A perturbative expansion scheme for supermembrane and matrix theory

Olaf Lechtenfeld[◇] and Hermann Nicolai[⋆]

[◇]Institut für Theoretische Physik and Riemann Center for Geometry and Physics
Leibniz Universität Hannover, Appelstrasse 2, 30167 Hannover, Germany

[⋆]Max-Planck-Institut für Gravitationsphysik (Albert-Einstein-Institut)
Am Mühlenberg 1, 14476 Potsdam, Germany

Abstract

We reconsider the supermembrane in a Minkowski background and in the light-cone gauge as a one-dimensional gauge theory of area preserving diffeomorphisms (APDs). Keeping the membrane tension T as an independent parameter we show that T is proportional to the gauge coupling g of this gauge theory, such that the small (large) tension limit of the supermembrane corresponds to the weak (strong) coupling limit of the APD gauge theory and its $\mathrm{SU}(N)$ matrix model approximation. A perturbative linearization of the supersymmetric theory suitable for a quantum mechanical path-integral treatment can be achieved by formulating a Nicolai map for the matrix model, which we work out explicitly to $\mathcal{O}(g^4)$. The corresponding formulæ remain well-defined in the limit $N \to \infty$, *i.e.* for the supermembrane theory itself. Furthermore we show that the map has improved convergence properties in comparison with the usual perturbative expansions because its Jacobian admits an expansion in g with a non-zero radius of convergence. Possible implications for unsolved issues with the matrix model of M theory are also mentioned.

1 Introduction and summary

The maximally supersymmetric supermembrane theory in spacetime dimension $D = 11$ [1, 2] is a model 'beyond' string theory that also incorporates $D = 11$ supergravity [3], and is thus a candidate theory for a non-perturbative formulation of superstring theory (besides there are three more classically consistent supermembrane theories for target-space dimensions $D = 4, 5, 7$ [2]). As shown long ago [4] the supermembrane in a flat (Minkowski) background and in the light-cone gauge can be reformulated as a one-dimensional maximally supersymmetric gauge theory of area preserving diffeomorphisms (APDs). Building on earlier results of [5, 6] it has been shown that this model can be equivalently obtained as the $N \to \infty$ limit of a maximally supersymmetric SU(N) matrix model [4]. Much later the very same model was re-interpreted in terms of D0 particle quantum mechanics [7], and proposed as a model of M theory in [8]. For reviews of supermembrane theory with many further references, see *e.g.* [9–11].

The main unsolved problem of (super-)membrane theory is its quantization. Unlike for string theory there exists no gauge which linearizes the equations of motion such that the determination of quantum correlators can be effectively reduced to free field theory computations. Likewise, in view of the non-linearities a covariant path-integral approach *à la* Polyakov appears hopeless for either the bosonic or the supersymmetric membrane. A more realistically feasible approach is based on (target-space) light-cone gauge quantization. Nevertheless, even with this preferred gauge choice the solution to the problem of quantization has so far remained elusive not only because one has to deal with a fully interacting theory on the world volume, but also because it is not obvious how to set up a perturbative expansion for the quantized supermembrane. These difficulties are mirrored by corresponding difficulties of the supersymmetric SU(N) matrix model for $N < \infty$, as a consequence of which key issues remain unresolved to this day. Apart from questions regarding the existence and properties of the $N \to \infty$ limit for the quantized theory, there are two main issues. One concerns the target-space Lorentz invariance of the quantized supermembrane (or the matrix model in the $N \to \infty$ limit). For the classical theory and for finite N, Lorentz invariance is in fact violated, but can be recovered in the $N \to \infty$ limit [12, 13]. However, there has been almost no progress on the quantized theory, which would first of all require a proper definition of the quantized Lorentz generators and ensuring their quantum consistency, before actually checking the Lorentz algebra. Amongst other things this involves the correct definition of the light-cone target-space coordinate X^- (which matrix theory by itself 'does not know about') as a quantum operator. Consequently, it also remains an open question whether $D_{\text{crit}} = 11$ is indeed the critical dimension for the supermembrane, eliminating the other classically consistent theories (see however [14] for some early results in this direction), unlike for the superstring where the well-known result $D_{\text{crit}} = 10$ can be established in more than one way.

A second key issue arises in connection with correlators and scattering amplitudes for the putative massless supermembrane excitations corresponding to the graviton, the gravitino and the 3-form field of $D = 11$ supergravity. In particular, there is no (super-)membrane analog known of the Veneziano and Virasoro–Shapiro amplitudes, as this would almost certainly involve higher order and non-perturbative contributions beyond the reach of conventional string technology. Remarkably, there do exist classical candidate expressions for vertex operators associated to these states [15], but like for the Lorentz generators, it has not been possible so far to turn them into well-defined quantum objects.

In this paper we wish to tackle the quantization of the supermembrane from a new and different perspective. A main ingredient here is the fact that the supersymmetric APD gauge theory (*alias* the supersymmetric SU(∞) matrix model) *is* the supermembrane. Our analysis leads us to the conclusion that the membrane tension T, made dimensionless, must be identified with the gauge coupling g of the APD gauge theory. This insight allows us to set up a systematic expansion scheme in terms of a path-integral formulation where the small (large) tension limit of the membrane theory corresponds to the weak (strong) coupling limit of the APD gauge theory – a result to be contrasted with the somewhat murky state of affairs with the zero-tension limit of string theory. This expansion is introduced by means of a Nicolai map [16, 17] (designated by $\mathcal{T}_g$) which we first construct for the finite-N theory up to and including quartic order $\mathcal{O}(g^4)$.[1] Our derivation is based on a systematic procedure that relies on very recent progress in perturbatively evaluating this map for supersymmetric Yang–Mills theories in higher dimensions, see especially [18, 19]. This prescription in principle allows for the determination of the map to *any* desired order. It then turns out that the pertinent formulæ all remain well-defined in the limit $N \to \infty$, via the straightforward replacement of SU(N) commutators by APD brackets, see especially (3.36).[2] These are two main results of this paper, which should eventually permit setting up an approximation scheme also for correlators and other quantities of physical interest. This might enable one to sidestep the finite-N approximation altogether and to deal directly with the limiting theory for $N = \infty$. Finally, we demonstrate that the expansion of the Jacobian in powers of g has a *non-zero* radius of convergence. This does not yet mean that the map itself has this property, but it represents strong evidence that the expansion is indeed better behaved than the usual perturbation expansions, and that it may likewise have a non-zero radius of convergence.

The structure of this paper is as follows. In Section 2 we review basic results on the light-cone gauge formulation of the supermembrane, mostly following the exposition in [4], and set up the path integral in Subsection 2.3. Section 3 explains the construction of the Nicolai map both for the matrix theory and the APD gauge theory. This section contains our main result, namely an explicit form of the map $\mathcal{T}_g$ in an expansion to quartic order in the coupling g. The question of the behavior of the map in the complex g plane (and thus the convergence properties of the expansion) is addressed in Section 4. Finally an Appendix provides details of two consistency checks on the main result derived in this paper.

2 Supermembrane basics

In the section we closely follow [4] to which we refer for further details of the derivation. The main difference is that we here keep the membrane tension T as an independent parameter, in order to expose the link with weakly and strongly coupled Yang–Mills theory.

[1] In the temporal gauge and actually including $O(g^5)$ since odd orders vanish at least up to this point.

[2] An accompanying N-dependent rescaling cancels only for the supermembrane, indicating that the $N \to \infty$ limit does not exist for the ordinary membrane.

2.1 Supermembrane in the light-cone gauge with variable tension

Classically consistent supermembranes exist for target-space dimensions $D = 4, 5, 7$ and 11, but in the remainder we will restrict attention mostly to the maximally supersymmetric case, for which $D = 11$. The target superspace coordinates $\{X^\mu, \theta\} \equiv \{X^\mu(\xi^i), \theta(\xi^i)\}$ with the range $\mu, \nu, \ldots = 0, 1, \ldots, D-1 = 10$ are then functions of the membrane world-volume coordinates

$$(\xi^i) \equiv (\tau, \boldsymbol{\sigma}) \equiv (\tau, \sigma^r) \qquad \text{where} \quad i, j, \ldots = 0, 1, 2 \quad \text{and} \quad r, s, \ldots = 1, 2 \; . \tag{2.1}$$

$\theta(\tau, \boldsymbol{\sigma})$ is a real 32-component Majorana spinor of SO(1,10) (we usually do not write out spinor indices). The target-space vielbein is

$$E_i{}^\mu \;=\; \partial_i X^\mu + \bar{\theta} \Gamma^\mu \partial_i \theta \; . \tag{2.2}$$

For $D = 11$, the real 32-by-32 Γ-matrices generate the SO(1,10) Clifford algebra, $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$. The world-volume metric is

$$\mathsf{g}_{ij} \;\equiv\; E_i{}^\mu E_j{}^\nu \eta_{\mu\nu} \; . \tag{2.3}$$

For the light-cone gauge we split the target-space coordinates as

$$\{X^\mu\} \equiv \{X^+, X^-, X^a\} \qquad \text{with} \qquad X^\pm = \tfrac{1}{\sqrt{2}}(X^{10} \pm X^0) \qquad \text{and} \qquad \{X^a\} \equiv \mathbf{X} \tag{2.4}$$

being the transverse components ($a, b, \ldots = 1, \ldots, 9$). We adopt the target-space light-cone gauge

$$X^+(\tau, \boldsymbol{\sigma}) \;=\; X_0^+ + \tau \; , \qquad \Gamma_+ \theta(\tau, \boldsymbol{\sigma}) \;=\; 0 \tag{2.5}$$

thus identifying the target-space light-cone coordinate X^+ with the world-volume time coordinate τ. With these gauge choices the induced metric on the three-dimensional world volume is

$$\begin{aligned}
\mathsf{g}_{rs} \;&\equiv\; \bar{\mathsf{g}}_{rs} \;=\; \partial_r \mathbf{X} \cdot \partial_s \mathbf{X} \; , \\
\mathsf{g}_{0r} \;&\equiv\; u_r \;=\; \partial_r X^- + \partial_0 \mathbf{X} \cdot \partial_r \mathbf{X} + \bar{\theta} \Gamma_- \partial_r \theta \; , \\
\mathsf{g}_{00} \;&=\; 2\partial_0 X^- + (\partial_0 \mathbf{X})^2 + 2\bar{\theta} \Gamma_- \partial_0 \theta \; .
\end{aligned} \tag{2.6}$$

The metric determinant is

$$\mathsf{g} \;\equiv\; \det \mathsf{g}_{ij} \;=\; -\Delta \, \bar{\mathsf{g}} \tag{2.7}$$

with

$$\bar{\mathsf{g}} \;\equiv\; \det \bar{\mathsf{g}}_{rs} \qquad \text{and} \qquad \Delta \;=\; -\mathsf{g}_{00} + u_r \bar{\mathsf{g}}^{rs} u_s \; , \qquad \bar{\mathsf{g}}^{rs} \bar{\mathsf{g}}_{st} = \delta^r_t \; . \tag{2.8}$$

The supermembrane Lagrangian then becomes

$$\mathcal{L} \;=\; T \left(-\sqrt{\bar{\mathsf{g}} \Delta} + \epsilon^{rs} \partial_r X^a \bar{\theta} \Gamma_- \Gamma_a \partial_s \theta \right) , \tag{2.9}$$

where we now include the membrane tension T as an independent parameter. In principle the membrane tension is of dimension $[\text{mass}]^3$, but we here find it convenient to render all variables dimensionless by rescaling them with appropriate powers of some reference mass scale (as was already implicitly assumed in (2.5)). This reference scale has no physical meaning in and by itself, as a proper identification of the gravitational coupling (Newton constant or Planck mass) and evaluating its relation to T will require the evaluation of a graviton scattering amplitude, as is the case in string theory. However, for the doubly dimensionally reduced supermembrane [20] such a relation can indeed be established by noting that $TR_{10} = (\alpha')^{-1}$, where R_{10} is the radius of the compactified 11th dimension. Because the latter is related to the string coupling by $R_{10} = g_s^{2/3}$ [21] (see also [22]), we see that

$$T \;=\; g_s^{-2/3}\,(\alpha')^{-1}\;. \tag{2.10}$$

In this way the parameter T ties together the two key parameters of string theory, and thus also with the APD gauge coupling via (2.23) below.

