_8)$.

The remaining parts of Theorem 1.8 on non-symplectic automorphisms are proved in Proposition 6.12. □

## 5 Symplectic automorphisms for low degree $K3$ surfaces

In this section we will discuss the case of $K3$ surfaces. As we have indicated, the classification of symplectic automorphisms for $K3$ surfaces was first systematically investigated by Nikulin [42] via lattice theory, and culminated in the celebrated result by Mukai [41] on a characterization of maximal finite symplectic groups of $K3$ surfaces via Mathieu group $M_{23}$. Kondō [29] simplified Mukai's proof by embedding the covariant lattice $S$ into a Niemeier lattice (an approach closely related to ours). Xiao [53] gave the complete list of finite symplectic automorphism groups of $K3$ surfaces by analyzing the combinatorial structures of the singularities of the quotient surface. Hashimoto [22] extended Kondō's lattice theoretic approach to give the complete list and analyze the possibilities of geometric realizations.

We briefly discuss here the case of symplectic automorphisms for low degree polarized $K3$ surfaces, along the lines of our analysis for cubic fourfolds. Our method is lattice theoretic and relies on the Höhn–Mason [24] classification. On the other hand, low degree $K3$ surfaces have projective models. For those $K3$ surfaces, one can study the automorphisms of the projective model via geometric methods; some partial results exist in the literature (e.g. [21], [9], [37]). Our discussion here only matches some of the maximal cases. A further analysis of the interplay between geometry and arithmetic would be interesting.

### 5.1 General discussion

As in the cubic fourfold case, the main point of our analysis is that for a $K3$ surface $Y$ with a faithful symplectic action of a finite group $G$, one gets a Leech pair $(G, S_G(Y)(-1))$ (see Lemma 4.2). The task now is to identify those that occur for $Y$ a polarized $K3$ surface of given degree. Similarly to our main criterion (Theorem 4.5) for cubic fourfolds, we obtain the following criterion for Leech pairs to arise from low degree $K3$ surfaces. Our arguments apply essentially verbatim as in the proof of Theorem 4.5 for the cases when there exists a Borcherds polarization on $Y$ (see Sect. 2.4) which is a root lattice. As already discussed, this is the case for degree 2 and 6. It is also true for the degree 4 case (e.g., [34, Sect. 1]). Finally, it also applies to elliptic $K3$ surfaces. By abuse of notation, we call an elliptic $K3$ surface *a degree* $0$ $K3$ *surface*, and we insist that the polarized symplectic automorphisms preserve the class of the fiber and of the section (i.e., the natural $U$ polarization for elliptic $K3$ surfaces is point-wise fixed by the automorphism).

**Theorem 5.1** *Suppose* $(G, S)$ *is a Leech pair. Let* $d \in \{0, 2, 4, 6\}$ *and* $R_d$ *be the root lattice* $E_8$, $E_7$, $D_7$, *or* $E_6 \oplus A_1$ *for* $d = 0, 2, 4, 6$ *respectively. The following three statements are equivalent:*

(i) *there exists a smooth degree* $d$ $K3$ *surface* $S$ *with a symplectic action* $G$ *which preserves the polarization, such that* $(G, S) \cong (G, S_G(X))$,
(ii) *there exists an action of* $G$ *on* $\mathbb{L}$ *with* $S = S_G(\mathbb{L})$ *and* $K = \mathbb{L}^G$, *such that there exists a primitive embedding of* $R_d$ *into* $K \oplus U^2$,
(iii) *there exists an embedding of* $S \oplus R_d$ *into the Borcherds lattice* $\mathbb{B}$, *such that* $S$ *has primitive image.*

The maximal rank for $S_G(Y)(-1)$ in the $K3$ case is 19 (or equivalently the orthogonal complement $K$ in the Leech lattice $\mathbb{L}$ has rank 5). From Höhn–Mason classification, we

**Table 2** Maximal Finite Symplectic Automorphism Groups of K3

| Number | Order | Group $\mathrm{Aut}^s(Y)$ | Discriminant form | $\deg(Y)$ in [41, Ex. 0.4] |
|---|---|---|---|---|
| 1 | 960 | $M_{20}$ | $2_{\mathrm{II}}^{-2}8_1^{+1}5^{-1}$ | 4 |
| 2 | 384 | $4^2.S_4$ | $4_7^{+1}8_6^{+2}$ | 4 |
| 3 | 360 | $A_6$ | $4_5^{-1}3^{+2}5^{+1}$ | 6 |
| 4 | 288 | $A_{4,4}$ | $2_{\mathrm{II}}^{+2}8_1^{+1}3^{+2}$ | 8 |
| 5 | 192 | $2^4 : D_{12}$ | $4_2^{-2}8_1^{+1}3^{-1}$ | 8 |
| 6 | 192 | $(Q_8 * Q_8) : S_3$ | $4_7^{-3}3^{+1}$ | 4 |
| 7 | 168 | $L_2(7)$ | $4_1^{+1}7^{+2}$ | 4 |
| 8 | 120 | $S_5$ | $4_3^{-1}3^{+1}5^{-2}$ | 6 |
| 9 | 72 | $M_9$ | $2_7^{-3}3^{-1}9^{-1}$ | 2 |
| 10 | 72 | $N_{72}(\cong 3^2 : D_8)$ | $4_1^{+1}3^{+2}9^{-1}$ | 6 |
| 11 | 48 | $T_{48}$ | $2_7^{+1}8_{\mathrm{II}}^{-2}3^{-1}$ | 2 |

identify the following 11 maximal cases in Table 2;[7] they correspond precisely to the 11 maximal cases of Mukai. It is interesting to note that all 11 cases have projective models of degree at most 8 (see [41, Example 0.4]).

**Notation 5.2** The notation of the finite groups appearing in Table 2 follows Mukai's appendix to [29] (N.B. there are some small typos in loc. cit.: the group $A_{4,4}$ has order 288, instead of 384). For reader's convenience, we recall that the group $M_{20}$ is isomorphic to $2^4 : A_5$, the group $M_9$ is isomorphic to $3^2 : Q_8$, the group $T_{48}$ is isomorphic to $L_2(3)$. The operator $*$ is the central product. Concretely, the group $Q_8 * Q_8$ is the quotient of $Q_8 \times Q_8$ by the center of $Q_8$ embedded diagonally, and it is isomorphic to an extraspecial group $2^{1+4}$.

Below, we discuss the maximal rank cases for $K3$ surfaces of degree 2 and 6 as those are connected to cubic fourfolds (as discussed, they correspond to "fake cubics", i.e., the Hassett divisors $\mathcal{C}_2$ and $\mathcal{C}_6$). The cases of degree 4 $K3$ surfaces and elliptic $K3$ surfaces are equally interesting, but less relevant to the core analysis in this paper. We point out however the classification of projective automorphisms of quartic $K3$ surfaces in [37], and the work [16] on automorphisms of elliptic $K3$ surfaces.

### 5.2 The degree 2 *K*3 case

The maximal symplectic cases for degree 2 $K3$ surfaces (analogue to Theorem 4.14 for cubics) are listed below.

**Theorem 5.3** *Let $Y$ be a $K3$ surface of degree 2 with a symplectic action of a finite group $G$. Suppose* $\mathrm{rank}(S_G(Y)) = 19$, *then* $(G, S_G(Y))$ *is one of the numbers* 3, 7, 9, 11 *in Table* 2. *In particular, the group $G$ can be* $A_6$ (*see* (5.1)), $L_2(7)$ (*see* (5.2)), $M_9$ (*see* (5.3)), *or* $T_{48}$ (*see* (5.4)).

[7] There are some typos in Höhn and Mason about the 2-parts of the discriminant forms. For instance, they write $4_3^{+1}8_2^{+2}$ in case 2, and $2_3^{+3}3^{-1}9^{-1}$ in case 9. These symbols are not allowed (see Sect. A.2). Corrected symbols for discriminant forms in this table can be found in [53] or [22].

***Proof*** Take a triple $(G, S, K)$ from Table 2. By Theorem 5.1, we need to check whether there exists an embedding of $S \oplus E_7$ into $\mathbb{B}$, such that the image of $S$ is primitive. We have $q_{E_7} = -q_{A_1} = 2_7^{+1}$. For numbers 1, 2, 4, 5, 6, the lattice $S \oplus E_7$ has no nontrivial saturation in which $S$ is primitive, and $l_2(S \oplus E_7) \geq 3$. For number 10, we have $l_3(S \oplus E_7) = l_3(K) = 3$. Therefore, in these cases, there are no embedding of $S \oplus E_7$ into $\mathbb{B}$ such that the image of $S$ is primitive. We next check the other cases one by one.

