

IPP Summer University for Plasma Physics



September 19 – September 23, 2016
Greifswald, Germany

Edited by Roberto Bilato, Ralf Kleiber

Contributions from:

C. Angioni, J. Baldzuhn, R. Bilato, A. Dinklage, M. Drevlak, O. Grulke,
D. Hartmann, M. Hirsch, H. Kersten, R. Kleiber, O. Maj, R. Pitts, T. Pütterich,
R. Stieglitz, W. Suttrop, U. v. Toussaint, T. Windisch, H. Zohm

List of speakers

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Prof. Dr. Richard Pitts	ITER Organization Building Headquarters Route de Vinon - CS90046 F-13067 Saint Paul Lez Durance
Prof. Dr. Holger Kersten	Christian-Albrechts-Universität zu Kiel Leibnizstraße 11-19 D-24098 Kiel
Prof. Dr. Robert Stieglitz	KIT-Zentrum Energie Hermann-von-Helmholtz-Platz 1 D-76344 Eggenstein-Leopoldshafen
Dr. Clemente Angioni	Max-Planck-Institut für Plasmaphysik
Dr. Roberto Bilato	Boltzmannstr. 2
Dr. Omar Maj	D-85748 Garching
Dr. Thomas Pütterich	
Dr. Wolfgang Suttrop	
Dr. Udo von Toussaint	
Prof. Dr. Hartmut Zohm	
Dr. Jürgen Baldzuhn	Max-Planck-Institut für Plasmaphysik
Dr. Andreas Dinklage	Wendelsteinstr. 1
Dr. Michael Drevlak	D-17491 Greifswald
Prof. Dr. Olaf Grulke	
Dr. Dirk Hartmann	
Dr. Matthias Hirsch	
Dr. Ralf Kleiber	
Dr. Thomas Windisch	

Cover page:

The experiment Wendelstein 7-X in Greifswald.

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Chapter 1

Basic plasma physics

*Monika Bessenrodt-Weberpals,
Hugo de Blank, Axel Könies,
Emanuele Poli, Ralf Schneider*

1.1 Introduction

A plasma is a hot gas consisting of charged and neutral particles which exhibit collective behaviour [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

The name *plasma* goes back to Langmuir (1929) who discovered electromagnetic oscillations in rarefied gases which he called *plasma oscillations*.

The plasma is often referred to as the fourth state of matter (solid – liquid – gas – plasma). This concept arises from a thermodynamic description but is not completely rigorous. The variation of the basic macroscopic physical quantities like density n and temperature T determines the properties of the matter in the plasma state (see Fig. 1.1). However, the transition from the liquid, gaseous or solid state into the plasma state is not abrupt but continuous. In a gas, for example, more and more of the atoms become ionized with rising temperature and a plasma forms.

We know now that 99% of the matter in the universe is in the plasma state. Plasma can be found in the interior of the stars as well as in the interstellar space and in the core of the planets. Plasma also occurs in gas discharges (*neon light*, lightning) as part of our daily life.

The transition from weakly interacting neutral particles to charged particles which interact over the long-range Coulomb force may alter the properties of the particle system considerably. Therefore, many-body interaction plays an important role. The possible charge separation gives rise to electromagnetic oscillations or waves in a plasma.

Note that today the term plasma is understood in a very broad sense and may be applied to all systems which consist of charged particles as e.g. electrons in a solid or, even more general,

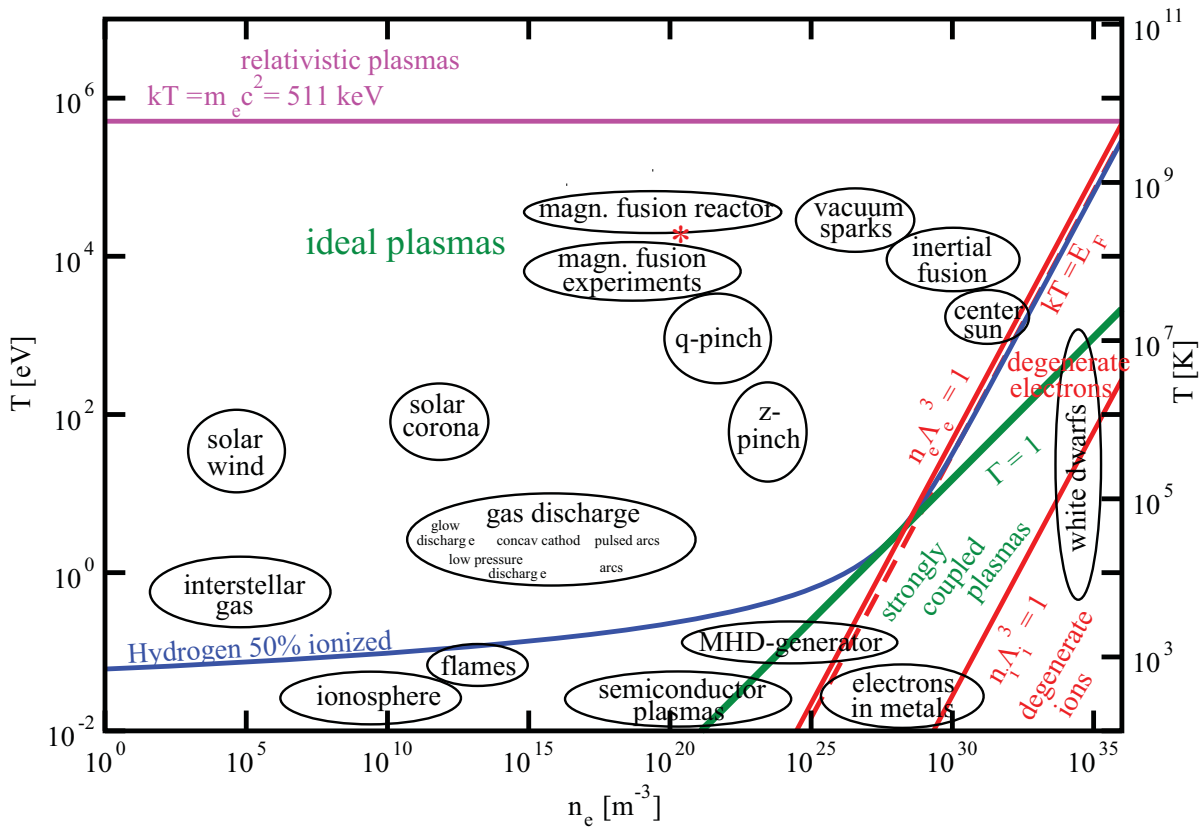


Figure 1.1: Temperatures and densities of astrophysical and laboratory plasmas. The asterisk indicates the fusion relevant parameters used in the examples: $T = 10^4$ eV, $n_e = 10^{20}$ m $^{-3}$, $B = 5$ T.

systems where the bound states of the constituents are broken. The quark-gluon-plasma with its *color* charged constituents, occurring shortly after big bang or at high energy nucleus-nucleus collisions, is a famous example for this concept.

In the course of these lessons we will focus on plasmas made up of ions and electrons which interact over the Coulomb force and which are globally neutral. In this Chapter, we will briefly discuss the types of plasmas in the next section. Then, we go over to the plasma parameters which provide a more quantitative access to the several types of plasmas shown in Fig. 1.1. We will discuss the fundamental concepts of plasma physics, namely the Debye screening, the plasma frequency and the so-called quasi-neutrality. Further we will touch plasma collisions and we will end the lecture discussing the plasma motion in a magnetic field which is of paramount importance for plasmas in magnetic fusion devices.

Finally, appendix A reports some characteristic plasma parameters and their numerical values for typical fusion plasmas ($T = 10^4$ eV, $n_e = 10^{20}$ m $^{-3}$, $B = 5$ T).

1.2 Types of plasmas

1.2.1 Astrophysical plasmas

There are a variety of astrophysical plasmas in nature. They cover a wide range of densities and temperatures. Here, we will mention only a few of them.

The interior of the sun and the stars consists of a very dense and very hot plasma where light atomic nuclei fuse to heavier ones and release the excess of binding energy according to Einstein's famous formula $E = mc^2$, see chap. 2. The plasma density and the plasma temperatures drop the closer one comes to the surface, thus the stellar atmosphere has completely different properties. The solar corona which is even more outside is a dilute magnetized plasma of several million degrees.

The sun emits an extremely dilute supersonic plasma, the solar wind, into its planet system. Near the earth the solar wind has $n_e \approx 5 \text{ cm}^{-3}$ and $T_e = 10^5 \text{ K}$. Because of its high temperature, the plasma has still a high conductivity, see sec. 1.6.

By interaction with the electromagnetic radiation from the sun the atoms of the upper atmosphere become partly ionized. We call this plasma which expands from about 60–2 000 km altitude the ionosphere.

A completely other type of plasmas are the degenerate plasmas within white dwarfs or neutron stars showing extremely high densities and thus behaving differently from the typical ideal plasmas we know from the majority of laboratory plasmas (ideal plasmas are defined below, sec. 1.3.4).

1.2.2 Laboratory plasmas

The tradition of laboratory plasma physics starts with the investigation of the weakly ionized plasmas of flames in the 18th century. Typical applications nowadays are plasma-aided welding and combustion for this type of plasma.

Since the plasma may carry an electric current, plasma discharges of various types are investigated in fundamental research and applied in industry. Low-pressure discharges like glow discharges carry small currents with cold electrodes. They serve for lightening, for gas lasers like the CO₂ laser or the HeNe laser, and for the wide-spread applications of plasma etching and deposition.

High-pressure discharges like arcs may carry larger currents and thereby attain higher temperature. They may also serve for lightening like the well-known high-pressure mercury lamp, for switches, and for plasma-material processing like melting, cutting, and welding.

In order to gain fusion energy in laboratories, high-temperature plasma physics has been started as a classified project around 1950. In 1958, the UN conference on the peaceful use of atomic energy has led to a declassification of fusion research, see chap. 2.

Note that free electrons in metal behave similarly to a high-density, low-temperature plasma (see Fig. 1.1).

1.3 Plasma parameters

To distinguish between different plasma states, let us look at the n - T diagram (see Fig. 1.1). The different physical behaviour is ordered by the four characteristic energies, namely the thermal energy $E_{th} = 3/2k_B T$, the Fermi energy $E_F = (\hbar^2/2m_e) \cdot (3\pi^2 n)^{2/3}$, the electrostatic interaction energy $E_{el} = e^2/(4\pi\epsilon_0 r)$, and the energy of Bohr's ground state $E_I = m_e e^4/(8\epsilon_0^2 \hbar^2)$. For simplicity, we concentrate here on hydrogen with charge $q = e$.

1.3.1 Ionization

First, with $E_{th} \approx E_I$ we can estimate the temperature T_I necessary for ionization from

$$\frac{3}{2}k_B T_I = m_e e^4 / (8\epsilon_0^2 \hbar^2)$$

we get

$$k_B T_I = \frac{m_e e^4}{8\epsilon_0^2 \hbar^2} \cdot 2/3 = 9 \text{ eV} . \quad (1.1)$$

Above this temperature, hydrogen plasmas are partially to fully ionized.

More exactly, the degree of ionization in a plasma in thermodynamic equilibrium is given by the Saha equation [14]

$$\frac{n_z n_e}{n_{z-1}} = 2 \frac{Z_z(T)}{Z_{z-1}(T)} \left(\frac{m_e k_B T}{2\pi\hbar^2} \right)^{3/2} \exp\left(-\frac{E_{I,z}}{k_B T}\right) \quad (1.2)$$

with n_z being the density of the z -th ionization stage and Z_z the corresponding partition function.

From Fig. 1.1, you can read the situation in a hydrogen plasma whose ionization degree is 50%. This shows that the hydrogen plasma is almost completely ionized at $k_B T > 1/10 E_I$. The Saha equation may be applied for low ionization stages and high-density plasmas where collision processes dominate over radiative transitions and thereby determine the equilibrium. For high-ionized atoms like heavy impurities in a fusion plasma, the full set of rate equations has to be solved including both collisional and radiative transitions, see chap. 7.

1.3.2 Degeneracy and thermal wave length

The electrons which are Fermions and thus obey the Pauli principle, have actually to be described by a Fermi distribution of the energy. However, in a wide range of plasma parameters ($E_{th} \gg E_F$) the Maxwell-distribution is an excellent approximation. This is the case for magnetic fusion plasmas, the sun, flames, gas discharges, interstellar plasmas ...

With increasing density or for low temperatures, if the Fermi energy approaches the thermal energy (see Fig. 1.1),

$$k_B T \approx E_F = \frac{\hbar^2}{2m} (3\pi^2 n)^{2/3} .$$

the electron gas becomes more and more degenerate. That means that the quantum statistical behaviour of the electrons does play a role and the Fermi distribution has to be used.

We can express the influence of the quantum character of the particles also using the ratio of the length scales of the thermal (de Broglie) wave length of the particles p

$$\Lambda_p = \frac{h}{\sqrt{2\pi m_p k_B T_p}} \quad (1.3)$$

to their average distance (also called Wigner-Seitz-Radius)

$$r_0 = \left(\frac{4\pi}{3} n \right)^{-1/3}. \quad (1.4)$$

The criterion of non-degeneracy for the particle species p is then given by

$$n_p \Lambda_p^3 \ll 1 \quad (1.5)$$

which is true if the de Broglie wave length is much less than the average particle distance.

1.3.3 Relativistic effects

If the thermal energy of the electrons is of the order of the energy corresponding to the electron rest mass

$$E_{th} \approx m_e c^2$$

at a temperature of approximately 511 keV, we have a relativistic plasma (see Fig. 1.1).

The plasma is called a relativistic degenerate plasma if the Fermi energy is of the order $m_e c^2$.

1.3.4 Ideal and non-ideal plasmas

If the thermal energy of a species of charged particles (e.g. electrons) is much larger than that of their Coulomb interaction, the plasma is called an ideal plasma.

To get a quantitative estimate, let us define the ratio of the Coulomb energy at the average interparticle distance to the thermal energy

$$\Gamma = \frac{Ze^2}{4\pi\epsilon_0 r_0 k_B T}. \quad (1.6)$$

The parameter Γ allows the classification of the degree to which the Coulomb interaction determines the plasma behaviour ($\Gamma \ll 1$ almost ideal, $\Gamma < 1$ weakly non-ideal and $\Gamma > 1$ strongly non-ideal).

The Landau length λ_L is defined as the value of r_0 at which the thermal energy equals the electrostatic energy ($\Gamma = 1$):

$$\lambda_L = \frac{Ze^2}{4\pi\epsilon_0 k_B T}. \quad (1.7)$$

This quantity will reappear in the calculation of the plasma resistivity.

A further parameter may be defined to estimate the particle-particle interaction, namely the ratio of the Landau length and the thermal de Broglie length

$$\xi_p = \sqrt{2\pi} \frac{\lambda_L}{\Lambda_p} = 2 \sqrt{\frac{E_I}{k_B T}}. \quad (1.8)$$

If the plasma parameter ξ is of the order of one then in microscopic scattering processes the quantum mechanical interaction does play an important role and the Born series approximation cannot be applied.

1.4 Plasma fundamentals

1.4.1 Quasi-neutrality

First, let us take a closer look at the electrostatic properties of plasmas. Indeed, plasmas are usually electrically *quasi-neutral*, i.e. local concentrations of charge or external potential are shielded out at a distance which is short compared with the typical system length. A simple estimate shows that a small charge separation causes a large restoring electric field. Consider a non-zero charge density $\rho_E = e(n_i - n_e)$ caused by a small separation x of the electrons from the ions (see Fig. 1.2 left). Application of Poisson's equation $\nabla \cdot \mathbf{E} = \rho_E / \epsilon_0$ yields ($\nabla \cdot \mathbf{E} \approx E/x$)

$$E \approx \frac{n_e e x}{\epsilon_0}. \quad (1.9)$$

A fully ionized plasma at 5 eV and atmospheric pressure has an electron density of $n_e = 6 \times 10^{22} \text{ m}^{-3}$. A charge separation of $x = 1 \text{ mm}$ would lead to a very strong electric field $E = 1 \times 10^{12} \text{ V/m}$ and an electric potential $\phi = 5 \times 10^8 \text{ V}$ over 1 mm. In fact, finite electric fields can be generated by very small deviations of the densities from quasi-neutrality.

1.4.2 Plasma frequency

The strong restoring electric field associated with deviations from quasi-neutrality causes a harmonic oscillation (Langmuir, 1929). In the absence of a magnetic field, electrons react to the electric field of (1.9) with

$$m_e \frac{d^2 x}{dt^2} = -e E = -\frac{n_e e^2 x}{\epsilon_0}.$$

The solution of this equation is an oscillation with the characteristic electron *plasma frequency*

$$\omega_{p,e} = \left(\frac{n_e e^2}{\epsilon_0 m_e} \right)^{1/2}. \quad (1.10)$$

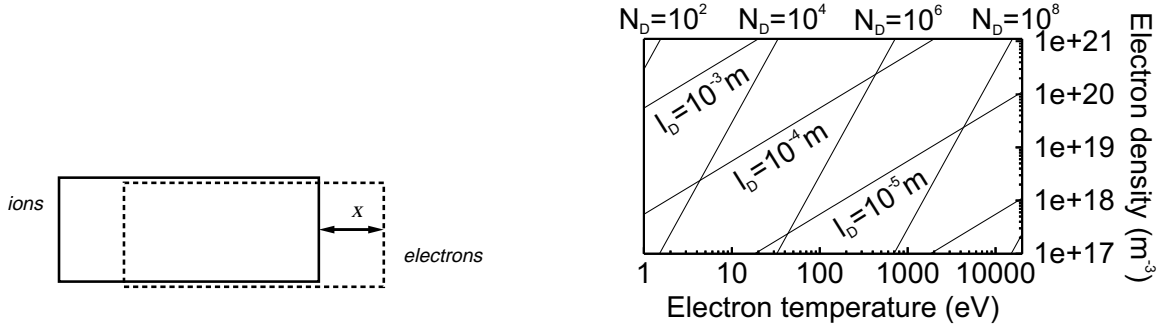


Figure 1.2: Left: Schematic illustration of charge separation in a plasma.
Right: Values of the Debye length λ_D and the plasma parameter N_D plotted against electron temperature and electron density.

The ions do not participate in these oscillations because of their high mass. The plasma frequency is temperature-independent. For typical fusion plasma densities (10^{20} m^{-3}) it is $\nu_{p,e} \approx 100 \text{ GHz}$. Electromagnetic waves with frequencies below $\omega_{p,e}$ cannot propagate in plasmas since their electric fields are screened by the Debye shielding. In this case, the waves are reflected. For larger frequencies, the inertia of the electrons allows the propagation of the waves. This is the reason why metals reflect visible light but not X-rays. The picture becomes more complicated if magnetic fields are present, cf. chap. 6. Analogously, the ion plasma frequency is defined as

$$\omega_{p,i} = \left(\frac{n_i Z^2 e^2}{\epsilon_0 m_i} \right)^{1/2} \quad (1.11)$$

and the total plasma frequency

$$\omega_p^2 = \omega_{p,e}^2 + \omega_{p,i}^2. \quad (1.12)$$

1.4.3 Debye shielding

Usually, ions are surrounded by electrons which screen the ion charge. Only at small distances a deviation from charge neutrality may be observed. In the presence of an electric potential ϕ , the electron and ion densities in thermal equilibrium are modified according to Boltzmann's distribution $\sim \exp(-E/k_B T) = \exp[(-mv^2/2 - q\phi)/k_B T]$ with $q = \pm e$. Integration over all velocities yields $\tilde{n}_q = n_q \exp(-q\phi/k_B T_q)$, where n_q are the densities for $\phi = 0$. For small exponents (ideal plasmas!), this can be approximated by $\tilde{n}_q = n_q(1 - q\phi/k_B T_q)$. For singly charged ions, Poisson's equation $\nabla^2 \phi = -\rho_E/\epsilon_0$ in spherical geometry reads

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) = \frac{e(\tilde{n}_e - \tilde{n}_i)}{\epsilon_0} = \frac{n_e e^2}{\epsilon_0} \left(\frac{1}{k_B T_e} + \frac{1}{k_B T_i} \right) \phi, \quad (1.13)$$

where the plasma approximation $n_e = n_i$ is used. With the definition of the *Debye length*

$$\lambda_{D,q} = \left(\frac{\epsilon_0 k_B T_q}{n_e e^2} \right)^{1/2}, \quad (1.14)$$

and the total Debye length

$$\lambda_D^{-2} = \lambda_{D,e}^{-2} + \lambda_{D,i}^{-2} \quad (1.15)$$

the solution of (1.13) gives the screened potential

$$\phi(r) = \frac{e}{4\pi\epsilon_0} \frac{\exp(-r/\lambda_D)}{r} \quad (1.16)$$

(the constant derives from assuming that ϕ matches the usual Coulomb potential for $r \rightarrow 0$). The Debye length describes the range of an electric potential inside a plasma, which is much shorter than the range in vacuum.

A more heuristic derivation of the Debye length is the following: Small charge separations arise as a result of thermal fluctuations. In a plasma the thermal energy density of the electrons per degree of freedom is $E_f = (1/2)n_e k_B T$. The electrostatic energy density results from a separation of charge over a length x (see Fig. 1.2 left and (1.9)) as $E_e = (\epsilon_0/2)E^2 = (\epsilon_0/2)(n_e e x / \epsilon_0)^2$. The comparison of E_f and E_e shows that substantial separation of charge can only occur over lengths of up to $x \approx (\epsilon_0 k_B T / n_e e^2)^{1/2}$, which is again the Debye length λ_D given in (1.14).

Averaged over many λ_D , the electron and ion charge densities are equal, i.e. $n_e = n_i$. The thermal electron velocity $v_{th,e} = (3k_B T_e / m)^{1/2}$ is found from (1.10) and (1.14) to be

$$v_{th,e} = \sqrt{3} \lambda_{D,e} \omega_{p,e} .$$

Therefore, $1/\omega_{p,e}$ can be interpreted as the time that a thermal electron needs to pass the Debye length.

The definition of an ideal plasma (see sec. 1.3.4) is equivalent to the call for a large number of particles in a Debye sphere, called the *plasma parameter* N_D

$$N_D = \frac{4}{3}\pi\lambda_D^3 n \gg 1 .$$

1.5 Particle collisions

1.5.1 Coulomb collisions and Rutherford scattering

Collisions in the plasma play an important role for collective effects like e.g. resistivity in the plasma. When an electron collides with a neutral atom, no force is felt until the electron is close to the atom; these collisions are similar to billiard-ball collisions. However, when an electron collides with an ion, the electron is gradually deflected by the long-range Coulomb field of the ion. Nonetheless, an effective cross-section may be derived. We will do it in a classical picture without considering quantum mechanical or relativistic corrections.

We will consider here the scattering of an electron by an ion. Because of the large ratio of ion to electron mass we can assume the ion location in the centre of mass of the two-particle system (see Fig. 1.3 left).

In the absence of Coulomb forces, the electron would have a distance of closest approach called the *impact parameter* ρ .

For the Coulomb interaction of two charges e_1 and e_2 , the potential reads $U(r) = \alpha/r$ with $\alpha = e_1 e_2 / (4\pi\epsilon_0)$.

Following textbooks of theoretical mechanics [17] we introduce cylindrical coordinates and employ the conservation of energy

$$E = \frac{m\dot{r}^2}{2} + \frac{L^2}{2mr^2} + U(r) \quad (1.17)$$

and angular momentum

$$L = mr^2\dot{\phi}. \quad (1.18)$$

Let us assume that the electron has the relative velocity v_∞ if it is at a very large distance from the scattering centre (ion). Then, the (conserved) energy must be equal to $E = mv_\infty^2/2$ and the (conserved) angular momentum $L = m\rho v_\infty$. For very long times the particle should again go to infinity. This allows us to calculate the angle ϕ_0 (see Fig. 1.3) as

$$\phi_0 = \int_{r_{min}}^{\infty} \frac{\rho \frac{dr}{r^2}}{\sqrt{1 - \frac{\rho^2}{r^2} - \frac{2U}{mv_\infty^2}}}. \quad (1.19)$$

The distance of closest approach r_{min} is reached when \dot{r} is zero, i.e. when the kinetic energy of the particle has been converted to energy in the effective potential $U(r) + L^2/(2mr^2)$ ($r = r_{min}$ is a root of the denominator in (1.19)). Integration gives

$$\rho^2 = \frac{\alpha^2}{m^2 v_\infty^4} \cot^2 \frac{\chi}{2} = \rho_0^2 \cot^2 \frac{\chi}{2}, \quad (1.20)$$

with $\chi = \pi - 2\phi_0$, cf. again Fig. 1.3. As we have usually an ensemble of particles which undergo the scattering we define the cross-section σ as

$$d\sigma = \frac{dN}{n}, \quad (1.21)$$

where N shall denote the number of particles being scattered by angles between χ and $\chi + d\chi$, while n is the number of particles per area unit. Assuming a homogeneous beam, we have $dN = n2\pi\rho d\rho$. Expressing this in terms of the solid angle Ω gives

$$d\sigma = \frac{\rho}{\sin\chi} \left| \frac{d\rho}{d\chi} \right| d\Omega. \quad (1.22)$$

Finally, the Rutherford cross-section becomes

$$d\sigma = \left(\frac{\alpha}{2mv_\infty} \right)^2 \frac{d\Omega}{\sin^4(\chi/2)}. \quad (1.23)$$

We can see that the number of collisions with small angles is larger than those of large angles. Furthermore, the value for the cross section diverges for large impact parameters ρ or small angles χ like χ^{-4} with the consequence of an infinite total cross-section. This is due to the infinite long range of the Coulomb interaction.

From the cross section we can infer an approximate expression for the electron-ion collision time if we define the collision time as the proportionality constant between the initial parallel momentum of the electron and its change during collisions with ions at all possible impact parameters ρ

$$\frac{d}{dt} \langle \Delta p_{\parallel} \rangle = -v_{ei} p_{\parallel} . \quad (1.24)$$

The initial parallel momentum is $p_{\parallel} = p = mv_{\infty}$ while $p_{\parallel}^f = p \cos \chi$, as it can easily be inferred from Fig. 1.3 left. Using trigonometric relations and (1.20) we get

$$\Delta p_{\parallel} = p(\cos \chi - 1) = -2p \frac{1}{1 + \cot^2(\chi/2)} = -2p \frac{1}{1 + (\rho/\rho_0)^2} . \quad (1.25)$$

In the following we will denote v_{∞} by v for brevity. For the net change of momentum per time unit of the electron we have to consider all the ions that the incoming electron can encounter in this time unit

$$\frac{d}{dt} \langle \dots \rangle = 2\pi n_i v \int_0^{\rho_{max}} \rho d\rho \dots . \quad (1.26)$$

The upper limit should actually be taken to be infinity. We will see, however, that the integral diverges for $\rho_{max} \rightarrow \infty$, which is the well known Coulomb divergence due to the r^{-1} dependence of the Coulomb potential. The problem can be cured considering the screening of the particle-particle interaction by the other charges in the system, resulting in the exponentially decaying Debye-potential we have derived in the last section. In a plasma, the typical radius of the interaction is the Debye-length. Therefore, we take this length as the upper cut-off and we obtain

$$\frac{d}{dt} \langle \Delta p_{\parallel} \rangle = -\frac{4\pi n_i}{mv^2} \alpha^2 \frac{1}{2} \left\{ \ln \left| 1 + \left(\frac{\rho_{max}}{\rho_0} \right)^2 \right| \right\} , \quad (1.27)$$

which gives

$$\frac{d}{dt} \langle \Delta p_{\parallel} \rangle \approx -\frac{n_i Z_i^2 e^4}{4\pi \epsilon_0^2 m v^2} \ln \frac{\lambda_D}{\rho_0} = -\frac{n_i Z_i^2 e^4}{4\pi \epsilon_0^2 m v^2} \ln \Lambda , \quad (1.28)$$

if we consider that the Debye length $\rho_{max} = \lambda_D$ is much larger than ρ_0 . The latter condition is true if the number of particles in a Debye sphere is large, i.e. if we have an ideal plasma, see discussion in Sec. 1.3.4. Compare ρ_0 , (1.20), with the Landau length, (1.7). The expression $\ln \Lambda = \ln(\lambda_D/\rho_0)$ is called the Coulomb logarithm.

The cross sections and collision times are fundamental in order to understand the behaviour of the plasma. In the next section we will relate them to the resistivity by very simple physical arguments. Note, however, that the derivation we gave here is heuristic and not rigorous at all. An average over the velocity distribution had actually to be included in the derivation. This is shown in chap. 5.

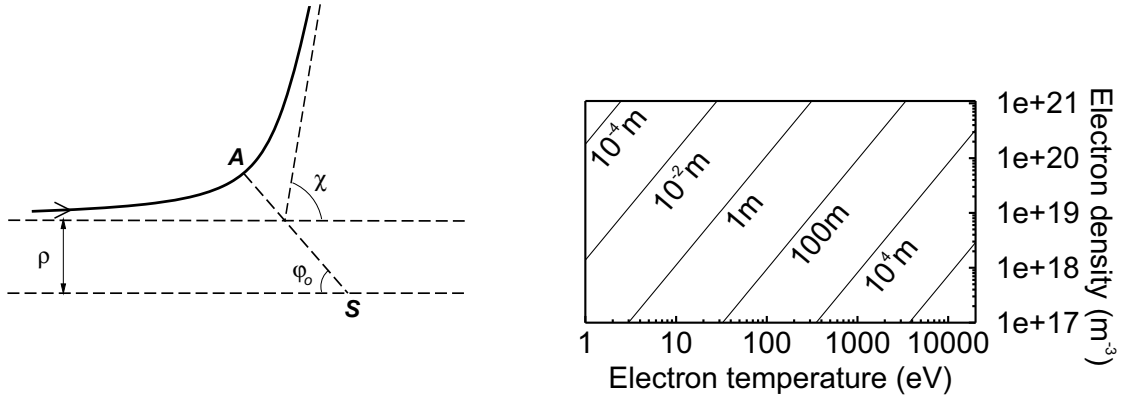


Figure 1.3: Left: Rutherford scattering; S is the centre of mass of the scattering partners. As the Coulomb force is a central force, the orbit is symmetric with respect to A and therefore $\chi = |\pi - 2\phi_0|$.
Right: Values of the mean free path of electrons λ_e plotted against electron temperature and electron density.

The rigorous derivation should start from a kinetic equation. There are a lot of approaches to non-equilibrium processes which give quantum mechanical expressions for cross sections or collision times. These expressions usually include the collective effects like screening which we had to introduce artificially by the cut-off parameter. The expression we have obtained here turns out to be the first order Born approximation of a quantum mechanical cross section. This makes clear that the approximation is valid for high energies of the electron only, i.e. when the thermal energy of a particle is much larger than the Coulomb interaction. Therefore, for non-ideal plasmas the simple classical picture can not be applied and the Coulomb logarithm is a poor approximation.

The collision time for electron-electron collisions follows from comparing (1.24) and (1.28)

$$\tau_{ei} = \frac{4\pi\epsilon_0^2 m_e^2 v_e^3}{Z^2 e^4 n_i \ln \Lambda} . \quad (1.29)$$

From the collision time the mean free path can be calculated as $\lambda_{mfp} = \tau_{ei} v_e$. Fusion plasmas possess a mean free path of about 15 km. This is much larger than the typical laboratory plasma dimensions, leading to the term *collision-less* plasma. The mean free path for fusion reactions is still four orders of magnitude larger at this temperature.

1.6 Electric resistivity

If an electric field is applied to the plasma, the electrons and ions are accelerated in opposite directions. The electrons speed up much faster due to their lighter mass. They deliver the gained energy to the ions by Coulomb collisions, yielding a *resistance* η which, according to Ohm's law, connects the electric field \mathbf{E} with the current density \mathbf{j}

$$\mathbf{E} = \eta \mathbf{j} . \quad (1.30)$$

The net change of electron momentum is supplied by the electric field and by the momentum loss due to collisions

$$\frac{d}{dt} \langle \Delta p_{\parallel} \rangle = -eE - v_{ei} p_{\parallel}. \quad (1.31)$$

Assuming an equilibrium between driving and dissipative forces, the term on the left hand side should vanish. If we further assume that the current is proportional to the relative velocity of electrons and ions $j = -en_e v = -en_e p_{\parallel} / m_e$, we obtain

$$-eE = -\frac{v_{ei} m_e j}{en_e} \quad (1.32)$$

From (1.30) and (1.32) the resistivity is

$$\eta = \frac{v_{ei} m_e}{n_e e^2} \propto T_e^{-3/2}, \quad (1.33)$$

where the temperature dependence follows from the estimate $v_e \sim v_{th} \sim \sqrt{k_B T_e}$ in (1.29). The exact value of the specific resistance was derived in [15] with the help of kinetic theory and the resulting *Spitzer resistivity* is a factor of about two smaller than our first approximation. For $T_e = 1.4$ keV, the plasma resistance is comparable to copper ($1.8 \times 10^{-8} \Omega\text{m}$). Since η is proportional to $T_e^{-3/2}$, fusion plasmas with $T \approx 10$ keV have resistance values which are an order of magnitude smaller than that of copper. The measurement of the resistance can be used to determine plasma temperatures since it depends only very weakly on the density via $\ln \Lambda$ chap. 8. In laboratory plasmas, the effects of different ion species (impurities) on the resistivity often has to be taken into account. For plasmas in strong magnetic fields, additional corrections have to be made to the electrical resistance. The specific resistance η_{\perp} is approximately twice the value of η_{\parallel} .

1.7 Motion of charged particles in electromagnetic fields

Both laboratory and astrophysical plasmas are often immersed in a magnetic field. The motion of a particle with electric charge q and mass m in electric and magnetic fields can be determined from the combined electrostatic and Lorentz force

$$\mathbf{F} = q (\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

1.7.1 Homogeneous magnetic field

For $E = 0$ and a homogeneous magnetic field, the kinetic particle energy remains constant because the Lorentz force is always perpendicular to the velocity and can thus change only its direction, but not its magnitude. The motion of charged particles is then described in terms of the velocity components parallel and perpendicular to the magnetic field, v_{\parallel} and v_{\perp} . Only v_{\perp} interacts with \mathbf{B} , leading to a circular motion perpendicular to \mathbf{B} . The superposition of this circular motion with a free streaming parallel to the magnetic field gives rise to the helical

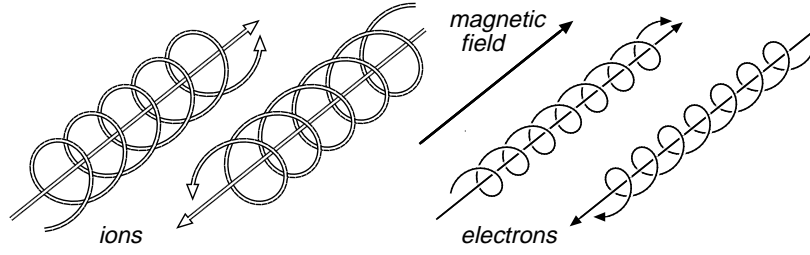


Figure 1.4: Orientation of the gyration orbits of electrons and ions in a magnetic field.

motion depicted in Fig. 1.4. The centrifugal force mv_{\perp}^2/r balances the Lorentz force $qv_{\perp}B$ for a gyration radius r equal to the *Larmor radius*

$$\rho_L = \frac{mv_{\perp}}{|q|B}. \quad (1.34)$$

If $mv_{\perp}^2/2 = k_B T$ is taken for the two-dimensional thermal motion perpendicular to \mathbf{B} , it is

$$\rho_L = \frac{(2mk_B T)^{1/2}}{|q|B}.$$

In a typical fusion plasma ($T = 10$ keV, $B = 5$ T), the electrons have a Larmor radius $\rho_{L,e} = 67 \mu\text{m}$ and deuterons have $\rho_{L,i} = 4.1$ mm. The frequency of the gyration, called *cyclotron frequency* Ω_q , follows from $v_{\perp} = \Omega \rho_L$ to be

$$\Omega_q = \frac{|q|B}{m_q}.$$

In fusion experiments, the electron cyclotron frequency Ω_e is of the same order of magnitude as the plasma frequency $\omega_{p,e}$.

The *Hall parameter*

$$x_q = \Omega_q \tau_{qq} \quad (1.35)$$

counts the number of cyclotron revolutions between succeeding collisions. A magnetized plasma is characterized by $x_e \gg x_i \gg 1$.

The *magnetic moment* μ is defined as the product of the current I times the area A which is surrounded by the current. Thus, the magnetic moment of a single gyrating particle is

$$\mu = I \cdot A = \frac{q\Omega}{2\pi} \cdot \pi\rho_L^2 = \frac{mv_{\perp}^2}{2B} = \frac{\mathcal{E}_{\perp}}{B}, \quad (1.36)$$

where \mathcal{E}_{\perp} is the perpendicular fraction of the kinetic energy. The magnetic field produced by the gyrating particles counteracts the externally imposed magnetic field \mathbf{B} (*plasma diamagnetism*).

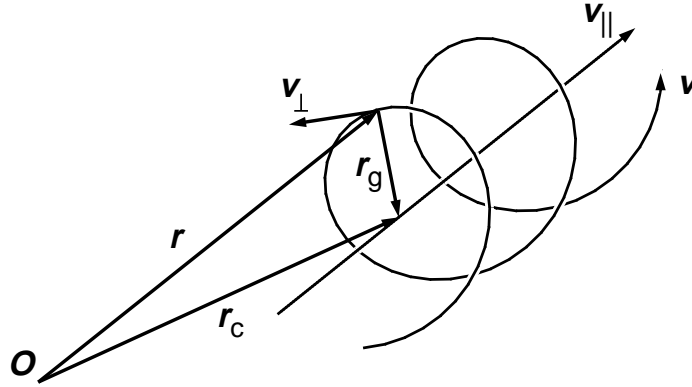


Figure 1.5: Definition of the guiding centre motion.

1.7.2 Motion in a constant magnetic field due to an additional force

If, in addition to the Lorentz force, a force \mathbf{F} acts on the charged particle, the equation of motion is

$$m \frac{d\mathbf{v}}{dt} = q (\mathbf{v} \times \mathbf{B}) + \mathbf{F} .$$

The motion of the particle under the influence of \mathbf{F} can be separated from the gyration due to \mathbf{B} by considering the *guiding centre* of the particle, as shown in Fig. 1.5. The guiding centre “c” is the centre of the gyration cycle. Its position \mathbf{r}_c can be written as

$$\mathbf{r}_c = \mathbf{r} + \mathbf{r}_g , \quad (1.37)$$

where \mathbf{r} is the position of the particle and \mathbf{r}_g the gyration radius vector

$$\mathbf{r}_g = \frac{m}{qB^2} \mathbf{v} \times \mathbf{B} .$$

In *strongly magnetized* plasmas such as in fusion experiments, the Lorentz force dominates other forces \mathbf{F} . Then, ρ_L is often much smaller than other length scales, and Ω much higher than other frequencies. Consequently, the particle orbit is well described by the motion of the guiding centre. With the assumption of a constant \mathbf{B} , the velocity of the guiding centre can be obtained by differentiating (1.37)

$$\mathbf{v}_c = \dot{\mathbf{r}}_c = \mathbf{v} + \frac{m}{qB^2} \frac{d\mathbf{v}}{dt} \times \mathbf{B} = \mathbf{v} + \frac{1}{qB^2} [q(\mathbf{v} \times \mathbf{B}) + \mathbf{F}] \times \mathbf{B} . \quad (1.38)$$

Using the vector relation

$$(\mathbf{v} \times \mathbf{B}) \times \mathbf{B} = (\mathbf{v}_\perp \times \mathbf{B}) \times \mathbf{B} = B(\mathbf{v}_\perp \cdot \mathbf{B}) - \mathbf{v}_\perp (\mathbf{B} \cdot \mathbf{B}) = -\mathbf{v}_\perp B^2 ,$$

the guiding centre motion follows to be

$$\mathbf{v}_c = \mathbf{v}_\parallel + \frac{\mathbf{F} \times \mathbf{B}}{qB^2} .$$

The velocity of the guiding centre can be split into components perpendicular and parallel to the magnetic field

$$\mathbf{v}_{c,\perp} = \frac{\mathbf{F}_\perp \times \mathbf{B}}{qB^2}, \quad \frac{dv_{c,\parallel}}{dt} = \frac{F_\parallel}{m}. \quad (1.39)$$

Any force with perpendicular components to the magnetic field leads to a motion perpendicular to \mathbf{B} and \mathbf{F} with the velocity $\mathbf{v}_{c,\perp}$.

1.7.3 $\mathbf{E} \times \mathbf{B}$ drift, gravitational drift

The most important additional force is a constant electric force $\mathbf{F} = q\mathbf{E}$, which leads to a drift

$$\mathbf{v}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2}. \quad (1.40)$$

This drift velocity does not depend on the sign of the charge or the mass of the particles. Thus, the $\mathbf{E} \times \mathbf{B}$ drift leads to a macroscopic movement of a plasma. More insight into the $\mathbf{E} \times \mathbf{B}$ drift can be gained from Fig. 1.6 left. If the particle accelerates in the electric field, the Larmor radius increases. On the other hand, if it slows down its Larmor radius decreases, leading to the non-closed trajectories shown in Fig. 1.6 left. The net effect is a drift velocity perpendicular to the electric and magnetic fields.

Another drift is due to the gravitation field. If $F_g = mg$ is perpendicular to \mathbf{B} , the drift velocity is $\mathbf{v}_g = mg/(qB)$.

1.7.4 Particle drift in inhomogeneous magnetic fields

For spatially varying magnetic fields, (1.39) can still be applied if the relative variation of B is small along one gyration of the particle, i.e. in the case of *slowly varying* fields. One important example is the particle drift in a magnetic field with a transverse gradient. The particle orbit has a smaller radius of curvature on that part of its orbit located in the stronger magnetic field. This leads to a drift perpendicular to both the magnetic field and its gradient. Quantitatively, the force on a particle with magnetic moment μ (see (1.36)) can be written as

$$\mathbf{F}_{\nabla B} = -\mu \nabla B,$$

leading to

$$\mathbf{v}_{\nabla B} = -\frac{mv_\perp^2}{2qB^3} (\nabla_\perp B) \times \mathbf{B}. \quad (1.41)$$

Let us consider a curved magnetic field in the case that the plasma current does not play a role, $\nabla \times \mathbf{B} = 0$ or, more generally, $\nabla \times \mathbf{B} \parallel \mathbf{B}$. As sketched in Fig. 1.6, there is a correlation between the field gradient and the curvature radius R_c

$$\frac{R_c}{R_c^2} = -\frac{\nabla_\perp B}{B}. \quad (1.42)$$

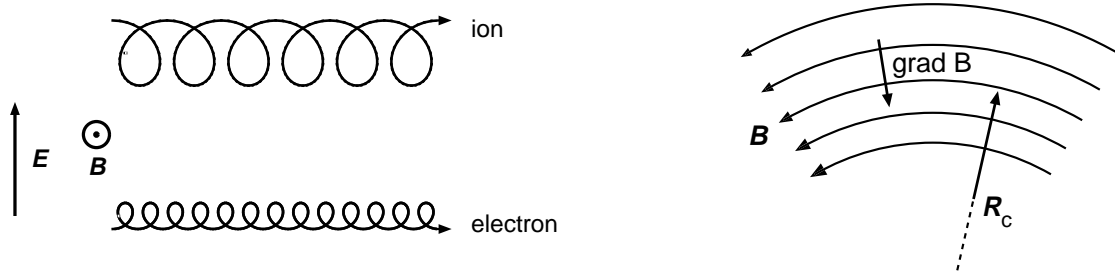


Figure 1.6: Left: $\mathbf{E} \times \mathbf{B}$ drift of ion and electron, $v_E = E/B$.

Right: Inhomogeneous magnetic field. Curvature radius and its relation to the field gradient in the absence of a plasma current ($\nabla \times \mathbf{B} = 0$).

Since the particles follow the curved magnetic field lines, a centrifugal force $F_c = mv_{\parallel}^2 \mathbf{R}_c/R_c^2$ is exerted, which is responsible for the drift velocity

$$\mathbf{v}_R = \frac{mv_{\parallel}^2}{qB^2} \frac{\mathbf{R}_c \times \mathbf{B}}{R_c^2}. \quad (1.43)$$

Combination of (1.41), (1.42), and (1.43), yields

$$\mathbf{v}_R + \mathbf{v}_{\nabla B} = \frac{m}{q} \frac{\mathbf{R}_c \times \mathbf{B}}{R_c^2} (v_{\parallel}^2 + \frac{1}{2}v_{\perp}^2).$$

In a very rough approximation $v_{\parallel}^2 + 1/2v_{\perp}^2 \approx v^2$, i.e. the resulting drift is proportional to the particle energy.

As a first application of these drifts, consider the electrons and protons captured in the earth's magnetic field (trapping in a magnetic field will be discussed in the next section). Due to the gradient and curvature of the earth's magnetic field, these electrons and protons drift around the equator, the electrons from west to east and the protons in the opposite direction, producing the so-called ring currents.

1.7.5 The guiding centre equations in general geometry

In fusion plasmas the geometry of the magnetic field is usually fairly complex. Therefore, it is mandatory to derive the guiding centre equations rigorously for arbitrary geometry.

Most elegantly this was done by Littlejohn [16]. He showed that the guiding centre equations can be derived from a generalized Lagrangian. We will outline the general idea and the results only. Littlejohn has shown that there is a non-canonical transformation such that the particle motion can be expressed independently from the gyroangle α to arbitrary order in a small parameter $\varepsilon_B = |\rho_L/L_B|$ (where $L_B = B/|\nabla B|$ is the scale length of the magnetic field). This set of phase-space coordinates $(\mathbf{R}, V_{\parallel}, V_{\perp}, \text{ or } \mathbf{R}, V_{\parallel}, \mu)$ is called guiding centre coordinates. In these coordinates, the magnetic moment constructed as

$$\mu = \frac{mV_{\perp}^2}{2ZeB(\mathbf{R})}$$

is conserved. Note, however, that the “true” $\mu = mv_{\perp}^2/(2ZeB)$ resulting from the velocity v_{\perp} in the real space is not conserved in general.

To first order in ε_B and for $d/(\Omega dt) \ll 1$ the following results have been obtained assuming the set of independent variables to be $\mathbf{R}, V_{\parallel}, \mu$

The time derivatives of \mathbf{R} , μ and V_{\parallel} can be obtained from the well known Lagrangian formalism [16] with the single-particle Lagrangian

$$L = Ze\mathbf{A}^* \cdot \dot{\mathbf{R}} + \frac{1}{Ze}\mu\dot{\alpha} - \frac{1}{2}mV_{\parallel}^2 - \mu B - Ze\phi \quad (1.44)$$

which gives the Euler-Lagrange equations

$$Ze(\mathbf{E}^* + \dot{\mathbf{R}} \times \mathbf{B}^*) = m\dot{V}_{\parallel}\mathbf{b} + \mu\nabla B, \quad (1.45)$$

$$\dot{\mu} = 0, \quad (1.46)$$

$$\dot{\alpha} = \frac{ZeB}{m}, \quad (1.47)$$

$$\dot{\mathbf{R}} \cdot \mathbf{b} = V_{\parallel}. \quad (1.48)$$

The modified field quantities are defined as in Littlejohn’s work [16]

$$\mathbf{A}^* = \mathbf{A} + \frac{m}{Ze}V_{\parallel}\mathbf{b}, \quad (1.49)$$

$$\mathbf{B}^* = \mathbf{B} + \frac{m}{Ze}V_{\parallel}\nabla \times \mathbf{b}, \quad (1.50)$$

$$\mathbf{E}^* = -\frac{\partial \mathbf{A}^*}{\partial t} - \nabla\phi, \quad (1.51)$$

$$\mathbf{b} = \frac{\mathbf{B}}{B}. \quad (1.52)$$

Finally we are left with the following equations of motion

$$\dot{\mathbf{R}} = \frac{1}{B_{\parallel}^*} \left[V_{\parallel}\mathbf{B}^* + \frac{1}{Ze}\mathbf{b} \times (\mu\nabla B - Ze\mathbf{E}^*) \right], \quad (1.53)$$

$$\dot{V}_{\parallel} = -\frac{1}{B_{\parallel}^*}\mathbf{B}^* \cdot (\mu\nabla B - Ze\mathbf{E}^*), \quad (1.54)$$

$$\dot{\mu} = 0, \quad (1.55)$$

$$\dot{\alpha} = \Omega = \frac{ZeB}{m}. \quad (1.56)$$

Note, in (1.54), the so-called mirror force (first term on the right-hand side) which is responsible for the *bounce* motion of a part of the particle population in regions of weaker magnetic field, as shown in Fig. 1.7 and Fig. 1.8. This phenomenon can be also understood in terms of adiabatic invariance of the magnetic moment, as explained below.

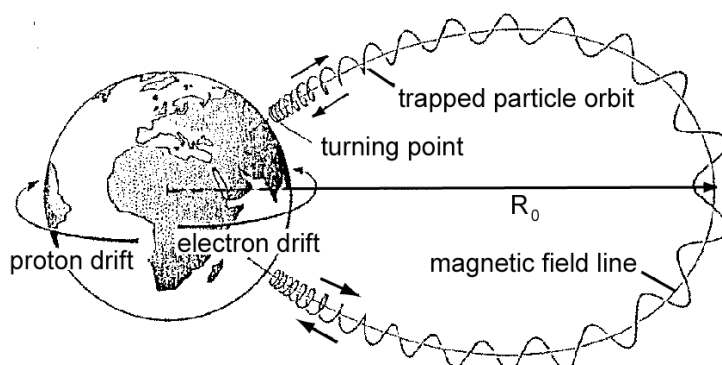


Figure 1.7: Drifts of electrons and protons in the earth's magnetic field.

1.8 Adiabatic invariants

The gyromotion and the bounce motion seen in the previous section are two examples of quasi-periodic motion of a charged particle in a magnetic field. The word *quasi*-periodic is used because it is assumed that the trajectory changes on a time scale which is much slower than the “period” of the motion. For instance, this implies that the Larmor orbit remains nearly unchanged over many gyroperiods $2\pi/\Omega$, or that the trajectory of a trapped particle is approximately the same during many bounce times. In this case, the quantity

$$I \equiv \oint p dq, \quad (1.57)$$

where p and q are conjugate canonical variables and the integral is taken over a complete cycle in q , is called an *adiabatic invariant* of the motion. The period T can be computed directly as $T = \partial I / \partial \mathcal{E}$, where \mathcal{E} is the energy of the system. Using this definition of I , we can immediately prove that the magnetic moment is an adiabatic invariant associated to the cyclotron motion. The relevant Hamiltonian is in this case $H(\mathbf{p}, \mathbf{q}) = (\mathbf{p} - q\mathbf{A})^2 / 2m$ and the canonical momentum $\mathbf{p} = m\mathbf{v} + q\mathbf{A}$. (1.57) becomes then

$$I_1 = \oint m\mathbf{v}_\perp \cdot d\mathbf{r}_\perp + \oint q\mathbf{A} \cdot d\mathbf{r}_\perp = 2\pi\rho m v_\perp - qB\pi\rho^2 = \frac{\pi m^2 v_\perp^2}{qB} \propto \frac{mv_\perp^2}{2B} = \mu,$$

where Stokes' theorem and (1.34) have been used. This shows that the magnetic moment μ , which is proportional to I_1 , is an adiabatic invariant.

We have met this adiabatic invariance in the section before and take the opportunity to recall: for a constant and homogeneous magnetic field guiding centre coordinates and real space coordinates agree what implies that μ is strictly conserved. For inhomogeneous and time dependent magnetic fields we observe that μ is conserved to a certain order in the ordering parameters ϵ_B and $d/(\Omega dt)$ only. Littlejohn's result that a phase space transform exists which finds a redefined μ as a conserved quantity can be regarded as formal expression of the term *adiabatic invariance*.

The invariance of μ is the basis for magnetic mirrors, which is schematically shown in Fig. 1.8. The parallel velocity of a particle in a (spatially or temporally) varying magnetic field is determined by $\mu = \text{const}$ and energy conservation $mv^2/2 = \text{const}$. The combination of these

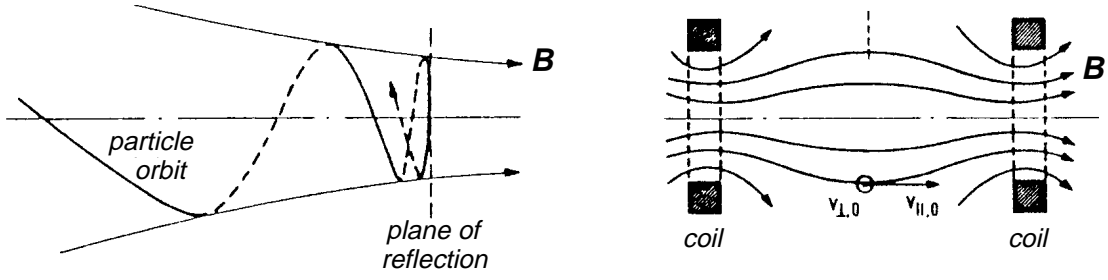


Figure 1.8: Magnetic field lines of a simple axisymmetric magnetic mirror.

conditions, $mv_{\parallel}^2/2 + \mu B = \text{const}$, reveals that v_{\perp}^2 increases with increasing B and hence v_{\parallel} decreases. The criterion for particle reflection ($v_{\parallel} = 0$) at the high field ends of the mirror machine is

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_{\parallel,0}^2 + \mu B_{\min} \leq \mu B_{\max}, \quad (1.58)$$

where $v_{\parallel,0}$ is the parallel velocity in the low field region. If the mirror ratio is defined as $R = B_{\max}/B_{\min}$ and (1.58) is divided by $\mu B_{\min} = 1/2mv_{\perp,0}^2$, the velocity ratio

$$\frac{v_{\parallel,0}}{v_{\perp,0}} \leq \sqrt{R-1} \quad (1.59)$$

describes the criterion for particle confinement. The earth's magnetic field is an example of a magnetic mirror. It forms two belts of confined charged particles originating from the solar wind (see Fig. 1.7). Obviously, particles which do not meet the condition (1.59) are lost from magnetic mirrors. This so called *loss cone* in velocity space is refilled by collisions and therefore pure mirror configurations have a very poor plasma confinement.

The second or *longitudinal invariant* is

$$J = m \oint v_{\parallel} d\ell.$$

Consider a particle between two magnetic mirrors: It bounces between them and therefore has a periodic motion at the bounce frequency. A constant of this motion is $\oint mv_{\parallel} d\ell$ with ℓ being the length between two mirrors and v_{\parallel} the longitudinal velocity. However, the motion is not exactly periodic, because the guiding centre drifts across field lines, and thus the constant of the motion becomes in general an adiabatic invariant. From this we can conclude that the particles will drift on $J = \text{const}$ contours.

If these contours are closed, the drift motion is also periodic and we observe another constant of motion – the so called third adiabatic invariant. Note, however, that to guarantee invariance the perturbations have to be even slower (see 1.8) than the drift period.

The third adiabatic invariant turns out to be the total magnetic flux Φ enclosed by the drift surface. Obviously, the particles will stay on a surface such that the total number of lines of force enclosed remains constant as B varies. This invariant has few applications because most fluctuations of B occur on a time scale short compared with the drift period.

Table 1.1: Characteristic times of motion for typical fusion parameters in a stellarator reactor similar to W7-X: ($B = 5$ T, large radius: 20 m, small radius 2 m, $T = 15$ keV, $n = 3 \cdot 10^{20} \text{ m}^{-3}$).

	gyration time τ_g	bounce time τ_{bounce}	drift time τ_{drift}
electrons	$\approx 7 \cdot 10^{-12} \text{ s}$	$\approx 10^{-6} \text{ s}$	$\approx 6 \cdot 10^{-2} \text{ s}$
deuterons	$\approx 3 \cdot 10^{-8} \text{ s}$	$\approx 10^{-4} \text{ s}$	$\approx 6 \cdot 10^{-2} \text{ s}$
α -particles	$\approx 3 \cdot 10^{-8} \text{ s}$	$\approx 3 \cdot 10^{-5} \text{ s}$	$\approx 10^{-4} \text{ s}$

Appendix A

1.8 summarizes characteristic plasma parameters and their numerical value for typical fusion plasmas, namely $T_e = T_i = 10^4$ eV, $Z = 1$, and $n_e = n_i = 10^{20} \text{ m}^{-3}$, $B = 5$ T for magnetically-confined experiments and $n_e = n_i = 10^{31} \text{ m}^{-3}$ for inertially-confined experiments. In the table, $a = e^4 \ln \Lambda / (6\pi\epsilon_0^2)$.

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Table 1.2: Characteristic plasma parameters and their numerical value for typical fusion plasmas.

Name	Formula	Magnetic fusion	Inertial fusion
Average distance	$\bar{r} \approx (\frac{4}{3}\pi n)^{-1/3}$	0.2 μm	3×10^{-11} m
Debye length	$\lambda_D = \sqrt{\epsilon_0 k_B T / (n_e e^2)}$	75 μm	2.4×10^{-10} m
Landau length	$\lambda_L = e^2 / (\pi \epsilon_0 k_B T)$	10^{-13} m	10^{-13} m
de Broglie length	$\Lambda_p = \hbar / (\sqrt{m_e k_B T})$	3×10^{-12} m	3×10^{-12} m
Mean free path	$\lambda_{qk} = v_{th,q} \tau_{qk}$	15 km	150 nm
Larmor radius	$\rho_{L,q} = m_q v_{\perp} / (q B)$	67 μm (e), 4.1 mm (i)	
Thermal velocity	$v_{th,q} = \sqrt{3k_B T_q / m_q}$	7.3×10^7 m/s (e), 1.7×10^6 m/s (i)	7.3×10^7 m/s (e), 1.7×10^6 m/s (i)
Plasma frequency	$\omega_{p,e} = (1/2\pi) \sqrt{n_e e^2 / \epsilon_0 m_e}$	90 GHz	2×10^{17} Hz
Cyclotron frequency	$\omega_{c,q} = q B / 2\pi m_q$	140 GHz (e), 40 MHz (i)	
Electron collision time	$\tau_{ee} = \frac{\sqrt{3}}{a} \sqrt{m_e} n_e^{-1} (k_B T_e)^{3/2}$	140 μs	5.4×10^{-15} s
Ion collision time	$\tau_{ii} = \frac{\sqrt{3}}{aZ^4} \sqrt{m_i} n_i^{-1} (k_B T_i)^{3/2}$	6 ms	2.3×10^{-13} s
Electrical resistance	$\eta = m_e / (n_e e^2 \tau_{ei}^E)$	1×10^{-9} Ωm	3×10^{-10} Ωm
Plasma parameter	$N_D = \frac{4}{3}\pi n \lambda_D^3$	2×10^8	500
Coulomb logarithm	$\ln \Lambda = \ln \left(\frac{9}{\sqrt{2}} N_D \right)$	17	4.4
Hall parameter	$x_q = \omega_{c,q} \tau_{qq}$	1×10^8 (e), 3×10^6 (i)	

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Chapter 2

Basic nuclear fusion

Hans-Stephan Bosch

2.1 Introduction

Any energy production from nuclear reactions is based on differences in the nuclear binding energy. Fig. 2.1 shows the nuclear binding energy per nucleon (proton or neutron). It has been derived from measurements of the masses of the nuclei, when it was observed that the masses of nuclei are always smaller than the sum of the proton and neutron masses which constitute the nucleus. This mass difference corresponds to the nuclear binding energy according to Einstein's energy-mass relation $E = \Delta m \cdot c^2$. An explanation of the structure of Fig. 2.1 was given by C.F. von Weizsäcker in 1935. Starting from the very limited range of the strong nuclear force he assumed that each nucleon just influences its nearest neighbours. The binding energy per nucleon would thus be constant. The smaller binding energies for smaller nuclei are due to the relatively large surface to volume ratio. The nucleons at the surface have missing partners and thus their contribution to the total binding energy of the nucleus is reduced. The decrease of binding energy per nucleon for nuclei beyond $A \approx 60$ is due to the repulsive coulomb force of the large amount of positive protons.

The finer structures in Fig. 2.1 are due to quantum mechanical effects, i.e. at certain so-called *magic* proton and neutron numbers the nucleus formed is a very stable configuration. This is roughly comparable to the stable electron configurations of the noble gases, where electron shells are completed. The first magic number is two, which is manifested as a most remarkable example of a local maximum in Fig. 2.1, i.e. the helium nucleus with two protons and two neutrons. From Fig. 2.1 it is clear that there are two ways of gaining nuclear energy:

1. By transforming heavy nuclei into medium-size nuclei: This is done by fission of uranium, which is not a topic of these lectures.
2. By fusion of light nuclei into heavier ones: In particular the fusion of hydrogen isotopes into stable helium offers the highest energy release per mass unit. Doing this in a controlled manner has been the goal of fusion research for about 40 years. Most of the following lectures are dedicated to this goal.

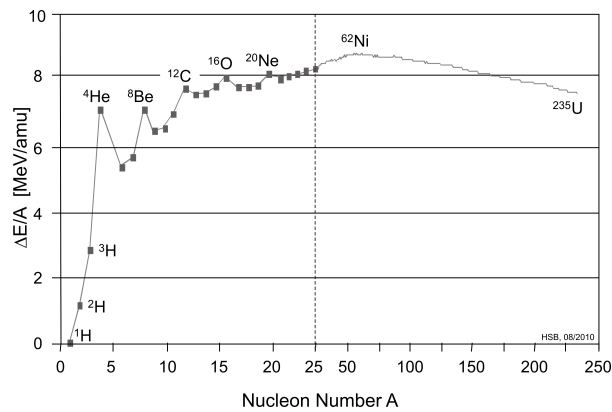


Figure 2.1: Nuclear binding energy per nucleon as a function of the nucleon number A .

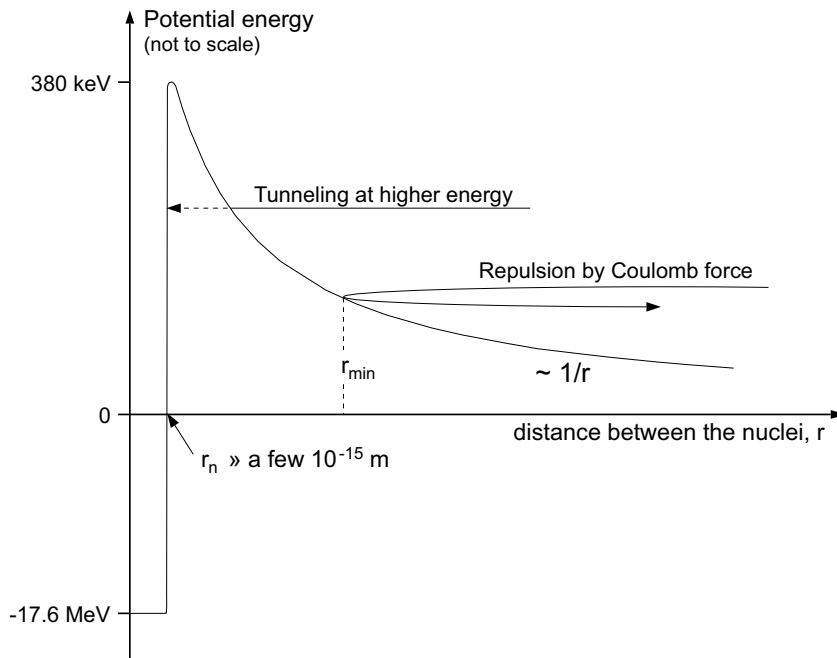


Figure 2.2: Schematic representation of the potential energy U of two nuclei as a function of their distance. The energies quoted are for a system of D ($=^2\text{H}$) and T ($=^3\text{H}$).

The energy release per nucleon is of the order of 1 MeV ($= 10^6$ eV) for fission reactions and in the order of a few MeV for fusion reactions. This is 6–7 orders of magnitude above typical energy releases in chemical reactions, which explains the effectiveness and attractiveness, but also the potential hazard of nuclear power.

All the nuclear reactions are mostly governed by the strong nuclear force which acts over very small distances in the order of the radius of the nuclei, but for distances above a few fermi (i.e. 10^{-15} m) the repulsive Coulomb force between the positively charged nuclei becomes dominant. The potential energy of two nuclei as a function the distance between these nuclei is shown in Fig. 2.2. The depth of the deep well at small radii is determined by the binding energy, discussed before, while the barrier at a few fermi is given by the Coulomb potential of

$Z_1 Z_2 e^2 / 4\pi\epsilon_0 r_m$ which is much smaller, but still poses a principal problem. For alpha-particle decay (where a ${}^4\text{He}$ nucleus separates itself out of a positive nucleus) as well as for fusion of lighter nuclei, this diagram demands a particle energy of the order of 500 keV, and this would make fusion processes almost impossible. However, it was known since the end of the last century, that α -particle decay occurs at room temperature, and in 1928 Gamov [1] explained this by the tunneling effect, which in turn also allows fusion reactions to occur at temperatures far below the Coulomb barrier: Due to quantum mechanical effects the minimum distance between the two nuclei is not fixed (as it is indicated by the repulsion in Fig. 2.2), but there is a finite probability for the nuclei to get closer, and eventually *tunnel* through the Coulomb barrier, as indicated by the dotted line in Fig. 2.2. In terms of wave functions, the amplitude is not zero for $r \leq r_{min}$, but it is finite and decays slowly for smaller radii. Therefore it can still be finite for $r \leq r_n$, i.e. the particles have a possibility to approach close enough for a fusion reaction to occur. This tunneling probability is a strong function of the relative velocity v of the reacting particles with charge Z_1, Z_2 ,

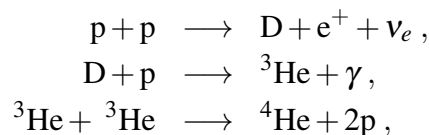
$$P_{tunneling} \sim \exp - \frac{2\pi Z_1 Z_2 e^2}{\hbar v} .$$

This equation shows that reaction partners with small mass (and charge Z) are preferred, and that the reaction probability increases strongly with the relative velocity v , i.e. the temperature.

In the light of the above discussion it becomes clear why fission energy has been much more readily obtained than fusion energy: Fission is triggered by thermal neutron capture, i.e. no force prevents the neutron from entering the uranium nucleus at room temperature and causing a fission reaction.

2.2 Energy production on the sun

Though still a somewhat exotic topic on earth, nuclear fusion is *the* energy source of our universe. The light of all stars is generated by fusion reactions. On the sun the main reactions are the following



where p denotes a proton, D a deuteron, a heavy hydrogen isotope with one proton and one neutron, ${}^3\text{He}$, ${}^4\text{He}$ are helium isotopes, γ stands for a high-energy photon, e^+ for a positron (anti-electron), and ν_e for an electron neutrino.

A very important feature of the energy production on the sun is the need for the weak interaction which transforms protons to neutrons (β^+ -decay), in the first of the above listed reactions. All weak interaction processes involve the emission of neutrinos (thereby keeping the lepton number constant). As the weak interaction is very slow, it causes the long lifetime of the sun and it is an essential condition for our evolution .

Further reactions which are important at temperatures above about 1 keV, produce ${}^7_4\text{Be}$, ${}^7_3\text{Li}$, ${}^8_5\text{B}$ and ${}^8_4\text{Be}$, which decays into two ${}^4_2\text{He}$ nuclei. Also in these reactions neutrinos are produced, however with a higher kinetic energy than those from the pp-reactions mentioned above.

These neutrinos ignore both the strong and electromagnetic interactions and therefore it is extremely hard to detect them experimentally. Almost all neutrinos produced in the sun travel through it without interaction and thus carry information from the reactions in the core of the sun. It was not till 1992 that the European GALLEX collaboration detected the low energy solar neutrinos from the main energy-producing reactions (pp-chain) listed above. This was the first experimental validation of our theoretical model of energy production in the stars.

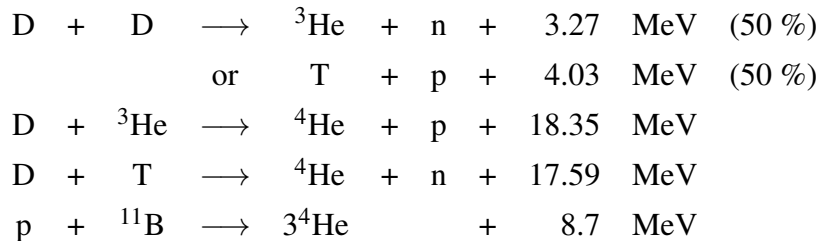
For much higher temperatures ($T \geq 2$ keV), i.e. in stars which with higher mass than the sun, fusion of four protons to ${}^4_2\text{He}$ can also occur in a catalytic process based on ${}^{12}_6\text{C}$. In this so-called Bethe-Weizsäcker cycle oxygen, nitrogen and ${}^{13}_6\text{C}$ are involved only in intermediate stages, and the net reaction again is $4\text{p} \rightarrow {}^4_2\text{He} + 2\text{e}^+ + 2\nu_e$.

2.3 Fusion on earth

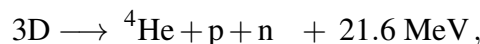
The fundamentals of fusion research are well described in great detail in two (German) books by J. Raeder [4] and U. Schumacher [5].

For energy production on earth the weak interaction has to be avoided since it would lead to unacceptably small reaction rates. The sun (and all other stars) overcome this problem by their huge mass, but a fusion reactor has to be considerably smaller in size than a star.

Possible candidates for using fusion energy on earth are the following reactions (T denoting tritium, the heaviest hydrogen isotope with two neutrons)



Because the kinetic energy of the reactants is much lower than the energy gained in the reaction, the distribution of the reaction energy onto the two product particles is inverse proportional to their mass, i.e. $E_1/E_2 = m_2/m_1$. In the DT-reaction, e.g. the α -particle has an energy of 3.54 MeV, and the neutron has 14.05 MeV. The first four reactions (for which the cross sections are shown in Fig. 2.3) can be summarized as



and therefore rely on deuterium as fuel only. Since the weight fraction of deuterium in water is 3.3×10^{-5} , the energy content of water is about 11.5 GJ per litre, which is about 350 times

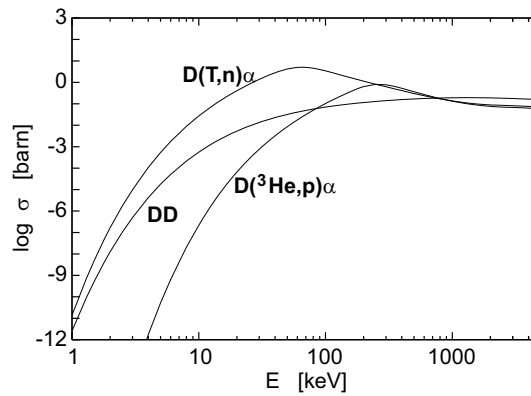
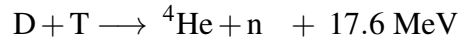


Figure 2.3: Measured cross-sections for different fusion reactions as a function of the center of mass energy [3]. The curve labeled DD shows the sum of the cross sections for both branches of this reaction. Reaction cross sections are usually measured in barn, where 1 barn = 10^{-28} m².

larger than the chemical energy density of gasoline. This demonstrates the huge potential afforded by nuclear fusion as an energy source.

All the reaction cross sections in Fig. 2.3 show a steep increase with the relative energy, as discussed before, but the D-T reaction



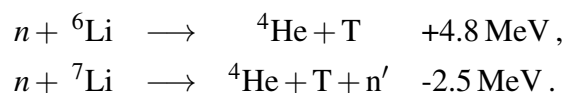
has by far the largest cross-section at the lowest energies. This makes the D-T fusion process the most promising candidate for an energy-producing system. The special role of D-T reactions becomes clear from the energy levels of the unstable ⁵He nucleus shown in Fig. 2.4. It has an excited state just 64 keV above the sum of the masses of deuterium and tritium. The D-T fusion cross-section therefore reaches its maximum at this energy difference, due to the resonance-like reaction mechanism. The DD reactions (see Fig. 2.4, left part) show no such resonances, and their cross sections are solely governed by the tunneling probability, showing a smooth increase without any maximum, while the D³He reaction also has a resonance at about 270 keV in the ⁵Li system, not shown here.

To be a candidate for an energy-producing system, the fusion fuel has to be sufficiently abundant. As mentioned earlier, deuterium occurs with a weight fraction of 3.3×10^{-5} in water. Given the water of the oceans, the static energy range is larger than the time the sun will continue to burn (a few billion years).

Tritium is an unstable radioactive isotope. It decays to



with a half-life of 12.3 years. Note the previously mentioned neutrino production of this β^- decay. Owing to the unstable character of tritium no significant amounts exist, but tritium can be produced with nuclear reactions of the neutrons from the D-T reaction and lithium



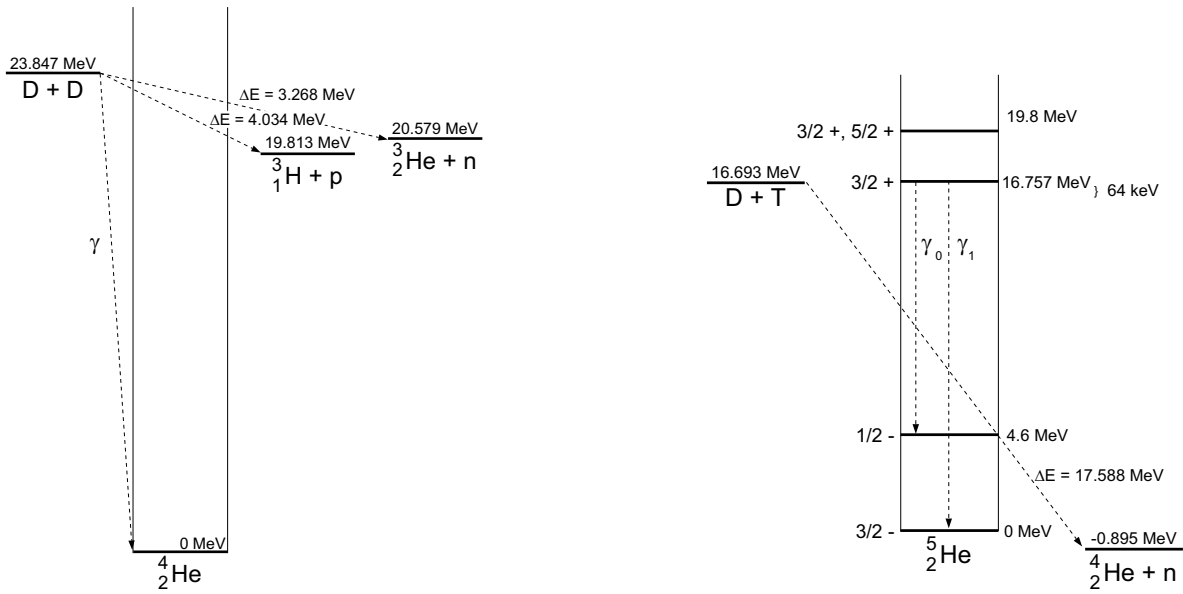


Figure 2.4: Left: Energy diagram of the ^4He nucleus and Right: Of the unstable ^5He nucleus (right), where the resonance at 16.76 MeV dominates the reaction cross section. The numbers left of the level scheme indicate the J -value and parity of the respective level.

The reaction with ^7Li is particularly important because it does not consume a neutron and opens the possibility for self-sufficient tritium production in a fusion reactor, i.e. each fusion neutron will produce at least one new tritium nucleus.

The ultimate fusion fuel will thus be deuterium and lithium. The latter is also very abundant and widespread in the earth's crust and even ocean water contains an average concentration of about 0.15 ppm (1 ppm = parts per million). Tab. 2.3 summarizes the estimated world energy resources.

2.4 Thermonuclear fusion

As discussed before, for a fusion reaction to occur, the two nuclei have to *touch* each other since the range of the nuclear force is of the order of the dimensions of the nuclei. The repulsive Coulomb force counteracts all attempts to bring them close together. This is what the difficult research on nuclear fusion is all about: How can the two reaction partners be brought into contact?

The simplest approach to realize the fusion reactions would be to accelerate the reactants to about 100 keV and bring them to collision. This does not lead to a positive energy balance, since the elastic Coulomb scattering as another reaction type has a much larger cross-section, as shown in Fig. 2.5. Thus the two particle beams would just scatter and diverge after one interaction.

A way of overcoming this problem is to confine a thermalized mixture of deuterium and tritium particles at energies of about 10 keV. Since the average energy of particles at a certain

Table 2.1: Estimated world energy resources. The figures are only indicative, being dependent on prices and subject to uncertainty because of incomplete exploration.

Present world annual primary energy consumption	3×10^{11}	
		gigajoules
Resources		
Coal	10^{14}	300 years
Oil	1.2×10^{13}	40 years
Natural Gas	1.4×10^{13}	50 years
^{235}U (fission reactors)	10^{13}	30 years
^{238}U and ^{232}Th (breeder reactors)	10^{16}	30000 years
Lithium (D-T fusion reactors):		
Land	10^{16}	30000 years
Oceans	10^{19}	3×10^7 years

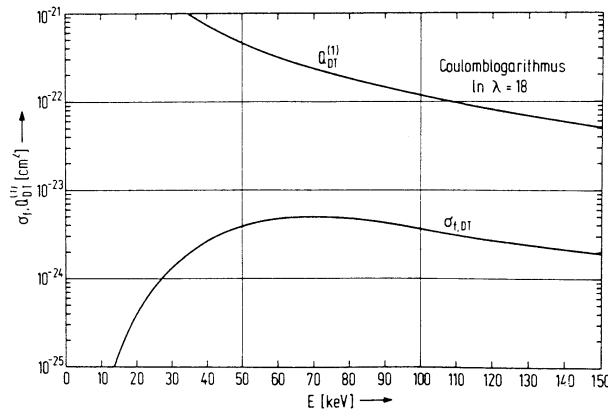


Figure 2.5: Comparison of the D-T fusion cross-section $\sigma_{f,DT}$ and the cross-section for momentum exchange by Coulomb scattering $\sigma_{Q,DT}$ in a fusion relevant plasma.

temperature is about kT , where k is the Boltzmann constant, temperatures are often given in electron volt units ($1 \text{ eV} \hat{=} 1.16 \times 10^4 \text{ }^\circ\text{C}$). At energies of 10 keV the hydrogen atoms are completely ionized and form a plasma of charged ions and electrons. The basic physics of plasmas are discussed in chap. 1. For now it should suffice to observe that in a plasma the particles thermalize as a result of many Coulomb scattering processes and thus entail a Maxwellian velocity distribution

$$f(v) = n \left(\frac{m}{2\pi kT} \right)^{3/2} \cdot \exp \left(-\frac{mv^2}{2kT} \right),$$

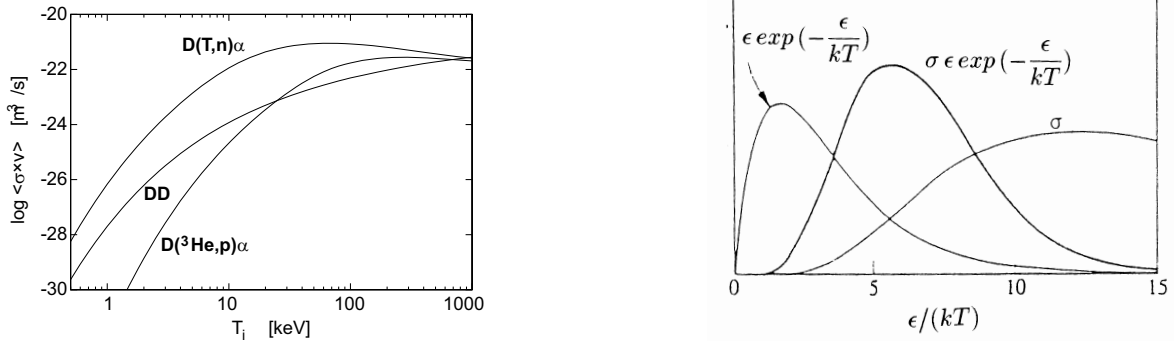


Figure 2.6: Left: Reaction parameter $\langle \sigma v \rangle$ as a function of ion temperature T_i for different fusion reactions [3].

Right: Graph of integrand of the reaction parameter equation and of its two factors $\sigma(\epsilon)$ and $\epsilon \cdot \exp(-\epsilon/kT)$ versus the normalized energy ϵ/T for a D-T plasma at $T = 10$ keV.

where f is the number of particles in the velocity interval between v and $v + dv$, n is the density of particles, m is their mass, and kT is their temperature.

The reaction rate per unit volume R can be written as

$$R = n_D \cdot n_T \cdot \langle \sigma v \rangle$$

with v now being the relative particle velocity and $\langle \sigma v \rangle$ being the reaction parameter, i.e. the average of the product of cross-section times velocity.

Calculation of the reaction parameter requires integration over the distribution function of deuterium and tritium. After some numerical transformations one obtains

$$\langle \sigma v \rangle = \frac{4}{(2\pi m_r)^{1/2} (kT)^{3/2}} \int \sigma(\epsilon_r) \cdot \epsilon_r \cdot \exp\left(-\frac{\epsilon_r}{kT}\right) d\epsilon_r,$$

where m_r is the reduced mass, and ϵ_r the relative kinetic energy.

Fig. 2.6 left shows the reaction parameter for some important fusion reactions. At temperatures of interest the nuclear reactions come predominantly from the tail of the distribution. This is illustrated in Fig. 2.6 right, where the integrand of the last equation is plotted versus ϵ_r/T together with the two factors $\sigma(\epsilon)$ and $\epsilon \cdot \exp(-\epsilon/kT)$ for a D-T plasma at a temperature of 10 keV.

Thermalization is thus not just a way of handling the large cross-sections for elastic Coulomb scattering, it also considerably increases the reaction rate considerably in relation to beam experiments with single particle energies.

2.5 Ignition

In the following the condition which a thermalized D-T plasma has to satisfy to serve as an energy producing system is investigated. Historically, in 1957 John D. Lawson deduced a

criterion for a positive energy balance using quantities such as the thermal cycle efficiency η of power reactors [2]. Today the approach has changed slightly to a more physics oriented condition: The aim is an ignited plasma where all energy losses are compensated by the α -particles from the fusion reactions, which transfer their energy of 3.5 MeV to the plasma while slowing down. The neutrons cannot be confined and leave the plasma without interaction.

Transport processes such as diffusion, convection, charge exchange and others are empirically described by an energy confinement time τ_E leading to the power loss term $3nkT/\tau_E$ with $3nkT$ as the inner thermal plasma energy (n is the electron density). Note that this is twice the ideal gas value since every hydrogen atom is split into two particles (electron and nucleus). Another loss mechanism is the bremsstrahlung, which becomes particularly important at high temperatures and impurity concentrations. The power loss due to bremsstrahlung can be written as

$$P_{\text{bremsstrahlung}} = c_1 \cdot n^2 \cdot Z_{\text{eff}} \cdot (kT)^{1/2}$$

with c_1 being the bremsstrahlungs constant ($c_1 = 5.4 \times 10^{-37} \text{ Wm}^3\text{keV}^{-1/2}$), and Z_{eff} the effective charge of the plasma, including all (impurity) species: $Z_{\text{eff}} = \sum_z n_z Z^2 / n$.

The energy balance now reads

$$\left(\frac{n}{2}\right)^2 \langle \sigma v \rangle \cdot \varepsilon_\alpha = 3nkT/\tau_E + c_1 n^2 Z_{\text{eff}} (kT)^{1/2}$$

and this can be rewritten to the ignition condition

$$n\tau_E = \frac{12 kT}{\langle \sigma v \rangle \varepsilon_\alpha - 4c_1 Z_{\text{eff}} (kT)^{1/2}},$$

where ε_α is the energy of the α -particle, 3.54 MeV. This equation shows that the product of the particle density and energy confinement time is only a function of the plasma temperature, with a minimum at about 13 keV. In the range of 10 keV the reaction parameter $\langle \sigma v \rangle$ is roughly proportional to T^2 , which motivated the definition of the so-called fusion product

$$n\tau T = \frac{12 kT^2}{\langle \sigma v \rangle \varepsilon_\alpha - 4c_1 Z_{\text{eff}} (kT)^{1/2}}$$

which has a flat minimum of about $35 \times 10^{20} \text{ s/m}^3\text{keV}$ around 10 keV. The fusion product dictates the strategy for developing fusion power as an energy producing system: One has to attain temperatures of around 10 keV (about 100 million °C) and achieve the required density and energy confinement time simultaneously. There are two distinct approaches

1. The hot plasma is confined by strong magnetic fields leading to maximum densities of about $1.5 \times 10^{20} \text{ m}^{-3}$, which is 2×10^5 times smaller than the atom density of a gas under normal conditions. With these densities, the energy confinement time required is in the range of 2–4 seconds. This approach, which is the main line in fusion research, will be covered by most of the following lectures, and is introduced in the next section.