With these conventions the (dimensionless) canonical momenta are

$$\begin{aligned} P^+ \;&=\; T\sqrt{\frac{\bar{\mathrm{g}}}{\Delta}}\;, \\ \mathbf{P} \;&=\; \frac{\delta\mathcal{L}}{\delta\partial_0\mathbf{X}} \;=\; T\sqrt{\frac{\bar{\mathrm{g}}}{\Delta}}\big(\partial_0\mathbf{X}-u_r\mathrm{g}^{rs}\partial_s\mathbf{X}\big) \;\equiv\; P^+\big(\partial_0\mathbf{X}-u_r\mathrm{g}^{rs}\partial_s\mathbf{X}\big)\;, \\ S \;&=\; \frac{\delta\mathcal{L}}{\delta\partial_0\bar\theta} \;=\; -T\sqrt{\frac{\bar{\mathrm{g}}}{\Delta}}\Gamma_-\theta \;\equiv\; -P^+\Gamma_-\theta\;. \end{aligned} \tag{2.11}$$

The last formula implies a second-class constraint (entailing the replacement of Poisson brackets by Dirac brackets). The formulæ (2.11) imply the first-class constraint

$$\phi_r \;=\; \mathbf{P}\cdot\partial_r\mathbf{X} + P^+\partial_r X^- + \bar S\partial_r\theta \;\approx\; 0\;, \tag{2.12}$$

which generates spatial diffeomorphisms on the membrane. This gauge freedom can be exploited to set $u^r = 0$ in (2.6), which in turn implies

$$\partial_r X^- \;=\; -\partial_0\mathbf{X}\cdot\partial_r\mathbf{X} - \bar\theta\,\Gamma_-\partial_r\theta\;. \tag{2.13}$$

To be able to solve this equation for X^- we must impose the integrability constraint

$$\phi \;\equiv\; \epsilon^{rs}\big(\partial_r\partial_0\mathbf{X}\cdot\partial_s\mathbf{X} + \partial_r\bar\theta\,\Gamma_-\partial_s\theta\big) \;\approx\; 0\;. \tag{2.14}$$

This constraint generates APDs on the membrane: while general (spatial) diffeomorphisms on the membrane are generated by vector fields $\delta\xi^r(\boldsymbol{\sigma})$, APDs are generated by divergence-free vector fields obeying $\partial_r(\sqrt{w}\delta\xi^r) = 0$ (where the reference density $\sqrt{w(\boldsymbol{\sigma})}$ coincides with the one introduced in (2.16) below). The latter are locally of the form $\sqrt{w}\delta\xi^r = \epsilon^{rs}\partial_s\delta\xi$ with a scalar parameter $\delta\xi(\boldsymbol{\sigma})$. On higher-genus membranes there are in addition topologically non-trivial diffeomeophisms formally generated by harmonic vector fields [12], and related to global diffeomeorphisms not contractible to the identity, which we will, however, disregard here.

With these gauge choices the (dimensionless) Hamiltonian density becomes (see also [23])

$$\begin{aligned} \mathcal{H}(\boldsymbol{\sigma}) \;\equiv\; -P^-(\boldsymbol{\sigma}) \;&=\; \mathbf{P}\cdot\partial_0\mathbf{X} + P^+\partial_0 X^- + \bar S\partial_0\theta - \mathcal{L} \\ &=\; \frac{\mathbf{P}^2 + T^2\bar{\mathrm{g}}}{2P^+} \;-\; T\epsilon^{rs}\partial_r X^a\bar\theta\Gamma_-\Gamma_a\partial_s\theta \end{aligned} \tag{2.15}$$

whose bosonic part was already derived long ago in [5,6] (for $T=1$). Here we see why we must choose the membrane tension to be positive; flipping the sign of T will change the sign of the kinetic part of the Hamiltonian by (2.11), hence result in an instability. This is, of course, in accord with expectations.

Because $P^+(\tau,\boldsymbol{\sigma})$ obeys the Hamiltonian equation of motion $\partial_\tau P^+(\tau,\boldsymbol{\sigma})=0$ and transforms as a density we can set [4]

$$P^+(\tau,\boldsymbol{\sigma}) \;=\; P_0^+\sqrt{w(\boldsymbol{\sigma})} \tag{2.16}$$

where $P_0^+>0$ is constant, and $\sqrt{w(\boldsymbol{\sigma})}>0$ is a reference density normalized to $\int \mathrm{d}^2\sigma\,\sqrt{w(\boldsymbol{\sigma})}=1$ (with an associated reference metric $w_{rs}(\boldsymbol{\sigma})$ on the membrane, which is however only needed when discussing target-space Lorentz invariance [12]). This leads to the (dimensionless) mass operator

$$\mathcal{M}^2 \;=\; -2P_0^+P_0^- - \mathbf{P}_0^2 \;=\; \int \mathrm{d}^2\sigma\; \Big([\mathbf{P}^2]' + T^2\bar{\mathrm{g}} - 2T\epsilon^{rs}\partial_r X^a\bar{\Theta}\,\Gamma_-\Gamma_a\partial_s\Theta\Big) \tag{2.17}$$

with $P_0^- = \int \mathrm{d}^2\sigma\, P^-(\boldsymbol{\sigma})$ and rescaled fermionic variables [3]

$$\Theta(\boldsymbol{\sigma}) \;\equiv\; \sqrt{P_0^+}\,\theta(\boldsymbol{\sigma})\;. \tag{2.18}$$

The prime in (2.17) indicates that zero modes have been removed from $\int \mathrm{d}^2\sigma\,\mathbf{P}^2(\boldsymbol{\sigma})$.

Finally we note that the fulfilment of the constraint (2.14) allows us to solve for the target-space coordinate X^-: we have

$$X^-(\tau,\boldsymbol{\sigma}) \;=\; -\int \mathrm{d}^2\sigma'\; G^r(\boldsymbol{\sigma},\boldsymbol{\sigma}')\Big(\partial_0\mathbf{X}\cdot\partial_r\mathbf{X}(\tau,\boldsymbol{\sigma}') - \bar{\theta}\,\Gamma_-\partial_r\theta(\tau,\boldsymbol{\sigma}')\Big) \tag{2.19}$$

with a suitable Green's function obeying $\partial_r G^r(\boldsymbol{\sigma},\boldsymbol{\sigma}')=\delta(\boldsymbol{\sigma},\boldsymbol{\sigma}')$ [5, 12]. This formula is needed for the target-space boost generators and for the verification of target-space Lorentz invariance in the classical limit [12, 13]. It is worth pointing out that this information is *not* available in the matrix model as such, where the Lorentz boost generators must either be "guessed" or deduced from the supermembrane matrix-model correspondence, as in [12].

2.2 APD gauge theory and matrix model

With the above formula for the mass operator the supermembrane theory can be reformulated as a one-dimensional supersymmetric gauge theory of area preserving diffeomorphisms [4]. This can be seen by exploiting the algebraic identity

$$\bar{\mathrm{g}} \;=\; \det\left(\partial_r\mathbf{X}\cdot\partial_s\mathbf{X}\right) \;=\; \{X^a,X^b\}\{X^a,X^b\}\;, \tag{2.20}$$

where the APD bracket of any two functions $A(\boldsymbol{\sigma})$ and $B(\boldsymbol{\sigma})$ on the membrane is defined by

$$\{A(\boldsymbol{\sigma}),\,B(\boldsymbol{\sigma})\} \;:=\; \frac{1}{\sqrt{w(\boldsymbol{\sigma})}}\,\epsilon^{rs}\partial_r A(\boldsymbol{\sigma})\partial_s B(\boldsymbol{\sigma})\;. \tag{2.21}$$

This is indeed a Lie bracket (obeying antisymmetry and the Jacobi identity) [5, 6].

Then (2.15) can be equivalently obtained from the supersymmetric Lagrangian [4]

$$\tfrac{1}{\sqrt{w}}\,\mathcal{L} \;=\; \tfrac{1}{2}(D_t\mathbf{X})^2 \;+\; \bar{\Theta}\Gamma_- D_t\Theta \;-\; \tfrac{1}{4}g^2\{X^a,X^b\}^2 \;+\; g\,\bar{\Theta}\Gamma_-\Gamma_a\{X^a,\Theta\} \tag{2.22}$$

[3] Which obey the canonical (Dirac) brackets $\{\Theta(\boldsymbol{\sigma}),\bar{\Theta}(\boldsymbol{\sigma}')\}_{\mathrm{DB}} = (4\sqrt{w(\boldsymbol{\sigma})})^{-1}\Gamma_+\delta^{(2)}(\boldsymbol{\sigma},\boldsymbol{\sigma}')$ [4].

[4] While τ is the time coordinate on the membrane world-volume, we denote the Yang–Mills time coordinate by t, but keep the erstwhile membrane coordinates σ^r as labels for the APD gauge group.

if we identify

$$T = g . \tag{2.23}$$

In view of our comments after (2.15) we must, however, restrict this identification to positive values of T and g, even though there appear to be no obstructions to continuing the APD gauge theory to negative couplings. Hence the small (large) tension limit of the supermembrane corresponds to the weak (strong) coupling limit of the supersymmetric APD gauge theory. The APD covariant derivative is given by

$$D_t f(t,\boldsymbol{\sigma}) := \partial_t f(t,\boldsymbol{\sigma}) + g\{\omega(t,\boldsymbol{\sigma}), f(t,\boldsymbol{\sigma})\} \tag{2.24}$$

with the APD gauge field $\omega(t,\boldsymbol{\sigma})$ which is here introduced *ad hoc*, as it is absent from the supermembrane action. The Lagrangian (2.22) is nothing but the dimensional reduction of maximally extended super-Yang–Mills theory [24] to one (time) dimension, with the identifications $\omega \equiv A_0$ and $X_a \equiv A_a$, and g the usual Yang–Mills coupling, but now with the infinite-dimensional APD gauge group. This works precisely in the dimensions where pure supersymmetric Yang–Mills theories exist, namely $D = 3,4,6,10$ [24], in agreement with the admissible target-space dimensions 4,5,7 and 11 for supermembranes.

The group of APDs on the membrane can be approximated by the finite-dimensional unitary groups SU(N), such that the full group of APDs is recovered in the limit $N \to \infty$ [5,6]. Replacing APDs by SU(N) gives the matrix model of M theory. For this approximation one expands all functions on the membrane into a complete orthonormal set of functions $Y^A(\boldsymbol{\sigma})$,

$$\int \mathrm{d}^2\sigma \, \sqrt{w(\boldsymbol{\sigma})}\, Y^A(\boldsymbol{\sigma}) Y^B(\boldsymbol{\sigma}) = \delta^{AB} , \tag{2.25}$$

where we separate off the zero modes,

$$\begin{aligned} X_a(t,\boldsymbol{\sigma}) &= X_a^{(0)}(t) + \sum_{A=1}^{\infty} X_a^A(t) Y^A(\boldsymbol{\sigma}) , \\ \omega(t,\boldsymbol{\sigma}) &= \omega^{(0)}(t) + \sum_{A=1}^{\infty} \omega^A(t) Y^A(\boldsymbol{\sigma}) , \\ \Theta(t,\boldsymbol{\sigma}) &= \Theta^{(0)}(t) + \sum_{A=1}^{\infty} \Theta^A(t) Y^A(\boldsymbol{\sigma}) . \end{aligned} \tag{2.26}$$

The zero modes $X_a^{(0)}(t)$ and $\Theta^{(0)}(t)$ decouple in (2.22), where $X_a^{(0)}(t)$ describes the center of mass motion of the membrane as a whole. Likewise, the gauge zero mode $\omega^{(0)}(t)$ drops out in the Lagrangian (as it acts effectively like a U(1) gauge field, which cannot couple because both $X_a^{(0)}$ and $\Theta^{(0)}$ are real). The remaining non-zero modes describe the 'internal' degrees of freedom of the supermembrane. The APD gauge group can thus be approximated by SU(N), as is most easily and explicitly done for S^2 [5,6] and T^2 [12,25,26], by cutting off the mode expansions at $N^2{-}1$ (ignoring topological modes) and replacing the APD-brackets by SU(N) commutators. In fact, as shown in [27] the SU(N) approximation works for *any* genus of the membrane. Consequently, we have

$$f_{\mathrm{APD}}^{ABC} \equiv \int d^2\sigma \sqrt{w(\boldsymbol{\sigma})}\, Y^A(\boldsymbol{\sigma}) \{Y^B(\boldsymbol{\sigma}), Y^C(\boldsymbol{\sigma})\} = \lim_{N\to\infty} f^{ABC}(N) \tag{2.27}$$

with SU(N) structure constants $f^{ABC}(N)$. Hence the expansion labels $A,B,\ldots=1,\ldots,N^2-1$ are thus turned into Yang–Mills indices, while $a,b,\ldots$ are transverse (for membrane) and space-like (for supersymmetric Yang–Mills) indices.