(1) For number 3 in Table 2, the discriminant form of $S \oplus E_7$ is $4_3^{-1}3^{+2}5^{+1} \oplus 2_7^{+1}$, and there is no nontrivial saturation of $S \oplus E_7$. Since $2 \times 4 \times 5 = 40 \equiv 1 \pmod 3$, there exists a unique negative rank 2 even lattice $T = -(12^0 30)$ with discriminant form $2_1^{+1}4_5^{-1}3^{+2}5^{+1}$. Thus there exists a primitive embedding of $S \oplus E_7$ into $\mathbb{B}$ with orthogonal complement $T$.
(2) For number 7 in Table 2, the discriminant form of $S \oplus E_7$ is $4_7^{+1}7^{+2} \oplus 2_7^{+1}$, and there is no nontrivial saturation of $S \oplus E_7$. Since $2 \times 4 = 8$ is a square in $\mathbb{Z}_7$, there exists a negative rank 2 even lattice $T$ with discriminant form $2_1^{+1}4_1^{+1}7^{+2}$. Thus there exists a primitive embedding of $S \oplus E_7$ into $\mathbb{B}$ with orthogonal complement $T$.
(3) For number 8 in Table 2, the discriminant form of $S \oplus E_7$ is $4_5^{-1}3^{-1}5^{-2} \oplus 2_7^{+1}$, and there is no nontrivial saturation of $S \oplus E_7$ in which $S$ is primitive. Since $2 \times 4 \times 3 = 24 \equiv -1$ is a square in $\mathbb{Z}_5$, there is no negative rank 2 even lattice with discriminant form $2_1^{+1}4_3^{-1}3^{+1}5^{-2}$. Thus there is no embedding of $S \oplus E_7$ into $\mathbb{B}$ for which the image of $S$ primitive.
(4) For number 9 in Table 2, the discriminant form of $S \oplus E_7$ is $2_5^{+3}3^{+1}9^{+1} \oplus 2_7^{+1}$, and there is a unique nontrivial saturation $\widetilde{S}$ of $S \oplus E_7$ in which $S$ is primitive. The discriminant form of $\widetilde{S}$ is $2_4^{+2}3^{+1}9^{+1}$. Since $2 \times 2 = 4$ is a square in $\mathbb{Z}_3$, there exists a negative rank 2 even lattice $T$ with discriminant form $q_{\widetilde{S}(-1)} = 2_4^{+2}3^{-1}9^{-1}$. Thus there exists a primitive embedding of $\widetilde{S}$ into $\mathbb{B}$ with orthogonal complement $T$.
(5) For number 11 in Table 2, the discriminant form of $S \oplus E_7$ is $2_1^{+1}8_{II}^{-2}3^{-1} \oplus 2_7^{+1}$, and there is a unique nontrivial saturation $\widetilde{S}$ of $S \oplus E_7$ in which $S$ is primitive. The discriminant form of $\widetilde{S}$ is $8_{II}^{-2}3^{-1}$. There exists a negative rank 2 even lattice $T$ with discriminant form $8_{II}^{-2}3^{+1}$. Thus there exists a primitive embedding of $\widetilde{S}$ into $\mathbb{B}$ with orthogonal complement $T$.

The claim follows. □

We discuss the geometric realizations for those maximal symplectic groups. The double cover of $\mathbb{P}^2$ branched along a sextic curve is a degree 2 $K3$. If a group acts on a plane sextic curve, it also acts on the corresponding degree 2 $K3$ surface. A classification of automorphism groups of plane sextic curves can be deduced from [21, Thm. 2.1]. It was discovered by Wiman [52] that the sextic curve

$$V(10x_1^3x_2^3 + 9x_3(x_1^5 + x_2^5) - 45x_1^2x_2^2x_3^2 - 135x_1x_2x_3^4 + 27x_3^6) \tag{5.1}$$

has an action of $A_6$. The corresponding degree 2 $K3$ surface also admits an action of $A_6$, which must be symplectic since $A_6$ is simple. In [9] the uniqueness of such a sextic curve (with an action of $A_6$) is proved.

The Klein sextic curve

$$V(x_1^5x_2 + x_2^5x_3 + x_3^5x_1) \tag{5.2}$$

has automorphism group $L_2(7)$. Therefore, the symplectic automorphism group of the corresponding degree 2 $K3$ surface is $L_2(7)$.

Another smooth plane sextic with large symmetry (see Remark 2.4 in [21]) is

$$V(x_1^6 + x_2^6 + x_3^6 - 10(x_1^3x_2^3 + x_2^3x_3^3 + x_3^3x_1^3)) \tag{5.3}$$

which has automorphism group isomorphic to the Hessian group $H_{216}$ of order 216 (this group can be represented as the affine special linear group $\mathrm{ASL}(2, \mathbb{F}_3)$, or as the projective unitary group $\mathrm{PU}(3, \mathbb{F}_2)$). Actually the degree 2 $K3$ surface corresponding to this sextic curve has symplectic automorphism group isomorphic to $M_9 \cong \mathrm{PSU}(3, \mathbb{F}_2)$ (cf. [41, Example 0.4]).

Finally, the group $T_{48}$ is realized by the double cover of $\mathbb{P}^2$ with branch curve

$$V(x_1^5 x_2 + x_2^5 x_1 + x_3^6), \tag{5.4}$$

(cf. [41, Example 0.4]).

### 5.3 The degree 6 $K3$ case

The maximal cases in the degree 6 case are listed below.

**Theorem 5.4** *Let $Y$ be a $K3$ surface of degree 6 with a symplectic action of a finite group $G$. Suppose $\mathrm{rank}(S_G(Y)) = 19$, then $(G, S_G(Y))$ is one of the numbers 3, 8, 10 in Table 2. In particular, the group $G$ can be $A_6$ (see (5.5)), $S_5$ (see (5.6)), or $N_{72}$ (see (5.7)).*

***Proof*** Take a triple $(G, S, K)$ in Table 2. By Theorem 5.1, we need to check whether there exists embedding of $S \oplus E_6 \oplus A_1$ into $\mathbb{B}$, such that the image of $S$ is primitive. We have $q_{E_6 \oplus A_1} = -q_{A_2} \oplus q_{A_1} = 2_1^{+1} 3^{+1}$. For cases with numbers 1, 2, 4, 5, 6, 11, we have $l_2(\widetilde{S}) \geq 3$ for any saturation $\widetilde{S}$ of $S \oplus E_6 \oplus A_1$ in which $S$ is primitive. For the case number 9, we have $l_3(\widetilde{S}) \geq 3$ for any saturation $\widetilde{S}$ of $S \oplus E_6 \oplus A_1$. Therefore, for those cases we cannot embed $S \oplus E_6 \oplus A_1$ into $\mathbb{B}$ with primitive image of $S$. We consider the other cases one by one.

(1) For number 3 in Table 2, the discriminant form of $S \oplus E_6 \oplus A_1$ is $4_3^{-1} 3^{+2} 5^{+1} \oplus 2_1^{+1} 3^{+1}$. We have a nontrivial saturation $\widetilde{S}$ of $S \oplus E_6 \oplus A_1$ with discriminant form $2_1^{+1} 4_3^{-1} 3^{-1} 5^{+1}$. There exists a negative rank 2 even lattice $T$ with discriminant form $2_7^{+1} 4_5^{-1} 3^{+1} 5^{+1}$. Thus there exists a primitive embedding of $\widetilde{S}$ into $\mathbb{B}$ with orthogonal complement $T$.
(2) For number 7 in Table 2, the discriminant form of $S \oplus E_6 \oplus A_1$ is $4_7^{+1} 7^{+2} \oplus 2_1^{+1} 3^{+1}$, and there is no nontrivial saturation of $S \oplus E_6 \oplus A_1$. Since $2 \times 4 \times 3 = 24 \equiv 3 \pmod 7$, and 3 is not a square in $\mathbb{Z}_7$, there is no negative rank 2 even lattice with discriminant form $2_7^{+1} 4_1^{+1} 3^{-1} 7^{+2}$. Thus there is no embedding of $S \oplus E_6 \oplus A_1$ into $\mathbb{B}$.
(3) For number 8 in Table 2, the discriminant form of $S \oplus E_6 \oplus A_1$ is $4_5^{-1} 3^{-1} 5^{-2} \oplus 2_1^{+1} 3^{+1}$. We have a nontrivial saturation $\widetilde{S}$ of $S \oplus E_6 \oplus A_1$ with discriminant form $2_1^{+1} 4_5^{-1} 5^{-2}$, in which $S$ is primitive. Since $2 \times 4 = 8$ is not a square in $\mathbb{Z}_5$, there exists a negative rank 2 even lattice $T$ with discriminant form $2_7^{+1} 4_3^{-1} 5^{-2}$. Thus there exists a primitive embedding of $\widetilde{S}$ into $\mathbb{B}$ with orthogonal complement $T$.
(4) For number 10 in Table 2, the discriminant form of $S \oplus E_6 \oplus A_1$ is $4_7^{+1} 3^{+2} 9^{+1} \oplus 2_1^{+1} 3^{+1}$. We have a nontrivial saturation $\widetilde{S}$ of $S \oplus E_6 \oplus A_1$ with discriminant form $2_1^{+1} 4_7^{+1} 3^{-1} 9^{+1}$, in which $S$ is primitive. Since $2 \times 4 = 8$ is not a square in $\mathbb{Z}_3$, there exists a negative rank 2 even lattice $T$ with discriminant form $2_7^{+1} 4_1^{+1} 3^{+1} 9^{-1}$. Thus there exists a primitive embedding of $\widetilde{S}$ into $\mathbb{B}$ with orthogonal complement $T$.