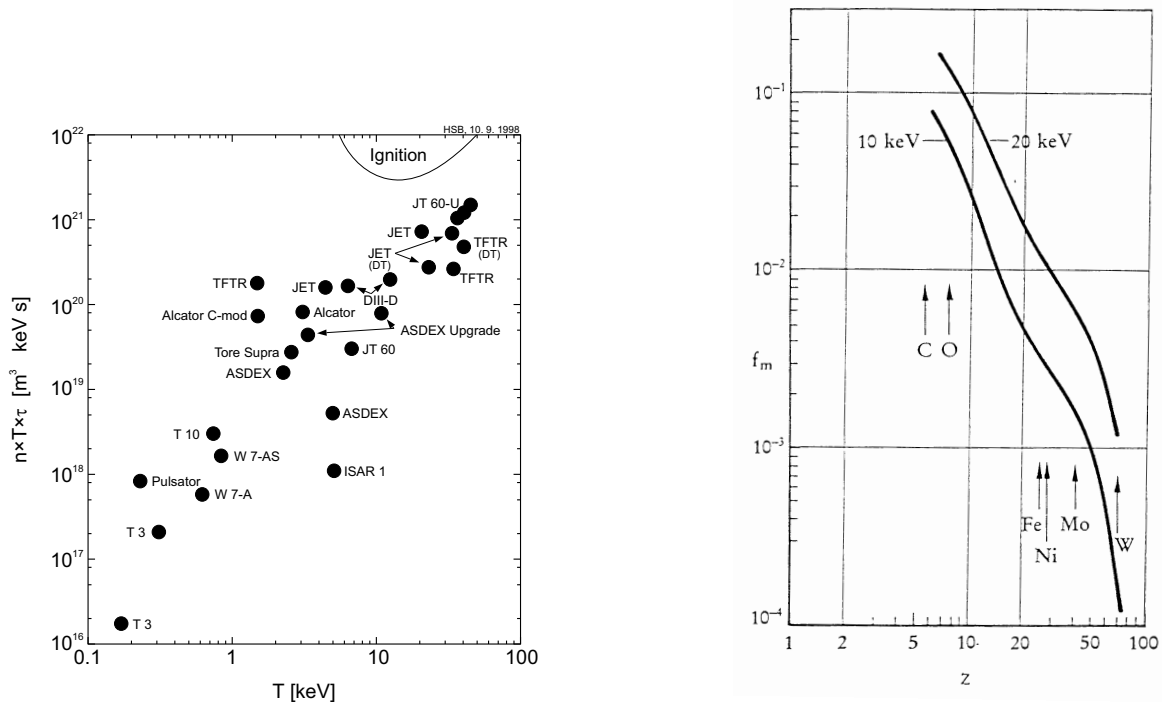


Figure 2.7: Left: Diagram of $nT\tau$ values versus T obtained in different experiments.

Right: Maximum tolerable impurity concentrations in a plasma to reach ignition.

2. The other extreme is to maximize the density. This can be done by strong, symmetric heating of a small D-T pellet. The heating can be done with lasers or particle beams and leads to ablation of some material causing implosion due to momentum conservation. It is clear that the energy confinement time is extremely short in this concept: it is the time required for the particles to leave the hot implosion center. Since it is the mass inertia which causes the finiteness of this time, this approach to fusion is often called *inertial fusion*. The density required is about 1 000 times the density of liquid D-T; the pressure in the implosion center reaches (at temperatures of 10 keV) that in the center of the sun. A detailed description of the actual status in this area can be found in references [7] and [8].

Fig. 2.7 left illustrates the progress of nuclear fusion research in approaching the required $n\tau T$ condition. Today a factor seven is missing for ignition, whereas in the mid-sixties the best experiments fell short of the required conditions by more than five orders of magnitude. However, it has to be kept in mind that achieving ignited plasmas is not sufficient for building fusion reactors. In addition, this plasma state has to be maintained for very long times to allow continuous energy production. One of the most difficult problems will be the interaction of the edge plasma with the surrounding structures and the removal of helium ash. Consequently, edge plasmas constitute an increasingly important research topic.

Impurities from the walls worsen the ignition condition in two ways. They dilute the fuel concentration and even in small amounts they can significantly enhance the radiation losses.

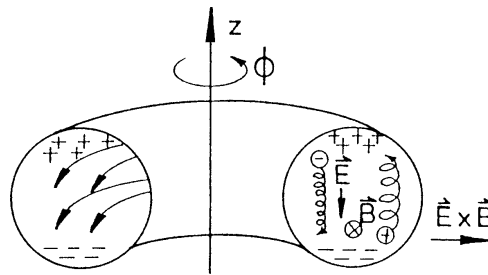


Figure 2.8: Vertical drifts and associated $E \times B$ in a toroidal field.

Fig. 2.7 right shows the maximum tolerable impurity concentration to reach ignition. Depending on the charge it ranges from a few % for light atoms to the few 10^{-4} level for high Z materials.

2.6 Magnetic confinement

As discussed in chap. 1, plasmas can be confined by magnetic fields, but in linear configurations the end losses are by far too large to reach the necessary energy confinement time τ_E of the order of some seconds. These end losses can be completely avoided in a toroidal system, but in a simple toroidal system with purely toroidal magnetic field, the magnetic field curvature and gradient (approximately as in Fig. 2.8) result in a vertical drift which is in opposite directions for ions and electrons. The resulting electric field causes an outward $E \times B$ drift of the whole plasma, and therefore such a simple magnetic field configuration will be unstable. To avoid this charge separation, it is necessary to twist the magnetic field lines by additional magnetic field components. Then, single field lines map out so-called flux surfaces. On these flux surfaces, plasma transport is fast, as it is always parallel to B , and therefore plasma parameters usually are constant on a given flux surface. Perpendicular to the flux surfaces, transport is hindered because particle motion perpendicular to B is restricted by the Lorentz force, and therefore plasma parameters can vary strongly in this direction.

Two different principles for twisting the magnetic field lines have been invented in the 50's, and are under investigation worldwide.

2.6.1 The stellarator

The stellarator was invented in 1951 by Lyman Spitzer, Jr. in Princeton. In a stellarator the twist of the field lines is created by external coils wound around the plasma torus, as shown in Fig. 2.9 left. Due to these external currents the plasma shape is not circular, but shows some indentation. In this case, with four coils (neighbouring coils carry opposite current), the plasma has an oval shape.

These external coils have the advantage that the current can be controlled from outside, and can flow continuously, but the configuration shown in Fig. 2.9 left, is very difficult from the

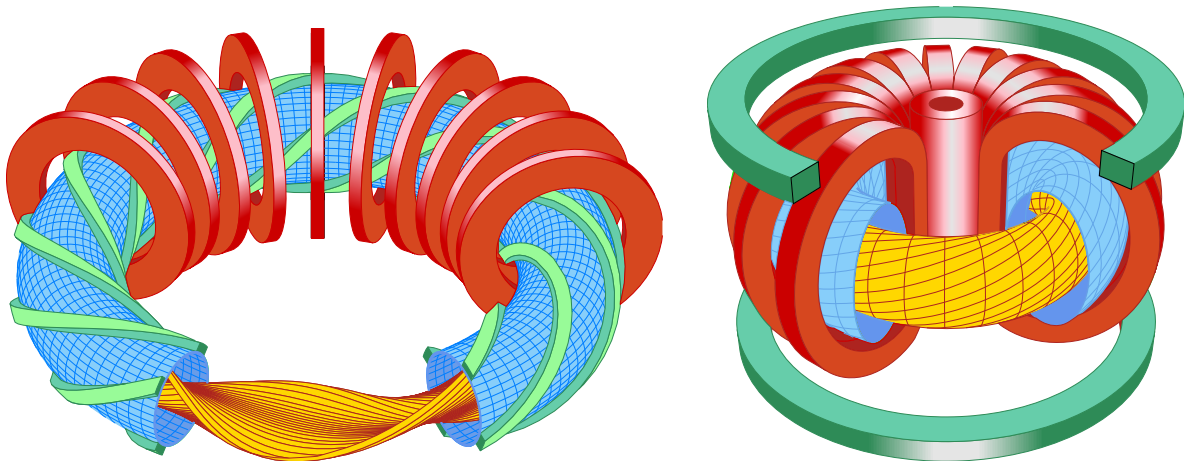


Figure 2.9: Left: Schematic view of a stellarator with four helical coils wound onto the vacuum vessel.
 Right: Tokamak concept. The innermost cylindrical coil is the transformer coil for inducing a plasma current. The toroidal coils above and below the machine create a vertical field for plasma shaping and position control.

engineering point of view, due to the interlinked magnetic coil systems. Also, these *classical* stellarators have shown a rather poor confinement of particles and energy. A solution for both problems has been found by means of numerical optimization of stellarator configurations. Following these studies, *classical* stellarators nowadays have been replaced by *modular* stellarators, where the planar toroidal coils and the helical coils have been replaced by one complex, but modular system of non-planar coils, as will be discussed in chap. 12 and chap. 13.

2.6.2 The tokamak

The second approach is the tokamak proposed by the two Russian physicists I. Tamm and A.D. Sakharov, in the year 1952 and realized by L.A. Artsimovich. The word *tokamak* itself is derived from the Russian words for toroidal chamber with magnetic field. The tokamak concept is outlined in Fig. 2.9 right. The toroidal magnetic field is provided by simple magnets and the necessary twist is produced by the plasma itself, by means of an electric current in the plasma which gives rise to the poloidal component of the twisted magnetic field. The current also serves for plasma build-up and heating. This current is produced by induction, the plasma acting as the secondary winding of a transformer. Tokamaks have proved to be very successful in improving the desired fusion plasma conditions and the today's best experiments are based on the tokamak principle. Of course, a transformer can induce the (dc-) plasma current only during a finite time, while, as mentioned before, a stellarator may principally run steady-state. For truly continuous tokamak operation, alternative current drive methods are being developed. Another disadvantage of the required large plasma current is the potential danger of so-called disruptions: Uncontrolled very fast (~ 10 ms) plasma current decays which can give rise to large forces on the machine.

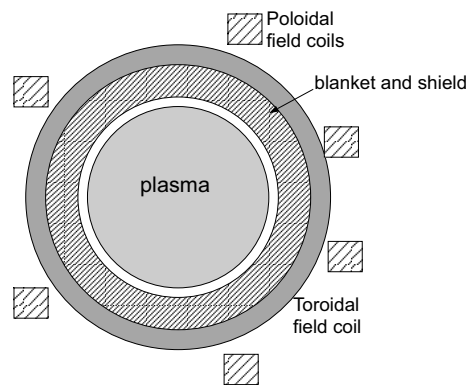


Figure 2.10: Layout of principal components in a conceptual fusion reactor.

More details on tokamak physics are described in chap. 10 and chap. 8 and a review on the status of tokamak research is given in [6].

2.6.3 The fusion reactor

The basic geometry of fusion reactors will be a torus (ring) for magnetically confined plasmas. A schematic cross-section of such a reactor is shown in Fig. 2.10. The hot plasma is surrounded by the first wall and blanket. The latter is filled with lithium to produce the tritium, as discussed before, and the majority of thermal energy of the plant is delivered here by neutron moderation. A shield is provided behind the blanket to stop the neutrons not captured by the blanket in order to reduce the heat and radiation loads to the cold structures of the superconducting magnets. The application of superconduction is mandatory for fusion reactors to obtain a positive energy balance.

2.7 Muon-catalyzed fusion

The real issue in achieving nuclear fusion is to overcome the repulsive Coulomb forces of light nuclei. The *conventional* approach is strong heating to reach the ignition conditions described above. Another, more exotic, way is to screen the electric charge by replacing the electron in the hydrogen atom with a muon [9]. This idea was first proposed by A.D. Sakharov in 1948. A muon is an elementary particle, a so-called lepton (*light* particle). This muon has the same properties as an electron, the only difference being that its mass is 200 times larger. Due to its larger mass the muon decays with a half-life of 1.5×10^{-6} seconds

$$\mu^- \longrightarrow e^- + \bar{\nu}_e + \nu_\mu .$$

Muons can be produced in accelerators which provide collisions between accelerated protons and some other material. In these collisions many pions are produced. A pion is the lightest elementary particle participating in the strong interaction (so-called hadrons). The three types

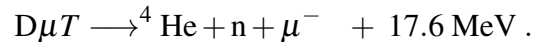
of pions (π^- , π^+ and π^0) are produced in equal amounts. A negative muon is formed from the decay of a negative pion

$$\pi^- \longrightarrow \mu^- + \bar{\nu}_\mu .$$

In a D-T mixture the muon slows down very fast ($\approx 10^{-9}$ s) and forms a $D\mu$ or $T\mu$ atom with a small Bohr radius of

$$a_\mu = a_e \cdot m_e / m_\mu \approx 2.5 \cdot 10^{-13} \text{ m} .$$

This reduced atomic radius is the key point for possible catalyzed nuclear fusion: If a $D\mu T$ molecule is formed, it takes just 10^{-12} s until quantum mechanical tunnelling triggers a fusion reaction



There are two limiting factors in this approach: the first is the time needed to form a $D\mu T$ molecule. This time can be influenced by some resonance mechanisms. The second limitation comes from the 0.6% probability that the muon will stick to the helium atom after the fusion reaction and will thus be lost for catalyzing more fusion reactions during its lifetime. Recently, experiments obtained fusion rates of more than 100 reactions per muon. For a pure energy producing system this rate is not sufficient, since about 3 GeV ($= 3 \times 10^9$ eV) is needed to produce one negative muon.

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Chapter 3

An introduction to MHD, or magnetic fluid dynamics

Bruce D. Scott

3.1 What MHD is

Magneto-hydro-dynamics (MHD) means magnetic fluid dynamics. It is a model system designed to treat macroscopic dynamics of an electrically neutral fluid which is nevertheless made up of moving charged particles, and hence reacts to magnetic fields. Because the magnetic field is in turn produced by electric currents – here, a relative drift between the two fluids of opposite charge density which permeate each other – the resulting dynamical system is rich in nonlinear character.

This is a heuristic introduction to the ideas of fluid dynamics and MHD. Each fluid – they may be termed electrons and ions – is separately treated as a perfect fluid which reacts in a dissipation-free way to the electric and magnetic fields. By perfect fluid, the concept of thermodynamic equilibrium is implied: The effects of dissipation in all forms are taken to be negligible. These include resistivity, thermal conductivity, and viscosity. The two fluids interact only through the electric and magnetic fields they induce. These fields in turn react to changes in the distribution of sources, which are the charge densities and currents represented by the two fluids. Externally imposed fields may also be present. An infinitesimal element of each fluid is assumed to contain an arbitrarily large number of charged particles of the corresponding species, but to be small compared to the spatial scale over which macroscopic thermodynamic or field quantity varies. The picture is a meaningful one if this scale is large compared to the mean-free path between particle collisions, and the radius of gyration of each particle about magnetic field lines is negligibly small. This is the *ideal two-fluid model* of plasma dynamics.

Then, because the mass of an individual electron is so much smaller than that of an individual ion, the contribution of the electrons to fluid inertia may be neglected. With a few additional assumptions which in effect define a parameter regime, the system is treated as a single fluid

which responds to a magnetic force of the Lorentz type, except that the plasma current enters instead of the fluid velocity. The magnetic field, responding to the electric field as usual, is actually advected by the velocity field, since the latter is related in a constitutive way to the electric field. This is the *ideal MHD model* of plasma dynamics, a subset of the two-fluid model.

3.2 The ideas of fluid dynamics

A fluid is in essence a continuous medium. Rather than an ensemble of particles, each individually treated, *populations* of particles are treated. The system is described in terms of a velocity flow field, a density field, and a few thermodynamic state variables. In the simplest picture, the fluid is an ideal gas, in which all state variables are simple relations of the density and temperature.

3.2.1 The density in a changing flow field – conservation of particles

We are first interested in how the density of the fluid evolves, when the flow field is known. This involves the deformation of volumes by the flow. Consider first a volume V , which is fixed in space. The conservation of particles follows simply from counting: The change of the number of particles N , in V is given by the number of particles entering V , less those that leave V . Consider that V is bounded by a surface S , an element of which is ΔS with unit normal \hat{e}_S . For each area element, the number of particles which cross it in a time interval Δt , is given by

$$\delta N = n \Delta x \Delta S, \quad \Delta x = v \cos \theta \Delta t = \mathbf{v} \cdot \hat{e}_S \Delta t,$$

where $n = N/V$ is the density of particles per unit volume, θ is the angle between the velocity vector and the unit normal, and Δx gives the thickness of the region containing particles which cross ΔS during Δt (a note on signs: The unit normal points outward, so a positive δN , indicating particles leaving V , contributes negatively to the change in the total number of particles ΔN).

Since S is a closed surface, we may add up all the contributions from the ΔS 's by performing a surface integral

$$\Delta N = - \oint_S \Delta S n \mathbf{v} \cdot \hat{e}_S \Delta t.$$

Since V is fixed, a change in N affects the density

$$\Delta N = \oint_V dV \Delta n,$$

so upon setting these two expressions equal, we have

$$\oint_V dV \Delta n = - \oint_S dS \cdot (n \mathbf{v}) \Delta t,$$

where ΔS and \hat{e}_S have been combined into dS . Note now that since S is the surface which encloses V , we may apply the Gaussian divergence theorem:

For any volume V enclosed by surface S , with directional element dS , any continuously differentiable vector F satisfies

$$\oint_V dV \nabla \cdot F = \oint_S dS \cdot F .$$

Therefore, we may write

$$\oint_V dV \Delta n = -\Delta t \oint_V dV \nabla \cdot n v .$$

This is valid for arbitrary V ; specifically, it is valid for an infinitesimal volume about any point. This means the integrands must themselves be equal, since in general they do not vanish. Further taking $\Delta t \rightarrow 0$, we obtain

$$\frac{\partial n}{\partial t} = -\nabla \cdot n v$$

as a statement of conservation of particles of number density n , advected by velocity v . It holds for a *fixed* reference frame. The partial time derivative refers to the fact that V is fixed in space.

So what is meant by divergence?

Suppose now that we take the same flow and particle distribution, but now let the boundary surface elements of the volume move with the fluid. Note now that no particles enter or leave V

$$\Delta N = 0 .$$

The volume does, however, change with time: since the density is given by $n = N/V$, its change is inverse to that in V

$$\frac{\Delta n}{n} = -\frac{\Delta V}{V} .$$

At each point, the boundary element moves with the velocity

$$\Delta x = v \Delta t .$$

The separation of points changes according to spatial variations in the velocity

$$\Delta x_2 - \Delta x_1 = (v_2 - v_1) \Delta t .$$

A volume may be expanded in a combination of two ways: Spreading or stretching. If the direction of a surface element dS is taken as the z -direction at some point, spreading is due to the expansion perpendicular to the element: $\partial v_x / \partial x + \partial v_y / \partial y$. Stretching is due to longitudinal variation: $\partial v_z / \partial z$. The sum of these is called the *divergence*, and the divergence of a velocity means the same thing as the expansion of the advected volume

$$\Delta V = V \nabla \cdot (\Delta x) ,$$

or, including Δt , which commutes with spatial derivatives

$$\frac{\Delta V}{\Delta t} = V \nabla \cdot \frac{\Delta x}{\Delta t} = V \nabla \cdot v .$$

With this for the volume change, the density change is given by

$$\frac{\Delta n}{\Delta t} = -\frac{n}{V} \frac{\Delta V}{\Delta t} = -n \nabla \cdot \mathbf{v} ,$$

or

$$\frac{dn}{dt} = -n \nabla \cdot \mathbf{v} ,$$

as a statement of conservation of particles of number density n , advected by velocity \mathbf{v} . It holds for the *co-moving* reference frame. Note that the total time derivative d/dt , gives changes in the frame moving with the fluid at each point, hence the name co-moving.

One final note: A common name for the equation for the conservation of particles is often called the *continuity equation*, since the fact that n and \mathbf{v} are continuously differentiable field quantities has been implicitly used to get it.

3.2.2 The advective derivative and the co-moving reference frame

We now have two statements of the general conservation of particles in terms of the density, n , and the velocity with which they are advected \mathbf{v} . One is in the fixed reference frame

$$\frac{\partial n}{\partial t} = -\nabla \cdot n\mathbf{v} .$$

The other is in the co-moving reference frame

$$\frac{dn}{dt} = -n \nabla \cdot \mathbf{v} .$$

These are actually the same statement in two different coordinate systems. Let the fixed system be denoted as (t, x, y, z) and the co-moving frame as (t', x', y', z') . The transformation between them, which is a *local* transformation, may be written

$$dx' = dx - v_x dt , \quad dy' = dy - v_y dt , \quad dz' = dz - v_z dt .$$

The time derivative transforms as

$$\begin{aligned} \frac{d}{dt} &= \frac{\partial}{\partial t} + \frac{dx}{dt} \frac{\partial}{\partial x} + \frac{dy}{dt} \frac{\partial}{\partial y} + \frac{dz}{dt} \frac{\partial}{\partial z} \\ &= \frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} , \end{aligned}$$

where the velocity components are not differentiated because the transformation is local. In general form

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla .$$

This is the time derivative in the co-moving frame, written in terms of the time derivative in the fixed frame and the local, instantaneous velocity. It is also known as the *advective derivative*, or in some texts the *substantiative derivative*. Its usefulness is that the laws governing the motion or constitutive change of fluid elements are often most easily formulated in the co-moving frame (or the fluid element's rest frame) and then transformed into the fixed frame. One can readily see that the two equations for the fluid density are equivalent.

3.2.3 Forces on the fluid – how the velocity changes

We now allow the velocity to change, and see how it is caused to change. We examine the forces on the fluid element in the co-moving frame, and then transform to the fixed frame. We consider only fluid pressure and electromagnetic (Lorentz) forces, neglecting such others as gravitation or the effects of moving boundaries.

Consider first the fluid pressure p . Each of the fluid particles is moving with a velocity w , in the co-moving frame, and the average of w over all the particles in a fluid element is zero. Nevertheless, the fluid exerts pressure on any surface, according to the rate at which momentum is transferred by the random particle motion. If p is prescribed, we may compute the net force F , on any fluid element by adding up the forces ΔF , on each surface element dS :

$$\Delta F = p dS .$$

Summing all the force elements over a closed surface gives an area integral

$$F = \oint_S dS p .$$

Since the force is exerted on the enclosed volume, V , we may apply another vector theorem

For any volume V enclosed by surface S , with directional element dS , any continuously differentiable scalar ζ satisfies

$$\oint_V dV \nabla \zeta = \oint_S dS \zeta .$$

It is important to get the correct sign: The force is exerted against the surface from the surrounding fluid; hence the signs in the above should all be negative (except for the theorem). Considering a cubical fluid element, oriented such that the pressure on the right is higher than that on the left: The forces on the top and bottom are equal and opposite, the forces on the front and back are equal and opposite, but the force from the right towards the left is greater than the force from the left toward the right. There is a net force from right to left, given by the pressure difference $\Delta p = p_{\text{right}} - p_{\text{left}}$, times the surface area ΔS . If the coordinates are set up such that z increases towards the right, we can see that the force from right to left is *opposite* to the directional gradient in the pressure. Since the size of the element is Δz , then the force is given by

$$F = -\Delta p \Delta S \hat{e}_z = -\Delta z \Delta S (\Delta p / \Delta z) \hat{e}_z = -\Delta V \nabla p \rightarrow -\oint_V dV \nabla p .$$

This force acts to accelerate the fluid element. The mass in the element is given by the mass of each particle m , times the number of particles N , in the volume V . The element accelerates according to

$$Nm \frac{dv}{dt} = F ,$$

or taking the volume to zero about a given point, the velocity change can be taken inside the integral

$$\oint_V dV nm \frac{dv}{dt} = -\oint_V dV \nabla p .$$

Dispensing with the volumes as before, we obtain

$$nm \frac{dv}{dt} = -\nabla p ,$$

in the co-moving frame, and

$$nm \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\nabla p ,$$

in the fixed frame.

This is the force law which determines how the fluid velocity field changes in reaction to gradients in its pressure. Standing alone, it is the force law for a neutral fluid.

Adding the Lorentz force is straightforward, since the force is a body force, one which acts on the element as a whole and not through the surface. The force on N particles of charge q in the fixed frame is

$$\mathbf{F} = Nq \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) = \oint_V dV nq \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) ,$$

so that the integrand is added to the neutral fluid force law to obtain the force law for a charged fluid

$$nm \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\nabla p + nq \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) .$$

Note that one could just as easily derive the Lorentz force term in the co-moving frame as well, with \mathbf{v} zero, and then transform to the fixed frame via the Lorentz transformation (the low-velocity form of which is the transformation used in sec. 3.2.2.

We now have equations describing the evolution of the density and velocity of a fluid in general, so we could build up the two-fluid system consisting of one set of these for each species, and then close the system with Maxwell's equations for the electric and magnetic fields. This would only be complete for an isothermal system, though, so we do need to discuss how the pressure changes.

3.2.4 Thermodynamics of an ideal fluid – how the temperature changes

The simplest system is the ideal fluid: One which evolves quasi-statically through successive states of local thermodynamic equilibrium. Changes are expected to be slow enough such that equilibrium is maintained, but fast enough so that fluid elements do not exchange entropy.

In thermodynamic equilibrium with only isentropic changes considered, the first law reads

$$\Delta E + p \Delta V = \Delta Q \rightarrow 0 .$$

E is the internal energy of a given fluid element of N particles. For the ideal gas law

$$E = \frac{3}{2} NkT .$$

V is the volume occupied by the N particles. Over infinitesimal time intervals

$$\frac{dE}{dt} + p \frac{dV}{dt} = 0,$$

in the co-moving frame.

We already know how the volume changes

$$\frac{dV}{dt} = V \nabla \cdot \mathbf{v}.$$

Given that the number of particles is kept fixed

$$\frac{3}{2} Nk \frac{dT}{dt} + pV \nabla \cdot \mathbf{v} = 0,$$

and since the density is $n = N/V$,

$$\frac{3}{2} nk \frac{dT}{dt} + p \nabla \cdot \mathbf{v} = 0.$$

Now, transforming to the fixed frame, we have the relation governing the change of the fluid's temperature given the flow field and all the thermodynamic state variables

$$\frac{3}{2} nk \left(\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) + p \nabla \cdot \mathbf{v} = 0.$$

3.2.5 The composite fluid plasma system

Under the preceding conditions: Perfect fluid with an ideal gas law, no reactions that create or destroy particles of any species, maintenance of local thermodynamic equilibrium...

The system of several fluids each made up of charged particles of species α evolves according to

$$\begin{aligned} \frac{\partial n_\alpha}{\partial t} + \nabla \cdot n_\alpha \mathbf{v}_\alpha &= 0, \\ n_\alpha m_\alpha \left(\frac{\partial}{\partial t} + \mathbf{v}_\alpha \cdot \nabla \right) \mathbf{v}_\alpha &= -\nabla p_\alpha + n_\alpha q_\alpha \left(\mathbf{E} + \frac{\mathbf{v}_\alpha}{c} \times \mathbf{B} \right), \\ \frac{3}{2} n_\alpha k \left(\frac{\partial T_\alpha}{\partial t} + \mathbf{v}_\alpha \cdot \nabla T_\alpha \right) + p_\alpha \nabla \cdot \mathbf{v}_\alpha &= 0, \end{aligned}$$

while the electric and magnetic fields evolve according to Maxwell's equations

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 4\pi \sum_\alpha n_\alpha q_\alpha, \\ \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} &= \nabla \times \mathbf{B} - \frac{4\pi}{c} \sum_\alpha n_\alpha q_\alpha \mathbf{v}_\alpha, \\ \nabla \cdot \mathbf{B} &= 0, \\ \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E}. \end{aligned}$$

Note that the charge density and the current have been specified in terms of the fluid variables

$$\rho_{ch} = \sum_{\alpha} n_{\alpha} q_{\alpha}, \quad \mathbf{J} = \sum_{\alpha} n_{\alpha} q_{\alpha} \mathbf{v}_{\alpha},$$

The above constitutes a closed system governing the evolution of several charged ideal fluids and the electric and magnetic fields they induce. Note that external fields may also be imposed.

3.3 From many to one – the MHD system

In many cases, the regime of parameters in which the plasma finds itself allows considerable simplification of the multi-fluid system. The fact that the electron mass is so much smaller than that of any ion allows one to neglect the electron inertia in comparison to that of the ions, with the result that the electron force equation becomes a relation for the electric field in terms of the velocity and magnetic field.

For purposes of illustration it is useful to consider that there is only one ion species.

The following assumptions are made:

1. The *displacement current*, which is the term involving the time derivative of \mathbf{E} , is negligible because the time for a light wave to cross the system is much shorter than any relevant dynamical scale.
2. The electron mass is sufficiently small that parallel force balance on the electrons is maintained at all times. This allows the neglect of electron inertia. For the ideal MHD system we assume that the parallel electron dynamics are generally negligible; this means the components of both ∇p_e and \mathbf{E} parallel to the magnetic field.
3. The drift velocity of the electrons relative to the ions due to the current is small compared to the ion velocity. This allows one to assume that all species move with the same velocity, when specifying velocities.
4. Pressure forces are negligible compared to Lorentz forces on all the fluids.
5. The system is approximately neutral, such that the total charge density is negligible compared to that of any constituent.

Note that this does not mean that $\mathbf{E} \rightarrow 0$, but that the spatial scale of any variation is large compared to the Debye length of the plasma, and that the plasma frequency is faster than any rate of change. This means, however, that we can use the relation of zero charge density as a good approximation for the electron density in terms of the ions. For one ion species with charge Ze , this means

$$n_e \approx Zn_i.$$

These assumptions will be checked once we explore the dynamical scales of the MHD system.

3.3.1 The MHD force equation

The electron and ion force equations appear as

$$\begin{aligned} n_i M_i \left(\frac{\partial}{\partial t} + \mathbf{v}_i \cdot \nabla \right) \mathbf{v}_i &= -\nabla p_i + n_i Z e \left(\mathbf{E} + \frac{\mathbf{v}_i}{c} \times \mathbf{B} \right), \\ 0 &= -\nabla p_e - n_e e \left(\mathbf{E} + \frac{\mathbf{v}_e}{c} \times \mathbf{B} \right), \end{aligned}$$

Adding these and using the charge neutrality relation, we obtain

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\nabla p + \frac{1}{c} \mathbf{J} \times \mathbf{B}.$$

This is the MHD force equation. To get it, we have written p for the total pressure $p = p_e + p_i$, and $\rho = n_i M_i$ for the mass density. We have further dropped the subscript on the ion velocity, using this as the bulk fluid velocity. The electron velocity is obtainable from \mathbf{v} and \mathbf{J} . Note that the current is

$$\mathbf{J} = n_i Z e \mathbf{v}_i - n_e e \mathbf{v}_e.$$

The electrons have two roles:

1. They provide pressure, and can be dominant in doing so (but only if the temperatures are allowed to be unequal).
2. They keep the system quasi-neutral through their ability to move arbitrarily fast along the magnetic field lines.

3.3.2 Treating several ion species

One can easily generalise the MHD force equation to a system of several ion species by defining the total velocity as the velocity of the *center of mass*

$$\rho \mathbf{v} = \sum_i n_i M_i \mathbf{v}_i,$$

where the sum is over all the ion species. The mass density evolves according to the continuity equations for all ion species

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0.$$

Provided all the temperatures are equal, this can be done for the energy equation as well

$$C_v \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) T + p \nabla \cdot \mathbf{v} = 0,$$

where the specific heat at constant volume is

$$C_v = \frac{3}{2} k \sum_{\alpha} n_{\alpha},$$

and the sum is over all species, including the electrons. Alternatively, this may be combined with the continuity equation to reflect the fact that p/ρ^γ , with $\gamma = 5/3$, is conserved in the dynamics, even for several species (see below).

One very important note: The system can be treated as ideal either if the heat exchange among the particle populations is (1) negligible or (2) so fast that the temperatures are all kept equal. In laboratory plasmas the first of these limits is usually well-satisfied for dynamics, and the second is usually valid for quasi-static equilibria.

3.3.3 The MHD kinematic equation

We need now only determine how the magnetic field evolves, and since the current is given by

$$\mathbf{J} = \frac{c}{4\pi} \nabla \times \mathbf{B} ,$$

that will be enough to close the system.

The magnetic field evolves according to

$$\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} ,$$

and from the electron force equation we have

$$\mathbf{E} + \frac{\mathbf{v}_e}{c} \times \mathbf{B} = -\frac{1}{n_e e} \nabla p_e .$$

For the dynamical scales of interest, it will become clear that the pressure gradient is negligibly small in this equation (recall we are neglecting parallel electron dynamics). This is assumption (4) above. We have already assumed that \mathbf{J} is small enough that the electron and ion velocities are equal to high accuracy (assumption 3). This implies that the component of the fluid velocity across the magnetic field is given by the $\mathbf{E} \times \mathbf{B}$ velocity

$$\mathbf{v} = \frac{c}{B^2} \mathbf{E} \times \mathbf{B} ,$$

which means that all the particles $\mathbf{E} \times \mathbf{B}$ drift together across the magnetic field lines, preventing any significant charge build-up. Substituting for \mathbf{E} in the equation for \mathbf{B} , we obtain

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) .$$

This is the MHD kinematic equation. Whether or not the magnetic field has important effect on the dynamics, it describes how the magnetic field is advected by the flow velocity. Much of the character of MHD springs from this equation, especially its most important consequence: Flux conservation. We will see what this is and what it means in a moment.

3.3.4 MHD at a glance

The complete ideal MHD system is collected here for clarity.

The continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0.$$

The MHD force equation

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\nabla p + \frac{1}{c} \mathbf{J} \times \mathbf{B}.$$

The adiabatic pressure equation

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) p + \frac{5}{3} p \nabla \cdot \mathbf{v} = 0.$$

The MHD kinematic equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}).$$

Ampere's law

$$\mathbf{J} = \frac{c}{4\pi} \nabla \times \mathbf{B}.$$

3.4 The flux conservation theorem of ideal MHD

There is an important result that arises immediately from the MHD kinematic equation. This is that the magnetic flux through any surface element advected by the fluid remains constant no matter what the flow field. Closely related is that the flux through the surface defined by any closed curve within the fluid is also conserved. The result gives rise to the concept of the magnetic flux tube.

3.4.1 Proving flux conservation

Consider an arbitrary, infinitesimal surface defined by a triangle of infinitesimal sides. Three points, x_0 , x_1 , x_2 , are given, and the surface element is

$$d\mathbf{S} = \frac{1}{2} (\mathbf{x}_1 - \mathbf{x}_0) \times (\mathbf{x}_2 - \mathbf{x}_0).$$

The magnetic flux through the surface is given by $\mathbf{B} \cdot d\mathbf{S}$, and it changes according to

$$\frac{d}{dt} \mathbf{B} \cdot d\mathbf{S} = \frac{d\mathbf{B}}{dt} \cdot d\mathbf{S} + \mathbf{B} \cdot \frac{d}{dt} d\mathbf{S}.$$

Remembering that the surface and magnetic field are advected by the fluid

$$\frac{d\mathbf{B}}{dt} = \frac{\partial\mathbf{B}}{\partial t} + \mathbf{v} \cdot \nabla\mathbf{B}.$$

Now find how the surface element changes

$$\frac{d}{dt}d\mathbf{S} = (\mathbf{x}_1 - \mathbf{x}_0) \times \frac{1}{2}(\mathbf{v}_2 - \mathbf{v}_1) + \frac{1}{2}(\mathbf{v}_1 - \mathbf{v}_0) \times (\mathbf{x}_2 - \mathbf{x}_1),$$

noting that $d\mathbf{x}/dt = \mathbf{v}$. Reform this in terms of contributions to $\Delta\mathbf{x} \times \mathbf{v}$

$$\begin{aligned} \frac{d}{dt}d\mathbf{S} &= -(\mathbf{x}_1 - \mathbf{x}_0) \times \frac{1}{2}(\mathbf{v}_1 + \mathbf{v}_0) - (\mathbf{x}_2 - \mathbf{x}_1) \times \frac{1}{2}(\mathbf{v}_2 + \mathbf{v}_1) \\ &\quad - (\mathbf{x}_0 - \mathbf{x}_2) \times \frac{1}{2}(\mathbf{v}_0 + \mathbf{v}_2) \\ &= -\sum_j \Delta\mathbf{x}_j \times \mathbf{v}_j, \end{aligned}$$

where j tracks the midpoint of each line segment. Employ a vector identity to replace this expression by

$$\frac{d}{dt}d\mathbf{S} = -(\mathbf{dS} \times \nabla) \times \mathbf{v}.$$

We require to further re-form this; using the component-index representation of the right side, we have

$$\begin{aligned} [(\mathbf{dS} \times \nabla) \times \mathbf{v}]_i &= \varepsilon_{ipq}(\varepsilon_{plm} \Delta S_l \partial_m) v_q \\ &= \varepsilon_{pqi}(\varepsilon_{plm} \Delta S_l \partial_m) v_q \\ &= \Delta S_q \partial_i v_q - \Delta S_i \partial_q v_q, \end{aligned}$$

which may be written in vector form as

$$\frac{d}{dt}d\mathbf{S} = \mathbf{dS}(\nabla \cdot \mathbf{v}) - (\nabla\mathbf{v}) \cdot \mathbf{dS}.$$

Inserting this back into the original expression for the flux evolution, we obtain

$$\begin{aligned} \frac{d}{dt}\mathbf{B} \cdot \mathbf{dS} &= \left(\frac{\partial\mathbf{B}}{\partial t} + \mathbf{v} \cdot \nabla\mathbf{B} \right) \cdot \mathbf{dS} + \mathbf{B} \cdot [\mathbf{dS}(\nabla \cdot \mathbf{v}) - (\nabla\mathbf{v}) \cdot \mathbf{dS}] \\ &= \mathbf{dS} \cdot \left(\frac{\partial\mathbf{B}}{\partial t} + \mathbf{v} \cdot \nabla\mathbf{B} - \mathbf{B} \cdot \nabla\mathbf{v} + \mathbf{B}\nabla \cdot \mathbf{v} \right) \\ &= \mathbf{dS} \cdot \left(\frac{\partial\mathbf{B}}{\partial t} - \nabla \times \mathbf{v} \times \mathbf{B} \right) = 0, \end{aligned}$$

since the expression in parentheses vanishes according to the MHD kinematic equation.

This proves the flux conservation theorem for an infinitesimal surface element advected by the fluid. It follows that the flux through any *surface* advected by the fluid is also conserved; simply add up all the surface elements.

3.4.2 Magnetic flux tubes

As a result of the fact that the magnetic flux through any surface advected by the fluid is conserved, we may find a group of field lines which serve as the boundary for a definite volume. Define a closed curve which is the boundary for a small but finite surface, the magnetic flux through which is not zero. Follow each field line an arbitrary distance away from the original curve, and define another curve which intersects the same field lines. This is a *magnetic flux tube*. Note that the magnetic flux through the sides of the tube is zero, and because the flux is conserved it stays zero.

Under advection of a flow which deforms the flux tube, the identity of the flux tube is maintained, even though the flux tube may be very greatly twisted and tangled with several other flux tubes. The consequence of this is that the field line topology is not allowed to change.

Consider two flux tubes which may be defined in a sheared magnetic field. Shear in the magnetic field may be thought of as follows: Consider a horizontal plane in which parallel lines are drawn. Now consider a plane immediately above or below the first one, in which parallel lines are also drawn. If the orientation of each set of parallel lines changes from plane to plane, then the field represented by the drawn lines is said to be *sheared*, and the shear can be quantified by giving the rate of change of this angle of orientation with perpendicular distance. Label two very narrow flux tubes, one lying in one such plane and another lying initially in a plane immediately below the first one, “a” and “b”, respectively. Due to the shear, when the flux tubes are brought together by a flow field, they cross. Propose that they might pass through each other as they are forced together. Before the interaction, the magnetic flux through the sides of both tubes is zero. If they are allowed to pass through each other, the field lines due to tube “a” would intersect the sides of tube “b”, and vice versa. The magnetic flux through the sides of the tubes would no longer be zero. This is in obvious contradiction to the flux conservation theorem, so the conclusion is that the flux tubes are never allowed to cross. Note that this conclusion holds as well for flux tubes of infinitesimal cross-section, and hence for individual field lines.

In ideal MHD, magnetic flux tubes and field lines cannot be advected through each other, because of the magnetic flux conservation theorem.

Below, we will explore how this constraint is relaxed by a finite plasma resistivity.

3.5 Dynamics, or the wires-in-Molasses picture of MHD

Note that the magnetic force term in the MHD force equation can be split into two pieces using an elementary vector identity

$$\frac{1}{c} \mathbf{J} \times \mathbf{B} = \frac{1}{c} \left(\frac{c}{4\pi} \nabla \times \mathbf{B} \right) \times \mathbf{B} = -\nabla \frac{B^2}{8\pi} + \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{4\pi}.$$

These two contributions denote *magnetic pressure* and *magnetic tension*, respectively. Magnetic pressure may be combined with gas pressure

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\nabla \left(p + \frac{B^2}{8\pi} \right) + \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{4\pi},$$

where one notes that it behaves like an energy density with two degrees of freedom (for which the pressure and energy density are equal).

The other contribution is magnetic tension. It is part of what gives MHD its unique character (flux conservation, or the advection of magnetic field lines is the other part). To see why it is called tension, consider the magnetic field due to a wire carrying current, surrounded by a vacuum. The field lines are described by loops centered upon the wire, and the forces are obviously zero because it is a vacuum. A cylindrical coordinate system may be defined, with z in the direction of the wire and r perpendicular to it. The only nonzero component of $\mathbf{B} \cdot \nabla \mathbf{B}$ is

$$\mathbf{B} \cdot \nabla \mathbf{B} = -\frac{B^2}{r} \hat{\mathbf{e}}_r.$$

This shows that magnetic tension is a force which acts in the direction of the curvature vector (towards the loop's center), a general result for a curved field line. Since the current outside the wire vanishes, the tension and pressure forces must be in balance: since $B \sim I/r$,

$$-\nabla \frac{B^2}{8\pi} = 2 \frac{B^2}{8\pi} \frac{1}{r} \hat{\mathbf{e}}_r = \frac{B^2}{4\pi r} \hat{\mathbf{e}}_r = -\mathbf{B} \cdot \nabla \mathbf{B}.$$

In such a situation, which can also exist in a plasma where the current is not zero, magnetic pressure and tension are in balance, and the configuration is termed *force-free*.

3.5.1 Magnetic pressure waves

Consider a compression in a magnetised plasma in equilibrium, perpendicular to the magnetic field. The velocity depends only on the direction perpendicular to the field, and it is itself directed perpendicular to the field. In this situation

$$\mathbf{B} \cdot \nabla \mathbf{B} = 0,$$

since \mathbf{B} is compressed perpendicular to its direction. The field lines are compressed together with the fluid, according to the MHD kinematic equation

$$\frac{\partial \mathbf{B}}{\partial t} = -\mathbf{v} \cdot \nabla \mathbf{B} - \mathbf{B} \nabla \cdot \mathbf{v},$$

where the third piece, $\mathbf{B} \cdot \nabla \mathbf{v}$, vanishes due to the geometry. This equation states that \mathbf{B} reacts exactly as would a density

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot \mathbf{v} \mathbf{B} = 0.$$

With the vanishing magnetic tension, the force equation reads

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\nabla \left(p + \frac{B^2}{8\pi} \right).$$

The result is exactly analogous to sound waves, since both the pressure and magnetic field are perturbed in the same way. Consider small perturbations of short wavelength, on which scale the equilibrium pressure and magnetic field is homogeneous. We *linearise* the equations by retaining terms only to first order in the perturbations, which are denoted by a tilde symbol. For example, B^2 becomes $B^2 + 2\mathbf{B} \cdot \tilde{\mathbf{B}}$, with the term quadratic in $\tilde{\mathbf{B}}$ neglected. The velocity requires no symbol, since it is understood to belong to the perturbations. Both pressure and magnetic perturbations are induced by compression in the velocity

$$\frac{1}{\gamma p} \frac{\partial \tilde{p}}{\partial t} = \frac{1}{B} \frac{\partial \tilde{B}}{\partial t} = -\nabla \cdot \mathbf{v},$$

with $\gamma = 5/3$. The perturbed force equation reads

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla \left(\tilde{p} + \frac{B\tilde{B}}{4\pi} \right),$$

and note the factor of two arising from perturbing B^2 . These may be combined into a wave equation

$$\frac{\partial^2 \tilde{\mathbf{B}}}{\partial t^2} - \left(\frac{B^2}{4\pi\rho} + \frac{\gamma p}{\rho} \right) \nabla_{\perp}^2 \tilde{\mathbf{B}} = 0,$$

where it is noted that the derivatives are all perpendicular to \mathbf{B} .

The second term in the parentheses will be recognised as the square of the adiabatic sound velocity v_s . The first term introduces the characteristic velocity of MHD in general, and the velocity of propagation of small magnetic disturbances in particular. It is the square of the *Alfvén velocity*

$$v_A^2 = \frac{B^2}{4\pi\rho},$$

after Hannes Alfvén, who is recognised as the founder of the MHD description of plasma fluid dynamics. Magnetic pressure waves, like sound waves, are longitudinal waves, but unlike sound waves they are in their pure form only when the disturbance propagates perpendicular to the magnetic field. In general, when both gas pressure and magnetic pressure are present, they both contribute to the actual wave speed: $v^2 = v_s^2 + v_A^2$. More on the ratio of gas to magnetic pressure shortly.

3.5.2 Alfvén waves: magnetic tension waves

Because the magnetic field also exhibits tension, transverse waves similar to waves on a taut string also occur in MHD. Assume now that there is a divergence-free perturbation of a magnetised plasma in equilibrium, still perpendicular to the magnetic field. The velocity now

depends only on the direction parallel to the field, although it is itself directed perpendicular to the field. In this situation

$$\nabla \cdot \mathbf{v} = \tilde{p} = \mathbf{B} \cdot \tilde{\mathbf{B}} = 0,$$

since the disturbance is a transverse-shear perturbation. The field lines are not compressed, but are bent, according to the MHD kinematic equation

$$\frac{\partial \mathbf{B}}{\partial t} = -\mathbf{v} \cdot \nabla \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{v},$$

where the divergence piece, $\mathbf{B} \nabla \cdot \mathbf{v}$, vanishes due to the geometry. Field line bending becomes clearer when this expression is re-cast in the co-moving frame

$$\frac{d\mathbf{B}}{dt} = \mathbf{B} \cdot \nabla \mathbf{v}.$$

The single contribution arises due to the fact that the field line is moved in alternate directions according to position along it.

This is a new situation, in which the pressure is unperturbed, and the magnetic field is perturbed only through the component perpendicular to the equilibrium field. For homogeneous perturbations

$$\begin{aligned} \frac{\partial \tilde{\mathbf{B}}_{\perp}}{\partial t} &= \mathbf{B} \cdot \nabla \tilde{v}_{\perp}, \\ \rho \frac{\partial \tilde{v}_{\perp}}{\partial t} &= \frac{\mathbf{B} \cdot \nabla \tilde{\mathbf{B}}_{\perp}}{4\pi}, \end{aligned}$$

where there is no factor of two since the perturbed field component is perpendicular. These may be combined into a wave equation

$$\frac{\partial^2 \tilde{\mathbf{B}}_{\perp}}{\partial t^2} - \frac{B^2}{4\pi\rho} \frac{(\mathbf{B} \cdot \nabla)^2}{B^2} \tilde{\mathbf{B}}_{\perp} = 0,$$

where note now that the derivatives are all parallel to \mathbf{B} .

This type of disturbance propagates parallel to \mathbf{B} , and it is a transverse wave. It is called an *Alfvén wave*, since it is the type of propagating wave which exists in MHD but not in neutral fluid dynamics. Its propagation speed is purely the Alfvén velocity. The detection of Alfvén waves in the solar wind by spacecraft in the 1960s gave evidence that MHD phenomena do occur in nature and are not a theoretical artifice.

The concept of field line tension is now clear, since the behavior of the field line in MHD is the same as that of a taut string: *Pluck it, and transverse waves run down the line*. The concept of *field line bending* is closely related: curvature of a field line gives rise to the magnetic tension force. However, field line bending should not be considered as a force in itself, but a cause of one.

A general description of MHD can be that of *molasses threaded by wires*. The wires are magnetic field lines, which exert force on the fluid and which are advected by the fluid as it moves. Additions to this are that the wires exert pressure as well as tension, tending to repel each other, and that in the ideal limit the fluid is not viscous.

One final note: The attractive force between two wires is due to magnetic tension overcoming magnetic pressure.

3.6 The validity of MHD

The Alfvén velocity as just introduced determines the natural time scale of any MHD phenomena of a confined system. With this in hand, we are in a position to judge the validity of MHD by providing *a posteriori* checks on its fundamental assumptions. The Alfvén times are discussed first, and then the checks are made.

3.6.1 Characteristic time scales of MHD

Although they were derived for small disturbances on a homogeneous background, the wave velocities for propagation perpendicular and parallel to the magnetic field indicate the characteristic time scales for adjustment to equilibrium for general perturbations of any confined MHD system. This is much the same as the way the time it takes for a sound wave to cross a neutral fluid in hydrostatic equilibrium under gravity gives the characteristic adjustment time for that fluid. The reason is that a global free oscillation is nothing more than the longest-wavelength limit of the appropriate wave. Examples of the neutral fluid case would be an ocean layer, the Earth's atmosphere, or the Sun. In that case, for a scale length a , the sound-wave transit time is

$$\tau_s = a/c_s ,$$

which is also the inverse of the frequency of the fundamental global mode of oscillation (cf. the five-minute oscillation observed on the Sun).

For a confined plasma in equilibrium the sound speed is replaced by the Alfvén velocity, v_A . It must be noted, though, that the geometry of the confined system is important, since different types of waves propagate parallel to and perpendicular to the magnetic field. With characteristic scale lengths L_\perp perpendicular to the field and L_\parallel parallel to the field, two time scales are of interest. Due to the propagation of magnetic pressure waves, we have the *fast Alfvén*, or compressional Alfvén time: $\tau_A = L_\perp/v_A$. From the propagation of magnetic tension waves (Alfvén waves) we have the *slow Alfvén*, or shear Alfvén time: $\tau_A = L_\parallel/v_A$.

Both of these are usually written as τ_A in the literature, and one has to extract the meaning from the context. The name *shear Alfvén time* originates from the fact that parallel length scales for many laboratory plasma instabilities arise from the existence of magnetic shear in the equilibrium configuration.

These time scales and characteristic velocities can be deduced directly from the MHD force and kinematic equations by *scaling* them. Assuming that the spatial scale is a , the kinematic equation yields

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) \quad \rightarrow \quad \frac{B}{\tau} \sim \frac{Bv}{a} ,$$

and the force equation yields

$$\rho \frac{dv}{dt} = -\nabla p + \frac{1}{c} \mathbf{J} \times \mathbf{B} \quad \rightarrow \quad \rho \frac{v}{\tau} \sim \frac{B^2}{4\pi a},$$

if the gas pressure is negligible. These similarity relations may be solved for v and τ , given B and a , and the result is $v \sim v_A$ with $\tau \sim a/v_A$.

3.6.2 Checking the assumptions

With a as a representative spatial scale and τ_A as the corresponding time scale, we may now examine the MHD assumptions listed in sec. 3.3, a little out of order.

1. The displacement current is neglected because $v_A \ll c$. This is easily satisfied for laboratory and space plasmas.
5. Quasineutrality depends additionally on both $\tau_A \omega_{pe} \gg 1$ and $\rho_i \gg \lambda_D$. The latter three parameters in this list are the electron plasma frequency ($\omega_{pe}^2 = 4\pi n e^2 / m_e$), the ion Larmor gyroradius ($\rho_i^2 = c^2 M_i T_i / e^2 B^2$), and the Debye screening length ($\lambda_D^2 = T / 4\pi n e^2$). The former inequality is needed to neglect the parallel electric field, as the time scale for the electrons to equilibrate charge is ω_{pe} . The latter is needed in order to use the Lorentz force in the fluid description, since the Lorentz force has the gyroradius as its implicit length scale and the particles are not supposed to individually interact.
3. Fluid elements of all species move with velocity v , with the relative drift implied by J negligible. This requires $J \ll n e v_A$, or using Ampere's law to express J in terms of B (as magnitudes), we find

$$\frac{\rho_i^2}{a^2} \equiv (\text{drift parameter})^2 \ll \text{plasma beta} = \beta \equiv \frac{8\pi p}{B^2}.$$

(The plasma beta is discussed below.) Neglecting ion inertia as a correction to the $\mathbf{E} \times \mathbf{B}$ velocity for ions depends on the same limit, since $\tau_A \Omega_i \gg 1$ is an equivalent statement ($\Omega_i = eB/M_i c$ is the ion gyrofrequency). The statement that the drift parameter be smaller than some limit is a requirement on how strongly the plasma is magnetised, and that it must be smaller than $\beta^{1/2}$ is usually well-satisfied in any fusion or space plasma application.

2. Electron inertia is negligible because $m_e \ll M_i$ in general. It is negligible compared to the pressure gradient if $n m_e v_A / \tau \ll p/a$, which implies $\beta \gg m_e / M_i$. This has no impact on the MHD kinematic equation, since ∇p is negligible to that anyway. But this serves as a check that electron inertia is negligible even when ion inertia is very strong compared to pressure corrections, which is why a limit on β makes sense (see below).
4. That ∇p is negligible compared to the Lorentz force term is implied as well by (3), since $\mathbf{J} \times \mathbf{B} / c$ is comparable to ∇p and $J \ll n e v_A$. If $\beta \ll 1$, then this is even more so.

3.6.3 A comment on the plasma beta

The *plasma beta*, defined by

$$\frac{\text{gas pressure}}{\text{magnetic pressure}} = \frac{8\pi p}{B^2} \equiv \beta ,$$

is very important in plasma fluid dynamics. It gives the relative importance of the gas pressure to the magnetic field as the restoring force to any disturbance. If $\beta \gg 1$, then the magnetic force has a negligible effect on the dynamics, but the magnetic field is still advected by the flow. This is called *MHD kinematics*. It is important in studies of the generation of a magnetic field by a conducting fluid undergoing motion forced by other means, such as convection in a gravitational field. The generation of a large-scale magnetic field by convective turbulence is called the *dynamo effect*, and it is thought to be the most likely scenario for the origin of planetary and stellar magnetic fields. (A note: Some treatments define β with 4π instead of 8π .)

In the opposite limit, $\beta \ll 1$, the gas pressure drops out of the MHD force equation to lowest order in β , but it remains as a slight correction to the geometry of any equilibrium. For such a *low-beta* plasma, the gas pressure is still important because it can break the tendency of magnetic pressure and tension to cancel. A consequence is that a finite gas pressure prevents the establishment of a force-free equilibrium, to which the plasma tends to relax in many important configurations, even in a situation with a nonzero current density. In addition, the gas pressure can cause instability in an equilibrium which would otherwise be stable to MHD perturbations (*MHD-stable*). In the tokamak configuration for a fusion plasma, for example, this fact is responsible for limiting the plasma beta to quite low values.

The tendency of magnetic pressure and tension to cancel is ultimately the reason that assumptions (3) and (5) have to be separately checked.

3.7 Parallel dynamics and resistivity, or relaxing the ideal assumption

One limit which was not examined above is the role of parallel dynamics: The general case of flows parallel to the magnetic field. If we compare the drift velocity implied by \mathbf{J} not to v_A but to v_s , we get a different limit: $\beta \gg \rho_i/a$ (the reader is invited to check this). The drop of one power of the drift parameter places a rather strict limit on how low the beta can be allowed to go. In solar plasmas the drift parameter is really very small, so this point can be ignored. In fusion plasmas, especially tokamak and stellarator plasmas in which β is quite limited by effects arising from the gas pressure, the limit can be violated, and it is nearly always violated in the boundary regions of the plasma. It is important to realise that a perturbation compressing the gas purely parallel to the magnetic field involves no disturbance of the field. A small disturbance of this type merely leads to longitudinal sound waves propagating parallel to the field lines. Such parallel effects may be neglected for any disturbance of the general equilibrium or violent instability, since these evolve on Alfvén time scales. However, transport

phenomena may give rise to force imbalances along the field lines, and these would relax on the parallel sound transit time scale. If phenomena on this scale are of global importance, then the kinematic MHD equation will be affected since although these effects are parallel, the forces they cause may have a nonzero perpendicular curl. When this is the case, studies of the consequences must treat the electron and ion fluids separately, on an equal footing. Further inquiry along these lines is beyond the scope of this introduction to MHD.

A simpler effect which breaks the ideal MHD constraints but still allows treatment of the system as MHD and as a single fluid is electrical resistivity. Resistivity means an exchange of momentum between electrons and ions as their respective fluids drift past each other. The details of the electron-ion collision process are complex, due to the fact that the angle through which an electron is scattered upon close approach to an ion depends strongly on the relative velocity. Nevertheless, by the inclusion of a simple momentum exchange term loosely based on a collision frequency, a qualitative picture of the most important consequence can be given: Magnetic field line diffusion, or how those flux tubes tangled together at arbitrarily small scale eventually relax. MHD with electrical resistivity is called *resistive MHD*.

Consider a plasma with a single ion species of charge e , and allow the ions and electrons to exchange momentum. The continuity and energy equations are unaffected, but the electrons lose momentum to the ions on a time scale given by a collision frequency ν_{ei}

$$0 = -\nabla p_e - n_e e \left(\mathbf{E} + \frac{\mathbf{v}_e}{c} \times \mathbf{B} \right) - n_e m_e \nu_{ei} (\mathbf{v}_e - \mathbf{v}_i),$$

in which inertia is still neglected. To conserve momentum, the same term appears with opposite sign in the ion momentum equation

$$n_i M_i \frac{d\mathbf{v}_i}{dt} = -\nabla p_i + n_i e \left(\mathbf{E} + \frac{\mathbf{v}_i}{c} \times \mathbf{B} \right) + n_e m_e \nu_{ei} (\mathbf{v}_e - \mathbf{v}_i).$$

We now add these two to obtain the resistive MHD force equation

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\nabla p + \frac{1}{c} \mathbf{J} \times \mathbf{B}.$$

Note that since the MHD force equation is one for *total* momentum, the addition of resistivity does not alter its form.

Now consider the MHD kinematics. Solving the electron momentum equation for \mathbf{E} , obtain

$$\mathbf{E} + \frac{\mathbf{v}_e}{c} \times \mathbf{B} = -\frac{\nabla p_e}{n_e e} - \frac{m_e \nu_{ei}}{n_e e^2} n_e e (\mathbf{v}_e - \mathbf{v}_i).$$

As before, we neglect the pressure force and assume that \mathbf{v}_e is \mathbf{v} . Note as well that $n_e e (\mathbf{v}_e - \mathbf{v}_i) = \mathbf{J}$. (The astute reader will note the slight complications that arise when the charge state of the ions differs from $+1$; let this be left as an exercise.)

The electron force balance, modified by resistivity, now reads

$$\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} = \eta \mathbf{J},$$

where $\eta = m_e v_{ei} / n_e e^2$ is the resistivity. Inserting this relation into the MHD kinematic equation, we find

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times \left(\frac{\eta c^2}{4\pi} \nabla \times \mathbf{B} \right).$$

Assuming for the moment that η is homogeneous, the double-curl operation may be reduced, re-casting the kinematic equation as

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{\eta c^2}{4\pi} \nabla^2 \mathbf{B}.$$

This has the form of a diffusion, which is the role that resistivity plays. The eventual fate of tangled magnetic flux tubes is now apparent. Supposing that

$$\frac{v_A}{a} \gg \frac{\eta c^2}{4\pi a^2},$$

or in terms of the *Lundquist number* S and *resistive decay time* τ_R

$$S \equiv \frac{\tau_A}{\tau_R} \gg 1, \quad \tau_R = \frac{a^2}{\eta c^2 / 4\pi},$$

the system evolves according to ideal MHD on large scales. When flux tubes are tangled on ever-smaller scales, however, some scale, λ , is reached at which $S(a \rightarrow \lambda) \sim 1$. At that scale the magnetic field lines lose their identity through diffusion and *re-connection*, and the tangles are smoothed out.

3.8 Towards multi-fluid MHD

Further relaxing the assumptions of MHD brings one eventually to the necessity of treating all the constituent fluids on an equal footing, especially when further *collisional* phenomena become important. This is in any case beyond the goal of this study, which is to introduce the ideas of MHD. Interested readers will no doubt find it stimulating both to explore MHD phenomena further, and to consider in more depth the different effects one finds in two- or more-fluid dynamics. For this purpose a set of references is provided.

References

A good MHD text, both for an introduction to MHD and for further study of the basic phenomena (it is the very best book containing reconnection and MHD turbulence): *Nonlinear Magnetohydrodynamics*, by D. Biskamp (Cambridge, 1993). It cites all of the references below.

A perhaps more introductory text which is less like a review is *Ideal Magnetohydrodynamics*, by J.P. Freidberg (Plenum Press, New York, 1987).

The standard text for astrophysical MHD applications is *Cosmical Magnetic Fields*, by E.N. Parker (Clarendon Press, Oxford, 1979). Convection of magnetic flux tubes and dynamo theory form the centerpiece of this book.

A tokamak-oriented text which includes the basic MHD problems as well as the more complicated phenomena one encounters is *Theory of Tokamak Plasmas*, by R.B. White (North Holland, Amsterdam, 1989).

The best source on two-fluid dynamics is still the review article by S.I. Braginskii, in *Reviews of Plasma Physics*, M.A. Leontovich, ed. (Consultants Bureau, New York, 1965), Vol. 1, p. 205. It contains a systematic derivation of the equations from kinetic theory as well as a clear introduction to the effects represented by the individual terms (and why they appear). One obtains a good understanding of the dependence of both the two-fluid and MHD models on the kinetic theory of plasmas, and their range of validity.

Chapter 4

Astrophysical plasmas

Harald Lesch

4.1 Introduction

99% of the luminous matter in the Universe is in plasma state, i.e. it consists of ions and electrons with some degree of ionization larger than 10^{-6} . This simple statement already proves the importance of plasma physics for astronomy. Astrophysical plasma systems appear everywhere in space: There are extragalactic jets made of relativistic particles and magnetic fields, flowing with almost the velocity of light, originating in the centres of quasars, which are 10^5 lightyears long, with pinch like magnetic field configurations. There are disk galaxies are magnetized spirals which rotate with a few hundred km per second of size of 10^5 lightyears. Supernovae explode with velocities of a few 10^4 km/sec, leaving a tiny nucleus of about 1.5 solar masses of 10 km in form of a strongly magnetized rapidly rotating neutron star. The interstellar medium is a highly filamented magnetized medium in which shear motions compress, stretch and twist the magnetic field within a highly conducting collisionless plasma. The intergalactic medium is a very thin (one particle per m^3), very hot plasma ($T \simeq 10^7\text{--}10^8$ K). The sun is surrounded by a corona which is hundred times hotter than its surface. The Earth ionosphere glows from time to time since plasma curtains emit the Aurora. We really live in a plasma universe.

4.2 Typical parameters of astrophysical plasmas

Typical examples for astrophysical plasmas are

- all the stars (about 10^{11} stars per galaxy),
- the interstellar medium in galaxies (about 10^{11} objects),
- the intergalactic medium between galaxies and galaxy clusters.

The particle number density covers an enormous range: From about one particle per cubicmeter in the intergalactic medium up to 10^{30} in the central regions of hydrogen burning solar like stars. The plasma temperature is only a few hundred Kelvin in cold, but still significantly ionized gas clouds in which stars form up to a few 10^8 Kelvin in the X-ray halos of galaxy clusters, emitting thermal bremsstrahlung. Every plasma in the universe is magnetized. The smallest magnetic field strengths of about 10^{-11} T have been deduced from the low-frequency radio emission in the intergalactic medium, which is due to synchrotron emission of relativistic electrons. The strongest magnetic fields at the surface of neutron stars are about 10^8 – 10^{10} T, measured by the cyclotron emission line in the X-ray regime. Neutron stars are only 10 km in size, the intergalactic medium has a typical size of millions of lightyears. One lightyear is about 10^{16} m. The length at which intergalactic magnetic fields are ordered is about a few thousand lightyears. Beside the population of thermal particles, astrophysical plasmas contain ultrarelativistic particles, electrons and nuclei, as well. This population is known as cosmic rays, although it consists of particles. The relativistic electrons radiate via synchrotron emission and if they are in compact systems with intense photon fields via inverse Compton scattering. We will discuss the different radiation processes in the next section. From observations of the electromagnetic spectrum we can estimate the electron energies required to provide the observed radiation. Especially from the most recent X-ray satellites CHANDRA and XMM it became clear that in active galactic nuclei, in extragalactic jets and in supernova remnants the electrons have energies up to 100 TeV. Typically at least a few hundred MeV electron population fills the space between the stars and galaxies. In most galaxies the electron population is energized globally up to 10 GeV.

Concerning the hadronic population our knowledge is not as complete, since these particles (protons and nuclei) do not undergo intense energy losses by radiation processes like the electrons. We know only the energy distribution of the cosmic ray hadronic population which encounters the earth. From cascades induced by cosmic rays hitting the earth atmosphere we can deduce maximum energies of about 10^{20} eV for the protons. Typical energies are in the range of MeV to GeV.

As stated in the introduction the velocity fields of astrophysical plasmas exhibit motions at all velocities. The plasma velocities are determined by gravity and rotation mainly. The interplay of these two forces explains the main part of astrophysical streamings. This is true for accretion disks around black holes and neutron stars, which accelerate plasmas almost to the speed of light. Close to galactic nuclei plasma clouds move with a few 10^3 – 10^4 km/s. In the interstellar medium clouds move with a few 10–100 km/s. Due to stellar explosions there are also plasma flows escaping with 20 000 km/s. In any case the velocity fields contain spatial gradients, i.e. they are always shear motions.

After that brief introduction we can already formulate the central issues of modern plasma astrophysics:

- Where do the magnetic fields come from?
- What is the origin of magnetized, relativistic jets?
- How are the particles accelerated to the ultrahigh energies?

- What is the interplay of acceleration processes and the radiative loss mechanisms?
- How do the astrophysical plasmas respond to the distortions by unsaturated externally driven shear flows?
- How do we have to describe such thin and highly conducting magnetized plasmas?

In that contribution we give an overview about the basic theoretical procedures and constituents necessary to tackle plasma astrophysical problems. Let us start with the source of information about the physical state of astrophysical plasmas the electromagnetic radiation.

4.3 Radiation processes

The electromagnetic radiation is the cosmic newspaper, thus we must know what kind of mechanism may be responsible for the photons which hit our detectors or telescopes. We observe the Universe in practically all bands of the electromagnetic spectrum, starting with the radio regime with which important information can be gained about the magnetic field strength and structure since most of the radio emission in space is of nonthermal origin, i.e. it is synchrotron radiation. Radio waves tell us about relativistic electrons gyrating in magnetic fields. The infrared photons mostly originating in star-forming regions are produced by scattered light from dust particles and by direct emission due to the rotational and vibrational transitions of molecules. IR-radiation is the indicator for star formation. Optical photons are due to stellar energy production. It is still the backbone of modern astrophysics observed by the largest ground based telescopes with mirrors up to 15 m and also in space with the Hubble Space Telescope. UV-photons can be observed only by satellites. They come from very hot stars and indicate young stars, since massive, hot stars which are 10–50 times heavier than the sun have lifetimes of only a few million years, whereas our sun lives ten billion years. The X-ray astronomy developed very rapidly in the last decade and gave deep insights into the hot plasmas in the intergalactic and interstellar medium, as well as about stellar X-ray sources and nonthermal X-ray emission from extragalactic jets. Finally the highest energies are now also available from the MeV-range up to 100 TeV. Here we gain information how particles are accelerated to the highest energies in the Universe. For example, the Crab nebula, the rest of a supernova explosion observed by Chinese astronomers in 1054, this supernova remnant emits gamma rays at energies of about 50 TeV, requiring electrons at least at this energy.

It is the central subject of astrophysics to translate the received electromagnetic radiation into models by using the theory of electromagnetic radiation.

There are in principle two classes of radiation mechanisms: The thermal and nonthermal processes. The latter are mechanisms like synchrotron radiation and inverse Compton scattering, where relativistic electrons emit photons due to the interaction with magnetic fields or external radiation fields. Nonthermal radiation is produced by particles with nonthermal energy distributions, mainly with power law distribution functions or almost monoenergetic distributions.

In the case of thermal radiation the energy distribution of the particles is a Maxwellian which defines a temperature. The particles will either emit lines (atomic transitions) or continuum ra-

diation (free-free emission, i.e. bremsstrahlung). From the blue and/or redshifts of the recombination lines we get knowledge about the plasma velocities and densities. We will not discuss the numberless line emission or absorption mechanisms but rather concentrate on the emission of a fully ionized plasma without recombination lines, etc. and we will also not consider the case of black-body radiation. This leads us to the mechanism of thermal bremsstrahlung, which gives information about the plasma temperature T and the electron density n_e .

4.3.1 Thermal bremsstrahlung

When a charge is accelerated or decelerated it emits electromagnetic radiation, and bremsstrahlung is the radiation emitted in all electromagnetic encounters between the charge and the nuclei of substance through which it passes. Of special importance is the thermal bremsstrahlung, where free electrons with a kinetic energy $1/2 m_e v_e^2$ equal to the thermal energy $k_B T_e$ move in a hydrogen plasma (k_B denotes the Boltzmann constant). The radiated energy is then a direct measure of the plasma temperature and the plasma density, since the energy loss rate of the plasma is

$$\left(\frac{dE}{dt} \right)_{brems} = 1.435 \times 10^{-27} T_e^{1/2} n_e n_i \text{ [erg cm}^{-3} \text{ s}^{-1}]. \quad (4.1)$$

For quasineutral plasmas with $n_e = n_i$, bremsstrahlung measures the electron temperature and electron (ion) density.

Its spectrum is characterized by a flat curve in the optically thin regime followed by an exponential break at $h\nu \simeq k_B T_e$. In the optically thick regime the spectrum rises proportional to ν^2 .

Thermal bremsstrahlung is an important radiation process for very hot gases radiating thermal X-rays (like supernova remnants and gas halos of galaxy clusters). Flat and Optically thick bremsstrahlung is emitted by HII-regions close to the regions of star formation in the radio regime.

4.3.2 Synchrotron radiation

Synchrotron emission is the radiation of a high energy particle gyrating in a magnetic field. It was originally noted in some early betatrons where high energy particles were first accelerated to ultrarelativistic energies. This same mechanism is responsible for the radio emission from the Galaxy, from supernova remnants and extragalactic radio sources. It is also responsible for the nonthermal optical and X-ray emission in the Crab Nebula and for the optical continuum emission of quasars and jets.

For nonrelativistic velocities the complete nature of the radiation is simple and is called cyclotron radiation. The frequency of emission is simply the frequency of gyration in the magnetic fields

$$\omega_g = \frac{zeB}{m_e}. \quad (4.2)$$

A useful figure to remember is that the gyrofrequency of an electron is

$$\nu_g = \frac{eB}{2\pi m_e} = 28 \text{ GHz} B[T]. \quad (4.3)$$

Particles emit only if their pitch angle θ (which is the angle between the particle velocity and the magnetic field line) is not zero. The energy loss rate of an electron is ($\sigma_T = 6.65 \times 10^{-29} \text{ m}^2$ denotes the Thomson cross section and γ is the particles Lorentz factor)

$$-\left(\frac{dE}{dt}\right) = 2\sigma_T c \gamma^2 \frac{B^2}{2\mu_0} \sin^2 \theta. \quad (4.4)$$

This result applies for electrons of a specific pitch angle θ . To get the average loss rate for particles of all pitch angles, we must average over solid angle, i.e. $P(\theta)d\theta = 1/2 \sin \theta d\theta$

$$-\left(\frac{dE}{dt}\right)_{\text{average}} = 2\sigma_T c \gamma^2 \frac{B^2}{2\mu_0} \frac{1}{2} \int_0^\pi \sin^3 \theta d\theta = \frac{4}{3} \sigma_T c \gamma^2 \frac{B^2}{2\mu_0}. \quad (4.5)$$

The typical frequency emitted by an electron is the nonrelativistic gyrofrequency times γ^2

$$\nu_{\text{syn}} \simeq \gamma^2 \nu_g \sin \theta. \quad (4.6)$$

The emitted radiation spectra depend on the energy distribution of the relativistic electrons. An almost monoenergetic (beam-like) distribution of relativistic electrons emit an optically thin synchrotron spectrum

$$I(\nu) \propto \nu^{1/3} \quad (\nu \ll \nu_{\text{syn}}) \quad (4.7)$$

and

$$I(\nu) \propto \exp\left(-\frac{\nu}{\nu_{\text{syn}}}\right) \quad (\nu \gg \nu_{\text{syn}}), \quad (4.8)$$

with a maximum frequency $\nu_{\text{max}} \simeq 0.29 \nu_{\text{syn}}$. Such a spectrum has been detected in several galactic nuclei and especially in the centre of our galaxy (SgrA*) and its immediate surrounding the arc. There the synchrotron emission is concentrated in long (150 light years) and thin (0.5 light years) highly polarized magnetic filaments. The magnetic field strengths in these filaments is estimated to be about 10^{-7} T , with the maximum frequency of about 100 GHz this indicates that some process has produced an electron energy distribution which is almost monoenergetic with a characteristic energy of about 40 GeV (Lesch and Reich 1992). In the central light year of the galactic centre, in SgrA* the monoenergetic electrons are at about several MeV (Lesch et al. 1988, Duschl and Lesch 1994).

Such monoenergetic spectra are rare, more typical are energy distributions of power law form

$$N(E)dE = \kappa E^{-x} dE, \quad (4.9)$$

where $N(E)dE$ refers to the number of particles per unit volume.

Such energy spectra emit optically thin radiation power laws in form of

$$I(\nu) \propto \nu^{(x-1)/2}. \quad (4.10)$$

The important rule is that *if the electron energy spectrum has a power law index x , the spectral index of the synchrotron emission of these electrons is $\alpha = (x - 1)/2$* . Such spectra are found in supernova remnants with $\alpha \simeq 0.5$ indicating an energy distribution of E^{-2} , in galaxies with $\alpha \simeq 0.83$, i.e. $E^{-2.66}$.

Synchrotron radiation is linearly polarized perpendicular to the magnetic field. Its degree of polarization in case of a power law is

$$\Pi = \frac{x + 1}{x + 7/3}, \quad (4.11)$$

giving a maximum linear polarisation of about 72%.

The optically thick synchrotron spectrum does not depend on the energy distribution and has a form of

$$I(\nu) \propto \nu^{5/2}. \quad (4.12)$$

Such optically thick synchrotron spectra are observed in nuclei of quasars which evolve at higher frequencies into a $\nu^{-\alpha}$ optically thin spectrum.

4.3.3 Inverse Compton scattering

Relativistic electrons lose energy when they collide with photons because the photons are scattered to much higher energies. This process is called *inverse Compton scattering* because in the usual Compton scattering interaction the photon gives energy to a particle initially at rest and thereby loses energy in the collision. The situation most often encountered in astrophysical plasmas is the limit in which the energy of the photon in the centre of momentum frame of the collision is much less than the rest mass energy of the electron, i.e. $\gamma h\nu \ll m_e c^2$. The energy loss rate due to inverse Compton scattering is

$$\left(\frac{dE}{dt} \right)_{IC} = \frac{4}{3} \sigma_T c \gamma^2 U_{rad}, \quad (4.13)$$

where U_{rad} denotes the energy density of the external photon field.

Note the great similarity of this result to that for synchrotron radiation all the way down to the factor of $4/3$. The reason for this is that in both cases, the particles being accelerated by the electric field which it sees. A particle does not care about the origin of this electric field. Due to this similarity it is clear what the typical emitted frequency in inverse Compton scattering must be

$$\nu_{IC} \simeq \gamma^2 \nu_0, \quad (4.14)$$

where ν_0 denotes the photon frequency of the external radiation.

In the limit $\gamma h\nu \ll m_e c^2$ inverse Compton scattering is Thomson scattering. If instead the Doppler shifted photon energy $\gamma h\nu$ in the rest frame of the electron is equal to the electrons rest mass energy $m_e c^2$ the cross section is given by the Klein Nishina cross section which is approximately $\sigma_{KN} \propto 1/\gamma \sigma_T$.

IC-process is important for compact objects like active galactic nuclei, in which very intense radiation fields are produced by accelerated plasma streams, for example accretion disks around black holes with masses of a few million solar masses. The disks are heated to temperatures of about 10^5 K, which can be deduced from line emission and thermal continuum radiation. Such disks emit intense radiation in the optical and UV at about 10^{13} – 10^{15} Hz. Some active galactic nuclei are intense emitters of TeV- radiation, i.e. 1 000 GeV. The only mechanism to produce such a high energy radiation without intense pair production (remember the rest mass of an electron plus a positron is about 1 MeV) is with optically thin inverse Compton scattering. One requires electrons with energies of several TeV, their corresponding Lorentz factor is about 10^6 which adds up $\nu_{IC} \simeq \gamma^2 \nu_0 \simeq 10^{25}$ Hz. This scattering works efficiently only for the low energy photons of the external radiation. For UV photons the Klein-Nishina limit holds and the cross section drops with $1/10^6$. since the photon energy in the rest frame of the electrons is comparable to the rest mass energy of the electrons.

Finally we give some interesting figures for the loss time scales of inverse Compton scattering and synchrotron radiation

For the IC-case we have a loss time

$$t_{loss}^{IC} = \frac{5 \times 10^8 \text{ s}}{\gamma U_{rad}}, \quad (4.15)$$

with U_{rad} in erg/cm^3 .

For synchrotron radiation we have

$$t_{loss}^{SYN} = \frac{3 \times 10^7 \text{ s}}{\gamma B^2}, \quad (4.16)$$

with B in 10^{-4} T.

For more information about radiation processes we refer to Rybicki and Lightman.

4.4 Cosmical magnetohydrodynamics

Since astrophysical plasmas are highly conductive the ideal magnetohydrodynamics (MHD) may be an appealing description for such magnetized gases. For large spatial scales this statement is correct, i.e. in cosmical plasmas magnetic fields obey the frozen in condition that the magnetic flux $\Psi = BR^2$ is conserved globally. In terms of Ohm's law an ideal MHD-plasma obeys $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$, i.e. induced electric fields force plasma and magnetic field to corotation and frozen-in flux. Large volumes in space like the interstellar medium in galaxies, the intergalactic medium or extragalactic jets can be almost perfectly described by ideal MHD.

It is not valid locally, i.e. for small spatial scales, in which magnetic boundary layers are build up by external forces like gravity, rotation or explosions. There the magnetic field is sheared, twisted and compressed leading to antiparallel field structures, that is to current sheets. Locally the ideal nondissipative MHD-description cannot be applied to cosmical plasmas. In the

boundary layers often such strong electrical currents flow that collective processes provide a local reduction of the conductivity that dissipation sets in via magnetic reconnection. According to Ohm's law the deviation from idealness is equivalent to a parallel electric field which may accelerate plasma particles to ultrarelativistic energies as it is required by the observations.

Let us start with the ideal MHD description and its consequences for the dynamics of astrophysical plasmas.

4.4.1 The ideal MHD equations

One important equation of ideal MHD describes the time evolution of magnetic fields due induction ($\mathbf{v} \times \mathbf{B}$). It can be derived from the Maxwell equation $\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t$, and Ohm's law $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$, leading to

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) . \quad (4.17)$$

According to this equation it is only the induction which changes the field strengths with time. This equation is synonymous with the frozen-in condition of the magnetic flux.

An equation which has profound implications for the energy transport in ideal MHD systems like astrophysical plasmas is

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} , \quad (4.18)$$

which can be rewritten as

$$\nabla \cdot \mathbf{j} = 0 . \quad (4.19)$$

In ideal MHD-plasmas the displacement-term $1/c^2 \partial \mathbf{E} / \partial t$ can be neglected because the conductivity is assumed to be high that electric fields are always much weaker than magnetic fields, i.e. $E \ll B$ and all velocities are smaller than the velocity of light. The condition that the divergence of the current density vanishes means that the current circuit has to be closed. If for example we distinguish between currents running parallel to the magnetic field \mathbf{j}_{\parallel} and currents perpendicular to the magnetic field lines \mathbf{j}_{\perp} than (4.19) presents an important relation between a generator, a load and the conductive wires

$$\nabla \cdot \mathbf{j}_{\parallel} = -\nabla \cdot \mathbf{j}_{\perp} . \quad (4.20)$$

Eq. (4.20) means that the perpendicular currents are sources for parallel currents and vice versa. If for example in a magnetized plasma forces are present which drive plasma motions perpendicular to the magnetic field, such a situation stores energy in the plasma. This region can be taken as a generator. It is connected with the outside material by magnetic field-aligned currents, which are force free (supposed that the external medium is of smaller density, i.e. the pressure forces outside of the generator are weaker than inside the generator). Force free currents are perfect for energy transport. They induce a filamentary magnetic field structure which stabilizes the plasma flow along the magnetic field and allows energy deposition far away from the generator. Such a scenario depends on the pressure gradients which may stabilize a given magnetic configuration. In the denser generator region pressure gradients will be definitely important.

In the external medium, which may be the corona of a star or the atmosphere of a galaxy or of an accretion disk around a black hole, the pressure may be weak and the pressure gradients small. The strong shear of the magnetic fields in the generator may be transported into that external medium by Alfvén waves, i.e. the magnetic field strengths can be enhanced by the kinetic energy of the plasma in the generator. The weakness of the pressure and the stronger magnetic field will both help to develop force-free filaments which are energetically preferred stages of magnetized plasmas in which currents are driven. Such force-free filaments should be a natural ingredients of any astrophysical plasma with low pressure.

In any case at least partially the stored energy in the filaments will be dissipated by either plasma heating or particle acceleration, both mechanisms should be observable in terms of thermal radiation in heated plasmas (like the solar corona) or in terms of nonthermal radiation where particles have been accelerated (like in solar flares). But the dissipation of the energy is not described by ideal MHD. We will discuss that important subject in the subsequent sections.

Here we list typical examples of such configurations in which an external force agitates a plasma perpendicular to the magnetic field lines, stores energy and transports this stored energy via magnetic field-aligned electric currents, thereby inducing magnetic filaments: Solar Flares – Solar Prominences – Earth Aurora – Interstellar Filaments – Stellar Jets – Extragalactic Jets.

This list is time dependent in that sense that it depends on the actual state of the observational techniques in astrophysics. The typical experience is the higher the spatial resolution of the instruments is the more spatial structures, i.e. inhomogeneities are detected. The more advanced astrophysical instruments are the more plasma physics becomes relevant for the interpretation of the observations.

For astrophysicists the plasma structure and the magnetic field structure is of course important for the understanding of stability and inhomogeneity. However, as mentioned above, the most important subject of plasma astrophysics concerns the electromagnetic radiation which can be expected from dynamical plasma processes in terms of plasma heating and particle acceleration. To investigate such problems we have to extend ideal MHD to resistive MHD, including the effects of a finite electrical conductivity.

4.4.2 The resistive MHD equations

The MHD equations taking into account the role of a finite electrical conductivity σ are quite different from the set of ideal MHD equations in many ways.