After these preparations, the matrix-model Lagrangian assumes the standard form [5]

$$\mathcal{L} \;=\; \tfrac{1}{2}(D_tX_a^A)^2 - \mathrm{i}\,\theta_\alpha^A D_t\theta_\alpha^A - \tfrac{1}{4}g^2(f^{ABC}X_b^BX_c^C)^2 - \tfrac{\mathrm{i}}{2}\,g\,f^{ABC}\theta_\alpha^A\gamma_{\alpha\beta}^a X_a^B\theta_\beta^C\;, \tag{2.28}$$

where we have now switched to SO(9) spinors θ_α^A with 16 real components and where

$$D_t\theta^a \;=\; \partial_t\theta^A + g\,f^{ABC}\omega^B\theta^C \tag{2.29}$$

is the SU(N) covariant derivative. The real symmetric 16-by-16 matrices γ_a generate the SO(9) Clifford algebra, $\{\gamma^a,\gamma^b\}=2\delta^{ab}$. Variation w.r.t. ω^A yields the constraint

$$f^{ABC}\big(X_a^B D_tX_a^C + \theta_\alpha^B\theta_\alpha^C\big) \;\approx\; 0 \tag{2.30}$$

which is equivalent to the canonical generator of SU(N) gauge transformations (after performing this variation we can put $\omega^A=0$ everywhere). The Lagrangian (2.28) is the one that underlies the M theory conjecture of [8], see also [31–33].

2.3 Setting up the path integral

Our goal is now to set up a path-integral formulation that should eventually permit the computation of correlators of physically relevant quantities, and complement the canonical quantization methods underlying many treatments of the matrix model. We shall thus be interested in evaluating correlation functions of the type

$$\begin{aligned}\Big\langle \mathcal{O}_1[\mathbf{X},\theta]\cdots\mathcal{O}_n[\mathbf{X},\theta]\Big\rangle_g \;=\; &\int\prod \mathcal{D}X_a(t,\boldsymbol{\sigma})\,\mathcal{D}\theta_\alpha(t,\boldsymbol{\sigma})\,\mathcal{D}\omega(t,\boldsymbol{\sigma})\,\mathcal{D}C(t,\boldsymbol{\sigma})\,\mathcal{D}\bar{C}(t,\boldsymbol{\sigma})\times\\ &\times\;\mathcal{O}_1[\mathbf{X},\theta]\cdots\mathcal{O}_n[\mathbf{X},\theta]\;\exp(\mathrm{i}\,\mathrm{S}_{\mathrm{tot}})\end{aligned} \tag{2.31}$$

where the precise form of the functionals $\mathcal{O}_i[\mathbf{X},\theta]$ need not be specified at this point. Because this is a gauge theory, the full action

$$\mathrm{S}_{\mathrm{tot}} \;=\; \mathrm{S} + \mathrm{S}' \tag{2.32}$$

with $\mathrm{S}=\int\mathrm{d}t\,\mathcal{L}$ must comprise a gauge-fixing part $\mathrm{S}'=\int\mathrm{d}t\,\mathcal{L}'$. For higher-dimensional Yang–Mills theories there are two preferred choices, namely the Lorenz gauge $\partial^\mu A_\mu=0$, and the axial gauge $n^\mu A_\mu=0$ (which includes the light-cone gauge for null vectors n^μ). In the reduction to one time dimension the axial gauge is necessarily identical with the temporal gauge. Consequently, we have two preferred choices for the gauge-fixing part, namely

$$\begin{aligned}\mathcal{L}' \;&=\; -\tfrac{1}{2\xi}(\partial_t\omega)^2 + \bar{C}\,\partial_tD_tC \qquad\qquad \text{(Lorenz gauge)}\;,\\ \mathcal{L}' \;&=\; -\tfrac{1}{2\xi}\omega^2 + \bar{C}D_tC \qquad\qquad\qquad \text{(temporal gauge)}\;.\end{aligned} \tag{2.33}$$

[5]For finite-dimensional gauge groups these supersymmetric matrix models were first obtained in [28–30].

A further peculiarity of one dimension is that the temporal gauge implies the Lorenz gauge

$$\omega(t,\boldsymbol{\sigma}) = 0 \qquad \Rightarrow \qquad \partial_t \omega(t,\boldsymbol{\sigma}) \equiv \dot{\omega}(t,\boldsymbol{\sigma}) = 0 \ . \tag{2.34}$$

$C(t,\boldsymbol{\sigma})$ and $\bar{C}(t,\boldsymbol{\sigma})$ are the usual Faddeev–Popov ghosts [34, 35], and ξ is a real parameter which will be eventually sent to zero to put the theory on the gauge hypersurface. After trading the $\boldsymbol{\sigma}$ dependence for SU(N) indices, we are left with a quantum mechanical path integral describing *finitely* many degrees of freedom. Because of the supersymmetry there is no need for a normalization factor in (2.31) (as can be easily checked for $g = 0$ with both gauge choices). In passing we note that we can of course equivalently switch to a Euclidean formulation by flipping the sign in the kinetic terms $(\dot{X}_a)^2$, $\dot{\omega}^2$ and for the ghosts, and by replacing the oscillatory exponent by $\exp(-\mathrm{S_{tot}})$; the factor i is then absent in the spinor kinetic term.

An important part of our construction is that we consider the path integral in a form where the fermions (and also the ghosts) are integrated out. For the temporal gauge and for finite N the integration over $\theta^A_\alpha(t)$ results in the Matthews–Salam–Seiler (MSS) determinant [36, 37]

$$\Delta_{\mathrm{MSS}}\big[\omega{=}0,\mathbf{X}\big] \ = \ \Big[\det\big(\delta^{AB}\delta_{\alpha\beta}\,\delta(t_1{-}t_2) \ + \ g\,K^{AB}_{\alpha\beta}(t_1.t_2)\big)\Big]^{1/2} \tag{2.35}$$

which is actually a Pfaffian because we are integrating over *real* fermions. The integral kernel appearing in this expression is

$$K^{AB}_{\alpha\beta}(t_1,t_2) \ := \ \varepsilon(t_1{-}t_2)\, f^{ACB}\gamma^a_{\alpha\beta} X^C_a(t_2) \ . \tag{2.36}$$

This is a real operator which is however not symmetric because hermitean conjugation also exchanges the arguments t_1 and t_2. Furthermore, we have taken out trivial factors of $\det(\partial_t)$ (which anyway cancel in the supersymmetric path integral). The free fermion propagator ε is just the Green's function for ∂_t,

$$\varepsilon(t-t') \ = \ \big[\partial_t^{-1}\big](t,t') \ = \ \int\frac{\mathrm{d}p}{2\pi}\,\frac{\mathrm{i}p}{p^2-\mathrm{i}\epsilon}\mathrm{e}^{-\mathrm{i}p(t-t')} \ = \ \Theta(t{-}t') - \tfrac{1}{2} \ = \ -\varepsilon(t'{-}t) \ . \tag{2.37}$$

This choice of integration constant implies $\varepsilon(0) = 0$ as well as

$$\int \mathrm{d}t \ \varepsilon(t{-}t') \ = \ 0 \ . \tag{2.38}$$

Our particular choice is important for the tests in the Appendix which otherwise cannot be satisfied (it is also consistent with the dimensional reduction of the usual Dirac propagator). In Section 4 we will study some properties of this determinant in more detail and prove in particular that the expansion of $\log(\Delta_{\mathrm{MSS}})$ in powers of g has a *non-zero radius of convergence* with suitable technical assumptions on the behavior of $X^A_a(t)$. We also note that we have no positivity statement about Δ_{MSS} (though the fermion determinant is non-negative for complex fermions!). Similarly, the determinant cannot be shown to be an even function of g because of the non-vanishing trace $\mathrm{tr}\,(\gamma^{a_1}\cdots\gamma^{a_9}) = 16\,\epsilon^{a_1\cdots a_9}$.

For the infinite-dimensional APD gauge group we must be a little more careful: while the kinetic term of (2.22) is local in $\boldsymbol{\sigma}$, the interaction term is not because it contains derivatives in $\boldsymbol{\sigma}$. To take into account this non-locality we can formally replace the integral kernel (2.36) by

$$K^{\mathrm{APD}}_{\alpha\beta}(t_1,t_2\,;\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) \ := \ \varepsilon(t_1{-}t_2)\,\gamma^a_{\alpha\beta}\,\frac{1}{\sqrt{w(\boldsymbol{\sigma_1})}}\,\epsilon^{rs}\,\frac{\partial X_a(t_2,\boldsymbol{\sigma}_1)}{\partial\sigma^r_1}\,\delta(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2)\,\frac{\partial}{\partial\sigma^s_2} \tag{2.39}$$

and the identity operator by $\delta_{\alpha\beta}\delta(t_1-t_2)\,\delta(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2)$, with the proviso that folding with this kernel now also contains an integral over $\boldsymbol{\sigma}$. When expanding the logarithm of the MSS determinant using log det = Tr log, we encounter for each trace a divergent factor $\delta(\boldsymbol{\sigma},\boldsymbol{\sigma})$, which can thus be uniformly factored out of the sum. To arrive at a sensible result in the $N\to\infty$ limit we must therefore remove this factor. This 'renormalization' corresponds to the replacement of the Cartan-Killing form $f^{ACD}f^{BCD}=N\delta^{AB}$ (whose limit $N\to\infty$ also diverges) in the matrix action by the finite action obtained from (2.22) by integrating $\frac{1}{\sqrt{w}}\mathcal{L}$ over the world volume with the measure $\mathrm{d}t\,\mathrm{d}^2\sigma\sqrt{w(\boldsymbol{\sigma})}$.

2.4 Physical correlators

With the path integral formalism at hand we can now in principle proceed to calculate gauge-variant and gauge-invariant correlators of suitable objects. But what are the physically relevant operators $\mathcal{O}[\mathbf{X},\theta]$? As in string theory, for the membrane the latter should be associated to vertex operators describing the emission or absorption of certain one-particle excitations from the membrane. As first shown in [15] there indeed exist the classical analogs of supermembrane light-cone vertex operators exciting the massless states of the supermembrane, which comprise the massless supermultiplet of maximal $D=11$ supergravity [3]. The related expressions must satisfy various consistency constraints (target-space and world-volume gauge invariance, linear and non-linear supersymmetry) which are explained at length in [15], corresponding to (but more complicated than) the ones known from type II superstring theory. In particular, in analogy with closed-string vertex operators they are to be integrated over the membrane world volume. For instance, for the transverse graviton components we have [15]

$$\mathcal{O}[\mathbf{X},\theta] \;=\; \int \mathrm{d}t\,\mathrm{d}\boldsymbol{\sigma}\; V_h[\mathbf{X},\theta] \tag{2.40}$$

with

$$\begin{aligned} V_h[\mathbf{X},\theta] \;=\; h_{ab}\Big[& D_tX^aD_tX^b \;-\; \{X^a,X^c\}\{X^b,X^c\} \;-\; \mathrm{i}\bar\theta\gamma^a\{X^b,\theta\} \\ & -\; \tfrac12 D_tX^a\,\bar\theta\gamma^{bc}\theta\,k_c \;-\; \tfrac12\{X^a,X^c\}\,\bar\theta\gamma^{bcd}\theta\,k_c \;+\; \tfrac12\bar\theta\gamma^{ac}\theta\,\bar\theta\gamma^{bd}\theta\,k_ck_d\Big]\;\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\mathbf{X}+\mathrm{i}k^-t} \end{aligned} \tag{2.41}$$

where h_{ab} is the transverse graviton polarization tensor, and $\{k_a\}=\boldsymbol{k}$ denotes the transverse components of the target-space momentum. For the light-cone gauge target-space momentum k^μ one must furthermore assume $k^+=0$ in order to avoid having to deal with the light-cone coordinate $X^-(\tau,\boldsymbol{\sigma})$ in the exponential (as is also customary in string theory [38]). Remarkably, and unlike for superstring theory, there do not appear to exist analogs of the string vertex operators for massive string states. This would be in accord with the fact that the supermembrane is not a first quantizable (*i.e.* one-particle) theory [10] and for finite N consistent with the D0-multiparticle interpretation of [8].