The theorem follows. □

The geometric realization of all these three groups can be found in Mukai [41, Example 0.4]. The group $A_6$ is the symplectic automorphism group of

$$Y = V(x_1 + \cdots + x_6) \cap V(x_1^2 + \cdots + x_6^2) \cap V(x_1^3 + \cdots + x_6^3) \tag{5.5}$$

(presented as a diagonal hyperplane section in $\mathbb{P}^5$). Similarly, the group $S_5$ is the symplectic automorphism group of

$$V(x_1 + \cdots + x_5) \cap V(x_1^2 + \cdots + x_6^2) \cap V(x_1^3 + \cdots + x_5^3) \tag{5.6}$$

(here the symplectic action is defined as follows: $g \in S_5$ acts on $(x_1, \ldots, x_5)$ by permutation, and $x_6 \to \mathrm{sgn}(g)x_6$; see [41, p. 188]). The group $N_{72}$ is the symplectic automorphism group of

$$V(x_1^3 + x_2^3 + x_3^3 + x_4^3) \cap V(x_1x_2 + x_3x_4 + x_5^2). \tag{5.7}$$

### 5.4 Uniqueness for *K*3 surfaces

While we don't investigate the uniqueness question here (i.e., analogues of Theorem 1.8), we point out that Hashimoto [22, Main Theorem] proved that for three of Mukai's maximal cases (specifically (3), (7), and (8), corresponding to groups $A_6$, $L_2(7)$, and $S_5$) there are exactly two primitive sublattices (up to conjugate) of $\Lambda_{K3}(-1)$ isomorphic to $S$ (where, as before, $S$ is the covariant lattice). Each of these cases has at least one realization for a $K3$ surface of degree 2 or 6 (see (5.5), (5.2), and (5.6) below). As Hashimoto works in the unpolarized case, the moduli space of $K3$ surfaces with symplectic automorphism groups in the above three cases has two connected component, both of dimension 1. The group $A_6$ is of special interest since it occurs for degree 2 and degree 6 cases (see (5.1) and (5.5)). Interestingly, the two cases are in two different components.

**Proposition 5.5** *The embeddings of S into $\Lambda_{K3}(-1)$ given by the two geometric realizations* (5.1) *and* (5.5) (*degree* 2 *and degree* 6) *have different orthogonal complements. In particular, these two $K3$ surfaces belong to different connected components of the moduli space of $K3$ surfaces with symplectic automorphism group $A_6$.*

***Proof*** Let $Y_1$ and $Y_2$ be the $K3$ surfaces of degree 2 and degree 6 with $A_6$ symplectic action respectively. Then the orthogonal complement of $S \cong S_{A_6}(Y_1) \hookrightarrow H^2(Y_1, \mathbb{Z})(-1)$ contains a vector with self-intersection $-2$, while the orthogonal complement of $S \cong S_{A_6}(Y_2) \hookrightarrow H^2(Y_2, \mathbb{Z})(-1)$ contains a vector with self-intersection $-6$. (Note that in our conventions we are scaling the cohomology by $-1$, making the polarization a negative vector. Furthermore, in these maximal cases, $S^\perp$ is negative definite of rank 3.) On the other hand, from Hashimoto [22, Table 10.3, item 79], the orthogonal complement $S^\perp$ of an embedding of $S$ into $\Lambda_{K3}(-1)$ can be either

$$\begin{pmatrix} -2 & -1 & 0 \\ -1 & -8 & 0 \\ 0 & 0 & -12 \end{pmatrix},$$

which contains $(-2)$-vector but does not contain any $(-6)$-vector, or

$$\begin{pmatrix} -6 & 0 & -3 \\ 0 & -6 & -3 \\ -3 & -3 & -8 \end{pmatrix},$$

which contains $(-6)$-vector but does not contain any $(-2)$-vector. The claim follows. □

### 5.5 A geometric relation to cubic fourfolds

Notice that the maximal symplectic automorphism groups for degree 2 (see Theorem 5.3) and degree 6 (see Theorem 5.4) also appear in case $\mathrm{rank}(S) = 19$ in Theorem 1.2. This is not a coincidence. The following proposition explains the geometry behind this phenomenon.

**Proposition 5.6** *Let $(G, S)$ be a Leech pair satisfying conditions of Theorem 5.1 for degree $d = 2$ or $6$, then $(G, S)$ is one of the 34 Leech pairs from Theorem 1.2. In each of these cases, the dimension of the corresponding moduli space of cubic fourfolds is one unit greater than that of the moduli space of $K3$ surfaces of degree 2 or 6.*

***Proof*** Let $(G, S)$ be a Leech pair such that there is an embedding of $S \oplus R_d$ into $\mathbb{B}$ with primitive image of $S$. Here $R_d = E_7$ if $d = 2$ and $R_d = E_6 \oplus A_1$ if $d = 6$. Notice that in both situations we have a natural embedding $E_6 \hookrightarrow R_d$. Thus we have an embedding $S \oplus E_6 \hookrightarrow S \oplus R_d \hookrightarrow \mathbb{B}$ such that the image of $S$ in $\mathbb{B}$ is primitive. Therefore, the Leech pair $(G, S)$ arises from symplectic actions of $G$ on certain smooth cubic fourfolds. The dimension of the moduli space of such cubic fourfolds is $20 - \mathrm{rank}(S)$, while the dimension of degree $d$ $K3$ surfaces with the corresponding symplectic action of $G$ is $19 - \mathrm{rank}(S)$. □

***Remark 5.7*** The above proposition implies that if we have a family of fake cubic fourfolds with symplectic action by a finite group $G$, then we can deform the fake cubic fourfolds into smooth ones, preserving the action of $G$. What we obtain is a family (of one more dimension) of cubic fourfolds with symplectic action of $G$ such that the generic fibers are smooth.

Let us briefly discuss the geometry behind Proposition 5.6 (and Remark 5.7). For simplicity, we restrict to the case of nodal cubic fourfolds (parametrized by the Hassett divisor $\mathcal{C}_6$). A singular cubic fourfold can be written as

$$X_0 = V(f_2(x_1, \ldots, x_5)x_6 + f_3(x_1, \ldots, x_5)) \subset \mathbb{P}^5 \tag{5.8}$$

for some homogeneous polynomials $f_2$, $f_3$ of degree 2 and 3 respectively. Note that the equation above singles out the singular point $p = (0, \ldots, 1) \in X$. The linear projection from $p$

$$\pi : X_0 \dashrightarrow \mathbb{P}^4$$

is a birational equivalence. The indeterminacy locus of the inverse map $\pi^{-1} : \mathbb{P}^4 \dashrightarrow X_0$ is the degree 6 $K3$ surface

$$Y = V(f_2(x_1, \ldots, x_5), f_3(x_1, \ldots, x_5)) \subset \mathbb{P}^4.$$

More precisely, assuming $Y$ is smooth, $X_0$ has a unique singular point $p$ which is either of type $A_1$ (if $V(f_2)$ is smooth) or type $A_2$ (if $V(f_2)$ is singular), and

$$\widetilde{X_0} = \mathrm{Bl}_p X_0 \cong \mathrm{Bl}_Y \mathbb{P}^4.$$

This establishes a Hodge correspondence (essentially an identification) between the Hodge structure on $H^4(X_0)$ (still pure) and $H^2(Y)(-1)$. Going on to automorphism, note that since the polarized automorphisms of $Y$ are induced from projective transformations, i.e., $G = \mathrm{Aut}(Y)_{pol} \subset \mathrm{PGL}(5)$, $G$ acts by automorphisms on $\widetilde{X}_0$. The group $G$ preserves the quadric $V(f_2) \subset \mathbb{P}^5$ and its strict transform $E$ in $\widetilde{X}_0 = \mathrm{Bl}_Y \mathbb{P}^4$. But then $E$ is precisely the exceptional divisor of $\widetilde{X}_0 = \mathrm{Bl}_p X_0 \to X_0$. We conclude that $G$ acts on $X_0$ by automorphisms preserving the singular point $p$.

Assuming that the equations $f_2$ and $f_3$ of $Y = V(f_2, f_3)$ can be chosen to be invariant with respect to $G$ (in general some character of $G$ might be involved), then the (pencil of) cubic fourfolds

$$X_t = V\left((f_2x_6 + f_3) + tx_6^3\right) \subset \mathbb{P}^5$$

admit $G$ as a group of automorphisms, with $G$ acting trivially on $x_6$. For general $t \in \mathbb{P}^1$, the above cubic is smooth. This allows us to lift the equations for maximal symmetric $K3$ surfaces of degree 2 and 6 to 1-parameter families of cubic fourfolds with large symmetry group (producing examples for most of the cases of Theorem 1.2(8)). The simplest example of such a lifting is the $A_6$ case (5.5). Specifically, the degree 6 $K3$ surface is

$$Y = V(x_1^2 + \cdots + x_5^2 + (x_1 + \cdots + x_5)^2, x_1^3 + \cdots + x_5^3 - (x_1 + \cdots + x_5)^3).$$

It can be lifted to the 1-parameter family of cubics $X_t = V(F_t)$ with $A_6$ symmetry, where

$$F_t = x_1^3 + \cdots + x_5^3 - (x_1 + \cdots + x_5)^3 + x_6\left(x_1^2 + \cdots + x_5^2 + (x_1 + \cdots + x_5)^2\right) + tx_6^3. \tag{5.9}$$

More symmetrically, we can write

$$X_t = V(x_0 + \cdots + x_5, x_0^3 + \cdots + x_5^3 + x_6(x_0^2 + \cdots + x_5^2) + tx_6^3).$$

In this particular case, the symplectic condition is automatic as $A_6$ is a simple group (see also Sect. 6.1 below).