First of all for a stationary current density \mathbf{j} Ohm's law is now

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \quad (4.21)$$

The electrical conductivity in its most simplest form is given by

$$\sigma = \frac{n_e e^2 \tau_{ei}}{m_e}, \quad (4.22)$$

where τ_{ei} denotes the collision time $\tau_{ei} \simeq 0.266 \times 10^6 T_e^{3/2} n_e^{-1} (\ln \Lambda)^{-1}$ (with the *Coulomb logarithm* $\ln \Lambda$ which is generally between 5–20 and has a weak dependence on temperature and density). In case of Coulomb collisions the conductivity is $\sigma = 1.53 \times 10^{-2} T^{3/2} n^{-1} (\ln \Lambda)^{-1}$. For temperatures higher than 10^3 K the conductivity is so high that the frozen-in condition is fulfilled. Nevertheless even in case of only Coulomb collisions the conductivity is finite and has to be taken into account if the plasma is known to be dissipative.

The ideal induction equation (17) changes now into

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times (\eta \nabla \times \mathbf{B}), \quad (4.23)$$

where $\eta = 1/(\mu_0 \sigma)$ is the *magnetic diffusivity*, which must not be confused with the electrical resistivity (σ^{-1} which is just the inverse of the electrical conductivity σ). If the magnetic diffusivity is uniform we get the equation usually used as induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}. \quad (4.24)$$

Eq. (4.24) for a vanishing plasma velocity \mathbf{v} describes the pure diffusion of magnetic field lines due to finite resistivity, i.e. finite conductivity. (4.23) with a spatial dependent diffusivity describes localized dissipation of magnetic field energy. Since most astrophysical plasmas are globally ideal and dissipative only on small spatial scales, we will continue with the consequences of (4.23).

What does that equation mean? According to the non-ideal Ohm's law a localized resistivity corresponds to a localized electrical current distribution. Strong currents imply strong gradients in the current induced magnetic fields. As long as only Coulomb collisions are the source for resistivity the field lines move with the plasma, since collisions are rare, i.e. the plasma is collisionless and behaves in an ideal way.

However, if the magnetic field lines are subject to strong plasma motions, the situation changes drastically. The frozen-in field lines may be twisted, stretched, compressed etc., and thereby strong currents are induced. Especially in the case of shear motions (like in the case of differential rotation), or turbulent gas flows, the magnetic energy is heavily dissipated via *magnetic reconnection*.

Reconnection is an intrinsic property of a magnetized plasma with shearing and/or turbulent motions. The encounter of magnetic field components with different polarity corresponds to parallel electric currents, which attract each other. In that sense any kind of plasma motion can trigger reconnection.

If the plasma pressure between opposite fields $\pm \mathbf{B}$ is insufficient to keep the fields apart (e.g. by pushing the different flux systems apart), the plasma squeezed from between them and the two fields approach each other. The field gradient steepens and eventually the current density $c/(4\pi) \nabla \times \mathbf{B}$ becomes so large that there is strong dissipation. The problem of reconnection is to know how the dissipation of the currents with density $j = en_e v_d$ is provided (e is the charge and v_d denotes the drift velocity of the electrons). If the current density is constant, enhanced dissipation is equivalent to the reduction of the electrical conductivity σ .

Thus strong dissipation not only means an enhancement of the current density but also a reduction of the conductivity. Indeed both effects support each other. This can be understood as follows: When the conductivity is high, the magnetic field lines are *frozen into* the plasma. Field lines embedded within a volume element of plasma are carried along by the moving plasma. Any two plasma elements that are threaded by the same lines of force will remain threaded in this way, i.e. two plasma elements that are threaded by different lines of force can never be threaded together. Thus a high conductivity prevents the field lines from merging. Reconnection occurs only in those plasma regions where the electrical conductivity is drastically reduced below its classical Coulomb value. The reduction of the electrical conductivity is provided by plasma instabilities, driven by the induced current. If the plasma is locally unstable, i.e. if the current density exceeds a critical value $j > j_{\text{crit}}$ or $v_d > v_{\text{crit}}$ (e.g. Huba 1985) microscopic instabilities will excite waves which enhance the collision frequency by wave-particle interactions; they lead to *anomalous conductivity*. The effective σ becomes anomalously low, greatly enhancing the dissipation and reconnection of the lines of force. Then the concept of *frozen in* field lines is no longer valid and the plasma moves relative to the field lines. When this occurs, strong electric fields along the reconnection length L are induced. It is the dissipation of the electric field E which transforms the current energy into particle energy.

Schindler et al. (1988, 1991) have shown that in three dimensions the reconnection process is always related to with a magnetic field aligned electric potential $U = - \int E_{\parallel} ds$. The induced electric field is always parallel to that field component which is not directly involved in the reconnection process. This result was used to consider the acceleration of particles in cosmical plasmas and it was concluded that parallel electric fields associated with nonideal plasma flows can play an important role in cosmic acceleration. The actual value of E_{\parallel} depends on the details of the microscopic instability, which is responsible for the deviations of the plasma from the ideal high conductivity state. It has been shown that in the case of the slow reconnection mode first proposed by Parker (e.g. Parker 1979) it is possible to get an estimate of the maximum Lorentz factor particles can attain via parallel electric potentials (Lesch and Reich 1992; Lesch and Birk 1997, 1998; Schopper et al. 1998, 1999).

Starting from the assumption of a stationary Ohmic dissipation in a three-dimensional reconnection sheet with an area $\sim L^2$ and a thickness l , the dissipation surface density in the sheet $l j^2 / \sigma$ is just to devour the influx of magnetic energy $u B_0^2 / 8\pi$. u is the approaching velocity of the field lines. Conservation of fluid requires that the net magnetic field inflow balances the outflow

$$uL = v_A l . \quad (4.25)$$

In terms of the magnetic Reynolds number $R_M = 2Lv_A/\eta$, where $\eta = c^2/(4\pi\sigma)$, one gets (Parker 1979)

$$l = \frac{2L}{\sqrt{R_M}} \quad (4.26)$$

and

$$u = \frac{2v_A}{\sqrt{R_M}} . \quad (4.27)$$

The thickness of a reconnection sheet is also defined by Maxwell's equation $\nabla \times \mathbf{B} = 4\pi/c \mathbf{j}$, which means $l \simeq cB/(4\pi j)$. The accelerating electric field is given by $E \simeq u/c B$. Now the

whole problem is shifted to the microscopic level of description. As mentioned above the particle collisions are not efficient enough to produce a significant magnetic diffusivity (or to decrease the electric conductivity). The collision frequency is too small. Current driven plasma instabilities are involved, whose excited frequency depends on the drift velocity between electrons and ions. We choose in the following the wave mode which has the lowest instability threshold where the drift velocity v_d is equal to the ion thermal velocity $v_{\text{thi}} = (k_B T_i / m_i)^{1/2}$ the *lower hybrid (LH) wave* $\omega_{\text{LH}} \simeq 4.2 \times 10^5 B$. LH waves depend on the magnetic field strength, which means they are also suitable for three-dimensional reconnection where the magnetic field is not zero in the reconnection zone. A fully developed LH-instability results in an effective collision frequency of the order of ω_{LH} (Sotnikov et al. 1978). We note that the central role of LH-waves in reconnection zones has been established for magnetospheric activity (e.g. Shapiro et al. 1994)

Inserting u into the electric field

$$E = \frac{\mathbf{v} \times \mathbf{B}}{c} \sim uB/c \quad (4.28)$$

and l into L, using $v_{\text{coll}} \simeq \omega_{\text{LH}}$ and with

$$\gamma m_e c^2 \simeq eEL \quad (4.29)$$

one obtains the maximum Lorentz factor electrons can achieve in a three-dimensional reconnection zone, where the conductivity is determined via lower-hybrid waves (Lesch 1991)

$$\gamma_{\text{LH}} \simeq 6 \times 10^3 B \simeq 6 \times 10^5 \left[\frac{B}{100 \text{ G}} \right]. \quad (4.30)$$

In the last years we have performed numerical simulations to create via three-dimensional MHD-calculations reconnection regions. We used a self-consistent description of the localized reduced current dependent conductivity and investigated by fully relativistic test-particle simulations within the MHD-structures the efficiency of such dissipative structures in terms of particle acceleration to relativistic energies, including energy losses via synchrotron and inverse Compton radiation (Lesch and Birk 1997, 1998; Schopper et al. 1998, 1999; Birk et al. 1999).

Our results can be summarized within one sentence: *Magnetic reconnection is a very efficient acceleration mechanism.* In reconnection regions due to sheared magnetic fields the radiation losses do not influence the acceleration, simply because the acceleration is much faster than the losses. Depending on the initial injection position particles experience different electric field strength and power laws appear. Depending on the isotropy of the injection spectrum, more less all low-energetic particles which enter the reconnection region are accelerated. In case of an anisotropic injection, the resulting energy distribution function of the electrons will be quasi-monoenergetic since the achievable energy depends on the length of the reconnection zone, i.e. it is a linear accelerator. Since the final energy distribution will show an accumulation of particles either at the maximum energy or at the energy where radiation losses and acceleration exactly cancel, we obtain an energy distribution which exhibits pronounced low and high energy cutoffs, i.e. a relativistic electron beam (REB) (Lesch and Schlickeiser 1987).

The energy given by (4.30) is close to the required energies for TeV-radiation. The particles would not lose their energies via synchrotron radiation in the strong magnetic field simply because they flow along the field lines, i.e. the energy distribution is highly anisotropic (Crusius-Wätzel and Lesch 1998).

In this section we have used conventional reconnection in terms of an enhanced collision frequency via plasma waves, excited by some phase-space instability. In a relativistic plasma the excitation of waves is much more difficult, the plasma is very stable due to the high energy of the particles. That is, if we still have external shear flows which twist and compress the magnetic field lines, thereby enhancing locally the current density the system starts to become *filamentary* (Wiechen et al. 1998; Lesch and Birk 1998). The plasma is now completely collisionless, there are even no plasma fluctuations to reduce the conductivity, i.e. we have no dissipation at all. In such a case, like for example in jets, where the plasma density is very low and only magnetic fields and relativistic particles react to the externally driven differential rotation of the disk, then the system reacts via strong filamentation and the inertia of the particles plays a very important role. We describe *the inertia-driven -reconnection* in the next section.

4.4.3 Inertia-driven-reconnection in AGN

How reacts an ideal collisionless, magnetized plasma which is strongly distorted by externally driven shear flows? The magnetic fields will be twisted more and more until tube like structures appear, in which at same spatial scale the magnetic field gradients will be so strong that magnetic reconnection sets in, which is equivalent to a current sheet. The magnetic field aligned electric field appearing in a reconnection zone is driven by the reduction of the local electrical conductivity. In principle the field strength of such a field cannot exceed the convective electric field, which is responsible for the shear, twist, etc., given by (4.28). The maximal shear velocity is the Alfvénic speed v_A .

Thus we get

$$E_{\parallel} = \zeta \frac{v_A}{c} B, \quad (4.31)$$

where ζ denotes the efficiency of the conversion of the generated electric field energy to particle energy.

The necessary condition for magnetic reconnection to operate is the local violation of the ideal Ohm's law

$$\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} = \mathbf{R}, \quad (4.32)$$

where $\mathbf{R} \neq 0$ is some unspecified nonideal term that gives rise to a nonvanishing electric field component parallel to the magnetic field.

Since the relativistic plasmas in AGN and especially inside the jets from AGN are highly collisionless, i.e. the dissipative term in (4.21)

$\mathbf{j}/\sigma = \eta \mathbf{j}$, where η is the resistivity (inverse of the conductivity) that would otherwise contribute to \mathbf{R} is negligible. Therefore, the question arises concerning the nature of the nonideality and also about the effectiveness ζ in a completely collisionless plasma. High energy

plasmas are very stable against the excitation of plasma waves since the wave excitation condition $v_{drift} > v_{the} \sim c$ is hardly to fulfill for relativistic energies.

If a plasma is completely collisionless only the particle inertia is left to introduce some resistance. With a time dependent current density Ohm's law reads as

$$\frac{m_e}{ne^2} \frac{\partial \mathbf{j}}{\partial t} = \mathbf{E} - \frac{\mathbf{j}}{\sigma_{coll}}, \quad (4.33)$$

where σ_{coll} denotes an assumed collisional conductivity.

From dimensional analysis we see from (4.33) that the inertial term is of order

$$\frac{m_e}{ne^2} \frac{\partial \mathbf{j}}{\partial t} \simeq \frac{m_e}{ne^2} \frac{\mathbf{j}}{\tau}, \quad (4.34)$$

where τ is the characteristic time of the system. Thus the inertial term can be large compared to the collisional term if $\tau \ll \tau_{coll}$, where $\tau_{coll} \simeq v_{coll}^{-1}$.

We can solve (4.34) explicitly (assuming zero initial current)

$$\mathbf{j} = \sigma_{coll} \left[1 - \exp\left(\frac{-\tau}{\tau_{coll}}\right) \right] \mathbf{E}. \quad (4.35)$$

We therefore define an effective conductivity due to the inertia of the particles σ_i , as

$$\sigma_i = \sigma_{coll} \left[1 - \exp\left(\frac{-\tau}{\tau_{coll}}\right) \right]. \quad (4.36)$$

The time τ in (4.36) is regarded as the lifetime of the particle in the system, that is the time during which the electric field can be effective in accelerating the particles. If $\tau \gg \tau_{coll}$ then the inertial conductivity σ_i , approaches the collisional conductivity σ_{coll} . In a system with $\tau \ll \tau_{coll}$ in which the characteristic time scales are much smaller than the collision time, i.e. in thin relativistic plasmas for example, then

$$\sigma_i(\tau) = \frac{\sigma_{coll} \tau}{\tau_{coll}} = \frac{ne^2 \tau}{m_e}, \quad (4.37)$$

and σ_i becomes arbitrarily small for $\tau \rightarrow 0$. Thus the lifetime of the particle in the system replaces τ_{coll} in the conductivity expression leading to this effective conductivity. Consequently the inertial term in Ohm's law, usually ignored in MHD becomes important when the lifetime of the particle in the system is small compared to a mean time between collisions.

The field aligned electric field supported by electron inertia is (e.g. Vasyliunas 1975)

$$E_{\parallel} \simeq \frac{m_e}{ne^2} \frac{\partial j}{\partial t} \simeq \frac{c}{\omega_{pe}^2} \frac{B}{L_{shear} \tau_A} = \frac{\lambda_{skin}^2}{L_{shear}^2} \frac{v_A}{c} B, \quad (4.38)$$

where L_{shear} and λ_{skin} denote the thickness of the current sheets of the filamentary current carrying magnetic flux tubes and the electron skin length, respectively. We now have a measure for the conversion efficiency ζ in (4.31)

$$\zeta \propto \frac{\lambda_{skin}^2}{L_{shear}^2}. \quad (4.39)$$

Since there are no collisions, the only characteristic time scales involve energy losses by radiation. For the central part we have strong inverse Compton losses (Birk et al. 1999) and for regions further away from the center and along the jets it is synchrotron radiation (Lesch and Birk 1998). The latter case allows a satisfying explanation for extended optical synchrotron emission in jets (Meisenheimer 1996) which requires continuous reacceleration along the jet flow until $\gamma \sim 10^6$. The first case gave us first hints toward a self-consistent accretion model for particles in the very centre of an active galactic nucleus towards TeV-energies.

Finally, some general comments may be appropriate. The fact that active galactic nuclei are magnetized plasma containing relativistic and nonrelativistic plasma streams which transport energy from a galactic nucleus into the jets, hot spots and lobes, together with the Ghisellini request for a nondissipative energy transport in the very centre (see introduction) suggests that force-free magnetic structures are very important for the plasma dynamics in AGN. Force-free means that no Lorentz force is exerted onto the plasma since the currents are *field-aligned*, i.e. they flow parallel to magnetic field lines: $\mathbf{j} \times \mathbf{B} = \nabla \times \mathbf{B} \times \mathbf{B} = 0$. This condition can be satisfied in three ways: $\mathbf{B} = 0$, $\nabla \times \mathbf{B} = 0$, i.e. $\mathbf{j} = 0$ or $\nabla \times \mathbf{B} = \phi \mathbf{B}$, where the scalar $\phi = \phi(r)$ in general. Such configurations tend to have a twisted or sheared appearance and represent the lowest state of magnetic energy that a closed system can achieve (Taylor 1986). This has two important consequences. It proves the stability of force-free fields with constant ϕ and shows that in a system in which the magnetic forces are dominant and in which there is a mechanism to dissipate the fluid motion, force-free fields with constant ϕ are the natural final configuration.

The advantages of force-free magnetic field systems are obvious: Force-free structures are globally stable and thus allow an almost unperturbed transport of energy from the generator region up to some volume in which dissipation takes place (only shearing instabilities at the edge of plasma streams with the ambient medium will distort the energy transport) until the jet energy is dissipated in the hot spots. The internal energy overshoot of the plasma in such a flux tube is dissipated via continuous particle acceleration. Since the currents flow parallel to magnetic field lines they will evolve into a filamentary structure simply by inducing local toroidal magnetic fields which isolate one filament from the other. In the context of acceleration it is then important to recognize the fact that the jets are collisionless. A probable shear of current carrying filaments cannot be opposed by some resistivity since the particles do not collide. The current filaments shrink to a size where the particle inertia becomes important, i.e. to filament with thicknesses of the order of c/ω_e . It is within these filaments the kinetic energy of the shear is dissipated via particle acceleration as described in our paper.

To summarize, the proposed highly filamentary structure of the plasma in AGN is a natural configuration of magnetohydrodynamic flows under the influence of external forces. The lowest energy state of force-free magnetic fields allows a simple explanation of the continuous dissipation via particle acceleration in the thin filaments along the jet flow. Since no collisional resistance can be build up in the flows only radiation losses of the relativistic particles can limit the acceleration.

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Chapter 5

Kinetic theory

Emanuele Poli

5.1 Introduction

The most complete description of a plasma composed by N particles would rely on the solution of the $3N$ equations of motion $\mathbf{F}_i = m_i \mathbf{a}_i$, where in the calculation of the force acting on the i -th particle one has to take into account the influence of all the particles in the system. The solution of this formidable set of coupled differential equations for realistic parameters is of course prohibitive from a computational point of view and would, moreover, require the knowledge of the $6N$ initial values of position and velocity of each particle. But perhaps even more important than this is the fact that a complete solution of Newton's equations would simply yield a tremendous amount of unnecessary information, since we will never perform an experiment aiming at determining the *microscopic* behaviour of each particle in the plasma. What we are going to measure are *macroscopic* quantities like density, temperature, currents and so on. To connect these two levels, a statistical approach, in the sense that will be specified in the next section, is required. This is the scope of kinetic theory.

The importance of kinetic theory is twofold. First of all, it constitutes a theoretical foundation for the macroscopic descriptions of a plasma, like fluid theory and magnetohydrodynamics. Moreover, processes whose treatment requires the knowledge of the velocity distribution of the particles in the plasma must be examined in the framework of kinetic theory. In these notes, the first point is not addressed. The focus will be rather on the second aspect. After sketching the derivation of the kinetic equation (see sec. 5.2) and of the Fokker-Planck term (see sec. 5.3), we will consider two of the most known applications of kinetic plasma theory, namely the Landau damping (see sec. 5.4) and the Coulomb collision operator (see sec. 5.5).

In order to be able to fit the content of this lecture in roughly fifteen pages, the standard derivation of the Landau damping, which can easily be found in standard textbooks, has been omitted. A short list of references is given at the end.

5.2 The kinetic equation for a plasma

The fundamental quantity in kinetic theory is the *distribution function*. It expresses the particle density in a six-dimensional phase space $\{\mathbf{r}, \mathbf{v}\}$, in a way which can be easily understood if we first introduce the so-called Klimontovitch distribution function

$$f_K(\mathbf{r}, \mathbf{v}, t) \equiv \sum_{i=1}^N \delta^{(3)}(\mathbf{r} - \mathbf{r}_i(t)) \delta^{(3)}(\mathbf{v} - \mathbf{v}_i(t)) , \quad (5.1)$$

where the sum runs on the position and velocity of the N plasma particles. As stated in the introduction, we do not want to be bound to the knowledge of position and velocity of each particle. Hence, we can introduce a smoother distribution function by averaging f_K over a volume $\Delta\mathbf{r}\Delta\mathbf{v}$ in phase space,

$$f(\mathbf{r}, \mathbf{v}, t) \equiv \frac{1}{\Delta\mathbf{r}\Delta\mathbf{v}} \int_{\Delta\mathbf{r}} d^3\mathbf{r} \int_{\Delta\mathbf{v}} d^3\mathbf{v} f_K = \frac{N_p}{\Delta\mathbf{r}\Delta\mathbf{v}} . \quad (5.2)$$

Here, N_p is the number of particles in the volume $\Delta\mathbf{r}\Delta\mathbf{v}$, which is taken to be large enough to contain a large number of particles (so that the averaging procedure implied by (5.2) is meaningful in a statistical sense); at the same time, this volume should be small compared to the macroscopic scale of the system, in such a way that the coordinates \mathbf{r} and \mathbf{v} can be still be thought as *local*. According to these considerations, we can denote the level of description expressed by the distribution function $f(\mathbf{r}, \mathbf{v}, t)$ as *mesoscopic*. Since the distribution function has the meaning of a phase-space density, its integral over the whole phase space gives the total number of particles in the system, $\int d^3\mathbf{r}d^3\mathbf{v}f = N$. Alternatively, the distribution function can be normalised as $\int d^3\mathbf{r}d^3\mathbf{v}f = 1$, in which case $f\Delta\mathbf{r}\Delta\mathbf{v}$ should be interpreted as the *probability* of finding a particle in the volume $\Delta\mathbf{r}\Delta\mathbf{v}$.

We turn now to the problem of determining an evolution equation for the distribution function. Being a particle density in phase space, f must satisfy a continuity equation, expressing the fact that the total number of particles must be conserved. Such an equation reads

$$\frac{\partial f}{\partial t} + \nabla_{\mathbf{r}, \mathbf{v}} \cdot [(\mathbf{v}, \mathbf{a})f] = 0 ,$$

where $\nabla_{\mathbf{r}, \mathbf{v}} \cdot$ is the divergence in the six-dimensional space. Writing $\mathbf{a} = \mathbf{F}/m$, the previous equation becomes

$$\frac{\partial f}{\partial t} + f \frac{\partial}{\partial \mathbf{r}} \cdot \mathbf{v} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + f \frac{\partial}{\partial \mathbf{v}} \cdot \frac{\mathbf{F}}{m} + \frac{\mathbf{F}}{m} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 . \quad (5.3)$$

At this point, we write the force \mathbf{F} as $\mathbf{F}^{\text{ext}} + \mathbf{F}^{\text{int}}$, i.e. we separate the forces *external* to the volume $\Delta\mathbf{r}\Delta\mathbf{v}$ (those acting equally on each particle in this volume), from the *internal* forces arising from the particle interaction (*collisions*) inside $\Delta\mathbf{r}\Delta\mathbf{v}$ (for these internal forces the point-like nature of the plasma is still important). Let us first neglect the contribution of the internal forces and suppose that, at the each *mesoscopic* point of the system, the Hamiltonian equations of motion $\mathbf{v} = \partial H / \partial \mathbf{p}$ and $\mathbf{F}^{\text{ext}} = \dot{\mathbf{p}} = -\partial H / \partial \mathbf{r}$ (with $\mathbf{p} = m\mathbf{v}$) are satisfied. It is then immediate to see that (5.3) simplifies to

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 \quad (5.4)$$

(the apex ^{ext} will be omitted from now on). This is the collisionless kinetic equation, also called the *Vlasov equation*. Introducing the total time derivative in phase space as $d/dt \equiv \partial/\partial t + \mathbf{v} \cdot \partial/\partial \mathbf{r} + (\mathbf{F}/m) \cdot \partial/\partial \mathbf{v}$, we see immediately that (5.4) can be written simply as $df/dt = 0$. This means that f is constant along trajectories (*characteristics*) satisfying the Hamiltonian equations of motion. Moreover, it conserves entropy.

The influence of the internal forces can be modelled simply by introducing on the right-hand side of (5.4) a collision term, which will be calculated explicitly later (cf. sec. 5.3 and sec. 5.5). The kinetic equation becomes then

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m} \cdot \frac{\partial f}{\partial \mathbf{v}} = \left. \frac{\partial f}{\partial t} \right|_{\text{coll}}. \quad (5.5)$$

In this form, the kinetic equation is also called the *Boltzmann equation*. It exhibits the following important properties, which are reported here without proof. First of all, Boltzmann's H-theorem holds, i.e. the time derivative of the quantity $H \equiv \int f \ln f d^3\mathbf{v}$ is never positive. It can be easily shown that H is (except for a constant term) proportional to $-s$, where s is the entropy density of the system. Moreover, at thermal equilibrium, the distribution function is a Maxwellian, which hence corresponds to the state with maximum entropy. Due to its importance, we report here its explicit form¹

$$f_M = n \left(\frac{m}{2\pi k_B T} \right)^{3/2} \exp(-mv^2/2k_B T), \quad (5.6)$$

where n is the density and T is the temperature. An important quantity connected with the exponent of f_M is the *thermal velocity*

$$v_{\text{th}} \equiv \sqrt{\frac{2k_B T}{m}}. \quad (5.7)$$

Despite their rather compact form, (5.4), (5.5) are by no means easy to solve. This is essentially due to the fact that the forces acting on the system can depend in some complicated way on the distribution function itself. This makes these equations, in general, nonlinear integro-differential equations. An explicit solution of the Vlasov equation is considered in sec. 5.4.

5.3 Markovian processes and the Fokker-Planck equation

In this section, we obtain an equation for the evolution of the distribution function under the influence of the microscopic (internal) forces. In the approach presented here, the effect of these forces is modelled as a stochastic process introducing a probability $\psi(\mathbf{Z}, \Delta\mathbf{Z})$ that the position \mathbf{Z} of a particle in *phase space* changes by an amount $\Delta\mathbf{Z}$ in a time Δt . This probability function does not depend explicitly on the time t , i.e. each transition is independent of the previous history of the particle. Such a process is called *Markovian*. The perhaps best-known example

¹The form reported in (5.6) corresponds to the normalisation $\int d^3\mathbf{r} d^3\mathbf{v} f = N$.

of Markovian process is the Brownian motion. Clearly, the function ψ must be normalised in such a way that its integral over all possible displacements $\Delta\mathbf{Z}$ is equal to one, $\int d(\Delta\mathbf{Z}) \psi = 1$.

The distribution function at a given position \mathbf{Z} and time t is supposed to be related to the distribution function at $\mathbf{Z} - \Delta\mathbf{Z}$ and $t - \Delta t$ by the Chapman-Kolmogorov equation

$$f(\mathbf{Z}, t) = \int d(\Delta\mathbf{Z}) f(\mathbf{Z} - \Delta\mathbf{Z}, t - \Delta t) \psi(\mathbf{Z} - \Delta\mathbf{Z}, \Delta\mathbf{Z}), \quad (5.8)$$

which represents a *sum* over all possible displacements $\Delta\mathbf{Z}$ leading to a given position \mathbf{Z} , each term of the *sum* being weighted with its probability ψ . If the changes $\Delta\mathbf{Z}$ are small compared to the size of the system, $|\Delta\mathbf{Z}| \ll |\mathbf{Z}|$, the right-hand side of (5.8) can be expanded as follows

$$f(\mathbf{Z}, t) = \int d(\Delta\mathbf{Z}) \left[f\psi - \Delta\mathbf{Z} \cdot \frac{\partial}{\partial\mathbf{Z}} (f\psi) - \Delta t \frac{\partial f}{\partial t} \psi + \frac{1}{2} \Delta\mathbf{Z} \Delta\mathbf{Z} : \frac{\partial^2 (f\psi)}{\partial\mathbf{Z} \partial\mathbf{Z}} \right], \quad (5.9)$$

where all quantities are evaluated at the position \mathbf{Z} and the dyadic notation has been used in the last term on the right-hand side of (5.9). In this expansion, it has been taken explicitly into account that ψ does not depend on time. Moreover, the expansion is performed up to the second order in \mathbf{Z} but only to the first order in t . This is consistent with the fact that, in a random walk ansatz, the mean square displacement is found to scale linearly with time.

Let us introduce the average changes in $\Delta\mathbf{Z}$ (*friction coefficient*) and $\Delta\mathbf{Z}\Delta\mathbf{Z}$ (*diffusion coefficient*) during the time Δt as

$$\langle \Delta\mathbf{Z} \rangle \equiv \frac{1}{\Delta t} \int d(\Delta\mathbf{Z}) \Delta\mathbf{Z} \psi(\mathbf{Z}, \Delta\mathbf{Z}), \quad (5.10)$$

$$\langle \Delta\mathbf{Z}\Delta\mathbf{Z} \rangle \equiv \frac{1}{\Delta t} \int d(\Delta\mathbf{Z}) \Delta\mathbf{Z}\Delta\mathbf{Z} \psi(\mathbf{Z}, \Delta\mathbf{Z}). \quad (5.11)$$

With these definitions, dividing both sides of (5.9) by Δt and recalling the normalisation condition for ψ (and that f is not a function of $\Delta\mathbf{Z}$) we obtain finally

$$\frac{\partial f}{\partial t} = - \frac{\partial}{\partial\mathbf{Z}} \cdot (\langle \Delta\mathbf{Z} \rangle f) + \frac{1}{2} \frac{\partial^2}{\partial\mathbf{Z} \partial\mathbf{Z}} : (\langle \Delta\mathbf{Z}\Delta\mathbf{Z} \rangle f). \quad (5.12)$$

This is the Fokker-Planck equation², which we will use in the following section to obtain a physical picture of Landau damping and in sec. 5.5 to calculate an explicit form for the collision operator.

5.4 Landau damping

The basic difference between a gas and a plasma is the fact that the plasma is composed by charged particles, which react to electromagnetic forces. As mentioned above, this makes the solution of the kinetic equation very complicated even in the collisionless limit, because

²This name is also used for the equation resulting from substituting (5.12) into the right-hand side of the Boltzmann equation (5.5).

the electromagnetic fields that determine these forces are functions of the charge and current distributions in the plasma, and hence of the distribution function itself.

As an example, we consider here the propagation of an electrostatic wave in a collisionless electron plasma (the ions will be supposed to give simply a neutralising static background). For these waves (which do not exist in vacuum, but can propagate in a plasma), the oscillating magnetic field is zero and the electric field is parallel to the wave vector, as can be deduced from the Maxwell equation for the curl of the electric field in the electrostatic limit, $\nabla \times \mathbf{E} = 0$, which yields in Fourier space $\mathbf{k} \times \mathbf{E} = 0$. To account for the interaction between this wave and the plasma, we need then the Vlasov equation (5.4), where the force $\mathbf{F} = -e\mathbf{E}$ must be calculated from

$$\nabla \cdot \mathbf{E} = \frac{\rho_{\text{ch}}}{\epsilon_0} = -\frac{e}{\epsilon_0} \int f d^3\mathbf{v}. \quad (5.13)$$

In the last step, the charge density ρ_{ch} has been expressed as an integral over velocity space of the distribution function. If we substitute (5.13) into the Vlasov equation, we see immediately that it becomes a nonlinear equation for f . However, if we suppose that the influence of the wave is just to produce a small perturbation of the background plasma, we can write the distribution function as $f = f_0 + f_1$, with $|f_1/f_0| \ll 1$, where f_1 represents the effect of the wave. Retaining then only the first-order terms in f_1 (i.e. in a *linear* picture), the coupled Vlasov-Maxwell set of equations becomes

$$\begin{aligned} \frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \frac{\partial f_1}{\partial \mathbf{r}} - \frac{e\mathbf{E}}{m} \cdot \frac{\partial f_0}{\partial \mathbf{v}} &= 0, \\ \nabla \cdot \mathbf{E} &= -\frac{e}{\epsilon_0} \int f_1 d^3\mathbf{v}. \end{aligned} \quad (5.14)$$

The solution derived by Landau (1946) for these equations can be found in many textbooks and is not reported here.

An instructive physical interpretation of the Landau damping can be obtained from a direct calculation of change in the velocity of an electron in the presence of an electrostatic wave and is discussed in the remainder of this section. Since the problem is one-dimensional, we simply write the wave field as

$$\mathbf{E} = E_0 \hat{\mathbf{x}} \cos(kx - \omega t),$$

where again the field amplitude is supposed to be small enough that it leads just to a small perturbation of the electron orbit. If this is the case, we can integrate the equation of motion

$$m \frac{dv}{dt} = -eE_0 \cos(kx - \omega t) \quad (5.15)$$

along the unperturbed orbit $x = x_0 + v_0 t$, to obtain

$$\begin{aligned} \Delta v &= -\frac{eE_0}{m} \int_0^t dt' \cos [kx_0 + (kv_0 - \omega)t'] \\ &= -\frac{eE_0}{m} \frac{1}{kv_0 - \omega} (\sin [kx_0 + (kv_0 - \omega)t] - \sin(kx_0)). \end{aligned} \quad (5.16)$$

The resonant denominator appearing in (5.16) indicates that the energy exchange between wave and particle is particularly effective if the velocity of the particle is close to the phase

velocity of the wave, $v_0 \approx \omega/k$, because in this case the particle does not experience a rapidly oscillating field (with a vanishing average effect), but rather a constant one. Whether Δv is then positive or negative depends on the relative phase kx_0 between wave and particle (in other words, it depends on whether the particle travels in phase with a positive or negative electric field).

At this point, we assume that it is possible to treat the wave-particle interaction as a Markovian process. This amounts to say that we suppose that there is some random process (e.g. binary collisions) running in the background that destroys the phase coherence between particles and wave on a typical time scale τ . We can then calculate the average values of Δv and $(\Delta v)^2$ using (5.10) and (5.11), assuming $\Delta t = \tau$ and that the values of the phase between wave and particle are uniformly distributed (i.e., that the probability ψ is constant). The integral over all the possible *displacements* Δv for a given electron is equivalent to an integral over all the initial phases between the wave and the electron. We obtain immediately

$$\langle \Delta v \rangle = \frac{1}{2\pi\tau} \int_0^{2\pi} d(kx_0) \Delta v|_{t=\tau} = 0$$

and, employing the trigonometric relation $\sin \alpha - \sin \beta = 2 \cos[(\alpha + \beta)/2] \sin[(\alpha - \beta)/2]$ in (5.16),

$$\langle (\Delta v)^2 \rangle = \frac{1}{2\pi\tau} \int_0^{2\pi} d(kx_0) (\Delta v)^2|_{t=\tau} = \frac{e^2 E_0^2}{m^2} \frac{2}{(kv_0 - \omega)^2 \tau} \sin^2 \left[\frac{(kv_0 - \omega)\tau}{2} \right]. \quad (5.17)$$

As we see, $\langle (\Delta v)^2 \rangle$ is proportional to E_0^2 . If we want to use (5.12) to determine the change in the distribution function due to the wave-particle interaction, we need for consistency to calculate $\langle \Delta v \rangle$ also to the second order in E_0 . This can be achieved by integrating the equation of motion, (5.15), along the orbit perturbation Δx given by

$$\Delta x = \int_0^t \Delta v(t') dt' = -\frac{eE_0}{m} \frac{1}{kv_0 - \omega} \left[\frac{-\cos[kx_0 + (kv_0 - \omega)t] + \cos(kx_0)}{kv_0 - \omega} - t \sin(kx_0) \right],$$

where Δv has been taken from (5.16). In calculating the second-order expression for Δv we make the further assumption that $k\Delta x(\tau) < 1$, which implies that the electric field in the equation of motion can be expressed as $E \approx (\partial E / \partial x)\Delta x$. The result is then

$$\begin{aligned} \Delta v = -\frac{e}{m} \Delta x \int_0^t dt' \frac{\partial E}{\partial x} &= \frac{e^2 E_0^2}{m^2} \frac{k}{(kv_0 - \omega)^2} \\ &\times \left[\frac{-\cos[kx_0 + (kv_0 - \omega)t] + \cos(kx_0)}{kv_0 - \omega} - t \sin(kx_0) \right] \\ &[\cos[kx_0 + (kv_0 - \omega)t] - \cos(kx_0)], \end{aligned} \quad (5.18)$$

which allows us to obtain finally

$$\langle \Delta v \rangle = \frac{1}{\tau} \frac{e^2 E_0^2}{m^2} \frac{k}{(kv_0 - \omega)^2} \left[\frac{-1 + \cos[(kv_0 - \omega)\tau]}{kv_0 - \omega} + \frac{\tau}{2} \sin[(kv_0 - \omega)\tau] \right]. \quad (5.19)$$

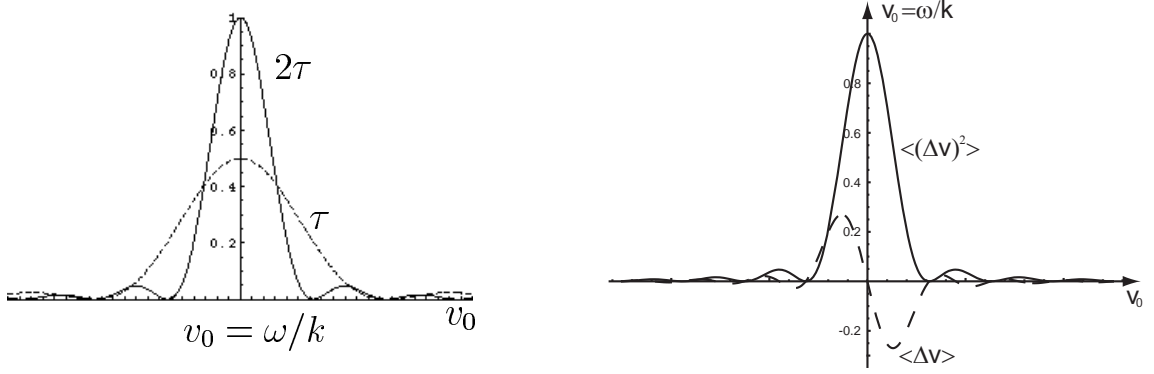


Figure 5.1: Left: The quantity $\langle(\Delta v)^2\rangle$ as a function of v_0 .
Right: The quantities $\langle\Delta v\rangle$ and $\langle(\Delta v)^2\rangle$ as functions of v_0 .

We can now make some comments on these results. First of all, we notice that $\langle(\Delta v)^2\rangle$ is peaked around $v_0 = \omega/k$, as shown in Fig. 5.1 left for two different values of the parameter τ . The fact that the peak value of $\langle(\Delta v)^2\rangle$ increases with τ but its width decreases can be explained as follows. When the interaction time becomes longer, more energy can be exchanged between the electron and the wave, but only if the velocity of the particle is closer to the resonance condition, because otherwise it will be kicked away from the resonance by the wave field. For this reason, the net energy exchange turns out to be independent of τ . In the limit $\tau \rightarrow \infty$ we have³ $\langle(\Delta v)^2\rangle \rightarrow \pi(eE_0/m)^2\delta(kv_0 - \omega)$ i.e. only particles which exactly satisfy the resonance condition contribute to the energy exchange, according to the standard result for Landau damping.

Before turning to the Fokker-Planck equation to determine the variation of the kinetic energy of the electrons interacting with the wave, we point out that the following relation holds

$$\langle\Delta v\rangle = \frac{1}{2} \frac{\partial}{\partial v_0} [\langle(\Delta v)^2\rangle] , \quad (5.20)$$

as can be verified from a direct differentiation of (5.17) employing the trigonometric identity $\sin^2(\alpha/2) = (1 - \cos \alpha)/2$. The dependence of $\langle\Delta v\rangle$ and $\langle(\Delta v)^2\rangle$ on v_0 is shown in Fig. 5.1 right.

The $\langle\Delta v\rangle$ term, which has been computed going up to the second order in the evaluation of wave-particle interaction, has also a simple meaning. Consider first the particles which are (slightly) slower than the wave: Those which have a phase such that they are accelerated by the wave field are brought into resonance and exchange energy for a longer time than those which are decelerated by the wave. On average, hence, slower particles are accelerated. Similarly, particles which are faster than the wave will, on average, lose energy while interacting with the wave.

Using (5.12) we can now determine the energy gained (or lost) by an electron distribution interacting with an electrostatic wave. In the picture given here, the distribution function f determines the distribution of the initial velocities v_0 . Let us now drop the subscript from v_0

³We recall the representation of the δ -function as $\delta(x) = \lim_{a \rightarrow 0} (a/\pi x^2) \sin^2(x/a)$.

for simplicity. The time derivative of the kinetic energy \mathcal{E} of the plasma is

$$\frac{\partial \mathcal{E}}{\partial t} = \int dv \frac{mv^2}{2} \frac{\partial f}{\partial t} = \int dv \left[mv \langle \Delta v \rangle + \frac{1}{2} m \langle (\Delta v)^2 \rangle \right] f, \quad (5.21)$$

where the first term on the right-hand side of (5.12) has been integrated once by parts and the second term twice.

We can now discuss the different mechanisms that contribute to the change in the particle energy.

1. The energy is quadratic in the velocity. Random changes in v lead to a non-vanishing change in the energy $m \langle (\Delta v)^2 \rangle / 2$.
2. The average $\langle \Delta v \rangle$ is odd in v around ω/k . However, $\int v \langle \Delta v \rangle dv < 0$. If f is not a function of v then the first term on the right-hand side of (5.21) will give a negative contribution. It turns out that in this case the negative contribution exactly cancels the positive contribution of $m \langle (\Delta v)^2 \rangle / 2$. This can be immediately seen if we substitute (5.20) into the first term on the right-hand side of (5.21) and then integrate this first term by parts to obtain $\partial \mathcal{E} / \partial t = -(m/2) \int dv \langle (\Delta v)^2 \rangle v \partial f / \partial v$, which is clearly zero if f is independent of v .
3. If f is a function of velocity such that there are more particles for which $v < \omega/k$ than for $v > \omega/k$ (which usually the case), there is a net increase in particle energy, i.e. an absorption of energy from the wave.

Summarising, there will be a net absorption of wave energy if the gradient of the distribution function at $v = \omega/k$ is negative, as we know from the standard picture of Landau damping. That the absorption is proportional to the gradient of the distribution function can also be seen by combining (5.12) with (5.20) to obtain

$$\frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial}{\partial v} \left[\langle (\Delta v)^2 \rangle \frac{\partial f}{\partial v} \right] \quad (5.22)$$

i.e. by writing the Fokker-Planck equation as a diffusion equation in velocity space. If the gradient of the distribution function is zero, no diffusion takes place and f is constant in time. If the gradient of the distribution function is positive, then the particles give energy to the wave. This will lead to an instability in which the wave amplitude is exponentially increasing.

Finally, let us briefly discuss the limits of applicability of our description of Landau damping. If the coherence time τ becomes too long, particles can make multiple bounces in the potential of the wave, i.e. they are trapped in this potential, making our (quasi-)linear ansatz invalid. Indeed, we decided to ignore such a process when we have assumed that $k\Delta x(\tau) < 1$. The absorption in the case of multiple bounces goes down. Another point deserves particular attention. The mechanism of Landau damping does not rely explicitly on collisions. However, there has to be some random process which destroys the coherence between the particles and the wave on a short enough time scale. As mentioned before, in a plasma this role is usually played by collisions. Since the amount of energy absorption is independent of the coherence time (if it is not too long), the collision frequency can determine the coherence time without appearing explicitly in the expression for the amount of damping of the wave.

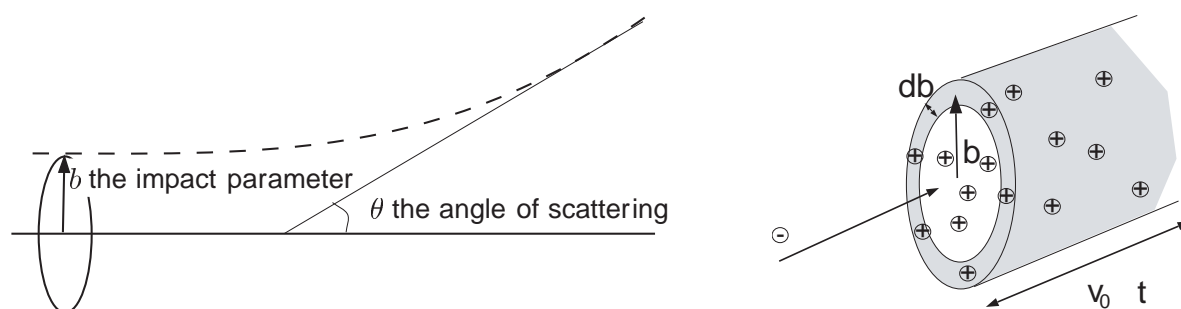


Figure 5.2: Left: Scattering of the particle in the center of mass frame.
Right: Scattering of an electron by ions as the electron travels for a time Δt .

5.5 The Fokker-Planck collision operator

As mentioned in sec. 5.2, the discreteness of the particles reveals itself through the appearance of *internal* forces, i.e. of collisions. Coulomb collisions can be regarded as a Markovian process leading to a change of the particle velocity, if we suppose that each collision event can be regarded as independent of the previous ones. So, we want to employ (5.12) to determine the right-hand side of (5.5). Before calculating the average changes in the particle velocity for a single electron according to (5.10) and (5.11), let us briefly recall some basic features of Coulomb scattering.

A Coulomb interaction between two particles labeled one and two leads to a scattering over an angle θ in the center of mass frame as sketched in Fig. 5.2 left. This angle is determined by the impact parameter b through the equation

$$\tan \frac{\theta}{2} = \frac{b_0}{b}, \quad (5.23)$$

where b_0 is the impact parameter for 90° angle scattering

$$b_0 = \left| \frac{e_1 e_2 (m_1 + m_2)}{4\pi\epsilon_0 m_1 m_2 |\mathbf{g}|^2} \right|, \quad (5.24)$$

where the relative velocity $\mathbf{g} \equiv \mathbf{v}_1 - \mathbf{v}_2$ has been introduced. In the following, we will denote with \mathbf{v}_1 the velocity of particle one before the collision and with \mathbf{v}'_1 its velocity after it.

What we want to calculate is the average value of $\Delta \mathbf{v}_1 \equiv \mathbf{v}'_1 - \mathbf{v}_1$. Introducing the centre-of-mass velocity $\mathbf{v}_{cm} \equiv (m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2) / (m_1 + m_2)$, and recalling that in the absence of external forces (neglected on the time scale of the collision process) $\mathbf{v}_{cm} = \mathbf{v}'_{cm}$, it is easy to show that

$$\Delta \mathbf{v}_1 = \frac{m_2}{m_1 + m_2} \Delta \mathbf{g} = \frac{m_2}{m_1 + m_2} g (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta - 1).$$

In the last step, we have chosen the reference frame in such a way that \mathbf{g} is in the direction of the z -axis, $\mathbf{g} = (0, 0, g)$ and we have employed cylindrical coordinates. As expressed by (5.23), there is a one-to-one correspondence between θ and b , so that an average in θ can be substituted by an average over b . In fact, the probability that particle one is scattered by particle

two between an angle θ and an angle $\theta + d\theta$ during the time Δt is given by the probability of finding particle two inside a cylindrical cell of radius between b and $b + db$ and of length $|\mathbf{v}_1 - \mathbf{v}_2|\Delta t = |\mathbf{g}|\Delta t$ (see Fig. 5.2 right for the case of an electron scattered by an ion population). We can hence write

$$\langle \Delta \mathbf{v}_1 \rangle = \frac{m_2}{m_1 + m_2} \int d\phi \int b db \int d^3 \mathbf{v}_2 |\mathbf{g}| f(\mathbf{v}_2) \Delta \mathbf{g}. \quad (5.25)$$

Before proceeding with the calculation of this integral, it is important to understand whether 90° -scattering processes are likely to occur, or if small-angle diffusion dominates. To this aim, we can calculate the mean value of b_0 over the velocity distribution, for instance in the case of a Maxwellian distribution (5.6). We can then compare this value with the typical distance between particles in the plasma, to know whether they come close enough to experience 90° -degree scattering. In general, the mean value of a quantity $A(\mathbf{v})$ over the distribution function is obtained from

$$\bar{A} \equiv \frac{\int A(\mathbf{v}) f_M(\mathbf{v}) d^3 \mathbf{v}}{\int f_M(\mathbf{v}) d^3 \mathbf{v}}.$$

In order to calculate \bar{b}_0 , according to (5.24) we have to evaluate the average distribution of the initial inverse squared velocities $1/v_0^2$

$$\overline{1/v_0^2} = \int \frac{f_M(v_0)}{v_0^2} d^3 \mathbf{v}_0 = 4\pi \int_0^\infty \frac{f_M(v_0)}{v_0^2} v_0^2 dv_0 = \frac{m}{k_B T},$$

where the integral over the polar angles has been performed in the first step and the formula $\int_0^\infty \exp[-x^2] dx = \pi^{1/2}/2$ has been used in the second. This yields

$$\bar{b}_0 = \frac{e_1 e_2}{4\pi \epsilon_0 k_B T} \equiv \lambda_L,$$

which is the definition of the *Landau length* λ_L . This quantity must be compared with the average distance of the plasma particles. This can be easily done by evaluating the number of particles within a sphere of radius λ_L , taking $e_1 = e_2 = e$. Recalling the definition of the Debye length $\lambda_D^2 = \epsilon_0 k_B T / e^2 n$ one obtains immediately

$$\frac{4\pi}{3} \lambda_L^3 n = \frac{1}{3} \left(\frac{1}{4\pi \lambda_D^3 n} \right)^2 = \frac{1}{27 N_D^2} \ll 1,$$

$N_D \equiv (4\pi/3) \lambda_D^3 n \gg 1$ being the number of particles inside the Debye sphere. This result shows that the particles are usually not close enough in order for large-angle scattering to occur, so that the plasma is dominated by small-angle collisions. In other words, the Coulomb potential is usually much smaller than the mean kinetic energy of the particles.

It follows from these considerations that the integral over b in (5.25) can safely be performed taking b_0 as the lower integration bound, because below this value no scattering process occurs. In the opposite limit, it is known that for distances larger than the Debye length the potential of a charged particle is screened by the others, so that also for $b > \lambda_D$ no scattering takes

place. Moreover, since small-angle collisions dominate, we can approximate $\sin \theta \approx \theta$ and $\cos \theta \approx 1 - \theta^2/2$. (5.25) becomes then

$$\begin{aligned}\langle \Delta \mathbf{v}_1 \rangle &= \frac{m_2}{m_1 + m_2} \int_0^{2\pi} d\phi \int_{b_0}^{\lambda_D} b db \int d^3 \mathbf{v}_2 f(\mathbf{v}_2) g^2 \left(\theta \cos \phi, \theta \sin \phi, -\frac{\theta^2}{2} \right) \\ &= -\frac{2\pi m_2}{m_1 + m_2} \int_{b_0}^{\lambda_D} b db \int d^3 \mathbf{v}_2 g \mathbf{g} \frac{\theta^2}{2}\end{aligned}$$

(remember that \mathbf{g} points by definition in the z -direction). Since from (5.23) it is $\theta^2/2 \simeq 2b_0^2/b^2$, using the definition of b_0 (5.24) we arrive finally at

$$\langle \Delta \mathbf{v}_1 \rangle = -\frac{e_1^2 e_2^2 \ln \Lambda}{4\pi \epsilon_0^2 m_1^2} \frac{m_1 + m_2}{m_2} \int d^3 \mathbf{v}_2 f(\mathbf{v}_2) \frac{\mathbf{g}}{g^3}, \quad (5.26)$$

where we have introduced the *Coulomb logarithm* $\ln \Lambda \equiv \ln(\lambda_D/b_0) \gg 1$.

The same exercise can be performed for $\langle \Delta v_i \Delta v_j \rangle$. First of all, let us write down the elements of the symmetric matrix $\{\Delta g_i \Delta g_j\}$

$$\begin{aligned}\Delta g_1 \Delta g_1 &= g^2 \theta^2 \cos^2 \phi, & \Delta g_2 \Delta g_2 &= g^2 \theta^2 \sin^2 \phi, & \Delta g_3 \Delta g_3 &= g^2 \frac{\theta^4}{4}, \\ \Delta g_1 \Delta g_2 &= g^2 \theta^2 \cos \phi \sin \phi, & \Delta g_1 \Delta g_3 &= -g^2 \frac{\theta^3}{2} \cos \phi, & \Delta g_2 \Delta g_3 &= -g^2 \frac{\theta^3}{2} \sin \phi.\end{aligned}$$

The only non-vanishing elements after the integration over ϕ are the elements “11” and “22” (neglecting terms of order θ^4), for which we have

$$\int_0^{2\pi} d\phi \Delta g_1 \Delta g_1 = \int_0^{2\pi} d\phi \Delta g_2 \Delta g_2 = \pi g^2 \theta^2, \quad (5.27)$$

where again we can approximate $\theta^2 = 4b_0^2/b^2$. This yields

$$\begin{aligned}\langle \Delta v_{1i} \Delta v_{1j} \rangle &= \left(\frac{m_2}{m_1 + m_2} \right)^2 \int_0^{2\pi} d\phi \int_{b_0}^{\lambda_D} b db \int d^3 \mathbf{v}_2 f(\mathbf{v}_2) g \Delta g_i \Delta g_j \\ &= \frac{e_1^2 e_2^2 \ln \Lambda}{4\pi \epsilon_0^2 m_1^2} \int d^3 \mathbf{v}_2 f(\mathbf{v}_2) \frac{g^2 \delta_{ij} - g_i g_j}{g^3},\end{aligned} \quad (5.28)$$

where the last expression derives from the fact that $g_1 = g_2 = 0$, $g_3 = g$. The two following identities, which can be checked employing the definition of \mathbf{g}

$$-\frac{\mathbf{g}}{g^3} = \frac{\partial}{\partial \mathbf{v}_1} \left(\frac{1}{g} \right) \quad \frac{g^2 \delta_{ij} - g_i g_j}{g^3} = \frac{\partial^2 g}{\partial v_{1i} \partial v_{1j}} \quad (5.29)$$

help us put the previous expressions in a more compact form

$$\langle \Delta \mathbf{v}_1 \rangle = \gamma \frac{\partial H}{\partial \mathbf{v}_1}, \quad (5.30)$$

$$\langle \Delta v_{1i} \Delta v_{1j} \rangle = \gamma \frac{\partial^2 G}{\partial v_{1i} \partial v_{1j}}, \quad (5.31)$$

where $\gamma \equiv e_1^2 e_2^2 \ln \Lambda / 4\pi \epsilon_0^2 m_1^2$ and

$$H(\mathbf{v}_1) \equiv \frac{m_1 + m_2}{m_2} \int d^3 \mathbf{v}_2 \frac{f(\mathbf{v}_2)}{|\mathbf{v}_1 - \mathbf{v}_2|} \quad G(\mathbf{v}_1) \equiv \int d^3 \mathbf{v}_2 f(\mathbf{v}_2) |\mathbf{v}_1 - \mathbf{v}_2| \quad (5.32)$$

are the *Rosenbluth potentials*. Substituting (5.30) and (5.31) into the Fokker-Planck equation we obtain the following form for the Coulomb collision term

$$\left. \frac{\partial f}{\partial t} \right|_{\text{coll}} = -\gamma \frac{\partial}{\partial \mathbf{v}_1} \cdot \left(f(\mathbf{v}_1) \frac{\partial H}{\partial \mathbf{v}_1} \right) + \frac{\gamma}{2} \frac{\partial^2}{\partial \mathbf{v}_1 \partial \mathbf{v}_1} : \left(f(\mathbf{v}_1) \frac{\partial^2 G}{\partial \mathbf{v}_1 \partial \mathbf{v}_1} \right). \quad (5.33)$$

The collision operator is local, i.e. all quantities are evaluated at the same position. Moreover, it is nonlinear in the distribution function, which appears once explicitly and once implicitly in the Rosenbluth potentials.

Before applying (5.33) to the case of a *test particle* in a uniform plasma in thermal equilibrium, we observe that the collision operator conserves particle number, momentum and energy. Furthermore, as stated in sec. 5.2, it satisfies Boltzmann's H-theorem.

Consider now the simple case in which a single particle with velocity $\mathbf{V}(t)$ collides with a thermal background, for which the distribution of the velocities \mathbf{v}_2 is Maxwellian

$$f(\mathbf{v}_2) = \frac{n}{\pi^{3/2} v_{\text{th}}^3} e^{-v_2^2/v_{\text{th}}^2} \quad (5.34)$$

with $v_{\text{th}}^2 \equiv 2k_B T/m$, see (5.7). The distribution function for the colliding particle is simply

$$f(\mathbf{v}_1) = \delta^{(3)}[\mathbf{v}_1 - \mathbf{V}(t)]. \quad (5.35)$$

The moments of (5.33) can be calculated to obtain expressions for the rate of change of velocity, energy, etc. as shown below. Let first multiply (5.33) by \mathbf{v}_1 and integrate over $d^3 \mathbf{v}_1$. On integrating twice by parts, the last term on the right-hand side of (5.33) vanishes. The other term can be integrated once by parts. The result is an evolution equation for the velocity of the test particle

$$\frac{\partial \mathbf{V}}{\partial t} = \gamma \frac{\partial H}{\partial \mathbf{V}}. \quad (5.36)$$

This confirms the interpretation of the first term on the right-hand side of the Fokker-Planck equation (5.12) as a dynamical friction decelerating the particle. Using (5.34), the Rosenbluth potential H and its derivative can be computed analytically as

$$H(\mathbf{v}) = \left(\frac{m_1 + m_2}{m_2} \right) \frac{n}{v} \phi \left(\frac{v}{v_{\text{th}}} \right),$$

where the index one has been dropped for simplicity and

$$\phi(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x dt e^{-t^2} \quad (5.37)$$

is the error function. Noting that H is in our case an isotropic function, we can write (5.36) as

$$\frac{\partial \mathbf{V}}{\partial t} = \gamma \frac{\partial H(\mathbf{V})}{\partial \mathbf{V}} \equiv -\mathbf{v}_f(\mathbf{V}) \mathbf{V}, \quad (5.38)$$

which defines the frictional coefficient ν_f . Computing explicitly the derivative of H we have

$$\nu_f = \frac{2\gamma}{V} \left(\frac{m_1 + m_2}{m_2} \right) \frac{n}{v_{th}^2} \psi \left(\frac{V}{v_{th}} \right), \quad (5.39)$$

where

$$\psi(x) \equiv \frac{\phi(x) - x\phi'(x)}{2x^2} \quad (5.40)$$

is the Chandrasekhar function. The functions ϕ and ψ have the following limits

$$\begin{aligned} \lim_{x \rightarrow 0} \phi(x) &= \frac{2x}{\sqrt{\pi}}, & \lim_{x \rightarrow \infty} \phi(x) &= 1, \\ \lim_{x \rightarrow 0} \psi(x) &= \frac{2x}{3\sqrt{\pi}}, & \lim_{x \rightarrow \infty} \psi(x) &= \frac{1}{2x^2} \end{aligned}$$

and they are constants of order one for $x = 1$.

From (5.39) we see that the frictional coefficient is proportional to density and inversely proportional to temperature. For very fast test particles, $V/v_{th} \rightarrow \infty$, ν_f is proportional to $1/V^3$, while it is independent of V at low speeds, $V/v_{th} \rightarrow 0$. Eq. (5.38) then implies that the frictional force increases with V at small speeds, but decreases as V increases in the opposite limit. This means that a current in a plasma is usually carried by fast electrons in the tail of the distribution function. Another interesting consequence is the phenomenon of the so-called *runaway electrons*, for which the friction becomes too small to balance a constant accelerating force, like for instance a constant electric field.

As a second example, we multiply (5.33) by $v_{1\perp}^2$ and integrate over d^3v_1 , assuming again that the distribution functions for the colliding particles are given by (5.34) and (5.35). Here, $v_{1\perp}^2$ is the sum of the squares of the components of v_1 perpendicular to $V(0)$. In this case, only the second term on the right-hand side of (5.33) is found to contribute and we obtain

$$\frac{\partial V_{\perp}^2}{\partial t} = \frac{2\gamma}{V} \frac{\partial G}{\partial V}. \quad (5.41)$$

For a Maxwellian plasma, the Rosenbluth potential G and its derivative can again be computed analytically. A *collision frequency* ν can be defined by imposing the right-hand side of (5.41) to equal to νV^2 , with the result

$$\nu = \frac{2\gamma}{V^3} n \left[\phi \left(\frac{V}{v_{th}} \right) - \psi \left(\frac{V}{v_{th}} \right) \right]. \quad (5.42)$$

In order to evaluate the collision frequency, (5.42), we will assume that a test electron travels at a speed $V = v_{th,e}$ and a test ion at $V = v_{th,i}$.

For the case of electron-electron and ion-ion scattering we take then $\phi(1) - \psi(1) \approx 1$ and recalling the definition of γ we get (taking the ion charge equal to absolute value of the electron charge)

$$\begin{aligned} \nu_{ee} &= \frac{ne^4 \ln \Lambda}{2\pi \epsilon_0^2 m_e^2 v_{th,e}^3}, \\ \nu_{ii} &= \frac{ne^4 \ln \Lambda}{2\pi \epsilon_0^2 m_i^2 v_{th,i}^3}. \end{aligned}$$

For the case of an electron scattered by ions and of an ion scattered by electrons we have to take in ϕ and ψ the limit for $x \rightarrow \infty$ and $x \rightarrow 0$, respectively

$$\begin{aligned} \nu_{ei} &= \frac{ne^4 \ln \Lambda}{2\pi\epsilon_0^2 m_e^2 v_{th,e}^3} \left[1 - \frac{1}{2(v_{th,e}/v_{th,i})^2} \right] \propto \frac{1}{m_e^2 v_{th,e}^3}, \\ \nu_{ie} &= \frac{ne^4 \ln \Lambda}{2\pi\epsilon_0^2 m_i^2 v_{th,i}^3} \frac{4}{3\sqrt{\pi}} \frac{v_{th,i}}{v_{th,e}} \propto \frac{1}{m_i^2 v_{th,e} v_{th,i}^2}. \end{aligned}$$

For nearly equal electron and ion temperatures, $T_e \approx T_i$, we have $v_{th,i} \approx v_{th,e}(m_e/m_i)^{1/2}$. The resulting ordering of these collision frequencies is then

$$\nu_{ee} : \nu_{ii} : \nu_{ei} : \nu_{ie} = 1 : \left(\frac{m_e}{m_i}\right)^{1/2} : 1 : \left(\frac{m_e}{m_i}\right). \quad (5.43)$$

The ions have a smaller (by a factor $(m_e/m_i)^{1/2}$) thermal velocity need more time than the electrons to meet and be deflected by each other. An even longer time is needed for a heavy ion to be deflected by the much lighter electrons.

As a last application of (5.33), we derive the so-called energy exchange times, which are related to the typical time scale required for a particle population to relax to a Maxwellian. Indicating by \mathcal{E} the kinetic energy of the test particle, we can write $\Delta\mathcal{E} = m[v_1^2 - V(0)^2]/2$. Following again the procedure outlined before, i.e. multiplying both sides of (5.33) by $(\Delta\mathcal{E})^2$ and integrating over v_1 , it is possible to derive an equation for the evolution of $\Delta\mathcal{E}$ in the form

$$\frac{\partial(\Delta\mathcal{E})^2}{\partial t} = \nu^E \left(\frac{mV^2}{2}\right)^2,$$

where the energy exchange frequency defined in the previous equation is found to be

$$\nu^E = \frac{8\gamma}{V^3} n \psi\left(\frac{V}{v_{th}}\right). \quad (5.44)$$

Assuming again v_1 of the order of the thermal speed, we first see that $\nu_{ee}^E \sim \nu_{ee}$ and $\nu_{ii}^E \sim \nu_{ii}$. Moreover, in analogy with (5.43), we can write an ordering of these frequencies for different species

$$\nu_{ee}^E : \nu_{ii}^E : \nu_{ei}^E (\sim \nu_{ie}^E) = 1 : \left(\frac{m_e}{m_i}\right)^{1/2} : \left(\frac{m_e}{m_i}\right). \quad (5.45)$$

In other words, thermal equilibration is reached first by electrons (on the electron-electron collision time scale), then by ions (on the ion-ion collision time), whereas equilibration between these two species takes place after a time which is longer than the electron thermalisation time by a factor m_i/m_e .

Acknowledgments

I wish to thank Dr. R. Bilato and Dr. A. G. Peeters for making available the scripts of their previous lectures on this subject.

References

Most of the material of this lecture is exhaustively covered in Chapters 7 and 8 of T.J.M. Boyd and J.J. Sanderson, *The Physics of Plasmas*, Cambridge University Press, and for Landau damping in Chapter 8 of T.H. Stix, *Waves in Plasmas*, AIP (1992). The physics of Coulomb scattering is described in H. Goldstein, *Classical Mechanics* (2nd edition), Addison-Wesley, 1980.

For a more advanced reading on the derivation of the collision operator the review article of Trubnikov in *Reviews of Plasma Physics*, Ed. M.A. Leontovich, Consultants Bureau, NY, 1965, Vol. 1 is still an undying masterpiece.

For insatiable readers, we recommend the seminal (original) papers of Landau on Coulomb interactions in Plasmas, *Zh. Eksper. Theor. Fiz.* **t**, p. 203 (1937), and on Landau Damping, *J. Phys (Moscow)* **10**, p. 25 (1946): They are examples of rare clarity and rigorous logic.

Chapter 6

Plasma heating

Dirk Hartmann

6.1 Introduction

In a fusion reactor that exploits the reaction $D+T \rightarrow {}^4\text{He} + n + 17.6\text{MeV}$ the plasma temperature is sustained by the heating done by the created and confined α -particles, i.e. ${}^4\text{He}$. The cross-section of this reaction is temperature dependent, in particular, it almost vanishes at low temperatures. Thus a temperature of about 20 keV is required for a fusion reactor to ignite. This temperature is much higher than what occurs during plasma start-up in tokamaks and stellarators. Therefore some additional heating is required to raise the plasma temperature until the fusion reaction commences to a sufficient extent and supplies the necessary heating.

Heating of the plasma means that the kinetic energy of the electrons and ions is increased. The heating schemes employed usually do not heat the different plasma species by the same amount. Rather only a fraction of the plasma particles is given kicks in velocity. Furthermore, this velocity increase is not isotropic. Subsequent collisions between the heated fraction and the bulk plasma, however, lead to randomization and equipartition of the additional energy. Generally, this energy causes enhanced radiation losses and particle losses to the walls. Eventually, the plasma reaches a new steady-state where the energy input per time balances the energy losses. This power flow is schematically shown in Fig. 6.1. There the conversion efficiency of the electric power into heating power (up to 50% in some cases) and the loss in the process of coupling the heating power to the plasma have been included.

Different heating methods have been developed and are successfully applied to plasmas:

- Ohmic heating (OH) assumes a singular role. It is the Joule dissipation of the toroidal plasma current that is driven in a tokamak. By itself this heating does not suffice to drive the plasma to reactor relevant temperatures. It is not usable for stellarators since it would unduly modify the magnetic field structure.
- Neutral beam injection heating (NBI) consists of injecting a beam of neutral fuel atoms at high energy that can cross the magnetic field lines into the plasma. By prudent choice

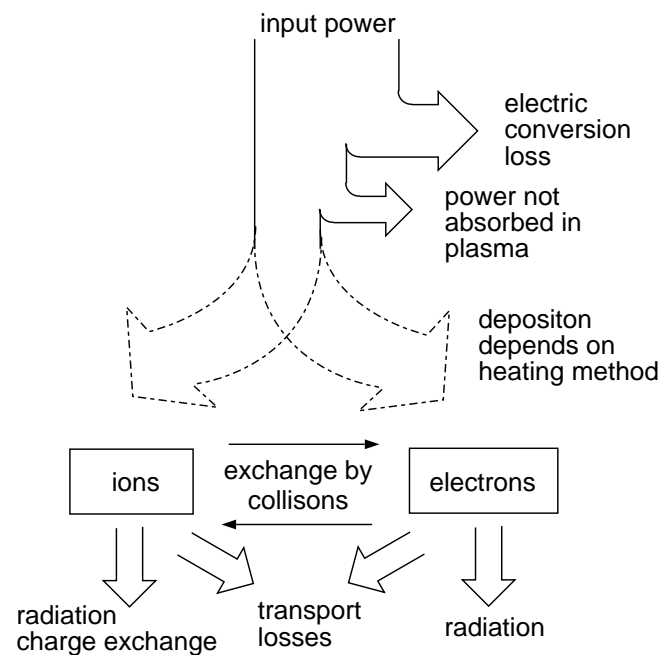


Figure 6.1: Principle of power flow for plasma heating.

of beam energy and plasma density one can achieve that the beam becomes ionized and therefore confined in the plasma center. Through collisions the beam ions transfer their energy to the bulk plasma.

- Radiofrequency heating (RF) consists of launching high power electromagnetic waves into the plasma. Depending on the choice of frequency, plasma density and composition different constituents of the plasma can be heated. RF heating is predominantly applied in the range of low frequency Alfvén waves (1–10 MHz), ion cyclotron frequencies (30–100 MHz), lower hybrid frequencies (1–10 GHz) and electron cyclotron frequencies (50–150 GHz).
- α -particle heating, finally, sustains the plasma temperature in a fusion reactor. Initially the α -particles carry 20% of the fusion energy, i.e. 3.5 MeV. But they need to be confined sufficiently long so that they can impart their energy onto the other constituents while thermalizing. Control of the current profile, then solely determined by the bootstrap current, might still require additional ohmic or non-inductive current.

If a neutral beam is injected at an oblique angle or if waves are launched predominantly in one direction with respect to main magnetic field then the current profile is also change due to the heating. This so called *non-inductive current drive* is desirable for tokamaks in order to extend the pulse duration and to access operation regimes of superior confinement which often depend on the current profile. Using non-inductive current drive is one of the advanced tokamak concepts. Current drive might also be necessary for stellarators in order to compensate for the bootstrap current.

6.2 Ohmic heating

A current flows in the plasma if an electric field is applied in the direction of the magnetic field. This current is predominantly carried by electrons since the ions are basically immobile. Due to the finite resistivity of the plasma the current is dissipated by Joule heating.

In the electric field E the electrons are accelerated up to a drift velocity at which the force due to the field is balanced by a frictional force due to collisions with the ions and other electrons. Ohm's law applies and the plasma resistivity is defined by $E = \eta j$, where j is the parallel current density. The plasma resistivity was calculated in an earlier chapter. It decreases with plasma temperature as $T_e^{-3/2}$. The local ohmic power density, p , then is given by $p(r) = \eta(r)j^2(r)$. Usually the current profile is peaked in the center where the electron temperature is highest. The total heating power is calculated by integration to be

$$P = 2\pi R \cdot 2\pi \int_0^a \eta(r)j^2(r)rdr .$$

Even though the resistivity decreases with temperature, this in itself does not rule out the use of ohmic heating to raise the plasma temperature till ignition. The plasma current, however, cannot be arbitrarily increased since it has been found, that magnetohydrodynamic stability requires the q -value at the plasma edge q_a to be greater than two. Thus with $q_a = aB_\phi/RB_\theta > 2$ and $B_\phi = \langle j \rangle \pi a^2 \mu_0 / 2\pi a$ one obtains for the average current density

$$\langle j \rangle < \frac{1}{\mu_0} \frac{B_\theta}{R} .$$

In ASDEX Upgrade this limits the plasma current to 1.6 MA. The central current could be higher, but this would also increase the MHD-activity (sawtooths).

The maximum achievable temperature can be estimated from power balance: $\eta j^2 = 3nT/\tau_E$, where τ_E is the energy confinement time which is dependent on plasma parameters. The empirical knowledge of many experiments and of different operating regimes has been condensed into statistical scaling laws for τ_E . For tokamak discharges it has been found that τ_E does not depend on plasma current, thus it is independent of heating power, whereas for other heating methods in tokamaks and stellarators alike it was found that $\tau_E \propto P_{tot}^{-0.5}$, where P_{tot} is the total auxiliary heating power. The proportionality factor and the details of the scaling depend on confinement regimes of stellarators and tokamaks, but even though the maximum expected temperatures of 3–4 keV are still too small for ignition.

Technically the toroidal electric field is induced in the tokamak by changing the magnetic flux ϕ through the center of the torus. This has been shown illustrated earlier when the tokamak concept was introduced. The torus itself is the single winding of a transformer on its secondary side. The flux through the center is changed by ramping the current on the primary side.

6.3 Neutral beam injection heating

A beam of plasma fuel atoms (H, D, T) at high velocities (≥ 50 keV) is injected into the plasma since only neutrals can cross the magnetic field lines. In the plasma the beam atoms are ionized through ionization by electrons (important only for low electron temperatures of 1 keV), charge exchange (dominant process below 90 keV for deuterium beam), and ionization by ions (dominant process above 90 keV for deuterium beam). In total these different processes lead to an approximately exponential decay of the beam neutral density along its path. In a reactorgrade plasma a 100 keV deuterium beam has a decay length of about 0.3 m. This would entail edge heating. Therefore it is desirable to go to higher injection velocities in order to penetrate deeper into the plasma.

The beam ions are slowed down by collisions with plasma ions and electrons. For typical values of $T_i = T_e = 5$ keV and 100 keV deuterium beam the velocities are ordered as $v_i^h \ll v_i^b \ll v_e^h$. At high beam energies collisions with electrons are the most frequent. This causes the beam to be slowed down without much change of the direction of propagation and causes the electrons to be dragged along. After the beam energy has become smaller than a critical energy E_c given by

$$E_c = 14.8 \frac{A_b}{A_i^{2/3}} T_e ,$$

where A_b, A_i are the charge states of the beam ions and plasma ions respectively, collisions with plasma ions become dominant. These lead to a change in velocity perpendicular to the initial beam velocity. Together, the decrease of the beam energy E_b is given by

$$\frac{dE_b}{dt} = -2 \frac{E_b}{\tau_s} \left[1 + \left(\frac{E_c}{E_b} \right)^{3/2} \right] ,$$

where τ_s is the slowing down time of an ion by an electron with faster velocity. The first term describes the slowing down on electrons and the second term describes the slowing down on ions. The total energy imparted on ions during the slowing down process of a beam with initial energy E_{b0} can be calculated by integrating the second part of the equation. The result is shown in Fig. 6.2 left. Nowadays many experiments have beam energies comparable to the critical energy, thus NBI equally heats ions and electrons. At ASDEX Upgrade the injection energy is 55 keV for deuterium and the critical energy of a 3 keV deuterium plasma is $E_c = 57$ keV.

Injecting the neutral beam perpendicular to the magnetic field lines, toroidal and helical ripple losses become important. In smaller devices there is also the problem that the beam is not completely absorbed by the plasma and impinges on the opposite wall causing sputtering and releasing impurities. Injection the neutral beam at an acute angle to the magnetic field lines is technologically more difficult because of the limited space between magnetic coils, but is essential in order to drive a plasma current.

The basic elements of a neutral beam injection system are shown in 6.2 right. It is necessary that particles are charged that need to be accelerated. These particles, say D^+ , are generated

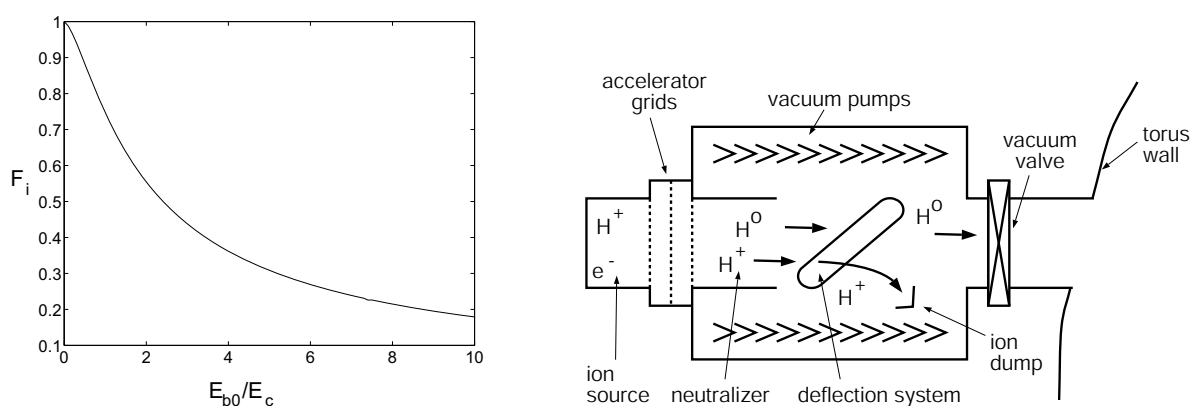


Figure 6.2: Left: Fraction F_i of the initial beam energy E_{b0} going to ions during the slow down process.
Right: Principle of neutral beam injection line.

in an RF or hot tungsten wire discharge. The ions are extracted through a grid and accelerated in an electric field of up to several hundred keV. The shape of the extraction grid allows some steering of the beamlets. By passing the beam through a gas target at low density the beam is partially neutralized through charge exchange processes. A magnet after the neutralizer separates out the unwanted fraction of still unneutralized ions and electrons and steers them onto a dump. The left over neutrals propagate into the plasma. Strong vacuum pumps are required to keep the pressure low in the beam duct. Usually the ion source also creates D_2^+ and D_3^+ . These molecules have a lower velocity corresponding to their higher mass and get ionized closer to the plasma edge. The initial beam ion energies in the plasma then are at E_{b0} , $E_{b0}/2$ and $E_{b0}/3$.

Beams with energies higher than 100 keV are necessary to penetrate deeper into the plasma. The neutralization efficiency of the gas target decreases rapidly with beam energy. It has its maximum of 60% at 19 keV. The neutralization efficiency of negative ions, D^- , depends less on the beam energy because the affinity of the additional electron to the deuterium is weak. However, extracting large beam currents of negative ions out of an ion source is difficult, even though it can be improved by adding cesium to the source plasma. A large fraction of electrons leaves the ion source together with the ions. The goal is to extract 40 A of ion current and accelerate it to 1 MeV. The neutralization efficiency is then about 50% and thus three beam injectors would suffice to couple 50 MW to ITER.

Neutral beam injection has proven very reliable on a number of fusion experiments. Some of its advantages are that the injectors can be tested separately on test-beds, the power deposition can be calculated to good accuracy and injecting neutrals into the center of the plasma is a means of particle refueling. Disadvantages are that the injectors have to be in the immediate vicinity of the torus thus being subjected to high neutron fluxes and tritium contamination. Present research on NBI focuses on the further development of injectors based on negative ion acceleration, new neutralizers based on plasma discharges or lasers, with the goal of high efficiency and continuous operation.

6.4 Radiofrequency heating

RF heating has been used successfully on many devices employing different schemes. In all cases low loss transmission lines exist that make it possible to transmit power from a remote generator to an antenna in the vessel, from which the power is radiated into the plasma. Even though the charged particles are accelerated and decelerated by the electric RF field the dissipation is usually very small because the RF frequency is much higher than the collision frequency of the particles. An exception are resonances that will be discussed at length.

In typical magnetic fusion plasmas the propagation of the RF waves is well described by the cold plasma dispersion relation derived in a preceding chapter, because the plasma pressure is much smaller than the magnetic pressure. The finite size of the plasma bounded by a conducting wall and the shape of the antenna imposes boundary conditions. In particular, the antenna typically fixes the values of k_{\parallel} of the launched waves, i.e. their k_{\parallel} -spectrum, for a given frequency, ω , of the RF generator. This together with the values of the plasma density and magnetic field determine the corresponding values of $n_{\perp} \equiv ck_{\perp}/\omega$ of the plasma waves. The extrema where $n_{\perp} = 0$ (cutoff) or $n_{\perp} = \infty$ (perpendicular wave resonance) are of particular interest. Cutoffs mark regions of the plasma where the wave cannot propagate and resonances mark regions where the phase and group velocity go to zero, thus where the wave energy is bound to pile up unless *something happens*. In the latter region the phase velocity becomes smaller than the thermal velocity of the plasma particles and the cold plasma dispersion relation is no longer sufficient to describe the plasma properly. Then thermal effects need to be included. This can be done in an orderly fashion through a finite Larmor radius (FLR) expansion where the smallness parameter is the ratio of the Larmor radius to the perpendicular wavelength. This increases the order of the dispersion relation and leads to the *appearance of new waves* that were not present in the solution of the cold plasma dispersion relation. At the former resonances coupling to the new waves occurs. The dispersion relation also predicts damping of the waves. This does not come as a surprise since the Kramers-Kronig relations state (based on causality) that resonances in the real part of the dispersion relation are connected with a non-vanishing imaginary part (usually predicting damping).

The damping process of the wave can be understood on a kinematic level. If a charged particle moves in the same direction and with about the same speed as the phase velocity of a wave, the wave electric field in the moving frame of the particle stays constant and the particle is accelerated or decelerated depending on its position with respect to the wave. If there are more particles that are decelerated than there are particles that are accelerated then there is net energy transfer from the wave to these resonant particles. The wave is damped and the average velocity of the particles is increased. For a Maxwellian velocity distribution wave damping can become significant for waves with phase velocities between one and three times the thermal velocity. Since collisions only partially restore the distribution function such an absorption process leads to a local deviation in velocity space.

With the addition of a static magnetic field the situation becomes slightly more complicated. The motion along the magnetic field still separates out and a wave-particle resonance is possible if the wave has an electric field component along the magnetic field and its phase velocity component ω/k_{\parallel} and the particle velocity component v_{\parallel} fulfill the requirement: $v_{\parallel} - \omega/k_{\parallel} \approx 0$. This is called Landau damping.

If the wave has circularly polarized component it can also couple to the gyro-motion of the particles. Resonant interaction requires $v_{\parallel} - \omega/k_{\parallel} - l\Omega/k_{\parallel} \approx 0$ where l is the number of the harmonic. This process is called Landau cyclotron damping. The resonance condition says that energy transfer between the wave and the resonant particles is possible not only if the Doppler-shifted wave frequency matches the gyro-frequency of the particle Ω but also if it is a multiple l of this frequency. If $l \neq 1$ it is necessary that the wave field has a gradient perpendicular to the magnetic field. Consider heating at the first harmonic, i.e. $l = 2$. Let's say that at one point particle and electric field point in the same direction and the particle is being accelerated. After the particle has rotated half a gyro-orbit, the electric field has already completed a full revolution. Thus the electric field points again in the same direction, yet the velocity direction of the particle is reversed and the particle is decelerated during the second half of its gyro-motion. On average no net energy transfer takes place. If, however, the electric field changes magnitude over the plane of the gyro-orbit of the particle, i.e. if the electric field has a gradient, net energy transfer does become possible.

Similar processes exist involving the interaction between the magnetic moment of the gyrating particle and the magnetic field of the wave. They are called transit time magnetic pumping (TTMP).

Concludingly, an RF wave that can propagate into a plasma and is not hindered by cutoffs from accessing a perpendicular wave resonance, can be absorbed there. The absorption coefficient depends on

1. the amount of the wave power that is converted into a wave that has the
2. proper polarization and phase velocity to be resonant with particles of the plasma and
3. the slope of the velocity distribution function of the resonant particles.

In addition, large field amplitudes can lead to a nonlinear enhancement of the wave absorption process.

The changes in the distribution function, in particular if a tail of fast, confined particles with decreased collision frequencies is created, can be useful for current drive if the toroidal symmetry is broken and the waves are predominantly launched into one direction.

In the cold plasma dispersion relation resonances do occur if n_{\perp}^2 goes to infinity which is the case when

$$\epsilon_{\perp} \equiv 1 - \sum_s \frac{\pi_s^2}{\omega^2 - \Omega_s^2} = 0,$$

where π_s is the plasma frequency and Ω_s is the signed cyclotron frequency of species s . This yields resonances at the lower hybrid frequency $\omega_{LH}^2 \equiv \pi_i^2 / (1 + \pi_e^2 / \omega_e^2)$, the upper hybrid frequency $\omega_{UH} \equiv \Omega_e^2 + \pi_e^2$ and at the ion-ion-hybrid frequency (close to the ion cyclotron frequencies). The latter occurs only in a multi species plasma. Surprisingly no resonances occur at the ion cyclotron or the electron cyclotron frequencies Ω_i and Ω_e , respectively. At these frequencies the cyclotron motion of the ions and electrons short-circuits the circularly polarized wave component that rotates in the same direction as the considered species, thus completely

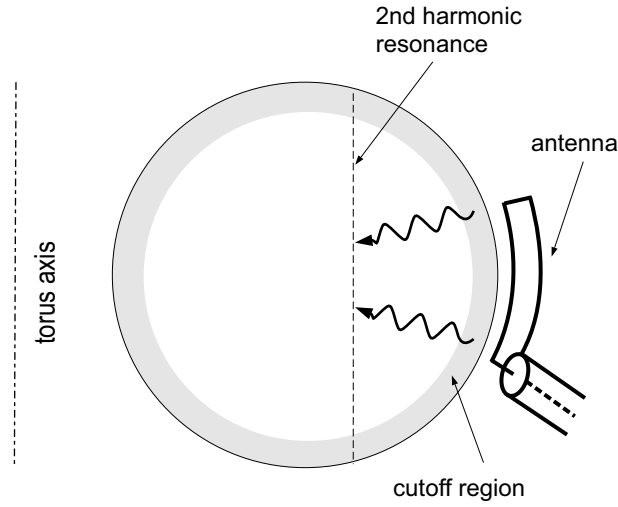


Figure 6.3: Principle of 2nd harmonic ion cyclotron heating.

quelling a possible resonance. As mentioned earlier, FLR contributions or relativistic corrections in the case of electrons need to be included to obtain the cyclotron resonances.