Because the light-cone vertex operators are given by complicated expressions, and because the measure in (2.31) is not Gaussian, no sustained attempt has been made, as far as we are aware, to evaluate their correlators. Neither has it been possible so far to set up a perturbative expansion, as this will also require understanding the quantum corrections (renormalizations) that are necessary for the vertex operators to remain well-defined in the quantized interacting theory. Our line of attack will therefore be a different one, in that we will reformulate the above path integral in terms of a Nicolai map. A main advantage of such an approach is that the formulæ to be presented below remain perfectly well-defined in the limit $N \to \infty$ and can thus be consistently implemented also in the APD path integral. Consequently, it may be possible in this way to sidestep the detour via the finite-N matrix model, and to directly tackle the $N = \infty$ theory right away. Possible applications of this technology to supermembrane vertices will, however, be left to future work.

3 The map to fourth order

The method that we propose here to tackle expressions like (2.31) is based on the Nicolai map $\mathcal{T}_g$ [16, 17, 39–43], exploiting recent progress in determining this map to higher orders in g [18, 19, 44, 45]. This map is a non-local and non-linear field transformation, which maps the theory to a free theory in such a way that after integrating out the fermions (gaugini and ghosts) the product of the resulting fermionic determinants equals the Jacobian of the map $\mathcal{T}_g$ at least locally in field space. For operators $\mathcal{O}_k(t_k)$ built from X_a (and ω) alone, this enables us to re-express the expectation value (2.31) in the matrix theory as a free-field correlator of transformed bosonic fields, *viz.*

$$\Big\langle \mathcal{O}_1(t_1)\cdots\mathcal{O}_n(t_n)\Big\rangle_g \;=\; \Big\langle \mathcal{T}_g^{-1}(\mathcal{O}_1(t_1))\cdots\mathcal{T}_g^{-1}(\mathcal{O}_n(t_n))\Big\rangle_0 \tag{3.1}$$

where integrating out the gaugini and ghosts is trivial on the right-hand side because the transformed operators are purely bosonic ones. We can therefore read this relation as one in the integrated-out theory as well as in the original one including the fermions. A key property of the map $\mathcal{T}_g$ is the equality of its functional Jacobian with the product of the fermionic determinants obtained by integrating out all anticommuting variables, to wit,

$$\det\left(\frac{\delta \mathcal{T}_g X}{\delta X}\right) \;=\; \Delta_{\mathrm{MSS}}[\omega,\mathbf{X}]\;\Delta_{\mathrm{FP}}[\omega,\mathbf{X}] \tag{3.2}$$

where Δ_{FP} and Δ_{MSS} are, respectively, the Faddeev–Popov determinant [34,35] and the MSS determinant (2.35) [36, 37]. We refer readers to [18, 19, 44–46] for recent progress in constructing the map $\mathcal{T}_g$ for pure supersymmetric Yang–Mills theories in all relevant dimensions. A crucial simplification follows from (2.34), since it allows us to largely ignore the distinction between 'on-shell' and 'off-shell' R-prescriptions in [18,19] that must be taken into account in more than one dimension.

3.1 Construction by dimensional reduction

The goal of this section is the construction of $\mathcal{T}_g$ [17,39–43] for the APD and SU(N) supersymmetric matrix models (2.22) and (2.28). This can be done either by repeating the construction procedure described in [39, 43] for this particular theory, or by dimensionally reducing the map for ten-dimensional super Yang–Mills theory to one-dimensional matrix mechanics. Let us first choose the second path.

Since we only have an on-shell formulation of supersymmetry in ten dimensions, we cannot employ the general scheme [18, 19] for arbitrary gauge fixing but have to stick to the Lorenz gauge, for which the map was presented on the gauge hypersurface in the critical spacetime dimensions $D=3,4,6$ and 10, to $O(g^3)$ in [45] and to $O(g^4)$ in [18]. In the dimensional reduction all quantities loose their coordinate dependence except for a dependence on time t, and the D components of the gauge potential become $(D-1)$ dynamical matrices $X_a(t)$ and one non-dynamical matrix $\omega(t)$. The Lorenz gauge reduces to $\partial_t\omega \equiv \dot{\omega} = 0$, hence the matrix ω is a constant on the gauge hypersurface. It will turn out that it is invariant under the map $\mathcal{T}_g$.

Let us recall the salient facts of the construction, keeping D arbitrary and denoting by r the dimension of the corresponding Majorana spinor representation. The map $\mathcal{T}_g$ is a nonlinear and nonlocal field transformation

$$\mathcal{T}_g : (X_a(t),\omega) \;\mapsto\; (X'_a(t),\omega') \; . \tag{3.3}$$

It affords to express the quantum correlator $\langle F\rangle_g$ of an arbitrary bosonic functional F at gauge coupling g in terms of a free correlator (g=0) of the same functional, but with its arguments transformed by the inverse map,

$$\Big\langle F[X,\omega]\Big\rangle_g \;=\; \Big\langle F[\mathcal{T}_g^{-1}X,\mathcal{T}_g^{-1}\omega]\Big\rangle_0 \; . \tag{3.4}$$

An infinitesimal (in g) version reads

$$\partial_g\Big\langle F[X,\omega]\Big\rangle_g \;=\; \Big\langle\big(\partial_g + R_g[X,\omega]\big)F[X,\omega]\Big\rangle_g \; , \tag{3.5}$$

where the "coupling flow operator" R_g is a linear functional integro-differential operator with a nonlinear and nonlocal dependence on X and ω. As the construction is perturbative in the coupling g,[6] we expand (note the index shift)

$$R_g[X,\omega] \;=\; \sum_{k=1}^{\infty} g^{k-1}\mathrm{R}_k[X,\omega] \;=\; \mathrm{R}_1[X,\omega] + g\,\mathrm{R}_2[X,\omega] + g^2\mathrm{R}_3[X,\omega] + g^3\mathrm{R}_4[X,\omega] + \ldots \; . \tag{3.6}$$

Integrating the infinitesimal flow equation (3.5) yields $\mathcal{T}_g^{-1}$ and finally

$$\begin{aligned}\mathcal{T}_g X_a \;=\; X_a \;-\; g\mathrm{R}_1 X_a \;-\; \tfrac{1}{2}g^2(\mathrm{R}_2-\mathrm{R}_1^2)X_a \;-\; \tfrac{1}{6}g^3(2\mathrm{R}_3-\mathrm{R}_1\mathrm{R}_2-2\mathrm{R}_2\mathrm{R}_1+\mathrm{R}_1^3)X_a \\ -\; \tfrac{1}{24}(6\mathrm{R}^4-2\mathrm{R}_1\mathrm{R}_3-3\mathrm{R}_2^2+\mathrm{R}_1^2\mathrm{R}_2-6\mathrm{R}_3\mathrm{R}_1+2\mathrm{R}_1\mathrm{R}_2\mathrm{R}_1+3\mathrm{R}_2\mathrm{R}_1^2-\mathrm{R}_1^4) \;+\; \ldots\end{aligned} \tag{3.7}$$

in terms of the flow operator's expansion coefficients. We have displayed the result to $O(g^4)$ since we shall evaluate the map to this order, and we omitted the analogous formula for ω because it reduces to $\mathcal{T}_g\omega = \omega$.

In order to avoid cluttering the equations with indices, we mostly suppress spinor and color indices as well as time dependence and employ the DeWitt summation convention (suppressing also time integrals) in the remainder of this section. We find it convenient to let the flow operator act (by functional differentiation) to the left. It is then given by a variation $\overleftarrow{\frac{\delta}{\delta X_a^A(t)}}$ followed by a string of matrices in color, spinor and coordinate space, such as

$$(X_a\times)^{AB}(t,t') = f^{AMB}X_a^M(t)\delta(t-t') \qquad \Rightarrow \qquad (X_c\times X_d)^A(t) = f^{AMN}X_c^M(t)X_d^N(t) \tag{3.8}$$

[6]There exists, however, a universal nonperturbative formula for the map, see [46].

and propagators G and S defined by

$$[D_t G]^{AB}_{\alpha\beta}(t,t') = \delta^{AB}\delta_{\alpha\beta}\delta(t-t') \ , \qquad [(D_t + g\gamma^a X_a\times)S]^{AB}_{\alpha\beta}(t,t') = \delta^{AB}\delta_{\alpha\beta}\delta(t-t') \tag{3.9}$$

with $D_t \equiv D_0 = \partial_t + g\omega\times$. We note that $\omega\times$ and $X_a\times$ are to be considered as matrices in color space. The product of all these objects is to be executed in canonical fashion, where we suppress obvious unit factors in the formulæ. Observe also that G is *not* the ghost propagator whose defining equation contains another derivative ∂_t. This is because in all relevant expressions the ghost propagator appears with a derivative ∂_t.

A careful dimensional reduction of the coupling flow operator eq.(1.19) of [45] then yields [7]

$$\overleftarrow{R} \ = \ -\tfrac{1}{r}\overleftarrow{\tfrac{\delta}{\delta X_a}}\,\mathrm{tr}\Big[(\gamma_a - gX_a\times G)\,S\,(\tfrac12\gamma^{cd}X_c\times X_d + \gamma^d\omega\times X_d)\Big] \tag{3.10}$$

where the explicit trace refers to the spinor space, and we have dropped a term proportional to $\frac{1}{g}\dot\omega$. Here, the first round bracket arises from a non-abelian projector [45] which in the Lorenz gauge reads ($\mu = (t,a)$)

$$P_\mu{}^\nu \ = \ \delta_\mu{}^\nu - D_\mu(\partial\cdot D)^{-1}\partial^\nu \qquad \overset{\text{reduction}}{\longrightarrow} \qquad \delta_\mu{}^\nu - D_\mu D_t^{-1}\partial_t^{-1}\partial^\nu \tag{3.11}$$

which obeys $\partial^\mu P_\mu{}^\nu = 0 = P_\mu{}^\nu D_\nu$ and yields

$$P_t{}^\nu = 0 \ , \qquad P_a{}^b = \delta_a{}^b \qquad \text{and} \qquad P_a{}^t = g\,X_a\times D_t^{-1} \ . \tag{3.12}$$

This shows that R does not contain a variation $\frac{\delta}{\delta\omega}$, *cf.* formula (1.19) in [45] (with $\mu = t$). The second round bracket is just the decomposition of $\frac12 A_\rho\times A_\lambda$ in the reduction.

3.2 Construction in the matrix model

Alternatively, we may take the first path and construct the map $\mathcal{T}_g$ directly for the matrix model, following the strategy of [39,43]. To this end, we implement the Lorenz gauge constraint $\dot\omega = 0$ by adding to the matrix model Lagrangian (2.28) [8]

$$\mathcal{L} \ = \ \tfrac12(D_t X_a)^2 \ - \ \tfrac14 g^2(X_c\times X_d)^2 \ - \ \tfrac{\mathrm{i}}{2}\theta\cdot(D_t + g\hat X\times)\theta \tag{3.13}$$

with $\hat X := \gamma^a X_a$ a "gauge-fixing term"

$$\mathcal{L}' \ = \ -\tfrac{1}{2\xi}\dot\omega^2 \ + \ \bar C\cdot\partial_t D_t C \tag{3.14}$$

with a real parameter ξ and ghost matrices C and $\bar C$. Taking the limit $\xi\to 0$ puts the theory on the gauge hypersurface.

We aim to directly derive a coupling flow operator R as in (3.5) for the matrix model, which will govern the infinitesimal change in the coupling g for the quantum correlator of an arbitrary

[7] Note that we have split $\mathcal{R} = \partial_g + R$.