## 6 Some remarks on the full automorphism groups for smooth cubic fourfolds

In this section we discuss about automorphisms and automorphism groups of smooth cubic fourfolds in general (i.e. without the symplectic assumption). We first discuss some general structure results in Sect. 6.1 (the same arguments apply to $K3$ surfaces or hyper-Kähler manifolds). In Sect. 6.2, we obtain some estimate on "how non-symplectic" the automorphism group of a cubic fourfold can be. Finally, in Sect. 6.3, we give some arithmetic conditions for smooth cubic fourfolds to admit non-symplectic automorphisms of order 2, 3 or 4, and then use this to find the full automorphism groups for smooth cubic fourfolds with $\mathrm{rank}(S) = 20$.

### 6.1 Basic structures of the full automorphism groups

Let $X$ be a smooth cubic fourfold, and $G = \mathrm{Aut}(X)$ the automorphism group. The induced action of $G$ on $H^{3,1}(X)$ gives a character $\chi : G \longrightarrow \mathbb{C}^\times$, with kernel the symplectic automorphism group $G_s = \mathrm{Ker}(\chi)$. The image of $\chi$ is a cyclic group which we denoted by $\overline{G}$. We have the following short exact sequence of finite groups:

$$1 \longrightarrow G_s \longrightarrow G \longrightarrow \overline{G} \longrightarrow 1.$$

As before, the symplectic part $G_s \subset \mathrm{Aut}(X)$ induces a Leech pair $(G_s, S)$. Denote by $T(X) \subset H^4(X, \mathbb{Z})$ the transcendental lattice of $X$. Note

$$T(X) \subset H^4(X, \mathbb{Z})^{G_s}_{prim} = S^\perp_{\Lambda_0}.$$

The induced action of the full automorphism group $G$ on $H^4(X,\mathbb{Z})$ (or $H^4(X,\mathbb{Z})_{prim}$) preserves the algebraic and transcendental lattices. Since $G_s$ acts trivially on $T(X)$, the action of $G$ on $T(X)$ factors through an action of $\overline{G}$ on $T(X)$. Clearly, the action of $\overline{G}$ preserves the Hodge structure on $T(X)$, and in particular it preserves the subspace $H^{3,1}\cong\mathbb{C}\subset T(X)$. Choosing a generator $\sigma$ of $H^{3,1}$ (i.e., the class of a (3, 1) form on $X$), we see that $\overline{G}$ acts on $\sigma$ by roots of unity, i.e., for any $1\neq\xi\in\overline{G}$, we have

$$\xi.\sigma=\zeta\sigma$$

for some root of unity $\zeta(\neq 1)\in U(1)\subset\mathbb{C}^*$. Thus the invariant subspace $T(X)^{\xi}$ is purely of Hodge type (2, 2). Since $T(X)$ is transcendental, we must have $T(X)^{\xi}=\{0\}$. In conclusion, we have:

**Lemma 6.1** *The induced action of $\overline{G}$ on $T(X)\backslash\{0\}$ is free.*

Denote by $n$ the order of $\overline{G}$ (i.e., $\overline{G}\cong n$). Lemma 6.1 and standard algebra leads to the following:

**Corollary 6.2** *We have $\varphi(n)\big|\mathrm{rank}(T(X))$. Here $\varphi$ is the Euler function.*

***Proof*** Let $\xi$ be a generator of $\overline{G}$, and $\zeta$ a primitive $n$-root of unity such that $\xi.\sigma=\zeta\sigma$ for $\sigma\in H^{3,1}(X)$. By Lemma 6.1, all the eigenvalues of $\xi$ on $T(X)$ are primitive $n$-roots of unity. The characteristic polynomial $p_\xi$ of $\xi$ (as an automorphism of $T(X)$) is rational. It follows that $p_\xi$ is a power of the cyclotomic polynomial. The claim follows. □

### 6.2 Order of the non-symplectic part

The list of smooth cubic fourfolds with a prime order automorphism is known. Specifically, according to [17, Theorem 3.8] there are 13 irreducible families[8] of cubics with a prime order automorphism. In particular,

**Proposition 6.3** *A prime factor of the order of the automorphism group of a smooth cubic fourfold can only be* 2, 3, 5, 7, *or* 11. *The only primes that can be an order of a non-symplectic automorphism of a smooth cubic fourfold are* 2 *and* 3.

***Proof*** The list of prime orders is a consequence of [17, Theorem 3.8]. The second part follows by noticing that 7 of the 13 cases were already identified in Theorem 4.15 as the symplectic cases (see also Remark 4.16). The symplectic cases cover all the cases involving the primes 5, 7, and 11. The claim follows. □

By Proposition 6.3, the order of $\overline{G}$ has only prime factors 2 or 3. Thus, we can write $n(=|\overline{G}|)=2^k3^l$. From Corollary 6.2 and the fact $T(X)\subset S^{\perp}_{\Lambda_0}$ we get:

$$\varphi(n)=\varphi(2^k3^l)\leq 22-\mathrm{rank}(S). \tag{6.1}$$

As mentioned, the induced action of $G$ on $H^4(X,\mathbb{Z})$ preserves the algebraic and transcendental lattices. In fact $G$ preserves also the covariant lattice $S(=S_{G_s}(X))$.

**Lemma 6.4** *The induced action of $G$ on $H^4(X,\mathbb{Z})$ leaves $S$ stable.*

[8] The case $\mathcal{F}_5^2$ in [17, Theorem 3.8] should be excluded, as the corresponding family contains only singular cubic fourfolds. This was pointed out in [6].

***Proof*** The subgroup $G_s$ is normal in $G = \mathrm{Aut}(X)$. Thus for any $g \in G$, $gG_sg^{-1} = G_s$. By definition, $S$ is the orthogonal complement of the invariant lattice $\Lambda^{G_s}$. Clearly $G_s = gG_sg^{-1}$ leaves every vector in $g\Lambda^{G_s}$ invariant. It follows that $g\Lambda^{G_s} = \Lambda^{G_s}$. By taking orthogonal complements, we see that $g$ leaves $S$ stable. □

The action of $G$ on $S$ induces a homomorphism $\pi : G \longrightarrow \mathrm{Aut}(q_S)$. Since $G_s$ acts trivially on $q_S$, the homomorphism $\pi$ descends to a morphism $\pi : \overline{G} \longrightarrow \mathrm{Aut}(q_S)$.

**Proposition 6.5** *When* $\mathrm{rank}(S) \geq 13$*, the homomorphism* $\pi : \overline{G} \longrightarrow \mathrm{Aut}(q_S)$ *is injective.*

***Proof*** Suppose $g \in G \setminus G_s$ acts trivially on $q_S$. Thus, the action of $g$ on $S$ is by isometries preserving the discriminant. As previously discussed, any such isometry of $S$ can be lifted to a symplectic automorphism of $X$. Thus, there exists $h \in G_s$, such that the restrictions of $g$ and $h$ to $S$ are the same. Replacing $g$ by $gh^{-1}$, we can assume (without loss of generality) that $g$ acts trivially on $S$.

Replacing $g$ by a power $g^k$, we can further assume that $g$ has prime order. By Proposition 6.3, we can assume that $g$ is of order 2 or 3.

By the classification in [17] and the discussion in [54, §6], there are two conjugacy classes of non-symplectic involutions with the corresponding moduli spaces being arithmetic quotients of type IV domains of dimensions 10 and 14 (the 14-dimensional case is discussed in detail in [35]). In particular, the invariant sublattice of $\Lambda_0$ (which contains $S$) is of rank 12 or 8 respectively, contradicting the assumption that $\mathrm{rank}(S) \geq 13$. The order 3 case is similar. Namely, there are 4 conjugacy classes of non-symplectic order three automorphisms, whose moduli spaces are arithmetic ball quotients of respective dimensions (the 10-dimensional case is [2]). Again, the automorphism $g$ cannot leave a sublattice of rank at least 13 of $\Lambda_0$ invariant, a contradiction. □

The proposition above is very useful in the cases where $S$ is of large rank, or equivalently $G_s$ is relatively large; this is the case of interest in this paper. In fact, note that most of the cases in Theorem 1.2 satisfy $\mathrm{rank}(S) \geq 13$. It would be interesting to classify the possible orders $n = 2^k3^l$ of non-symplectic automorphisms on a cubic fourfold, especially we do not know which largest $n$ may occur. These cases will have essentially trivial symplectic automorphism group, thus they should be handled by different methods.