These resonances define the predominant frequency ranges used for RF plasma heating in today's large magnetic confinement experiments. They will be discussed in detail in the following. For completeness also Alfvén waves will be discussed briefly even though none of the large experiments employs them in a heating scheme.

6.4.1 Heating in the ion cyclotron range of frequencies

Near the plasma edge an arrangement of current carrying poloidal conductors, called current straps, excites fast waves in the plasma. They travel almost perpendicular to the magnetic field lines into the plasma center where they can be absorbed at an ion cyclotron resonance or an ion-ion-hybrid resonance. The location of these resonances depends predominantly on the magnetic field strength. In a tokamak, for example, the toroidal magnetic field decreases radially proportional to $1/R$, thus the resonant zones are vertical layers where $\Omega_i(R) = \omega$.

In this frequency range $\omega \approx \Omega_i$ except at very low density the slow wave is evanescent. Then $|\epsilon_{\parallel}| \gg |\epsilon_{\perp}|, |\epsilon_{xy}|$ and the fast wave dispersion relation is given by

$$n_{\perp}^2 = \epsilon_{\perp} - n_{\parallel}^2 - \frac{\epsilon_{xy}^2}{\epsilon_{\perp} - n_{\parallel}^2}.$$

This wave is elliptically polarized with a small component of the electric field in the direction of the magnetic field. There is a cutoff at the plasma edge for densities $< 2 \times 10^{18} \text{ m}^{-3}$. Note that, as said before, for a single species plasma this dispersion relation does not have a resonance even though ϵ_{\perp} goes to infinity at the ion cyclotron resonance. The situation changes if finite Larmor radius effects are taken into account or the plasma is allowed to have more than one ion species.

After FLR contributions have been added to the fast wave dispersion relation one finds that due to the parallel electric field component and for higher electron temperatures there is weak Landau and TTMP damping of the electrons even if no ion resonance is in the plasma. Using this process is called *fast wave heating* and is often used to drive a current in the plasma by phasing the current straps such that the radiated k_{\parallel} -spectrum is skewed. The current is not only an effect of the local changes in the velocity distribution function but also due to the decreasing collision frequency at higher velocities that alleviates sustaining the changes in the distribution function.

With the addition of the FLR contribution to the dielectric tensor elements absorption zones appear in the vicinity of the harmonics of the ion cyclotron layers and Landau cyclotron damping can occur. The resonance width is usually small compared to the minor radius of the plasma. Using this process is called *cyclotron harmonic damping*. To determine the strength of the damping also the polarization of the wave in the vicinity of the resonance has to be included. Even with FLR effects the damping at the fundamental resonance is small due to the unfavorable polarization. It has its maximum at the second harmonic and decreases to higher harmonics. Fig. 6.3 shows the evanescent zone near the plasma edge where $n_{\perp}^2 < 0$ and the resonance layer for 2nd harmonic heating in a tokamak-like cross-section.

At the fundamental the change in the distribution is uniform over all velocities, at the higher harmonics a tail of fast particles is created that becomes more anisotropic with velocity since the particles predominantly gain perpendicular energy.

If the plasma consists of another species of low concentration, say the omnipresent hydrogen in a deuterium plasma, absorption at the hydrogen resonance becomes very strong due to Landau cyclotron damping on the hydrogen. This is possible because in the vicinity of the hydrogen resonance a considerable fraction of the wave power is left circularly polarized because the polarization is determined by the majority deuterium. This scheme is called *minority heating*. Because all the power is absorbed by the minority ions, a strong tail in the distribution function is created. The energies are often higher than the critical energy E_c and the tail relaxes mostly due to collisions with electrons.

For a higher concentration of the second species the beneficial effect on the polarization in the vicinity of the resonance decreases and so does the damping. However, above a critical concentration of a few per cent a new resonance and an associated cutoff occur: The ion-ion-hybrid resonance. The location of cutoffs and resonances are shown schematically in Fig. 6.4 right for the case of hydrogen in deuterium. The resonance calls again for a more detailed analysis using the FLR expansion. It is found that near this resonance a mode-conversion of the fast wave to an ion-Bernstein wave occurs. The ion-Bernstein wave has an electric field component in the direction of the magnetic field and is strong Landau damped on electrons. Now the location of the antenna becomes important. If it is located on the low field side the wave is only weakly damped at the cyclotron resonance. At the cutoff it is partly reflected and it partly tunnels to the resonance layer where it then couples to an IBW. There is an upper limit of about 25% of the power of the fast wave that can mode convert to an IBW on a single pass. If the antenna is located on the high field side direct coupling to the IBW is possible. Except for some stellarator-like experiments RF antennas are usually installed on the low field side for practical reasons.

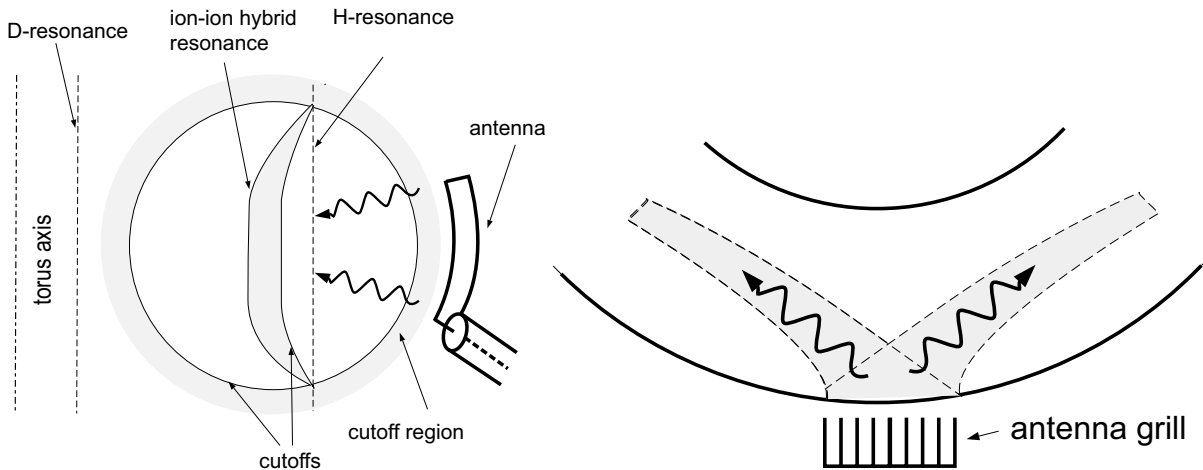


Figure 6.4: Left: Principle of mode-conversion heating.
Right: Wave propagation for lower hybrid heating.

The direct heating of the electrons through this mode-conversion scheme makes it again possible to use it for current drive. However, the location of the mode-conversion layer is critically dependent on the concentration of the minority species thus the repeatability of the experiments very much depends on the gas inventory of the apparatus walls.

Low loss coaxial transmission lines and generators (that were developed for radio communications etc.) are readily available. Usually in antennas the radiated power is only a small fraction of the reactive power. Therefore they are coupled with other reactive elements to form a resonant circuit. This causes large electric fields in the antenna that need to be screened from the plasma.

Good absorption efficiencies, in particular in dense and hot plasmas, have been obtained. The large electric fields in the antenna and the dependence of the plasma coupling on the plasma properties intrinsically link the RF system to the plasma such that separate testing and conditioning are of limited use only.

Present research focuses on a better understanding of the antenna plasma coupling, more detailed investigation of the absorption mechanisms and the development of higher power cw generators.

6.4.2 Lower hybrid heating

A slow wave is launched with a wave guide. In the plasma it travels at a small angle with respect to the magnetic field lines, this causes the wave to circle the torus several times until the center of the plasma is reached. Landau damping along the path often becomes more important than ion damping near the lower hybrid resonance.

In the lower hybrid frequency domain the ordering is $\Omega_i \ll \omega_{LH} \ll \Omega_e$. The dispersion relation of the slow mode is approximately given by

$$n_{\perp}^2 = \epsilon_3(1 - n_{\parallel}^2/\epsilon_1).$$

As in the preceding RF heating schemes the wave has to tunnel through an evanescent region in front of the antenna until a minimal density given by the perpendicular Alfvén resonance is reached. A typical cutoff density is 10^{18} m^{-3} .

Depending on the k_{\parallel} of the launched spectrum the slow wave can couple to the fast wave and since the group velocity of the fast wave is such as to transport the power back to the plasma edge the wave energy then becomes trapped in the plasma periphery. This occurs for n_{\parallel} smaller than a critical density n_c given by

$$n_c^2 = \frac{1}{1 - \omega^2 / (|\Omega_e \Omega_i|)}.$$

For typical parameters this means that the parallel wavelength needs to be shorter than 0.2 m. For slow waves that well fulfill this requirement the ratio of group velocity perpendicular and parallel to the magnetic field is independent of k_{\parallel} , thus the wave energy travels into the plasma along well collimated beams mainly along the toroidal direction. With approach of the resonance the angle to the magnetic field lines decreases further. This behavior is schematically shown in Fig. 6.4 right.

Even though the wave might not yet have reached the resonance its electric field component parallel to the toroidal magnetic field causes Landau damping on the electrons. Since the k_{\parallel} -spectrum of the launched waves is often quite wide (limited by the accessibility condition and the upper end of the antenna spectrum) the resonant velocities extend from the electron thermal velocities up to a substantial fraction of the speed of light. Thus an electron tail is created extending to very high energies. Near the lower hybrid resonance strong ion damping occurs, predominantly through nonlinear stochastic damping on the ion cyclotron motion.

The antenna arrays consist of a phased array of open rectangular waveguides arranged with the RF electric field aligned with the magnetic field. The shape of the antenna face resembles the plasma shape to be able to move the antenna as close as possible to the plasma surface. However, experiments have shown that the antenna creates its own plasma, thus coupling is usually not at problem. The RF generators used are klystrons.

The creation of the long tail in the electron distribution function and the natural occurrence near the plasma edge have sparked interest in lower hybrid current drive. Today's technologies of antenna grill launchers allow the necessary tailoring of the k_{\parallel} -spectrum. Experimentally the current drive efficiencies are the best achieved, particularly at low densities and high magnetic fields.

6.4.3 Electron cyclotron resonance heating

In contrast to the two preceding RF heating methods, at electron cyclotron frequencies no evanescent region between the antenna and the plasma exists. Thus the proper polarization can be chosen at the antenna and the desired plasma wave can be excited with good conversion ratio. Only since the mid 70s generators, called gyrotrons, have become available that operate in the frequency range of the electron cyclotron resonance. In this range only electrons respond to the waves since $\omega \approx \Omega_e \approx \pi_e \gg \Omega_i, \pi_i$.

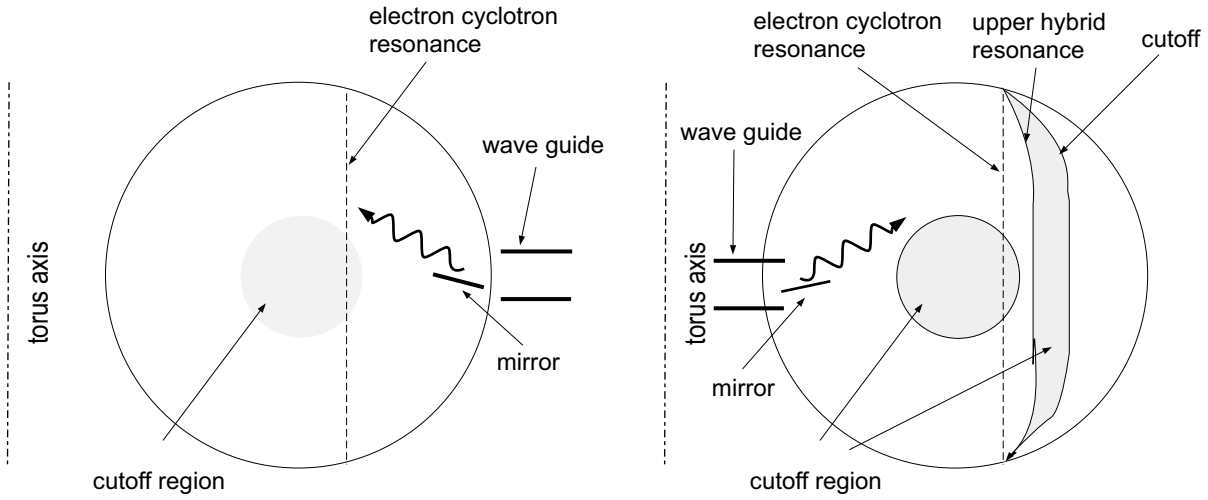


Figure 6.5: Left: Principle of fundamental O-mode heating.
Right: Principle of fundamental X-mode heating from high field side.

Two plasma modes exist: The O-mode and the X-mode. The O-Mode is linearly polarized with $E \parallel B$ and the X-mode is elliptically polarized with $E \perp B$. For perpendicular propagation ($n_{\parallel} = 0$), the dispersion relations are given by

$$\text{O-mode:} \quad n_{\perp}^2 = 1 - \frac{\pi_e^2}{\omega^2}, \quad (6.1)$$

$$\text{X-mode:} \quad n_{\perp}^2 = \left(1 - \frac{\pi_e^2}{\omega^2} - \frac{\Omega_e}{\omega}\right) \left(1 - \frac{\pi_e^2}{\omega^2} + \frac{\Omega_e}{\omega}\right) \left(1 - \frac{\pi_e^2}{\omega^2} + \frac{\Omega_e^2}{\omega^2}\right)^{-1}. \quad (6.2)$$

The O-mode has a cutoff at the electron plasma frequency. The X-mode has a cutoff below the electron plasma frequency and above the upper hybrid frequency. Fig. 6.5 left and Fig. 6.5 right schematically show the location of resonances and cutoffs for O- and X-mode. For O-mode heating the electron cyclotron resonance is only accessible for densities below the cutoff given by $\pi_e^2 = \omega^2$. For X-mode heating, because of the additional cutoff, the electron cyclotron resonance is only accessible from the high field side. For perpendicular propagation the polarization of the X-mode nearly linear. Efficient absorption requires oblique propagation, whereby the X-mode attains an increasingly right-handed polarization. Strong absorption is obtained at the second harmonic. The absorption process is dependent on thermal (FLR) and relativistic corrections, yet the absorption region is usually very narrow and much smaller than the plasma radius, thus making ECRH also useful for localized heating in order to suppress MHD activity.

The limitation to use ECR heating at densities higher than cutoff has recently been overcome by coupling from an O-mode to an X-mode that finally mode-converted to an Electron-Bernstein-wave. This tricky scheme is possible at a particular angle of incidence. The experimental proof is a fine example of the predictive power of the warm plasma dispersion relation as well as how sophisticated ECRH experiments have become.

The RF power in the electron cyclotron range of frequencies is generated using gyrotrons. In these devices an electromagnetic wave is coupled to spiraling electrons in a steady magnetic field. The bunching of the electrons in phase space amplifies the wave. Care is taken to amplify

only one waveguide mode, e.g. the TE_{02} . Output powers of up to 1 000 kW for continuous operation at frequencies of 28–170 GHz have been obtained and research is aiming at 2 MW cw gyrotrons. The RF power is coupled out of the gyrotrons using ceramic and lately diamond windows. Latter have very low absorption and high thermal conduction coefficients allowing edge cooling.

Transmission lines consist of waveguides with different mode converters to change the gyrotron output mode into an TE_{11} -mode for O-mode heating or an HE_{11} for X-mode heating. Transmission of Gaussian beams using quasi-optical lines have recently also become available. Finally inside of the torus the direction of the beam can be changed using movable mirrors, allowing off-axis heating without changing the frequency or magnetic field.

The pinpoint accuracy of ECRH due to its narrow deposition profile and the possibility of ray-tracing calculations in an inhomogeneous plasma have made ECRH a valuable and reliable tool. Technical development and testing can be done independently of the plasma. Thus the *power horse* of W7-X will be ECRH with the ambitious goal of 10×1 MW cw gyrotrons.

6.5 α -particle heating

The collisional energy transfer between the 3.5 MeV α -particles to the bulk plasma is the process that will sustain the plasma temperature in a fusion reactor, since the neutrons leave the plasma without interaction. However, even the α -particles might not be confined sufficiently long by the magnetic field to thermalize completely. Possible loss mechanisms involve the toroidal field ripple and interaction with coherent MHD activity (toroidal Alfvén eigenmodes). In tokamaks the α -particle confinement improves with plasma current and in stellarators the magnetic field configuration can be optimized. First experimental studies of α -particles were done during D,T shots in fusion experiments TFTR and JET basically confirming the understandings of the loss mechanisms.

6.6 Non-inductive current drive

The discussed heating schemes usually heat only a small fraction of the plasma particles. This can be taken advantage of by asymmetric wave launching or by tangential neutral particle injection to drive a current in the plasma. This is useful in order to reduce or eliminate the necessity of ohmic current in a tokamak or to compensate for the pressure-driven bootstrap current in a stellarator.

A current drive efficiency γ_{th} can unambiguously be defined as the ratio of local parallel current density, j , and the local absorbed power density p : $\gamma_{th} = j/p$. Different experiments are often compared using a scaled figure of merit γ_{ex} defined as

$$\gamma_{ex} = \frac{\bar{n}_e [10^{20} m^{-3}] R [m] I [A]}{P [W]},$$

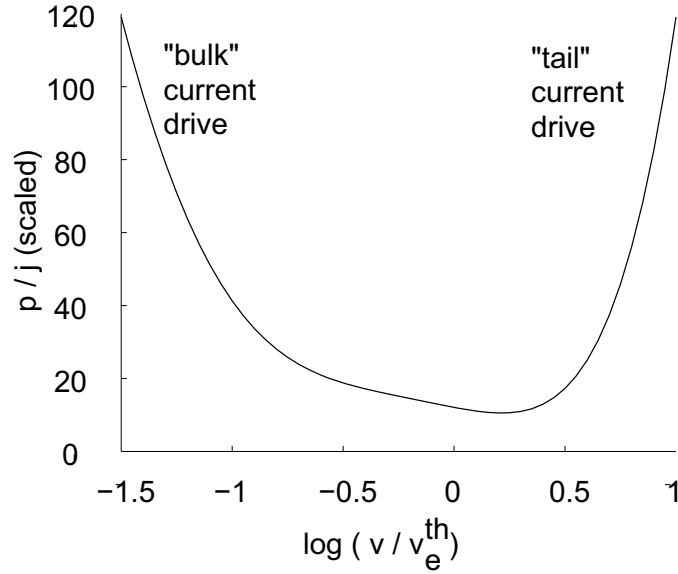


Figure 6.6: Normalized current drive efficiency for Landau damping.

where \bar{n}_e is the line-averaged density, R is the major radius of the plasma, i.e. the distance between the torus axis and the center of the plasma column, I is the plasma current and P is the total absorbed power. With $I \approx \pi a^2 j$ and $P = \pi a^2 \cdot 2\pi R \cdot p$ and $j/p \propto 1/\bar{n}_e$ it is easy to confirm that $\gamma_{th} \propto \gamma_{ex}$.

A number of physical concepts can be used to drive a current in a plasma. The following will be mentioned briefly: Pushing of electrons, asymmetric electron collisionality and difference in electron and ion drift.

Electrons are being pushed along the magnetic field lines using Landau damping of waves that are launched in the plasma. This changes the velocity distribution of the resonant electrons until the collisional relaxation balances the driving force. If the antenna current straps are phased properly the waves are launched predominantly in one direction inside the torus, thus a net electron current results. For $\omega/k_{\parallel} \ll v_e^{th}$ the absolute change in electron momentum for a given loss of energy of the wave is large, but a large fraction of the electrons are trapped in the toroidal field ripple and the collision frequency is large. For $\omega/k_{\parallel} \gg v_e^{th}$ the collision frequencies are small but there are also few electrons in the distribution function. The compilation of these two effects leads to a theoretical curve shown in Fig. 6.6 identifying the regions of *bulk* and *tail* current drive. Lower hybrid, fast wave and ion-Bernstein wave mode-conversion change the tail of the electron distribution function. Alfvén wave current drive is expected to change the bulk with good efficiency.

Electron cyclotron heating predominantly changes the perpendicular temperature of the electrons and does not impart much parallel momentum. Still current drive is possible since the electrons that fulfill the resonance condition $v_{\parallel} = (\omega - \Omega_e)/k_{\parallel}$ (relativistic effects are neglected) and are heated will attain a lower collision frequency due to their higher velocity. Therefore their collisional relaxation will be reduced. Actually, this effect is also of great importance for current drive based on Landau damping. Yet there is a small caveat. Because of the radial dependence on the magnetic field, this resonance occurs only a small region next to

the location where $\omega = \Omega(r)$. Electrons on the high field side of this resonance will also meet the requirement $v_{\parallel} = (\omega - \Omega_e)/k_{\parallel}$, albeit for reversed velocity. Therefore it is necessary that the wave is strongly damped. Otherwise one creates a shear layer in the current profile with without any change in the total current.

If a neutral beam is injected at an angle, the slowing down of the beam ions on the electrons drags the electrons along. This electron current only partially cancels the ion current for two reasons. Firstly, if the plasma consists of two species of different charge states the different ion-electron collision times cause the electron drift to match more closely the ions of the highest charge state. Secondly, electron trapping in the toroidal magnetic field ripple reduces the electron drift. Current drive efficiencies of $\gamma_{ex} \approx 0.03$ have been obtained.

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Chapter 7

Plasma wall interaction in nuclear fusion devices

Karl Krieger

7.1 Overview

A major issue for design and construction of a nuclear fusion reactor with a magnetically confined plasma is the interaction of the hot plasma with the material components of such a device. On the one hand, the plasma facing vessel components represent a sink for energy and particles released by the plasma. On the other hand, the particle bombardment of the material surface may lead to release of wall atoms and of previously implanted fuel atoms which in turn may enter the plasma.

The contamination of the plasma by impurities released from the vessel structure is one of the main problems caused by plasma wall interaction processes. An additional problem is the alteration of the material structure by the particle bombardment and the high energy flux which may limit the lifetime of the plasma facing components significantly. These problems must be solved under the constraint that the generated power has to pass through the vessel components at some location. The wall may further act as a reservoir for the hydrogen fuel isotopes leading to an uncontrollable additional source of fuel atoms, which may cause problems in maintaining stationary discharge conditions.

In the following we will first introduce a basic model of the edge plasma region in contact with the walls, the so-called scrape-off layer (SOL). Then the basic interaction processes between plasma and material walls will be discussed. This includes in particular wall erosion processes leading to impurity production and mechanisms involved in the formation of hydrogen wall inventories and the corresponding recycling processes. Finally, the main diagnostic methods for investigation of plasma wall interaction processes will be discussed.

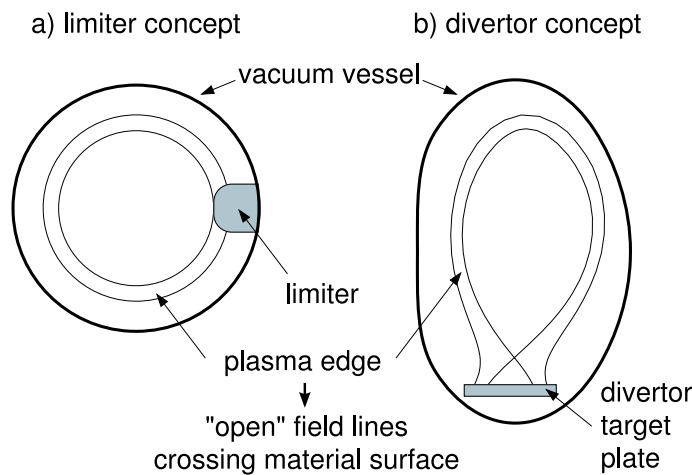


Figure 7.1: Poloidal cross-section of a limiter type (a) and a divertor type (b) plasma configuration with principal plasma facing wall components.

7.1.1 Plasma facing components

In a fusion reactor, we find several components facing the plasma directly. The largest surface consists of the first wall which surrounds the bulk region of the plasma torus. The plasma shape may be restricted by additional limiters to protect the vessel wall or equipment like antennas for radio-frequency heating which cannot withstand excessive heat loads. Finally, a very important part of the plasma facing components in current and future fusion devices are the divertor target plates. In a diverted plasma configuration these plates provide the main plasma-surface interaction zone. The fraction of the fusion power carried by the produced α -particles is coupled out to a large extent through these areas.

Fig. 7.1 shows poloidal cross-sections of a limiter type (a) and a divertor type (b) plasma configuration. In a divertor configuration the distance of the plasma wetted surface areas to the confined plasma region is much larger than for limiter configurations. Due to the correspondingly lower penetration probability of eroded material, it is much easier to maintain a low impurity level in diverted plasma discharges.

7.1.2 Bombardment of the walls with plasma particles

If the plasma is in direct contact with wall components, electrons and ions will hit the surface. This particle bombardment leads to release of atoms by collisions and, for certain materials by chemical reactions. In addition the wall material will be heated by the corresponding energy transfer. Finally, electron impact may produce secondary electrons which are reemitted back into the plasma.

The incident plasma ions will be neutralized with a fraction of them being reflected, while the remaining part will be implanted and may be released with some delay. Neutralized particles entering the plasma are ionized again by electron impact or by charge exchange processes with plasma ions. Charge exchange processes in hot plasma regions will produce neutral particles at high energies, which can escape the plasma hitting also plasma facing components without

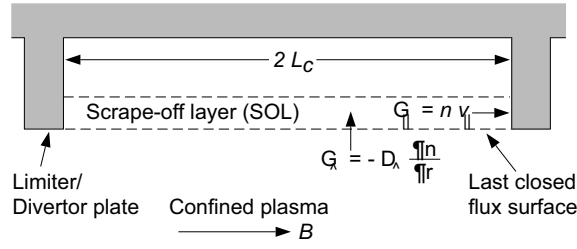


Figure 7.2: Schematic view of the scrape off layer in a magnetically confined plasma [1]. a is the minor plasma radius, B is the magnetic field, L_c is the connection length, Γ_{\perp} is the diffusive flux of particles perpendicular to B and Γ_{\parallel} is the flux of particles streaming along B .

direct plasma contact. high energetic particles and about one half of these will hit the walls again and might be reemitted with some probability.

In fusion plasmas facing components are also subject to intense irradiation by neutrons, which leads to modifications of the material structure and to corresponding degradation of mechanical stability and thermal conductivity.

Wall atoms eroded by the particle bombardment and, in case of excessive heat loads, by evaporation processes, will enter the plasma as impurities with detrimental effects on plasma performance. The erosion of wall surfaces will also limit the lifetime of the respective components. The eroded material will, however, migrate through the plasma until it is finally redeposited, which may balance the erosion to a certain extent.

7.2 The scrape-off layer

As shown in Fig. 7.1 the plasma edge in a magnetically confined plasma is either defined by a material limiter or, in the case of a diverted plasma, by a magnetic separatrix (see Fig. 7.2). Inside the so defined boundary, the magnetic surfaces are closed while in the region between the boundary and the wall surface, the so called scrape-off layer, the field lines intersect material components. The particle exhaust and the α -particle fraction of the produced power (as well as the additional heating power during start-up) are coupled out to a large extent through this region and transferred to the limiters or divertor plates.

The variation of plasma density and temperature in the scrape-off layer as a function of the minor radius can be approximately described by an exponential decay. Using a simple model of particle balance with purely diffusive transport perpendicular to the magnetic field and sonic flow along the field lines, one can estimate the decay length λ_n as a function of the perpendicular diffusion coefficient, D_{\perp} , and the ion sound velocity, c_s . The rate \dot{N}_{\perp} of particles per unit poloidal length W diffusing across the separatrix into the SOL must be equal to the rate $2\dot{N}_{\parallel}$ of particles per unit poloidal length streaming along the magnetic field lines towards the two

limiters or target plates. \dot{N}_\perp and \dot{N}_\parallel are given by

$$\begin{aligned} \dot{N}_\perp &= D_\perp \partial n / (\partial r)|_{r=a} L_c W &= D_\perp n(a) / \lambda_n L_c W &= \Gamma_\perp L_c W &\text{ and} \\ \dot{N}_\parallel &= \int_a^\infty n(a) \exp(-(r-a)/\lambda_n) dr c_s W &= n(a) c_s \lambda_n W &= \Gamma \lambda_n W, \end{aligned} \quad (7.1)$$

where $2L_c$ is the connection length and $c_s = ((T_e + T_i)/m_i)^{1/2}$. Solving $\dot{N}_\perp = 2\dot{N}_\parallel$ one obtains an exponential decrease of n with a characteristic decay length of $\lambda_n = (D_\perp L_c / c_s)^{1/2}$. A similar analysis holds for the temperature assuming convective energy transport. Inserting typical parameters ($D_\perp = 1 \text{ m}^2/\text{s}$, $L_c = 10\text{--}100 \text{ m}$, $T_{e,i} = 10\text{--}100 \text{ eV}$), one obtains a decay length in the range from 1–5 cm which is in good agreement with measured values. The decay length of the power flux is smaller because it is determined by the product of n and T .

Since the total energy and particle flux leaving the plasma is concentrated in a very narrow region of the target plates, the incident flux will be strongly amplified over the flux across the plasma boundary. Assuming a circular shaped plasma torus with major radius R and minor radius a , one obtains a torus surface of $4\pi^2 aR$. The plasma wetted surface of a divertor or limiter in contact with the plasma boundary is $2\pi R\lambda$. The radial flux crossing the boundary of the confined plasma is therefore amplified by the ratio of these surfaces $\pi a/\lambda \simeq 100$ with $a = O(1 \text{ m})$ and $\lambda = O(1 \text{ cm})$. For a fusion machine like ITER, the α -particle heating power is $\simeq 200 \text{ MW}$. With a major plasma radius of $\simeq 6 \text{ m}$ one obtains a heat flux at the target plates of $\simeq 100 \text{ MW/m}^2$, which exceeds by far the engineering limits of any material. It is therefore necessary, to distribute the power to a larger area of the vessel, for example by increasing edge radiation from impurities and charge exchange processes.

7.3 Electric coupling of plasma and wall

When the magnetic field intersects a solid surface, the surface will charge up negatively with respect to the plasma. Consider a small plasma cube with one side limited by a wall and with electrons and ions in thermodynamic equilibrium i.e. with a Maxwellian velocity distribution. For the particle flux to the wall one obtains $\Gamma_e = n\bar{v}_e/4$ and $\Gamma_i = n\bar{v}_i/4$. With $\bar{v} = (8kT/\pi m)^{1/2}$ one sees readily that $\Gamma_e = \Gamma_i (m_i/m_e)^{1/2} \gg \Gamma_i$. A repelling potential for the electrons will build up until the fluxes are balanced which is necessary to maintain the quasi neutrality of the plasma. Because of the screening effect of the plasma, the potential is restricted to a sheath with a width of some Debye lengths $\lambda_D = (kT_e/4n_e^2)^{1/2}$.

To determine the value of the potential, we consider the balance of ion and electron flux along the field lines. The ions enter the electrostatic sheath with ion sound velocity, thus $\Gamma_i = nc_s$. The electron flux to the wall is modified by the sheath potential

$$\Gamma_e = \frac{n_e}{4} \sqrt{\frac{8kT_e}{\pi m_e}} \exp(-e\Phi/kT_e). \quad (7.2)$$

The electron hitting the wall material will produce secondary electrons which reenter the plasma with a flux $\Gamma_{es} = -\gamma\Gamma_e$ with γ being the secondary electron emission coefficient. From

the balance of ion and electron flux $\Gamma_i = (1 - \gamma)\Gamma_e$ we obtain

$$e\Phi = -\frac{kT_e}{2} \ln \left(\frac{2\pi m_e}{m_i} \left(1 + \frac{T_i}{T_e} \right) \frac{1}{(1 - \gamma)^2} \right). \quad (7.3)$$

For a cold hydrogen plasma ($\gamma \ll 1$) with $T_i \simeq T_e$ one obtains $\Phi \simeq 3kT_e/e$. Obviously, if γ increases, the argument of the logarithm will increase as well and become undetermined for $\gamma = 1$. In reality, for $\gamma \geq 0.8$, an electron space charge layer will form at the surface inhibiting any further secondary emission. In that case, the equation above does not hold anymore.

Because only the fast electrons will reach the wall, the electrons are cooled in the sheath. The ions, however, gain additional energy because they are accelerated in the sheath potential. For a Maxwellian plasma the energy flux in one direction is given by $Q = 2kT\Gamma$. The convective energy fluxes to the wall surface are then given by

$$Q_e = 2kT_e\Gamma_e = 2kT + e\frac{\Gamma_i}{1 - \gamma}, \quad Q_i = \left(2kT_i + \frac{kT_e}{2} - e\Phi \right) \Gamma_i, \quad (7.4)$$

where the contribution of $kT_e/2$ for the ions has its origin by an additional acceleration in a pre-sheath potential drop. Obviously, the major part of the energy flux is deposited by the ions because of the acceleration by the sheath potential. With increasing edge temperatures the secondary electron emission may come closer to one, thus leading to very high energy fluxes to the wall through the electron channel.

In addition the energy deposition is increased by the neutralization energy of the ions and, for hydrogen, by the recombination energy of the hydrogen atoms.

7.4 Hydrogen fuel cycle

The hydrogen isotopes used as fuel for nuclear fusion interact with most of the relevant reactor materials. This is important in particular because of

- radiological reasons: The storage of tritium in the first wall is responsible for the major fraction of the tritium inventory in fusion reactors;
- engineering aspects: Hydrogen in solids may alter their mechanical properties in an unfortunate way;
- density control of the plasma: The hydrogen inventory in the walls during a stationary discharge will be much higher than the hydrogen content of the plasma itself, leading to possible difficulties in maintaining a stable plasma density.

At low plasma temperatures, neutral atoms and ions hitting the wall will usually be reflected after transferring a part of their kinetic energy to the material.

For high plasma temperatures, the particles can penetrate deeply into the material. In this case, the recycling of the particles will be determined by kinetic reflection, but in addition, the particles may be implanted and reemitted after diffusion back to the surface. In the following, these processes will be discussed in more detail.

7.4.1 Reflection

Ions and neutral particles hitting the wall material will not be reflected right at the solid surface. Usually they will penetrate the material and be scattered by the atoms and electrons in the crystal lattice. Collisions with nuclei will lead to energy transfer and deflection while collisions with electrons will only lead to energy loss. If the path of the penetrating particles leads them back to the surface, they will be reemitted. The particle reflection coefficient R_n and energy reflection coefficient R_E defined by

$$R_n = \frac{\text{number of reflected particles}}{\text{number of incident particles}} \quad \text{and} \quad R_E = \frac{\text{energy of reflected particles}}{\text{energy of incident particles}} \quad (7.5)$$

can be determined by computer simulations where the particle trajectory and the energy transfer in the scattering processes is calculated assuming binary collisions. Experimental results are obtained from backscattering experiments, which are, however, very difficult at energies below 10 eV due to strongly decreasing detector sensitivities.

Fig. 7.3 left shows the reflection coefficients of hydrogen for two reactor relevant materials, carbon and tungsten, as a function of the energy of the incident hydrogen atoms. The reflection is mainly determined by backscattering processes and therefore increases with the target atom mass. With increasing energy the incident particles will penetrate more deeply into the bulk material, which leads to the observed decrease of the reflection coefficients towards higher energies. The simulations assume an attractive surface potential, which leads to the strong decrease of the reflection coefficients at energies below 1 eV. The determination of the precise form of the surface potential is, however, a very difficult problem. In addition, adsorbed hydrogen at the surface will strongly modify the surface potential function.

7.4.2 Implantation, diffusion and reemission

The fraction of implanted hydrogen ions which are not reflected, i.e. which do not find their way back to the surface right after penetration, are slowed down by subsequent scattering and will eventually be trapped, either at grain boundaries or at vacancies in the bulk. In addition they may be bound in form of a metal hydride.

When the ions enter the solid surface they recombine to atoms and the fraction which is not trapped within the course of the thermalization process may then travel through the material by diffusion. If the atoms reach a surface, they will recombine to molecules and may be released by thermal desorption.

Metals

Diffusive propagation of neutral hydrogen in a metal is described approximately by a transport equation for the concentration $c(x)$ of free atoms in a depth x

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2} + Q - \frac{\partial c_T}{\partial t}, \quad (7.6)$$

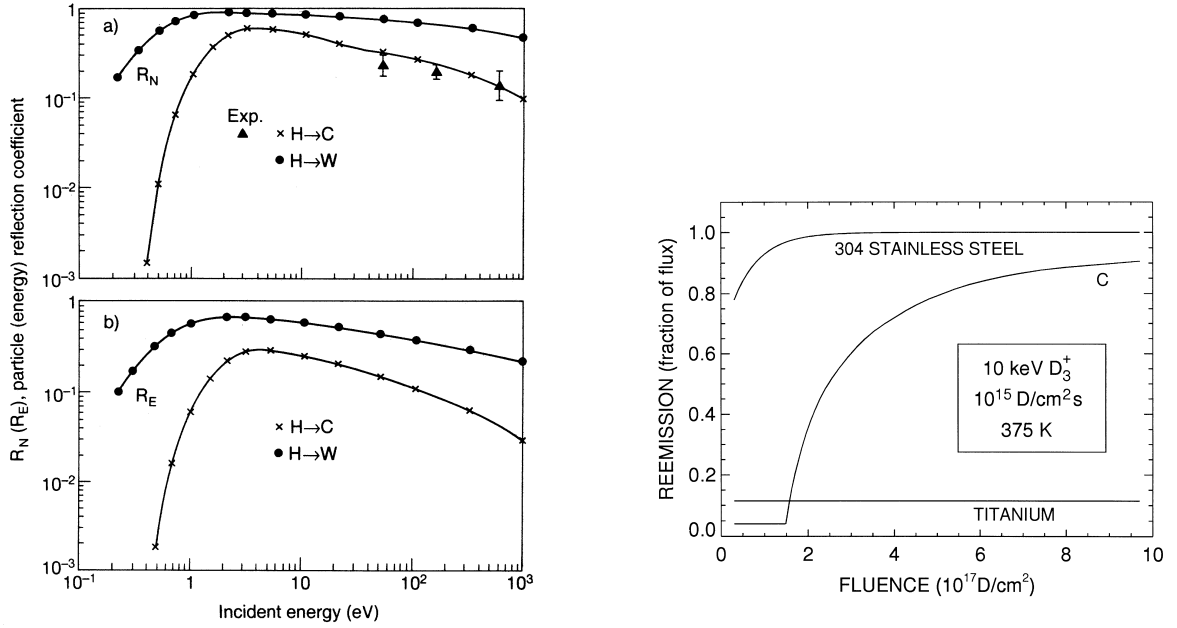


Figure 7.3: Left: Particle (a) and energy reflection coefficients (b) for hydrogen ions incident on carbon and tungsten solid surfaces as a function of the impact energy.

Right: Reemission as a function of fluence (total number of implanted atoms) for different materials. The reemission is expressed in units of the implantation flux.

where Q is the source function of implanted hydrogen atoms and c_T is the concentration of trapped atoms. The maximal concentration will be found in a depth approximately equal to the mean ion range R_p . From there, the atoms diffuse back to the surface as well as deeper into the bulk material. At the surface, the rate of release is determined by the recombination rate coefficient to H_2 because energetically only molecules can be released. For the H_2 -flux one obtains

$$\Gamma_{H_2} = -D \frac{\partial c}{\partial x} \Big|_{x=0} = Kc(x=0)^2, \quad (7.7)$$

where K is the molecular recombination rate constant.

The final equilibrium distribution depends further on the solubility of the hydrogen in the material. For metals in which hydrogen dissolves exothermically like titanium, the surface is a potential barrier. Therefore, the reemission of hydrogen will be very low, the hydrogen is collected in Ti, forming TiH_2 (see Fig. 7.3 right).

In metals like steel, where hydrogen is dissolved endothermically, the reemission depends on the temperature. At low temperatures the diffusion coefficient is small and the reemission is determined by the diffusive transport. A steady state concentration builds up until the flux of implanted atoms is equal to the reemission flux (see Fig. 7.3 right). After stopping the implantation process, the hydrogen inventory in the material will be reemitted. This situation is typical for present fusion experiments with steel walls where the wall temperature remains low because of the short discharge times.

With higher temperatures (500°C for steel) the diffusion increases strongly. In that case, the reemission is mainly determined by the recombination to H_2 at the surface. This situation would occur in a fusion reactor and is important with respect to the tritium fuel. The tritium inventory in the wall will be small but on the other hand, a considerable fraction of the tritium atoms will diffuse through the wall and must be captured for example by a double wall structure with a pumped interspace.

Carbon

For carbon as wall material, the situation is different to the metals. For temperatures 400°C like in present fusion experiments with graphite as wall material, there is practically no diffusion. Implanted hydrogen will be collected until a saturation level of 0.4 H/C is reached. If the implantation continues, the additional hydrogen will be reemitted immediately (see Fig. 7.3 right). After stopping the implantation, the reemission will be much smaller than for metals. In present fusion experiments, the saturation level is reached after some discharges which may cause problems in controlling the plasma density. With so called conditioning discharges using helium, one can decrease the hydrogen inventory if necessary.

In a fusion reactor with wall temperatures above 500°C, diffusion and recombination to H_2 become important also for carbon with similar problems for the tritium handling as for the metals.

A further process leading to trapping of hydrogen in carbon will occur in devices with divertor plates made of carbon. Carbon atoms produced by erosion of the plate surfaces may be deposited at other locations where they will collect more hydrogen. This leads to a further increase of the hydrogen inventory in the vessel.

7.5 Impurity production

As mentioned in sec. 7.1, the bombardment of the wall elements by particles escaping the plasma or streaming along field lines which intersect a solid surface will produce in turn a source of neutral atoms from the wall material. Depending on the location of the surface and the plasma properties in the vicinity of the source, a certain fraction of the wall atoms will penetrate the plasma becoming ionized there. The presence of these impurities may degrade severely the plasma properties necessary for nuclear fusion. In this section, we will discuss the most important processes responsible for the contamination of the plasma.

7.5.1 Sputtering

When an ion or neutral atom is hitting a solid surface, it follows a complicated trajectory because of collisions with the lattice atoms in the solid. During this collision cascade the incident particle will be reflected or trapped as described in the previous section. In addition, if a surface atom receives sufficient energy from a collision, it is sputtered away from the solid

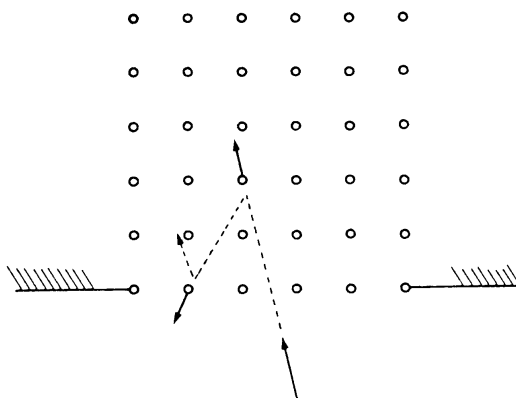


Figure 7.4: Schematic view of the physical sputtering process. The ion incident from the left is reflected in a first collision to the surface where it releases a lattice atom in the second collision.

(see Fig. 7.4). This is only possible if the energy of the incident particle is above a certain threshold E_{th} . The sputtering yield Y is defined by

$$Y = \frac{\text{average number of sputtered particles}}{\text{incident ion}} . \quad (7.8)$$

For central collisions, the fraction of the kinetic energy E_0 of an incident particle with mass m_1 which is transferred to a target atom with mass m_2 is given by $\gamma = 4m_1m_2/(m_1 + m_2)^2$. For light projectiles sputtering will occur only if the incident particle is reflected in a first collision towards the surface as shown in Fig. 7.4. Its energy after the collision is $E_0(1 - \gamma)$. To sputter a surface atom, the transferred energy in the following collision $E_0\gamma(1 - \gamma)$ must be high enough to exceed the surface binding energy E_s . This leads to a threshold energy of

$$E_{th} = \frac{E_s}{\gamma(1 - \gamma)} . \quad (7.9)$$

For most materials E_s is in the range 3.5–9 eV. Therefore, for the same projectile mass, the threshold energy is mainly determined by the mass of the target atoms. However, if m_1m_2 , e.g. if the projectile and the target atoms are the same (self sputtering), the collision cascades are much larger and the model described above does not hold any longer. In that case, the threshold energy is $E_{th} = 4E_s$.

The left part of Fig. 7.5 shows the sputtering yields Y of different targets for bombardment with D atoms. Apart from the described dependence of the threshold energy from the mass of the target atoms the position of the maximum in Y increases to higher target masses.

To determine the sputtering yield of a material exposed to a plasma, one has to integrate the sputtering yields over the energy distribution function of the incident plasma ions. In the plasma the ions usually are close to a Maxwellian distribution. Ions impinging on a solid surface are, however, accelerated by the sheath potential, which significantly increases their

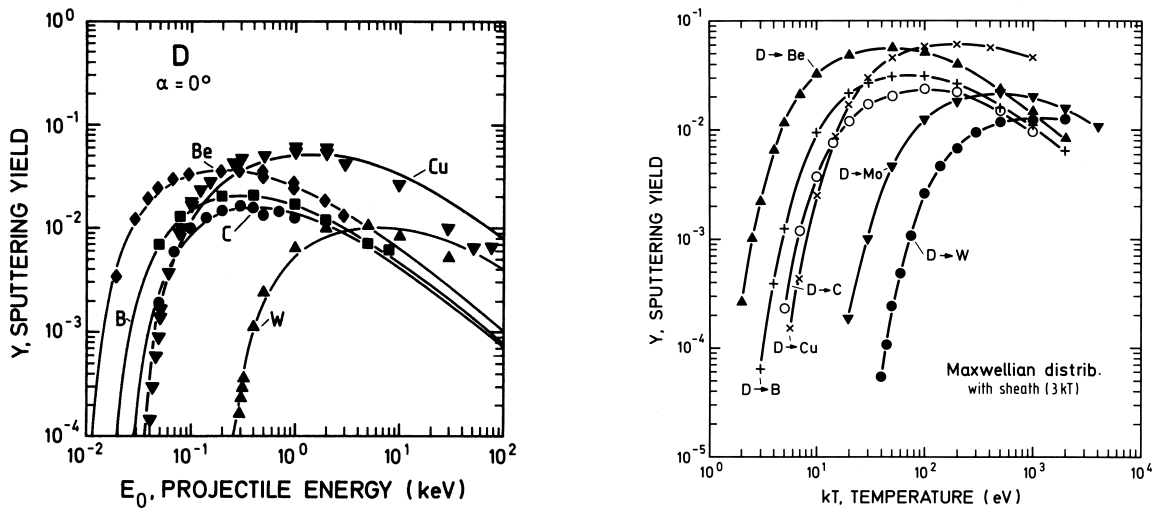


Figure 7.5: Left: Calculated sputtering yield Y of various target materials for bombardment by D ions as a function of impact energy.
Right: sputtering yields for exposure to a D-plasma as a function of the plasma temperature including acceleration by the sheath potential.

impact energy. The right plot of Fig. 7.5 shows results for a D-plasma. The ions from the high energy tail of the temperature distribution and the effect of the sheath acceleration lead to significant sputtering even at plasma temperatures fairly below the monoenergetic sputtering threshold energy.

For heavy projectiles, in particular in the case of self sputtering, the yield may even become greater than one which will lead to an unlimited increase of the impurity content in the plasma.

For light projectiles Y_{\max} is always < 1 . This is, however, only true for normal incidence. If the angle of incidence increases from the normal, the sputtering yield increases as well and may become > 1 .

The sputtered wall particles are practically all neutral. Except for single crystal targets their angular distribution is given approximately by a $\cos(\theta)$ distribution. Their energy distribution has typically a maximum at $0.5E_s \simeq 2-3$ eV. Therefore, they can deeply penetrate the plasma before they become ionized which gives them a high probability to become confined in the plasma.

7.5.2 Chemical erosion

Another erosion process will be found if the incident particles produce volatile compounds with the target material. This is observed in particular for carbon surfaces with hydrogen plasmas or with oxygen as a plasma impurity. In the first case, methane and higher carbohydrates are produced while in the latter case, production of carbon monoxide (CO) can be observed. These gases are set free very easily from the surface by thermal desorption or collisions.

For the most important case $D^+ \rightarrow C$, chemical erosion by high energy particle bombardment will occur mainly at surface temperatures in the range $500-600^\circ\text{C}$. However, if the energy of

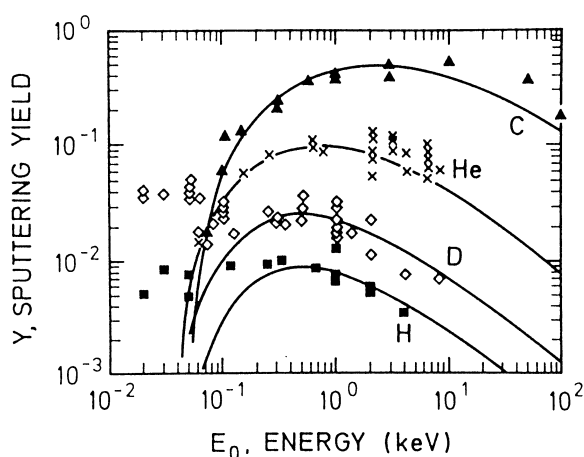


Figure 7.6: Dependence of the sputtering yield of graphite for H, D, He and C ions at normal incidence. The solid curves represent results from simulations of physical sputtering.

the incident deuterons is smaller than 100 eV, chemical erosion processes will dominate the physical sputtering processes. Fig. 7.6 shows sputtering yields for hydrogen, deuterium and helium impact on a carbon surface. The solid lines represent results from calculations with a simulation code for physical sputtering. Obviously, for hydrogen and deuterium sputtering/erosion occurs even below the threshold energy for physical sputtering.

E results from ASDEX Upgrade which is equipped with graphite wall tiles and divertor plates, confirm that the carbon content of the plasma is indeed to a large extent produced by chemical erosion of the main chamber graphite tiles.

7.5.3 Sublimation, evaporation

In a closed system, sublimation from a solid surface or evaporation of fluid material is in equilibrium with the condensation from the gas phase. The equilibrium is determined by the temperature and the steam-pressure of the material. For steam pressures above 10^{-6} mbar, sublimation or evaporation will be the dominant process. For beryllium which is considered as a possible wall material, the corresponding temperature is 800°C while sublimation of carbon will only occur for temperatures $> 2\,000^{\circ}\text{C}$.

Furthermore, laboratory experiments show that in carbon exposed to high levels of particle influx, interstitial atoms are produced which can reach the surface by diffusion where they sublime. This process starts at much lower surface temperatures ($\gtrsim 800^{\circ}\text{C}$) and could lead to increased carbon production in a fusion reactor.

7.5.4 Desorption

The surfaces inside the vacuum vessel are covered with thin layers of water, cleaning agents and residue of manufacturing processes. Depending on their surface binding energy, they will

be released from the wall by the particle bombardment from the plasma thus producing an additional source of plasma impurities. It is possible to remove the weakly bound layers by glow discharges, however, the remaining oxide layers and adsorbed water molecules cannot be removed and are the main source for the oxygen contamination of the plasma.

7.5.5 Arcing and melting

If metals are used as wall material, the surface can be destroyed by melting, if the heat load from the plasma becomes so high that it cannot be removed by conduction and cooling. Droplets of molten material from the upper vessel regions can fall into the plasma where the sudden rise of the radiation losses may cause a break down of the discharge.

As less spectacular erosion process can be observed under certain discharge conditions. Because of the potential difference between edge plasma and wall, electric arcs are ignited by field emission from ridges and peaks in the surface which last for times $1 \mu\text{s}$ with a size of some mm. After ignition the cathode spot on the material surface is heated up by ion bombardment from the arc and cooled by the melting and conduction. The cathode spot grows because of the increasing erosion and after reaching a critical size the arc breaks down and may ignite again in the vicinity.

7.5.6 Positive and negative aspects of impurity contamination

The impurity contamination of the plasma in a fusion reactor can only be tolerated below a critical level which depends mainly on the charge number of the impurity atoms. However, if one can restrict the zone with higher impurity concentration to the plasma edge and the divertor plasma, the presence of impurities can also have a beneficial effect as described below.

High-Z – cooling of the central reaction zone

Heavy impurities like tungsten, molybdenum and to a certain extent nickel and iron are not fully ionized in a fusion plasma. The power loss by their line radiation is a function of the plasma temperature which has for high-Z elements a maximum in the temperature range of 10^6 – 10^7 K relevant for nuclear fusion. Therefore, concentrations higher than 0.01% (for W, Mo) and higher than 0.1–1% (for Fe, Ni) will lead to a power losses which inhibit the ignition of the plasma.

Fuel dilution

Light impurities like beryllium, boron and carbon and also helium as a product of the fusion process will be fully ionized in the plasma core. Therefore, they do not emit line radiation in the central region in contrast to the cooler edge regions. The radiation losses in the outer plasma zone will not inhibit the ignition and the burning of the plasma. However, they still

can degrade the overall energy balance of the fusion reactions. A further negative effect is the dilution of the fuel ions in the core because of the quasi neutrality of the plasma which will decrease the fusion rate. For these reasons, the concentrations of light impurities in the confined plasma must not be higher than 2–5%.

Edge cooling and power exhaust

Under certain plasma conditions, in particular at high densities, it turned out that it is possible to confine the impurities in the divertor region of a tokamak plasma. There are attempts to use the intense line radiation produced by light impurities in this cooler zone as a means to spread the power flux generated from the plasma onto a larger area of the vessel surface thus decreasing the power load of the divertor target plates. The impurities can be introduced into the divertor plasma either by sputtering from the walls or by additional puffing of impurities in gas form. This concept is known as *radiative divertor*.

Another concept to make use of radiation to decrease the power load of the divertor plates consists basically in trying to increase the divertor density to very high values and to puff in additional gas. Then, a cold plasma zone will establish above the divertor plates in which the plasma ions streaming towards the plates are cooled by charge exchange radiation. However, in these processes, as mentioned above, fast neutral atoms are produced which may increase the wall sputtering. This concept is known as *cold gas target*.

7.6 Diagnostics for plasma surface interaction

Many diagnostics are needed to provide the information for a more detailed understanding of the edge plasma and plasma-wall interaction. A detailed description would break the size limit for this manuscript, therefore only a short overview is given over diagnostics used directly in fusion experiments and diagnostic methods used in laboratory experiments for example to analyze wall materials and surfaces.

7.6.1 In situ diagnostics

- *Electron density and temperature* are measured by Thomson scattering and from spectral emission of neutral atom beams injected into the plasma edge. A further possibility is the use of electrical Langmuir probes which allow the measurement of n_e and T_e and in addition the *floating potential* of the plasma.
- *Impurity densities and fluxes* and the *densities of molecules* are measured by spectroscopy and laser resonance fluorescence. With good spectral resolution, *impurity temperatures* can be measured as well. However, there is no good method to obtain the *temperature of the plasma ions*.
- *The composition of the neutral gas* is determined by mass spectrometry. This gives valuable information for example on the molecules produced by chemical erosion.

- *Velocity spectra of escaping neutrals* are measured by time of flight spectrometers. These provide information on the ion temperature in the plasma and allows to estimate the effects of the neutral wall bombardment.
- *Surface temperatures and heat flux deposited on surfaces* are measured by infra red cameras.
- *Surface erosion and deposition* are investigated with long term marker probes where materials are vaporized as thin layers onto surface elements. To analyze single discharges, collector probes are used which allow in addition the determination of *impurity flux, mean energy and mass* and *plasma ion fluxes and mean energy*. After exposure to the plasma for a certain time, the probes are removed and analyzed by surface analytical techniques described in the next paragraph. As a further method to measure *erosion and redeposition of thin material layers* one can measure the thickness of such layers by interferometric methods.

7.6.2 Laboratory experiments

- *Analysis of collector probes and long term exposition probes.* The surface structure of the probes is examined by raster electron microscopy. The electron beam of the microscope will produce bremsstrahlung when hitting the examined surface which is characteristic for each element and allows the determination of the material composition. A quantitative determination of the distribution of an element in the material is possible by scanning the surface with high energy ion beams (in the MeV range) of protons, deuterons or α -particles. One can measure the energy distribution of the ions from Rutherford backscattering processes (RBS) or, in case of a proton beam, the spectrum of the proton-induced X-ray emission (PIXE). A further possibility is the detection of products of nuclear reactions induced by the incident ion beam which are also characteristic for a certain element. To measure the erosion of the probe surface, a trace element is implanted in a surface layer with known thickness. The erosion rate is determined by measuring the thickness of the remaining layer after exposition of the probe.
- *Hydrogen recycling* is studied by implanting protons or deuterons in materials until steady state is reached and measuring the pressure of the reemitted atoms as a function of time after switching off the beam.
- *Sputtering coefficients* can be measured by bombarding a probe with an ion beam and measuring the weight loss after a certain exposition time. In addition, the sputtered material can be analyzed by laser resonance fluorescence.
- *Reflection coefficients* are determined by estimating the amount of trapped atoms or by detecting the reflected particles using ion spectroscopy.

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Chapter 8

Introduction to plasma diagnostics

Hans-Jürgen Hartfuß

8.1 Introduction

Investigating hot magnetized plasmas in the frame of fusion research, a large number of different diagnostic methods must be applied. Common to all diagnostic techniques is that plasma probing is conducted without any material contact between the measuring instrument and the plasma. Material probes can only be used in the very plasma edge and only for short moments.

The physics principles underlying the various techniques originate from almost all fields of physics. They are complementing one another with respect to dynamic range or concerning certain information gained by other diagnostic systems necessary to interpret the measurements. The whole diagnostic system has redundancy which means that a given physical quantity, say the electron temperature, is measured simultaneously by a number of different measuring systems which are based on different physics principles, ensuring that systematic errors do not remain unrecognized.

Although a large number of diagnostic methods is well developed and established, growing new fields of research demand for additional information, or higher spatial and temporal resolution, stimulating the development of new methods and even more refined techniques. Recent developments are often related to turbulence characterization, measuring fluctuating plasma parameters with high time and space resolution and determining correlations between various fluctuating quantities.

Typically more than 50 different diagnostic systems are installed at modern large fusion experiments. They provide the data which form the basis for the scientific work conducted, and they provide as well the bulk of information necessary for the safe operation of the fusion experiment and the protection of its components. Diagnostic data rate of the whole system typically exceeds 100 Mbyte per second. The data must reliably be stored in such a way that data access remains possible even after many years.

To characterize the plasma in an adequate way and to understand its complex behaviour as complete as possible, a large number of different plasma parameters must be determined simultaneously. These are the density of the electrons and the ions, the composition and density

of certain impurities in the plasma, the temperature of the electrons and of the ions which can differ significantly depending on the heating scenarios conducted, the total stored energy, the plasma pressure, the radiation loss, currents and electric fields in the plasma and many others. Most of them are local quantities varying with the radial coordinate. Examples are the electron temperature and density being typically maximal at the plasma axis and approaching low values at the plasma edge varying over orders of magnitude. Most of the quantities are time dependent with characteristic times varying significantly. While fluctuating quantities are characterized by time scales of the order of $1 \mu\text{s}$, certain currents in the plasma or the energy content vary slower by five orders of magnitude. The demanded accuracy of all the diagnostic measuring systems is typically 1–10%. The spatial resolution aimed at is 1–5 cm and the time resolution of the order of 0.1 ms. This short introduction to the subject concentrates on the physics behind the most important diagnostic methods, factoring out completely the wide field of technical realizations.

8.2 Categorization

The various diagnostic methods can be arranged according to different ordering principles. A natural distinction divides into active and passive methods whereas in each case waves or particles must be considered. Moreover Langmuir-probes form a separate class of techniques which could be assigned to both passive and active particle diagnostics. Another class is formed by magnetic diagnostics.

A more physics oriented categorization distinguishes the physics processes involved: Electron or ion processes, processes leading to the emission of electromagnetic radiation, light scattering processes, processes connected with the refractive index of the plasma and so on. Other organizing principles used in literature are based on the techniques involved, like laser-, spectroscopic-, X-ray-, microwave-, atomic beam techniques, etc. We will adapt an ordering scheme according to the first one given above. The chapters following directly reflect this choice.

8.3 Sightlines

Plasma probing with waves or by particles occurs along a sightline through the plasma determined by the experimental conditions of the measuring system.

- In a single sightline arrangement one cannot distinguish from which point of the sightline the information is coming from. Only line integrated information can be gained (examples: spectroscopy, interferometry).
- Using 10 to 15 sightlines simultaneously, however, local information can be derived from a reconstruction process called Abel-inversion (multi-channel interferometry).

- Modern computer based reconstruction methods allow for the application of tomography. Local information is obtained from a large number of typically several hundreds of line-integrated signals from pinhole camera systems (soft X-ray bremsstrahlung, bolometry).
- A special case of passive wave diagnostics, gives local information from a single sight-line since the origin of radiation is connected with the frequency of the emission that mapping between frequency and location is possible (electron cyclotron emission).
- Applying active techniques with atomic or laser beams, the origin of the information detected is defined by the crossing volumes of primary beam and detection beams, resulting in local information (Thomson scattering, neutral hydrogen diagnostic beam).

All diagnostic methods out of the categories active and passive wave or particle diagnostics belong to one of these five different observation schemes. Using Langmuir-probes the location is defined by the tip of the material metallic probe. Magnetic diagnostics have no spatial resolution since they measure the total flux changes enclosed by loops outside the plasma column.

8.4 Magnetic diagnostics

Magnetic measurements are conducted with induction loops wound around the torus in various orientations. The voltage V obtained at the clamps of a loop with N turns and area A is determined by the temporal flux change enclosed by the loop

$$V = - \int_F \frac{\partial B}{\partial t} dF = -NAB\dot{.} \quad (8.1)$$

The magnetic field B is obtained by integration accomplished electronically. The most important magnetic diagnostics reveal the total plasma current with the Rogowski-coil, the loop voltage with the voltage-loop, and the total stored energy in the plasma with the diamagnetic-loop. The basic arrangements are shown in Fig. 8.1.

8.4.1 Rogowski-coil

The Rogowski-coil is a solenoid enclosing the whole plasma current i.e. the torus in one poloidal plane. It is sensitive only to the poloidal field generated by the net current I_p consisting of electron and ion currents in opposite directions. No information is gained on the current distribution. The plasma current I_p is derived by integrating the measured voltage V over time which is for a coil with n windings per unit length and area A given by

$$V = \dot{\Phi} = nA\mu_0\dot{I}_p \quad (8.2)$$

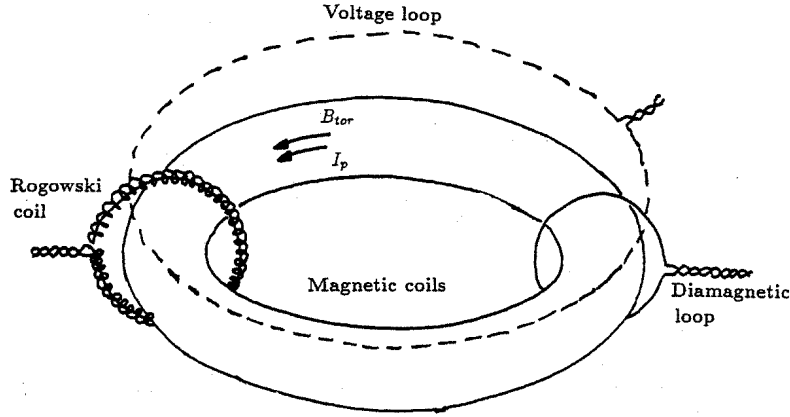


Figure 8.1: Arrangement of the most important inductive loops forming the set of magnetic diagnostics in a toroidal fusion device: The Rogowski-coil to measure the total plasma current, the voltage-loop to measure the current driving loop voltage, and the diamagnetic-loop which allows to determine the total stored energy in the plasma.

8.4.2 Voltage-loop

In tokamak experiments the plasma current is driven by the voltage generated by the external transformer. This so-called loop voltage is measured by an inductive loop consisting of a single wire which encloses the total flux generated by the transformer. The voltage loop lies parallel to the torus ring. Knowledge of the loop voltage V_l and the plasma current I_p are connected through the plasma conductivity and since the conductivity is temperature dependent, also the average electron temperature can be derived

$$\frac{I_p}{V_l} \propto \sigma \propto T_e^{3/2}. \quad (8.3)$$

8.4.3 Diamagnetic-loop

The average plasma pressure or equivalently the average energy density can also be determined from purely inductive measurements. In a magnetized plasma electrons and ions are gyrating around the field lines in a way that the external field is weakened giving rise to the plasma diamagnetism. The flux change $\Delta\Phi$ caused by the plasma built up in the magnetic field is proportional to the average plasma pressure $\langle p \rangle$ equivalently to the average energy density

$$\Delta\Phi \propto \langle p \rangle = k_B \langle n_e T_e + n_i T_i \rangle. \quad (8.4)$$

It is measured with the simple flux loop wound around the torus in poloidal direction. It must be integrated in time to get the actual stored *diamagnetic* energy.

8.5 Wave diagnostics

Wave diagnostics gain information either by analyzing the spectral composition of the emitted radiation from the plasma (standard spectroscopy) or by measuring the changes in amplitude, phase, polarisation or k -vector a wave undergoes when passing the plasma.

The hot magnetized plasma is emitting radiation in a very wide spectral range. To gain the full information contained, passive probing must therefore be done at wavelengths extending from about 10 mm to 0.1 nm over about eight orders of magnitude covering the ranges from micro-waves to X-rays. Basically also absorption spectroscopy can be done in this range. It has gained, however, only minor importance.

In active probing experiments the wave particle interaction is almost completely governed by the mobile electron component of the plasma. The electron density determines a cut-off frequency below which no propagation is possible and beyond which the refractive index of the plasma varies in a characteristic way. Largest changes occur in the mm and sub-mm spectral regions which therefore have the highest diagnostic potential.

At frequencies far above this cut-off, the refractive index approaches the vacuum value, however, electrons can still be accelerated by the wave's E -field causing wave scattering which carries important information about the scattering centres.

8.5.1 Active wave measurements

Microwave diagnostics

Propagation of an electromagnetic wave through the plasma is described by the dispersion relation connecting the wave k -vector to the frequency $k = k(\omega)$, from which the refractive index N can be derived.

Neglecting ion motion and collisions between particles in a non-magnetized plasma, the dispersion relation and the refractive index are

$$\omega^2 - \omega_p^2 = k^2 c^2; \quad N = \sqrt{1 - \left(\frac{\omega}{\omega_p}\right)^2}. \quad (8.5)$$

ω_p is the plasma frequency determined by the electron density

$$\omega_p^2 = n_e \frac{e^2}{\epsilon_0 m_e}. \quad (8.6)$$

Propagation is only possible for $\omega > \omega_p$. Since electron densities are typically in the 10^{19} – 10^{20} m^{-3} range, propagation is only possible beyond about $f = \omega/2\pi > 100 \text{ GHz}$. It is the mm-wave range where the strongest changes of the refractive index occur, which is therefore used for probing (see Fig. 8.2 left).

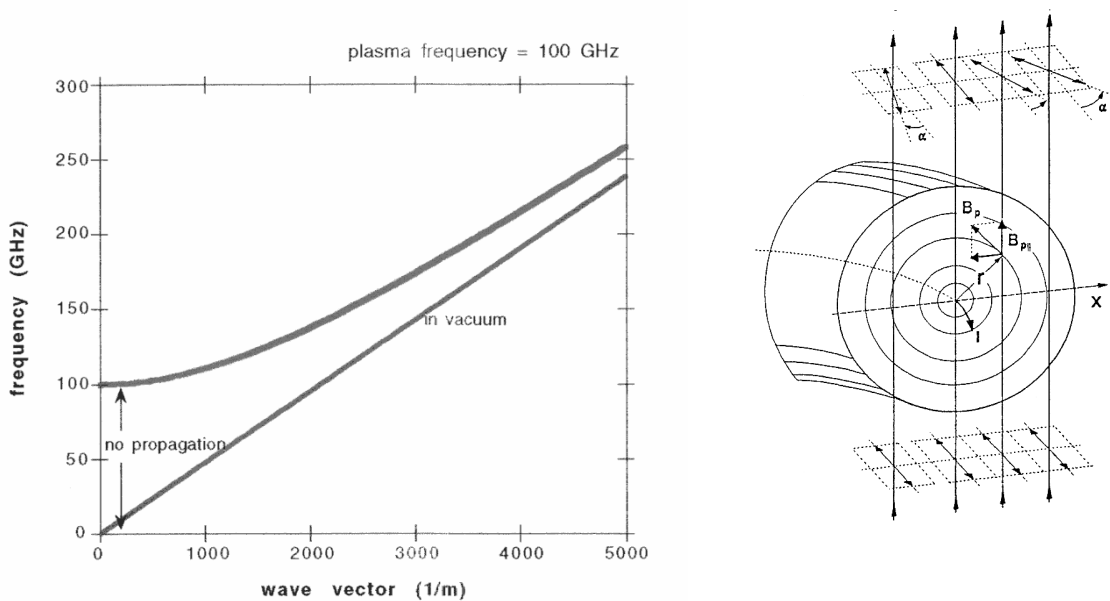


Figure 8.2: Left: Dispersion relation of a non-magnetized plasma (red). The plasma frequency is chosen 100 GHz corresponding to an electron density of $1.26 \times 10^{20} \text{ m}^{-3}$. At frequencies below the plasma frequency no propagation is possible.

Right: Principle arrangement of a polarimetry system to measure the poloidal field generated by the plasma current. The Faraday-effect gives rise to a rotation of the polarization plane by the angle α . The rays through the plasma can be used as well as interferometer channels to determine simultaneously the electron density (H. Soltwisch, Transactions of Fusion Technology **25**, 304 (1994)).

Contrary to this a magnetized plasma is a strongly anisotropic medium. General expressions for the dispersion relation are complicated.

However, in case a linear polarized wave propagates perpendicular to the B -field through the plasma with its E -field parallel to B (ordinary mode) the dispersion relation is identical to that one of the non-magnetized plasma, as given above. The relation is the basis for two important diagnostic systems interferometry and reflectometry operated under the above described conditions.

1. Interferometry

Since the refractive index depends on the electron density via the plasma frequency, measurements based on the refractive index give information on the electron density. The measurement is done by comparing the optical path difference between a microwave signal through the plasma and one through free space which can easily deduced from the phase difference of the two signals. If the probing frequency is large compared to the plasma frequency, the phase difference is proportional to the electron density integrated

along the line of sight

$$\varphi = \frac{2\pi}{\lambda} \int_{z_1}^{z_2} (N_V - N(z)) dz \propto \int_{z_1}^{z_2} n(z) dz. \quad (8.7)$$

The integral quantity determined in this way is called the line-density and is one of the most important parameters in fusion diagnostics. Local information can be gained by operating simultaneously a number of interferometers with parallel sightlines covering the plasma cross section and by applying mathematical inversion techniques (Abel-inversion).

2. Reflectometry

While interferometry uses frequencies in the range 200 GHz to 30 THz far above the plasma frequency, in reflectometry the probing frequency range is chosen just to meet it. As mentioned before plasma frequencies range from about 30–100 GHz. Reflectometry then launches a wave in this frequency range and determines the position inside the plasma where $N(\omega)$ approaches zero and from where the launched wave is reflected. The position of the reflecting cut-off layer is determined applying RADAR techniques by measuring the time delay to the cut-off layer and back. This can be done electronically in various ways, either with short microwave pulses of sub-ns duration or by sweeping the frequency of a continuous wave and measuring the resulting phase change. Other methods are basically combinations of these two. The local electron density $n_e(r)$ is determined from a scan of the probing frequency relating time delay τ and cut-off frequency $\omega(\tau)$.

3. Polarimetry

The dielectric properties of the magnetized plasma give rise to additional effects which can favourably be used for diagnostic applications. Considering here only active wave probing with propagation parallel and perpendicular to any B -field of a magnetized plasma. In case of parallel propagation, the two characteristic waves are the left and the right hand circular polarized waves. Due to the gyration motion of the electrons with frequency ω_c , the magnetized plasma is circular birefringent, which means the refractive indices of the left and right hand circular waves N_L and N_R differ. A magnetized plasma therefore shows the Faraday-effect: The polarization plane of a linear polarized wave propagating along the B -field is rotated by a small angle α given by

$$\alpha = \frac{1}{2} \frac{2\pi}{\lambda} \int_{z_1}^{z_2} (N_L - N_R) dz \approx \frac{1}{2c} \int_{z_1}^{z_2} \omega_p^2 \omega_c \frac{dz}{\omega^2} \propto \int_{z_1}^{z_2} n_e(z) B_{\parallel}(z) dz. \quad (8.8)$$

The measurement is a line-integrated measurement in an arrangement almost identical to that of an interferometer. Therefore often the two are combined to form an interfero-polarimeter arrangement measuring simultaneously the phase to determine the electron density and the rotation of polarization to determine the parallel B -field component along the line of sight. If the line of sight is chosen within one poloidal plane, the parallel B -field is the one generated by the plasma current. In case many sightlines are used simultaneously, the plasma current density can be determined (see Fig. 8.2 right).

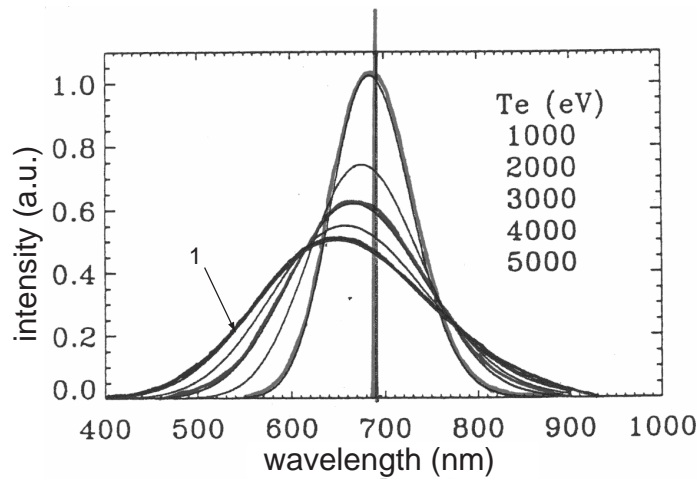


Figure 8.3: The spectra of the Thomson scattered radiation for various temperatures T_e . The primary laser line is at about 700 nm. With increasing temperature relativistic effects cause an asymmetry in the scattering spectra (line 1).

In case of propagation perpendicular to a B -field field, the two characteristic waves are linear polarized waves with the E -field parallel to B , the one used for interferometry, called ordinary wave (O-mode) and the extraordinary wave (X-mode) polarized perpendicular to B . Again the refractive indices for the two waves differ, which means the magnetized plasma is linear birefringent as well, it shows the Cotton-Mouton-effect. This means the ellipticity of a linear polarized wave composed of an X- and an O-mode component is changed when passing the plasma. Analysis of the polarization state gives line-integrated information on the electron density and the perpendicular B -field components involved.

Thomson scattering

At frequencies very large compared to the plasma frequency, the plasma refractive index approaches the vacuum value $N = 1$. However, individual plasma electrons are periodically accelerated in the E -field of a propagating wave launched as a laser beam for example. The oscillating charges are then origin of secondary radiation. The total intensity observed is composed of contributions of the individual electrons. The process is called incoherent Thomson scattering. Since the scattering centres are moving, the secondary waves have characteristic frequency shifts reflecting the particle's velocity, so carrying information about the velocity distribution function of the electrons. With a Maxwellian velocity distribution characterized by the temperature T_e of the electrons, a continuous spectrum around the primary laser frequency

$$I_s \propto \sqrt{\frac{m_e}{2\pi T_e}} n_e e^{-m_e \omega^2 / (2k^2 T_e)} \quad (8.9)$$

is observed from which the temperature can be derived (see Fig. 8.3). The scattering intensity I_s is proportional to the number of scattering centres, i.e. proportional to the electron density.

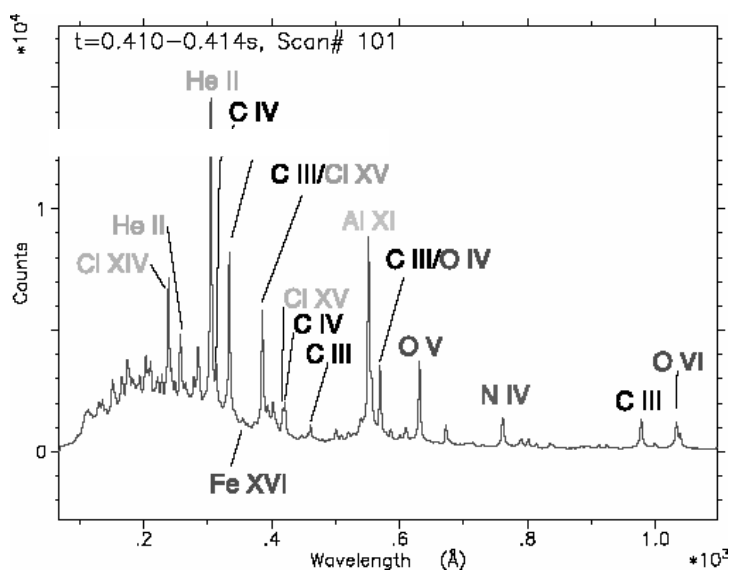


Figure 8.4: Typical VUV spectrum emitted from an impurity contaminated plasma. Characteristic lines of various charge states (roman numbers) of various elements like He, C, O, Cl, N, Al, and Fe could be identified. A continuous radiation background is also present (courtesy R. Burhenn, W7-AS).

Observation geometry and laser beam define the scattering volume. The arrangement therefore gives local information about electron density and temperature.

8.5.2 Passive wave measurements

Electromagnetic radiation emitted by fusion plasmas is composed of parts created in many different physical generation processes. In a fully ionized pure hydrogen plasma there is continuum radiation caused by bremsstrahlung owing to the accelerated motion of electrons in the field of the ions, cyclotron radiation due to the gyration motion of the charged particles in the external magnetic field, and recombination radiation after electron capture of ions. This continuous emission is superimposed by line radiation of recombined hydrogen atoms at the cool very edge of the plasma (Balmer-lines).

In addition to these unavoidable emission processes of a magnetically confined ideal hydrogen plasma, emission caused by impurities in the plasma must be considered. The impurities are ions of higher atomic number $Z > 1$. If these ions are fully ionized, they contribute to the emission with the processes listed before.

In case they are not fully ionized, which is true in particular for high Z ions, collision processes in the plasma lead to inner shell electronic excitation followed by spontaneous emission of characteristic line radiation with quantum energies typically in the keV range. Since this emission is connected with a significant energy loss, the impurity concentration of a fusion plasma must be kept sufficiently small. With regard to a future fusion reactor, impurity concentration of light atoms like C or O might reach the 1% level, however, the concentration of high- Z materials like W or Mo must not exceed the 0.1% level.

The small unavoidable impurity content can favourably be used for diagnostic purposes. The analysis of this radiation is done with regard to the wavelength which is characteristic for the element and its charge state. Analyzing also the line width and an eventual Doppler-shift, the ion temperature and collective drifts of the ions can be evaluated.

Line radiation

Main aim when analyzing the spectrum of line radiation is the determination of the impurity composition and the concentration of the various elements. To enable this, additional information is necessary. From the wavelength the element can uniquely be determined. The intensity of the emission depends on the concentration of this element but also on the excitation conditions, in particular on the local electron temperature and density. This makes the application of extensive code calculations necessary considering all excitation and de-excitation processes, to interpret the measured line intensities of the spectra.

The observation is typically along a sightline through the plasma centre. Since the plasma is optically thin a line integrated measurement results. Without additional knowledge no localization is possible.

A gross estimation, however, can be based on the fact that since the electron temperature profile is maximal in the plasma centre, the individual ionization states are not equally distributed in the plasma cross section but rather distributed like an onionskin in layers of a certain temperature range resulting in a localization of the individual charge states of an ion. The known electron temperature profile allows in this way an allocation of the emission of different charge states observed.

One of the most direct interpretations is based on the evaluation of the Doppler-broadening of the spectral lines. The line width reflects the ion temperature of the majority of ions of the bulk plasma since due to collisional interaction full thermalization is normally obtained (see Fig. 8.5 left). Measuring the line widths of elements in various charge states, the ion temperature profile can be reconstructed, at least estimated.

Continuum radiation

Continuum radiation is generated in the plasma due to the action of electric and magnetic forces on the electrons. The acceleration of the ions is small. Their emission can usually be neglected.

1. Bremsstrahlung

The accelerated motion of an electron in the field of an ion is the classical process to generate bremsstrahlung. Since the electron is before and after the interaction in a free state the process is called free-free transition. In case the electron is bound in its final state, recombined with an ion, the process is a free-bound transition. Both processes lead to the emission of continuous radiation, forming a continuous background in the spectral range from mm to X-ray regions. While recombination is dominant in the cool plasma edge, at the higher temperatures in the plasma bulk free-free transitions are dominant.

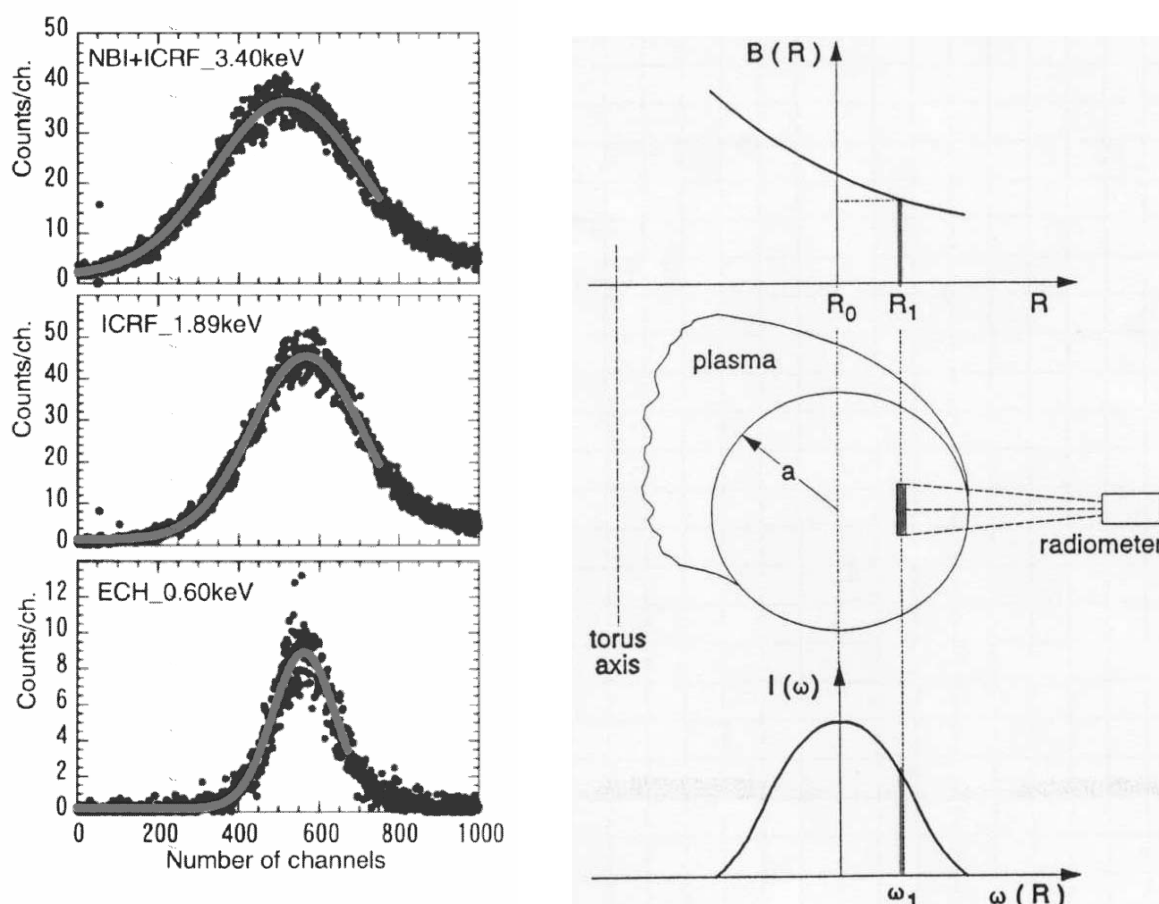


Figure 8.5: Left: The line width of a Doppler-broadened spectral line of the 20-fold ionized Ti for various heating scenarios NBI + ICRF, ICRF, and ECRH resulting in ion temperatures of 3.4, 1.89, and 0.6 keV. Measurements (dark) and simulations of the width assuming a Maxwellian energy distribution nicely agree (courtesy of S. Morita, LHD, Japan). Right: In a toroidal confinement device the B -field monotonically decreases with R across the plasma cross section. As a result the gyration frequency varies with R too. The frequency of the emission is therefore connected with the position of generation. The intensity of the emission is proportional to the local temperature.