[8] Here and below, the $\cdot$ denotes a contraction in color space, i.e. $P\cdot Q := \delta^{AB}P^A Q^B$.

bosonic matrix functional $F[X,\omega]$. Keeping in mind the g-dependence of the functional integral weight $\mathrm{e}^{\mathrm{i}\int(\mathcal{L}+\mathcal{L}')}$, we compute (suppressing the subscript in $\langle\cdots\rangle_g$)

$$\begin{aligned}\partial_g\Big\langle F\Big\rangle &= \Big\langle\partial_g F+F\,\partial_g\int\mathrm{i}(\mathcal{L}+\mathcal{L}')\Big\rangle\\ &= \Big\langle\partial_g F+F\ \mathrm{i}\int\Big[D_tX_a\cdot\omega\times X_a-\tfrac12 g(X_c\times X_d)^2-\tfrac{\mathrm{i}}{2}\,\theta\cdot(\omega+\hat X)\times\theta+\bar C\cdot\partial_t(\omega\times C)\Big]\\ &= \Big\langle\partial_g F+F\ \mathrm{i}\Big[\delta_\alpha\Delta_\alpha+\mathrm{i}q\int\theta\cdot(\omega+\hat X)\times\theta+\int\bar C\cdot\partial_t(\omega\times C)\Big]\Big\rangle\end{aligned}\tag{3.15}$$

where

$$\Delta_\alpha\ =\ -\tfrac{1}{r}\int\mathrm{d}t\ (\gamma^d\theta)_\alpha\cdot\omega\times X_d\ +\ \tfrac{1}{2r}\int\mathrm{d}t\ (\gamma^{cd}\theta)_\alpha\cdot X_c\times X_d\ .\tag{3.16}$$

With the supersymmetry transformations

$$\delta_\alpha\omega=-\mathrm{i}\theta_\alpha\ ,\qquad\delta_\alpha X_a=-\mathrm{i}\,(\theta\gamma_a)_\alpha\ ,\qquad\delta_\alpha\theta_\beta=-\gamma^d_{\alpha\beta}D_tX_d-\tfrac{g}{2}\gamma^{cd}_{\alpha\beta}X_c\times X_d\tag{3.17}$$

one confirms that indeed

$$\delta_\alpha\Delta_\alpha\ =\ \int\mathrm{d}t\ \Big[D_tX_a\cdot\omega\times X_a\ -\ \tfrac12 g(X_c\times X_d)^2\ -\ \mathrm{i}\,\tfrac{D-1}{r}\,\theta\cdot(\omega+\hat X)\times\theta\Big]\ .\tag{3.18}$$

Therefore, $\delta_\alpha\Delta_\alpha$ in (3.15) reproduces $\partial_g\int\mathcal{L}$ but with a mismatch in the coefficient of the Majorana term, which thus still appears there but with a coefficient

$$q\ =\ \tfrac{D-1}{r}-\tfrac12\ =\ \tfrac1r\qquad\text{for}\quad D=3,4,6,10\ .\tag{3.19}$$

It is noteworthy that for a temporal gauge this mismatch is absent,

$$A_0=0\qquad\Rightarrow\qquad\omega=0\quad\text{and}\quad D_t=\partial_t\qquad\Rightarrow\qquad\delta_\alpha\big(\Delta_\alpha|_{\omega=0}\big)\ =\ \partial_g\int\mathcal{L}\ ,\tag{3.20}$$

since effectively $D\to D{-}1$ and the ghosts decouple.

Next, we employ the broken supersymmetric Ward identity $\langle\delta_\alpha Y\rangle=-\mathrm{i}\langle(\delta_\alpha\int\mathcal{L}')\,Y\rangle$ together with

$$\delta_\alpha\int\mathcal{L}'\ =\ -s\,\delta_\alpha\Delta_{\mathrm{gh}}\qquad\text{for}\quad\Delta_{\mathrm{gh}}\ =\ \int\bar C\,\dot\omega\tag{3.21}$$

and the Slavnov variations

$$s\,\omega=D_tC\ ,\qquad s\,X_a=g\,X_a\times C\ ,\qquad s\,\theta=g\,\theta\times C\ ,\qquad s\,C=-\tfrac{g}{2}C\times C\ ,\qquad s\,\bar C=\tfrac1\xi\dot\omega\tag{3.22}$$

to rewrite

$$\begin{aligned}\partial_g\Big\langle F\Big\rangle &= \Big\langle\partial_g F+\mathrm{i}\,\Delta_\alpha\delta_\alpha F\Big\rangle+\Big\langle F\ \Big[\Delta_\alpha\,s\,\delta_\alpha\Delta_{\mathrm{gh}}-q\int\theta\cdot(\omega+\hat X)\times\theta+\mathrm{i}\int\bar C\cdot\partial_t(\omega\times C)\Big]\Big\rangle\\ &= \Big\langle\partial_g F+\mathrm{i}\,\Delta_\alpha\delta_\alpha F-\Delta_\alpha(\delta_\alpha\Delta_{\mathrm{gh}})\,s\,F\Big\rangle\\ &\quad+\ \Big\langle F\ \Big[(s\,\Delta_\alpha)(\delta_\alpha\Delta_{\mathrm{gh}})-q\int\theta\cdot(\omega+\hat X)\times\theta-\mathrm{i}\int\dot{\bar C}\cdot(\omega\times C)\Big]\Big\rangle\ ,\end{aligned}\tag{3.23}$$

where in the last step we used the BRST Ward identity $\langle s\,Y\rangle=0$.

For the flow equation (3.5) to hold, the last correlator has to vanish for any bosonic functional F. Writing out

$$s\,\Delta_\alpha \;=\; \tfrac{1}{r}\int(\hat{X}\times\theta)_\alpha\cdot\dot{C} \qquad \text{and} \qquad \delta_\alpha\Delta_{\text{gh}} \;=\; -\mathrm{i}\int\dot{\bar{C}}\cdot\theta_\alpha \tag{3.24}$$

and performing the functional integrations over the fermions and the ghosts, this requirement becomes

$$0 \overset{!}{=} -\tfrac{\mathrm{i}}{r}\int(\hat{X}_{\alpha\beta}\times\theta_\beta)\cdot\dot{C}\int\dot{\bar{C}}\cdot\theta_\alpha \;-\; q\int(\hat{X}_{\alpha\beta}\times\theta_\beta)\cdot\theta_\alpha \;-\; q\int(\omega\times\theta_\alpha)\cdot\theta_\alpha \;+\; \mathrm{i}\int(\omega\times C)\cdot\dot{\bar{C}}\;, \tag{3.25}$$

where the contractions stand for the fermionic and ghost propagators

$$\theta^B_\beta(t)\;\theta^A_\alpha(t') \;=\; -S^{BA}_{\beta\alpha}(t,t') \qquad \text{and} \qquad C^B(t)\;\dot{\bar{C}}^A(t') \;=\; \mathrm{i}\,G^{BA}(t,t')\;, \tag{3.26}$$

respectively (note the time derivative on $\bar{C}^A$). With $\partial_t G(t,t')^{BA} = \partial_{t'}G^{AB}(t',t) =: \partial G^{AB}(t',t)$ the condition (3.25) reads

$$0 \;\overset{!}{=}\; -\tfrac{1}{r}\,\mathrm{Tr}\big[(\hat{X}\times S)\,\partial G\big] + q\,\mathrm{Tr}\big[\hat{X}\times S\big] + q\,\mathrm{Tr}\big[\omega\times S\big] - \tfrac{1}{r}\,\mathrm{Tr}\big[\omega\times G\big]\;, \tag{3.27}$$

where the trace here refers to spin, color and time altogether. Abbreviating the unit operator by the symbol $\mathbb{1}$, and inserting the useful identities

$$\partial G \;=\; \mathbb{1} - g\,\omega\times G \qquad \text{and} \qquad S \;=\; G - g\,G\,(\hat{X}\times S)\;, \tag{3.28}$$

into the first and third term, respectively, we cancel the second and fourth terms (provided $q=\frac{1}{r}$) and remain with

$$0 \;\overset{!}{=}\; \tfrac{1}{r}\,g\,\mathrm{Tr}\big[(\hat{X}\times S)\,(\omega\times G)\big] \;-\; q\,g\,\mathrm{Tr}\big[(\omega\times G)\,(\hat{X}\times S)\big]\;, \tag{3.29}$$

which indeed holds in the critical dimensions.

We return to (3.23) and integrate out the fermions and ghosts to read off the flow operator

$$\begin{aligned} R_g \;&=\; \mathrm{i}\Delta_\alpha\delta_\alpha - \Delta_\alpha(\delta_\alpha\Delta_{\text{gh}})\,s \\ &=\; \Delta_\alpha\int\theta_\alpha\cdot\frac{\delta}{\delta\omega} \;+\; \Delta_\alpha\int(\theta\gamma_a)_\alpha\cdot\frac{\delta}{\delta X_a} \\ &\quad -\;\mathrm{i}\Delta_\alpha\int\theta_\alpha\cdot\dot{\bar{C}}\int D_tC\cdot\frac{\delta}{\delta\omega} \;-\; \mathrm{i}\Delta_\alpha\int\theta_\alpha\cdot\dot{\bar{C}}\int(g\,X_a\times C)\cdot\frac{\delta}{\delta X_a}\;. \end{aligned} \tag{3.30}$$

Since $D_tG=\mathbb{1}$, the two variations w.r.t. ω (first and third terms) cancel, and we are left with

$$R_g \;=\; \Delta_\alpha\int\theta_\beta\cdot\big[(\gamma_a)_{\beta\alpha}\,\mathbb{1} \;+\; g\,\delta_{\beta\alpha}G\times X_a\big]\cdot\frac{\delta}{\delta X_a}\;. \tag{3.31}$$

In the curly brackets we recognize the (dimensionally reduced) non-abelian projector (3.12). Recalling Δ_α from (3.16), inserting the fermion propagator (3.26) and reversing the multiplication order, one again arrives at the flow operator presented in (3.10).

3.3 The map to third and fourth order

For the perturbative power series we need the expansion of the propagators,

$$
\begin{aligned}
G &= \varepsilon - g\,\varepsilon\omega\times\varepsilon + g^2\varepsilon\omega\times\varepsilon\omega\times\varepsilon - g^3\varepsilon\omega\times\varepsilon\omega\times\varepsilon\omega\times\varepsilon \pm \ldots\ , \\
S &= \varepsilon - g\,\varepsilon(\omega+\hat{X})\times\varepsilon + g^2\varepsilon(\omega+\hat{X})\times\varepsilon(\omega+\hat{X})\times\varepsilon \mp \ldots\ ,
\end{aligned}
\tag{3.32}
$$

with the free fermion propagator (2.37). Because spin traces vanish for an odd product of gamma matrices (for less than nine factors), the expansion coefficients R_k displayed here carry only even/odd powers of ω for k being even/odd. This parity extends to the map $\mathcal{T}_g$ itself. Carrying out the spin traces, one gets

$$
\begin{aligned}
\overleftarrow{\mathrm{R}_1} &= -\overleftarrow{\tfrac{\delta}{\delta X_a}}\,\varepsilon\omega\times X_a\ , \\
\overleftarrow{\mathrm{R}_2} &= \overleftarrow{\tfrac{\delta}{\delta X_a}}\,\varepsilon\omega\times\varepsilon\omega\times X_a + \overleftarrow{\tfrac{\delta}{\delta X_a}}\,\varepsilon X_b\times\varepsilon X_b\times X_a\ , \\
\overleftarrow{\mathrm{R}_3} &= -\overleftarrow{\tfrac{\delta}{\delta X_a}}\,\varepsilon\omega\times\varepsilon\omega\times\varepsilon\omega\times X_a - \overleftarrow{\tfrac{\delta}{\delta X_a}}\,\varepsilon\omega\times\varepsilon X_b\times\varepsilon X_b\times X_a - \overleftarrow{\tfrac{\delta}{\delta X_a}}\,\varepsilon X_b\times\varepsilon\omega\times\varepsilon X_b\times X_a \\
&\quad - \overleftarrow{\tfrac{\delta}{\delta X_a}}\,\varepsilon X_a\times\varepsilon X_b\times\varepsilon\omega\times X_b + \overleftarrow{\tfrac{\delta}{\delta X_a}}\,\varepsilon X_b\times\varepsilon X_a\times\varepsilon\omega\times X_b - \overleftarrow{\tfrac{\delta}{\delta X_a}}\,\varepsilon X_b\times\varepsilon X_b\times\varepsilon\omega\times X_a \\
&\quad + \overleftarrow{\tfrac{\delta}{\delta X_a}}\,X_a\times\varepsilon\varepsilon X_b\times\varepsilon\omega\times X_b\ ,
\end{aligned}
\tag{3.33}
$$

and so on. Inserting all these into (3.7), performing the functional derivatives and observing various cancellations, we arrive at

$$
\begin{aligned}
\mathcal{T}_g X_a &= X_a + g\,\varepsilon\omega\times X_a - \tfrac{1}{2}g^2\varepsilon X_b\times\varepsilon X_b\times X_a \\
&\quad + \tfrac{1}{6}g^3\Big[2\,\varepsilon X_b\times\varepsilon\omega\times\varepsilon X_b\times X_a + 2\,\varepsilon X_a\times\varepsilon X_b\times\varepsilon\omega\times X_b - 2\,X_a\times\varepsilon\varepsilon X_b\times\varepsilon\omega\times X_b \\
&\qquad - \varepsilon X_b\times\varepsilon X_a\times\varepsilon\omega\times X_b + \varepsilon X_b\times\varepsilon X_b\times\varepsilon\omega\times X_a + \varepsilon(\varepsilon X_b\times X_a)\times(\varepsilon\omega\times X_b)\Big] \\
&\quad + O(g^4)\ .
\end{aligned}
\tag{3.34}
$$