***Remark 6.6*** A major difference between the lattice theoretic methods in the symplectic and anti-symplectic cases is that the covariant lattice $N$ for an anti-symplectic automorphism contains the transcendental lattice $T(X)$, and thus (except the case $\mathrm{rank}(T(X)) = 2$) $N$ is indefinite (in particular, $O(N)$ is typically infinite).

### 6.3 Maximal cases

We conclude our discussion of the automorphism groups of cubic fourfolds with a discussion of the full automorphism group for the 8 maximal cases (with respect to symplectic automorphisms) identified in Theorem 1.8. These are the most interesting cases from the perspective of this paper, and they are particularly suitable for classification (see Proposition 6.4 and Remark 6.6, and note $\mathrm{rank}(S) = 20$, $\mathrm{rank}(T) = 2$).

Since we assume $\mathrm{rank}(S) = 20$, the transcendental lattice $T(X)$ is the orthogonal complement of $S(-1)$ in $H^4(X, \mathbb{Z})_{prim}$ and has rank 2. From Equation (6.1) we deduce that the possible orders for the non-symplectic part $\overline{G}$ are $n = 2, 3, 4$, or 6. We discuss first the case of anti-symplectic involutions.

An involution on a cubic $X$ can be diagonalized to one of the following three types: $\mathrm{diag}(-1, 1, 1, 1, 1, 1)$, $\mathrm{diag}(-1, -1, 1, 1, 1, 1)$, and $\mathrm{diag}(-1, -1, -1, 1, 1, 1)$ (see also [17]). The involution $\mathrm{diag}(-1, -1, 1, 1, 1, 1)$ is symplectic, while the other two are anti-symplectic.

***Remark 6.7*** (Eckardt points) An essential ingredient in the geometric classification of the automorphism groups of cubic surfaces are the *Eckardt points* (see [12] and [46]). The Eckardt points can be defined for cubics of any dimension (e.g. see [35]). From the perspective of automorphisms, a smooth cubic $n$-fold $V \subset \mathbb{P}^{n+1}$ has an Eckardt point iff it is invariant with respect to an involution $\iota$ that fixes a hyperplane (thus of type $\mathrm{diag}(-1, 1, \ldots, 1)$); the Eckardt point is the isolated fixed point of $\iota$. Explicitly, $V$ is defined by a cubic polynomial $F(x_2, \ldots, x_{n+2}) + x_1^2 L(x_2, \ldots, x_{n+2})$, where $\deg(F) = 3$ and $\deg(L) = 1$; $[1 : 0 : \cdots : 0 : 0] \in V$ is an Eckardt point. We refer to [35] for further details.

We have the following necessary condition for a smooth cubic fourfold with maximal symplectic symmetry to admit an anti-symplectic involution.

**Proposition 6.8** *Let $X$ be a smooth cubic fourfold with* $\mathrm{rank}(S) = 20$. *Suppose there exists an anti-symplectic involution on $X$, then the composition of $S \oplus E_6 \hookrightarrow H_0^4(X, \mathbb{Z}) \oplus E_6 \hookrightarrow \mathbb{B}$ is not primitive.*

***Proof*** By Lemma 6.4, the induced involution $\iota^*$ on $H_0^4(X, \mathbb{Z})$ preserves $S = S_{G_s}(X)$. Since $\iota^*$ acts as $-id$ on the orthogonal complement of $S$ in $H^4(X, \mathbb{Z})$, the invariant sublattice $M = H_0^4(X, \mathbb{Z})^{\iota^*}$ of $H_0^4(X, \mathbb{Z})$ is contained in $S$. Suppose $j \colon S \oplus E_6 \hookrightarrow \mathbb{B}$ is primitive, then the inclusion $j \colon M \oplus E_6 \hookrightarrow \mathbb{B}$ is also primitive.

On the other hand, the involution $\iota^*$ on $H_0^4(X, \mathbb{Z})$ extends to an involution on $\mathbb{B}$ with trivial restriction to $E_6$. The invariant sublattice of $\mathbb{B}$ under the action of $\iota^*$ is $M \oplus E_6$. This is a contradiction, because the invariant sublattice (in a unimodular lattice) of an involution has 2-group as its discriminant group, while $|A_{E_6}| = 3$. □

In particular, this allows us to distinguish the two cases of Theorem 1.8(2) with symplectic automorphism group $A_7$. Namely, we note that cubic fourfold with $A_7$ automorphisms identified by Höhn–Mason has an extra symplectic involution, while the other does not have.

**Corollary 6.9** *Let* $X = V(x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 - (x_1 + x_2 + x_3 + x_4 + x_5 + x_6)^3)$ *with symplectic automorphism group $A_7$ (cf. [25, Table 2]). Let $S$ be the covariant sublattice of $H^4(X, \mathbb{Z})$ with respect to the induced action of $A_7$. Then the orthogonal complement $T$ of $S$ in $H_0^4(X, \mathbb{Z})$ is* $-(2^1 18)$, *and* $q_T = 5^{+1}7^{+1}$.

***Proof*** This is the (Clebsch) diagonal cubic, and thus its automorphism group is $S_7$. Obviously, $\overline{G} \cong S_7/A_7 \cong 2$ (one can easily verify that exchanging $x_1, x_2$ is an anti-symplectic involution of $X$). By Proposition 6.8, the inclusion $j \colon S \oplus E_6 \hookrightarrow \mathbb{B}$ is not primitive. From the proof of Theorem 1.8, we conclude $T = -(2^1 18)$ and $q_T = 5^{+1}7^{+1}$. □

Similarly, we get:

**Corollary 6.10** *The cubic fourfold $X^2(A_7)$, and those with symplectic automorphism groups $G_s = L_2(11)$ and $M_{10}$, have no anti-symplectic involution (equivalently, the order of $\overline{G}$ is odd).*

We now switch our attention to the case of anti-symplectic automorphisms of order 3 and 4. The main point here is that in these cases $T(X)$ has a decomposition into two conjugate eigenspaces, and in fact it acquires the structure of a (Hermitian) lattice over $\mathbb{Z}[\omega]$ or $\mathbb{Z}[i]$, Eisenstein or Gaussian integers respectively. This fact is the starting point of multiple works by Kondō (e.g. [10]) and Allcock–Carlson–Toledo (e.g. [2]). In our situation, $T(X)$ is of rank 2, and thus of rank 1 as Eisenstein/Gaussian lattice. This allows us to obtain the following simple criterion for $|\overline{G}|$ to be a multiple of 3 or 4.

**Lemma 6.11** *Let $T$ be a positive definite rank 2 even lattice. Then $T$ admits an automorphism of order 3 if and only if then there exists a positive integer $a$ such that $T \cong A_2(a)$, and $T$ admits an automorphism of order 4 if and only if there exists a positive integer $a$ such that $T \cong A_1^2(2a)$.*

***Proof*** The lattice $A_2 = (1^2 1)$ admits an order 3 automorphism, explicitly $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$. The lattice $A_1^2 = (2^0 2)$ admits an order 4 automorphism, explicitly $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Thus we have necessity.

Suppose $T$ admits an automorphism $\rho$ of order 3. Choose $v \in T$ with minimal norm. Take $a$ such that $(v, v) = 2a$. A nontrivial order 3 automorphism on $T$ is fixed-point free, hence $v + \rho(v) + \rho(\rho(v)) = 0$. Thus

$$(v, v) = (\rho(\rho(v)), \rho(\rho(v))) = (v + \rho(v), v + \rho(v)) = 2(v, v) + 2(v, \rho(v))$$

which implies that $(v, \rho(v)) = -a$. We claim that $(v, -\rho(v))$ is a basis for $T$. If not, then we can find non-zero numbers $\lambda, \mu \in [\frac{1}{2}, \frac{1}{2}]$ such that $\lambda v + \mu\rho(v) \in T$. But $(\lambda v + \mu\rho(v), \lambda v + \mu\rho(v)) = 2\lambda^2 + 2\lambda\mu + 2\mu^2 < 2(|\lambda| + |\mu|)^2 \leq 2$. This contradicts the fact that $v$ has minimal norm. We conclude that $T \cong A_2(a)$.

Suppose $T$ admits an automorphism $\rho$ of order 4. Since $\rho$ is rational, it has two eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$. Thus $\rho^2 = -1$. Now take $v \in T$ with minimal norm $2a$. Then $(v, \rho(v)) = (\rho(v), \rho(\rho(v))) = (\rho(v), -v)$, which implies that $(v, \rho(v)) = 0$. Similarly to the order 3 case, $(v, \rho(v))$ is a basis for $T$. We conclude that $T = A_1^2(2)$. □

We conclude with the computation of the non-symplectic part $\overline{G}$ for the 8 maximal cubic fourfolds appearing in Theorem 1.8.