The total power emitted within a certain finite spectral interval is proportional to the local densities of ions and electrons, the ion charge squared and to the square root of the electron temperature

$$P \propto Z^2 n_e n_i \sqrt{T_e}. \quad (8.10)$$

Due to this strong dependence on quantities which are in most cases constant on flux surfaces, bremsstrahlung emission can be used to reconstruct the flux surfaces, their shift under the action of plasma pressure as well as to determine MHD-phenomena. To limit the bremsstrahlung power loss, high- Z impurities must be avoided in a future fusion reactor.

The measurements are made in the soft X-ray spectral region using multi sightline pin-hole camera systems with hundreds of sightlines. The local emissivity is reconstructed from the line integrated measurements applying tomographic reconstruction techniques. The spectrum shows a characteristic exponential decay with frequency ω

$$I_\omega d\omega \propto e^{-(\hbar\omega)/(kT_e)} d\omega \quad (8.11)$$

from which the electron temperature can be determined without absolute calibration of the detection system.

2. Cyclotron radiation

In a magnetically confined plasma the Lorentz-force causes the electrons to gyrate around the field lines with a frequency

$$\omega_0 = \frac{e}{m_e} B . \quad (8.12)$$

This accelerated motion leads to the emission of electromagnetic radiation at the cyclotron frequency ω_0 and its harmonics. At B -fields of the order of a few T emission is in the mm-wave range. Since the energy density of the emission is proportional to the acceleration squared, the intensity of the emission caused by the ion gyration motion at the same energy is completely negligible. Only the electron cyclotron emission (ECE) has therefore gained diagnostic importance. With the line of sight in a poloidal plane perpendicular to the plasma axis, the width of the emission spectrum is dominated by the B -field variation through the plasma, resulting in a continuous spectrum with a width of typically several 10 GHz. At sufficiently high electron temperature and density, the plasma is optically thick in the range of cyclotron frequencies. The intensity of the emission then reaches the black-body level which, through Plancks law, is solely determined by the electron temperature. Since the frequency is determined by the known local field $B(R)$, a local electron temperature $T_e(R)$ can be deduced from the measured intensity $I(\omega)$. The classical Rayleigh-Jeans approximation of Plancks law then gives a linear relation between electron temperature T_e and the intensity I

$$I_\omega d\omega = \frac{\omega^2}{8\pi^3 c^2} \frac{\hbar\omega}{\exp(\hbar\omega/(kT_e)) - 1} d\omega \approx \frac{\omega^2}{8\pi^3 c^2} kT_e d\omega . \quad (8.13)$$

In this simplified picture the energy of individual electrons of a thermalized population has not been considered. It causes, as a result of the relativistic mass increase with kinetic energy, a downshift of the emission frequency, or -accordingly- at given observation frequency a shift of the emitting layer away from the so-called cold resonance into regions of higher B -field.

As a result, the emission of the electrons at the various energies of a Maxwellian distribution is smeared out over a certain radial range behind the cold resonance. However, due to the multi-step emission and reabsorption processes of the radiation on the way to the observer, radiation can escape only from a layer right behind the cold resonance whose temperature is being reflected in the observed blackbody intensity. The mechanism only

holds in case the B -field is monotonically decreasing to the observer, which is the necessary condition of the method. A radiometer calibrated against a blackbody source of known physical temperature determines basically the ECE radiation temperature T_{rad}

$$T_{rad} = T_e [1 - e^{-\tau}] \quad (8.14)$$

approaching the electron temperature T_e in case the optical depth τ is sufficiently high. The optical depth is obtained integrating the absorption along the line of sight. The leading parameters determining the absorption are the local electron density and temperature. Densities in the 10^{19} m^{-3} and temperatures of about 50 eV are sufficient to fulfil the conditions.

8.6 Particle diagnostics

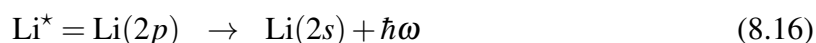
Atoms and ions are injected to diagnose the plasma in various ways. While impurity pellets are injected to study their excitation, ionization, their dwell time, and motion with the passive spectroscopic diagnostics described, well collimated low and high energy H, He, or Li atomic beams are used to investigate local phenomena occurring along the beam as a result of its interaction with the plasma particles. Neutral atoms are used to avoid deflection in the magnetic field. However, also ions of high energy and mass are sent into the plasma to probe its local electric potential.

Passive particle diagnostics analyze neutral atoms escaping from the plasma after charge exchange interaction with fast H-atoms injected by a so-called diagnostic injector. Also the diagnostic of neutrons produced in fusion reactions fall into this category.

8.6.1 Active particle diagnostics

Li-beam

A well collimated beam of neutral Li-atoms is being injected into the plasma, directed to the plasma axis. Beam energy and electron density in the plasma determine the penetration depth. With energies of some 10 keV it is of the order of a few 10 cm. In electronic collision the beam atoms are excited



and the intensity of the resonance line from the $2p$ to $2s$ transition is being detected perpendicular to the beam. Arrays of detectors allow for the simultaneous observation along the beam. The intensity reflects almost directly the electron density, which can be determined in this way with high spatial resolution in the very important edge region of the plasma where density gradients are often very high and where its exact knowledge is of great importance.

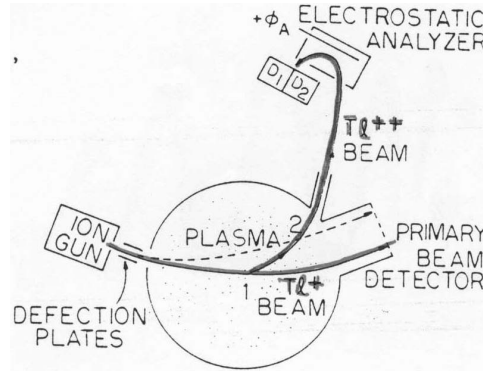
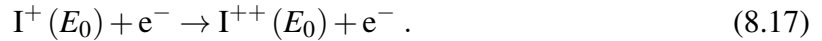


Figure 8.6: Experimental arrangement of a heavy ion beam probe diagnostic. Tl^+ ions are generated in an ion gun and launched into the plasma (trajectory 1). If the ion is further ionized to Tl^{++} , the gyro-radius is decreased (trajectory 2). From the energy change of the Tl^{++} ions, the local plasma potential at the position of the ionization can be deduced (T. Dolan, Fusion Research, p. 246).

Heavy ion beam probe

High energy, heavy mass singly charged ions are shot into the plasma in a well collimated beam of small divergence. Typical energies range from 100 keV to several MeV. Ions used for this purpose are typically Cs^+ , Ta^+ or Au^+ . Due to the Lorentz-force the ions gyrate around the field direction, with a gyro-radius larger than the plasma radius. Along the only slightly bended beam path, in electronic collisions ionization processes change the charge q of the primary beam ions from e to $2e$



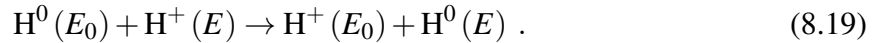
As a result the ion gyro-radius

$$r_g = \frac{vM_I}{qB} \quad (8.18)$$

is decreased by a factor of two right at the location where the ionization occurs. The new trajectory of the *secondaries* now differs clearly from the primary injected ions. If the plasma has a locally varying finite electric potential $\Phi(s)$ along the beam path s , the energy of the ion passing the plasma changes by $e\Phi(s)$ reaching the initial value E_1 again when leaving it, since the plasma is electrically neutral and energy gained when entering is lost when leaving the plasma or vice versa. If, however, at position s_0 higher-ionization by one step occurs, the two-fold ionized particle leaves the plasma with an energy $E_2 = E_1 + e\Phi(s_0)$, reflecting the local potential at position s_0 . The energy difference $E_2 - E_1$ is determined via deflection in electric fields. Scanning the primary beam and the detection geometries, the potential distribution across the plasma can be determined, which is of importance to interpret transport processes in the plasma.

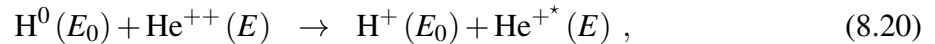
High energy neutral hydrogen injection

A beam of high energy neutral hydrogen atoms H^0 generated in a system called diagnostic injector with sufficient energy $E_0 > 50$ keV to penetrate the plasma as much as possible produces along its trajectory due to charge exchange processes with the hydrogen bulk ions neutral hydrogen atoms which can leave the plasma almost unopposed



The velocity distribution of these atoms leaving the plasma reflects the velocity distribution of the plasma ions since in the interaction process to a good approximation only charge has been transferred, not energy and momentum. Energy and momentum analysis of escaping atoms in electric and magnetic fields after a new external ionization therefore gives detailed local information about the ion energy distribution i.e. the ion temperature in the plasma. The position is defined by the overlap of the primary beam and detection geometry.

The diagnostic injector can successfully be used in a further application. For this purpose a small amount of He atoms is introduced into the hydrogen plasma as a tracer element. At the typical electron temperatures He is fully ionized. In a similar process as described before, the He^{++} ions exchange charge with the neutral hydrogen atoms of the injected beam to result in electronically excited singly charged He^+ ions



with $\lambda = 468.4$ nm . Again, since no energy is transferred in this process, the excited He^+ ions still reflect the He^{++} initial energy distribution. The Doppler-width of the spectral line at wavelength λ emitted in the de-excitation process then reflects the ion temperature of the plasma bulk ions since thermal equilibrium can be considered. The crossed beam-sightline geometry enables a local measurement of the ion temperature.

8.6.2 Passive particle diagnostics

Charge exchange neutrals leaving the plasma are already discussed together with the active diagnostics. Langmuir-probes extracting particle from the plasma are discussed in a separate chapter. Only fusion neutrons need to be briefly discussed here.

Neutron diagnostics

Neutron diagnostics will play a major role in future burning plasma experiments (BPX) like ITER where the analysis of the 14.2 MeV fusion neutrons becomes most important. In existing fusion experiments operating with mixtures of deuterium and hydrogen or with pure deuterium

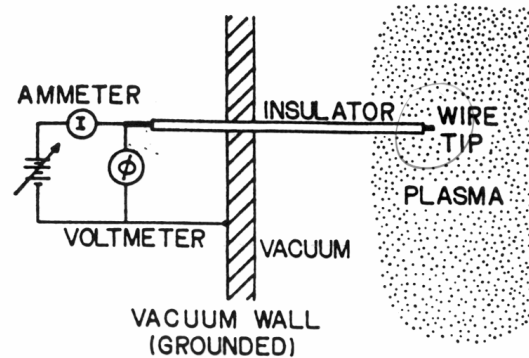
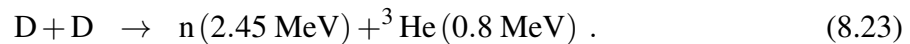
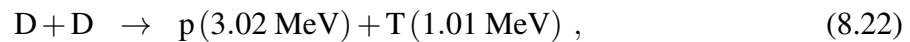


Figure 8.7: Langmuir-probes consist of a tiny electrode of about 1 mm length which is in direct contact with the plasma edge. Depending on the bias voltage, either ions or electrons are extracted from the plasma. From the measured I–V characteristic information can be gained about the electron and ion energy distribution function as well as about the plasma potential. (T. Dolan, Fusion Research, p. 239).

plasmas, neutrons are produced in two different reaction schemes with the kinetic energies of the reaction products given in brackets



The first scheme produces charged products only which can leave the plasma in nowadays medium sized experiments due to their high energy but will stay confined in future large machines. The neutrons from the second reaction, however, will leave the plasma. Energy analysis of the Doppler-broadened line at 2.45 MeV gives information about the energy of the reacting ions in the plasma.

8.7 Langmuir-probes

Langmuir-probes are counted among the oldest plasma diagnostics with still high importance in both fields fusion high temperature and in particular low temperature plasma diagnostics. The Langmuir-probe is a tiny metallic electrode in direct contact with the plasma. Retroaction to the plasma can therefore not completely excluded. In fusion experiments probes are moved into the plasma only for some 10 ms to avoid overheating of the device. Dependent on the potential applied, the small electrode either injects or extracts charged particles from the plasma. Currents to the probe are governed by the electron density and the electron temperature as well as the plasma potential. All three quantities can be derived from the measured current-voltage characteristic of the probe, which is obtained conducting a fast voltage sweep.

Table 8.1: Main methods and some important quantities. Diagnostic techniques given in the rows, most important parameters in the columns.

	f_e	f_i	n_e	n_i	n_Z	T_e	T_i	p	E	B
Magnetics						×		×		×
Particle flux	×	×	×	×		×	×		×	
Refractive index			×							×
Cyclotron emission	×					×				
Bremsstrahlung			×		×	×				
Line radiation					×		×		×	×
Scattering	×		×			×		×		
Charge exchange		×			×		×			
Nuclear reaction		(×)		×			×			

8.8 Summarizing remarks

To summarize the various techniques, Tab. 8.8 connects the main methods and some important quantities they are aiming at. The diagnostic methods briefly discussed in the frame of this introduction to the field are linked matrix-like with the physical quantities the measuring systems are diagnosing.

The various diagnostic techniques given in the rows are based on measurements of magnetic flux changes and particle fluxes from the plasma, measurements of the plasma dielectric properties, the radiometry of the cyclotron emission, the spectroscopy of the continuous bremsstrahlung and the discrete impurity line emission, on laser light scattering and charge exchange processes and based as well on nuclear reactions resulting in neutrons leaving the plasma.

The most important parameters as given in the columns are: $f_{e,i}$, energy distribution function of electrons and ions, $n_{e,i}$, electron and ion densities, $T_{e,i}$, electron and ion temperatures, p plasma pressure, E and B electric and magnetic fields. This is of course only a small part of the plasma parameters to be measured. However, to list them all would definitely go beyond the scope of this introductory chapter.

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Chapter 9

Safety and environmental aspects of fusion

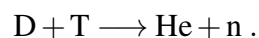
Hans-Werner Bartels

9.1 Introduction

Future fusion reactors will operate with radioactive materials: Tritium is radioactive and very mobile and the neutrons produced in the fusion reactions will activate the components near the plasma. To assess the radiological hazards of fusion, the inventories and possible release mechanisms are discussed. The behaviour of the plant during normal operation and accident situations is analyzed together with the waste management. The control of the mobile tritium is a key issue in reducing the effluents in normal operation. Associated with this is the design goal of minimizing the tritium inventories which can be released in accidents. The absence of chain reactions and the moderate decay heat densities in a fusion reactor are inherent safety-enhancing differences to fission plants. Since the neutron-induced radioactivity strongly depends on the materials used in a fusion-reactor, appropriate material selection offers the possibility of significantly influencing the amount of radioactivity produced. Generally, the decay times of fusion waste are much shorter than those of fission waste. Recycling of reactor materials after 50 to 100 years of cooling time is a potential waste management option instead of geological disposal.

9.2 Inventories

The safety implications of nuclear fusion can immediately be seen from the fundamental reaction equation



Two distinct classes of radioactive inventories follow from this:

- Tritium as part of the fuel is radioactive.

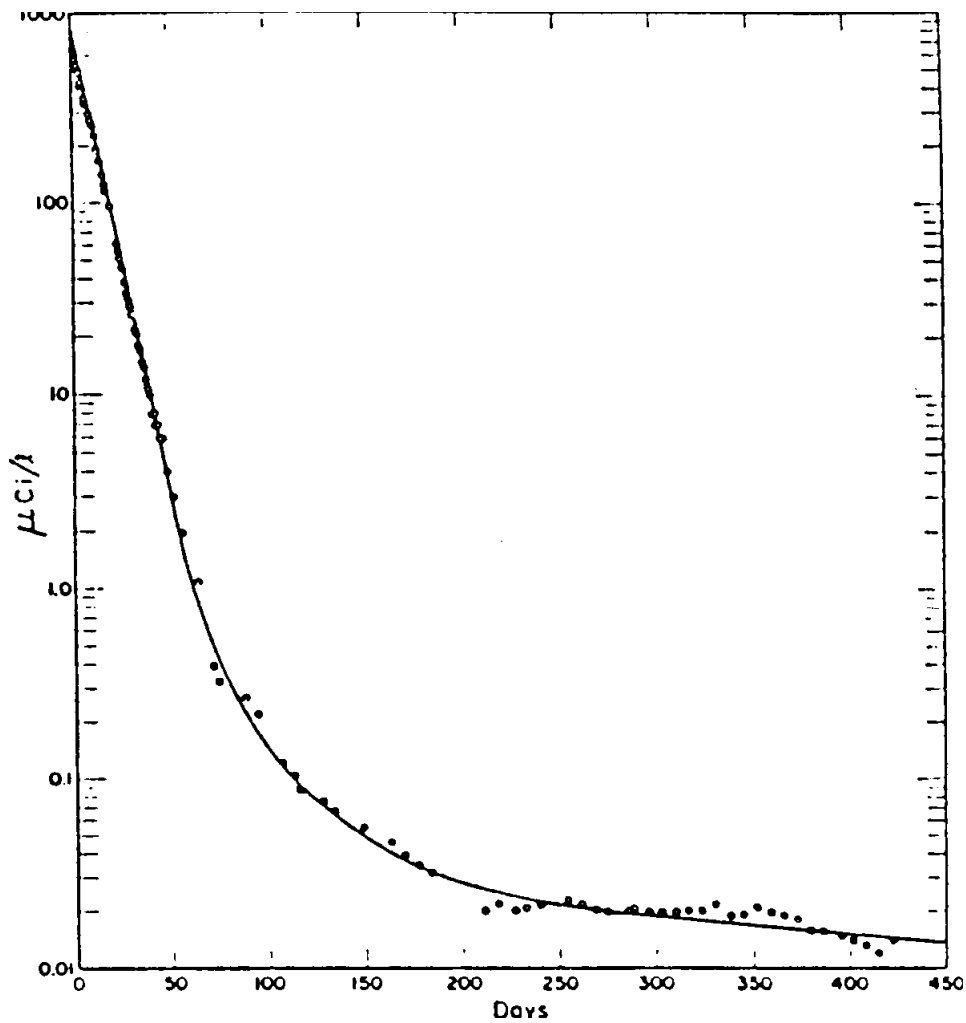


Figure 9.1: Time-dependent tritium concentration in a human after a single accidental incorporation of HTO.

- Neutrons leaving the plasma do not just produce tritium by nuclear reactions with lithium (breeding). They also perform nuclear reactions with other materials in the reactor, mainly those of structures near the plasma (first wall). Many of these reactions lead to radioactive end-products.

Tritium

The expected tritium inventory of a fusion plant amounts to a few kg. Although just 0.5 g of tritium will be inside the hot plasma, much larger amounts will be present in other systems such as the blanket, pumping system, tritium processing plant and storage.

Tritium is a weak β^- emitter with a half-life of 12.3 years. Its mean β^- energy is 5.7 keV, its maximum energy 18.5 keV. Due to the rather low energy, its mean range in organic matter is below $6 \mu\text{m}$, whereas the human skin is covered by a horny layer which is $70 \mu\text{m}$ thick. Thus tritium does not pose any hazard as an external radiator. The situation changes when tritium

is incorporated into the body via ingestion, inhalation, or skin absorption. The most common chemical species of tritiated molecules are HTO (tritiated water) and HT (tritiated hydrogen gas). Both forms are very mobile since their behaviour is similar to the non-tritiated species H_2O and H_2 . Especially HTO can easily be absorbed into the human body. Once in the body it is rapidly dissolved in the human water cycle.

Fig. 9.1 shows the decrease of tritium in the body of a technician working in a tritium laboratory who accidentally inhaled a significant amount of HTO. The measurements show a decay with three time constants. The dominating decay time of about 10 days reflects the recycling in the water compartment of the human body. A reference person weighing 70 kg has a water compartment of 42 l and a daily water exchange of about 3 l leading to a biological half-life of ten days. The longer components of about 30 and 300 days in Fig. 9.1 pertain to the exchange between water and the organic compartment. On the average the slow component accounts for roughly 10% of the tritium dose. Thus the *effective* biological half-life of tritium in the human body is 11 days.

The intake of a certain amount of tritium measured in activity A (unit: Bq = decays per second) leads to the following energy deposition in the human body

$$\varepsilon = A \cdot \tau \cdot \langle E_{\beta^-} \rangle \cdot m^{-1}$$

with ε is the energy density [J/kg], A the activity [Bq], τ the effective biological lifetime [s], $\langle E_{\beta^-} \rangle$ the average β^- energy [J] and m the mass of reference person [kg].

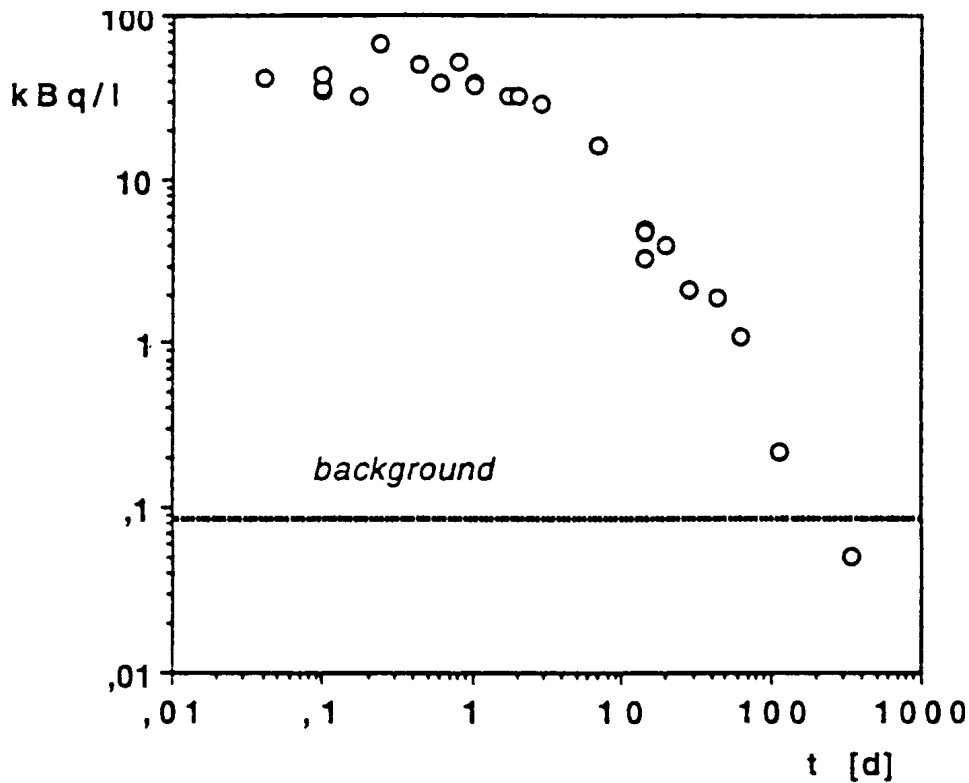
Generally, the energy dose has to be multiplied with a quality factor to account for radiation forms more hazardous than γ -rays, e.g. α -radiation is assigned a quality factor of 20. The resultant quantity is the radiation *dose* measured in units of Sievert (Sv) and allows quantification of radioactive radiation hazards to humans. For tritium the international consensus is to use $Q = 1$, but some experiments tend in the direction of $Q = 2-3$ for tritium since the low β -energy leads to localized ionization.

With $Q = 1$, the intake of 1 Bq HTO leads to a dose of 1.8×10^{-11} Sv. Intake of tritium in organic form, e.g. by eating contaminated food, would increase the dose by a factor of 2.5. This is due to a larger fraction of tritium transferred into the body compartments with longer biological lifetimes. For a given air concentration, tritiated gas (HT) is 10 000 times less hazardous than HTO since the uptake and incorporation into the body is strongly reduced. Tab. 9.2 shows a comparison of doses for different radioisotopes. It can be clearly seen that the amount of radioactivity measured in units of activity (Bq) is not a good indicator of the associated hazard. Tritium produces smaller radiation doses per Bq because the average energy per release is much smaller than for other isotopes and the effective lifetime in the human body is also shorter.

The special concern with tritium is its high volatility and the associated problems in keeping it inside the fusion plant. Once released, tritium is still very volatile. In Fig. 9.2 the specific tritium activity in soil surface water is shown versus time after a controlled release experiment in Canada. In the first year the tritium concentration dropped by a factor of 1 000 and reached the background values which were measured before the experiment. Thus strong contamination of large areas which lasts for many decades is not expected in the event of a major tritium

Table 9.1: Dose conversion factors for different isotopes.

Isotope	Dose Conversion Factor	Pathway	Relation to Tritium
Tritium	1.8×10^{-11} Sv/Bq	ingestion	1
	1.8×10^{-11} Sv/Bq	inhalation	1
Cs-137	1.4×10^{-8} Sv/Bq	ingestion	800
Pu-239	1.0×10^{-7} Sv/Bq	ingestion	5 500
	1.0×10^{-4} Sv/Bq	inhalation	5 500 000

**Figure 9.2:** Time-dependent specific activity in the soil humidity of the top 20 cm after a controlled tritium release experiment (Canada 1987).

release. Instead, most tritium will end up in the oceans, which carry a natural tritium inventory of a few kg. This tritium is formed by nuclear reactions of cosmic rays entering the earth's atmosphere (e.g.: $n + {}^{14}\text{N} \rightarrow {}^{12}\text{C} + \text{T}$). During the period of extensive nuclear weapon testing in the fifties and sixties, the earth's inventory was increased to about 500 kg. Today the residual inventory has been reduced by decay of tritium to about 80 kg.

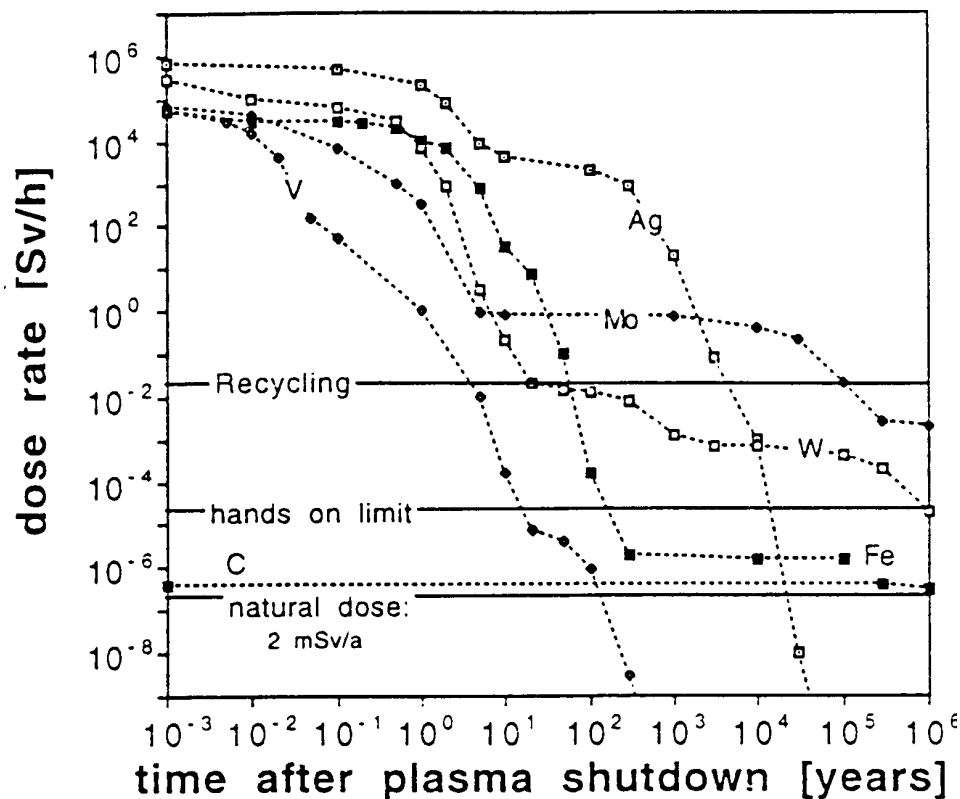


Figure 9.3: Dose rate of several elements after neutron irradiation of 12.5 MWa/m².

Activation products

The expected neutron wall loading is in the range between 2–5 MW/m², which will lead to considerable radiation damage of the components near the plasma. Thus a regular exchange of these components after 2.5–5 years of operation is envisaged. A target for fusion materials is a possible neutron fluence of about 12.5 MWa/m². The development of such materials is a very challenging task.

The amount of activation products in future fusion reactors strongly depends on the choice of materials. Fig. 9.3 shows the dose rates of some elements in the first wall. The total amount of radioactivity in a fusion plant will be governed by the activation products. If conventional steels are used, a level of about 50% of the radioactivity in fission plants is reached.

There is a clear distinction between the inventories of tritium and activation products, i.e. the volatility. The radioactivity inside the structural material is very hard to mobilize by definition since the structure has to withstand large forces during normal operation.

Special attention has to be given to the first-wall dust eroded by the plasma, because it can easily be mobilized. The use of very low activation materials such as carbon, beryllium, or boron would strongly reduce the hazard of this dust (see Fig. 9.3 for the dose rate of C). If high-Z materials have to be used in the plasma facing components (e.g. tungsten in the divertor), the dust inventory could pose problems similar to those of the tritium inventory. As indicated, a main issue of the activation products is waste management, which will be treated

later. The environmentally most attractive elements for structural materials in a fusion plant are vanadium, chromium, and titanium.

9.3 Normal operation effluents

During normal operation, it is the volatile tritium which has to be properly controlled. Experience of tritium handling in large facilities is available from a special type of fission reactor working with natural uranium and heavy water as moderator, the Canadian CANDU reactors. In this type of fission reactor, inventories of a few hundred grams are produced from n-capture in D₂O. Experience with these CANDU reactors allows the extrapolation that with present technology tritium effluents can be limited to 2 g per year or 50 Ci/day (1 Ci = 3.4 × 10¹⁰ Bq). In Los Alamos, USA, the largest fusion-oriented tritium laboratory, with a maximum inventory of 200 g, is in operation. The experience from this laboratory (TSTA: Tritium Systems Test Assembly) has demonstrated that one might even do better than the extrapolation above. This laboratory was closed for three month after an unexpected release of 0.4 Ci HT, which demonstrates the high safety level of the plant.

If we assume a daily release of 50 Ci in the more hazardous form HTO from a 100 m high stack, the resultant dose to the most exposed individual (MEI) living at the fence has to be calculated. It is assumed that food and sometimes also water originate from the location of the MEI. A conservative estimation of the dose to the MEI is very simple: The source term S [Bq/s] will be diluted by atmospheric dispersion. For a 100 m high stack, the plume will reach the ground at about $R_{max} = 1\,000$ m, leading to a maximum ground level air concentration c_{air} of

$$c_{air} = S \cdot \frac{1}{v_{wind}} \cdot \frac{f}{2\pi R_{max}} \cdot \frac{1}{2h_{stack}}$$

with S being the source term in [Bq/s], v_{wind} the wind speed in [m/s], f a factor accounting for nonuniform wind rose, R_{max} the downwind distance from the source in [m], and h_{stack} the height of the source in [m].

A conservative assumption is $f = 3$ and a typical average wind speed is 6 m/s at a height of 100 m. Given the air concentration, one can transform this to a concentration in the air humidity

$$c_{H_2O,air} = c_{air} / \rho_{H_2O,air} ,$$

$\rho_{H_2O,air} = 10 \text{ g/m}^3$ is a typical value for the Central European growing season.

Assuming complete equilibration of the specific tritium activity of the air humidity with that of soil water, plant water, animals, and humans, the average tritium concentration in the MEI is equal to $c_{H_2O,air}$ leading to a dose rate D of

$$D = c_{H_2O,air} \cdot \langle E_{\beta^-} \rangle .$$

Table 9.2: Energy inventories and characteristic time scales for their release in a fusion reactor (water volume in the primary loop: 400 m³, heat of evaporation at 1 bar: 900 GJ).

Source	Energy [GJ]	Time Scale
Plasma	2	< 1 s
DT in Plasma	250	1 min
Coils	100	1 min
Cooling Water	300	10 min
Decay Heat	80	1 hour
	550	1 day
	2 000	1 week

For a source term of 50 Ci/d this simple approach results in an annual dose of 23 $\mu\text{Sv/a}$, which is about 1% of the dose received from natural sources of radioactivity. More sophisticated models predict doses to the MEI which are 2–3 times smaller.

9.4 Accidents

With respect to safety in the event of accidents, the most important characteristic of fusion is self-termination of the fusion process under such conditions. So-called *passive safety* appears to be achievable in fusion power plants because of the absence of chain reactions. The energy sources that could drive accidents are low compared with those in fission, whereas the volumes and surface areas are large. Tab. 9.4 lists some relevant energy inventories together with typical time scales of release.

After shutdown of the reactor, the decay of radioactive nuclei constitutes a heat source, known as decay heat. Passive removal of the decay heat is a key safety feature. In the event of loss of coolant accidents (LOCA), the core of a fission reactor is prone to melting if no additional cooling can be applied. In the case of fusion, the decay heat is much smaller and is distributed over large surfaces. If all cooling media are lost, just thermal conduction and radiation are left as heat removal mechanisms. Fig. 9.4 shows a detailed analysis of such a situation for the ITER device. This is a good example of passive safety since the removal of the decay heat is managed by physical mechanisms that do not rely on engineering systems, which can always fail with a certain probability. Thermal radiation will never fail. The maximum temperatures in the fusion device stay well below the melting point of the steel structures (1 400°C).

The safety studies in fusion are prompted by the large radioactive inventories involved, but the fundamental difference between fusion and fission reactors is the fact that in the former major accidents are not driven by the nuclear system itself. Instead most safety studies focus on conventional hazards. The most important accident initiators are

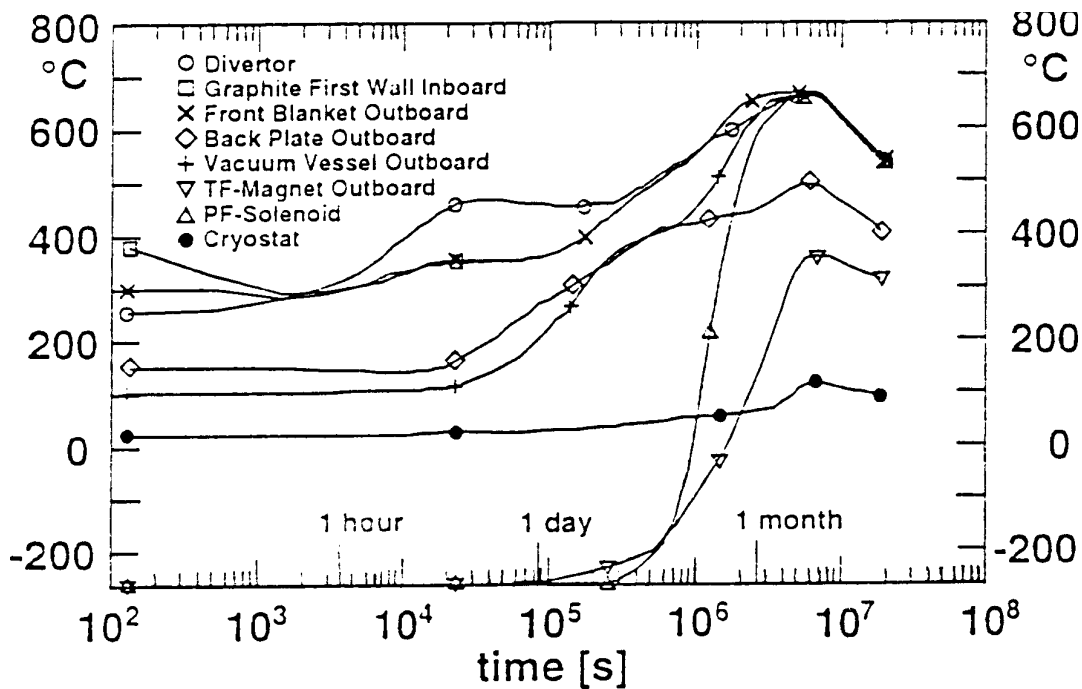


Figure 9.4: Temperature transient up to seven months for loss of all active cooling accident in ITER.

- loss of vacuum accidents with air ingress into the plasma chamber;
concerns: chemical reactions (fire), transport of tritium
- loss of coolant accidents;
concerns: pressurization, formation of hydrogen (explosion hazard)
- loss of offsite electric power;
concerns: loss of coolant flow, quench of magnets
- electric arcs in the coil system;
concerns: melting of adjacent systems, tritium release to the building
- coil deformation and displacement;
concerns: damage of adjacent components, loss of vacuum, missile generation
- volume change of cryogenic helium;
concern: pressurization
- explosions in the hydrogen isotopic separation system;
concerns: damage of building, tritium release
- fire in the long-term tritium storage beds;
concern: tritium release

The most detailed analyses have been performed for the ITER conceptual design phase. One example is air ingress into the plasma chamber, which is of concern since then one major

confining barrier is breached. For a design where the majority of the first wall is covered by carbon, chemical reactions of this material were investigated. No self-sustained fire can develop in such a case since the plasma would definitely extinguish and the first-all cooling would very effectively reduce the carbon temperatures. Even if the active cooling was simultaneously, the inertia of the cooling system would be sufficient to cool the carbon below its ignition temperature. The mobilized material, mainly tritium in the carbon dust and pumping system, would be released into the adjacent rooms. Here it would be confined in the second of three confinement barriers.

Generally, the aim of fusion safety studies is to ensure that future fusion plants are designed so that any internal accident and all conceivable external events should entail consequences confined inside the fusion plant by passive means. No internal accident should be able to destroy the buildings. The strategy is to provide a robust multi-confinement system, reduce all radioactive and energy inventories as far as possible, apply passive safety features and use high-quality nuclear components. Active systems can also play an important role to reduce any damage inside the plant.

In fusion reactors large amounts of lithium will be used to breed the necessary tritium. This could be done very efficiently by using liquid lithium. In this case the lithium could also serve as coolant since it has very good thermal properties which would simplify the design. Technical problems are still presented by the large pressure losses of liquid metals flowing in magnetic fields (MHD effect). Another serious drawback are safety concerns arising from violent chemical reactions of liquid lithium with air, water, nitrogen, and concrete. The potential energy release of the lithium inventory is about 7 000 GJ. Any contact with water produces large amounts of hydrogen with the unpleasant tendency to trigger detonations, which can seriously threaten the confinement buildings.

An alternative approach is the use of lithium ceramics such as Li_2O , Li_4SiO_4 , Li_2ZrO_3 , and other candidates which eliminate the chemical hazard caused by elemental lithium. Unfortunately, the need for a self-sufficient tritium breeding ratio calls for the neutron multiplier, beryllium (${}^9\text{Be}(n,2n)2\text{He}$), in almost all ceramic breeder designs. Beryllium is a very toxic material which builds up significant tritium inventories during neutron irradiation. In addition, there are questions about the abundance of beryllium on earth. At temperatures above 600–700°C some violent chemical reactions can also occur with air and water. Consequently, attempts are being made to push the design of ceramic breeder blankets to the extreme so as to eliminate the beryllium. Li_2O is the best candidate for these studies since it has the highest lithium atom density of all lithium ceramics. The lithium density in Li_2O is even higher than in metallic lithium. In addition the neutron activation of Li_2O is extremely small.

Another option is to use $\text{Li}_{17}\text{Pb}_{83}$, which has a melting point of 235°C and is thus a suitable liquid metal. Its chemical reactivity is much smaller than that of liquid lithium, but the use of lead poses technical problems because of its weight and there are safety concerns because of the neutron activation of lead.

The final choice of the breeder material will strongly influence the safety and environmental properties of future fusion plants.

Table 9.3: Specific dose to the most exposed individual of the public (MEI) at a distance of 1 km from the release point.

Source	Specific Dose	Type of Dose
tritium as HTO	0.5 mSv/g-T	early
	2.5 mSv/g-T	chronic
tritium as HT	0.04 mSv/g-T	early
AISI-316 first-wall steel	0.03 mSv/g	early
ITER conditions	2 mSv/g	chronic

Accidental doses to the public

The early dose is defined as the dose received during the first week after an accident, excluding the ingestion pathway. This dose definition is the basis for considering evacuation. Above 100 mSv evacuation is triggered in many countries. The chronic accidental dose integrates over 50 years of exposure time and includes all pathways. In Germany 50 mSv is the legal upper limit for public exposure after *design basis accidents*, i.e. accidents which have to be controlled by design.

Detailed studies for atmospheric accidental tritium and activation product releases have been performed. Specific doses for the most exposed individual of the public (MEI) at a distance of 1 km from the release point are presented in Tab. 9.4. The dose due to inhalation and skin absorption is 0.5 mSv/g-T (early dose) for ground level releases in the chemical form of HTO. If ingestion is included, the dose values are about five times higher (chronic accidental doses). For gaseous releases (HT), doses are at least a factor of ten lower. The earlier mentioned difference in radiotoxicity of 10 000 between HTO and HT is levelled out by the fact that HT released to the environment will diffuse into the soil, where it is rapidly oxidized by micro-organisms and subsequently re-emitted as HTO. Specific doses for steel dust envisaged during the conceptual design of ITER are about one order of magnitude below the tritium values, but for the chronic accidental doses similar values result.

Under the German licensing procedure, it has to be shown that in design basis accidents the releases stay below 20 g for tritium in the form of HTO for ground level releases. This is achievable as long as the confinement building remains intact. To avoid evacuation, tritium releases have to be kept below 200 g in all circumstances. This is the reason for setting up the safety target of keeping the tritium inventory below 150 g for all single components.

If we compare the potential hazard of the tritium and steel dust of a fusion plant with the hazard of the J-131 and Cs-137 inventories of a fission plant, fusion is 500 times more benign.

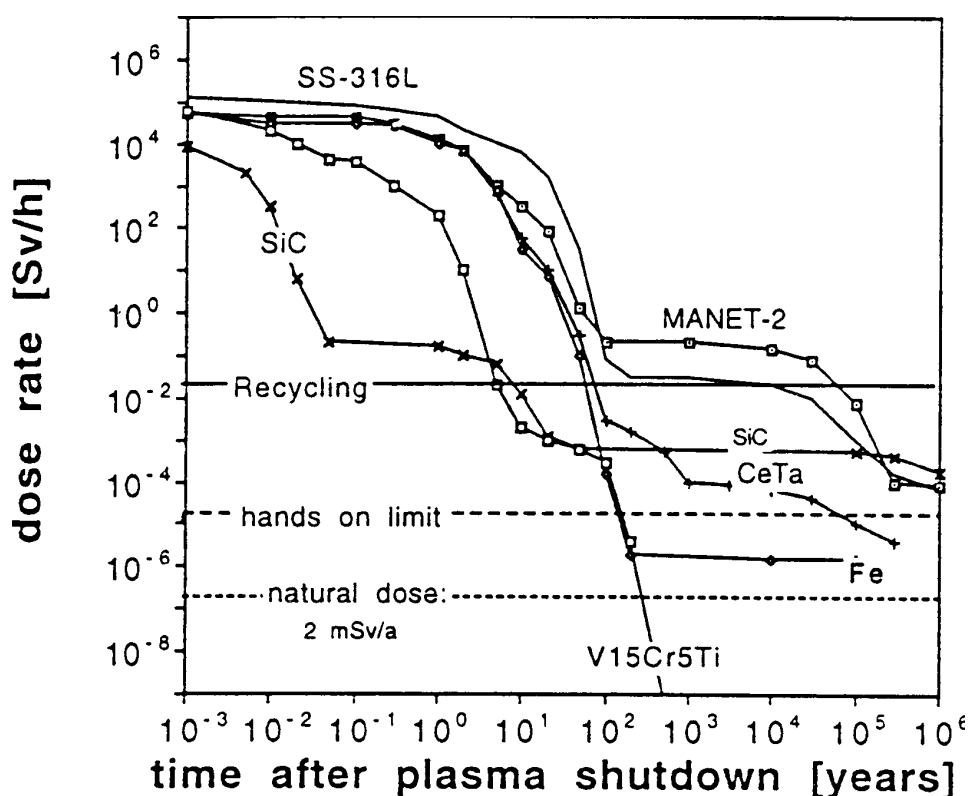


Figure 9.5: Time-dependent dose rates for technologically important materials irradiated by a neutron fluence of 12.5 MW/m^2 ; a standard steel (SS-316L), a fusion optimized steel (CeTa, KfK), the theoretical lower limit for iron based steels (Fe), a ceramic (SiC), and a vanadium alloy (V15Cr5Ti) are shown.

9.5 Waste management

The majority of the radioactive waste in a fusion reactor will arise from replacement of divertor, first wall, and blanket segments and decommissioning of the whole tokamak, including the vacuum vessel and the magnets. These components will be activated by the intense flux of 14 MeV neutrons. Due to the shielding effect of the materials involved, there will be a steep gradient in the activation when going away from the plasma.

The relatively short life of the majority of the radioisotopes produced and the absence of very toxic actinides (α -emitters) causes the ingestion hazard index of the fusion waste to be more than 1 000 times less than for fission waste after 100 years of cooling time. The basis of comparison is the total waste produced during the lifetime of the plants.

The volume of packed waste is estimated to add up to about $25\,000 \text{ m}^3$ for a fusion reactor life. This is comparable to the amount of packed waste from a typical PWR plant life, including reprocessing. Without reprocessing the volume of packed fission waste will be 30–50% smaller. The fission waste still requires much more repository consumption due to the significantly larger heat production in the waste, which limits the packing density in geological repositories.

Fig. 9.4 shows the dose rates of some technologically important materials irradiated at the first wall. Given moderate progress in the development of fusion-optimized materials, the reuse of activated material in new reactors is a potential option of waste management. For reduced-activation steels, recycling of components is estimated to be feasible up to contact dose rates of 25 mSv/h by remote handling techniques. The most activated component, which is the first wall of the plasma chamber, would reach this level 50–100 years after being dismantled from the reactor.

If new classes of low-activation materials, such as vanadium alloys, could be used in fusion reactors, *hands-on recycling* might be feasible after a few decades of cooling time. The benefit will be strong simplification of recycling procedures and thus greater economic attractiveness. However, great care must be taken to keep undesirable impurities to a low level in order to retain the radiological attractiveness, e.g. below 0.1 ppm for silver (see Fig. 9.3).

9.6 Conclusion

A board of independent scientists from different fields are regularly asked to review the European Fusion Programme. The following extract from the last *Fusion Programme Evaluation Board* (Colombo report, July 1990) is quoted to conclude this overview:

It is essential to expand the effort on technology, not only on the Next Step, but also on longer term issues, particularly in relation to the new emphasis on environmental and economic constraints. Environmental and safety criteria should as of now be considered as essential elements in governing the evolution of the Fusion Programme. Problems concerning materials, the use of tritium, the maintainability of the reactor should all be faced in time avoiding freezing of design or engineering concepts which might lead to environmentally unacceptable fusion reactors or make their design excessively complex.

Environment and safety must assume high priority in the European Fusion Programme and in its wider international extension...

It would be particularly important for fusion to qualify in two respects:

1. It must be clearly shown that the worst possible fusion accident will constitute no major hazard to populations outside the plant perimeter that might result in evacuation.
2. Radioactive wastes from the operation of a fusion plant should not require isolation from the environment for a geological timespan and therefore should not constitute a burden for future generations.

Chapter 10

Tokamaks: equilibrium, stability and transport

Hartmut Zohm

10.1 Introduction

The goal of confining a plasma in a magnetic field can be achieved in various geometries. The quality of the magnetic confinement is characterized by different criteria. For nuclear fusion, the following points are of special interest:

- The ratio of kinetic plasma pressure, averaged over the plasma volume, to magnetic pressure $\beta = \langle p \rangle / (B^2 / (2\mu_0))$. This number is a measure of the economic efficiency of the confinement since the fusion power output roughly scales as p^2 and B is the magnetic field which has to be provided externally. Usually, β is limited by the occurrence of MHD (Magneto-Hydrodynamic) instabilities.
- The energy confinement time $\tau_E = W_{pl} / P_{heat}$ which is one of the quantities appearing in the Lawson criterion $n_i \tau_E T_i \geq f(T_i)$ and characterizes the quality of the heat insulation, i.e. the transport properties of the configuration.

The simplest approach is a linear configuration (mirror, pinch) which suffers from losses appearing at the ends. These points are optimized in devices with toroidal geometry. Today, the *Tokamak* configuration is the best developed; the *Stellarator* is expected to show confinement properties of similar quality.

10.2 Plasma equilibrium

10.2.1 The steady state MHD-equations

The equilibrium of a plasma in a magnetic field can be described by the MHD equations. It is described by neglecting the temporal derivatives. If, in addition, no flow is considered ($v = 0$),

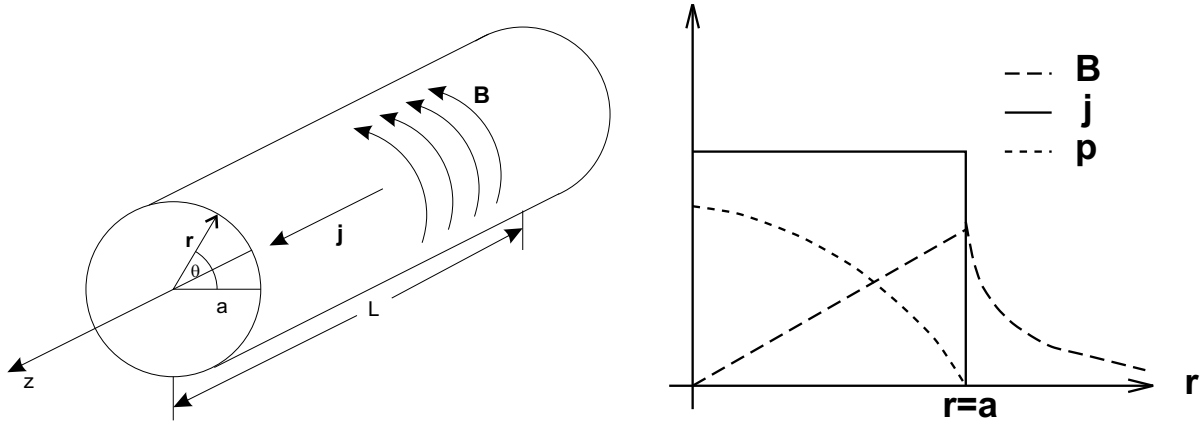


Figure 10.1: The linear z -pinch: geometry and radial profiles of j_z, B_θ and p for the example shown in the text.

the equations reduce to

$$\nabla p = \mathbf{j} \times \mathbf{B} , \quad (10.1)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} , \quad (10.2)$$

$$\nabla \cdot \mathbf{B} = 0 . \quad (10.3)$$

The first equation (10.1) is the force balance which states that a current flowing perpendicular to the magnetic field exerts a force on the fluid element. In equilibrium, this force balances the plasma kinetic pressure, i.e. everywhere in the plasma, the local pressure gradient is balanced by the Lorentz force. In particular, for $\nabla p = 0$, the fields \mathbf{j} and \mathbf{B} are collinear (*force free region*). Eq. (10.2) and (10.3) are Maxwell's equations for a static magnetic field.

10.2.2 Equilibrium in the linear pinch

A simple analytic solution of (10.1)–(10.3) can be given in the case of a linear cylindrical configuration, the so-called pinch. The simplest case is a z -pinch, i.e. a cylinder carrying a current in the z -direction (see Fig. 10.1).

Here, the magnetic field is purely azimuthal and the force balance reads

$$\frac{dp}{dr} = -j_z \cdot B_\theta . \quad (10.4)$$

To explicitly calculate the radial profile $p(r)$, we have to specify a current density profile $j_z(r)$. For simplicity, we chose $j_z = \text{const.} = j_0$ inside the plasma of radius a and $j_z = 0$ outside. Then we can solve (10.2) in cylindrical geometry to obtain

$$B_\theta = \frac{\mu_0 I_p}{2\pi a^2} r \quad \text{for } r \leq a , \quad (10.5)$$

$$B_\theta = \frac{\mu_0 I_p}{2\pi r} \quad \text{for } r > a , \quad (10.6)$$

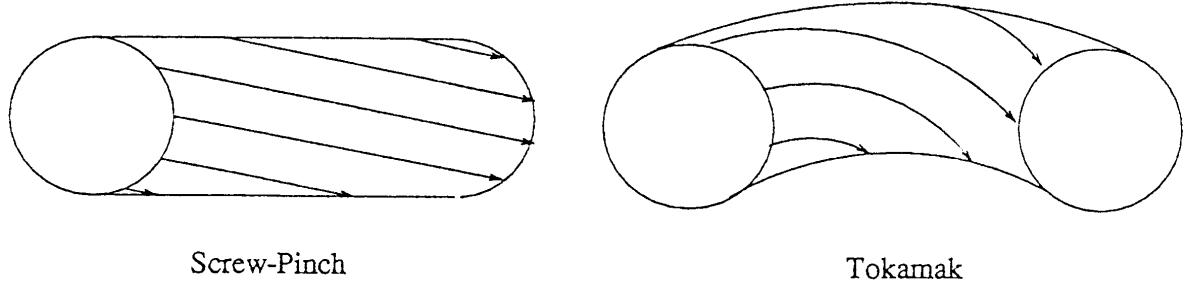


Figure 10.2: Magnetic configuration of the screw-pinch and analogous tokamak.

where $I_p = j_0 \pi a^2$ is the total current flowing in the plasma. Using this result, we can integrate (10.4) to yield

$$p(r) = \frac{\mu_0 I_p^2}{4\pi^2 a^2} \left(1 - \left(\frac{r}{a}\right)^2\right). \quad (10.7)$$

Fig. 10.1 shows a plot of the profiles. Calculating β_p , i.e. the ratio of average kinetic pressure and magnetic pressure associated with the poloidal field

$$\beta_p = \frac{2\mu_0 \langle p \rangle}{B_\theta^2} \quad (10.8)$$

we find $\beta_p = 1$ for this particular example. It can be shown that this is a general result for the z -pinch, independent of the profiles specified.

This simple equilibrium calculation does not include a stability analysis, i.e. we have shown that a state with $\partial t \rightarrow 0$ exists, but we do not know whether this is a stable solution. This will be done in the next section. For the z -pinch, one can show that it is unstable unless we add a magnetic field in the z -direction. Fig. 10.2 shows this magnetic configuration (*screw pinch*).

As can be seen, the magnetic field lines helically wind around the magnetic axis. If we bend the cylindrical screw-pinch to form a torus, we obtain a tokamak geometry. For large aspect ratio (i.e. ratio of major and minor radius $A = R/a$), a tokamak can be approximated by a screw-pinch periodic in z with period $2\pi R$. The properties of the field lines are characterized by the safety factor q defined by

$$q = \frac{\text{number of toroidal windings}}{\text{number of poloidal windings}} \quad (10.9)$$

of a field line on the torus. In the limit of the screw-pinch, the relation $q = (r/R)(B_t/B_\theta)$ holds. From stability considerations it follows that $q \geq 1$ (see below), so that $B_t \approx B_\theta q R/r$. Inserting typical tokamak values (e.g. ASDEX-Upgrade $a = 0.5$ m, $R = 1.65$ m and $q = 3$), we arrive at $B_t \approx 10B_\theta$, i.e. for this screw pinch $\beta_t = 2\mu_0 \langle p \rangle / B_t^2 \approx 1\%$. Thus stability requirements have a severe impact on the economic properties of the device.

In a screw pinch or tokamak, there is the possibility of poloidal currents that exert a force on the plasma. For $\mathbf{j}_p \times \mathbf{B}_t \uparrow \uparrow \mathbf{j}_t \times \mathbf{B}_p$, the plasma pressure balanced by the magnetic field is higher than in the pure z -pinch; in this case $\beta_p > 1$ and the poloidal current weakens the original B_t , therefore the plasma is said to be *diamagnetic*. In the opposite case, $\mathbf{j}_p \times \mathbf{B}_t \uparrow \downarrow \mathbf{j}_t \times \mathbf{B}_p$, we can sustain less pressure, and $\beta_p < 1$ (*paramagnetic* plasma).

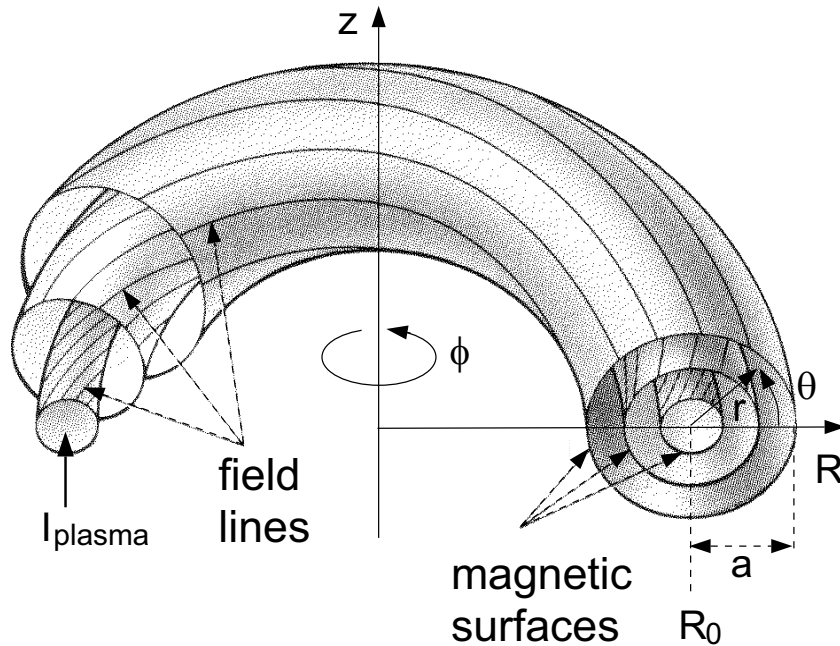


Figure 10.3: Co-ordinates used for the Grad-Shafranov equation.

10.2.3 The Grad-Shafranov equation

For an axisymmetric system like the tokamak, it is convenient to re-write the force balance (10.1). For this purpose, we need the following ideas:

- As $\nabla p = \mathbf{j} \times \mathbf{B}$, $\nabla p \cdot \mathbf{j} = \nabla p \cdot \mathbf{B} = 0$. Therefore the field lines of \mathbf{j} and \mathbf{B} lie in the surfaces of constant pressure.
- The flux integrals $\int \mathbf{B} dA$, $\int \mathbf{j} dA$ have a constant value on the $p = \text{const}$ surfaces for an arbitrary curve C on this surface, since \mathbf{j} and \mathbf{B} lie in this surface and thus any part of the integrals on the surface vanishes. These surfaces are therefore called *flux surfaces* and can be labelled by the (scalar) fluxes. Also, p is only a function of the fluxes.
- As $\nabla \cdot \mathbf{B} = \nabla \cdot \mathbf{j} = 0$, any flux integral $\int \mathbf{B} dA$, $\int \mathbf{j} dA$ has a constant value if the arbitrary surface A is surrounded by the same curve C and thus we are free to choose the surface A . This means that each flux surface can be assigned a unique value of the flux, independent of the geometry in which the flux was calculated.

On a torus, there are two topologically different types of curves: Those winding around the torus in the toroidal direction and those winding around the torus in the poloidal direction. If we choose a curve winding around in the toroidal direction, integration over the domain enclosed by this surface yields the poloidal magnetic flux function Ψ and the total poloidal current I_{pol} . Both functions are constant on the flux surface and the magnetic field can be

calculated as

$$B_\phi = \frac{\mu_0 I_{pol}}{2\pi R}, \quad (10.10)$$

$$B_R = -\frac{1}{2\pi R} \frac{\partial \Psi}{\partial Z}, \quad (10.11)$$

$$B_Z = \frac{1}{2\pi R} \frac{\partial \Psi}{\partial R}, \quad (10.12)$$

where the co-ordinates from Fig. 10.3 have been used. Note that with this definition, Ψ denotes the poloidal flux per unit length in toroidal direction ($[\Psi] = \text{Vs/m}$). Using these quantities, the force balance can be re-written as

$$-\Delta^* \Psi \equiv -\left[R \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial \Psi}{\partial R} \right) + \frac{\partial^2 \Psi}{\partial z^2} \right] = \mu_0 (2\pi R)^2 p(\Psi)' + \mu_0^2 I_{pol}(\Psi) I_{pol}'(\Psi) \quad (10.13)$$

known as the Grad-Shafranov equation. The prime denotes the derivative of a flux surface quantity with respect to Ψ . Eq. (10.13) is usually nonlinear in Ψ . To solve it, we can e.g. specify $p(\Psi)$ and $I_{pol}(\Psi)$ and then solve for $\Psi(R, Z)$. Usually, one will specify boundary conditions. If the plasma is surrounded by a perfectly conducting vacuum vessel, the vessel is a flux surface and therefore $\Psi = \text{const}$ at the vessel determines shape and position of the plasma boundary. To satisfy these conditions, we have to add a solution of the homogeneous equation, i.e. a function Ψ_{ext} with $\Delta^* \Psi_{ext} = 0$. Such a field is produced by external coils, i.e. the solution of the Grad-Shafranov equation with fixed boundary and profile functions tells us how to arrange external control currents to maintain the plasma in equilibrium.

10.2.4 The tokamak

The tokamak principle was already introduced in chap. 2. The plasma acts as a secondary winding of a transformer. Ramping the current in the primary coil induces a loop voltage inside the discharge vessel; after breakdown, a toroidal current flows in the plasma. This current heats the plasma via its resistance (Ohmic Heating (OH), the primary coil is also called the OH coil). The current itself creates the poloidal magnetic field, the toroidal field is produced by external coils (a stellarator produces all its field components by external coils). The poloidal field coils maintain the plasma in force equilibrium as described by (10.13). A sketch of the tokamak principle is shown in Fig. 2.9.

Fig. 10.4 shows typical examples for poloidal cross-sections calculated with (10.13). One can see that the flux surfaces are not centered to one axis, but are slightly shifted. This *Shafranov shift* is due to the forces trying to expand the torus radially outward:

- The kinetic pressure of the plasma.
- The expansion force of a current loop due to the $\mathbf{j} \times \mathbf{B}$ force.

The sum of these two forces is called *hoop force*. In Fig. 10.4 b), one can see that the plasma cross-section does not necessarily have to be circular; in fact, elongation and triangularity

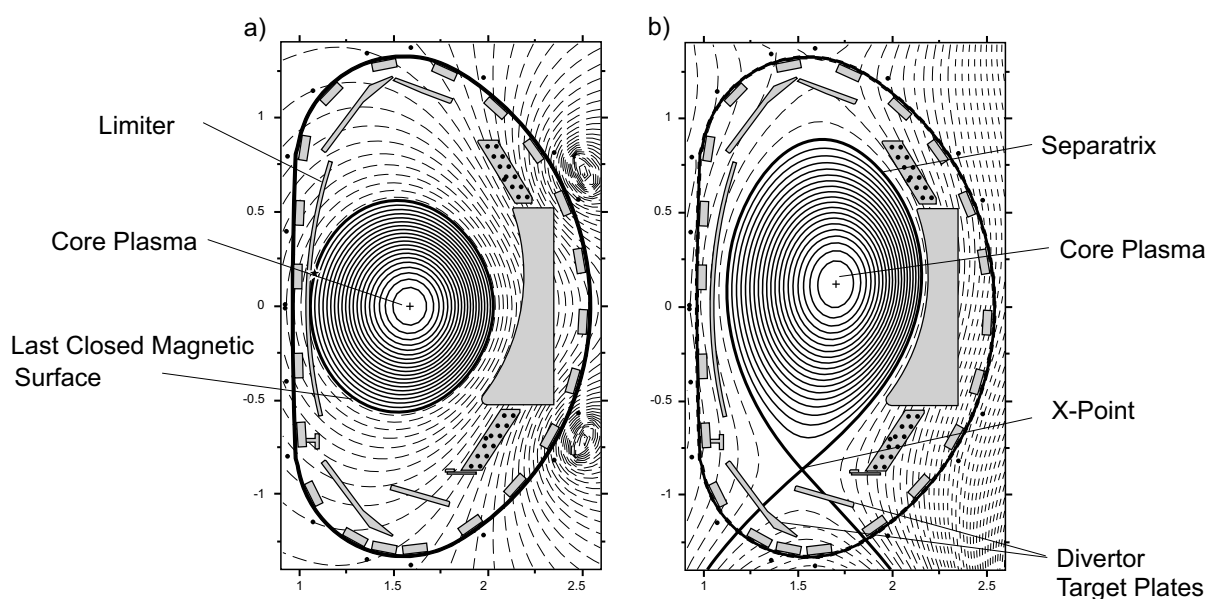


Figure 10.4: Typical poloidal cross-sections of a tokamak.

produced with additional poloidal field coils enhance plasma performance (see next section). One also sees the different ways to limit a plasma: Fig. 10.4 a) shows a so called *limiter* case, where a material limiter defines the last closed flux surface, in Fig. 10.4 b), this last closed flux surface is defined magnetically by a so-called *separatrix*; plasma crossing the separatrix flows along field lines to the material plates at the bottom of the vessel. This configuration is called a magnetic *divertor*.

10.3 Stability of an equilibrium state

As has been mentioned before, the Grad-Shafranov (10.13) describes a force equilibrium, but it does not tell us if the equilibrium is stable, i.e. if a small variation of the plasma parameters or the external control currents will lead to another equilibrium or to an unstable situation. Therefore, we have to analyze the stability properties of a configuration by a separate treatment.

MHD instabilities can deteriorate the plasma performance due to various effects: Large-scale instabilities can lead to a loss of plasma control (e.g. during a *disruption*, the plasma current collapses in an uncontrollable way) whereas small scale perturbations and MHD turbulence can significantly enhance radial transport of particles and energy. Nevertheless, in some cases instabilities may help to control parameters; e.g. ELMs (Edge Localized Modes) periodically expel impurities from the plasma edge and help keep the plasma clean.

10.3.1 Methods of stability analysis

From MHD theory, there are different ways to analyze the stability of a given equilibrium. Often, the analysis is done by introducing a perturbation of the equilibrium configuration as

a displacement ξ of a fluid element. Two methods are especially useful to check the stability against such displacements:

- The energy principle: This approach calculates the energy W_{MHD} of the configuration as a functional of the displacement vector. Stability is obtained if the second order change in energy $\delta^2 W_{MHD}\{\xi\}$ is positive for an arbitrary ξ (the linear variation is zero due to the equilibrium condition). This method is useful to prove that a configuration is *not* stable, because it is sufficient to find one unstable ξ ; on the other hand, stability is hard to prove.
- Eigenmode analysis: Here, the time-dependent MHD equations are solved with an eigenmode ansatz for the displacement ξ . An ansatz often used is the Fourier decomposition in time and space. For the screw pinch, the spatial Fourier ansatz means periodicity in poloidal and toroidal co-ordinate θ and ϕ . Note that in a real torus, symmetry in the θ direction is only an approximation and coupling of poloidal harmonics will occur. For linearized MHD equations, this leads to an eigenvalue problem for each single Fourier mode

$$\xi = \xi_{mn}(r)e^{i(m\theta - n\phi)}e^{\gamma t}. \quad (10.14)$$

The indices m and n are called poloidal and toroidal mode-numbers, for real γ , stability means $\gamma < 0$. If γ has an imaginary part, the eigenmode is called overstable and will exhibit a temporal oscillation. If γ is real and positive, it can be interpreted as a linear growth rate. The saturated amplitude of an instability can only be found by introducing nonlinear effects such as changes in the equilibrium introduced by the perturbation.

Perturbations are easily excited if the field lines do not ergodically cover a flux surface but close upon themselves after few revolutions in the torus. These flux surfaces are the ones with rational safety factor q ; they are called *resonant surfaces* and a standing wave with mode numbers $q = m/n$ can occur.

10.3.2 Classification of instabilities

In a confined plasma, an instability is driven by the free energy contained in the equilibrium configuration. In a tokamak, there are two main sources of free energy: The kinetic energy of the plasma and the energy of the magnetic field generated by the plasma. Instabilities can therefore be driven by the radial gradient of either the pressure or the current profile. At low β , the magnetic energy is much higher than the kinetic energy and the instabilities will mainly be current-driven; at high β , we expect the pressure driven instabilities to become significant.

In cylindrical approximation, the perturbed magnetic field has a radial component. The growth of this field is governed by Faraday's law. Inserting Ohm's law for the plasma yields in the linearized case

$$\frac{\partial B_{1r}}{\partial t} = -\frac{1}{r} \frac{\partial E_{1z}}{\partial \theta} = \frac{1}{r} \frac{\partial}{\partial \theta} \left(\eta j_{1z} + (\mathbf{v}_1 \times \mathbf{B}_0)_z \right), \quad (10.15)$$

where the subscript "1" refers to the perturbed quantities and the subscript "0" to the equilibrium quantities. From this we deduce that there are two ways for a perturbation to grow:

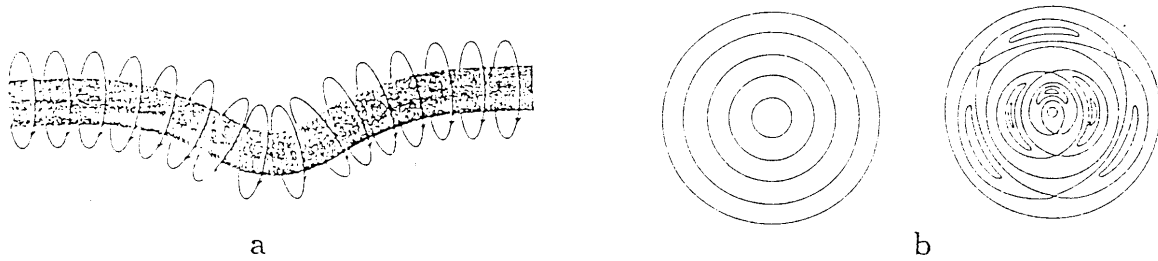


Figure 10.5: Ideal kink (a) and resistive tearing (b) instability. In (b), magnetic islands are formed.

- Ideal case (no resistivity, i.e. $\eta \rightarrow 0$): Flow of plasma perpendicular to the field lines; in this case, $\mathbf{v} \times \mathbf{B}$ balances the E -field and in the fluid frame ($\mathbf{v} = 0$), magnetic flux is conserved, i.e. \mathbf{B} moves with the fluid. This means that there is no change in the *topology*.
- Resistive case ($\eta \neq 0$): A current flows along the equilibrium field lines and generates the B -field; in this case, the topology is changed (*tearing* and *reconnection* of field lines).

Fig. 10.5 shows two examples: The so-called *kink* as a typical ideal instability, the *tearing mode* as a resistive one. In the case of the tearing mode, so-called *magnetic islands* form. Note that the magnetic island is a nonlinear solution of the MHD equations.

10.3.3 Examples

A simple example of an unstable situation is the case of an elongated plasma. In order to vertically elongate a plasma cross-section, we have to run coil currents of the same sign and order of magnitude than the plasma current. For a perfectly centered plasma, the net force is zero. However, any shift δz of the plasma current center increases the attractive force of the coil towards which the plasma moves, so that further acceleration results. The plasma experiences a vertical displacement event (VDE). As the plasma has negligible mass, this is a very fast event (on the order of the Alfvén time scale, i.e. $\approx \mu\text{s}$). In order to feedback control this instability, one therefore has to provide conducting structures that, via induction, slow down the motion to a ms timescale.

Other MHD phenomena have been shown to influence the achievable β of a configuration. An example is the *external kink* driven by current gradients near the edge imposing restrictions on the possible current profiles. Restrictions on the pressure profile come from the so-called interchange instability. Here, neighbouring flux tubes can change place, analogous to the Rayleigh-Taylor instability of a normal fluid, if the field line can shorten and thus lose magnetic energy due to this process. As can be seen in Fig. 10.6, this is the case when the radius of curvature is parallel to the pressure gradient (so-called *bad curvature*). Conversely, if the radius of curvature is anti-parallel to the pressure gradient, an interchange of plasma and magnetic field will increase the magnetic energy and thus be stabilising (*good curvature*). In a screw pinch, only

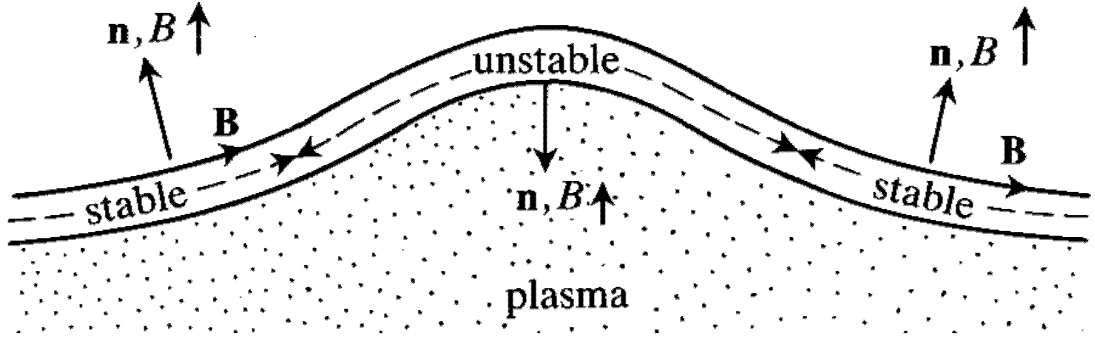


Figure 10.6: Interchange instability in a curved magnetic field.

the poloidal field is curved and its curvature is *bad*, i.e. potentially destabilising. Stability is then achieved by sufficient shear dq/dr so that adjacent field lines have a different pitch angle and are not easily interchanged. In a torus, the toroidal field is also curved, exhibiting an area of good curvature on the inside and bad curvature on the outside of the torus. Thus, stability has to be determined by integrating along a field line and summing up the various stabilising and destabilising contributions. This leads to the Mercier criterion

$$\left(\frac{q'}{q}\right)^2 > -\frac{8\mu_0 p'}{rB_0^2}(1-q^2). \quad (10.16)$$

As $p' < 0$, this means stability for all pressure profiles with $q > 1$; for $q \leq 1$, the shear has to be nonzero to provide stability. At high β , however, a new type of mode, the so-called *ballooning instability*, can develop. This mode has a small perturbation component along the magnetic field, which allows a variation of the amplitude such that it concentrates on the outside of the torus, i.e. in the bad curvature region. Above a (shear dependent) critical pressure gradient, the energy gained by concentrating there is higher than the energy needed to bend the equilibrium field lines, and the mode becomes unstable.

A simultaneous optimization of pressure and current profiles yields a limit for the achievable β for given safety factor q_a and with $q_0 > 1$

$$\beta_{max}[\%] = c \frac{I[\text{MA}]}{a[\text{m}]B_t[\text{T}]} . \quad (10.17)$$

This limit is known as the *Troyon limit*, the constant c depends only on the plasma shape and varies between \approx three and five. Theory predicts an increase in c with elongation and triangularity; this is confirmed by various experiments.

Other examples for MHD instabilities are sawteeth and disruptions, each of them linked to a special rational surface ($q = 1$ and $q = 2$, respectively). This is treated in the section on experimental results of the tokamak.

10.4 Transport in a fusion plasma

10.4.1 Transport equations

For one single particle in a toroidal device, the confinement is perfect. In reality, collisions, drifts and MHD turbulence lead to a radial transport of particles and energy (i.e. perpendicular to the magnetic flux surfaces). It is this transport which determines the global energy and particle confinement times τ_E and τ_p .

To analyse the nature of transport, we define the particle flux Γ as the number of particles passing through a magnetic surface per unit area and time. For Γ we make the following ansatz

$$\Gamma = -D\nabla n + nv \quad (10.18)$$

stating that there is a diffusive part driven by the density gradient (characterized by the diffusion coefficient D) due to the statistical random walk of particles and a convective part due to a directed motion v . The equation of continuity links Γ to the temporal change in density via

$$\frac{\partial n}{\partial t} = -\nabla\Gamma + S, \quad (10.19)$$

where $S(r,t)$ is a source term describing the change in plasma density due to ionisation or recombination (i.e. processes which cannot be described as a flux into or out of the volume element). Finally, in equilibrium, n does not change in time and we obtain

$$\nabla(D\nabla n) + \nabla \cdot (nv) = S. \quad (10.20)$$

For a known source function and with given D and v , we can therefore determine the plasma density profile. On the other hand, measured density profiles are used to determine D and, with some additional assumptions, v . Transport of other quantities as e.g. energy can be treated analogously.

10.4.2 Transport coefficients

In a magnetized plasma, we distinguish between transport coefficients parallel and perpendicular to the magnetic field, e.g. D_{\parallel} and D_{\perp} . In general, the condition $D_{\parallel} \gg D_{\perp}$ holds. Since the confinement properties of a magnetic configuration are governed by the perpendicular coefficients, we will therefore focus on those.

The simplest approach to calculate diffusion coefficients comes from the random-walk ansatz. Here, we assume that, due to collisions with other particles, a particle makes a step Δx perpendicular to the magnetic field after a time Δt . If the step can be done in either direction $\pm\Delta x$ with equal probability, the process is purely diffusive and the diffusion coefficient D is given by

$$D \approx \frac{\Delta x^2}{2\Delta t}. \quad (10.21)$$

In order to estimate D , we have to evaluate Δx and Δt . The latter is given by the inverse of the collision frequency ν_c . With appropriate averaging we obtain the following relations for a 90° scattering (made up of many small angle scatterings) for electron-electron (ee), electron-ion (ei) and ion-ion (ii) collisions (we assume a pure hydrogen plasma, i.e. $Z = 1$)

$$\nu_{ee} \approx \nu_{ei} \propto \frac{ne^4}{\sqrt{m_e T_e}^{3/2}}, \quad (10.22)$$

$$\nu_{ie} = \left(\frac{m_e}{m_i}\right) \nu_{ee}, \quad (10.23)$$

$$\nu_{ii} = \left(\frac{m_e}{m_i}\right)^{1/2} \left(\frac{T_e}{T_i}\right)^{3/2} \nu_{ee}. \quad (10.24)$$

In the so-called classical approach to transport, we take Δx to be the Larmor radius r_L

$$\Delta x = r_L = \frac{\sqrt{2mkT}}{eB}. \quad (10.25)$$

However, the location \mathbf{R} of the guiding centre of the gyro-orbit is given by

$$\mathbf{R} = \frac{\mathbf{p} \times \mathbf{B}}{q_C B^2}, \quad (10.26)$$

with q_C denoting the charge of the particle. In a collision, momentum balance requires $\Delta \mathbf{p}_a = -\Delta \mathbf{p}_b$, and thus, for like-particle collisions, $\Delta \mathbf{R}_a = -\Delta \mathbf{R}_b$. Therefore, at each collision, the particles just change place so that like-particle collisions do not contribute to particle transport (note that heat can be transferred that way, because a hot particle may change place with a cold one).

The situation changes for electron-ion collisions. Here, $q_a = -q_b$ and therefore $\Delta \mathbf{p}_a = -\Delta \mathbf{p}_b$ leads to $\Delta \mathbf{R}_a = \Delta \mathbf{R}_b$. Thus diffusion is ambipolar, electrons and ions make a step of equal magnitude and direction per collision. The diffusion coefficient is then given by

$$D_{e,class} = \nu_{ei} r_{L,e}^2 = \nu_{ie} r_{L,i}^2 = D_{i,class}. \quad (10.27)$$

For the thermal conductivities, similar arguments can be applied to show that the relation $\chi_i \approx (m_i/m_e)^{1/2} \chi_e \approx 40 \chi_e$ should hold. However, experimentally determined diffusion coefficients are larger by a factor of $\approx 10^5$. Also, the electron heat conductivity is found to be comparable to that of the ions. Therefore, classical transport cannot be the dominating transport process in a fusion plasma.

So far, we did not consider the effects of the toroidal geometry. This is the subject of the so-called *neoclassical* transport theory. The main differences to classical transport theory arise from two facts:

- Along a magnetic field line, $|B|$ is not constant; a particle moving along a field line sees a higher field on the inside of the torus and, for sufficient v_\perp/v_\parallel , may be reflected in this magnetic mirror (see chap. 1). Therefore we have to distinguish between *trapped* and *passing* particles. It can be shown that the number of trapped particles is given by $n_t/n = \sqrt{2\varepsilon}$, where $\varepsilon = r/R$ is the inverse aspect ratio of the flux surface. A simple result of this effect is that the electrical conductivity is lowered in neoclassical theory as trapped particles cannot contribute to the toroidal current.

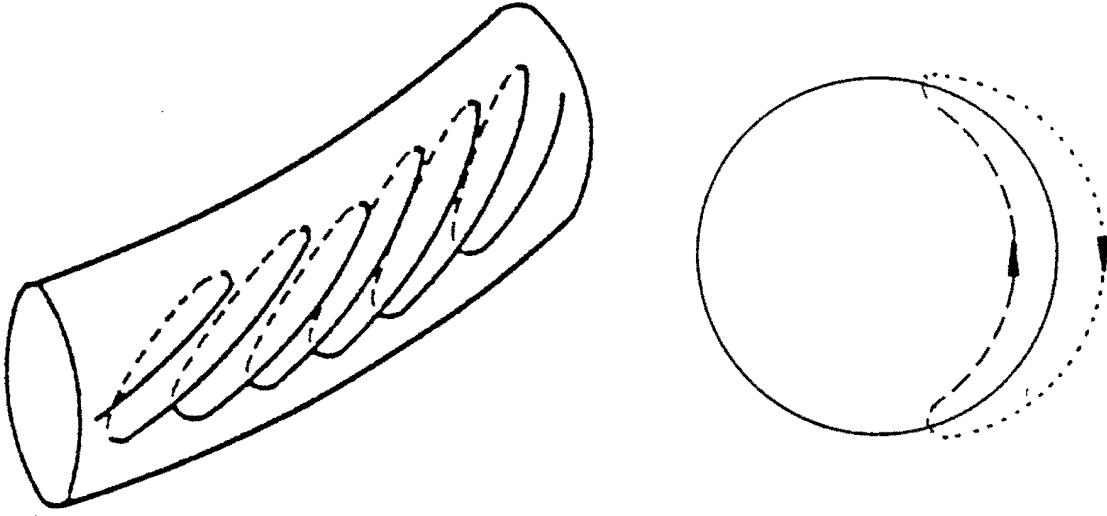


Figure 10.7: Banana Orbits in a toroidal configuration.

- Due to ∇B and curvature drifts, the path of a trapped particle deviates from the magnetic surface. Therefore, the projection of the trapped particle orbit in a poloidal plane is not a sector of a circle, but rather has a banana-like shape (see Fig. 10.7). These orbits are known as *banana orbits*. One can show that the banana width is given by

$$r_B = \frac{r_L q}{\sqrt{\varepsilon}}. \quad (10.28)$$

For trapped particles, an *effective* collision frequency $\nu_{eff} \approx \nu_c/(2\varepsilon)$, where ν_c can be any of the collisionalities discussed above, is defined. This takes into account, that a trapped particle becomes a passing one when its pitch angle has changed by $(2\varepsilon)^{1/2}$. Finally the ratio $\nu_* = \nu_{eff}/\nu_t$, where ν_t is the inverse of the time a particle needs to transit the banana, determines whether the particle can complete a banana orbit between two collisions.