For the practitioner's convenience we spell this out with our shorthand notation fully expanded,

$$
\begin{aligned}
\mathcal{T}_g X_a^A(t) &= X_a^A(t) + g\,f^{ABC}\!\int\!\mathrm{d}s\,\varepsilon(t{-}s)\,\omega^B X_a^C(s) \\
&\quad - \tfrac{1}{2}g^2 f^{ABC}f^{CDE}\!\int\!\mathrm{d}s\,\mathrm{d}u\,\varepsilon(t{-}s)\,X_b^B(s)\,\varepsilon(s{-}u)\,X_b^D X_a^E(u) \\
&\quad + \tfrac{1}{6}g^3 f^{ABC}f^{CDE}f^{EMN}\!\int\!\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}v\,\varepsilon(t{-}s)\,X_b^B(s)\,\varepsilon(s{-}u)\,\Big[\\
&\quad\ 2\omega^D(u)\,\varepsilon(u{-}v)\,X_b^M X_a^N(v) - X_a^D(u)\,\varepsilon(u{-}v)\,\omega^M X_b^N(v) + X_b^D(u)\,\varepsilon(u{-}v)\,\omega^M X_a^N(v)\Big] \\
&\quad + \tfrac{1}{3}g^3 f^{ABC}f^{CDE}f^{EMN}\!\int\!\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}v\,\varepsilon(t{-}s)\,\big[X_a^A(s) - X_a^A(t)\big]\,\varepsilon(s{-}u)\,X_b^D(u)\,\varepsilon(u{-}v)\,\omega^M X_b^N(v) \\
&\quad + \tfrac{1}{6}g^3 f^{ABC}f^{BDE}f^{CMN}\!\int\!\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}v\,\varepsilon(t{-}s)\,\big[\varepsilon(s{-}u)\,X_b^D X_a^E(u)\big]\big[\varepsilon(s{-}v)\,\omega^M X_b^N(v)\big] \\
&\quad + \mathcal{O}(g^4)\ .
\end{aligned}
\tag{3.35}
$$

As a check, beyond $\mathcal{O}(g)$ all terms linear in X_a^A (and thus of maximal power in ω^A) cancel out, a feature that can be proven to hold in general. Also, in the temporal gauge $\omega^A{=}0$ the map drastically simplifies and admits only even powers in g at least up to the order considered. Moreover, to the order displayed here all terms share the "linear tree" topology of the flow operator, except for the last term in $O(g^3)$, which is the first "branched tree". We have checked that this result is consistent with the final result of [18]. It is straightforward though tedious to extend the above computation to higher orders. Equivalently, the $\mathcal{O}(g^4)$ result can be read off by dimensionally reducing the result of [18], but we refrain here from spelling out this formula with non-vanishing ω because it is rather lengthy and not very illuminating.

However, the result simplifies greatly in the temporal gauge $\omega{=}0$. Furthermore, given our transcription rule (2.27), it is straightforward to write it down right away with the APD brackets (2.21). Suppressing the common argument $\boldsymbol{\sigma}$, we arrive at

$$\begin{aligned}
\mathcal{T}_g X_a(t) \;=\; & X_a(t) \;-\; \tfrac12 g^2 \int\!\mathrm{d}s\,\mathrm{d}u\,\varepsilon(t{-}s)\,\varepsilon(s{-}u)\Big\{X_b(s)\,,\,\{X_b(u),X_a(u)\}\Big\} \\
& +\; \tfrac18 g^4 \int\!\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w\;\varepsilon(t{-}s)\,\varepsilon(s{-}u)\,\varepsilon(u{-}v)\,\varepsilon(v{-}w)\,\Big[\\
& \qquad 6\,\Big\{X_b(s)\,,\Big\{X_c(u)\,,\,\{X_{[a}(v)\,,\,\{X_b(w),X_{c]}(w)\}\}\Big\}\Big\} \\
& \qquad +\; 2\,\Big\{X_b(s)\,,\Big\{X_{[b}(u)\,,\,\{X_{|c|}(v)\,,\,\{X_{a]}(w),X_c(w)\}\}\Big\}\Big\} \\
& \qquad +\; 2\,\Big\{X_a(s)-X_a(t)\,,\Big\{X_b(u)\,,\,\{X_c(v)\,,\,\{X_b(w),X_c(w)\}\}\Big\}\Big\}\Big] \\
& +\; \tfrac18 g^4 \int\!\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w\;\varepsilon(t{-}s)\,\varepsilon(s{-}u)\,\varepsilon(s{-}v)\,\varepsilon(v{-}w)\,\times \\
& \qquad \Big\{\{X_a(u),X_b(u)\}\,,\Big\{X_c(v)\,,\,\{X_b(w),X_c(w)\}\Big\}\Big\} \;+\; \mathcal{O}(g^6)\;.
\end{aligned} \tag{3.36}$$

This expression is perfectly well-defined for well-behaved functions $X_a(t,\boldsymbol{\sigma})$, whence the $N\to\infty$ limit of (3.35) is equally well-defined. At higher orders we will encounter more nested APD brackets, but the expansion stays well-defined to arbitrary order. It is noteworthy that, while (3.35) contains both even and odd powers in g, the expansion (3.36) with the temporal gauge $\omega=0$ contains only even powers in g. This can only change in higher orders (starting with R_7 in (3.6), to be completely precise) when we encounter γ-traces such as $\mathrm{tr}\,(\gamma^{a_1}\cdots\gamma^{a_9}) = 16\,\epsilon^{a_1\cdots a_9}$.

Finally, as an independent check, in the Appendix we also demonstrate that the "free-action condition"

$$\tfrac12(\partial_t\mathcal{T}_g X_a)^2 \;\stackrel{!}{=}\; \tfrac12(D_t X_a)^2 - \tfrac14 g^2 (X_b{\times}X_c)^2 + \text{ total derivative} \tag{3.37}$$

as well as the "determinant matching condition" [9]

$$\mathrm{Tr}\log\Big(\frac{\delta\mathcal{T}_g X}{\delta X}\Big) \;\stackrel{!}{=}\; \tfrac12\mathrm{Tr}\log(D_t+g\hat{X}\times) \;+\; \mathrm{Tr}\log\partial_t D_t \;-\; \tfrac{D}{2}\mathrm{Tr}\log\partial_t^2 \tag{3.38}$$

[9]For the temporal gauge the last two terms on the r.h.s. are replaced by $\big[\mathrm{Tr}\log D_t - \frac{D-1}{2}\mathrm{Tr}\log\partial_t^2\big]$, but the cancellations remain the same, of course.

for the Jacobian, Matthews–Salam–Seiler [36, 37] and Faddeev–Popov [34] determinants are both fulfilled up to and including $O(g^3)$ by the maps (3.34) and (3.36), provided

$$D-2 \overset{!}{=} \frac{r}{2}\ , \tag{3.39}$$

as happens to be the case for the critical dimensions $D=3,4,6$ and 10. As already mentioned, the determinants are more subtle in the APD gauge theory directly: like for the APD integral kernel (2.39) we can in each APD bracket (2.21) separate the two $\boldsymbol{\sigma}$ arguments by inserting δ-functions together with integrals over $\boldsymbol{\sigma}$ variables. When expanded, the Jacobian of the map $\mathcal{T}_g$ then contains exactly the same divergent factor $\delta(\boldsymbol{\sigma},\boldsymbol{\sigma})$ that we encountered in the expansion of $\Delta_{\rm MSS}$ and which can thus be dropped for the same reason.

The results of this section should be considered as a generalization of the polynomial map that obtains in supersymmetric quantum mechanics (see *e.g.* [39]), where the perturbative expansion terminates after the first step and gives rise to a closed expression (see also [47, 48] for attempts to find polynomial maps in higher dimensions). Such a feature cannot be expected for the APD gauge theory or matrix model. However, our expressions (3.35) and (3.36) are almost as good, because they can be obtained from a universal formula for $\mathcal{T}_g$ in terms of a path-ordered exponential [46]. This formula furnishes an algorithmic procedure to work out the expansion of $\mathcal{T}_g$ systematically to any given order in g, a calculation that can be automated and implemented on a computer. Again the result will be much simpler with the temporal gauge $\omega=0$. On the technical side it is worth emphasizing that because of (2.34) the differences between the axial and the Lorenz gauge choices almost disappear in one dimension, together with the considerable complications accompanying gauge choices different from the Lorenz gauge in higher dimensions.

4 The Jacobian has a non-zero radius of convergence

One main difference between the present approach and more conventional perturbative expansions of the path integral is that the series expansion for $\mathcal{T}_g$ has better convergence properties (here we are not referring to UV divergences, but to the non-summability of what would be the renormalized perturbation expansion in higher dimensions). That the convergence properties should be better was already anticipated in [39] but never actually proven. Here we present further evidence for this conjecture by showing that with suitable technical assumptions the Jacobian of the map admits a non-zero radius of convergence when expanded around $g=0$ in the complex g plane. This we can do by exploiting the equality (3.2) of the Jacobian with (the product of) the fermionic determinants. Since we are actually only interested in the statement for the remporal gauge let us therefore set $\omega=0$ for which $\Delta_{\rm FP}$ is trivial, and consider the MSS determinant (2.35). To this aim we expand the logarithm of $\Delta_{\rm MSS}$ and make use of the triangle inequality,

$$\Big|\log\det{}^{1/2}(\mathbb{1}+gK)\Big| = \tfrac{1}{2}\Big|\sum_{n=1}^{\infty}\tfrac{(-1)^{n-1}}{n}\,g^n\,\mathrm{Tr}\,K^n\Big| \ \leq\ \tfrac{1}{2}\sum_{n=1}^{\infty}\frac{|g|^n}{n}\,|\mathrm{Tr}\,K^n|\ , \tag{4.1}$$

where the kernel K is defined in (2.36). Let us have a look at the individual terms: we have

$$\operatorname{Tr} K^n = \int dt_1 \cdots \int dt_n \; \varepsilon(t_1-t_2)\cdots\varepsilon(t_n-t_1)\; \operatorname{tr}(\gamma^{a_1}\cdots\gamma^{a_n}) \times \\ \times \operatorname{tr}(T^{A_1}\cdots T^{A_n})\; X^{A_1}_{a_1}(t_1)\cdots X^{A_n}_{a_n}(t_n) \tag{4.2}$$

where T^A are the $\mathrm{SU}(N)$ generators in the adjoint representation. We can now derive an upper bound on the absolute value of this expression by using $|\varepsilon(t)| \le \frac{1}{2}$, together with

$$\left|\operatorname{tr}(\gamma^{a_1}\cdots\gamma^{a_n})\right| \le r \qquad \text{and} \qquad \left|\operatorname{tr}(T^{A_1}\cdots T^{A_n})\right| \le c^n \tag{4.3}$$

where $c \equiv c_N$ is an N-dependent positive constant. Furthermore introducing the L^1-norm

$$\|\mathbf{X}\|_1 := \sum_{a,A}\int dt \, |X^A_a(t)| \tag{4.4}$$

we can majorize the individual terms to obtain

$$(4.1) \le \frac{r}{2}\sum_{n=1}^{\infty}\left(\frac{c}{2}\right)^n \frac{|g|^n}{n}\,\|\mathbf{X}\|_1^n \,. \tag{4.5}$$

This series converges for $|g| < 2c^{-1}\|\mathbf{X}\|_1^{-1}$. Consequently if we constrain the functions $X^A_a(t)$ to belong to the Lebesgue space $\mathrm{L}^1(\mathbb{R}^k)$ (for $k = 9(N^2-1)$) the series always has an (X-dependent) non-zero radius of convergence.