**Proposition 6.12** (1) *For the Fermat cubic fourfold $X(3^4 : A_6) = V(x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3)$ the order of $\overline{G}$ is $n = 6$.*
(2) *For $X^1(A_7) = V(x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 - (x_1 + x_2 + x_3 + x_4 + x_5 + x_6)^3)$ we have $n = 2$; for $X^2(A_7)$ we have $n = 1$.*
(3) *For the cubic fourfold with symplectic automorphism group $G \cong 3^{1+4} : 2.2^2$, we have $n = 4$.*
(4) *For $X^1(M_{10})$ and $X^2(M_{10})$, we have $n = 1$.*
(5) *For $X(L_2(11)) = V(x_1^3 + x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_6 + x_6^2x_2)$ we have $n = 3$.*
(6) *For $X(A_{3,5}) = V(x_1^3 + x_2^3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_6 + x_6^2x_3)$ we have $n = 6$.*

***Proof*** By Theorem 1.8, we know the transcendental lattices of the 8 cubic fourfolds. By Lemma 6.11, we identify the cubic fourfolds which have order 3 or 4 non-symplectic automorphisms. Combining with Corollary 6.10 we conclude the proof. □

***Remark 6.13*** One easy way to produce geometrically a non-symplectic automorphism of order 3 is to consider a cubic threefold $Y = V(f(x_2, \dots, x_6)) \subset \mathbb{P}^4$. Then the Allcock–Carlson–Toledo [2] construction associates to $Y$ the cubic fourfold $X = V(f + x_1^3) \subset \mathbb{P}^5$

with an order 3 anti-symplectic automorphism ($x_1 \to \omega x_1$). Of the six items of Proposition 6.12, note that items (1), (5), and (6) are of Allcock–Carlson–Toledo type. ((1) and (6) also have an anti-symplectic involution given by switching $x_1 \to x_2$.) In other words, they are obtained from highly symmetric cubic threefolds.

Kondō [30] proved that the $K3$ surface

$$V(x_1^4 + x_2^4 + x_3^4 + x_4^4 + 12x_1x_2x_3x_4) \tag{6.2}$$

has finite automorphism group of maximal possible order 3, 840. Here we conclude an analogue of Kondō's result. Namely, the Fermat cubic fourfold has maximal order for the automorphism group, namely $|3^4 : A_6| \times |\mathbb{Z}/6| = 174{,}960$.

**Corollary 6.14** *The maximal possible order for automorphism groups of smooth cubic fourfolds is* 174,960*, which is reached only by the Fermat cubic fourfold.*

***Proof*** The order of automorphism group $G$ for a smooth cubic fourfold is given by the product of $|G_s|$ and $n = |\overline{G}|$. The value of $n$ is bounded by (6.1). The claim follows by a straightforward inspection of Theorem 1.2, Theorem 1.8, and Proposition 6.12. □

**Acknowledgements** Most of the work was done while the second author visited Stony Brook during the Spring 2018 semester. His stay was supported by Tsinghua Scholarship for Overseas Graduate Studies. He thanks Stony Brook for hosting him and he is grateful to his advisor, Eduard Looijenga, for constant support and helpful discussions on related topics. The research of the first author was partially supported by NSF grants DMS-1254812 and DMS-1802128.

**Funding** Open Access funding enabled and organized by Projekt DEAL.



## Appendix A: Some lattice theory

We review some of the basic results of Nikulin [43] on lattices and discuss the standardized notation of Conway–Sloane [8] (which is less familiar in algebraic geometry). The Conway–Sloane notation is quite efficient and precise, and it is used in one of our primitive references [24]. Thus, we are using it systematically throughout the paper. This appendix aims to set up the basics as used in our paper (for further details, we refer to [43] and [8]).

### A.1: Lattices

We introduce some notation and results in lattice theory. Recall that a lattice over an integral ring $R$ is a free $R$-module of finite rank together with a non-degenerate bilinear form valued in $R$. An integral lattice is a lattice over $\mathbb{Z}$. An integral lattice is called *even* if the norms of all elements are even numbers; called *odd* if it is not even. Once an ordered basis for an $R$-lattice is chosen, there is an associated symmetric Gram (or intersection) matrix. The *discriminant* of an $R$-lattice is the absolute value of the determinant of the intersection matrix. The discriminant does not depend on the choices of the basis. An $R$-lattice is called *unimodular* if its discriminant is 1. An integral lattice $M$ can be diagonalized as $\mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$ over $\mathbb{R}$. Let $n_1$ be the multiplicity of 1, and $n_2$ that of $-1$. Then $n_1 + n_2$ is the rank of $M$, and $(n_1, n_2)$ is called the *signature of* $M$.

An element $v$ in an $R$-lattice $M$ is called *primitive* if $v$ is non-zero and for any integer $n \geq 2$, the quotient $v/n$ is not in $M$. A sublattice $N$ of $M$ is called *primitive*, if there does not exist an element $v \in M \setminus N$ and a positive integer $n \geq 2$ such that $nv \in N$. An embedding of lattices $N \hookrightarrow M$ is called *primitive* if the image is a primitive sublattice.

We use $\langle n \rangle$ to denote the rank one lattice such that the norm of the generator is equal to $n$. For an $R$-lattice $M$ and $n \in \mathbb{Z}$, we define $M(n)$ to be an $R$-lattice obtained from $M$ by multiplying the bilinear form by $n$. In the category of $R$-lattices, we have naturally direct sum $\oplus$. For a Dynkin diagram $A_k$, $D_k$ or $E_k$, there is the associated intersection matrix, which defines a positive integral lattice, still denoted by the same symbol $A_k$, $D_k$ or $E_k$. We use $U$ to denote the hyperbolic lattice, given by the intersection matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

We have a classification of integral unimodular lattices (e.g. [47, Chapter 5]):

**Theorem A.1** *(Milnor) An odd indefinite unimodular integral lattice of signature* $(n_1, n_2)$ *is isomorphic to* $\mathrm{I}_{n_1,n_2} = (1)^{n_1} \oplus (-1)^{n_2}$*. An even indefinite unimodular integral lattice of signature* $(n_1, n_2)$ *exists if and only if* $n_1 \equiv n_2 \pmod 8$*, and when this holds, the lattice is isomorphic to* $\mathrm{II}_{n_1,n_2} = E_8^{\frac{n_1-n_2}{8}} \oplus U_2^{n_2}$ *or* $E_8(-1)^{\frac{n_2-n_1}{8}} \oplus U_2^{n_1}$.

***Remark A.2*** The structure theory in the definite case is much more complicated. For example, we have 24 Niemeier lattices (see Theorem 3.2), all of which are positive definite, unimodular, even and of rank 24.

### A.2: Classification of p-adic lattices, and Conway–Sloane's notation

For any prime $p$, we use $\mathbb{Z}_p$ for the ring of $p$-adic integers, and $\mathbb{Q}_p$ for the field of $p$-adic numbers. We next discuss about the classification of $\mathbb{Z}_p$-lattices, and the standard notation of Conway and Sloane [8]. We also call a lattice over $\mathbb{Z}_p$ a *p-adic lattice*. Let $Qu(\mathbb{Z}_p)$ be the semigroup of $p$-adic lattices (with respect to $\oplus$).

For $\theta \in \mathbb{Z}_p^*/(\mathbb{Z}_p^*)^2$, denote by $K_\theta(p^k)$ the $p$-adic lattice determined by the matrix $\langle \theta p^k \rangle$. For $p$ an odd prime, $\mathbb{Z}_p^*/(\mathbb{Z}_p^*)^2$ contains two elements. For $p = 2$, $\mathbb{Z}_2^*/(\mathbb{Z}_2^*)^2$ has four elements represented by $1, 3, 5, 7 \in \mathbb{Z}_2^*$. For the case $p = 2$, we need to also consider lattices $U(2^k) = \begin{pmatrix} 0 & 2^k \\ 2^k & 0 \end{pmatrix}$ and $V(2^k) = \begin{pmatrix} 2^{k+1} & 2^k \\ 2^k & 2^{k+1} \end{pmatrix}$ for any $k \geq 0$.

**Proposition A.3** [43, Proposition 1.8.1] *For $p$ an odd prime, the semigroup $Qu(\mathbb{Z}_p)$ is generated by $K_\theta(p^k)$. For $p = 2$, the semigroup $Qu(\mathbb{Z}_2)$ is generated by $K_\theta(2^k)$, $U(2^k)$ and $V(2^k)$.*

For $p$ odd, any $p$-adic lattice $K$ can be written as a direct sum of rank one $p$-adic lattices. Explicitly, the quadratic form $q$ can be decomposed as

$$K = K_1 \oplus pK_p \oplus p^2 K_{p^2} \oplus \cdots \oplus lK_l \oplus \cdots,$$

where $l$ are powers of $p$, the determinant of each $K_l$ is coprime to $p$. Here the p-adic lattice $lK_l$ can be write as a direct sum of several p-adic lattices of the form $K_\theta(l)$. Following [8, Chapter 15, §7], *the p-adic quadratic form $lK_l$ is denoted by $l^{\epsilon_l n_l}$*. Here $n_l$ is the rank of $K_l$, and $\epsilon_l$ is $+$ if $\det(K_l)$ is a square in $\mathbb{Z}_p^*$; is $-$ otherwise. Then the $p$-adic form $K$ is written as $1^{\epsilon_1 n_1} p^{\epsilon_p n_p} \cdots l^{\epsilon_l n_l} \cdots$. We call this the *Conway–Sloane expression* for $K$.