There are three different regimes of neoclassical transport:

- Collisional or Pfirsch-Schlüter regime: Here, $\nu_* > 1$, particles do not complete their banana orbits but are scattered before. In this case, D is evaluated to be

$$D_{PS} = q^2 r_L^2 \nu_c = q^2 D_{class}. \quad (10.29)$$

As we have seen, q typically ranges between two and five, so transport is strongly increased.

- Banana regime: Here, $\nu_* < \varepsilon^{3/2}$, particles follow the banana orbit several times before they are scattered. We can apply the random walk argument for the $\sqrt{2\varepsilon}$ trapped particles with step size r_B and typical time interval $1/\nu_{eff}$

$$D_B \propto \sqrt{2\varepsilon} r_B^2 \nu_{eff} \approx \frac{r_L^2 q^2}{\varepsilon^{3/2}} \nu_c = \frac{q^2}{\varepsilon^{3/2}} D_{class}. \quad (10.30)$$

For typical values, e.g. $\varepsilon = 1/3$ and $q = 5$, this yields $D_B = 130 D_{class}$.

- Plateau regime: Here, $1 > v_* > \epsilon^{3/2}$ and one can derive $D_{pl} \approx v_{th} r_L^2 q/R$. This result is independent of v_c (hence the name). Numerical values for D_{pl} are in between the Pfirsch-Schlüter and the Banana regimes, so that, with rising collisionality, a smooth transition occurs between the regimes.

These neoclassical terms have to be added to the classical terms, however, they completely dominate the latter. We have seen that, due to neoclassical effects, transport coefficients can be increased by two orders of magnitude, nevertheless, they still cannot explain the experimental observations, especially for the electron channel. Today, theory tries to explain this so-called *anomalous* transport by nonlinear MHD turbulence models. It is found both theoretically and experimentally that the plasma exhibits turbulent eddies of a radial extension Δx of a few cm and a lifetime Δt of several $100 \mu s$. If we calculate the diffusion coefficient according to (10.21), we arrive indeed at typical values of $1 \text{ m}^2/\text{s}$ which could explain the experimental observations. However, the precise identification of the responsible small-scale MHD instabilities is a subject of ongoing research in the field of fusion plasma physics. In particular, it has become clear that a simple description in terms of one dominant mode does not exist and several contribution must be taken into account, for the different species and the large variation of plasma parameters from the edge to the core.

Chapter 11

Experimental results from tokamaks

Wolfgang Suttrop

11.1 Tokamak plasmas

11.1.1 The tokamak principle

Nuclear fusion can be achieved in plasmas which are composed of a mixture of deuterium and tritium. At the high temperatures necessary for nuclear fusion, in the vicinity of 10 keV or 100 Million degrees K, the plasma is fully ionised, i.e. consists only of ions and electrons. In order to reach these temperatures, good thermal insulation is necessary, which can be provided by a magnetic field. The Lorentz force makes charged particles spiral (*gyrate*) around magnetic field lines. Electrons and ions can freely move along field lines. If magnetic field lines close themselves inside the plasma volume, the free parallel motion does not result in particle losses from the plasma. Consequently, toroidal field geometry (as sketched in Fig. 11.1 left is most effective for confinement of high temperature plasmas and in fact most magnetic fusion concepts involve this topology.

The toroidal shape of the plasmas implies a non-uniform magnetic field strength. This results in a particle drift (Grad-B and curvature drift) perpendicular to the magnetic field and its gradient. This drift originates from a different gyro-radius at the low and high field sides of the particle orbit. The drift is directed parallel to the torus (vertical) axis in different directions for electrons and ions and hence leads to a separation of charge and a vertical electrical field. The electrical field induces an additional $E \times B$ drift which leads to fast radial losses of electrons and ions (see Fig. 11.1 right). Hence, a simple toroidal magnetic field is not sufficient for plasma confinement.

The losses induced by these drifts can be eliminated by introduction of an additional poloidal magnetic field. The resulting helical field directs field lines successively below and above the equator of the device so that vertical drifts result in inward and outward motion, respectively. As a result, the trajectories of the centers of the gyromotion (*gyrocenters*) are displaced from the magnetic field lines but are also closed in themselves. The magnetic moment of the gyromotion is conserved and with increasing magnetic field strength the velocity parallel to the

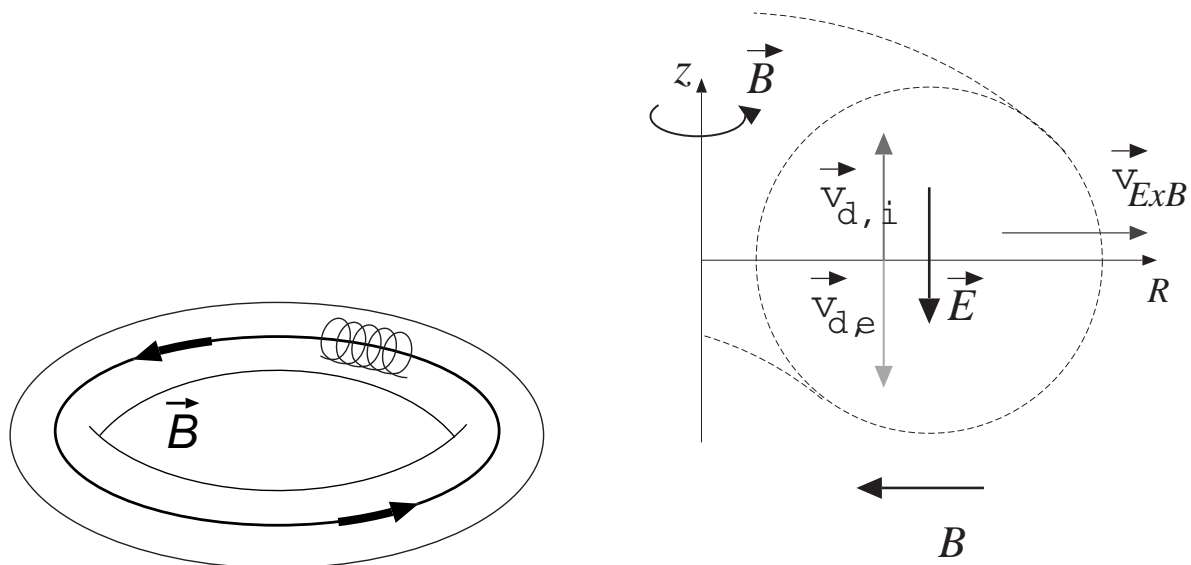


Figure 11.1: Left: Closed field lines in toroidal geometry.
Right: Outward particle drift in toroidal geometry.

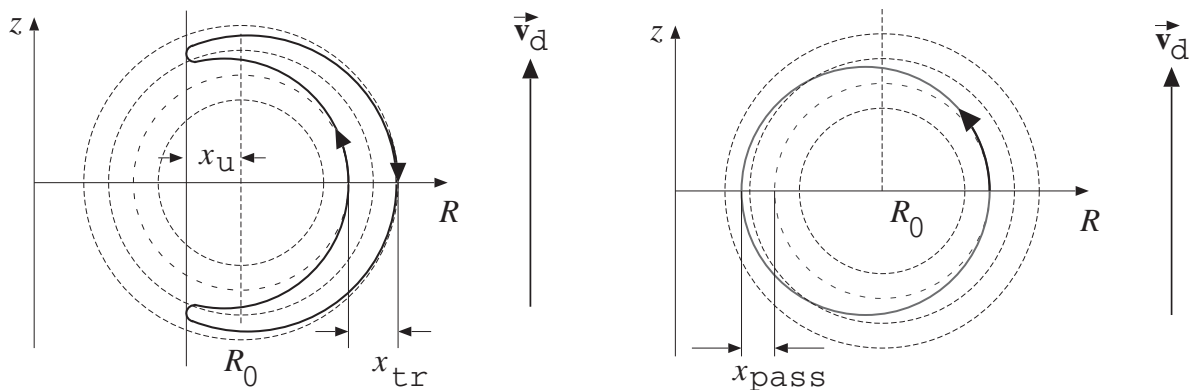


Figure 11.2: Particle trajectories in toroidal geometry:
Left: Trapped particles,
Right: Passing particles.

field reduces. Fig. 11.2 shows poloidal sections of two basic types of gyro-center trajectories: *Trapped particles*, see Fig. 11.2 left do not have a sufficiently large parallel velocity at the two field side to reach the location of maximum magnetic field strength. They are confined to the magnetic low field side and move in poloidally and toroidally opposite directions on the inner and outer halves of their orbit. The characteristic *banana* shape as viewed in a poloidal section gives rise to the nick-name *banana orbits* for these trajectories (see section on *magnetic mirror* above). *Passing particles* have a sufficiently large parallel velocity to continue their motion on the high field side, Fig. 11.2 right. The associated radial displacement with respect to field lines, Δx , is larger by a factor $(R/a)^{1/2} \equiv 1/(\epsilon)^{1/2}$ for trapped particles than for passing particles, where R is the *major radius* (distance from torus axis to plasma center) and a the *minor radius* (half of the diameter of the plasma column) of the toroidal plasma.

The effects of toroidicity on particle orbits have several important consequences. Cross-field

diffusion by Coulomb collisions is enhanced by the larger radial displacement of the particle orbits compared to their gyroradius (*neo-classical* diffusion). In presence of a pressure or density gradient, the average toroidal velocity of trapped particles at a given radial location is not zero. Momentum is transferred by Coulomb collisions to passing particles resulting in a plasma current (*bootstrap current*).

The additional poloidal field needed for confinement can be created in different ways. *Stellarators* generate a helical field entirely by external field coils. *Tokamaks* use a plasma current to produce the poloidal field component, while the toroidal field is generated by an axisymmetric arrangement of external coils. The word *tokamak* stems from a Russian acronym for *Toroidal chamber with magnetic coils*. In the following, we concentrate on tokamaks.

11.1.2 Components of a tokamak

The main components of a tokamak are illustrated in Fig. 11.3 left. Helical magnetic field lines span nested surfaces of constant magnetic flux (*flux surfaces*). The plasma is enclosed in a vacuum chamber (not shown) which is needed to remove air and control the hydrogen pressure outside the plasma. The axisymmetric toroidal field is produced by poloidal currents in a set of magnetic coils surrounding the plasma. A central solenoid is used to produce a time-varying poloidal magnetic flux which induces a toroidal loop voltage, and due to finite plasma resistivity, a plasma current. The plasma is subject to a poloidally and radially outward directed force which results from a finite plasma pressure gradient. This force is balanced by the $j \times B$ force acting on the plasma current due to the helical field and an additional vertical field produced by toroidal currents in external conductors (*vertical field coils*).

As an example, the poloidal cross section of ASDEX Upgrade (Axi-Symmetric Divertor Experiment) at Max-Planck-Institut für Plasmaphysik, Garching (Germany), is shown in Fig. 11.3 right. This machine can produce plasmas with major radius $R = 1.65$ and horizontal minor radius $a = 0.5$ m. An array of vertical field coils is installed which do not only produce the vertical field for compensation of the radial outward force but also can create suitable poloidal fields to create a wide range of plasma shapes. Typically, vertically elongated plasmas are produced by toroidal currents in upper and lower shaping coils which have the same direction as the plasma current. This configuration is inherently unstable against vertical displacements. A small upward displacement from the equilibrium position will result in a larger attracting force on the plasma current of the upper coil current than of the lower coil, thereby amplifying the perturbation. This position instability is damped by induced currents in a passive saddle-loop near the plasma (*passive stabilizing loop*). On time scales longer than the L/R time constant of the saddle loop (about 1 ms), the vertical position is actively controlled with an external radial field, i.e. by a differential current in the upper and lower vertical field coils.

11.1.3 Divertor tokamaks

Quadrupole currents in the poloidal field coils are also used to form a *magnetic divertor* configuration. Fig. 11.4 illustrates two basic magnetic configurations of a tokamak plasma.

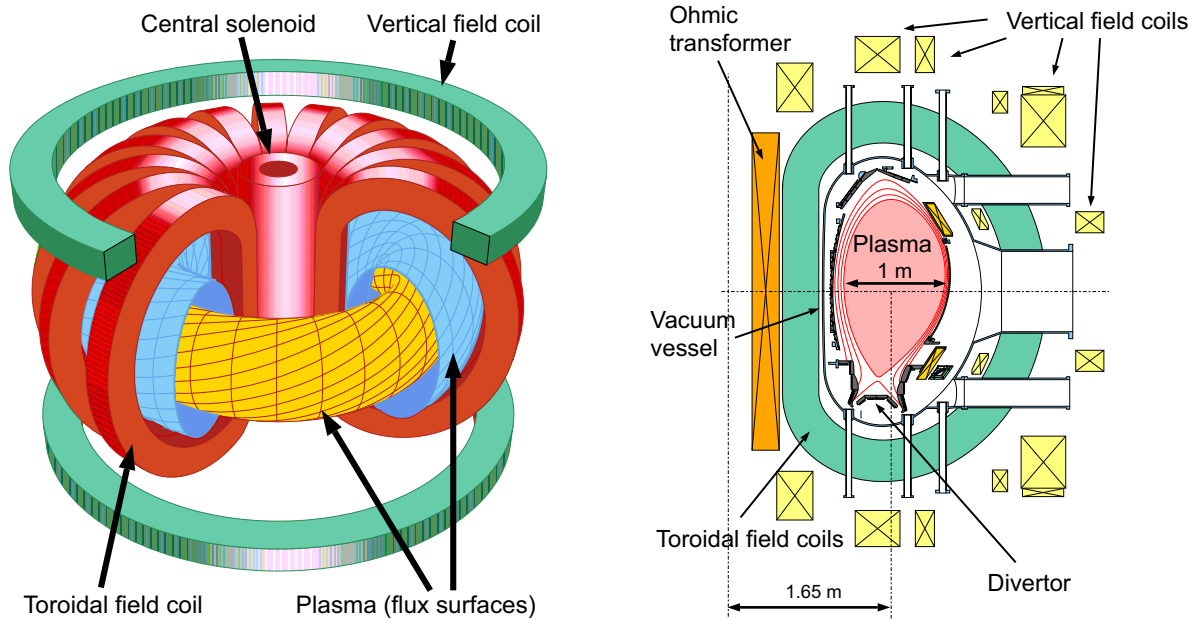


Figure 11.3: Left: Tokamak: Schematic view of main components. Right: Cross section of ASDEX Upgrade (Axi-Symmetric Divertor EXperiment), installed at Max-Planck-Institut für Plasmaphysik, Garching, Germany.

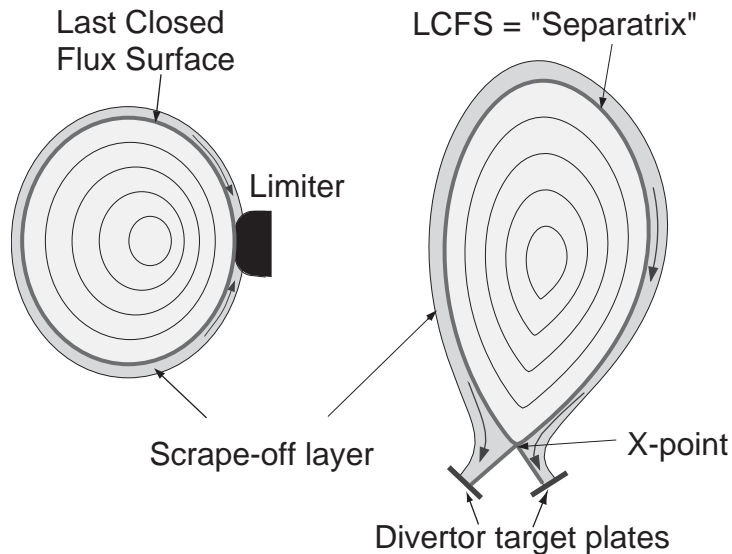


Figure 11.4: Limiter and divertor configurations.

Radial diffusion of particles and heat across the magnetic field will lead to expansion of the plasma until a material wall is intersected. Without a magnetic divertor, this first material contact will be close to the main confined plasma volume, and hence be subjected to strong heat load and particle fluxes. This region, called the *limiter*, must be appropriately designed in order to prevent excessive materials erosion. With a suitable axi-symmetric quadrupole field a magnetic X-point is introduced which defines a magnetic surface (*separatrix*) that intersects the material wall only at a distance from the main plasma. Particles and energy from the main

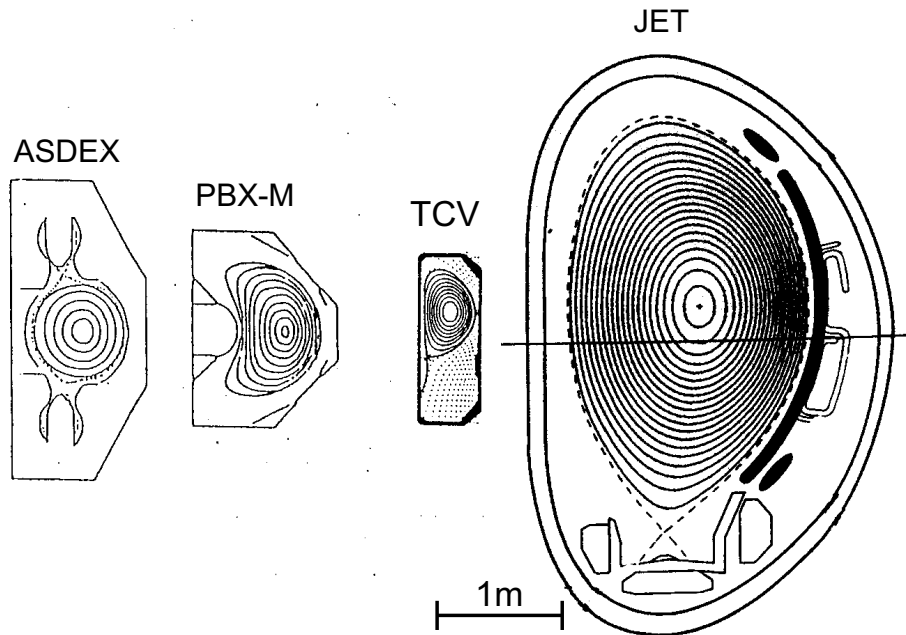


Figure 11.5: Variations of magnetic configurations (poloidal cross sections) in different experiments in shape (ASDEX, PBX-M) and plasma size (TCV, JET).

plasma flow along field lines to a separated divertor region where particles are neutralized at the material wall (*target plates*). A fraction of the neutral particles is reflected towards the separatrix, re-ionised and is streaming back again to the target plates. This recycling flux effectively increases the plasma density in the divertor region. Under normal conditions, the plasma pressure is constant along magnetic field lines, hence the plasma temperature in the divertor is smaller than that at the separatrix near the main plasma. As a result, the erosion damage at the material wall produced by ions can be strongly reduced and cleaner plasmas are obtained compared with a limiter configuration. This favourable property of divertors is exploited in most modern magnetic fusion devices, both tokamaks and stellarators. Fig. 11.5 shows cross sections of four tokamaks: ASDEX (Max-Planck-Institut für Plasmaphysik, Garching, Germany), PBX-M (Princeton Plasma Physics Laboratory, Princeton, New Jersey, USA), TCV (Centre de Recherche en Physique des Plasmas, Lausanne, Switzerland) and JET (Joint European Torus, Culham, United Kingdom). These examples illustrate a variety of plasma sizes, cross sections and divertor geometries used in tokamaks.

11.1.4 Magnetic configuration

At the densities and temperatures reached in fusion plasmas, significant kinetic pressure is obtained in the plasma core and large pressure gradients produce strong forces on the plasma column. The ∇p forces are balanced by $j \times B$ forces arising from the magnetic field interacting with internal (toroidal and poloidal) plasma currents. This force balance is often called the *magnetic equilibrium* of a plasma configuration. (It has to be stressed that this terminology

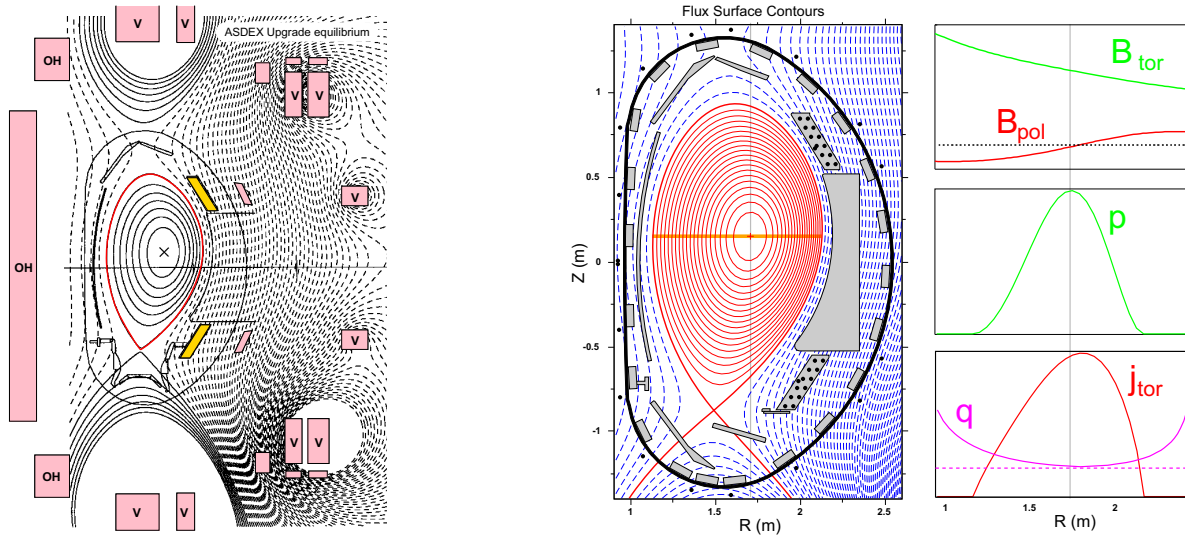


Figure 11.6: Left: Poloidal magnetic flux in and around a shaped tokamak plasma (poloidal section), which is created by toroidal currents – plasma current and a set of external poloidal field coils. Right: Tokamak equilibrium: Flux surfaces, radial midplane profiles of B_{tor} , B_{pol} , pressure p , toroidal plasma current j_{tor} and safety factor q .

does not imply the existence of a thermodynamic equilibrium state. In the contrary, because of the presence of strong heat flows into and out of the plasma, fusion plasmas are always open thermodynamic systems, far from thermodynamic equilibrium).

The poloidal magnetic field components B_R , B_z can be expressed by the derivatives of the poloidal magnetic flux ψ through a circle co-axial with the torus axis $B_R = -(1/2\pi R)\partial\psi/\partial z$, $B_z = (1/2\pi R)\partial\psi/\partial R$. From Ampere's law (toroidal derivatives vanish due to axi-symmetry) the toroidal current density can be expressed as

$$j_{tor} = -\frac{1}{2\pi\mu_0} \left[\frac{1}{R} \frac{\partial^2 \psi}{\partial z^2} + \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial \psi}{\partial R} \right) \right] \equiv -\frac{1}{2\pi\mu_0} \frac{1}{R} \Delta^* \psi. \quad (11.1)$$

The magnetic field lines span nested toroidal surfaces of constant plasma pressure which can be labelled with the flux ψ (*flux surfaces*). Expressing the toroidal field B_t by the *poloidal* current I_{pol} within a flux surface, $(\partial(RB_t)/\partial R) = (\mu_0/2\pi) (\partial(I_{pol})/\partial R)$, and using Ampere's law again, one obtains the famous Grad-Shafranov-Schlüter equation

$$\Delta^* \psi = -4\pi^2 \mu_0 R^2 p' - \mu_0^2 I_{pol} I'_{pol}, \quad (11.2)$$

where the prime ($'$) denotes the derivative with respect to ψ . Solutions of (11.2) describe the magnetic flux and toroidal current density for an axisymmetric magnetic equilibrium configuration characterized by toroidal current density $j_{tor}(R, z)$ (which contains plasma currents and external shaping coil currents), plasma pressure gradient p' and poloidal current profile I_{pol} .

Fig. 11.6 left shows (in a poloidal cross-section) contours of constant magnetic flux for an experimental magnetic equilibrium of a plasma in ASDEX Upgrade. The toroidal plasma

current generates poloidal magnetic flux which, in superposition with the flux generated by the poloidal field coils, defines a region of magnetic surfaces which are closed in themselves and have no intersection with a material wall. The boundary of this region (*last closed flux surface*, or *separatrix*) is highlighted in Fig. 11.6 left. Measurements of the magnetic flux are taken at many positions inside the vacuum vessel to reconstruct the experimental magnetic surfaces using (11.2). The shape of these surfaces can be controlled (feed-forward or feed-back) by the currents in the shaping coils.

Fig. 11.6 right shows a close-up of the vacuum vessel, the region of closed magnetic surfaces and profiles of the toroidal magnetic field B_{tor} , poloidal magnetic field $B_{pol} = (B_R^2 + B_z^2)^{1/2}$, plasma pressure p and toroidal current density j_{tor} along a horizontal section across the plasma center (indicated in the figure). Magnetic field lines follow a helical path on flux surfaces around the torus. A field line returns to a given poloidal position at a certain increment of the toroidal angle $\Delta\Phi$, which depends on B_t and B_{pol} along the field line. The *safety factor* $q = \Delta\Phi/2\pi$ counts how often a field line passes toroidally around the torus before it returns to the same poloidal position. For cylindrical cross-section and $R/a \gg 1$, $q(r)$ can be approximated as $q = (r/R)(B_t/B_{pol})$. The profile of q is also shown in Fig. 11.6 right. For a sufficiently peaked j_{tor} profile, q rises monotonically from the plasma center to the edge. This is typically the case if the plasma current is driven entirely by the induced loop voltage so that j_{tor} is determined by the radial resistivity profile. For irrational q , field lines span the entire flux surface. For rational q , field lines join up into themselves after a finite number of passes around the torus, so that helical perturbations can easily occur. Magnetic flux surfaces with integer or low rational q values are important for the stability of tokamak plasmas. In particular, q at the plasma edge has to be above a value of two to avoid a large scale disruptive instability (see sec. 11.2.3). This defines a minimum toroidal field for any given plasma current. The safety factor in the plasma center is usually clamped to values near or above unity by sawtooth oscillations (see sec. 11.2.3) or other types of magnetohydrodynamic activity.

11.1.5 Plasma heating

In order to balance heat losses from the plasma, a continuous heat source must exist to maintain the temperatures needed for fusion reactions to occur.

A heat source inherent for tokamaks is the Joule heat ηj^2 which originates from the plasma current and the low but finite resistivity η of a high temperature plasma (*Ohmic heating*). The plasma resistivity is proportional to $T_e^{-3/2}$ so that the Ohmic heating power reduces with increasing plasma temperatures, while the heat loss increases with increasing temperatures and their gradients. In practice Ohmic heating is limited to temperatures up to about 1 500 eV, significantly below the temperatures needed to initiate fusion reactions. Neutral particle injection (NBI) or absorption of radio-frequency waves can be used to heat the plasma to temperatures significantly higher than those achieved with Ohmic heating.

For NBI heating, a beam of ionized deuterium is produced and accelerated in an electric field. Typical particle energies are 60–140 keV and beam currents range up to 100 A per NBI source. The deuterium beam is guided through a neutralizer chamber filled with neutral deuterium gas. Charge exchange reactions occur for 60–90% of the ions (depending on beam energy).

Particles not neutralized are deflected by a magnetic field towards the wall of the chamber where they recombine and are pumped out. The fast neutral particles can pass through the tokamak periphery without being affected by the magnetic field. In the plasma, the neutral beam particles undergo charge exchange reactions which produce a population of fast ions. The plasma is heated by collisions of the fast ions with the thermal particles.

Radio frequency heating makes use of resonances in the dispersion relation for waves in plasmas, in particular resonances with the cyclotron motion of ions and electrons (*cyclotron resonance*). Electromagnetic waves are absorbed at the cyclotron resonance (and their harmonics) and the wave energy is converted into increased kinetic energy perpendicular to the magnetic field of the resonant particles. Collisions lead to thermalization of the energy distribution at increased temperature. Due to the different mass of ions and electrons different frequencies must be used. While for ion cyclotron heating (ICRH) frequencies in the range of several ten MHz are used, electron cyclotron resonance heating (ECRH) requires frequencies of the order of 100 GHz. ICRH heating systems typically use vacuum tube amplifiers, coaxial conductors and loop antennas mounted inside the vacuum vessel close to the plasma surface. ECRH power is generated by strong electron beams in a magnetic field with properly tuned magnetic fields (in devices called *gyrotrons*). The microwave power is transmitted in oversized waveguides or by quasi-optical imaging. Antennas are steerable mirrors which launch Gaussian beams that can be directed for variable power deposition positions.

11.1.6 Anatomy of a tokamak plasma

Figure 11.7 left shows for a typical tokamak discharge (example taken from the ASDEX Upgrade tokamak) several measurements as a function of time. After start-up of the discharge, the plasma current I_p is ramped up within one second to its flat-top value of 1 MA. The discharge is terminated after 6.5 s by a ramp down of I_p . The same panel shows I_{OH} , the current in the central solenoid needed to achieve the requested plasma current: Before the start of the pulse, I_{OH} is set to 40 kA. Initially, a large loop voltage (second panel) is needed for break-down. This is achieved with a fast step in I_{OH} . During current ramp up, large flux is consumed due to the inductance of the plasma. During current flat top, flux consumption depends on plasma resistivity. In our example, neutral beams are injected during two time intervals (1.5–3 s and 5–6.5 s) resulting in an increase of plasma temperature and consequently a decrease of resistivity. This results in the measured reduction of the loop voltage and reduced ramp rate of I_{OH} during the time intervals with auxiliary heating.

Details of the start-up phase of this discharge are shown in Fig. 11.7 right. A carefully dosed amount of deuterium gas (*gas prefill*) is introduced into the vacuum chamber and then a fast step of I_{OH} is applied to generate a large loop voltage. This results in ionisation of the neutral gas. The resulting free electrons are accelerated and help to further ionize the neutral gas. The plasma density (bottom panel) rises along with the plasma current. After about $t = 100$ ms, the plasma current is feedback-controlled and slowly ramped up. After $t = 200$ ms the plasma density is also feed-back controlled with the gas fuelling rate.

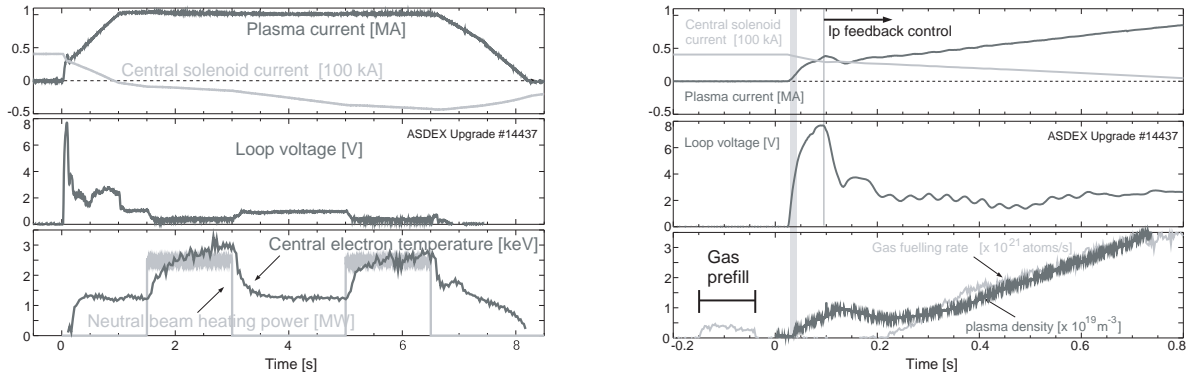


Figure 11.7: Left: Measured time traces for a tokamak plasma during start and plasma current ramp-up phase, flat-top phase, and ramp-down phase. Right: Start-up phase of a tokamak plasma: A high loop voltage, generated by a fast central solenoid current ramp, ionises an initial neutral gas prefill.

11.2 Experimental results

11.2.1 Confinement and transport

Confinement time

One of the main requirements for a nuclear fusion plasma is a sufficiently large *energy confinement time*, defined as $\tau_E = W_{th}/P_{heat}$, where W_{th} is the thermal stored energy and P_{heat} the heating power required to obtain this energy. The confinement time is a major figure of merit for tokamak performance and is relatively easy to measure. The heating power is the sum of ohmic heating ($V_{loop} \times I_p$) and auxiliary heating power, plus alpha particle power (small in today's tokamaks). The thermal stored energy can be obtained from density and temperature profiles, or approximated by the total stored energy (e.g. from diamagnetic measurements), corrected for the measured or estimated energy of fast (non-thermal) particles, which mainly depends on heating method, heating power and plasma density.

Fig. 11.8 shows a scaling of the confinement time (experimental vs. predicted τ_E) for a relatively large number of tokamak experiments with different plasma size. This scaling, termed IPB98(y), is used to predict the performance of the planned International Thermonuclear Experimental Reactor (ITER) from present-day machines and has the form

$$\tau_{E,IPB98(y)} = 0.0365 I_p^{0.97} B_t^{0.08} P_{heat}^{-0.63} n_e^{0.41} M^{0.2} R^{1.93} \epsilon^{0.23} \kappa^{0.67}, \quad (11.3)$$

$\tau_{E,IPB98(y)}$ is the predicted confinement time in seconds, I_p plasma current in MA, B_t toroidal magnetic field in T, P_{heat} auxiliary heating power in MW, n_e line averaged electron density (10^{19} m^{-3}), M ion mass number, R major radius (m), $\epsilon = a/R$, κ is the elongation (ratio of vertical and horizontal minor plasma radius). Energy confinement depends most strongly on plasma current I_p , plasma size (radius R) and elongation κ .

The experimental data base spans more than two orders of magnitude of τ_E in similar plasma types (High confinement mode, see sec. 11.2.2 below). ITER is designed for $\tau_E \approx 6$ s, which

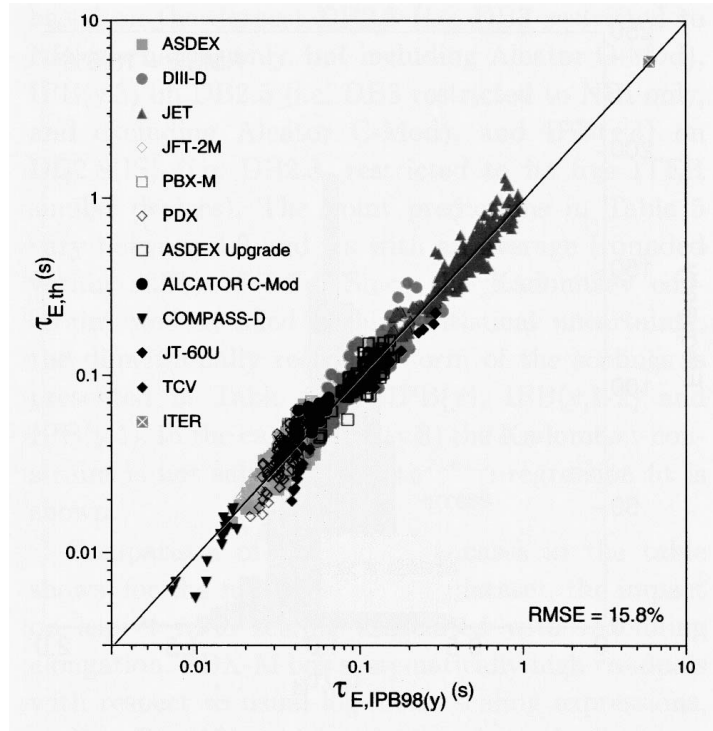


Figure 11.8: Energy confinement time scaling of the International confinement database for a large set of tokamaks.

is less than a factor of ten above the largest confinement times obtained in tokamaks to date. Naturally there are uncertainties with such extrapolations, originating from variance in the data base, and possible physics regime changes from current experimental parameters to a reactor-size plasma. It is therefore interesting to study the physics of heat transport in more detail.

Heat diffusion

Heat transport can be expressed in terms of a local power balance for the electron and ion heat conduction channels

$$q_{\perp e,i} = -n_{e,i} \chi_{e,i} \text{grad} T_{e,i} \pm Q_{e,i}, \quad (11.4)$$

where $\chi_{e,i}$ is the *heat diffusivity*, $n_{e,i}$ the density, $\text{grad} T_{e,i}$ the temperature gradient and $q_{\perp e,i}$ the heat flux of the considered species (electrons or ions). $Q_{e,i}$ is the collisional heat transfer between electrons and ions.

For *random walk* radial diffusion dominated by Coulomb collisions, one expects a diffusivity of the order of $\chi_{e,i} \approx \Delta x^2 / \tau_{e,i}$ where Δx is the radial step size (gyroradius for classical diffusion, banana orbit width for neo-classical diffusion in a toroidal plasma) and $\tau_{e,i}$ is the electron-ion collision time. However, it is found that the actual diffusivity in tokamak plasmas is normally significantly larger than the neo-classical value. In addition, heat diffusivity increases with increased heat flux. Historically, this behaviour has been termed *anomalous* transport and it is now known to originate from small-scale turbulence which creates fluctuating poloidal

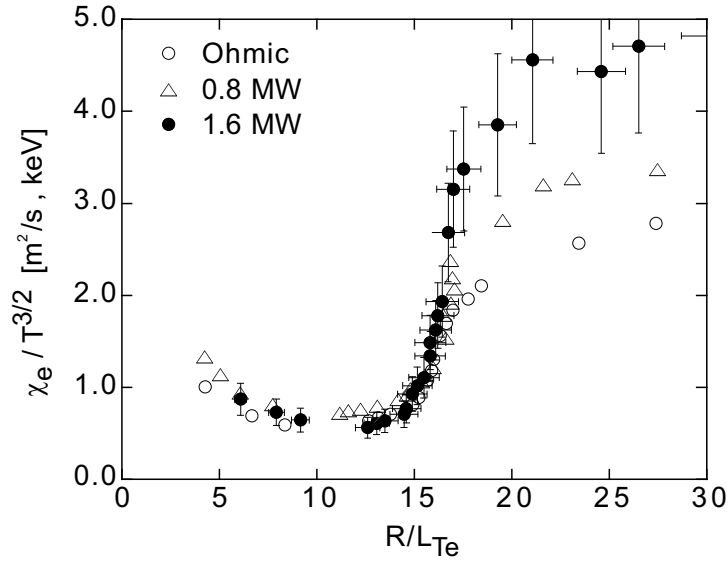


Figure 11.9: Normalized electron heat diffusivity $\chi_e/T_e^{3/2}$, measured as a function of normalized electron temperature gradient $(\nabla T_e)R/T_e = R/L_{T_e}$.

electrical fields. As a result, fluctuating radial $E \times B$ drifts occur which exchange plasma between hot and cold regions.

Various instability mechanisms exist that can drive transport-relevant turbulence. A common feature is that destabilization occurs if the temperature gradient length $L_T = T/\nabla T$ drops below a critical threshold $L_{T,crit}$. The heat diffusivity strongly depends on $L_T/L_{T,crit}$ and scales with $T^{3/2}$ (*gyro-Bohm* dependence). This behaviour is indeed found in experiments, as shown for electron by measurements of χ in ASDEX Upgrade (evaluated using (11.4)). Fig. 11.9 is a plot of electron heat diffusivity χ_e (normalised by $T_e^{3/2}$) as a function of inverse gradient length $R/L_{T_e} = R\nabla T_e/T_e$ (R is the major radius at midplane on the low-field side). Three different levels of electron heating power are employed: Ohmic heating and ECRH with 0.8 MW and 1.6 MW RF power. The data points correspond to measurements at different radial locations, central points have low heat flux and low inverse gradient length R/L_T (lower left), plasma edge data appears at high diffusivity (upper right corner of the diagram). The experiment shows the predicted increase of χ_e at $R/L_T \approx 15$. Furthermore, data with different heating power (and different temperature) is described by the same normalisation of χ_e by $T_e^{3/2}$.

The existence of a critical threshold for χ_e above which heat transport increases strongly leads to the formation of *canonical* temperature profiles, where L_T is near $L_{T,crit}$, almost independently of the heating power. This is the origin of the strong deterioration of confinement time τ_E with heating power (11.3). For fixed gradient length, the central temperature depends strongly on the edge temperature, which is limited by edge instabilities (see sec. 11.2.2 and sec. 11.2.3 below). Consequently, high edge pressure and high edge temperature help to achieve high stored energy in the core of a tokamak plasma.

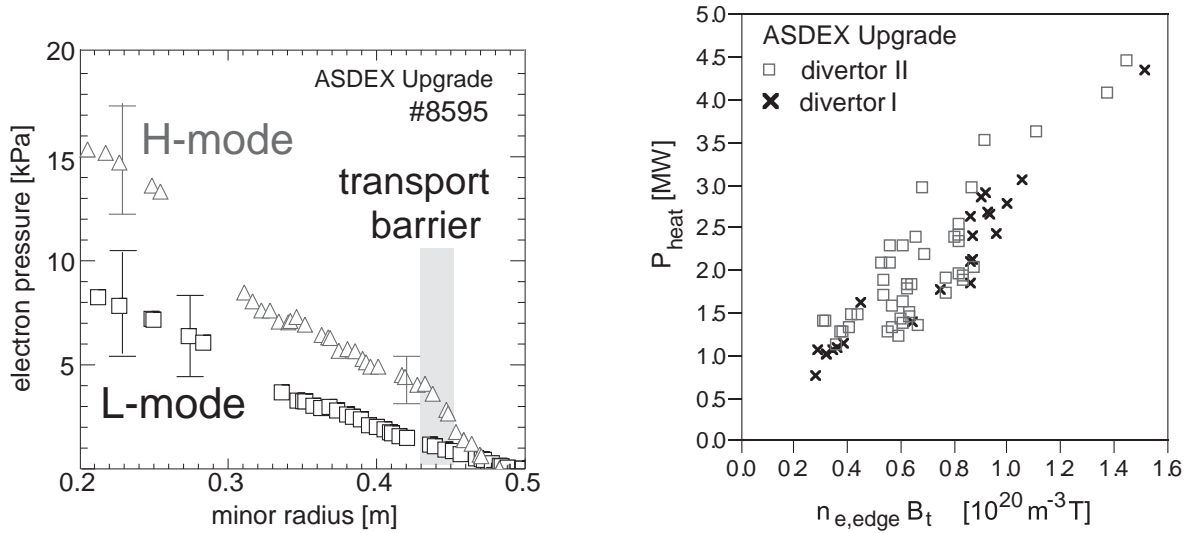


Figure 11.10: Left: Comparison of electron pressure profiles in L-mode and H-mode.
Right: Experimental power threshold for barrier formation.

11.2.2 Confinement improvement

High-confinement mode

The level of radial particle and energy transport found in tokamaks is mainly caused by small-scale turbulence. Nevertheless, suppression of this turbulence can be observed in specific cases, leading to a reduction of the particle and heat diffusivity in a restricted plasma volume. This behaviour is often called the formation of a *transport barrier*. With the introduction of a divertor, spontaneous formation of a transport barrier at the plasma edge has been found in the ASDEX tokamak (Garching, 1984). This regime has been termed the *high-confinement mode (H-mode)*. The effect of the edge transport barrier is demonstrated in Fig. 11.10 left for a discharge of the ASDEX Upgrade tokamak before and after transition to H-mode. Since the discovery of H-mode, the plasma state before the transition to H-mode is usually termed *L-mode (low confinement mode)*. In H-mode, the pressure gradient steepens up at the plasma edge (r between 0.43–0.45 m for the profile shown), giving rise to a pressure *pedestal*, i.e. additional stored energy in the plasma from the pressure offset at the edge. The temperature just inside the transport barrier is significantly higher than in L-mode, allowing (for the same gradient length $L_T = T/\nabla T$) a higher plasma core temperature gradient than in L-mode, resulting in an additional confinement benefit.

The condition to enter High-confinement mode is a minimum heating power across the plasma boundary. Empirically, the threshold power depends on the product of plasma edge density and magnetic field, $P_{\text{heat}} \propto n_{e,\text{edge}} \times B_t$. Fig. 11.10 right illustrates this dependence for the case of two divertor geometries in ASDEX Upgrade. Little influence of the divertor geometry (and other parameters such as heating method) is found.

A typical time history around the transition to H-mode is shown in Fig. 11.11 left. The heating power is being stepped up to just above the H-mode threshold at $t = 1.65$ s. Large heat flux in

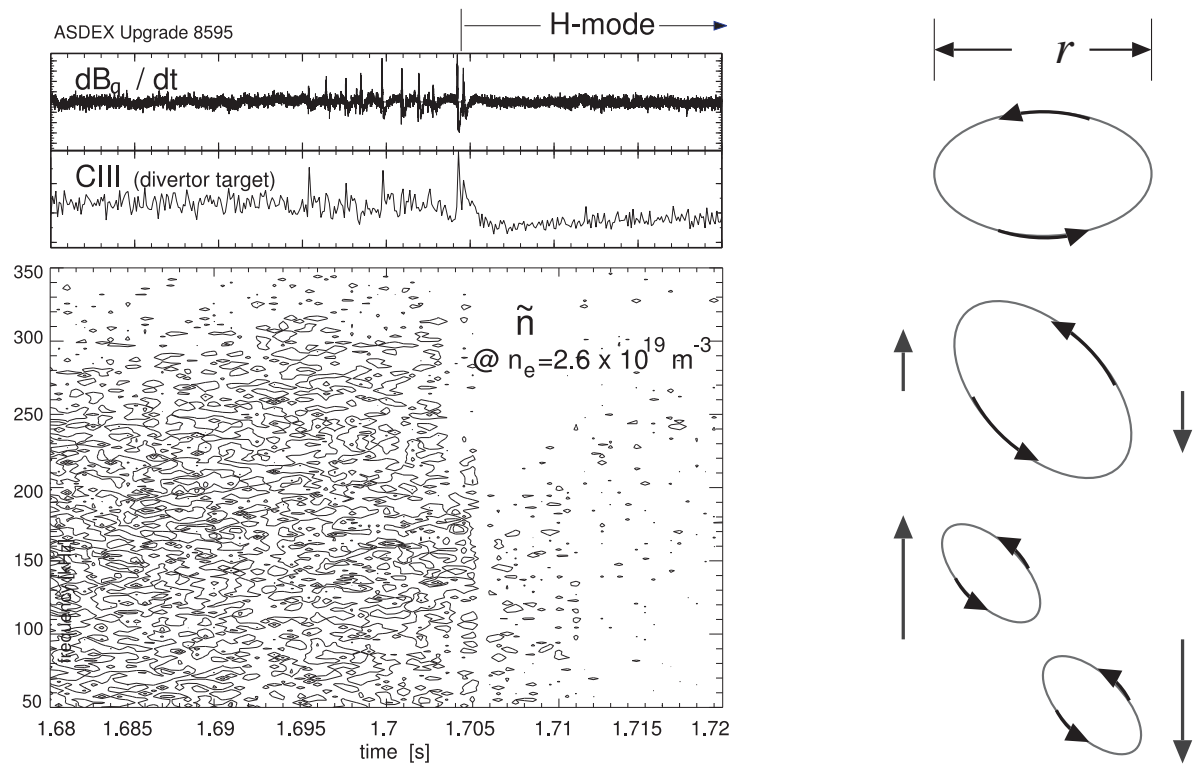


Figure 11.11: Left: Fluctuations are reduced at the transition from L-mode to H-mode.

Right: H-mode model: $E \times B$ shear flow reduces radial size of turbulent eddies.

L-mode leads to turbulence that can be seen as broad-band density fluctuations in the \tilde{n} spectrogram contour plot. Transport occurs in bursts that appear in the divertor heat flux (C-III spectral line intensity) and main plasma magnetic probes (\dot{B}). After the transition to H-mode (at $t = 1.705$ s in the example), the level of density fluctuations is much reduced and the C-III signal in the divertor drops.

While in L-mode additional heating power creates steeper gradients and thus generates more free energy to drive turbulence and enhanced losses, it can also trigger a transition to H-mode, a self-organising process that results in a bifurcation to a state of reduced transport. This transition to H-mode is accompanied with the creation of a radial electrical field gradient, corresponding to sheared rotation of the centers of the particle gyro-motion. It is believed that de-correlation of turbulent $E \times B$ convection cells by sufficiently fast sheared rotation is the origin of the transport reduction (cf. cartoon Fig. 11.11 right). However, although a number of possible mechanisms has been proposed the physics that drives the radial electrical field is still unknown and subject to ongoing research effort.

Internal transport barriers

Transport barriers can exist not only at the plasma edge, but also in the plasma core, causing steep temperature and/or density profiles, thereby increasing the central pressure. *Internal*

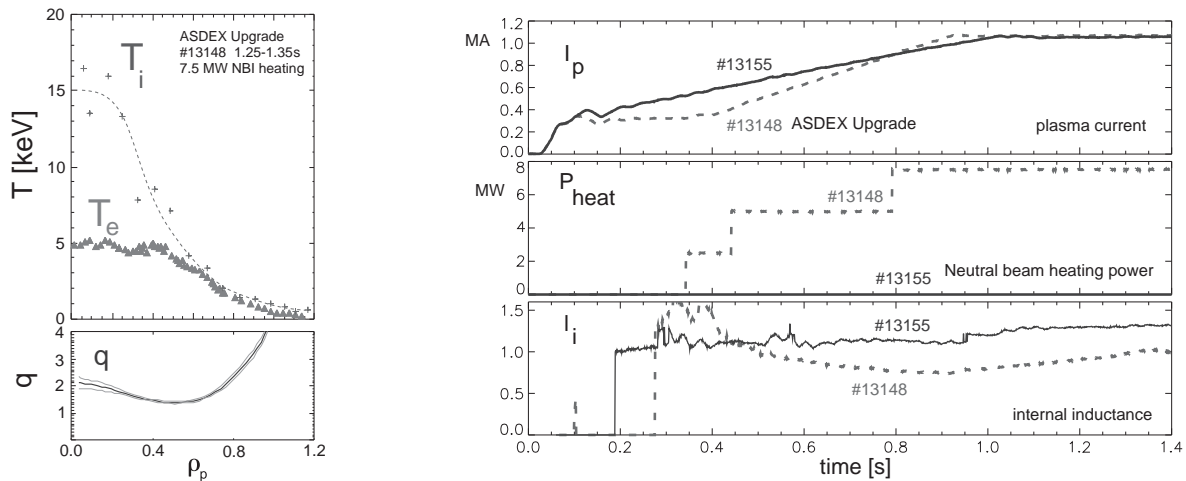


Figure 11.12: Internal transport barrier example:

Left: Profiles of T_i , T_e and safety factor q .

Right: Comparison of discharges with (13148) and without (13155): Early heating during the current ramp leads to broader current profile as seen by a reduced value of l_i .

transport barriers (ITB) arise from an influence of the magnetic shear $s = (r/q) dq/dr$ on the growth of micro-instabilities and thereby the heat diffusivity. In conventional tokamak plasmas without ITB, q increases monotonically from the center towards the edge, i.e. the magnetic shear is positive across the entire plasma radius. It is observed that heat transport can be reduced at low or negative magnetic shear.

Fig. 11.12 left shows profiles of the electron and ion temperature and the safety factor q in a discharge with ITB in the ASDEX Upgrade tokamak. The q profile is flat or slightly reversed inside a normalized radius of $\rho = 0.6$. A strong ion temperature gradient is maintained around $\rho = 0.4$, leading to a central ion temperature of 15 keV. This temperature, obtained in a tokamak much smaller than an envisaged fusion reactor, is near the optimum temperature for D-T fusion reactions. Fig. 11.12 right shows how this result is achieved. A conventional plasma (shot #13155) is compared with an ITB discharge (#13148). In the ITB pulse, heating power is applied early in the plasma current ramp, resulting in higher T_e . Due to the increased resistive skin time the initial flat or slightly hollow plasma current profile remains *frozen* for the duration of the experiment. This is seen by the slow evolution of the *internal inductance* l_i which indicates by its lower value after $t = 0.45$ s a broader (less peaked) current profile.

The advanced tokamak

For truly stationary tokamak operation the plasma current has to be driven entirely non-inductively, for example by the bootstrap effect due to the pressure gradient in the plasma. This leads to the interesting concept of the *Advanced Tokamak*, where the bootstrap current profile maintains the low or reversed shear profile required for the transport reduction which causes the ITB. Fig. 11.13 compares profiles of the pressure, toroidal current density and safety factor q (assuming pure ohmic or pure bootstrap current) for both a *conventional* (non-ITB) and an ITB plasma.

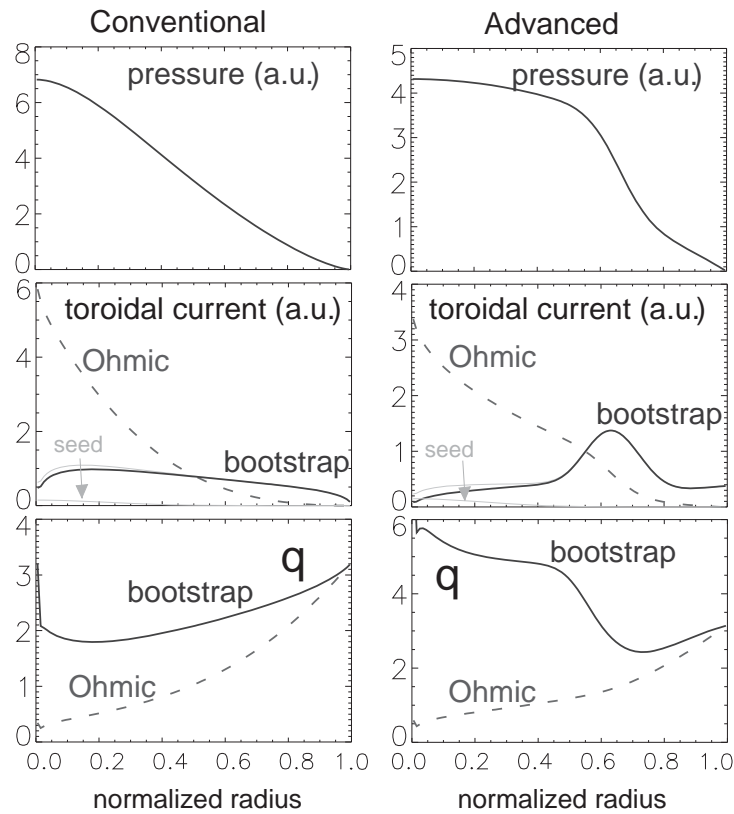


Figure 11.13: Profiles of plasma pressure p , current density j and safety factor q comparing conventional and advanced scenarios. A conventional plasma has a monotonously rising q profile. Flat or reversed reversed q profile can lead to a transport barrier. In an ideal advanced scenario, the resulting steep pressure gradient creates a bootstrap current that maintains the q -profile non-inductively in steady state. Ohmic and bootstrap contributions to j and q are shown separately.

In the conventional scenario the ohmic (inductively driven) current dominates. The ohmic current density profile is fixed by the conductivity (electron temperature) profile. A sufficiently strong transport barrier can, in principle, sustain a reversed shear profile. The strong pressure gradient produced by the transport reduction creates a strong off-axis bootstrap current. The resulting non-monotonic current profile maintains the weak or negative magnetic shear profile that allows to sustain the transport barrier. A tokamak reactor with an ITB and this type of *self-generated* plasma current could be built smaller than a conventional tokamak and would allow true steady-state operation. A crucial condition is the alignment of the reversed shear region with the region of actual strong pressure gradient, which if not matched, leads to rapid changes of the radial position of the transport barrier. A (small) current profile correction can be applied non-inductively by external current drive techniques, e.g. by electron cyclotron or lower hybrid wave heating with toroidal component of the wave vector. Advanced tokamak scenarios with a variety of current drive methods are an important topic in most current tokamak research programs.

11.2.3 Stability and operational limits

Instability occurs if a plasma perturbation amplifies itself and grows until the plasma configuration is severely modified. Free energy sources for instability growth arise for instance from large gradients of pressure and currents present in a fusion plasma. Many instabilities involve displacements of magnetic surfaces (*MHD instabilities*), but also other types of instabilities exist, for example radiation instability originating from increased radiated power with decreasing temperatures. MHD instabilities are distinguished by their scale: Micro-instabilities occur on the gyro-radius scale and are relevant for radial particle and heat transport. Macroscopic instabilities affect the plasma on the scale of the plasma dimensions and can have large effects on confinement or even terminate a plasma discharge. In toroidal geometry one can characterize unstable modes by their integer toroidal (n) and poloidal (m) mode numbers. Most macroscopic MHD modes can become unstable only at *resonant* surfaces, i.e. magnetic surfaces where $q = m/n$, and n and m are small numbers. At finite aspect ratio, modes with different poloidal mode number m couple, giving rise to complex mode number spectra.

Instabilities are important as they define the accessible plasma parameter range for stable tokamak operation. Subsequently, we discuss the basic tokamak operation range and examine several important types of instabilities and their effects.

Tokamak operational limits

For stable tokamak operation, there are upper bounds for the plasma current and the plasma density. Fig. 11.14 shows a schematic diagram of the accessible parameter range in a plot of $1/q_a$, the reciprocal edge safety factor vs. $\bar{n}_e R/B_t$ (known as *Hugill diagram*). We note that $1/q_a \propto I_p/B_t$, so the vertical axis is proportional to the plasma current. For $1/q_a \rightarrow 0.5$ the $q = 2$ surface approaches the plasma edge and a $m = 2, n = 1$ *kink* instability occurs. This is a fast-growing large scale ideal MHD mode which leads to a quick termination of the plasma discharge.

Empirically, a density limit is found which is proportional to the plasma current density. In the Hugill diagram (see Fig. 11.14) this is expressed by an upper bound of the line averaged density $\bar{n}_e R/B_t \propto I_p/aB_t$ or $\bar{n}_e \propto I_p/aR \propto I_p/a^2$ (for constant aspect ratio R/a). The physics of the density limit is quite complex, and some of its aspects are outlined in the next section.

The accessible operation range of tokamaks is bounded by these limits. High plasma density is desirable for good performance and to facilitate power exhaust with a divertor. For a tokamak reactor it is therefore desirable to operate at highest possible plasma current density. The maximum value of B_t is defined by technical limitations for toroidal field coil system. Hence, the $q = 2$ limit sets an upper bound to I_p . From these considerations, the optimum operation point is in the upper right corner of the Hugill plot.

Density limit disruptions

The density limit in tokamaks can assume quite complex phenomenology. A typical history of a density limit in a discharge with only ohmic heating is shown in Fig. 11.15 left. In this

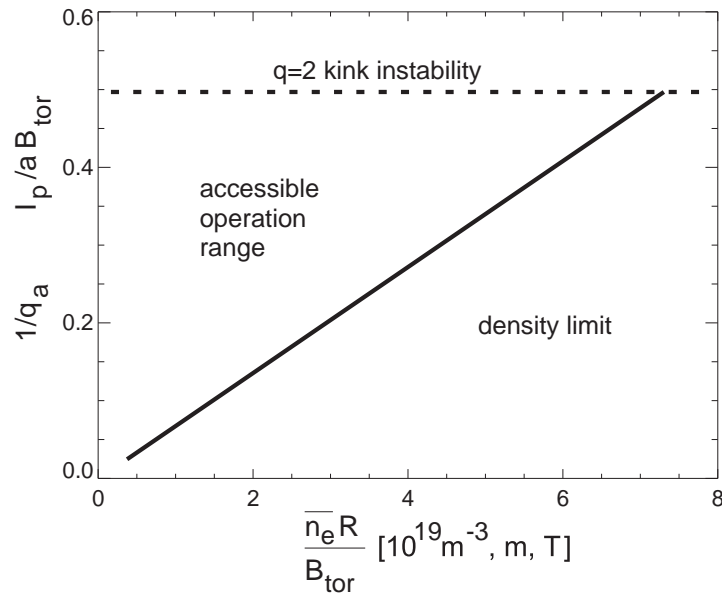


Figure 11.14: Hugill-Diagram showing the basic tokamak operational range. An upper limit to the plasma current is given by the onset of $n = 1, m = 2$ kink instability for $q = 2$ at the plasma edge. A density limit exists at $\bar{n}_e \propto B_t / (Rq) \propto I_p / a^2$.

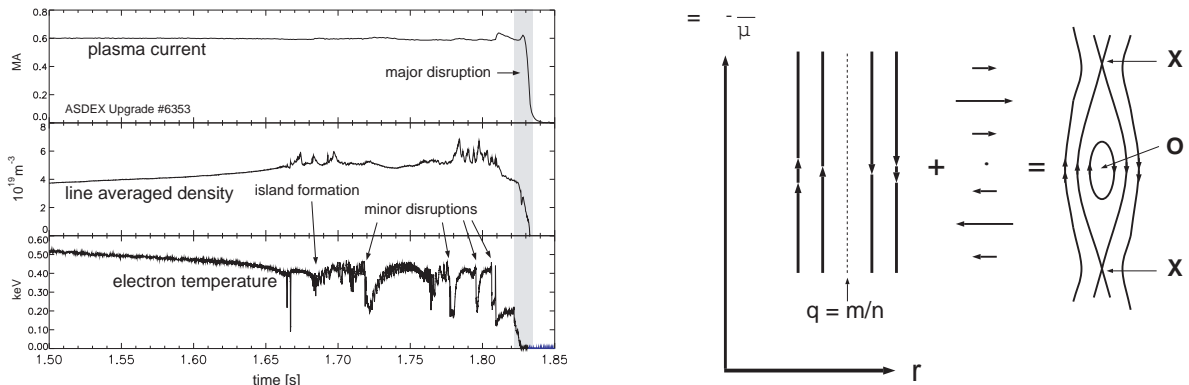


Figure 11.15: Left: History of a density limit disruption. With increasing density the plasma edge is cooled and magnetic islands grow. Finally a *major* disruption occurs with thermal and plasma current quench. Right: Formation of magnetic islands by superposition of an unperturbed sheared magnetic field and a helical B_r field.

experiment, the line averaged density is slowly raised by gas puffing. The edge electron temperature (bottom trace) reduces with increasing density. When the edge has sufficiently cooled down, *tearing mode* instability occurs which results in the formation of magnetic islands by reconnection.

Fig. 11.15 right shows the structure of magnetic islands (in helical and radial coordinates). Magnetic islands result from helically periodical perturbation currents which give rise to a radial magnetic field with helical symmetry. They can grow only at resonant rational surfaces,

i.e. where $q = m/n$. At the density limit, several islands with $m/n = 4/1, 3/1$ and $2/1$ appear. Since heat transport is faster along magnetic field lines than perpendicular to the field, radial heat flux effectively bypasses the island interior (*O-point*) and concentrates near the X-point. Consequently the temperature is flat inside the island, while near the X-point a gradient is maintained. Plasma rotation makes this modulation visible as an oscillation of the temperature (bottom panel in Fig. 11.15 left).

After islands have grown to a certain radial extent, fast confinement loss events (*minor disruptions*) occur, which lead to a flattening of the temperature profile around a resonant surface. The volume inside this surface cools down and islands on inner resonant surfaces begin to grow. After several minor disruptions, several island chains can exist simultaneously and the rotation velocity reduces to zero (*mode locking*). A *major disruption* occurs, with a fast loss of plasma energy (*energy quench*) and subsequent quench of the plasma current. Because of the inductivity of the plasma, the plasma current quench results in *halo currents* in the vacuum vessel wall, which lead to very large forces on the material structure of a tokamak. The vacuum vessel and the support structure of all magnetic coils must be designed to withstand these forces.

Nevertheless, the disruptive termination of a plasma can be avoided or at least mitigated. If the density is kept below the density limit, the phenomenology described above does not occur. If the density limit is encountered, one can delay tearing mode growth by localized electron cyclotron current drive which replaces the missing helical current in the O-point. Several techniques have been successfully applied to delay or avoid mode locking. Finally, during a disruption the kinetic and magnetic energy in the plasma can be disposed by radiation from injected impurities leading to a more even distribution of energy deposition on the walls, and a reduction of halo currents.

The physics mechanism at the onset of the density limit are still under investigation. Several observations can be made depending on the confinement regime: Increase of radial transport at the edge leading to edge cooling and breakdown of the H-mode barrier; *detachment* of the divertor, i.e. loss of the parallel pressure balance between main chamber edge and divertor due to radial momentum transfer, and formation of a poloidally asymmetric condensation instability (*MARFE*) which leads to edge cooling. Several mechanisms can drive magnetic island growth: Deformations of the current profile, and helical current perturbations due to radiation losses from the island interior or loss of the bootstrap current because of pressure profile flattening around the O-point. The interplay of these mechanisms is being studied in experiments and by complex computer simulations.

Sawtooth oscillations

In many tokamak discharge scenarios, the central temperature undergoes a characteristic oscillation: After a slow linear rise (of few to several 100 ms duration) the temperature crashes very quickly (within about 100 μ s) and the cycle repeats. This behaviour has been first observed in the central soft X-ray emissivity and has been named *sawtooth* oscillations for its peculiar time history (see Fig. 11.16 left, top panel). X-ray tomography can be used to reconstruct a two-dimensional image of the central emissivity which reveals the structure of the underlying perturbation. The lower part of Fig. 11.16 left shows the reconstructed X-ray emissivity

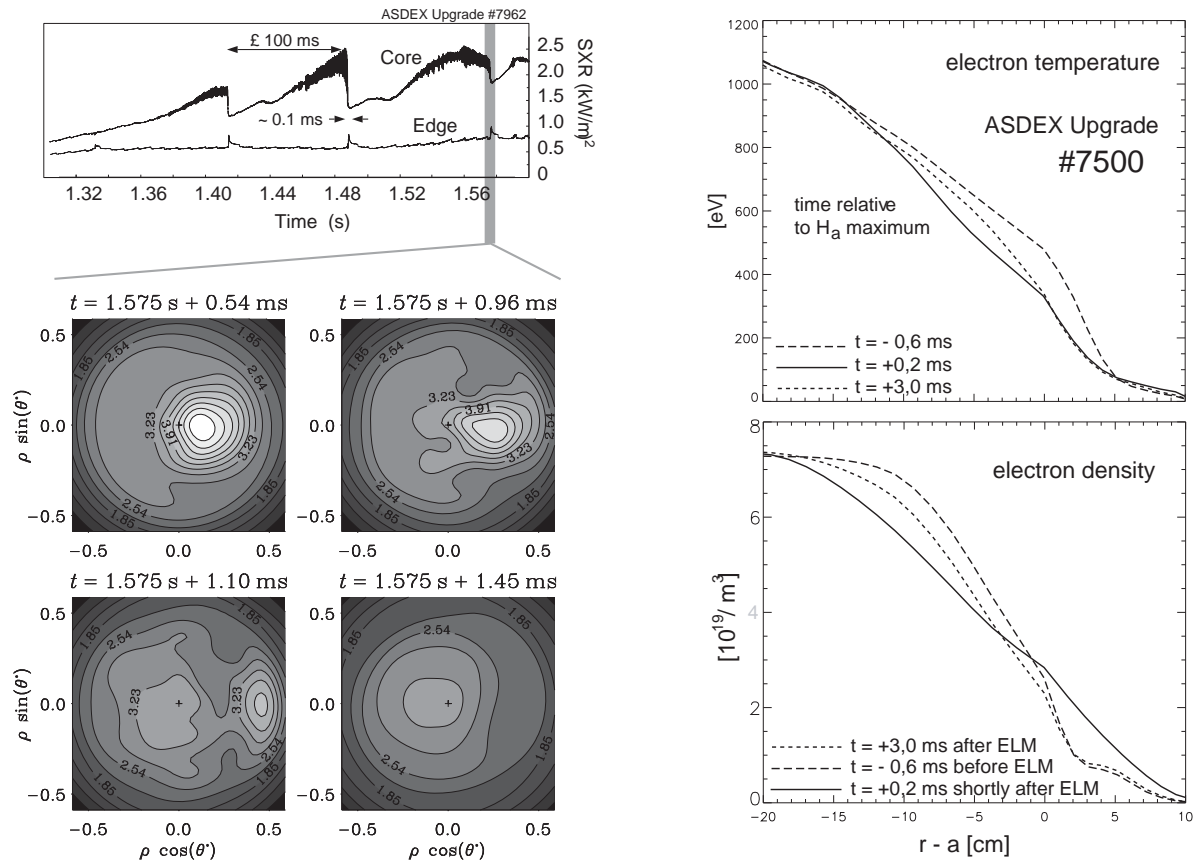


Figure 11.16: Left: Sawtooth oscillations as seen by soft X-ray measurements. Bottom right: Tomographic reconstruction of soft X-ray emissivity before, during and after the sawtooth crash. Right: Edge electron temperature and density profiles (A) before an ELM, (B) shortly after an ELM, (C) in the recovery phase in between ELMs.

for four phases before, during and after the sawtooth crash. In between sawtooth crashes the plasma center is heated and the central temperature increases. If $q(0) < 1$, an $n = 1, m = 1$ kink mode is destabilised. Because of good radial confinement the interior of this region is hot ($t = +0.54$ ms). Because of the toroidal plasma rotation, the difference of temperatures inside and outside the kink structure appears as fast oscillations (black areas) in the X-ray signals (see Fig. 11.16 left top). With increasing central current density the structure becomes unstable and reconnection sets in ($t = +0.96$ ms and $t = +1.1$ ms). Hot plasma streams into a larger volume and the peak temperature and emissivity are reduced, but the 1/1 mode re-appears and the cycle starts over ($t = +1.45$ ms).

Fig. 11.17 left schematically shows the temporal evolution of the profiles of the central current density and associated safety factor q . In between sawtooth crashes, $j(0)$ (and $T_e(0)$) both rise, resulting in a drop of central $q(1)$. Eventually $q(0)$ falls significantly below unity (2). A quick reconnection event flattens the central j (and T_e) profile and q rises again above one; the cycle starts over.

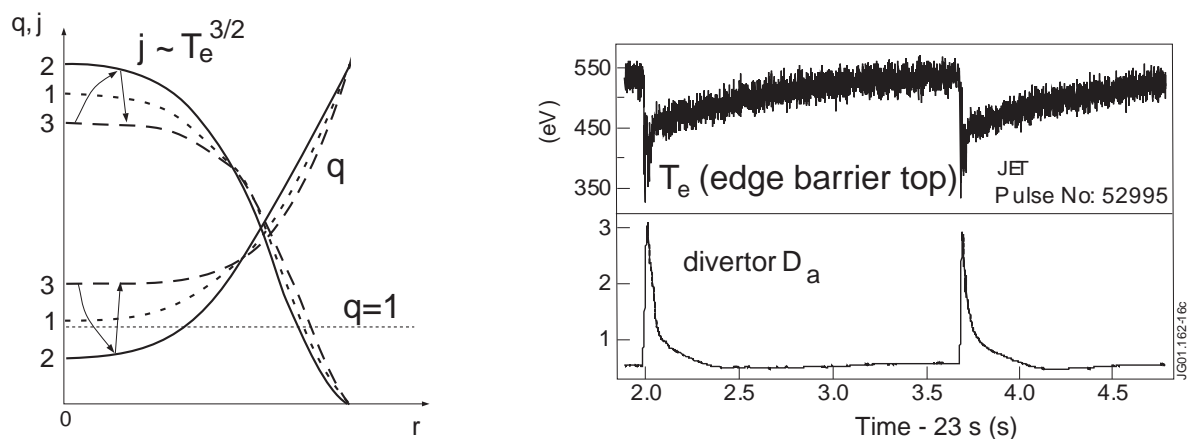


Figure 11.17: Left: Profiles of j during a sawtooth cycle.

Right: Time traces of edge electron temperature and divertor D_α showing repetitive losses at the edge and increased recycling during ELMs.

Edge localized modes

Edge Localized Modes (ELMs) are instabilities that occur in H-mode where a transport barrier leads to a steep pressure gradient at the plasma edge. The edge temperature (shown in Fig. 11.17 right, top time trace) and edge density (not shown) increase until sudden heat and particle losses occur. These losses can be detected as a characteristic spike in the D_α line radiation intensity in the divertor due to increasing recycling particle flux and increasing ionisation because of the heat loss (Fig. 11.17 right, bottom trace).

The particle and heat losses are also seen as a sudden partial erosion of the edge temperature and density profiles (see Fig. 11.16 right). Steep profiles (A) in the edge barrier region before ELMs flatten within few $100 \mu\text{s}$ (B) and recover slowly (C) by radial heat and particle transport from within the plasma core until the maximum pressure is reached again and a new ELM occurs. This results in a characteristic oscillation cycle which repeats itself many times, often hundreds of times in one plasma discharge. Along with hydrogen, also impurities are lost which otherwise tend to accumulate in the main plasma due to the very slow out-diffusion across the H-mode transport barrier. While this feature of ELMs allows for stationary H-mode plasmas, the fast energy loss during the collapse phase can present a large peak power load to the divertor structure which can cause significant erosion of the divertor structure. For this reason, ELMs with small losses occurring at a high frequency are preferred over slow and large ELMs.

The precise physical origin of ELMs and the nature of the instability are still under investigation. The phase of enhanced transport during ELMs is characterized by a broad spectrum of MHD modes with a duration of about $100 \mu\text{s}$. ELMs set in near values of the total edge pressure gradient where one expects *ballooning* instability, i.e. short wavelength modes driven by the pressure gradient at the magnetic low field side where the curvature of the magnetic field is not stabilizing. Fig. 11.18 left shows a stability diagram for ballooning modes. The attainable normalized pressure gradient (first stability limit) $\alpha = 2\mu_0 R (\nabla p) q_a^2 / B_t^2$ increases with magnetic shear $s = (r/q) dq/dr$. Interestingly, at low shear a second stable regime can exist. Low

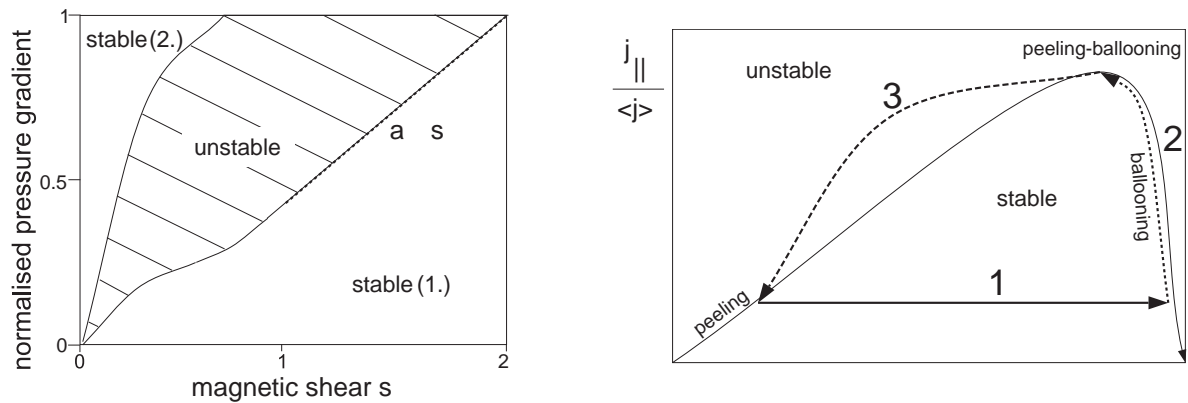


Figure 11.18: Left: Stability diagram for ballooning modes. Stability occurs for high magnetic shear (*first* stable regime) or very low magnetic shear (*second* stable regime). Access to the second stable regime is often blocked by an unstable region as shown. Right: Schematic model for the ELM cycle: Different time scales for evolution of edge pressure gradient α and edge current density j_{\parallel} result in several stability limitations during the cycle (see text).

local shear at the magnetic low field side can be produced by suitable plasma shaping or by high edge current density.

Ballooning modes are typically small scale modes, most unstable for high mode numbers and thus do not explain the large perturbations associated with ELMs. Furthermore, it is often observed experimentally that the pressure gradient saturates well before an ELM occurs indicating a more complex physics process involved. Recent models for ELMs consider a variety of instabilities, including modes driven by the large toroidal current gradient at the plasma edge. A large edge current density originates from the bootstrap current driven by the large edge pressure gradient in the H-mode transport barrier. In this picture the ELM cycle can be described as follows (see Fig. 11.18 right).

In between ELMs (1) the (normalized) pressure gradient α rises on the transport time scale (few ms for heat and particle diffusion over a few cm) until the ballooning pressure gradient limit is reached. Small scale instability leads to enhanced losses, effectively clamping the pressure gradient while the edge current density j_{\parallel} builds up within the resistive skin time (10–100 ms for the edge of current tokamaks). Ultimately the stability limit for coupled peeling (i.e. medium- n kink) and ballooning modes is reached, leading to macroscopic instability, the ELM crash (3). Pressure and current are expelled and the cycle repeats itself. At present, the ELM cycle and stability limits are qualitatively and quantitatively described. Current theoretical and experimental work focuses to understand and predict the magnitude of the ELM losses and to tailor the plasma edge in order to reliably obtain ELM types with smaller energy losses.

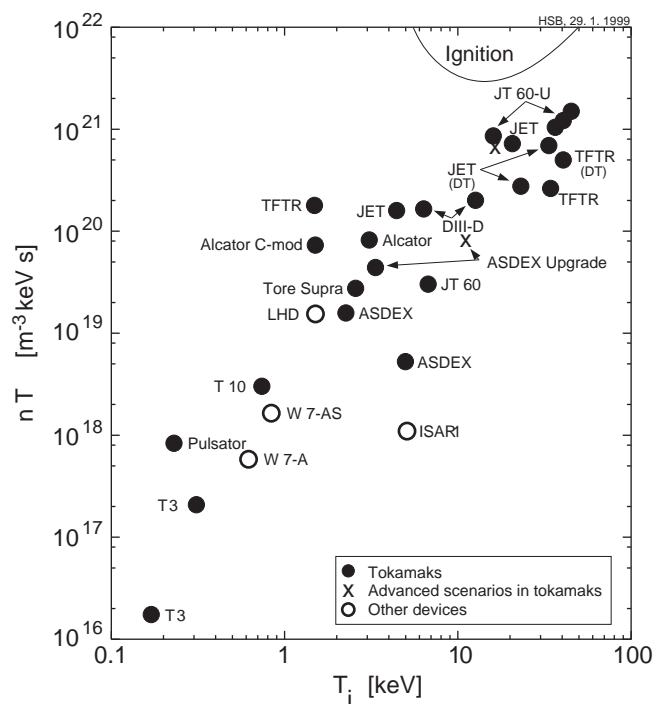


Figure 11.19: Historical performance of tokamaks and other devices, fusion triple product $nT_i \tau_E$ as a function of T_i .

11.3 Summary and outlook

Tokamaks can confine a nuclear fusion plasma with a combination of external toroidal field and poloidal magnetic field created by an internal plasma current. The plasma current can be driven inductively by a magnetic flux ramp created with a central solenoid which limits the duration of tokamak discharges. Non-inductive current drive by externally driven current or bootstrap (pressure gradient current) current can extend the pulse duration and possibly allow stationary tokamak operation. Heat and particle transport in tokamaks is usually dominated by fluctuating $E \times B$ convection from micro-instabilities which are driven by the strong temperature gradients. However, transport barriers at the plasma edge and in the plasma interior can be produced which reduce the heat diffusivity to near a level expected for collisional transport. The strong pressure and current density gradients present in a well-confined tokamak plasma can also drive a range of macroscopical instabilities, which often take the form of limit-cycle oscillations. Examples are sawtooth oscillations, associated with central $m = 1, n = 1$ kink modes that undergo fast reconnection events and a flattening of the central temperature, and Edge Localized Modes, driven by the steep pressure gradient of an edge transport barrier resulting in repetitive energy and particle losses. While these instabilities do not prevent stationary plasma operation and in some cases even help avoiding accumulation of impurities, other instabilities lead to severe confinement deterioration or even a termination of the plasma. The accessible tokamak parameter range is bounded by the conditions to avoid these instabilities.

The historical progress of nuclear fusion research can be measured by the fusion triple product $nT_i \tau_E$ which is plotted in Fig. 11.19 as a function of T_i for tokamaks and other magnetic confinement devices (mostly stellarators). The values of $nT_i \tau_E$ have increased over several orders

of magnitude in the last few decades and plasma temperatures and densities needed for a fusion reactor are obtained in contemporary tokamaks. In order to reach ignition (self-sustained plasma heating by alpha particles resulting from fusion reactions) an increase of the energy confinement time to values of several seconds is needed. This can be achieved by a larger plasma size, such as that of the planned International Thermonuclear Experimental Reactor (ITER), a tokamak with major radius $R = 6.2$, about twice as large as that of the presently largest tokamak, JET ($R = 3$ m). The step to a reactor-scale experiment with large fusion power and significant neutron production will allow to study dominant plasma heating by fast alpha particles, confinement and plasma-wall interaction, and test first-wall and low-activation structural materials needed for an integrated power plant concept. Alternative confinement concepts such as stellarators will further develop and demonstrate their attractive potential in new experiments, for example for steady state operation. Nevertheless, one can expect that tokamaks will remain at least for some foreseeable future the reliable *workhorse* and universal test-bed for magnetic fusion research they have been in the last three decades.

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Chapter 12

Introduction to stellarator theory

Ralf Kleiber, Håkan Smith

12.1 Introduction

12.1.1 Some history

The stellarator concept was introduced in 1951 by the astrophysicist Lyman Spitzer. In order to introduce rotational transform (see sec. 12.1.2) in a toroidal configuration he proposed to twist a torus into a shape resembling the shape of the number eight (figure-eight stellarator).

Some years later the Model C Stellarator experiment was built in Princeton but was not successful: The plasma was lost very quickly. As was understood later this was mainly due to an incomplete knowledge about the behaviour of resonances in magnetic surfaces. While the Princeton stellarator experiment was still plagued by a low confinement time (a few Bohm times¹ at $T_e \approx 100$ eV) Russian scientists at the 1968 IAEA Novosibirsk conference showed that, with a confinement concept called tokamak, it was possible to achieve an electron temperature of about 1 keV for 30 Bohm times.

This historical situation resulted in the fact that tokamaks became the main line of fusion research in the world while the stellarator concept was pursued only in some places, mainly at the IPP Garching and at the University in Kyoto. Today there is a renewed interest in the stellarator concept as an alternative to the tokamak: A large stellarator of the torsatron² type (Large Helical Device (LHD)) started its operation 1998 in Japan at the NIFS. The German-EU experiment Wendelstein 7-X is being constructed in Greifswald and scheduled to start operation in 2015. Other (smaller) stellarators are in use in the USA (HSX), Spain (TJ-II) and Australia (H-1).

¹The Bohm time $\tau_B = a^2/D_B$ (a : minor radius) is defined via the Bohm diffusion coefficient $D_B = 1/16k_B T_e/(eB)$.

²The stellarator concept appears under different names, e.g. heliotron, torsatron, heliac, which mainly distinguish the way the coil systems were historically constructed. In this text we will not elaborate on this, instead we will use just the collective notion *stellarator*.

12.1.2 Some basic notions

Fig. 12.1 shows the basic torus geometry of a fusion device with the poloidal and toroidal angles ϑ and φ . The ratio $A = R/a$ of major to minor radius is called the aspect ratio.

If for a given magnetic field $\mathbf{B}(\mathbf{r})$ a function $\Psi(\mathbf{r})$ exists in a region of space such that $\mathbf{B} \cdot \nabla \Psi = 0$ then the surfaces $\Psi(\mathbf{r}) = \text{const}$ are called flux surfaces (i.e. the magnetic field is tangent to these surfaces). The existence of nested³ flux surfaces (Fig. 12.1 shows three such surfaces), at least in the main volume of the device, is an essential prerequisite for plasma confinement by a magnetic field. The innermost flux surface that is degenerated into a line is called the magnetic axis. If the equilibrium consists only of nested flux surfaces they are distinguished by a flux surface label $s \in \mathbb{R}$.