While the fact that the Jacobian has a non-zero radius of convergence as a function of g does not imply that the map itself has this property, it strongly constrains the series expansion for $\mathcal{T}_g$, regardless of the precise form of the functions $X^A_a(t)$. The main reason that makes the argument work is that, unlike for higher-dimensional Yang–Mills theories, the supersymmetric matrix model has no UV divergences which would necessitate infinite subtractions (as in [37]). With appropriate UV and IR regularizations the above statements remain valid for supersymmetric Yang–Mills theories in higher dimensions, at least with the axial gauge choice (for which, however, $\mathcal{T}_g$ is considerably more complicated than for the Lorenz gauge [18, 19]). So in that case *both* regulators are necessary for the MSS determinant to make sense in a more rigorous context.

5 Outlook

We hope that the present investigations will open some new and so far unexplored avenues for addressing several outstanding key problems of supermembrane and matrix theory. Among the topics for future investigation we have already highlighted two of these, namely the question of quantum target-space Lorentz invariance, and the problem of computing physically relevant correlations functions. Here our approach provides a perturbative expansion scheme of a type that has not been available in the literature so far. Finally we note that our methods may also turn out to be applicable to matrix string theory [49, 50], which corresponds to the reduction of maximally extended super-Yang–Mills theory to two spacetime dimensions.

Acknowledgments. We thank Hannes Malcha for help in matching (3.36) with the dimensional reduction of the $\mathcal{O}(g^4)$ result of [18], and Daniele Dorigoni for discussion.

A Appendix: tests

A.1 Free action test

Writing

$$\mathcal{T}_g X_a \;=\; X_a \;+\; g\,\mathrm{T}_1 X_a \;+\; g^2 \mathrm{T}_2 X_a \;+\; g^3 \mathrm{T}_3 X_a \;+\; O(g^4) \tag{A.1}$$

we read off from (3.34) the concrete expressions for T_k. The free-action condition (3.37) then breaks up into

$$\begin{aligned}
&\dot{X}_a\cdot(\mathrm{T}_1X_a)^{\cdot} \;\stackrel{!}{=}\; \dot{X}_a\cdot(\omega\times X_a)\ , \\
&\tfrac12(\mathrm{T}_1X_a)^{\cdot}\cdot(\mathrm{T}_1X_a)^{\cdot}+\dot{X}_a\cdot(\mathrm{T}_2X_a)^{\cdot} \;\stackrel{!}{=}\; \tfrac12(\omega\times X_a)^2-\tfrac14(X_a\times X_b)^2\ , \\
&(\mathrm{T}_1X_a)^{\cdot}\cdot(\mathrm{T}_2X_a)^{\cdot}+\dot{X}_a\cdot(\mathrm{T}_3X_a)^{\cdot} \;\stackrel{!}{=}\; 0\ ,
\end{aligned} \tag{A.2}$$

modulo total derivatives in t.

The first condition is fulfilled since $(\mathrm{T}_1X_a)^{\cdot}=\omega\times X_a$. This also matches the first terms on either side of the second condition. Its remainder is also fulfilled because

$$\begin{aligned}
\dot{X}_a\cdot(\mathrm{T}_2X_a)^{\cdot} \;&=\; -\tfrac12\dot{X}_a\cdot(X_b\times\varepsilon\,X_b\times X_a) \;=\; -\tfrac12(\dot{X}_a\times X_b)\cdot\varepsilon\,(X_b\times X_a) \\
&=\; -\tfrac14(X_a\times X_b)^{\cdot}\cdot\varepsilon\,(X_b\times X_a) \;=\; -\tfrac14(X_a\times X_b)^2 \;+\; \partial_t(\dots)\ .
\end{aligned} \tag{A.3}$$

The third condition is more involved. The left-hand side reads (suppressing total derivatives)

$$\begin{aligned}
&-\;\tfrac12(\omega\times X_a)\cdot X_b\times\varepsilon\,(X_b\times X_a) \;+\; \tfrac16\dot{X}_a\cdot(\varepsilon\,X_b\times X_a)\times(\varepsilon\,\omega\times X_b) \\
&+\;\tfrac13\dot{X}_a\cdot\Big\{X_b\times\varepsilon\,\omega\times\varepsilon\,(X_b\times X_a) \;+\; X_a\times\varepsilon\,X_b\times\varepsilon\,(\omega\times X_b) \;-\; \partial_t\{(X_a\times\varepsilon\varepsilon\,X_b\times\varepsilon\,(\omega\times X_b)\} \\
&\qquad -\;\tfrac12 X_b\times\varepsilon\,X_a\times\varepsilon\,(\omega\times X_b) \;+\; \tfrac12 X_b\times\varepsilon\,X_b\times\varepsilon\,(\omega\times X_a)\Big\} \\
=&-\;\tfrac12(\omega\times X_a)\times X_b\cdot\varepsilon\,(X_b\times X_a) \;-\; \tfrac16 X_a\cdot\partial_t\{(\varepsilon\,X_b\times X_a)\times(\varepsilon\,\omega\times X_b)\} \\
&+\;\tfrac13\dot{X}_a\times\Big\{X_b\cdot\varepsilon\,\omega\times\varepsilon\,(X_b\times X_a) \;-\; \dot{X}_a\cdot\varepsilon\varepsilon\,X_b\times\varepsilon\,(\omega\times X_b) \;-\; X_b\cdot\varepsilon\,X_{[a}\times\varepsilon\,(\omega\times X_{b]})\Big\} \\
=&-\;\tfrac12(\omega\times X_a)\times X_b\cdot\varepsilon\,(X_b\times X_a) \;-\; \tfrac16 X_a\cdot(X_b\times X_a)\times\varepsilon\,(\omega\times X_b) \;+\; \tfrac16 X_a\cdot(\omega\times X_b)\times\varepsilon\,(X_b\times X_a) \\
&+\;\tfrac16(X_a\times X_b)^{\cdot}\cdot\varepsilon\,\omega\times\varepsilon\,(X_b\times X_a) \;-\; \tfrac16(X_a\times X_b)^{\cdot}\cdot\varepsilon\,X_a\times\varepsilon\,(\omega\times X_b) \\
=&-\;\tfrac12(\omega\times X_a)\times X_b\cdot\varepsilon\,(X_b\times X_a) \;-\; \tfrac16 X_a\times(X_b\times X_a)\cdot\varepsilon\,(\omega\times X_b) \;+\; \tfrac16 X_a\times(\omega\times X_b)\cdot\varepsilon\,(X_b\times X_a) \\
&-\;\tfrac16(X_a\times X_b)\cdot\omega\times\varepsilon\,(X_b\times X_a) \;+\; \tfrac16(X_a\times X_b)\cdot X_a\times\varepsilon\,(\omega\times X_b) \\
=&-\;\tfrac16\Big\{3\,(\omega\times X_a)\times X_b \;-\; X_a\times(\omega\times X_b) \;+\; (X_a\times X_b)\times\omega\Big\}\cdot\varepsilon\,(X_b\times X_a) \\
=&-\;\tfrac16\Big\{3\,(\omega\times X_{[a})\times X_{b]} \;+\; (\omega\times X_{[b})\times X_{a]} \;+\; (X_a\times X_b)\times\omega\Big\}\cdot\varepsilon\,(X_b\times X_a) \\
=&-\;\tfrac16\Big\{(\omega\times X_a)\times X_b \;+\; (X_b\times\omega)\times X_a \;+\; (X_a\times X_b)\times\omega\Big\}\cdot\varepsilon\,(X_b\times X_a) \;=\; 0\ ,
\end{aligned} \tag{A.4}$$

where in the last line the Jacobi identity was applied. Several times we employed partial integration and $\partial_t\,\varepsilon=\mathbb{1}$ as well as $A\cdot(B\times C)=(A\times B)\cdot C$ and the complete antisymmetry of the structure constants, $A{\times}B=-B{\times}A$. Furthermore, for the first equality we cancelled part of the ∂_t term with the term preceding it, for the second equality we dropped a term $\sim\dot{X}_a{\times}\dot{X}_a=0$, for the fourth equality the second and fifth terms cancelled, and for the fifth equality the index antisymmetry in the final factor $X_b{\times}X_a$ was used. We note that the value D of the spacetime dimension played no role here.

A.2 Determinant matching test

Since the determinants match in the free theory, it suffices to bring their logarithms to a form

$$\log\det\Delta(g) \;=\; \log\det\Delta(0) \;+\; \mathrm{Tr}\log\big(\mathbb{1}+M(g)\big) \;=\; \mathrm{const}+\mathrm{Tr}\,M-\tfrac{1}{2}\mathrm{Tr}\,M^2+\tfrac{1}{3}\mathrm{Tr}\,M^3+O(g^4) \tag{A.5}$$

since $M(g)$ is of order g, and to compare the expressions in the orders g, g^2 and g^3 of the perturbative expansion. The Tr symbol refers to a trace in position, color and spinor space, while below we reserve the tr symbol for the trace in position and color space only, after having explicitly performed the gamma traces.

For the Faddeev–Popov determinant we have

$$\Delta=\partial_t D_t=\partial_t(\partial_t+g\,\omega\times) \qquad\Rightarrow\qquad M=g\,\partial_t^{-1}\omega\times=g\,\varepsilon\,\omega\times \tag{A.6}$$

which, since there are no spin degreees of freedom, leads to

$$\mathrm{tr}\log\big(\mathbb{1}+M(g)\big) \;=\; g\,\mathrm{tr}\,(\omega{\times}\varepsilon) \;-\; \tfrac{1}{2}g^2\,\mathrm{tr}\,(\omega{\times}\varepsilon\,\omega{\times}\varepsilon) \;+\; \tfrac{1}{3}g^3\,\mathrm{tr}\,(\omega{\times}\varepsilon\,\omega{\times}\varepsilon\,\omega{\times}\varepsilon) \;+\; O(g^4)\;. \tag{A.7}$$

The Matthews–Salam–Seiler determinant produces

$$\Delta=D_t+g\,\hat{X}\times=\partial_t+g\,(\omega+\hat{X})\times \qquad\Rightarrow\qquad M=g\,\varepsilon\,(\omega+\hat{X})\times \tag{A.8}$$

which, with a factor of $\frac{1}{2}$ from the Majorana property, yields

$$\begin{aligned}\tfrac{1}{2}\mathrm{Tr}\log\big(\mathbb{1}+M(g)\big) \;=\;& \tfrac{r}{2}g\;\mathrm{tr}\,(\omega{\times}\varepsilon) \;-\; \tfrac{r}{4}g^2\,\mathrm{tr}\,(\omega{\times}\varepsilon\,\omega{\times}\varepsilon \;+\; X_a{\times}\varepsilon\,X_a{\times}\varepsilon) \\ &+\; \tfrac{r}{6}g^3\,\mathrm{tr}\,(\omega{\times}\varepsilon\,\omega{\times}\varepsilon\,\omega{\times}\varepsilon \;+\; 3\,X_a{\times}\varepsilon\,X_a{\times}\varepsilon\,\omega{\times}\varepsilon) \;+\; O(g^4)\;,\end{aligned} \tag{A.9}$$

where only even powers of $\hat{X}$ survived the spin trace, which produces a factor r for the dimensionality of the spinor representation. Each of the trace terms can be represented by a loop diagram, with bosonic propagators ε and external "legs" ω or X_a. Because $\varepsilon(t{-}t)=\varepsilon(0)=0$, single-leg loops vanish, and we only have to consider the orders g^2 and g^3 in the matching.