**Proposition A.4** *For $p$ odd, a $p$-adic lattice $K$ has a unique Conway–Sloane expression $1^{\epsilon_1 n_1} p^{\epsilon_p n_p} \cdots l^{\epsilon_l n_l} \cdots$.*

For $p = 2$ the notation is more complicated. In this case, any 2-adic lattice $(M, q)$ can be written as a direct sum of rank one 2-adic lattices or rank two 2-adic lattices of the forms $\begin{pmatrix} 2^k a & 2^k b \\ 2^k b & 2^k c \end{pmatrix}$, where $a, c$ are even and $b$ is odd. Explicitly, the 2-adic quadratic form $K$ can be decomposed as

$$K = K_1 \oplus 2K_2 \oplus 2^2 K_{2^2} \oplus \cdots \oplus lK_l \oplus \cdots,$$

where $l$ are powers of 2, and the determinant of each $K_l$ is odd. By [8, Chapter 15, §7], the 2-adic quadratic form $lK_l$ is written as $l_{S_l}^{\epsilon_l n_l}$. Here $n_l$ is the rank of $K_l$, and $\epsilon$ is $+$ if $\det(K_l)$ is congruent to 1 or 7 modulo 8; is $-$ otherwise. A 2-adic lattice is called even, if the norm of each vector is even; odd otherwise. If the 2-adic lattice $K_l$ is even, then $S_l = \mathrm{II}$, and we say $lK_l$ are of even type. When $lK_l$ is of even type, it can be decomposed as a direct sum of 2-adic lattices of the form $U(l)$ or $V(l)$. If $K_l$ is odd, then $S_l = \mathrm{Tr}(K_l) \in \mathbb{Z}/8\mathbb{Z}$, and we say $lK_l$ are of odd type. When $lK_l$ is of odd type, it can be decomposed as a direct sum of 2-adic lattices of the form $K_\theta(l)$, for $\theta \in \mathbb{Z}_2^*/(\mathbb{Z}_2^*)^2$. The 2-adic form $K$ can be written as $1_{S_1}^{\epsilon_1 n_1} 2_{S_2}^{\epsilon_2 n_2} \cdots l_{S_l}^{\epsilon_l n_l} \cdots$. The ways to express a 2-adic form as above are not unique, but there is a canonical way to do this (see [8, Chapter 15, §7.6]).

***Remark A.5*** The following conditions must hold for any 2-adic constituent $l_{S_l}^{\epsilon_l n_l}$ of rank $n$:

(1) If $n = 0$, then $S_l = \mathrm{II}$ and $\epsilon_l = +$.
(2) If $n = 1$, then the form is of odd type. In this case, if $\epsilon_l = +$, then $S_l$ is congruent to 1 or 7 modulo 8; if $\epsilon = -$, then $S_l$ is congruent to 3 or 5 modulo 8.
(3) if $n = 2$ and the form is of odd type, then $\epsilon = +$ implies that $S_l$ is congruent to 0, 2 or 6 modulo 8, $\epsilon = -$ implies that $S_l$ is congruent to 2, 4 or 6 modulo 8.

For further discussion, we refer to [8, Chapter 15, §7.8] (esp. [8, Table 15.5]).

Two integral lattices are said to have the same *genus* if they have the same signature and are equivalent over the $p$-adic integers for all $p$. Under mild conditions, for indefinite lattices, there exists a single isometry class in a given genus. For definite lattices, typically there are multiple isometry classes in a genus (e.g. compare Theorems A.1 and 3.2 in the unimodular case).

### A.3: Conway–Sloane's expression for finite quadratic forms

One of the main tools in the Nikulin's theory [43] is the systematic use of finite discriminant forms. Here we review the basics, and we connect it with the Conway–Sloane notation.

Given an integral lattice $M$, we denote $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, *the dual lattice*. We have naturally $M \hookrightarrow M^* \hookrightarrow \mathrm{Hom}_{\mathbb{Q}}(M_{\mathbb{Q}}, \mathbb{Q})$, where the first map sends $x$ to $(x, \cdot)$. Define *the discriminant group* of $M$ to be $A_M = M^*/M$, which is a finite group of order the discriminant of the lattice. For a finite abelian group $A$, we denote by $l(A)$ the minimal number of generators in $A$. The bilinear form on $M$ induces a bilinear form $b_M$ on $A_M$ valued in $\mathbb{Q}/\mathbb{Z}$, by sending

$[v], [w] \in A_M$ (with $v, w \in M^*$) to $[(v, w)] \in \mathbb{Q}/\mathbb{Z}$. If $M$ is even, we can define a quadratic form

$$q_M : A_M \to \mathbb{Q}/2\mathbb{Z},$$

by sending $[v] \in A_M$ to $[(v, v)] \in \mathbb{Q}/2\mathbb{Z}$. The quadratic form $q_M$ determines $b_M$ via the relation:

$$b_M([v], [w]) = \frac{1}{2}(q_M([v + w]) - q_M([v]) - q_M([w])).$$

The quadratic form $q_M$ is called *the discriminant form* of $M$. We sometimes write $q_M$ instead of $(A_M, q_M)$. In the rest of the Appendix, we restrict ourselves to the case of even lattices.

Any finite quadratic form $(A, q)$ has a unique decomposition $(A, q) = \oplus_p (A_p, q_p)$. Here $A_p$ is the group of elements in $A$ whose orders are $p$-powers, and $q_p$ is the restriction of $q$ to $A_p$. The finite quadratic form $q_p$ takes value in $\mathbb{Q}_p/\mathbb{Z}_p \cong \mathbb{Q}^{(p)}/\mathbb{Z}$ if $p$ is odd, and in $\mathbb{Q}_2/2\mathbb{Z}_2 \cong \mathbb{Q}^{(2)}/2\mathbb{Z}$ if $p = 2$.

Let $qu(\mathbb{Z}_p)$ be the semigroup of finite quadratic forms on an abelian group with order a $p$-power. Denote by $q_\theta(p^k)$ the discriminant form of the $p$-adic lattice $K_\theta(p^k)$. Denote by $u(2^k)$, $v(2^k)$ the discriminant form of the 2-adic lattices $U(2^k)$, $V(2^k)$ respectively.

**Proposition A.6** [43, Proposition 1.8.1] *The semigroup $qu(\mathbb{Z}_p)$ is generated by $q_\theta(p^k)$ if $p$ is an odd prime; by $q_\theta(2^k)$, $u(2^k)$ and $v(2^k)$ if $p = 2$.*

The following theorem (see [43, Theorem 1.9.1]) tells that except very special cases (which occur only for $p = 2$), a finite quadratic form over $\mathbb{Z}_p$ is induced uniquely by a $p$-adic form:

**Theorem A.7** (Nikulin) *Let $p$ be a prime and $(A, q) \in qu(\mathbb{Z}_p)$. There exists a unique $p$-adic lattice $K(q) \in Qu(\mathbb{Z}_p)$ of rank $l(A)$ whose discriminant form is isomorphic to $q$, except in the case when $p = 2$ and $q$ is $q_\theta(2) \oplus q_2'$ for some $\theta \in \mathbb{Z}_2^*/(\mathbb{Z}_2^*)^2$.*

*If $q = q_\theta(2) \oplus q_2'$, there are precisely two 2-adic lattices $K_{\alpha_1}(q)$ and $K_{\alpha_2}(q)$ of rank $l(A)$ whose discriminant forms are isomorphic to $q$. Here $disc(K_{\alpha_i}(q)) = \alpha_i |A| (\mathbb{Z}_2^*)^2$ for $i = 1, 2$, where $\alpha_1, \alpha_2 \in \mathbb{Z}_2^*/(\mathbb{Z}_2^*)^2$ and $\alpha_1\alpha_2 = 5(\mathbb{Z}_2^*)^2$.*

Given $q \in Qu(\mathbb{Z}_p)$, we have then a $p$-adic lattice $K(p)$ of rank $l(q)$ whose discriminant form is $q$. The Conway–Sloane expression of the $p$-adic lattice $K(q)$ is used also to denote $q$. Notice that when $p = 2$ and $q = q_\theta(2) \oplus q'$, the expression of $q$ is not unique. A finite quadratic form $q \in Qu(\mathbb{Z})$ can be uniquely decomposed as a direct sum of finite quadratic forms over $\mathbb{Z}_p$. Putting together the Conway–Sloane expressions for those sub forms of $q$, we get a Conway–Sloane expression for $q$.

### A.4: Nikulin's criterions

We repeatedly use in our arguments two key results of Nikulin: the criteria for existence and uniqueness of embeddings of lattices into unimodular lattices (in our case, the relevant unimodular lattices are the Leech lattice $\mathbb{L}$ and the Borcherds lattice $\mathbb{B} = \mathbb{L} \oplus U^2$). Specifically, the following is Nikulin's far-reaching generalization of Theorem A.1 for even lattices.