A field line on a given flux surface can be specified by a (non-periodic) function $\vartheta = f(\varphi)$. If φ increases by $2\pi n$ ($n \in \mathbb{N}$) the angle ϑ changes by an amount ϑ_n . The quantity ι defined by $\iota := \lim_{n \rightarrow \infty} \vartheta_n / (2\pi n)$ is called the rotational transform⁴ and has the same value for each field line on a given flux surface; in general ι is a function of the flux surface label. The s -derivative of the rotational transform is called shear ι' . A flux surface s_0 where the rotational transform is rational $\iota(s_0) = n/m$ ($m, n \in \mathbb{Z}$) is called a rational surface⁵. On a rational surface all field lines are closed lines while on a non-rational surface one field line fills the surface ergodically.

Even if magnetic surfaces do not exist one can obtain a good impression of the field structure by following a field line for many turns and marking with a point the positions where it intersects an arbitrarily chosen poloidal plane. The collection of these points is called a Poincaré plot (examples can be seen in Fig. 12.7, top). In such a plot a flux surface shows up by points forming a closed curve.

12.1.3 Tokamak, stellarator

The way in which the rotational transform ι is generated can be used to distinguish two classes of fusion devices: Tokamaks and stellarators. In a tokamak the plasma current, which is e.g. driven by the externally induced electric field, causes the poloidal field that in combination with the toroidal field from the (toroidal) field coils leads to a twisting of the magnetic field lines and thus to a rotational transform. Since usually the current density is highest at the center of the torus ι decreases with increasing radius.

In a stellarator the rotational transform is caused solely by a coil system externally to the plasma without any total plasma current J , i.e.

$$J := \int_{A_p} \mathbf{j} \cdot d\mathbf{A} = 0, \quad (12.1)$$

³Be M_S the set of points enclosed by a flux surface S . Then nested means that for all pairs (S_1, S_2) $M_{S_1} \subseteq M_{S_2}$ or $M_{S_2} \subseteq M_{S_1}$.

⁴In the simplest approximation ι can be regarded as the pitch of the field line on the flux surface.

⁵Since the set of rational numbers \mathbb{Q} is dense in \mathbb{R} the same is true for the set of rational surfaces with respect to all flux surfaces.

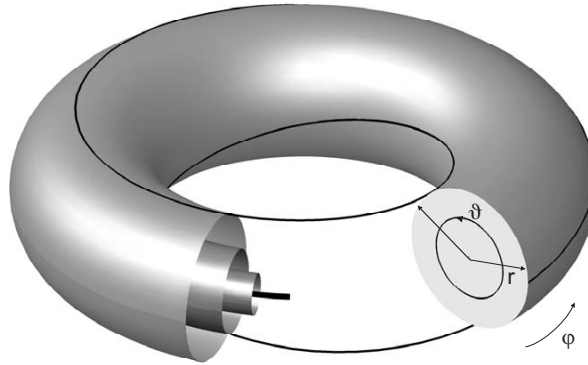


Figure 12.1: Toroidal geometry with nested flux surfaces, magnetic axis and a field line (red) with $\iota = 0.38$.

where j is the current density and A_p is a poloidal plasma cross section. Since the influence of the coils is strongest at larger radii the rotational transform increases with increasing radius.

So one main difference between tokamak and stellarator is the existence or absence of a total toroidal plasma current.

The beneficial role of a total plasma current for confinement in a tokamak has two main accompanying disadvantages: Firstly it can lead to violent instabilities called disruptions which destroy the plasma confinement and secondly – in a standard⁶ tokamak – the operation time is limited by the necessity of inducing an electric field. An advantage of the tokamak concept is that a tokamak possesses rotational symmetry and thus is a two-dimensional configuration.

The main advantages of a stellarator are that because of $J = 0$ it is inherently disruption free and can be used to achieve steady state operation. It is easy to show that the condition $J = 0$ cannot be satisfied for an axisymmetric configuration and thus a stellarator must have three-dimensional geometry. This makes stellarators geometrically and computationally more complex than tokamaks but, as we will see in the following sections, offers much more flexibility in their design which can be used for their optimisation.

Although in principle the three-dimensional geometry of a stellarator need not have any symmetries one usually imposes two symmetry requirements (see Fig. 12.2):

- Periodicity: Invariance against a rotation by $2\pi/P$ in the toroidal direction, where P is the number of – so called – field periods.
- Stellarator symmetry: Invariance against flapping around certain lines.

In a tokamak the rotational transform is of first order in the poloidal field component B_p : $\iota \sim B_p/B_0$, (B_0 is the total field). For a stellarator the first order effect of the helical field

⁶This is not the case for the so called advanced tokamak where bootstrap current (driven by density and temperature gradients) and current drive are responsible for ι .

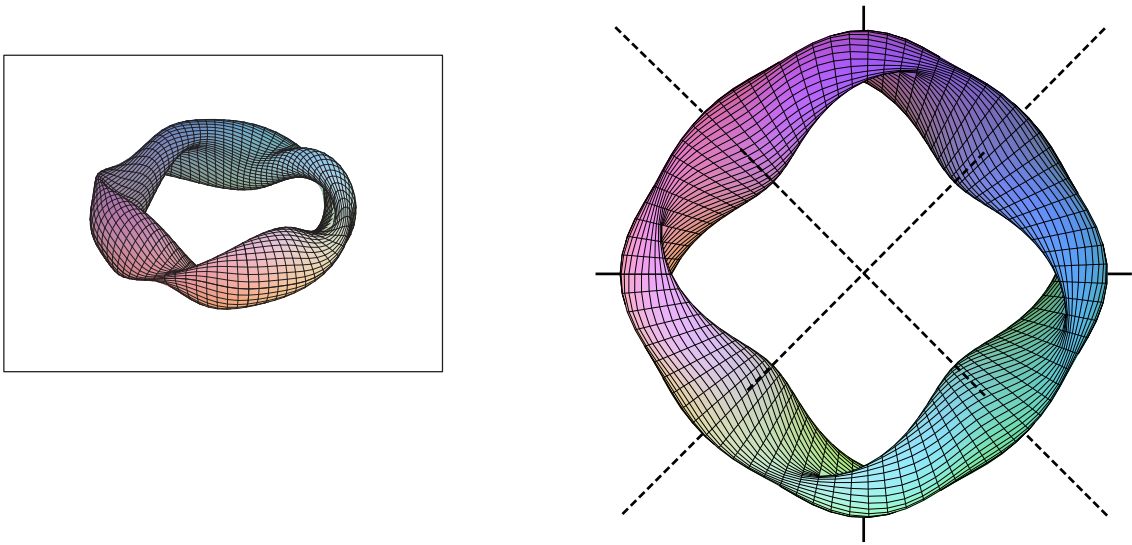


Figure 12.2: Side (left) and top (right) view of a stellarator with $P = 4$. The field periods and the lines for the stellarator symmetry are denoted by short and dashed lines, respectively (here both types of lines do have the respective complementary feature).

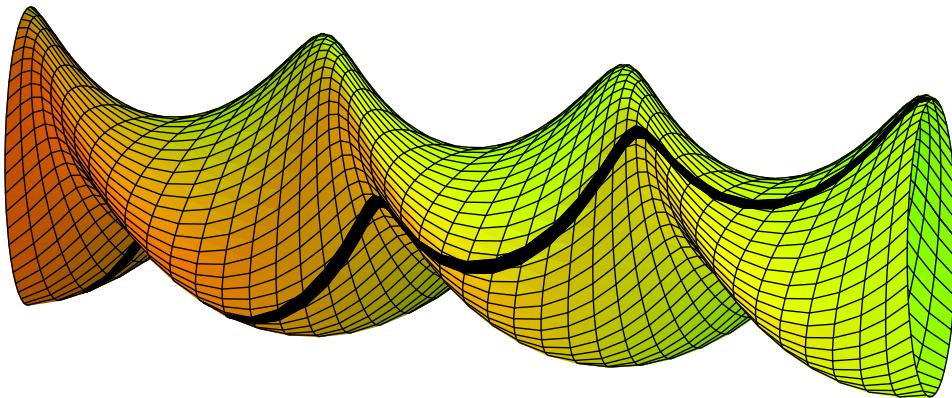


Figure 12.3: Three-dimensional view of one flux surface of a straight $\ell=2$ stellarator (three field periods). The thick black line shows one field line.

component B_h causes the field line to wiggle while the rotational transform, i.e. the poloidal shift that remains after averaging, is only of second order: $\iota \sim (B_h/B_0)^2$. This behavior of the field line can be understood if one rewrites (12.1) (using $\mu_0 \mathbf{j} = \nabla \times \mathbf{B}$) as a line integral around a poloidally closed curve ∂A_p giving $J = \int_{\partial A_p} B_p dl = 0$. For this integral to be zero the poloidal field component B_p has to change sign which implies an oscillating behavior of the field line.

Both effects are illustrated for a straight $\ell=2$ stellarator in Fig. 12.3; the wiggling is conspicuous while the rotation of the field line reflects the rotational transform.

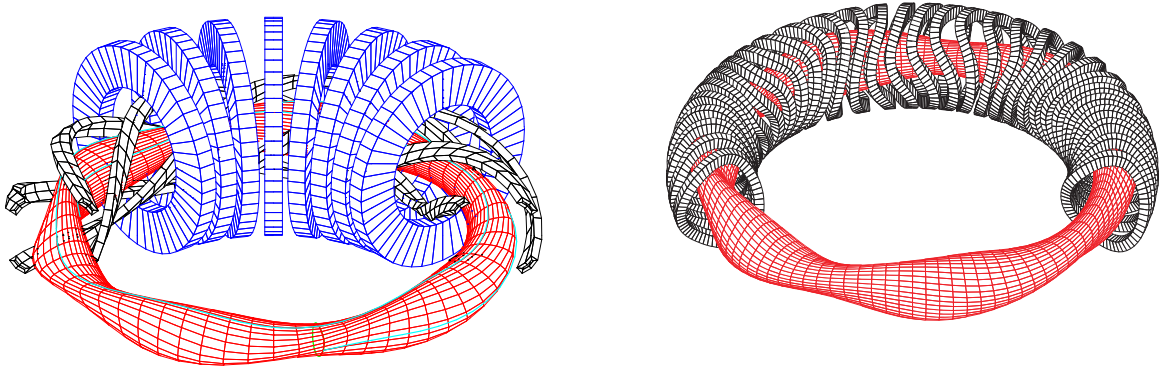


Figure 12.4: Classical (left) and modular (right) coil system for an $\ell=2$ stellarator.

12.1.4 Coil system

An important part in the engineering design of a stellarator is the question how a coil system must look like in order to produce the desired magnetic field. The classical coil system for e.g. an $\ell=2$ stellarator consists of groups of interlinked coils (Fig. 12.4 left): A set of planar coils generating the toroidal field component and four helical coils carrying currents running in alternating directions in neighboring coils⁷ responsible for the helical field components. The drawback of this kind of coil system is that strong forces can arise between the interlinked coils thus making it difficult to build. Also it can only be used for relatively simple configurations due to its low flexibility with regard to design.

Another possibility for finding a suitable coil system comes from the following observation: If in a toroidal volume V a magnetic field is given that on the boundary surface ∂V is tangential it is possible to calculate a surface current distribution on a second surface outside of ∂V that generates this field; discretizing this current distribution then leads to a coil system⁸. This way of obtaining a coil system is crucial for the design of modern optimised stellarators since it allows to start from a given plasma shape, which e.g. is obtained by stellarator optimisation (see sec. 12.6), and then calculate the corresponding – so called – modular coil system, which consists of a set of non-planar coils.

⁷A configuration with currents flowing always in the same direction is called a torsatron; it does not have toroidal field coils, but instead needs two coils generating a vertical field component.

⁸Since the corresponding mathematical problem does not have a unique solution there are many different coil systems giving nearly the same magnetic field. This allows to take into account engineering constraints in an optimisation procedure which combines aspects of field accuracy and coil flexibility.

12.2 Equilibrium

12.2.1 Equilibrium equations

The system of equations to be solved for a macroscopic fluid equilibrium follows from the general magnetohydrodynamic (MHD) equations by assuming $\partial/\partial t = 0$ and $\mathbf{v} = 0$

$$\mathbf{j} \times \mathbf{B} = \nabla p, \quad (12.2)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}, \quad (12.3)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (12.4)$$

Eq. (12.2) gives the balance between Lorentz force and pressure force while (12.3) to (12.4) are Maxwells equations. One immediate conclusion is $\mathbf{B} \cdot \nabla p = 0$, i.e. the pressure is constant along a field line and consequently – assuming the existence of flux surfaces – it must be constant on each flux surface. From $\mathbf{j} \cdot \nabla p = 0$ it follows that the current can only flow in a flux surface.

12.2.2 Straight stellarator

It was mentioned in the last section that a stellarator cannot be axisymmetric, so the simplest possible stellarator is a helical symmetric configuration, i.e. all the physical quantities (using cylindrical coordinates) depend only on r and $\vartheta - kz$, where k is the parameter determining the length of a period.

For a vacuum field one has to solve the equation $\Delta\Phi = 0$ to obtain the magnetic potential (b_ℓ are free coefficients)

$$\Phi = B_0 z + \frac{1}{k} \sum_{\ell=1}^{\infty} b_\ell I_\ell(\ell kr) \sin \ell(\vartheta - kz).$$

From this follows an expression for the magnetic surfaces for a general straight stellarator

$$\Psi = B_0 \frac{kr^2}{2} - r \sum_{\ell=1}^{\infty} b_\ell I'_\ell(\ell kr) \cos \ell(\vartheta - kz) = \text{const}. \quad (12.5)$$

Fig. 12.5 shows the cross section of stellarators each having only one ℓ component; due to the helical symmetry the curves rotate around the origin for increasing z . Note that the $\ell=1$ stellarator is the only one with a non-straight magnetic axis.

Since straight configurations are only of theoretical interest we will in the following always assume toroidal topology.

12.2.3 Magnetic coordinates

Typical for a fusion plasma is the extremely high degree of anisotropy introduced by the magnetic field. As an order of magnitude estimate for the perpendicular and parallel length scale

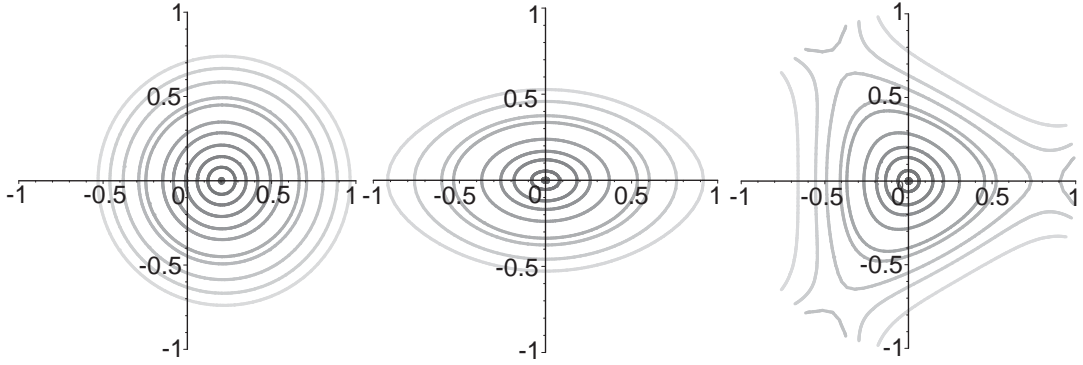


Figure 12.5: Intersection of $\Psi = \text{const}$ surfaces with the $z = 0$ plane for an $\ell = 1, 2, 3$ stellarator (from left).

one can take the ion gyroradius which is of the order of some 10^{-2} m and the parallel mean free path of a particle which is around some 10^3 m, respectively. Thus it is highly advantageous to have at one's disposal a coordinate system in which the field line can be described by a simple functional form and a separation of the perpendicular and parallel direction is easy.

The so called magnetic coordinates (s, ϑ, φ) are a special curvilinear coordinate system fulfilling these requirements. Here $s \in [0, 1]$ is the flux surface label – using magnetic coordinates requires the assumption that nested flux surfaces exist everywhere – and $\vartheta, \varphi \in [0, 2\pi]$ are angle-like coordinates. The geometric complexity of the configuration thus becomes hidden in the metric tensor g_{ij} . For calculations in this coordinate system one can use the formalism of tensor calculus.

In magnetic coordinates the magnetic field has the simple form

$$\mathbf{B} = \frac{1}{\sqrt{g}} (F'_T \mathbf{r}_{,\varphi} + F'_P \mathbf{r}_{,\vartheta}),$$

where F'_T, F'_P is the s derivative of the toroidal ($F_T(s)$) and poloidal ($F_P(s)$) magnetic flux, $g^{1/2}$ is the determinant of the metric tensor and $\mathbf{r}_{,\varphi}, \mathbf{r}_{,\vartheta}$ are the covariant basic vectors. This expression shows that in magnetic coordinates the field lines are just straight lines $\vartheta = \iota \varphi$ with the rotational transform given by $\iota = F'_P/F'_T$.

12.2.4 Magnetic islands

One vital question for a fusion device is if it possesses nested flux surfaces or not.

Because of axisymmetry this question is simple to answer for a tokamak. Its symmetry allows to reduce the MHD equilibrium equations to one elliptic equation for the flux function $\Psi(r, z)$: the Grad-Shafranov-Equation

$$\left(r \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right) \Psi + \mu_0 r^2 \frac{\partial p(\Psi)}{\partial \Psi} + \frac{\mu_0}{8\pi^2} \frac{\partial I^2(\Psi)}{\partial \Psi} = 0, \quad (12.6)$$

where $p(\Psi)$ and the current distribution $I(\Psi)$ are arbitrary functions. The fact that p only depends on Ψ ensures a constant pressure on each flux surface. Specifying profiles for p and

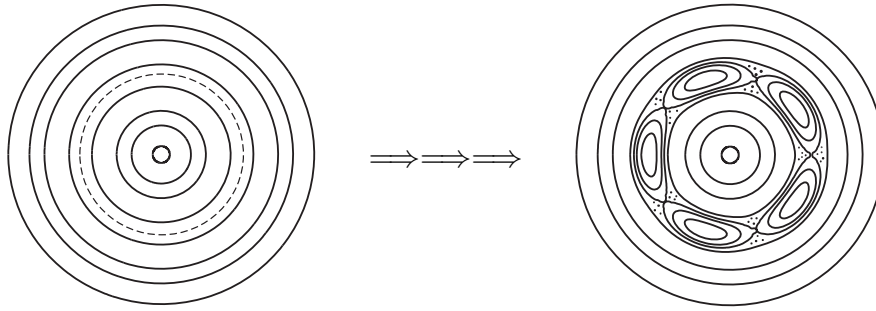


Figure 12.6: Left: Equilibrium with nested flux surfaces (the resonant surface is shown dashed).
Right: Equilibrium with perturbation, ergodic regions are indicated with dots.

I and imposing boundary conditions on Ψ (e.g. $\Psi = 0$ at the given plasma boundary) this (in general) nonlinear equation can be solved and directly gives the flux surfaces. Thus there is no problem for axisymmetric equilibria.

For three-dimensional configurations it is generally not possible to show that a flux function exists, which makes it very complicated to obtain information about the existence of flux surfaces.

The complications arising in connection with three-dimensionality can be seen by the following gedankenexperiment: Imagine bending a straight stellarator, which always possesses flux surfaces (see (12.5)), into a torus. This process can be regarded as adding perturbations with Fourier amplitude \mathcal{P}_{mn} to the original straight equilibrium in order to produce the new toroidal one. It can be shown that the surface which is in resonance with the perturbation \mathcal{P}_{mn} will break up into a chain of islands separated by X-points⁹. The latter are surrounded by ergodic regions where the field lines move chaotically filling out a volume instead of a surface¹⁰. This is illustrated in Fig. 12.6 where the effect of an $m = 5$ perturbation on an equilibrium with flux surfaces is sketched.

How this looks like for a real equilibrium is shown in Fig. 12.7. Here a chain with large islands (light grey) is located around the flux surface where the ι profile crosses 5/5 (note that the ι profile is constant inside the region occupied by the islands). For a shifted ι profile that does not contain 5/5 these islands disappear.

The width w of an island can be estimated by

$$w = 4\sqrt{\frac{R}{m\iota'}\mathcal{P}_{mn}}. \quad (12.7)$$

⁹These islands are called *natural* in order to distinguish them from island caused by effects external to the equilibrium, e.g. small errors in the coils. For natural islands the Fourier number n must be a multiple of the number of field periods.

¹⁰All this is analogous to nonlinear dynamics where it is shown that perturbations added to an integrable system lead to a destruction of the resonant tori in phase space (see the Poincaré-Birkhoff and Kolmogorov-Arnold-Moser (KAM) theorem). This analogy can be put on rigorous footing by noticing that the field lines in a torus can be described by a Hamiltonian function.

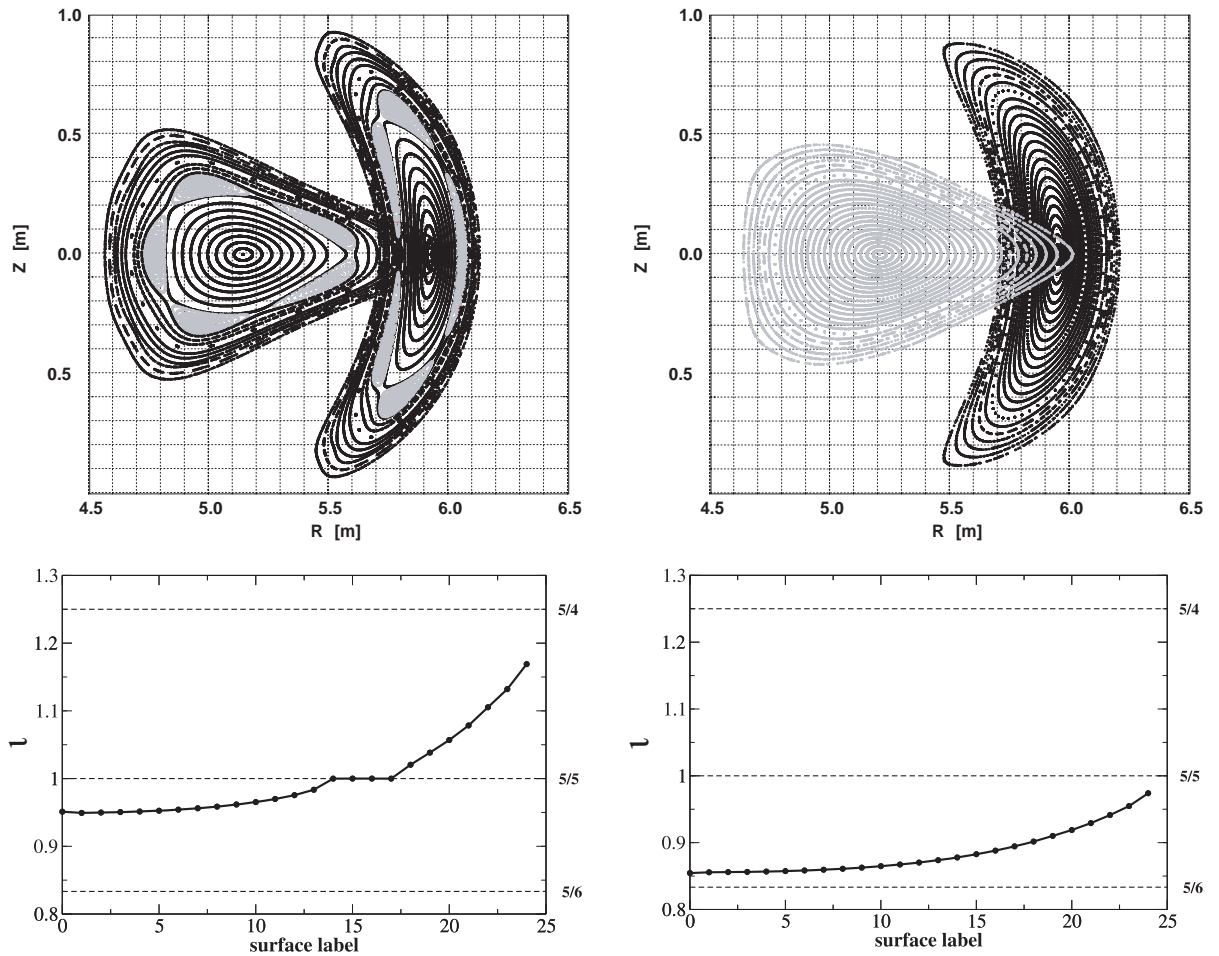


Figure 12.7: Poincaré plots of two Wendelstein 7-X variants with different ι profiles.

Left: ι profile including the 5/5 resonance.

Right: The same calculation but with a downward shifted profile avoiding this resonance.

Using e.g. values typical of Wendelstein 7-X and assuming a perturbation amplitude $\mathcal{P}_{mn} \approx 10^{-4}$ one finds that this perturbation can lead to islands of width $w/a \approx 0.2$ showing that even a small perturbation can create relatively large islands. The danger of islands lies in the fact that they normally lead to a strong deterioration of plasma confinement by allowing particles to quickly cross the region occupied by the island chain. The existence of islands in an equilibrium is an inherent feature of three-dimensional systems, so it is of major interest for stellarator theory to design configurations in such a way that islands are so small that they have nearly no effect on confinement.

Eq. (12.7) shows that rational numbers n/m with low m – so called low order rationals – produce larger islands than rationals with high m and that high shear leads to smaller islands than low shear. Hence there are two strategies to arrive at three-dimensional equilibria with small islands:

- High shear minimises the influence of low order rationals but makes it necessary for the ι profile to cross many rationals. This strategy is followed in e.g. LHD.
- Low shear allows to place the flat ι profile in a region where there are no low order rationals. This line is followed by Wendelstein 7-X.

12.2.5 3D equilibrium codes

In order to describe an equilibrium (without bootstrap current) one has to specify the radial profile of the pressure and the shape of the outer flux surface \mathcal{S} (along with the boundary condition that \mathbf{B} must be tangent to \mathcal{S}). For a stellarator the condition $J = 0$ is imposed – the ι -profile then follows – while for a tokamak either the current or the ι -profile must be given. Solution of (12.2) to (12.4) then gives the magnetic field inside \mathcal{S} .

The shape of \mathcal{S} is usually described using the parametric representation

$$\begin{aligned} R &= \sum_{m,n} R_{m,n} \cos(mU - nV), \\ Z &= \sum_{m,n} Z_{m,n} \sin(mU - nV), \\ \phi &= V, \end{aligned}$$

where $U, V \in [0, 2\pi]$ are parameters and (R, Z, ϕ) are cylindrical coordinates, together with the set of Fourier coefficients $R_{m,n}, Z_{m,n}$.

Since in three dimensions the existence of flux surfaces is not guaranteed, the structure of a solution can only be found a posteriori making it non-trivial to obtain pressure profiles which are constant on each flux surface.

The numerical calculation of stellarator equilibria is e.g. carried out with the help of two tools: The Variational Magnetic Equilibrium Code (VMEC) and the Princeton Iterative Equilibrium Solver (PIES).

VMEC

In this approach one *assumes* that the equilibrium has nested flux surfaces, i.e. one ignores the possibility of islands. As we have seen this assumption is generally not true for three-dimensional equilibria. Thus the solution given by the code serves only as an approximation to the exact equilibrium and can be rather bad if this has large islands¹¹. Whether the approximation is a good one can only be tested afterwards.

Under this assumption making the energy functional

$$W = \int \left(\frac{B^2}{2\mu_0} + \frac{p}{\gamma - 1} \right) dV$$

¹¹Instead of islands the approximative solution shows divergences in the parallel current density j_{\parallel} at the resonant surfaces. In the exact solution j_{\parallel} stays finite everywhere.

(γ : adiabatic coefficient) stationary, together with the additional constraint that the mass inside each flux surface does not change by the variation, is equivalent to solving the equilibrium (12.2) to (12.4).

The attractiveness of this approach lies in the fact that it is computationally much faster than computing an equilibrium allowing for islands. Thus VMEC has become the standard tool for calculating stellarator equilibria.

PIES

The PIES code permits islands in the equilibrium solution¹². It solves the MHD equilibrium equations employing a Picard iteration, i.e. starting with an initial guess \mathbf{B}^0 – obtained with e.g. VMEC – it solves the equations

$$\nabla \times \mathbf{B}^{n+1} = \mu_0 \mathbf{j}(\mathbf{B}^n), \quad \nabla \cdot \mathbf{B}^{n+1} = 0,$$

where $\mathbf{j}(\mathbf{B}^n)$ is found from solving

$$\mathbf{j} \times \mathbf{B}^n = \nabla p, \quad \nabla \cdot \mathbf{j} = 0.$$

The drawback of this approach is its extremely slow convergence making the calculation of an equilibrium very time consuming¹³.

12.3 Pfirsch-Schlüter current

One goal of fusion power plant design is to achieve $\beta := 2\mu_0 p/B^2$ values of about 5% (averaged in the confinement region). Connected with an increase in the plasma pressure is a shift of the plasma center with respect to the outer plasma surface (Shafranov shift). If this shift is too large the plasma center comes close to the outer plasma surface which may cause problems. So, cautiously one limits β to some value β_{eq} such that the plasma center keeps a safe distance from its boundary.

For an axisymmetric equilibrium and classical stellarators the Shafranov shift Δ is given approximately by $\Delta/R = \beta/\iota^2$. Usually one defines β_{eq} to be reached if $\Delta = a/2$, resulting in $\beta_{\text{eq}} \sim \iota^2/A$. Typical values are $A \approx 3$, $\iota \lesssim 1$ for a tokamak and $A \approx 10$, $\iota < 1$ for a stellarator. While this definition leads to a useful β -limit for a tokamak, a stellarator with small ι and large A has a β_{eq} which is too low for fusion applications.

The way out of this difficulty is to use the freedom given by the three-dimensional structure of a stellarator to design $|\mathbf{B}|$ in such a way as to reduce the Shafranov shift.

First we have to understand what causes the Shafranov shift: From the MHD equilibrium relation $\mathbf{j} \times \mathbf{B} = \nabla p$ one obtains the diamagnetic current $\mathbf{j}_\perp = \mathbf{B} \times \nabla p/B^2$. Splitting the quasineutrality relation $\nabla \cdot \mathbf{j} = 0$ into a parallel and perpendicular component gives $\nabla \cdot \mathbf{j}_\parallel = -\nabla \cdot \mathbf{j}_\perp$. The

¹²A second code allowing for islands is the HINT code which uses the time dependent MHD equations.

¹³An equilibrium calculation for the W7-X configuration using VMEC needs a few minutes computing time on an SX-5 vector computer; the analogous calculation using PIES takes approximately one week.

diamagnetic current thus gives rise to a parallel current called the Pfirsch-Schlüter (PS) current. As a result one gets

$$\nabla \cdot \mathbf{j}_{\parallel} = -\frac{2}{B^3} (\nabla p \times \mathbf{B}) \cdot \nabla |\mathbf{B}| = -\frac{2|\nabla p|}{B} \kappa_g$$

with κ_g the geodesic curvature¹⁴. Note that ∇p is normal to the flux surface and, as a consequence, only the derivative of $|\mathbf{B}|$ in the flux surface enters this expression. The geodesic curvature will also play an important role in the next section and thus will show up as a quantity of main importance for stellarators.

For a simple axisymmetric configuration this equation can easily be solved and gives the PS current as

$$j_{\parallel} = \frac{\partial p}{\partial r} \frac{2}{\iota B_0} \cos \vartheta .$$

This parallel current density – flowing in different directions at the inside respectively outside of the torus – leads to a vertical magnetic field that in combination with the toroidal and poloidal field results in an outward shift of the plasma: The Shafranov shift.

For a helically symmetric configuration with $\mathbf{B} = B(r, \vartheta - kz)$ one finds $j_{\parallel} \propto 1/(\iota - 1)$. It follows for $\iota \ll 1$ that the magnitude of j_{\parallel} (and consequently the Shafranov shift) in a helically symmetric configuration is much smaller than in an axisymmetric one. Using the VMEC code it is possible to construct quasi-helically symmetric stellarator equilibria¹⁵ for which even a β of 40% leads to nearly no Shafranov shift.

This observation shows the strategy how to reduce the Shafranov shift and increase β_{eq} for a stellarator: Using the freedom given by the three-dimensional structure one can design $|\mathbf{B}|$ and with it the geodesic curvature in such a way as to minimize the Shafranov shift (more details on this will be given later in sec. 12.6). Taking the volume averaged ratio of the parallel to the diamagnetic current $\langle j_{\parallel}^2 / j_{\perp}^2 \rangle$ as a measure for the importance of the PS current one can see the progress of stellarator design with time: While for the classical (non-optimised) stellarator W7-A this ratio is around 12 it can be reduced to three for the partly optimised W7-AS and then to 0.5 for the fully optimised W7-X (corresponding to $\beta_{\text{eq}} \approx 5\%$). This is illustrated in Fig. 12.8, where W7-AS for $\beta \approx 2\%$ shows a noticeable Shafranov shift while it is hardly visible in W7-X for $\beta \approx 5\%$.

12.4 Neoclassical transport

12.4.1 Neoclassical transport

The main aim of neoclassical (transport) theory is to calculate the collisional transport of particles and heat – e.g. radial diffusion coefficients for particle and heat fluxes – as functions

¹⁴Using a triad $(\mathbf{b}, \mathbf{n}, \mathbf{n} \times \mathbf{b})$ such that \mathbf{b} is the unit B-field vector and \mathbf{n} is the unit normal vector of the flux surface the curvature $\kappa := (\mathbf{b} \cdot \nabla) \mathbf{b}$ can be written as $\kappa = \kappa_n \mathbf{n} + \kappa_g \mathbf{n} \times \mathbf{b}$ with the geodesic and normal curvature κ_g, κ_n . In contrast to the normal curvature the geodesic curvature is an intrinsic property of the surface geometry.

¹⁵In toroidal topology helically symmetric equilibria do not exist; one has to resort to so called quasi-helically symmetric equilibria (see sec. 12.4.2).

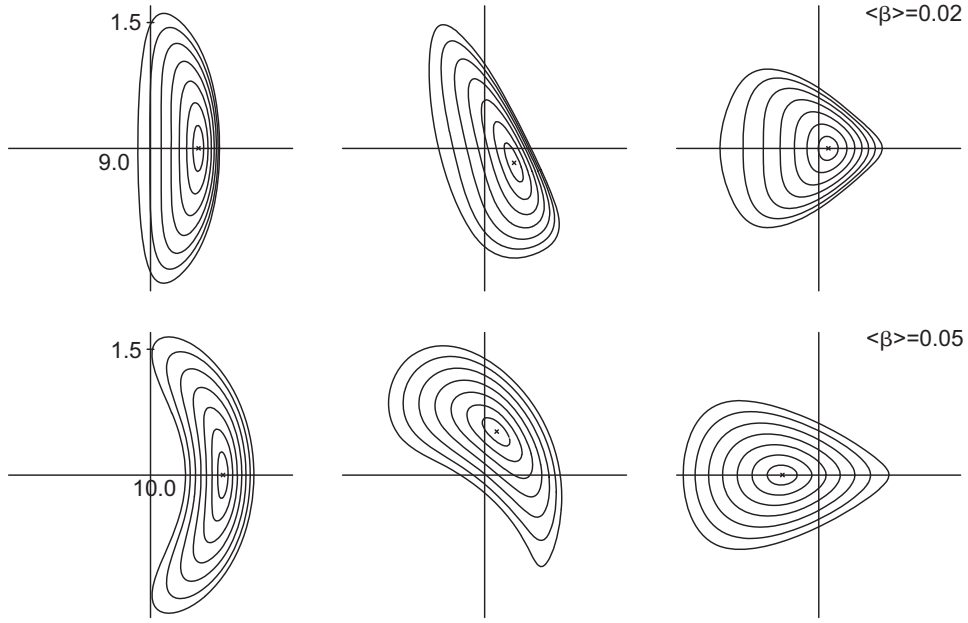


Figure 12.8: Toroidal cuts (at $\phi = 0, 2\pi/20, 2\pi/10$) for W7-AS (top) and W7-X (bottom).

of the collision rates and the electric field taking into account the toroidal geometry of a fusion device¹⁶. The theoretical base for neoclassical calculations is collisional kinetic theory especially its drift kinetic simplification.

Fig. 12.9 gives an overview on the neoclassical transport for tokamaks and stellarators. Depending on ν_{ei} (ion-electron collision frequency) one finds different regimes: The Pfirsch-Schlüter, plateau and banana regime for high, intermediate and low collisionality, respectively. The three-dimensional geometry of a stellarator can lead to regimes not present in a tokamak. In the long mean free path regime the helical variation of $|B|$ (helical ripple) for a classical stellarator leads to the so called $1/\nu$ regime¹⁷ which is of paramount importance since it can lead to extremely large neoclassical losses potentially dangerous for a stellarator fusion device.

For the PS regime a fluid treatment of radial diffusion is possible: For a cylinder the equilibrium equation and Ohm's law ($\eta = m_e/(ne^2)\nu_{ei}$)

$$\mathbf{j} \times \mathbf{B} = \nabla p, \quad \mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j}$$

give the perpendicular velocity of the fluid as

$$\mathbf{v}_\perp = \frac{\mathbf{E} \times \mathbf{B}}{B^2} - \frac{1}{B^2} \eta \nabla p$$

consisting of the $\mathbf{E} \times \mathbf{B}$ -drift and a radial drift induced by collisions. Using the last term to calculate the radial particle flux $\Gamma = n\mathbf{v}_\perp$ and comparing this with Fick's law $\Gamma = -D\nabla n$ leads to the classical diffusion coefficient (ρ_e : electron gyroradius)

$$D_c = \left(1 + \frac{T_i}{T_e}\right) \rho_e^2 \nu_{ei}. \quad (12.8)$$

¹⁶In cylinder geometry one speaks of classical transport theory.

¹⁷If an electric field is included a $\nu^{1/2}$ regime can also be present.

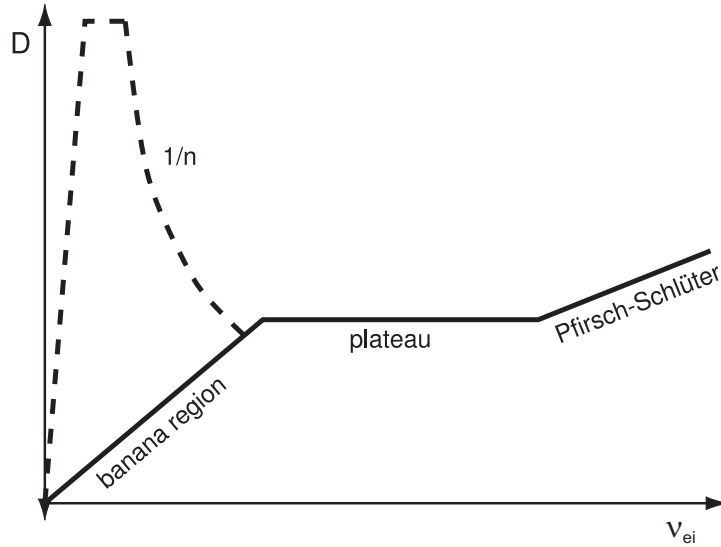


Figure 12.9: Neoclassical diffusion coefficient as a function of collision rate. Black: tokamak. Dashed: additional regimes for a stellarator.

A more complicated calculation in toroidal geometry¹⁸ including the PS current gives an additional factor (PS factor) and leads to the larger neoclassical diffusion coefficient $D_{nc} = (1 + \iota^{-2})D_c$.

To understand the origin of the banana and $1/\nu$ regime one has to look at the motion of particles. For this task it is sufficient to use the guiding center equations of motion¹⁹

$$\dot{\mathbf{R}} = v_{\parallel} \mathbf{b} + \mathbf{v}_d, \quad (12.9)$$

$$\dot{v}_{\parallel} = -\mu \mathbf{b} \cdot \nabla |\mathbf{B}|, \quad (12.10)$$

$$\dot{\mu} = 0 \quad (12.11)$$

with the perpendicular drift velocity ($\mathbf{b} := \mathbf{B}/|\mathbf{B}|$ and $\mu := v_{\perp}^2/(2B)$ the magnetic moment per unit mass)

$$\mathbf{v}_d = \frac{\mu \mathbf{B} + v_{\parallel}^2}{B\Omega} \mathbf{b} \times \nabla |\mathbf{B}|. \quad (12.12)$$

From these equations²⁰ follows the very important fact that particles with low enough energy can be trapped around a minimum of $|\mathbf{B}|$ while for higher energies they can stream (nearly) freely. A particle trapped in a minimum of B shows a fast bouncing motion superposed onto a slow drift. This slow drift motion can be separated by averaging over the fast bouncing motion (bounce average) and gives the motion of the bounce center²¹ $\langle \mathbf{v} \rangle = 1/\tau_b \oint \mathbf{v} dt$ (τ_b : bounce time).

¹⁸This can most easily be seen by using the general relation $\Gamma = n \int \eta j^2 dS / (|\nabla p|)$ (with $\eta_{\perp} \approx 2\eta_{\parallel}$) connecting the particle flux Γ with the resistive dissipation integrated over one flux surface. Thus the additional dissipation caused including parallel (PS) currents leads to an increase in the particle flux by the PS factor.

¹⁹See the lecture "Introduction to fusion plasmas".

²⁰Note that (12.10) follows from (12.9), (12.11) and energy conservation.

²¹To obtain the drift kinetic theory the fast gyrating particle had been replaced via averaging by its slowly drifting guiding center. Now drift kinetic is further reduced to a theory with the bounce center as the slowly moving basic entity obtained by averaging over the comparatively faster bounce motion of the guiding center.

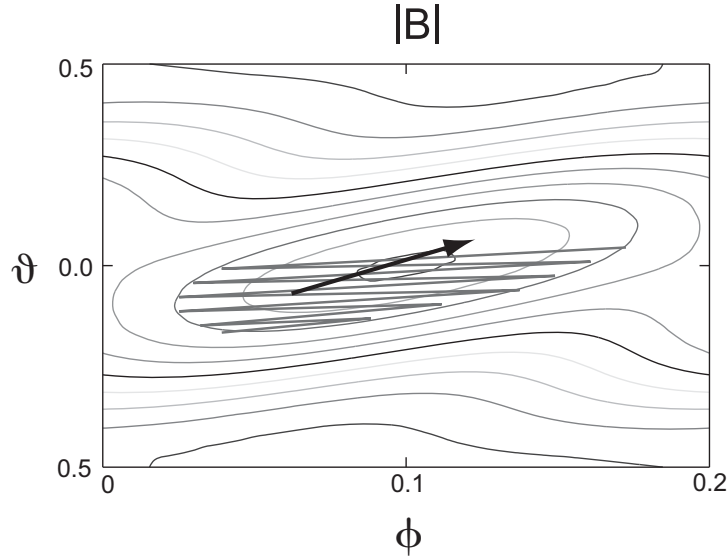


Figure 12.10: Motion of a trapped particle near a minimum of B (contour lines). The red line shows the actual motion of the guiding center while the black arrow shows the motion of the bounce center.

In a tokamak trapped particles move on so called banana orbits with radial width Δ_b . The center of a banana orbit slowly drifts toroidally but not radially, i.e. $\langle v \rangle$ has only a toroidal component. Collisions cause particles to jump from one banana orbit to another thus leading to a random walk with step size Δ_b resulting in a diffusion coefficient (given by $D = l^2/\tau$, where l is the diffusion step length and τ the collision time²²) in the banana regime as $D_b = \Delta_b^2 v_{ei}$.

As a very simple model for $|B|$ in a stellarator one often uses

$$B = B_0 (1 - \varepsilon_t \cos \vartheta - \varepsilon_h \cos(\ell \vartheta - P\phi)) ,$$

where $\varepsilon_{t,h}$ measures the strength of the toroidal or helical component, respectively.

For $\varepsilon_t = 0$ the bounce center drifts along the helical minimum (helically trapped particle) and $\langle v \rangle$ has no component in the radial direction.

Adding a toroidal component ($\varepsilon_t \neq 0$) leads to a net radial component of $\langle v \rangle$ which can be computed by performing the bounce average of

$$v^r = v_d \frac{\nabla r}{|\nabla r|} = \frac{\mu B + v_{\parallel}^2}{\Omega} \kappa_g .$$

This result can be used to estimate a diffusion coefficient by noting that the diffusion step length is $l = \langle v^r \rangle / (v_{ei})$.

$$D_{l/v} = f_t \frac{\langle v^r \rangle^2}{v_{ei}} \sim f_t \frac{T^{7/2}}{nB^2} \kappa_g^2 \quad (12.13)$$

(f_t : fraction of trapped particles).

²²One can show that only collisions between unlike particle species lead to particle diffusion, while for heat diffusion collisions between like particle species are important.

This equation also shows the crucial role played by the geodesic curvature; its importance for the PS-current we have already seen in sec. 12.3. Thus decreasing the size of the geodesic curvature in the region where particles are trapped is beneficial for minimizing the neoclassical transport.

Summarizing the above picture one can say that caused by the toroidal curvature a helically trapped particle in a stellarator in general acquires a radial drift eventually leading to its loss; these particles are responsible for the $1/\nu$ regime.

One can argue more generally: Due to its axial symmetry the toroidal momentum is a conserved quantity for particle motion in a tokamak and it follows as a consequence that the maximal deviation of a (trapped or untrapped) particle from the flux surface is limited; this results in a relatively low neoclassical transport. The situation is different for a stellarator: Caused by its three-dimensional structure a conserved quantity and the related constraint on the orbit width does in general not exist²³ possibly leading to rapid particle losses showing up as the $1/\nu$ regime.

The strong dependence of $D_{1/\nu}$ on T is the cause why the $1/\nu$ regime is dangerous for a (classical) stellarator: The high temperature necessary for fusion leads to losses which are too high for a fusion power plant to work. So in order to construct a viable stellarator power plant it is of utmost importance to reduce this kind of neoclassical losses. From (12.13) one can see the possible ways how to do this: Designing a stellarator in such a way that the particles get trapped in a region with small geodesic curvature leads to a small neoclassical radial transport. This path is followed in the design of W7-X and as a result neoclassical losses in W7-X are small enough to be compatible with fusion power plant requirements.

12.4.2 Quasi symmetry

Using magnetic coordinates (s, ϑ, φ) (see sec. 12.2.3) the guiding center equation of motion for $\dot{\mathbf{R}}$ (12.9) can be written as

$$\dot{s} = \frac{1}{\sqrt{g}D} \frac{e\rho_{\parallel}}{m} (\rho_{\parallel, \vartheta} I - \rho_{\parallel, \varphi} J), \quad (12.14)$$

$$\dot{\vartheta} = \frac{1}{\sqrt{g}D} \frac{e\rho_{\parallel}}{m} \left[-F'_p - (I\rho_{\parallel})_{,s} + \left(\tilde{\beta}\rho_{\parallel} \right)_{,\varphi} \right], \quad (12.15)$$

$$\dot{\varphi} = \frac{1}{\sqrt{g}D} \frac{e\rho_{\parallel}}{m} \left[-F'_T + (J\rho_{\parallel})_{,s} - \left(\tilde{\beta}\rho_{\parallel} \right)_{,\vartheta} \right] \quad (12.16)$$

with $\rho_{\parallel} := mv_{\parallel}/(eB)$, $D := 1 + \rho_{\parallel} \mathbf{B} \cdot \mathbf{j} / B^2$, $\sqrt{g} := -(JF'_p + IF'_T) / B^2$ and $\tilde{\beta}$ a function that can be calculated from B .

This formulation shows that the motion of the guiding center formulated in the special coordinate system of magnetic coordinates depends *only* on the function $B(s, \vartheta, \varphi)$ (magnetic topography) but not on the spatial structure of the flux surfaces which would show up as the

²³Quasi-symmetric configurations (see next section) do have conserved quantities.

metric tensor. Formulated sloppily one can say that in magnetic coordinates the particles do not *see* the real geometry of the configuration but only the magnetic topography. This observation leads to the notion of quasi-symmetry²⁴ as a symmetry of the magnetic topography formulated in magnetic coordinates; the configuration does not need to possess a symmetry in real space.

Configurations with $B = B(s, \vartheta)$ or $B = B(s, k\vartheta - P\varphi)$ (k, P : integer parameters) are called quasi-axisymmetric or quasi-helical symmetric, respectively. They can be realized in a torus and examples are known (e.g. see Fig. 12.11). On the other hand the quasi-symmetries $B = B(s, \varphi)$ and $B = B(s)$, which would guarantee that a particle stays exactly at a flux surface²⁵, can not be achieved for toroidal systems.

For a configuration possessing a quasi-symmetry it follows from the guiding center equations of motion that there exists a conserved quantity²⁶. This conserved quantity limits the maximal excursion of a particle from the flux surface (as noted in the last section) and thus eliminates the cause for the existence of the $1/\nu$ regime. Detailed calculations indeed show that e.g. a quasi-helical symmetric stellarator shows similar neoclassical transport as a corresponding tokamak.

As an illustration Fig. 12.11 shows one flux surface of a quasi-helical symmetric configuration with six field periods. In real space it has a complicated three-dimensional structure and shows no apparent symmetries other than the ones described in sec. 12.1.3. After transformation to magnetic coordinates the helical symmetry of B becomes manifest.

Quasi-helical symmetric configurations have a bootstrap current which endangers their usefulness. Thus one considers another class of configurations, called quasi-isodynamic, where the second adiabatic invariant $J := \int v_{\parallel} dl$ (l : length along a field line) is nearly constant on the flux surfaces. In these configurations (without symmetry of B) the bounce center drifts poloidally. W7-X belongs to the class of quasi-isodynamic stellarators.

12.5 MHD stability

Since the vast field of stability quickly becomes very technical we will only give a short overview of the basic concepts of MHD stability.

Having found a plasma equilibrium configuration that meets the requirements described in the former sections it remains to show if a small perturbation applied to the system grows or decays with time. In the first case the equilibrium configuration is unstable and plasma confinement may be destroyed quickly.

Formally one uses time dependent equations, symbolically written as $\partial g / \partial t = \mathcal{M}g$ with g denoting the dynamical variables and \mathcal{M} a nonlinear operator. The solution is split into a

²⁴The qualifier *quasi* is meant to indicate that this symmetry only exists for B in a specially chosen coordinate system. Additionally it hints that quasi-symmetry is attainable in a three-dimensional configuration only to some approximation but not exactly.

²⁵A configuration fulfilling $B = B(s)$ exactly would be called isodynamic.

²⁶For e.g. quasi-helical symmetry the quantity $F_T - F_P - \rho_{\parallel}(I + J)$ is conserved.

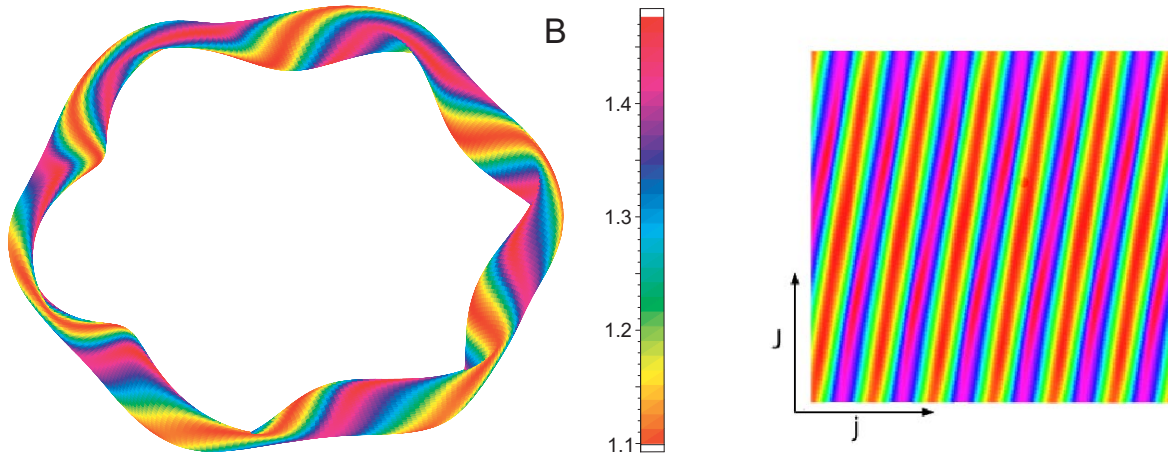


Figure 12.11: Quasi-helical symmetric stellarator in real space (left) and magnetic coordinates (right) (one flux surface is shown, color indicates the magnitude of B).

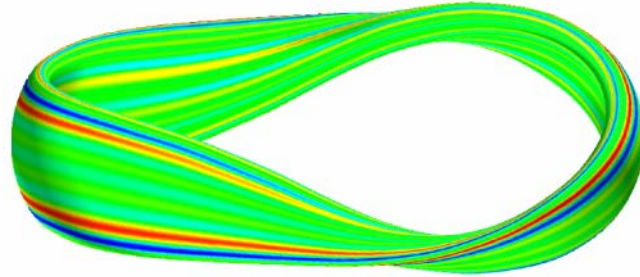


Figure 12.12: Eigenfunction for a ballooning mode. Shown is the pressure perturbation on a flux surface (high/low pressure is colored with red/green).

time independent part representing the equilibrium under investigation and a time dependent part describing the perturbation: $g(\mathbf{r}, t) = g_0(\mathbf{r}) + g_1(\mathbf{r}, t)$. Assuming small perturbations $g_1 \ll g_0$ the nonlinear operator \mathcal{M} is linearized resulting in the linear operator \mathcal{L} . Usually one is interested only in solutions behaving exponentially in time²⁷ justifying the ansatz $g_1 = \tilde{g}(\mathbf{r})e^{-i\omega t}$ (with the complex eigenvalue ω). Specification of boundary conditions on \tilde{g} then leads to the eigenvalue problem $\mathcal{L}\tilde{g} = \omega\tilde{g}$. The equilibrium is unstable if $\text{Im}(\omega) > 0$ showing that the initially infinitesimal perturbation is growing exponentially with time.

Since for MHD the operator \mathcal{L} is self-adjoint its eigenvalue problem only has solutions with $\omega^2 \in \mathbb{R}$ and can be formulated as a variational problem: The system is unstable if a perturbation $\xi(\mathbf{r})$ in the position of the plasma exists such that $\delta^2 W[\xi] < 0$ where the functional²⁸ $\delta^2 W$ is given by

$$\delta^2 W[\xi] = -\frac{1}{2} \iiint [S^1 + S^2 + S^3 + S^4 + S^5] d^3 r.$$

²⁷Perturbations growing algebraically in time may exist but because of their slow growth normally are of no interest.

²⁸ $\delta^2 W$ is the second variation of the functional W from sec. 12.2.5.

The expressions for $S^1(\xi) \dots S^5(\xi)$ are complicated and instead of giving the details here it suffices to note their general properties: S^1 , S^2 and S^5 describe the influence of field line bending, field compression and fluid compression, respectively; they act stabilizing on $\delta^2 W$. In contrast to this S^3 and S^4 are indefinite and depend on the pressure gradient p' and the parallel current density j_{\parallel} ; these two terms can cause $\delta^2 W$ to become negative. The instabilities caused by these terms are called pressure or current driven.

From the large zoo of MHD instabilities we briefly look at the kink and the ballooning instabilities: The kink instability is driven by a parallel current and consequently does not exist in a stellarator. Pressure is the energy source for the ballooning instability which bears this name since e.g. in a tokamak it has its maximal amplitude in the region of unfavorable curvature (i.e. at the outside of the torus). This instability disappears for fixed pressure gradient if the pressure is low enough and thus provides a stability β -limit. W7-X is designed in such a way that it is stable against the ballooning instability for $\beta \lesssim 5\%$. For a tokamak, large scale kink and ballooning instabilities limit the operation regime, while for a stellarator only ballooning modes have been found experimentally, hitherto. As an illustration Fig. 12.12 shows a ballooning instability in a quasi-axially symmetric stellarator above the stability β -limit. This picture also demonstrates a very general structural property of modes in a magnetized plasma: The wavelength in the direction perpendicular to the field lines is much smaller than the wavelength along the field line. This high degree of anisotropy of the instability reflects the anisotropy introduced by the magnetic field.

12.6 Stellarator optimisation

In sec. 12.2 to sec. 12.5 we have discussed four points which are of vital importance for a stellarator as a fusion device. Stellarator optimisation is the comprehensive framework where these points (and other) are put together in order to arrive at a viable design for a fusion device. Crucial for optimisation is the high degree of freedom provided by the three-dimensional structure of a stellarator.

Since the main goal of stellarator optimisation is to find equilibria that have good confinement properties one first has to quantify what one understands by *good confinement properties*. This is done by assigning to each equilibrium configuration from the configuration space Con ²⁹ a real number as a quality measure by providing a function $\mathcal{Q} : \text{Con} \mapsto \mathbb{R}$. The construction of this function requires the consideration of the physics one regards as relevant for confinement and the design targets.

In the design of W7-X one required:

- Nested magnetic surfaces with small islands;
- Small Shafranov shift for $\beta \approx 5\%$;

²⁹Since we have seen in sec. 12.2.5 that an equilibrium (except for its profiles) can be uniquely defined by specifying its outer flux surface the space Con can be identified with the set of Fourier coefficients used in the representation of this surface.

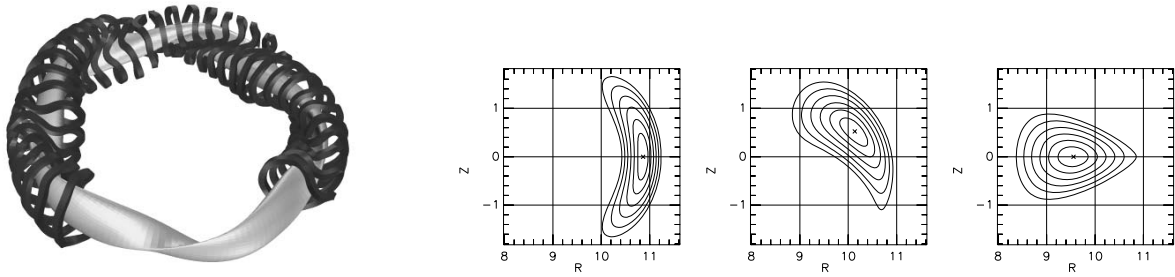


Figure 12.13: Left: Wendelstein 7-X plasma configuration with modular coils.
Right: Toroidal cuts through the equilibrium configuration at $\phi = 0, 2\pi/20, 2\pi/10$.

- Small neoclassical transport;
- MHD stability, especially stability against ballooning modes for $\beta \approx 5\%$;
- Good α -particle confinement:
 α -particles are needed for the heating of the plasma and thus they must not leave the fusion device in a time shorter than their slowing down time;
- Small bootstrap current:
This is another current driven by the pressure gradient. In contrast to the PS-current it gives rise to a total toroidal current causing a shift in t with increasing β . Since t control is crucial for the confinement properties (see e.g. sec. 12.2.4) such a shift was considered to be undesirable³⁰;
- Feasible modular coils:
For actually building a device constraints emerge motivated by engineering considerations regarding the design of the modular coils (e.g. a minimal distance between neighboring coils must be kept, the coil curvature cannot be too large). Some of them can be taken into account in the plasma optimisation.

In the optimisation procedure, which is carried out computationally, first an outer flux surface is specified from which (using VMEC) the corresponding equilibrium configuration is computed; based on this equilibrium the function \mathcal{Q} is calculated. This procedure is repeated – employing an optimisation algorithm – with changed outer flux surface until \mathcal{Q} has reached a maximum. When the optimised equilibrium has been found the actual coil system is calculated subsequently.

Depending on the criteria and the design goals many different configurations can be found via optimisation. One possible result from this procedure, the optimised stellarator Wendelstein 7-X, is shown in Fig. 12.13.

³⁰Quasi-axisymmetric configurations are an exception to this consideration.

References

There exists no good textbook about stellarator physics (as e.g. the book by Wesson for tokamaks). So everything is scattered in articles which generally are difficult to understand for the beginner. For this reason I here give references to some of the few more basic articles.

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Very good introduction, recommended.
- F. Wagner, “Topics in toroidal confinement”, *Plasma Physics and Controlled Fusion*, vol. 39, p. A23 (1997).
Experimentally oriented introduction.
- M. Wakatani, “Stellarator and Heliotron Devices”, Oxford University Press (1998).
One of the few books about stellarators but written very technically.

Chapter 13

Stellarator experiments

Thomas Windisch

13.1 Introduction

The stellarator concept was introduced by Lyman Spitzer Jr. in 1951. In a far-sighted article Spitzer described the idea of magnetic flux surfaces, i.e. a toroidal surface enclosing a certain magnetic flux $\psi = \text{const.}$, which is densely covered with magnetic field lines that lie in the surface $\mathbf{B} \cdot \nabla \Psi = 0$. If the field lines do not close they cover the toroidal surface ergodically and since they cannot cross each other, the field must proceed in the poloidal direction as it proceeds in the toroidal direction encircling the surface helically (see Fig. 13.1).

Hence, the magnetic field needs a certain twist or rotational transform ι , which is the ratio of poloidal and toroidal turns. The key element of magnetic confinement are nested flux surfaces

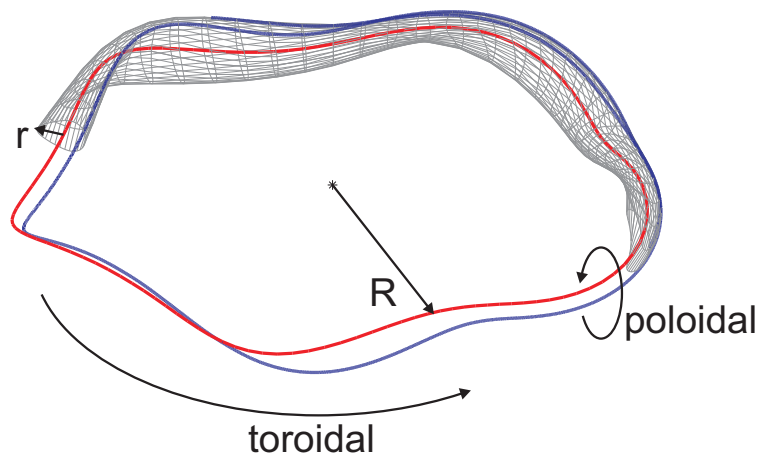


Figure 13.1: Toroidal magnetic confinement: flux surface (grey), helical magnetic axis (red) and open magnetic field line (blue).

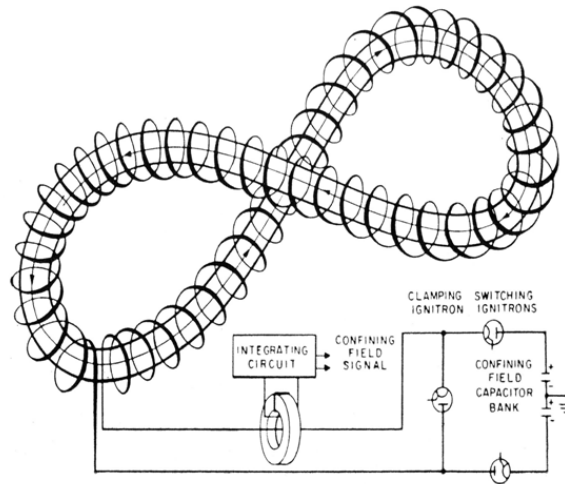


Figure 13.2: Figure-8 stellarator as proposed by Spitzer in 1951 [1].

with a small distance between the surfaces such that $|\nabla\Psi| \neq 0$ except on the magnetic axis, where the flux surface collapses to a single magnetic field line. As Spitzer recognized there are three possibilities to produce a rotational transform:

1. by a non-planar magnetic axis,
2. by an elongation and poloidal rotation of the flux surfaces, e.g. due to a transverse magnetic field whose direction rotates with distance along the magnetic axis ,or,
3. by driving a toroidal current.

Since a stellarator uses the first two concepts it allows for a large variety of non-axisymmetric helical confinement devices in which the magnetic field is solely produced by external coils without plasma currents. In contrast, the tokamak - conceptually developed also in 1951 by Tamm and Sakharov - relies only on the third method. The toroidal current is induced using ohmic transformers which makes tokamaks pulsed. Other non-inductive current drive mechanisms do exist but they are economically not efficient (a large fraction of the power produced in a fusion reactor is needed to sustain the current drive). Moreover, the tokamak suffers from current-driven instabilities leading in the worst case to disruptions. In contrast, the stellarator has a steady-state capability without plasma currents and without a hard upper density limit as a tokamak (Greenwald limit) but suffers from unconfined particle orbits as described below and a complex three dimensional geometry.

Spitzer proposed a figure-8 stellarator (see Fig. 13.2), where the rotational transform ($t/2\pi = 0.5$) is produced by the geometry. His idea was that the curvature and ∇B -induced particle drifts

$$\mathbf{v}_D = - (W_{\perp} + 2W_{\parallel}) \frac{\nabla \mathbf{B} \times \mathbf{B}}{qB^3}, \quad (13.1)$$

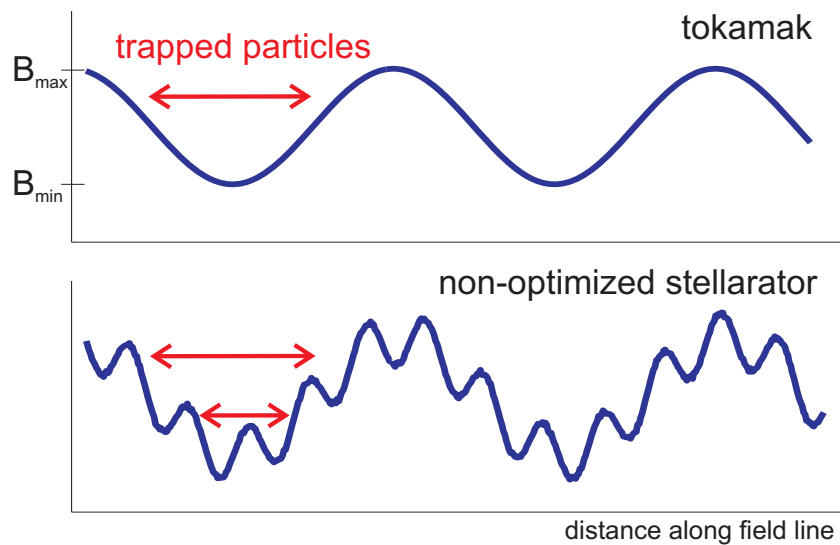


Figure 13.3: Magnetic field B along a magnetic field line for a tokamak and stellarator configuration.

where W denotes the parallel and perpendicular component of the kinetic energy, in one u-bend are compensated by the second u-bend. Without compensation the charge separation caused by the drift gives rise to a vertical electric field that leads to a rapid $\mathbf{E} \times \mathbf{B}$ -loss of the plasma.

For practical reasons the realized figure-8 stellarators, e.g. the largest being the Model-C stellarator in Princeton (1961-1969), had a racetrack-type geometry with consecutive straight and curved sections. These early stellarator experiments suffered from a small confinement time in the order of only a few Bohm times $\tau_{Bohm} = a^2/D_B$, with a being the minor radius and $D_B = T_e/(16B)$ is the Bohm diffusion coefficient and $T_e \approx 100$ eV. This was a major drawback for the stellarator concept since russian scientist reported in 1968 a tokamak confinement time of 30 Bohm times at much higher electron temperatures ($T_e = 1$ keV). The poor confinement in the early stellarator experiments were most likely caused by the large error fields induced by the transitions from straight to curved sections leading to a large fraction of unconfined particle orbits. It was soon realized that in the absence of axisymmetry of a three-dimensional stellarator field new classes of trapped particles do exist.

This is illustrated in Fig. 13.3, where the modulus of the magnetic field along a field line is displayed for a tokamak and stellarator geometry. For a tokamak, $B = |\mathbf{B}|$ varies in a periodic manner. Particles with a small velocity parallel to the magnetic field, $v_{\parallel} \leq v_{\perp}$, can be trapped in a magnetic mirror as they move from the low-field side to the high-field side. During their bounce motion between the mirror points the particles suffer from radial drift \mathbf{v}_D , which lead to so-called banana orbits in the poloidal projection of their bounce motion (see Fig. 13.4). Due to the toroidal symmetry of the tokamak magnetic field the inward and outward drift balance each other. In contrast to the axisymmetric (two-dimensional) tokamak, the stellarator exhibits in addition complex helical magnetic field ripples (cf. Fig. 13.3), i.e. local minima of magnetic field strength.

These helically trapped particles can suffer from unconfined drift orbits with large radial drifts up to a large fraction and hence spoil the confinement. As it was early realized this could be

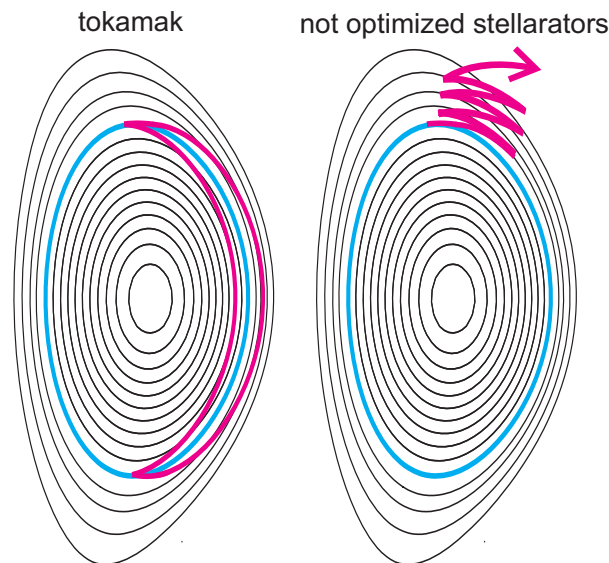


Figure 13.4: Poloidal cut of a tokamak banana trajectory and an unconfined particle orbit of a helically trapped particle in a stellarator.

a severe problem for a fusion reactor. The fast alpha particles born in the fusion process are lost to the wall and damage it before dissipating their energy by collisions with bulk plasma. But also thermal electrons and ions may be lost from the hot core and limit the energy confinement time. In the absence of turbulence, the energy and particle fluxes associated with unconfined particle orbits are described within the framework of neoclassical transport theory (see sec. 13.5). In the subsequent years following the early experiments in the 1970s one of the main goals in stellarator research was to minimize the fraction of particles with unconfined orbits by using the freedom of the three-dimensional magnetic field topology. Note, that in a stellarator the fraction can only be minimized since in the absence of turbulence and collisions the guiding-center drift orbit of every particle is confined in axisymmetric plasmas only (known as *Tamm's theorem*). An in this sense perfectly optimized stellarator would be *omni-genous*, i.e. the time-averaged radial guiding center drift of trapped particles vanishes. For passing particles this is automatically fulfilled as for particle orbits in tokamaks. This imposes some constraints on the magnetic field: In an *omni-genous* field the minimum and maximum B on any flux surface form closed curves rather than isolated points. As these curves can either close toroidally, poloidally or helically a large variety of optimized stellarator configurations is possible. Early experiments and the recent quasi-symmetric developments are introduced in the following section.

13.2 Stellarator concepts

In a *classical stellarator* (see Fig. 13.5) the poloidal magnetic field is produced by $2l$ toroidally closed helical windings with opposite current in adjacent windings, while the toroidal magnetic field is produced by additional field coils. The magnetic axis is circular. Experimental flexibility is achieved since the helical and toroidal field can be varied independently. Exam-

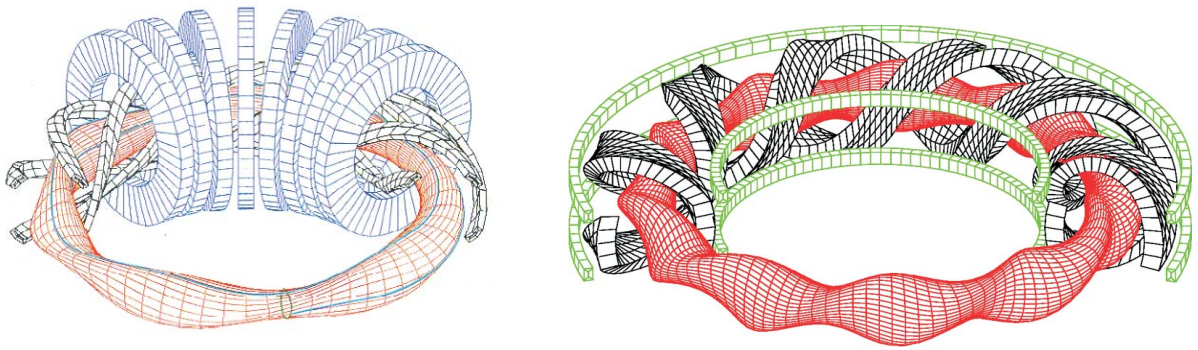


Figure 13.5: Left: A classical $l = 2$ stellarator, (W7-A). Right: *Heliotron* configuration.

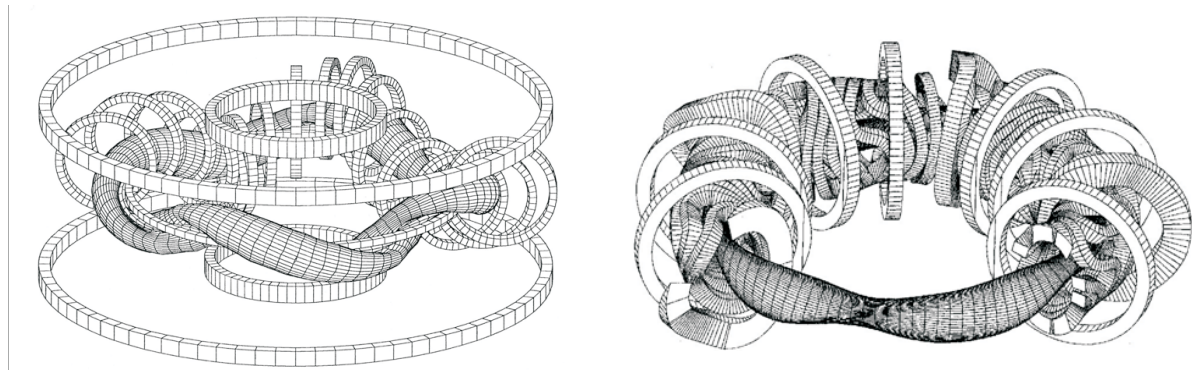


Figure 13.6: Left: Heliac configuration. Right: Advanced stellarator W7-AS.

ples are W2-A (major radius $R = 0.5$ m, $a = 0.02$ m, $l = 2$, number of field periods (toroidal turns of the helical windings) $M = 5$), and the largest ever constructed W7-A ($R = 2$ m, $a = 0.1$ m, $l = 2$, $M = 5$). The interaction between the toroidal magnetic field coils and the helical windings can lead to large radial forces acting on the helices which need heavy structural support making the concept unattractive for a fusion reactor.

In *torsatron/heliotrons* (Fig. 13.5) the poloidal and toroidal field components are produced by l helical windings with unidirectional currents. The vertical field that arises from the net toroidal current is compensated by additional poloidal field coils. Examples are *ATF* ($l = 2$, $M = 5$, $R = 2.1$ m, $a = 0.27$ m), *heliotron-J* ($l = 1$, $M = 4$, $R = 1.2$ m, $a = 0.2$ m), *CHS* ($l = 2$, $M = 8$, $R = 1$ m, $a = 0.2$ m) and largest operating helical device *LHD* ($l = 2$, $M = 10$, $R = 3.9$ m, $a = 0.6$ m, superconducting).

In a *heliac* configuration the rotational transform is produced solely by a three dimensional magnetic axis (see Fig. 13.6), i.e. circular coils placed around a helical magnetic axis. An example is TJ-II ($M = 4$, $R = 1.5$ m, $a = 0.2$ m).

In order to overcome the optimization limitations introduced by the specific concept and the large forces acting on the helical windings the concept of modular coils has been developed

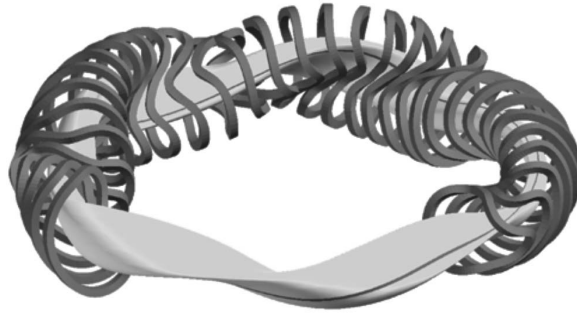


Figure 13.7: Quasi-isodynamic stellarator W7-X.

by Rekher and Wobig in 1972. The modular coil concept allows to approximate an arbitrary vacuum magnetic field inside a domain Ω tangential to the surface $\delta\Omega$ by a current sheath on this surface. The current sheath can be discretized into current lines which yield a set of modular and poloidally closed coils which than in turn reproduce this magnetic field in the domain Ω . At the same time a massive increase in computation power allowed to consider also other physics optimization criteria (see Sec. 13.3.2). The first stellarator experiment in which the magnetic configuration was the starting point to design the coils was W7-AS (see Fig. 13.6), where AS stands for advanced stellarator ($M = 5$, $R = 2$ m, $a = 0.18$ m).

As outlined in sec. 13.1 symmetries are of special importance for neoclassical confinement. The introduced quasi-symmetry of B is often only visible in certain special coordinate systems (magnetic coordinates). In a quasi-symmetric devices¹ B varies on a flux surface only through a fixed linear combination of toroidal and poloidal angles: $B(\psi, \theta, \varphi) = B(\psi, M\theta - N\varphi)$, where ψ is a flux surface label (radial), θ is the poloidal and φ the toroidal angle, respectively. Possible options are *quasi-axisymmetric* $N = 0$ (e.g. NCSX: $M = 3$, $R = 1.4$ m, $a = 0.33$ m, construction has been stopped), *quasi-poloidal symmetric* $M = 0$ and *quasihelical symmetric* $M \neq 0$ and $N \neq 0$ (HSX: $M = 1$, $N = 4$, $R = 1.2$ m, $a = 0.15$ m). In HSX it has been demonstrated that the neoclassical transport drops to tokamak level.

A different subclass of *omni-genous* optimization relies on the *quasi-isodynamic* principle, in which the parallel Pfirsch-Schlüter currents are minimized (see sec. 13.3.2). The latter increases the pressure limit and optimizes MHD stability. The *isodynamic* optimization principle has been followed in the design of W7-X ($M = 5$, $R = 5.5$ m, $a = 0.5$ m, Fig. 13.7), the successor of W7-AS. Thereby, the maximum volume averaged pressure increases from $\langle\beta\rangle = 3\%$ (W7-AS) to 5% (W7-X), making the stellarator concept economically more attractive for a fusion reactor because the fusion power scales as $P_{fus} \sim \beta^2$.

¹Quasi-symmetric devices are automatically omni-genous, meaning that all collisionless particle orbits are confined.

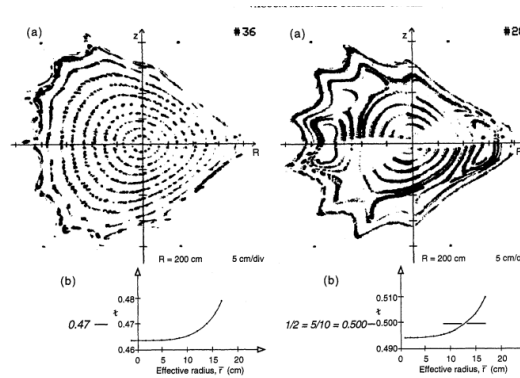


Figure 13.8: Poloidal cut of measured flux surfaces in the *W7-AS* stellarator for two different configurations of the rotational transform ι . The radial ι -profile is shown below the Poincaré plots [2].

13.3 Equilibrium

13.3.1 Vacuum magnetic field

In contrast to a tokamak, flux surfaces in a stellarator already exist in vacuum and can be measured using an electron emitter (e.g. a hot filament) and a fluorescent mesh or movable rod. The emitted electrons follow the field lines until the entire flux surface is covered (if it exists) and the fluorescence light is recorded with a camera. A poloidal cut of measured flux surfaces of two *W7-AS* scenarios with slightly different rotational transform ι are illustrated in Fig. 13.8.

On the left is a case, in which ι does not assume any low order rational value (e.g. $\iota = 1/3, 1/2, 2/3, \dots$) and unperturbed flux surfaces exist. On the right, the ι -profile covers the $\iota = 1/2$ resonance and magnetic island are clearly visible. If ι is a rational, the field lines do not cover the flux surfaces ergodically but close. Hence magnetic islands appear, if radial magnetic perturbation field with Fourier components $B_{n,m}$ (here $n = 1, m = 2$) are in resonance with the rational field line. These perturbations can be caused by external error fields (as presumably in the early figure-8 devices) or by inherent symmetries. *W7-AS* has a five-fold toroidal symmetry leading to $5/m$ -Fourier components. For this reason one observes 10 small islands ($5/10 = 1/2$) in addition to the two large islands arising from the $1/2$ -perturbation. Magnetic islands are detrimental for confinement. They offer a short circuit for radial plasma transport since they connect plasma volumes at different radii. Especially large magnetic islands can destroy the confinement due to a deterioration of flux surfaces. The width of the islands is given by $w = 4\sqrt{RB_{mn}/(m\iota')}$, where $\iota' = d\iota/dr$ is the magnetic shear. Low order rationals (low m) lead to larger island widths. The strategy of avoiding large magnetic islands is either based on a large magnetic shear (e.g. *LHD*) or by a flat iota profile that does not cover a low order rational (e.g. *Wendelstein* devices). A large magnetic shear keeps the island size small but at the same time the distance between the islands decreases and a large number of overlapping islands may exist which leads to an edge ergodic region with poor confinement. Typical radial ι -profiles of different devices are illustrated in Fig. 13.9. In stellarators the shear

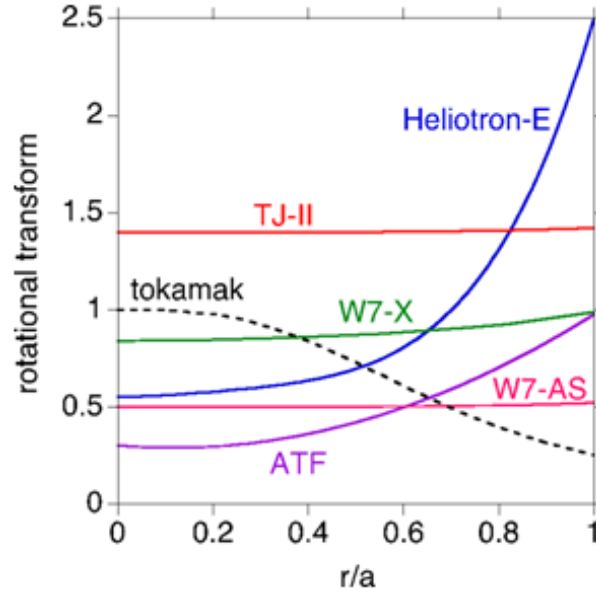


Figure 13.9: Radial profile of the rotational transform ι for different helical devices compared to a standard tokamak configuration (dashed line).

increases usually from the magnetic axis to the edge due to its generation by external coils. This corresponds to the reversed shear scenarios in tokamaks, where the shear is caused by the toroidal current in the center.

13.3.2 Plasma Equilibrium and MHD stability

In the presence of a plasma the vacuum magnetic field is altered by the currents flowing in the plasma. The MHD equilibrium of a magnetically confined plasma is determined by the radial force balance between the pressure gradient ∇p and the Lorentz force of the perpendicular current density \mathbf{j}_\perp ,

$$\nabla p = \mathbf{j} \times \mathbf{B} \quad \text{with} \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{j} \quad \nabla \cdot \mathbf{B} = 0, \quad (13.2)$$

which in turn determines the perpendicular current density $\mathbf{j}_\perp = \mathbf{B} \times \nabla p / B^2$, which is called diamagnetic since it reduces the magnetic field inside the plasma. The quasi-neutrality condition $\nabla \cdot \mathbf{j} = \nabla \cdot \mathbf{j}_\perp + \nabla \cdot \mathbf{j}_\parallel = 0$ implies the existence of a parallel current in toroidal geometry, the so-called Pfirsch-Schlüter (PS) current. For an axisymmetric configuration it reads $j_{PS} \approx (2j_\perp / \iota) \cos(\theta)$. The diamagnetic and PS currents are no real currents since none of them is divergence free. In the particle picture the PS currents balance the charge separation of electrons and ions that arises from the curvature drifts (see Fig. 13.10).