Finally, considering the Jacobian of $\mathcal{T}_g$, we must in each tree of the expression (3.34) "differentiate away" one "leaf" X in all possible ways. This results in an expression of the form

$$\frac{\delta\mathcal{T}_g X_a^A(t)}{\delta X_b^B(t')} \;=\; \delta^{AB}\delta_{ab}\delta(t{-}t') \;+\; \big(g\,M_1+g^2M_2+g^3M_3+O(g^4)\big)_{ab}^{AB}(t,t')\;, \tag{A.10}$$

which can be viewed as a string starting from the tree root and ending at the cut leaf location, possibly with branches attached to it. Inserting this expansion into (A.5) we find

$$\log\det\Big(\frac{\delta\mathcal{T}_g X}{\delta X}\Big) \;=\; \mathrm{const} \;+\; g\,\mathrm{Tr}\,M_1 \;+\; g^2\big(\mathrm{Tr}\,M_2-\tfrac{1}{2}\mathrm{Tr}\,M_1^2\big) \;+\; g^3\big(\mathrm{Tr}\,M_3-\mathrm{Tr}\,M_1M_2+\tfrac{1}{3}\mathrm{Tr}\,M_1^3\big)\;. \tag{A.11}$$

Under each trace we glue together the strings in the product and then short-circuit the total string by identifying the end points and summing over the corresponding indices (including integration over time). As a result we collect

$$\begin{aligned} \mathrm{Tr}\, M_2 &= -\tfrac{1}{2}(D-2)\,\mathrm{tr}\left(X_a\times\varepsilon\, X_a\times\varepsilon\right)\ , \\ -\tfrac{1}{2}\mathrm{Tr}\, M_1^2 &= -\tfrac{1}{2}(D-1)\,\mathrm{tr}\left(\omega\times\varepsilon\,\omega\times\varepsilon\right) \end{aligned} \tag{A.12}$$

and, after several cancellations,

$$\begin{aligned} \mathrm{Tr}\, M_3 &= (\tfrac{D}{2}-\tfrac{2}{3})\,\mathrm{tr}\left(X_a\times\varepsilon\, X_a\times\varepsilon\,\omega\times\varepsilon\right) \ +\ \tfrac{1}{3}\,\mathrm{tr}\left(\varepsilon\, X_a\times\varepsilon\right)\times\varepsilon\,\omega\times X_a \\ &\quad -\ \tfrac{1}{3}\,\mathrm{tr}\left(\varepsilon\,\varepsilon\, X_a\times\right)\times\varepsilon\,\omega\times X_a \ -\ \tfrac{1}{3}\,\mathrm{tr}\left(X_a\times\varepsilon\, X_a\times\omega\times\varepsilon\,\varepsilon\right) \\ &= (\tfrac{D}{2}-1)\,\mathrm{tr}\left(X_a\times\varepsilon\, X_a\times\varepsilon\,\omega\times\varepsilon\right) \ +\ \tfrac{1}{3}\,\mathrm{tr}\left(\varepsilon\, X_a\times\varepsilon\right)\times\varepsilon\,\omega\times X_a\ , \\ -\mathrm{Tr}\, M_1 M_2 &= (\tfrac{D}{2}-1)\,\mathrm{tr}\left(X_a\times\varepsilon\, X_a\times\varepsilon\,\omega\times\varepsilon\right)\ , \\ \tfrac{1}{3}\mathrm{Tr}\, M_1^3 &= \tfrac{1}{3}(D-1)\,\mathrm{tr}\left(\omega\times\varepsilon\,\omega\times\varepsilon\,\omega\times\varepsilon\right)\ . \end{aligned} \tag{A.13}$$

In the four contributions to $\mathrm{Tr}\, M_3$, the fourth term is of the same form as the first one because ω being constant can be moved past ε. The other two contributions are loops with a branch attached. The third term vanishes because the trace is proportional to $\partial_t^{-2}(0)$ which gets regularized to zero. Finally, the second term is of the form

$$f(t',t'')\int \mathrm{d}t\ \varepsilon(t-t')\,\varepsilon(t'-t)\,\varepsilon(t-t'') \ =\ -\tfrac{1}{4}\, f(t',t'')\int \mathrm{d}t\ \varepsilon(t-t'') \ =\ 0\ . \tag{A.14}$$

Collecting all remaining contributions, we end up with two 2-leg loops at $O(g^2)$ and two 3-leg loops at $O(g^3)$:

expression	FP	MSS	Jac
$g^2\,\mathrm{tr}\left(\omega\times\varepsilon\,\omega\times\varepsilon\right)$	$-\frac{1}{2}$	$-\frac{r}{4}$	$\frac{1}{2}(1-D)$
$g^2\,\mathrm{tr}\left(X_a\times\varepsilon\, X_a\times\varepsilon\right)$	0	$-\frac{r}{4}$	$\frac{1}{2}(2-D)$
$g^3\,\mathrm{tr}\left(\omega\times\varepsilon\,\omega\times\varepsilon\,\omega\times\varepsilon\right)$	$\frac{1}{3}$	$\frac{r}{6}$	$\frac{1}{3}(D-1)$
$g^3\,\mathrm{tr}\left(X_a\times\varepsilon\, X_a\times\varepsilon\,\omega\times\varepsilon\right)$	0	$3\,\frac{r}{6}$	$D-2$

Here, "FP", "MSS" and "Jac" denote the weight of the individual expressions contributing to the logarithm of the Faddeev–Popov, Matthews–Salam–Seiler and Jacobian determinant, respectively. Fortunately, the sum of the FP and MSS columns agrees with the Jac column provided that again $D-2=\frac{r}{2}$, singling out the critical dimensions once more. This provides a nontrivial check on the expression (3.35) of the Nicolai map, which formally is guaranteed to work out by the construction scheme. Finally, we remark that the matching also works in the temporal gauge, since the Faddeev–Popov determinant becomes trivial but the first and third expression in the table vanish for $\omega=0$ anyway.

References

[1] E. Bergshoeff, E. Sezgin and P.K. Townsend, *Phys. Lett. B* **189** 75.

[2] E. Bergshoeff, E. Sezgin and P.K. Townsend, *Ann. Phys.* **185** (1988) 330.

[3] E. Cremmer, B. Julia and J. Scherk, *Phys. Lett. B* **76** (1978) 409.

[4] B. de Wit, J. Hoppe and H. Nicolai, *Nucl. Phys. B* **305** (1988) 545.

[5] J. Goldstone, unpublished.

[6] J. Hoppe, MIT PhD Thesis, 1982; in: Proc. Int. Workshop on *Constraint theory and relativistic dynamics*, eds. G. Longhi and L. Lusanna (World Scientific, 1987).

[7] E. Witten, *Nucl. Phys. B* **460** (1996) 335, `arXiv:hep-th/9510135`.

[8] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, *Phys. Rev. D* **55** (1997) 5112, `arXiv:hep-th/9610043`.

[9] M.J. Duff, TASI 1996 lectures, `arXiv:hep-th/9611203`.

[10] R. Helling and H. Nicolai, ICTP Spring School 1998 lectures, `arXiv:hep-th/9809103`.

[11] B. de Wit, 1998 Corfu Workshop lectures, `arXiv:hep-th/9902051`.

[12] B. de Wit, U. Marquard and H. Nicolai, *Comm. Math. Phys.* **128** (1990) 39.

[13] K. Ezawa, Y. Matsuo and K. Murakami, *Prog. Theor. Phys.* **98** (1997) 485, `arXiv:hep-th/9705005`.

[14] U. Marquard, R. Kaiser and M. Scholl, *Phys. Lett. B* **227** (1989) 234.

[15] A. Dasgupta, H. Nicolai and J.C. Plefka, *JHEP* **05** (2000) 007, `arXiv:hep-th/0003280`.

[16] H. Nicolai, *Phys. Lett. B* **89** (1980) 341.

[17] H. Nicolai, *Nucl. Phys. B* **176** (1980) 419.

[18] H. Malcha and H. Nicolai, *JHEP* **06** (2021) 001, `arXiv:2104.06017 [hep-th]`.

[19] O. Lechtenfeld and M. Rupprecht, *Phys. Lett. B* **819** (2021) 136413, `arXiv:2104.09654 [hep-th]`.

[20] M.J. Duff, P.S. Howe, T. Inami and K.S. Stelle, *Phys. Lett. B* **191** (1987) 70.

[21] E. Witten, *Nucl. Phys. B* **433** (1995) 85.

[22] P.K. Townsend, ICTP summer school 1996 lectures, `arXiv:hep-th/9612121`.

[23] E. Bergshoeff, E. Sezgin and Y. Tanii, *Nucl. Phys. B* **298** (1988) 187.

[24] L. Brink, J.H. Schwarz and J. Scherk, *Nucl. Phys. B* **121** (1977) 77.

[25] J. Hoppe, *Phys. Lett. B* **215** (1988) 706.

[26] D. Fairlie, P. Fletcher and C.N. Zachos, *Phys. Lett. B* **218** (1989) 203.

[27] M. Bordemann, E. Meinrenken and M. Schlichenmaier, *Commun. Math. Phys.* **165** (1994) 281, `arXiv:hep-th/9309134`.

[28] M. Claudson and M.B. Halpern, *Nucl. Phys. B* **250** (1985) 689.

[29] M. Baake, M. Reinicke and V. Rittenberg, *Journ. Math. Phys.* **26** (1985) 1070.

[30] R. Flume, *Ann. Phys.* **164** (1985) 189.

[31] D. Bigatti and L. Susskind, Cargese NATO ASI 1997 lectures, `arXiv:hep-th/9712072`.

[32] T. Banks, TASI 1999 lectures, `arXiv:hep-th/9911068`

[33] W. Taylor, Iceland NATO summer school 1999 lecture notes, `arXiv:hep-th/0002016`.

[34] L.D. Faddeev and A. Popov, *Phys. Lett. B* **25** (1967) 29.

[35] G. 't Hooft, *Nucl. Phys. B* **33** (1971) 173.

[36] T. Matthews and A. Salam, *Nuovo Cim.* **12** (1954) 563.

[37] E. Seiler, *Commun. Math. Phys.* **42** (1975) 163.

[38] M.B. Green and J.H. Schwarz, *Nucl. Phys. B* **198** (1982) 252.

[39] H. Nicolai, in: *Supersymmetry*, eds. K. Dietz et. al., *Plenum Press* (1984).

[40] R. Flume and O. Lechtenfeld, *Phys. Lett. B* **135** (1984) 91.

[41] K. Dietz and O. Lechtenfeld, *Nucl. Phys. B* **255** (1985) 149.

[42] K. Dietz and O. Lechtenfeld, *Nucl. Phys. B* **259** (1985) 397.

[43] O. Lechtenfeld, *Construction of the Nicolai mapping in supersymmetric field theories*, Doctoral Thesis, Bonn University (1984).

[44] S. Ananth, H. Nicolai, C. Pandey and S. Pant, *J. Phys. A* **53** (2020) 174001, `arXiv:2001.02768 [hep-th]`.

[45] S. Ananth, O. Lechtenfeld, H. Malcha, H. Nicolai, C. Pandey and S. Pant, *JHEP* **10** (2020) 199, `arXiv:2005.12324 [hep-th]`.

[46] O. Lechtenfeld and M. Rupprecht, *Phys. Rev. D* **104** (2021) L021701, `arXiv:2104.00012 [hep-th]`.

[47] O. Lechtenfeld, *Nucl. Phys. B* **274** (1986) 633.

[48] V. de Alfaro, S. Fubini and G. Furlan, *Phys. Lett. B* **163** (1985) 176.

[49] L. Motl, `arXiv:hep-th/9701025`.

[50] R. Dijkgraaf, E. Verlinde and H. Verlinde, *Nucl. Phys. B* **500** (1997) 43, `arXiv:hep-th/9703030`.