**Theorem A.8** (Nikulin [43, Thm. 1.10.1]) *An even lattice of invariant $(n_1, n_2, A, q)$ exists if and only if the following conditions are fulfilled:*

(1) $n_2 - n_1 \equiv \operatorname{sig}(q)$ *(mod 8),*
(2) $n_1 \geq 0, n_2 \geq 0, n_1 + n_2 \geq l(A),$

(3) $(-1)^{n_2}|A| \equiv disc(K_{q_p})$ *(mod* $(\mathbb{Z}_p^*)^2$*) for all odd primes* $p$ *with* $n_1 + n_2 = l(A_p)$,
(4) $|A| = \pm disc(K_{q_2})$ *(mod* $(\mathbb{Z}_2^*)^2$*) if* $n_1 + n_2 = l(A_2)$ *and* $q_2 \neq q_\theta(2) \oplus q_2'$.

An embedding of a lattice $M$ into a unimodular lattice exists iff a lattice with complementary invariants (most notably discriminant form $-q_M$) exists. The above theorem allows one to settle this question. If an embedding exists, the uniqueness of the embedding can be often settled by the following result.

**Theorem A.9** (Nikulin [43, Thm. 1.14.4]) *Let* $S$ *be an even lattice of signature* $(n_1, n_2)$ *and let* $M$ *be an even unimodular lattice of signature* $(l_1, l_2)$*. There exists a unique primitive embedding of* $S$ *into* $M$ *if the following conditions are fulfilled:*

(i) $l_1 > n_1, l_2 > n_2$,
(ii) $l_1 + l_2 - n_1 - n_2 \geq l(A_S) + 2$.

## Appendix B: Finite groups

In this appendix, we make a quick review of some facts on finite groups relevant for the present paper. We follow the notation of Mukai [41] and Höhn–Mason [24,25].

### B.1: Extensions of finite groups

Let $N$, $Q$ be two finite groups. An extension of $Q$ by $N$ is a finite group $E$ with a short exact sequence:

$$1 \longrightarrow N \longrightarrow E \xrightarrow{p} Q \longrightarrow 1. \tag{B.1}$$

Suppose there is a group homomorphism $r\colon Q \longrightarrow E$ with $p \circ r = id$, then the sequence (B.1) is called split. In this case, $E$ is denoted by $N \rtimes Q$, which is called the a semidirect product of $N$ and $Q$. Semidirect products of $N$ and $Q$ are not unique, and are uniquely determined by group homomorphisms $Q \longrightarrow \mathrm{Aut}(N)$. Following [24,25], we use $N : Q$ to represent a semidirect product of $N$ and $Q$ that is not a direct product, and use $N.Q$ to represent an extension of $Q$ by $N$ for which we are not sure whether it is split or not.

### B.2. Mathieu groups

The series of Mathieu groups consists of five sporadic groups denoted by $M_{11}$, $M_{12}$, $M_{22}$, $M_{23}$, $M_{24}$. This is the first series of sporadic groups, which was found by Mathieu (1861, 1873). No other sporadic groups were found until 1965, when the first Janko group was found. There are also Mathieu groups $M_8$, $M_9$, $M_{10}$, $M_{20}$, $M_{21}$, with the first four not simple and the last one isomorphic to $\mathrm{PSL}(3, \mathbb{F}_4)$, which is simple but not sporadic.

We give the definition of the largest Mathieu group $M_{24}$ (see [13, Appendix B] for further details). Let $N$ be the Niemeier lattice with root lattice $L$ of type $24A_1$. Then $L \subset N \subset L^*$ and $L^*/L \cong (\mathbb{F}_2)^{24}$, where $\mathbb{F}_2$ is the field with 2 elements. The quotient $\mathscr{G} = N/L \subset \mathbb{F}_2^{24}$ is a 12-dimensional vector space over $\mathbb{F}_2$, characterized by the property that each nonzero vector in it has at least 8 nonzero coordinates. This vector subspace $\mathscr{G}$ is known as the extended binary Golay code. The permutation group $S_{24}$ acts on $\mathbb{F}_2^{24}$ by permuting the 24 coordinates, and the Mathieu group $M_{24}$ is defined to be the subgroup of $S_{24}$ that leaves $\mathscr{G}$ stable. The smaller Mathieu groups $M_{24-i}$ can be defined as the subgroups of $M_{24}$ stabilizing $i$ coordinate axes in $\mathbb{F}_2^{24}$, where $i = 1, 2, 3$ or 4.

***Remark B.1*** For $K3$ surfaces, Kondō's approach [29] to the Mukai's classification of symplectic automorphisms reduces the problem to one about subgroups of $M_{24}$ [and in fact $M_{23}$] (and involves the associated Niemeier lattice of type $24A_1$). For hyper-Kähler manifolds, it is necessary to pass to the Conway group $\mathrm{Co}_0$ (N.B. $M_{24}$ can be embedded into $\mathrm{Co}_0$) and the associated Leech lattice $\mathbb{L}$ (e.g. see [27]). Furthermore, the description in terms of Leech lattice is more uniform; this is the point of view taken in this paper.

A dodecad of $\mathscr{G}$ is a vector with exactly 12 vanishing coordinates. The Mathieu group $M_{12}$ is by definition the stabilizer of a dodecad in $\mathbb{F}_2^{24}$ under the action of $M_{24}$. Then $M_{12}$ is a subgroup of the permutation group $S_{12}$ acting by permutations on the 12 coordinates that vanish on the dodecad. The Mathieu group $M_{12-i}$ can be constructed as stabilizer in $M_{12}$ of $i$ coordinate axes among those 12 which are permuted by $M_{12}$. The action of $M_{12-i}$ on the remaining $12-i$ coordinates is sharply $(5-i)$-transitive.

For $k+l \leq 12$ and $l \geq 8$, $M_{k,l}$ is the subgroup $M_{k+l} \cap S_k \times S_l$ of $M_{k+l}$. This is well-defined since $M_{k+l}$ is $k$-transitive. Moreover, we have an exact sequence:

$$1 \longrightarrow M_l \longrightarrow M_{k,l} \longrightarrow S_k \longrightarrow 1,$$

and thus $|M_{k,l}| = |M_l| \cdot k!$. Let us briefly discuss the groups $M_{3,8}$ and $M_{2,9}$, as they are relevant for our study. The group $M_8$ is isomorphic to the quaternion group $Q_8$, and we have a semidirect product $M_{3,8} \cong Q_8 : S_3$. Mukai [41] denotes this group by $T_{48}$. The group $M_9$ is isomorphic to $\mathrm{PSU}_3(\mathbb{F}_2)$ (see §B.4), and $M_9 \cong 3^2 : Q_8$. We have $M_{2,9} \cong 3^2 : \mathrm{QD}_{16}$. It is natural to expect that $M_{2,9}$ is exactly the group in item 15 of Table 1, but we have not checked all the details.

### B.3: Extraspecial group

For $p$ prime, recall that a $p$-group is a finite group of order a power of $p$.

**Definition B.2** An extraspecial group is a non-abelian $p$-group $G$ with center $Z(G) \cong p$ and an elementary abelian quotient $G/Z(G)$.

Every extraspecial group has order $p^{1+2k}$ with $k$ a positive integer. Conversely, for any prime number $p$ and positive integer $k$, there exist two extraspecial groups of order $p^{1+2k}$. By convention, the symbol $p^{1+2k}$ represents an extraspecial group of order $p^{1+2k}$. For $p=2$ and $k=1$, the two extraspecial groups $2^{1+2}$ are the dihedral group $D_8$ and quaternion group $Q_8$.

### B.4: Linear and projective groups over finite fields

Linear and projective groups over a field $K$ refer to Zariski-closed subgroups of $\mathrm{GL}(n, K)$ or $\mathrm{PGL}(n, K)$. When $K$ is a finite field, these groups are finite and play an important role in the classification of finite simple groups. In the final section we collect such kinds of groups related to our classifications.

We introduce the unitary groups over finite fields. For a finite group $\mathbb{F}_{q^2}$ where $q = p^r$ and $p$ is a prime number, there is an $\mathbb{F}_q$-linear involution $\alpha\colon \mathbb{F}_{q^2} \longrightarrow \mathbb{F}_{q^2}$ sending $x$ to $x^q$ (this is the $r$-th power of the Frobenius automorphism of $\mathbb{F}_q$). Let $V$ be an $n$ dimensional vector space over $\mathbb{F}_{q^2}$, then there is a unique $\mathbb{F}_q$-bilinear form (called Hermitian form over finite field) $H\colon V \times V \longrightarrow \mathbb{F}_{q^2}$ satisfying $H(w, v) = \alpha(H(v, w))$ and $H(v, cw) = cH(v, w)$

for any $c \in \mathbb{F}_{q^2}$. Explicitly,

$$H(v, w) = \sum_{i=1}^{n} v_i^q w_i$$

The unitary group[9] $U(n, q)$ is the automorphism group of the Hermitian space $(V, H)$. We note that the projective special unitary group $\mathrm{PSU}(3, \mathbb{F}_2)$ is isomorphic to the Mathieu group $M_9$, and appears as symplectic automorphism group of a degree 2 $K3$ surface.

The group $\mathrm{PSL}(2, \mathbb{F}_{11})$ is simple and appears as the automorphism group of the Klein cubic threefold $V(x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_4 + x_4^2 x_5 + x_5^2 x_1)$ (see [1]). As shown in Theorem 1.8, there is a unique cubic fourfold with an order 11 automorphism which is a triple cover of $\mathbb{P}^4$ branched along the Klein cubic threefold.

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