The PS currents flow at different directions at the inside respectively outside of the torus. This dipole structure gives rise to a vertical field that causes a horizontal shift of the plasma axis relative to the outermost flux surface, the so-called Shafranov shift Δ . This means that the PS currents lead to a pressure induced change of magnetic field configuration. Reconstructed

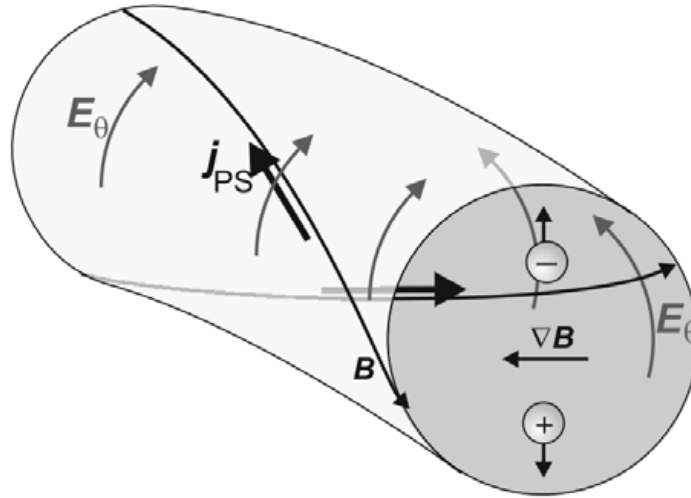


Figure 13.10: Pfirsch-Schlüter currents in a toroidal plasma [3].

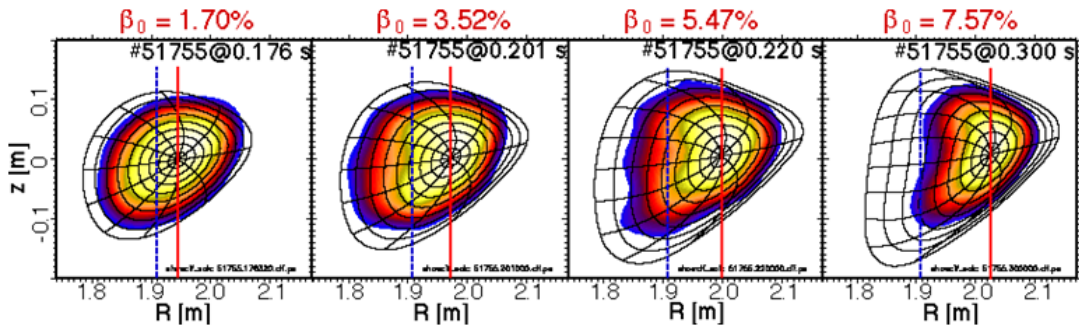


Figure 13.11: Shafranov shift observed in W7-AS for increasing plasma pressure β [Weller].

W7-AS flux surfaces for increasing plasma pressure β are shown in Fig. 13.11. The Shafranov shift of the magnetic axis is clearly observed. For a low-beta, large aspect ratio the Shafranov shift is given by $\Delta/a = R/a(\epsilon_{01}/\epsilon_t)^2 \langle \beta \rangle / (2\iota^2)$, where ϵ_{01} is the Fourier harmonic of the equilibrium field inhomogeneity associated with the toroidal curvature, $\langle \dots \rangle$ denotes the volume average and $\epsilon_t = r/R$ is the inverse aspect ratio. For a tokamak $\epsilon_{01} = \epsilon_t$. Hence, the Shafranov shift can be either reduced by reducing the aspect ratio R/a , increasing the rotational transform ι (e.g. as in torsatrons/heliotrons) or by decreasing the toroidal curvature ϵ_{01} (as e.g. in the Wendelstein devices). Along the Wendelstein optimization the factor ϵ_{01}/ϵ_t decreased from 1 (W7-A) via 0.7 (W7-AS) to 0.4 (W7-X). The reduction of the toroidal curvature can be achieved by an increase of the average elongation $\kappa = b/a$ (with b and a being the major and minor axes of an ellipse) of the flux surfaces (3D-shaping). The latter reduces the poloidal variation of B on a flux surface, which in turn also reduces neoclassical ripple losses. A result of 3D-shaped flux surfaces is shown in Fig. 13.12 for W7-AS at $\beta = 2\%$ and W7-X at $\beta = 5\%$.

Applying these optimization criteria an additional vertical magnetic field for plasma position

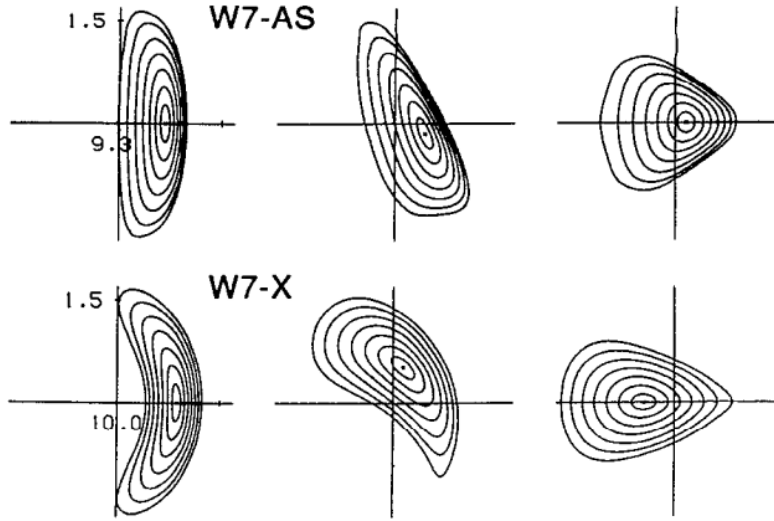


Figure 13.12: A comparison of computed flux surfaces for W7-AS ($\beta = 2\%$) and its successor W7-X ($\beta = 5\%$) [4].

control might be not necessary and the equilibria become stiff with respect to the plasma pressure. The Shafranov shift sets an operational limit for the maximum plasma pressure due to a progressive deterioration of flux surfaces. This β -limit is defined as $\Delta = a/2$, i.e. when the Shafranov shift of the magnetic axis reaches half the minor radius.

Concerning stellarator optimization small PS currents are desirable, which results in the limit $j_{PS} = 0$ in the net-current free case $\nabla \cdot \mathbf{j}_{\perp} = 0$ determined by $(\mathbf{B} \times \nabla \mathbf{B}) \cdot \nabla p = 0$ (Palumbo limit). In this *isodynamic* limit (s. Sec. 13.1) the neoclassical transport would be zero. However, the criterion cannot be fulfilled in toroidal geometry. Reducing the PS currents means reducing the geodesic curvature κ_g , since $\nabla \cdot \mathbf{j}_{PS} = -2|\nabla p| \kappa_g / \mathbf{B}$. κ_g describes the projection of the curvature vector onto the magnetic surfaces. The geodesic curvature of a magnetic field line vanishes if it is a geodesic (such as shortest airplane flight paths between two different locations on earth). In a *quasi-isodynamic* configuration the lines of constant current, i.e. the sum of the PS and diamagnetic current, $\mathbf{j}_{PS} + \mathbf{j}_{\perp}$, are poloidally closed and the net toroidal current vanishes. The situation changes at low collisionality, where the so-called bootstrap current gives rise to a net toroidal current that can alter the magnetic field configuration. The bootstrap current is caused by the pressure gradient and relies on the momentum transfer from trapped particles to passing particles, $j_b \sim (f_t/f_p) \nabla p$, where f_t and f_p denotes the fraction of trapped and passing particles, respectively. While the bootstrap current increases ι in a tokamak and thus helps to achieve a rotational transform, helically trapped particles in a stellarator lead to a j_b -component that lowers ι and thus changes the magnetic shear ι' which in turn leads to the formation of a magnetic well.² In the ATF torsatron (see Fig. 13.13) it has been demonstrated that the bootstrap current can be controlled by modifying the Fourier components of the mag-

²A magnetic well exist if the average magnetic field on a magnetic flux surface increases with enclosed volume, or in other words, if the flux surface has an average favorable curvature.

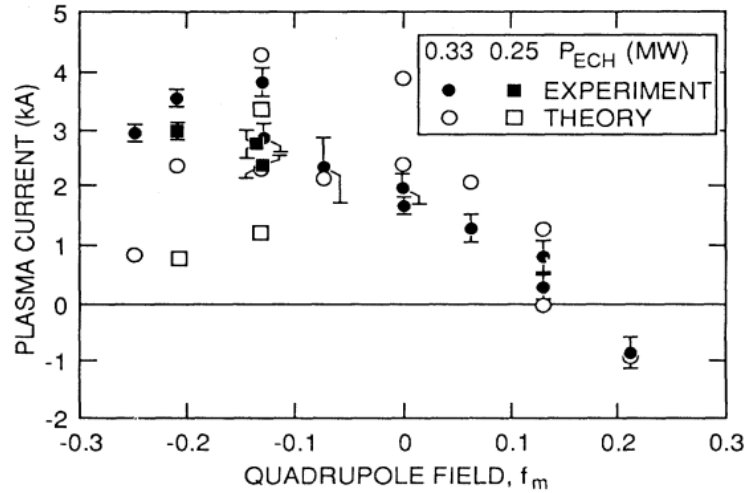


Figure 13.13: Control of the bootstrap current by modifying the Fourier components of the magnetic field in the ATF torsatron [5].

netic field such that a toroidal net-current free plasma can be achieved. The minimization of the bootstrap current was one criteria of the *W7-X* drift optimization.

The gradients in the pressure and parallel currents are sources of free energy that can drive MHD instabilities. The tokamak suffers from multiple current-driven instabilities, such as disruptions, resistive wall modes, or kink and tearing modes, which strongly affect the equilibrium and limit the operation. For example, a typical tokamak is limited to have $\iota < 1/2$ at the edge in order to avoid disruptions. Most of these MHD instabilities are absent in a stellarator and the most important remaining are the interchange and ballooning modes. The Mercier stability criteria reduces for an optimized low-shear stellarator to a local condition $D_{well} + D_{geo} > 0$, where the stabilizing D_{well} term represents the magnetic well and the negative (destabilizing) D_{geo} represents the geodesic curvature κ_g . Hence, a reduction of κ_g reduces PS currents and thereby improves MHD stability. The increase of the magnetic well at higher pressures described above is also beneficial for MHD stability, since it compensates the missing magnetic shear stabilization in low-shear devices. Controlled disruptions have been studied in the *W7-AS* stellarator by inducing an Ohmic toroidal current as in a tokamak. The induced Ohmic current was used for start-up and heating before ECRH became a success, but also to balance the residual bootstrap currents and to control the magnetic configuration. Although similar disruption-like features like a drop of the electron temperature have been observed, the equilibrium could be maintained as it is dominated by the vacuum magnetic field.

13.4 Neoclassical transport

The discouraging confinement times of the early stellarator devices were mainly caused by the large neoclassical particle losses due to error fields. Classical and neoclassical transport are both diffusive processes that can be described by a random walk model, i.e. the diffusion

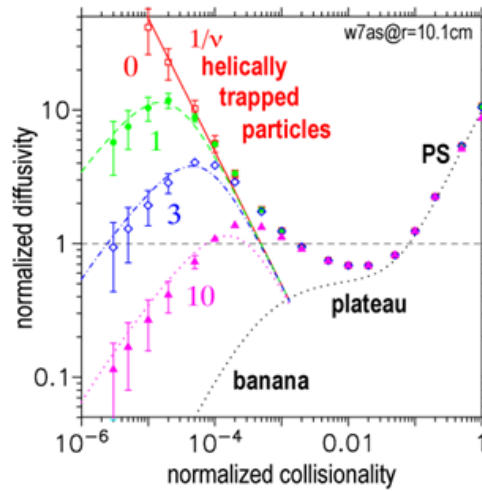


Figure 13.14: Theoretically predicted normalized neoclassical diffusion coefficient for a stellarator and tokamak configuration. The different curves in the low collisionality $lmfp$ -regime correspond to different values of the radial electric field.

coefficient is $D = d^2\nu$, where d is the step size and ν the collision frequency. The particle flux Γ is then given by Fick's law, $\Gamma = -D\nabla n$. While classical diffusion describes the situation in a homogeneous magnetic field, in which the step size is determined by the Larmor radius, the additional guiding-center drift v_D of inhomogeneous magnetic fields lead to larger step sizes. The transport associated with the guiding-center drifts in toroidal magnetic fields is described by the neoclassical theory. For a fusion reactor the long mean free path ($lmfp$) regime with high temperatures (low collisionality) is most relevant. In a stellarator the radial heat flux caused by the radial $\nabla\mathbf{B}$ -drift of ripple-trapped particles scales in the $lmfp$ -regime for small electric fields as $D \sim T^{9/2}$. At the same collisionality the heat flux in a tokamak scales in the banana regime only with $D \sim \sqrt{T}$. At higher collisionality the diffusivities in tokamaks and stellarators are comparable. A comparison of the particle diffusion coefficient as a function of normalized collisionality is shown in Fig. 13.14. In a tokamak the diffusion coefficient increases linearly with collisionality in the banana and Pfirsch-Schlüter regime. In the banana-regime toroidally trapped particles bounce between the reflection points (cf. Fig. 13.3) and can get de-trapped by a collision that displaces the banana particle by the banana width, which determines the step size. In the high-collisional Pfirsch-Schlüter regime passing particles cannot complete their poloidal orbit before experiencing a collision and thus the step size is determined by the displacement of the particles by the gradient and curvature drifts v_D , $D = v_D\tau/\tau^2$, with the collision time $\tau = 1/\nu$.

The transition region (plateau regime) is determined by a gradual decrease of the step size with increasing collisionality since the particles cannot complete their banana orbit before experiencing a collision. In a stellarator the plateau and Pfirsch-Schlüter regimes are recovered, but in the low collisionality $lmfp$ regime a $1/\nu$ -regime exists due to helically ripple trapped

particles. In the $1/\nu$ -regime the diffusion coefficient scales as³

$$D_{1/\nu} \sim (\varepsilon/\varepsilon_t)^2 \varepsilon_h^{3/2}, \quad (13.3)$$

where ε_h is the helical ripple modulation, ε is the poloidal variation of B and ε_t the inverse aspect ratio. Hence, decreasing the average toroidal curvature by elongating the plasma helps to reduce the diffusivity in the $1/\nu$ -regime. The presence of a radial electric field E_r can compensate the large diffusivity in the $1/\nu$ -regime, since the associated poloidal $\mathbf{E} \times \mathbf{B}$ -drift partly averages out the $\mathbf{B} \times \nabla \mathbf{B}$ -drift. The reduction of the diffusivity in the *lmfp*-regime for increasing E_r can clearly be observed in Fig. 13.14.

The ambipolar radial electric field is in the absence of impurities determined from the balance of thermal electron and ion fluxes $\Gamma_e = Z_i \Gamma_i$ with

$$\Gamma_\alpha = -n_\alpha \left[D_{11}^\alpha \left(\frac{\nabla_r n_\alpha}{n_\alpha} - \frac{q_\alpha E_r}{T_\alpha} \right) + D_{12}^\alpha \frac{\nabla_r T_\alpha}{T_\alpha} \right], \quad (13.4)$$

with $\alpha = e, i$ and the neoclassical radial transport coefficients D_{11} (diffusive transport) and D_{12} (flux driven by temperature gradients). In dependence on the heating method other externally driven fluxes may contribute (neutral beam injection *NBI*). In the *lmfp*-regime the solution of the ambipolarity condition has multiple roots, because the transport coefficients depend on E_r , $D_{ik} = f(E_r, \nu, r)$. For $T_e \gg T_i$ (peaked core electron temperature profiles and flat ion temperature profiles) the electron fluxes exceed the ion fluxes if $E_r = 0$. Hence, a positive E_r arises in the core that reduces the high electron flux Γ_e to the ion level. This is the so-called electron root. This regime is typically observed at low-density ECRH discharges, i.e. when the energy balance is dominated by the electrons and the ions are energetically decoupled by low collisionality leading to $T_e \gg T_i$. In contrast, at high densities (higher collisionality, $T_e \sim T_i$) the ion-root with moderate E_r in the core and strong negative E_r in the plasma edge is observed. The latter is caused by large fluxes induced by ion temperature gradients that are balanced by the negative electric field. In Fig. 13.15 a W7-AS discharge is shown that reflects the electron-root regime in the core. As in Fig. 13.15 c, often multiple solutions of the ambipolarity condition are possible (red symbols). The measured E_r follows the electron root in the core and the ion-root in the edge.

The condition of core electron-root (positive E_r) and edge ion-root (negative E_r) can lead to a large shear of the radial electric field and the associated poloidal $\mathbf{E} \times \mathbf{B}$ -drift. This is further supported by the electron flux to target plates or limiters in the region of open magnetic field lines, which causes a positive E_r . Sheared flows are an important turbulence reduction mechanism by decorrelating turbulent structures. This reduces turbulent transport, which usually exceeds the neoclassical transport in the far edge. The influence of this large turbulent, so-called anomalous, transport is e.g. observed in Fig. 13.15 d, where the estimated heat diffusivity χ_e clearly exceeds the neoclassical predictions in the edge ($r > 13$ cm).

For a fusion reactor the ion-root regime with $T_e \sim T_i$ and moderate E_r in the core is most relevant. This is somehow in contradiction to the need to overcome the poor $1/\nu$ -scaling in the *lmfp*-regime by a finite E_r . However, since the radial electric field cannot be well controlled

³For a classical stellarator the magnetic field can be written as $B/B_0 = 1 - \varepsilon \cos \theta - \varepsilon_h \cos(l\theta - M\phi)$. For a tokamak $\varepsilon \approx r/R = \varepsilon_t$ and $\varepsilon_h = 0$.

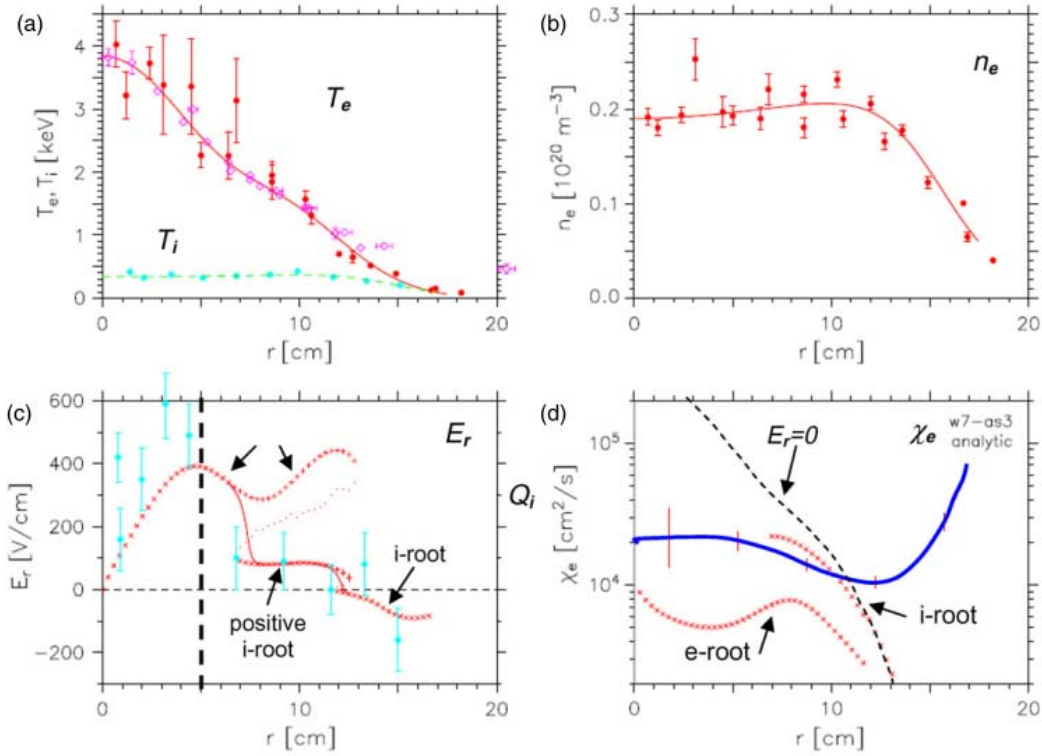


Figure 13.15: Low density core electron-root conditions observed in *W7-AS*. Shown are radial profiles of (a) temperatures, (b) density, (c) radial electric field and (d) electron heat diffusivity (blue: measured, dashed: neoclassical prediction) [6].

experimentally recent stellarators are optimized to have small $1/\nu$ -transport per se. An important cross-machine measure is the effective helical ripple ϵ_{eff} , that can be estimated from the D_{11} coefficient, $D_{11} \sim \epsilon_{eff}^{3/2}$. Calculated radial profiles of ϵ_{eff} are shown Fig. 13.16 for *LHD*, *W7-AS* and *W7-X*. The huge reduction of ϵ_{eff} by optimization is nicely observed in *LHD*, where the magnetic axis can be shifted. With inward shifted axis the ripple decreases leading to smaller neoclassical losses as compared to the standard configuration. From *W7-AS* to *W7-X* ϵ_{eff} is reduced by a factor of 10.

13.5 Confinement

As outlined in the previous section, the turbulent (or anomalous) transport can significantly contribute and even exceed the neoclassical losses. Since the detailed mechanisms behind this anomalous transport are not fully understood empirical scaling laws are used to describe the dependence of the observed transport on some leading parameters. The derived *ISS04* scaling for the global energy confinement time $\tau_e = W/(P - dW/dt)$, where W is the plasma energy and P is the heating power, reads

$$\tau_e^{ISS04} = f \cdot 0.134 a^{2.28} R^{0.64} \bar{n}_e^{0.54} B^{0.84} (t_{2/3}/(2\pi))^{0.41}, \quad (13.5)$$

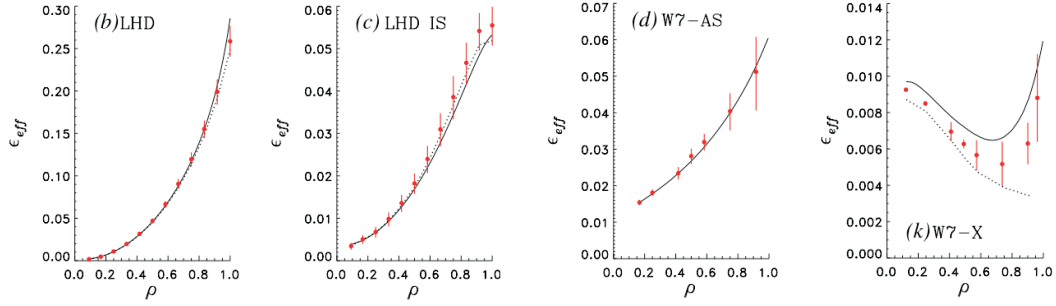


Figure 13.16: Calculated effective helical ripple ε_{eff} in the $1/\nu$ -regime. From left: *LHD* (standard), *LHD* (inverse shifted magnetic axis), *W7-AS* and *W7-X* [7].

where ι is taken at $2/3$ of the minor radius. The configuration factor f summarizes the magnetic configuration details. The maximum (best confinement) value is 1. *W7-AS* at low ι reaches $f = 1$ and *LHD* with inward shifted configuration $f = 0.93$. Similar to tokamaks a power degradation is observed: τ_e decreases with heating power. The energy confinement time of different helical devices is compiled in Fig. 13.17, together with typical tokamak data. Stellarator and tokamak have comparable confinement times. The influence of the magnetic configuration on the stored plasma energy is illustrated in Fig. 13.18 for net-current free discharges in *W7-AS*. The plasma energy shows pronounced maxima if boundary value of ι is close to a low-order rational ($\iota = 1/3, 2/5, 1/2$). An empirical model relying on an enhanced electron heat conductivity on rational surfaces is able to explain the observations (solid line in Fig. 13.18) but leaves the involved physical processes open. An explanation of the radial transport processes near or at rationals is missing.

The high confinement (*H*-mode) regime first observed in tokamaks also exists in stellarators as demonstrated first in *W7-AS* in 1992. In general, the *H*-mode is related with a sheared radial electric field at the plasma edge that suppresses turbulence and hence increases the confinement typically by a factor of two. As a result steep gradients of the plasma parameter develop at the edge. The necessary shear of the radial electric field can be driven, for example, by neo-classical fluxes. The *H*-mode goes along with an improved particle and impurity confinement which lead in the absence of ELMs (edge localized modes) to a degradation of the plasma energy if the impurities radiate a fraction 50–60% of the energy. ELMs relax the pressure gradient and expel particles and energy from the core. For a large distinct ELM the particle and energy losses are typically 5%. The so-called quiescent *H*-mode without ELMs is prone to impurity accumulation leading to a radiative collapse. An example of a *W7-AS* quiescent *H*-mode accessed during a density ramp is shown in Fig. 13.19. At the transition the edge radial electric field decreases dramatically leading to a large shear rate. In the following the plasma energy W increases but also the radiation (obtained from bolometry) due to improved impurity confinement. If the radiation reaches a certain threshold the stored energy decreases and the quiescent *H*-mode falls back to an ELMy *H*-mode. Since the large particle and energy fluxes associated with ELMs can severely damage the plasma facing components in a fusion reactor ELM-free high confinement modes as the High Density *H*-Mode (*HDH*-mode) are envisaged. The *HDH*-mode mode became accessible in *W7-AS* with successful island divertor operation

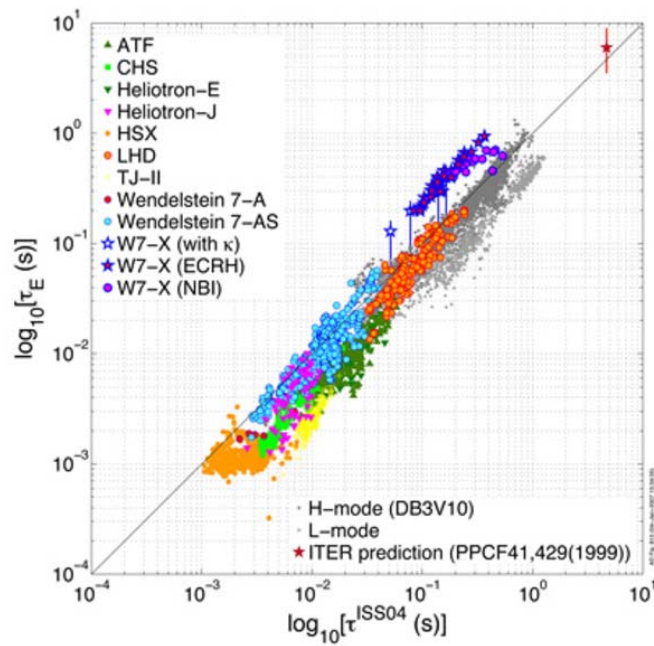


Figure 13.17: Energy confinement data of helical devices and properly rescaled tokamak configuration vs. the ISS04 scaling [<http://ipp.mpg.de/ISS>].

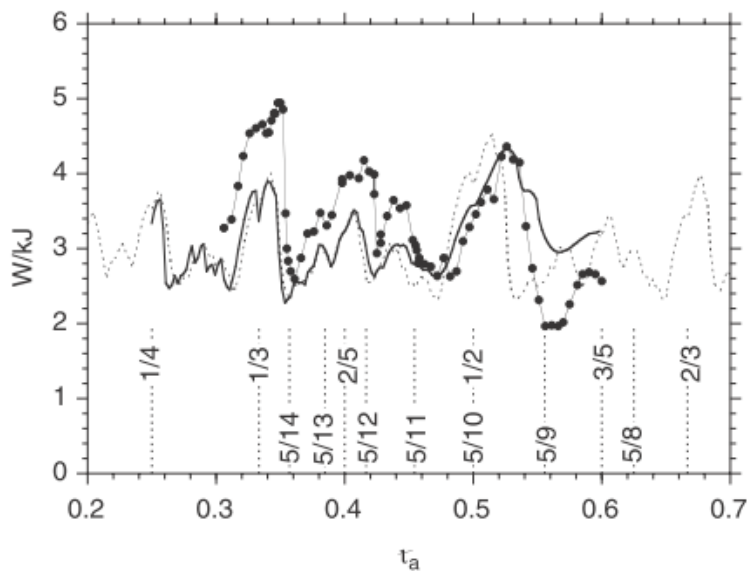


Figure 13.18: Plasma energy as a function of the edge τ value in net-current free W7-AS discharges (dotted: measurement, solid: model) [8].

at densities $n > 10^{20} \text{ m}^{-3}$. The *HDH*-mode is characterized by large decrease of the impurity confinement time due to impurity screening as indicated in Fig. 13.20.

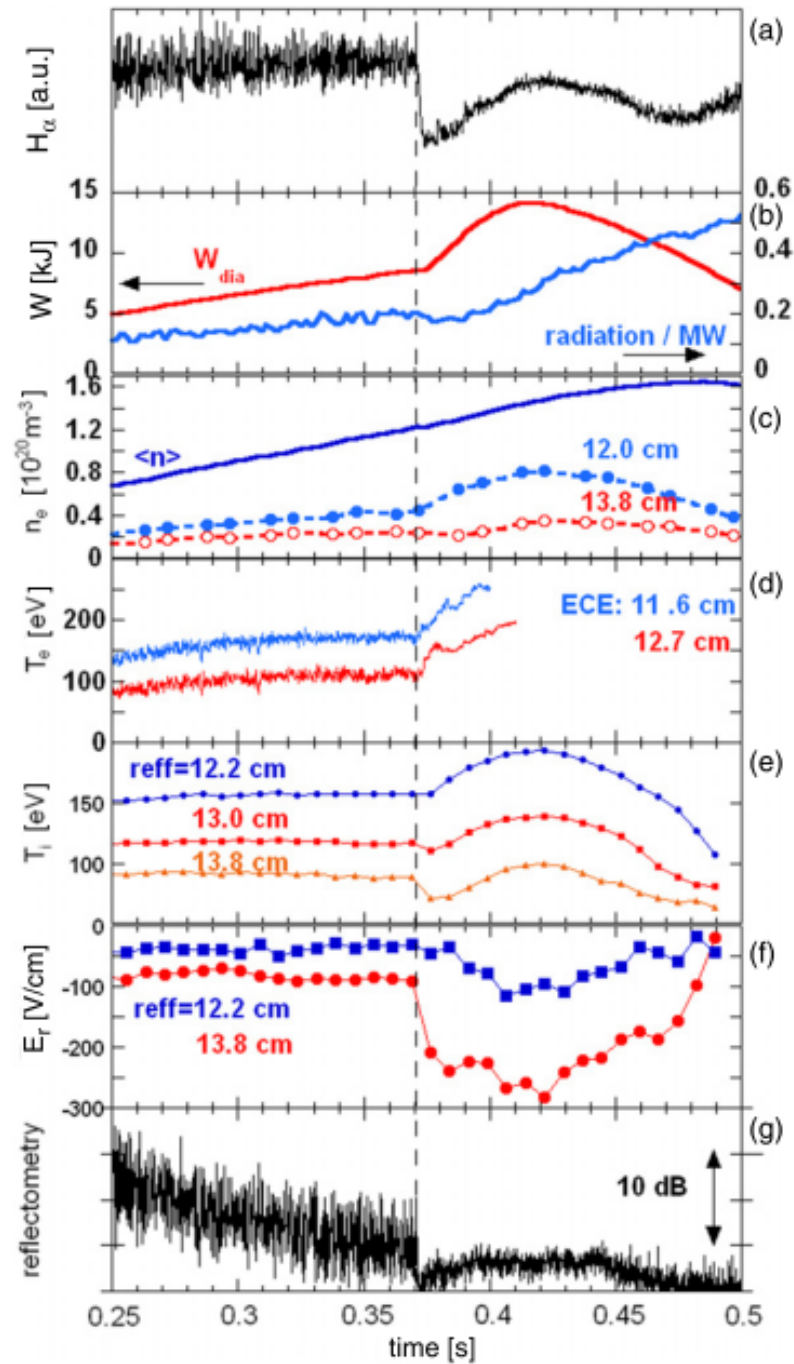


Figure 13.19: Transition into a quiescent H -mode by ramping the density. The transition time is indicated by the dashed line [9].

13.5.1 Impurity transport and divertor concepts

As discussed in the last section radiation losses due to impurity accumulation limits the pulse length and acts as a soft density limit in stellarators. For a steady-state device these losses have to be avoided. The impurity dynamics is determined by three different interaction zones:

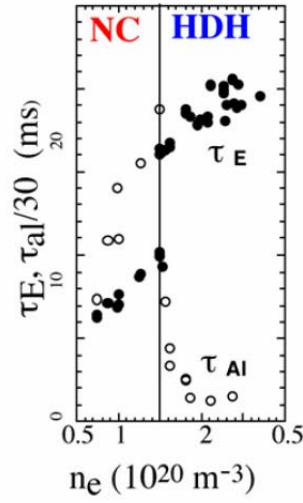


Figure 13.20: Energy confinement time τ_e and Aluminum impurity confinement time τ_{Al} during the transition from normal confinement (NC) to the HDH-mode [10].

1. Plasma-facing components with recycling and erosion of wall material,
2. The region of open flux surfaces (the so-called scrape-off layer (*SOL*)), which regulates the impurity influx to the confined region with closed field lines and
3. the plasma core.

The impurity influx from the wall can be reduced by proper wall conditioning and the flux across the *SOL* can be partially controlled by the magnetic configuration and edge density. However, the impurity transport in the confined region is determined by neoclassical transport

$$\Gamma_z = -n_Z D_z \left[\frac{\nabla_r n_Z}{n_Z} + \gamma_Z \frac{\nabla_r T_i}{T_i} - Z \left(\frac{\nabla_r n_i}{n_i} + \gamma_i \frac{\nabla_r T_i}{T_i} \right) \right], \quad (13.6)$$

where E_r has been replaced by the standard ion-root solution (cf. (13.4)). While the factor γ_i in front of the thermal gradient can be negative for a tokamak (impurity or temperature screening if ions are in the banana regime) it is almost always positive for stellarators. This means that for peaked ion temperature profiles impurity accumulation is always predicted. This does not hold for the low-density electron-root, but this is not important as this scenario has no fusion reactor relevance. A possible answer to the problem is to shield the impurity influx to the core in the *SOL*. An effective impurity retention can be achieved by a proper divertor. A divertor regulates the particle and power exhaust by channeling the fluxes to cooled target plates, where the neutralized particles can be pumped. The divertor concept has been proposed by Spitzer already in 1951 in order to overcome the direct contact of the plasma with a limiter and to increase the pumping efficiency of neutrals, i.e. the helium ash in fusion reactor, by an increased neutral pressure in the divertor chamber.

Different divertor concepts are compiled in Fig. 13.21 a. In all concepts particles leaving the confined plasma core with closed flux surfaces are diverted in the open field line region along magnetic field lines that terminate on target plates. Tokamaks have additional coils that produce one or two Null point(s) (X-point) in the poloidal plane (single or double null configurations). In low-shear stellarator experiments naturally occurring islands provide a natural separation between the confined and unconfined region (local island divertors). In large shear experiments (e.g. LHD) a stochastic ergodic layer exists (see sec. 13.1). After passing this region the particles follow one of the four divertor legs produced by the two helical coils (helical divertor). The subtle divertor dynamics in tokamaks and stellarators is quite different, which is mainly caused by the different diversion field line pitches Θ at the divertor target plates. For a tokamak $\Theta = B_{pol}/|\mathbf{B}| \approx 0.1$ while for an island divertor $\Theta = w r' a/R \approx 0.001$. Due to the smaller pitch angle the connection length L_c , i.e. the distance the particles have to travel before hitting the target, can become much larger in a stellarator. At the same time the perpendicular distance from the X-point to the target is much smaller. While in a tokamak the SOL transport is mainly determined by parallel heat conduction q_{\parallel} along the open field lines, the larger L_c and smaller target-to-core distance of a stellarator enhance the perpendicular transport across the magnetic islands.

In reducing the neutral inventory two effects are important: First, the islands screen the core from penetrating neutrals. If the island density (associated with the density n_{es} upstream from the targets) is sufficiently large and the island temperature is kept above the ionization energy the neutral penetration length becomes very small, i.e. the neutrals are ionized in the island. Hence, a barrier exists for the neutrals. If n_{es} is low the islands are transparent to the neutrals, which then can enter the confined region. As a consequence of this high n_{es} barrier fewer neutrals reach the confined region and the neutral core-fueling rate decreases. In order to further increase the core plasma density this island neutral screening effect has to overcome. This can be achieved by further increase n_{es} to cool the islands and decreasing the ionization rate in the islands. In the experiment this is achieved by strong gas puffing, which has to be applied to enter the HDH regime. A second important aspect to understand the HDH regime is the cross-field heat flux associated with magnetic islands as mentioned above. The cross-field transport can short-circuit the parallel temperature gradients between the target plates and the plasma upstream from the target plates. In reducing the parallel thermal force ($\sim \nabla_{\parallel} T_i$), which drives the neutrals into the SOL, the friction force ($\sim n v(v_{\parallel}^i - v_{\parallel}^z)$) becomes dominant in the parallel momentum balance and the impurities are dragged back to the targets. Numerical modeling results of the W7-AS local island divertor show a large decrease of the carbon impurity density in the SOL with increasing upstream density. When the density is further increased beyond the HDH-threshold, a large fraction (80-90%) of the power in the SOL is radiated by impurities within a radiation layer that shifts from the target plates towards the confined region (detachment). For a detached divertor the energy and particle fluxes to the target plates nearly vanish, which is beneficial for a fusion reactor. However, due to the complex three-dimensional island dynamics a stable detachment has not been achieved in W7-AS and LHD. The typical power deposition on a target plate as observed by a thermographic camera is shown in Fig. 13.22 for the attached and partially detached regime. The total power flux to the target is highly reduced in the detached regime although a hot spot remains.

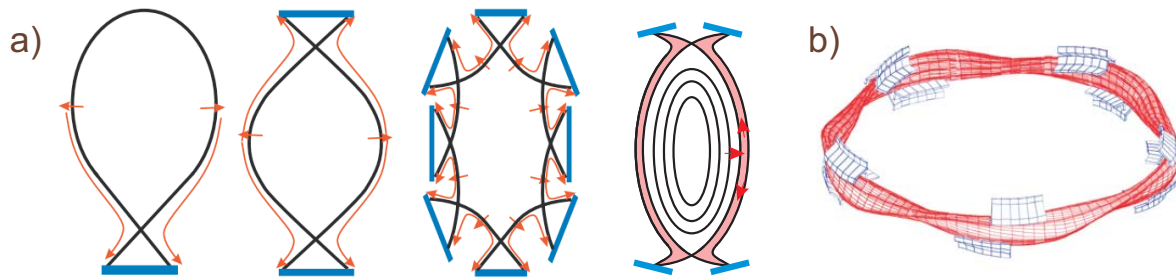


Figure 13.21: (a) Overview of divertor configurations (from left): tokamak single null, tokamak double null, island divertor, helical divertor (*LHD*). The target plates are shown in blue. (b) Local island divertor of W7-AS utilizing the 5/9 island [adapted from [11].

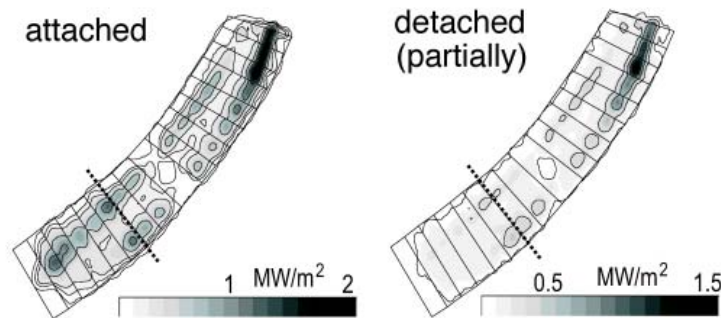


Figure 13.22: Thermographic measurement of the W7-AS local island divertor in the attached and partially detached regime [11].

13.6 Helical reactor concepts

Due to the potential to operate in steady state without current drive and the optimization success of the latest devices, the stellarator concept became (again) attractive to realize a fusion reactor. Since mechanical stability problems that arise from forces acting on the coil windings (see sec. 13.1) are limiting the possible configuration, only the heliotron and modular coil concept remain as a serious option. The developed force-free heliotron reactor (*FFHR*) relies on an upscaled *LHD* concept. In contrast, the *HELIAS* reactor relies on the modular coil concept and is an upgraded version of *W-7X*. The main parameters of *LHD*, *FFHR*, *W7-X* and the *HELIAS* reactor (*HSR 5/22*) are compiled in Tab. 13.6. The high magnetic field $B = 12\text{ T}$ in the *FFHR* reactor allows to operate at fairly small β -values leading to stiff confinement properties.

parameter	<i>LHD</i>	<i>FFHR</i>	<i>W7-X</i>	<i>HSR 5/22</i>
major radius R [m]	3.9	20	5.5	22
av. minor radius a [m]	0.65	2	0.52	1.8
plasma volume V [m ⁻³]	32.5	1579	30	1407
magnetic field B [T]	4	12	3	4.75
av. plasma- β [%]	5	0.7	5	4.24
fusion power P [GW]	0	3	0	3

13.7 Summary

Starting with enthusiastic developments in the 1960's and 1970's, the stellarator concept experienced many drawbacks with respect to plasma confinement that raised skepticism. While the first devices were limited by neoclassical transport losses the outlined route of optimization lead to a dramatic increase in the confinement time. Nowadays, optimized stellarators have confinement times that are equivalent to tokamaks of comparable size. More recently, attractive confinement regimes became accessible using the island divertor concept, which is an option for power and particle exhaust in a fusion reactor. With the operation of W7-X (the commissioning of the device started already) the next optimization step towards a fusion relevant steady state plasma will (hopefully) be realized.

Recommended literature

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Chapter 14

Computational plasma physics

Michael Drevlak

14.1 Computational physics – why bother?

As a student one is exposed to countless exercise problems which can be solved with paper and pencil. While this may be both adequate and instructive for educational purposes, this luxury is not representative of real world applications. Especially the physics of (fusion) plasmas, due to its ramifications into many different branches of theoretical physics, is riddled with many difficulties:

1. High dimensionality of many problems.
Example: plasmas, especially fusion plasmas, may be far away from thermal equilibrium. Fully kinetic description leads to a density function in six dimensions.
2. Non-linearity makes many standard tools (Fourier transform etc.) useless.
Example: Even MHD theory, the simplest formalism to describe a plasma on large scales, has nonlinearities.
3. Input data may not be available in analytic form.
Example: Computational plasma physics codes may be used with experimental data as input in order to improve understanding of *real life* plasma shots.
4. Geometrically complex boundary conditions make even simple equations analytically intractable.
Example: Maxwells equations are known to solve readily in orthogonal coordinate systems, when boundary conditions and discontinuities conform with these coordinates. Any deviation normally requires a mesh oriented field solver package.
5. Simulations of complex systems involving different processes, maybe on disparate scales of space and time.
Example: Physics of the plasma edge combines processes like radiative and convective heat transport, ionisation and neutral particles etc.

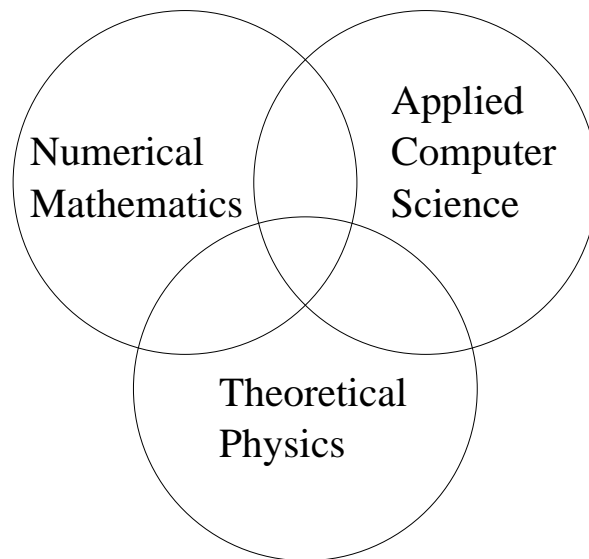


Figure 14.1: Computational plasma physics is located at the crossroads between theoretical physics, numerical mathematics and applied computer science.

6. Non-local effects may have drastic consequences.
Example: Magnetic field lines may form closed flux surfaces, islands or behave stochastically. Only field line integration can tell which is the case.
7. The overwhelming majority of relevant problems amenable to analytic investigation has been solved already.
Reminder: The famous *Reviews of Plasma Physics* [1], written by some of the finest scientists in the field, should give a rough idea of how far analytic investigation can carry us.

Computational plasma physics is located at the crossroads between theoretical physics, numerical mathematics and applied computer science (see Fig. 14.1).

Neither can be learned from reading a chapter of this script and attending a 90 minute lecture. Instead, it is attempted to give the reader an overview of what this field is about and pass on a few pieces of good advice that may be useful in other areas, too.

The investigation of plasma physical phenomena with computational techniques is rather an iterative process than a linear one. It can be divided into several steps (see Fig. 14.2). First, the physical process of interest needs to be cast into a mathematical description. This may include an examination as to which physical effects should be included and what can be neglected. Then a numerical scheme must be devised or selected by which the problem might be solved. In a next step, this numerical procedure is implemented as a computer code. This is often a tedious and protracted process. Moreover, it is not seldom that the analytic process associated with the design of a piece of software reveals a necessity to modify the numerical approach. Finally, the new software is put to use in order to make specific predictions about plasma physical processes or in order to obtain improved understanding of experimental re-

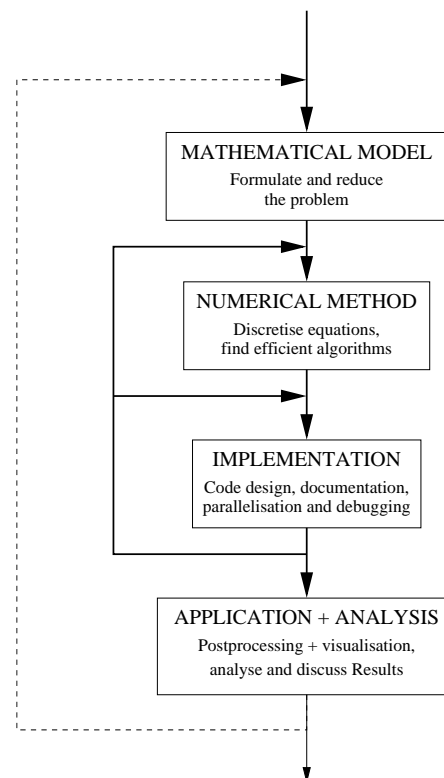


Figure 14.2: The investigation of plasma physical phenomena with computational techniques is rather an iterative process than a linear one. It can be divided into several steps.

sults. Again, the outcome of the code application may raise new questions, commanding a return the development process and a refinement of mathematical models or numerical tools.

The following sections of this article will address issues relating to all four of the above mentioned steps involved in computational plasma physics. While there will be no review of plasma physics – this lecture series is offering enough of the latter – emphasis will be laid on the importance of picking efficient algorithms. Moreover, a brief overview will be given of some programming tools and techniques, of the computer architectures commonly used in computational physics, and on the production of parallel code. Finally, two examples of scientific codes and their results will be presented.

14.2 Numerical examples – algorithms matter

Most of plasma physics is described by differential equations in one or more dimensions. The problems encountered are very diverse, and so are the algorithms and approaches used to solve them. It is not possible to convey an exhaustive overview of numerical techniques in the brevity of this introduction. Still, we want to point out a few simple applications in order to point out that the choice of algorithm can have great influence on the accuracy and efficiency of a code.

14.2.1 Differential equations – field line tracing

Integration of ordinary differential equations is a bread and butter application in many areas of physics. An application coming to mind in fusion plasmas is the integration of magnetic field lines. It is known that magnetic field lines can form closed surfaces, islands or behave stochastically. Many properties of a configuration depend crucially upon the topology of the magnetic field lines and, if surfaces exist, on their shape.

Computing a magnetic field line boils down to solving the ODE

$$\frac{dx}{dt} = f(t, x), \quad \text{with } f = \frac{\mathbf{B}}{|\mathbf{B}|}.$$

The simplest algorithm for integrating this differential equation is the explicit single step method

$$x_{n+1} = x_n + f(t_n, x_n) \cdot \Delta t. \quad (14.1)$$

The algorithm outlined above is rarely used as better results can be obtained using more sophisticated methods.

One of the simplest algorithms doing much better than (14.1) is the Adams-Bashforth-method (see [8] for instance). In this method, the derivatives at the preceding $N - 1$ points are kept. Together with the derivative computed at x_n they are used to fit a Legendre polynomial to the derivative of $f(x)$, yielding

$$x_{n+1} = x_n + \int \sum_{k=0}^P f(t_{n-k}, x_{n-k}) \prod_{l=0, l \neq k}^P \frac{\tau - t_{n-l}}{x_{n-k} - x_{n-l}} d\tau. \quad (14.2)$$

The integrals of the Legendre polynomials can be solved analytically. For $P = 4$ this reduces to a very convenient expression

$$x_{n+1} = x_n + \frac{55}{24}f(x_n) + \frac{-59}{24}f(x_{n-1}) + \frac{37}{24}f(x_{n-2}) + \frac{-9}{24}f(x_{n-3}). \quad (14.3)$$

Fig. 14.3 shows a comparison between Poincare data obtained with method (14.1) (left) and with Adams method (right), applied to the vacuum field excited by the coil system of W7-X. It can be seen that the single step method yields entirely unsatisfactory results, while Adams method reveals an accurate picture of an excellent closed magnetic surface.

14.2.2 Monte Carlo methods

Assume, for simplicity, a one dimensional integral $F = \int_{x_a}^{x_b} f(x)dx$ is to be computed using a Monte Carlo approach. This can be achieved by sampling $f(x)$ at N random locations $x_n \in [x_a, x_b]$ and approximating

$$F_N \approx \frac{x_b - x_a}{N} \sum_N f(x_n).$$

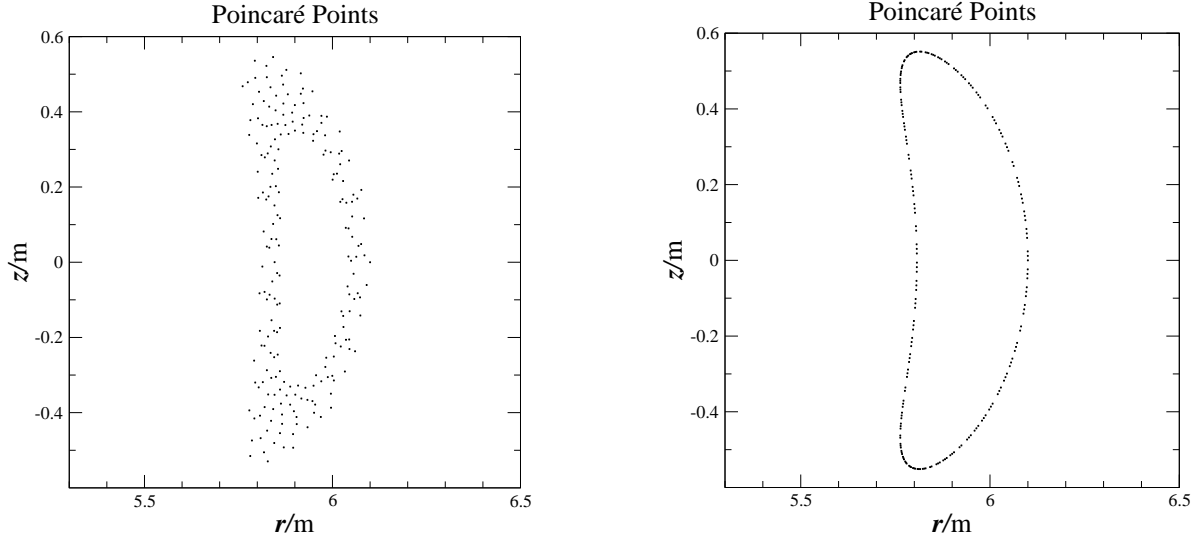


Figure 14.3: Poincaré data for W7-X obtained with algorithm (14.1) (right) and with (14.3) (left).

The error of this estimate scales as

$$|F_N - F| \sim \frac{x_b - x_a}{\sqrt{N}}.$$

Significant improvement can be achieved if $f(x)$ can be approximated using a known function $g(x) \sim f(x)$ that can be integrated analytically

$$G = \int_{x_a}^{x_b} g(x) dx.$$

Exploiting the known integral for $g(x)$ one can write

$$F_N \approx \frac{x_b - x_a}{N} \sum_N (f(x_n) - g(x_n)) + G.$$

This results in a significant improvement of accuracy as the contribution of G is exact and only the error of the difference between F and G scales as

$$|(F_N - G_N) - (F - G)| \sim \frac{1}{\sqrt{N}}.$$

The 1D integration presented here is an educational example. Nobody would use an algorithm with an error scaling like $1/N^{1/2}$ when there are countless other algorithms delivering vastly superior results.

This is different when a kinetic description of a plasma, for instance with test particles, is desired. Since in a fully kinetic picture the plasma is described by a density function in six

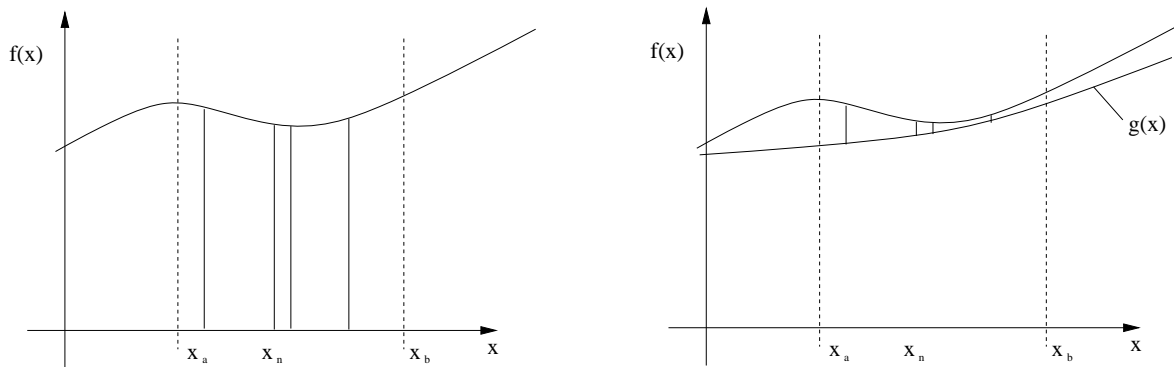


Figure 14.4:

dimensions, a very large number of test particles would be required in order to get reasonable coverage of this phase space.

Often, however, one is interested in situations where the energy distribution deviates only slightly from the thermal equilibrium. Hence, significant savings can be achieved by approximating the plasma by a Maxwellian to 0th order and modeling only the deviation from a Maxwellian using test particles. Approaches of this kind are referred to as δf -methods. This procedure is similar in nature to the ansatz outlined above and results in enormous improvement of accuracy and computational effort.

NOTE: The choice of your algorithm can have a drastic impact on the accuracy of your solution and the efficiency of your code.

14.3 Programming languages – ugly surprise

Scientific computing is entirely dominated by a small number of computer languages: FORTRAN and C/C++.

14.3.1 FORTRAN

is a general-purpose, procedural, imperative computer programming language designed for numerical applications [2]. Developed in the 1950, it was released in 1957 by IBM. This was the first complete compiler, and it constituted an important milestone in doing away with manual coding.

The language definition for FORTRAN has seen several revisions over the years, ranging from FORTRAN ... FORTRAN V, over FORTRAN 66 and 77 to Fortran 2003 which is becoming available with recent compiler releases.

An important feature contributing to the success of FORTRAN was an exceptionally high degree of standardisation. A piece of code compliant with this standard can safely be expected to compile and run successfully on any other platform.

NOTE: If your code is any good, somebody (you) will port it. Always comply with the language standard. No exceptions!

14.3.2 C/C++

C is a general-purpose, block structured, procedural, imperative computer programming language developed in 1972 by Dennis Ritchie at the Bell Telephone Laboratories. C is a very lean language [3], allowing full access to machine level while retaining all language elements required for structured programming (object oriented programming in the case of C++ [4, 5, 6, 7]). Higher level operations (e.g. string manipulation etc.) are provided by standardised libraries. The operating systems UNIX and LINUX are written in C. Like FORTRAN, C/C++ provides a certain degree of safety derived from that fact that these languages have been around for a couple of decades now, and they are not going away anytime soon.

NOTE: If used with pragmatism, C++ delivers *precisely* the same performance as FORTRAN. Just keep potentially time-consuming object-oriented features out of the low-level numerics.

The computational scientist can safely expect to find mature, thoroughly tested, standard compliant and efficient compilers for FORTRAN/C/C++ on any platform. Starting a scientific software project in any other language involves a significant risk.

14.4 Numerical libraries – a few should do

Computational plasma physics makes heavy use of numerical methods. While there may be many cases where scientists have to devise new algorithms to tackle specific problems, it is often possible to draw on preexisting numerical libraries. These libraries may in some cases be very extensive and cover practically everything that would be found in most textbooks.

There are many reasons why library functions should be preferred over home grown code:

- Library functions can save a lot of work.
- They have been tested by a large community of users and are less likely to suffer from programming glitches.
- Many library functions are the result of man-months, maybe man-years of cutting edge mathematical research. Most of them are hard to beat, both in performance and accuracy.
- The BLAS (Basic Linear Algebra Subroutines) library is available in tailor made versions optimised for individual computer architectures and even processors.

This said, the computational scientist will find himself writing plenty of numerical code when a problem involves special requirements, or when special properties of a problem can be exploited.

Table 14.1: Overview of a few example libraries that have become very common in scientific computing.

Name of Library	Purpose	Description	Availability
BLAS	Low level linear algebra	Basic vector/matrix operations	free, often optimised for indiv. platforms, www.netlib.org/
LAPACK	Higher level linear algebra	Eigenvalue problems, linear equations, ...	free www.netlib.org
GSL (GNU Scientific Library)	General purpose	extensive, growing	free (www.gnu.org) C, \exists wrappers for other languages
PETSc	PDEs, ODEs, linear algebra, non-lin. equations, ...	Parallel	free www.mcs.anl.gov
NETLIB	general purpose	very extensive various languages	free, www.netlib.org
IMSL (International Mathematical and Statistical Library)	General purpose	very extensive, dates back to 1970ies	commercial, C, Java, C#.NET, FORTRAN
NAG (Numerical Algorithms Group)	General purpose	very extensive dates back to 1970ies	commercial, FORTRAN, C/C++

NOTE: If your code is any good, someone (you) will port it. Don't use too many different libraries and stick with those you expect to be available on other platforms. If you don't, you'll be rewriting your code over and over again.

Most numerical libraries are written in FORTRAN, so they can easily be called from FORTRAN, C and C++.

NOTE: If you are writing your code in C/C++, make sure you structure your data so as to facilitate access to numerical routines written in FORTRAN.

Here are a few example libraries that have become very common in scientific computing (see Tab. 14.4). Every computational physicist will sooner or later be exposed to one or several of these packages.

14.5 Computers

Computational physics has and still is being done on various types of computing machines.

14.5.1 Scalar architecture

Scalar computers are the most common type of computing machines. Both instructions and data are executed sequentially (SISD = Single Instruction, Single Data). Mainframe computers of the early days used to be scalar machines. The same is true for most personal computers and, until a couple of years ago, most lower budget workstations.

Pros:

- + simple, often cheap machines
- + common device
- + easiest to program

Cons:

- modest performance compared with vector and parallel machines
- annoying maintenance compared with centralised machines

14.5.2 Vector computers

The processor of a vector computer has an array register, often referred to as *the vector*, that facilitates parallel identical arithmetic operations on multiple data.

Most vector machines use interlaced memory in order to speed up memory access, usually by a factor 8 or more. This implies that the performance of the machine is exploited only if identical operations are performed on large chunks of consecutive data.

However fast these machines may be with their vector facility, they are often rather slow for scalar operations. Therefore, interactive access to these machines is often provided by front end computers providing software like editors, cross compilers and debuggers.

Well managed computing centres measure the vector performance of their users codes and ban inefficient users/codes.

Pros:

- + high performance for many problems of relevance, especially linear algebra, mesh oriented problems etc.
- + programming is relatively easy, simple problems may require no human assistance under FORTRAN.
- + huge memory on single processor.

Cons:

- Data and data access must be arranged so as to support sequential access to interlaced memory banks.
- often poor scalar performance.
- conditional execution may be show stopper to vectorisation.
- vectorisation of C/C++ code requires human assistance; not difficult, though.

14.5.3 Parallel computers

In a parallel environment, several processors jointly work on the solution of a problem. The individual processors need not run in strict synchronicity and may in fact be assigned to rather

unrelated aspects of a computation. This mode of operation is therefore characterised as *Multiple Instructions, Multiple Data* (MIMD). Here are a couple of important examples for parallel machines:

- Multicore PC: These devices have processors with several, usually 2 or 4, cores, each capable of acting like an independent processor. All processor cores have equal access to on board memory and all other hardware components. No network is required for interprocess communication.
- Clusters: Clusters are arrays of regular personal computers usually connected by a serial network. No exotic hardware is required. It is common to operate these computer clusters under LINUX, both because it is free and there is ample experience with this approach in the community.
- Blade Servers: These machines consist of so called *Blades* or *Nodes*, boards usually holding several processors. As in the case of Linux clusters, the blades are connected by a very fast network. Communication within one blade is provided by the memory shared.

Pros:

- + currently cheapest way to get enormous computing power
- + works especially well for problems, whose subproblems require little or no communication, for instance Monte Carlo, PIC, etc.

Cons:

- not all problems scale well.
- in many environments, parallel execution must be programmed manually.
- code gets harder to test.
- often distributed memory, so in large problems the data may have to be distributed.
- connection between processors (*Backbone*) may be a performance bottleneck.

The boundaries between these types of computers, scalar, vector and parallel, is often blurred. Any parallel machine, for instance, will allow execution of scalar code simply by running it on a single processor. Most vector computers have several processors, each equipped with its own vector facility. Usually these processors can be used in parallel.

All of the above-mentioned hardware architectures are currently in use at the major fusion research institutes. The choice of any type of hardware may have important consequences on the architecture of a new code to be developed.

14.6 Parallelisation – everywhere

“If you were plowing a field, which would you rather use:

Two strong oxen or 1024 chickens?”
Seymour Cray

This famous quip by one of the most notable pioneers of supercomputer design related to the choice between parallel computers with simple processors on one hand and machines with fewer, more powerful vector processors on the other. Most of scientific computing currently would go for the *1024 chickens*. With current technology, this is the cheapest way to get large computational power.

On the downside, the user needs to write additional code designed to make a large number of processors cooperate. As the complexity of this task is comparable with generating the computational code proper, there are in general no fully automated tools to do this.

On UNIX/LINUX platforms, the parallelisation of codes is achieved using preexisting, free libraries or special, modified compilers. The most important ones are the following.

14.6.1 OpenMP

OpenMP is an environment consisting of special/modified compilers and additional run-time libraries (see for instance [10]). It works well on shared-memory architectures. The example code below shows its ease of use. It is a simple loop over the elements of an array, loading each element with a computed value.

```
#define N 100000
int main(int argc, char *argv[])
{
    int i, a[N];
#pragma omp parallel for
    for (i=0;i<N;i++)
        a[i]= 2*i;
    return 0;
}
```

Note that that code remains fully portable to scalar architectures. Parallelisation of the code is done in a transparent fashion, the user need not provide any detailed instructions. The simplicity of this example is representative of the way OpenMP is often used.

14.6.2 MPI

The **Message Passing Interface** is a set of libraries facilitating the communication of different processes running on different processors. A useful introduction to this tool can be found in [11, 12]. Attached below is an example code. The user must determine how the workload is to be divided and organised between the different processors. The resulting parallelisation strategy

needs to be implemented with detailed instructions. The example below is an implementation of a master-slave architecture. The master, processor 0, sends a string to all slave processors. Each slave processor adds an individual message to the string received and sends it back to the master. The master, finally, prints all strings collected from its slaves. Observe the increased complexity.

```

#include <mpi.h>
#include <stdio.h>
#include <string.h>

#define BUFSIZE 128
#define TAG 0

int main(int argc, char *argv[])
{
    char idstr[32];
    char buff[BUFSIZE];
    int numprocs;
    int myid;
    int i;
    MPI_Status stat;

    MPI_Init(&argc,&argv); /* all MPI programs start with MPI_Init */
    MPI_Comm_size(MPI_COMM_WORLD,&numprocs);
                          /* how many PE's are there? */
    MPI_Comm_rank(MPI_COMM_WORLD,&myid);
                          /* get ID of this PE */

    if(myid == 0)          /* executed by master PE 0 */
    {
        printf("We have %d processors\n", numprocs);

        strcpy (buff, "Hello World: ");
        MPI_Bcast(buff, BUFSIZE, MPI_CHAR, TAG, MPI_COMM_WORLD);
                          /* send string buff to all other PE's */

        for(i=1; i<numprocs; i++)
            {MPI_Recv(buff, BUFSIZE, MPI_CHAR, 0, TAG, MPI_COMM_WORLD, &stat);
              /* collect modified string buff from
              other PE's, one by one */
            printf ("%d", buff);
            }
    }
    else                  /* executed by slave PEs */
    {
        MPI_Recv(buff, BUFSIZE, MPI_CHAR, 0, TAG, MPI_COMM_WORLD, &stat);
                          /* receive string buff from master PE 0: */
    }
}

```

Table 14.2: The GOURDON code.

Written:	1968
First published:	1970
Authors:	C. Gourdon
Language:	FORTRAN66, now FORTRAN90
Libraries used:	/
Source:	~ 1200 lines
First platform:	at IPP: CRAY XMP
Execution time:	several minutes on 2007 PC

```

    sprintf(idstr, "Processor %d ", myid);
                                /* modify recieved string */
    strcat(buff, idstr);
    strcat(buff, "reporting for duty\n");

    MPI_Send(buff, BUFSIZE, MPI_CHAR, 0, TAG, MPI_COMM_WORLD);
                                /* send back modified string buff
                                to master PE 0: */
}

MPI_Finalize(); /* MPI Programs end with MPI Finalize */
return 0;
}

```

This code example owes its elevated complexity both from the explicit coding of communication and the division of the work load into different occupations (master/slave). This is representative of many applications of MPI.

The MPI library is written in C. It can also be called from FORTRAN and C++.

14.7 Example codes – tools for a lifetime

14.7.1 Tiny: GOURDON

The so-called GOURDON code (see Tab. 14.7.1) is one of the oldest workhorses in stellarator physics. Its purpose is to compute the field lines resulting in any given coil configuration. The coil system is described by a set of polygons, the magnetic field is computed using Biot-Savarts law.

As output data the code produces a set of poincare cuts of the magnetic field lines traced.

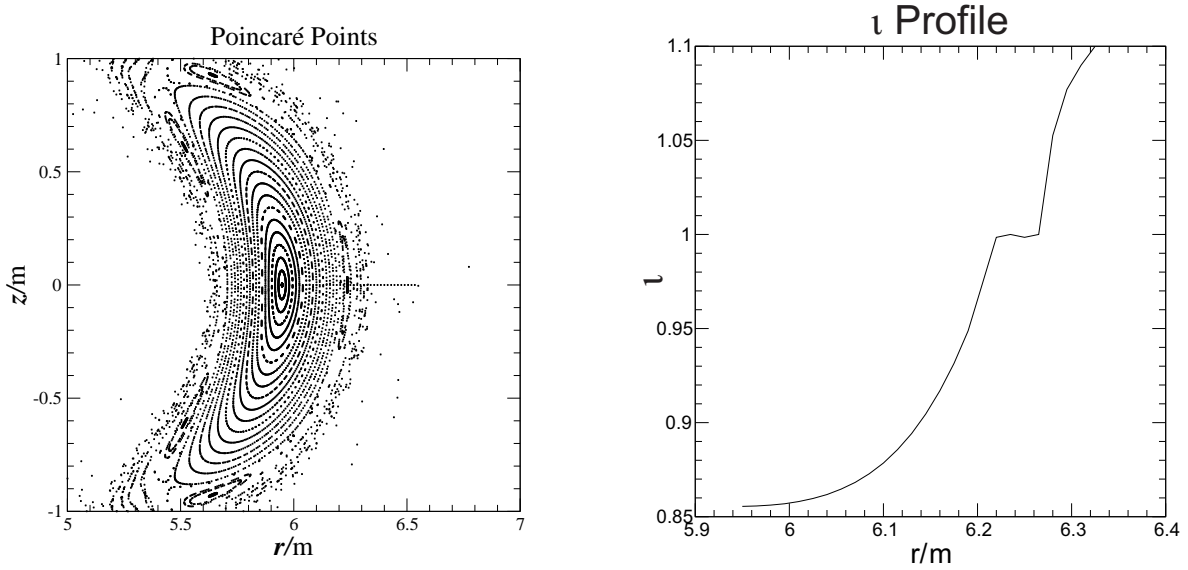


Figure 14.5: Poincaré data and t -profile obtained from the vacuum field of W7-X using GOURDON.

14.7.2 Big: EUTERPE

Gyrokinetic theory

Gyrokinetic (GK) theory for ions is the standard theory used for describing the core (where temperature and density are high). In the following example (see Tab. 14.7.2) it will be applied to the Ion Temperature Gradient (ITG) instabilities.

GK theory is a simplification of full kinetic theory where the Vlasov equation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_r f + \frac{q}{M} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f = 0 \quad (14.4)$$

(q, M : ion charge, mass; \mathbf{E}, \mathbf{B} : electric, magnetic field) describes the time evolution of a six-dimensional distribution function $f(\mathbf{r}, \mathbf{v})$ in phase space. Solving this equation numerically would be very costly; thus approximations are required in order to make a numerical simulations more feasible.

Simplification 1: Strictly, the dynamics of f should be governed by the electric field \mathbf{E} and the magnetic induction \mathbf{B} . In the following, only $\mathbf{E} = -\nabla\phi$ is allowed to change while \mathbf{B} will be held constant.

Simplification 2: The motion of a charged particle in a magnetic field can be decomposed into a fast gyration with the gyrofrequency $\Omega_i := qB/M$ along a circle – called the gyroring –, with radius $\rho := v_\perp/\Omega_i$ centered at the slowly drifting gyrocenter \mathbf{R} , and a tangential motion of the gyrocenter along the magnetic field lines. The position of the particle can thus be written as $\mathbf{x} = \mathbf{R} + \rho(\alpha)$ with $\rho(\alpha) = \rho(\cos(\alpha)\mathbf{e}_{\perp 1} + \sin(\alpha)\mathbf{e}_{\perp 2})$ and $\mathbf{e}_{\perp 1,2}$ two base vectors perpendicular to \mathbf{B} . The velocity \mathbf{v} is decomposed into components parallel (v_\parallel) and perpendicular (v_\perp) to \mathbf{B} . α is called the gyrophase.

If the characteristic frequency ω of the processes in question is much smaller than Ω_i – such is the case for turbulence and ITG instabilities – it is possible (under assumptions on the space variation of the quantities) to average over the fast gyromotion. Due to the averaging procedure the gyrophase α disappears and the result is the gyrokinetic equation [14] now describing the evolution of a distribution function $f(\mathbf{R}, v_{\parallel}, \mu)$ in a five-dimensional phase space spanned by the gyrocenter position \mathbf{R} and two velocity components (here the magnetic moment per unit mass $\mu := v_{\perp}^2/(2B)$ is used instead of v_{\perp})

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{\mathbf{R}} \cdot \nabla f + \dot{v}_{\parallel} \frac{\partial f}{\partial v_{\parallel}} + \dot{\mu} \frac{\partial f}{\partial \mu} = 0. \quad (14.5)$$

The equations of motion for the gyrocenter (for simplicity a vacuum magnetic field is assumed) are

$$\dot{\mathbf{R}} = v_{\parallel} \mathbf{b} + \frac{1}{B} \mathbf{b} \times \nabla \langle \phi \rangle + \frac{\mu B + v_{\parallel}^2}{B \Omega_i} \mathbf{b} \times \nabla B, \quad (14.6)$$

$$\dot{v}_{\parallel} = -\mu \mathbf{b} \cdot \nabla B - \frac{q}{M} \left(\mathbf{b} + \frac{v_{\parallel}}{B \Omega_i} \mathbf{b} \times \nabla B \right) \cdot \nabla \langle \phi \rangle, \quad (14.7)$$

$$\dot{\mu} = 0 \quad (14.8)$$

with $\mathbf{b} := \mathbf{B}/B$. Eq. (14.6) gives the motion of the gyrocenter as a parallel motion plus $\mathbf{E} \times \mathbf{B}$ -drift and grad- B /curvature-drift. The first term in the equation for \dot{v}_{\parallel} shows the mirror effect, μ is a constant of motion.

The equations of motion do not depend on the electrostatic potential directly but on its average over the gyroring described by the gyroaveraging operator $\langle \cdot \rangle$

$$\langle \phi \rangle(\mathbf{R}) := \frac{1}{2\pi} \int_0^{2\pi} \phi(\mathbf{R} + \boldsymbol{\rho}(\alpha)) d\alpha. \quad (14.9)$$

The space density n_i of ions is given by the gyroaveraged ion density $\langle n_i \rangle$ defined as

$$\langle n_i \rangle(\mathbf{x}) := \int f(\mathbf{R}, v_{\parallel}, \mu) \delta(\mathbf{R} + \boldsymbol{\rho}(\alpha) - \mathbf{x}) d\mathbf{R} dv \quad (14.10)$$

plus a correction term called the polarization density (n_0 denotes the equilibrium ion density and ∇_{\perp} the gradient perpendicular to the magnetic field). For small $k_{\perp} \rho_i$ this yields

$$n_i = \langle n_i \rangle + \nabla_{\perp} \cdot \left(\frac{n_0}{B \Omega_i} \nabla_{\perp} \phi \right). \quad (14.11)$$

Since the gyroradius of the electrons is much smaller than that of the ions the former can be described by a simple approximation called adiabatic electrons: $n_e = n_0 + en_0/(k_B T_e) \phi$. Together with charge neutrality $n_e = n_i$ this leads to the final Helmholtz equation determining ϕ

$$\frac{en_0}{k_B T_e} \phi - \nabla_{\perp} \cdot \left(\frac{n_0}{B \Omega_i} \nabla_{\perp} \phi \right) = \langle n_i \rangle - n_0. \quad (14.12)$$

Eq. (14.5) to (14.10) and (14.12) constitute a closed system for ϕ and f .

Table 14.3: The EUTERPE code.

Written:	derived from GYGLES written in 1996
First published:	2000
Authors:	G. Jost
Language:	FORTRAN90
Libraries used:	PETSc, FFTW
Source:	20000 lines
Purpose:	GK theory, ITG modes
First platform:	parallel UNIX/LINUX systems
Execution time:	2h-12h on 16 PEs Athlon 2800MHz

GK theory can be understood intuitively by a simple physical picture: Due to the time–scale separation between gyration and drift motion the ion can be replaced by a ring of charge with radius ρ centered around R . f now becomes the distribution function of these rings located in a five-dimensional phase space. Eq. (14.6) to (14.8) describe the motion of such a ring in this phase space. As a consequence the electric field does not act at the center of the ring R but on the ring itself; thus to get its action it must be integrated along the ring what is expressed by (14.9). Eq. (14.10) can be understood by noting that the charge at one specific space point is the sum of the charge from all the rings with different ρ passing through it. Finally, the polarization density can only be understood by deriving the gyrokinetic theory rigorously using the theory of Lie transformations [14].

The PIC method

Useful solutions of the gyrokinetic equation have been obtained by particle in cell (PIC) simulations. This method becomes particularly efficient if combined with a δf approach. An example in place is the EUTERPE code, which is derived from a line of PIC codes originating from the CRPP at the EPFL Lausanne. The dedicated reader may find further details (linear/nonlinear numerical implementation, geometry or extensions for non-adiabatic electrons, electromagnetic effects) in the literature (see e.g. [16, 15, 17, 18, 19, 20]).

The fluctuation amplitude of ITG turbulence in a plasma is much smaller than the magnitude of the equilibrium quantities. For a simulation using particles this means that most of the particles are wasted representing the equilibrium and only a small remainder actually describes the fluctuations. Since the numerical noise depends on the particle number N as $1/N^{1/2}$ this results in poor statistical behaviour. This can be overcome by employing the δf approach outlined in sec. 14.2 and described in detail for instance in [21, 22, 23]; here the time dependent distribution function f is split into a fluctuating part δf and a time–independent part f_0 representing the equilibrium: $f(t) = f_0 + \delta f(t)$. If now f_0 is given analytically (usually it is assumed to be a Maxwellian distribution) all the particles can be used to represent the fluctuation δf only, resulting in highly improved statistics.

Using this ansatz in (14.5) one gets $d/dt \delta f = S$ (S is a source term resulting from the application of d/dt on f_0). This PDE can be solved formally using the method of characteristics: Given an initial function $\delta f(t=0)$ integration of the ODE $d/dt \delta f = S$ along the characteristics determined by (14.6) to (14.8) gives the general solution $\delta f(t)$.

A gyrokinetic PIC code rests on four procedures subsequently executed at each time step:

1. Particle pushing: Integrating the equations of motion for each particle in a given potential.
2. Gyro-averaging: The gyroaverage $\langle \phi \rangle$ of the electrostatic potential ϕ needs to be calculated by numerically approximating the integral in (14.9).
3. Charge assignment: the potential equation (14.12) is discretised on a grid but the particles are distributed irregularly; thus a prescription must be provided specifying how to generate from the particles (and their weights) a density defined on the grid (in gyrokinetics one has also to take into account that one does not deal with point particles but with gyrorings). Charge assignment can thus be regarded as the link connecting the Lagrangian (particles) with the Eulerian part (grid).
4. Potential solver: given the gyroaveraged density on the grid the Helmholtz equation must be solved in order to obtain the potential.

Introducing N marker particles each one carrying a weight w_p the quantity δf is discretised by writing it as a sum of delta functions

$$\delta f = \sum_{p=1}^N \frac{1}{J} w_p(t) \delta(\mathbf{R} - \mathbf{R}_p(t)) \delta(v_{\parallel} - v_{\parallel p}(t)) \delta(\mu - \mu_p(t)) \quad (14.13)$$

($J = 2\pi B$ is the phase space Jacobian) resulting in the equation

$$\dot{w}_p = S|_{\mathbf{R}_p, v_{\parallel p}, \mu_p} \quad (14.14)$$

for the weights.

For the numerical integration of (14.6) to (14.8) and (14.14) with e.g. a Runge-Kutta- or a Predictor-Corrector-method $\nabla \langle \phi \rangle$ must be known at each time step. In order to compute this one uses $-\nabla \langle \phi \rangle \approx \langle \mathbf{E} \rangle$ and approximates the integral by an average over N_a points ρ_j evenly spaced on the gyroring

$$\langle \mathbf{E} \rangle = \frac{1}{N_a} \sum_{j=1}^{N_a} \mathbf{E}(\mathbf{R} + \rho_j). \quad (14.15)$$

ϕ is obtained by solving the Helmholtz equation (14.12) discretised on a grid; also the right hand side $\langle n_i \rangle$ must be computed by the charge assignment process. An elegant way that combines the charge assignment and the discretisation of the Helmholtz equation uses B-splines as

finite elements [16]. The potential is discretised by writing it as a sum over the finite element basis

$$\phi(x) = \sum_v \phi_v \Lambda_v(x) \quad (14.16)$$

where Λ_v is a tensor product B-spline of order k . B-splines are continuous non-negative functions with finite support piecewise defined by polynomials of order k . Their most interesting property here is that B-splines provide a partition of unity i.e. $\sum_v \Lambda_v(x) = 1$ [24]; this guarantees the conservation of charge during charge assignment.

Using (14.16) and the particle discretisation (14.13) in (14.12) one can derive the following matrix equation for the coefficient vector ϕ_v

$$\sum_{v'} A_{vv'} \phi_{v'} = n_v \quad (14.17)$$

with the matrix

$$A_{vv'} = \int \left(\frac{n_0}{B\Omega_i} \nabla_{\perp} \Lambda_v \cdot \nabla_{\perp} \Lambda_{v'} + \frac{en_0}{k_B T_e} \Lambda_v \Lambda_{v'} \right) dx \quad (14.18)$$

that is the same at each time step and thus can be precomputed.

The charge assignment process defines the vector n_v

$$n_v = \sum_{p=1}^N w_p \frac{1}{2\pi} \int_0^{2\pi} \Lambda_v(\mathbf{R}_p + \boldsymbol{\rho}_p) d\alpha, \quad (14.19)$$

where the last integral is again discretised by an average over N_a points on the gyroring (analogously to (14.15)).

This completes the short description of the ingredients needed for a numerical solution of the gyrokinetic equation. We now demonstrate the application with a linear example.

Example results

If $\delta f \ll f_0$ is assumed then the gyrokinetic equation can be linearized. The EUTERPE code [18] solves this equation globally in full three-dimensional geometry; thus it can be used to investigate stellarator configurations (e.g. Wendelstein 7-X [13]).

As a simple example the result of a calculation for a circular tokamak configuration is shown in Fig. 14.6 (the ion temperature gradient driving the instability is chosen such that it is maximal near half the minor radius of the configuration). In this simulation one follows the time evolution of small initial fluctuations imposed onto an equilibrium plasma. In the beginning ($t \lesssim 4$) the plasma oscillates randomly but for $t > 4$ a coherently oscillating mode is established and grows exponentially in time. The slope of the blue line gives the growth-rate while the frequency of the mode can be obtained from the red/black curve. The mode structure at the end of the simulation is presented in Fig. 14.7, (since the configuration is axially symmetric only one poloidal cut is depicted). The structure shown rotates poloidally which its envelope always has its maximum at the right hand side (corresponding to the low field side of a tokamak); thus it is a typical toroidal ballooning mode. This kind of mode is characterized by small scale structures perpendicular to the magnetic field and long structures nearly following the field lines; this reflects the high anisotropy of the system caused by the magnetic field.

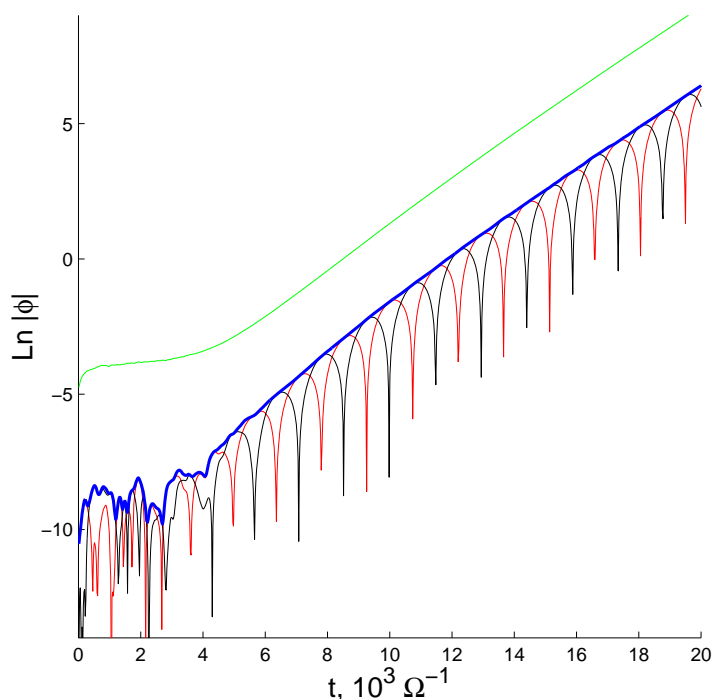


Figure 14.6: Linear ITG mode in a tokamak. Time traces of $Re(\phi)$, $Im(\phi)$, $|\phi|$ (red, black, blue) and field energy (green).

Acknowledgements

The author gratefully acknowledges assistance and advice from Ralf Kleiber, Carolin Nührenberg and Erika Strumberger.

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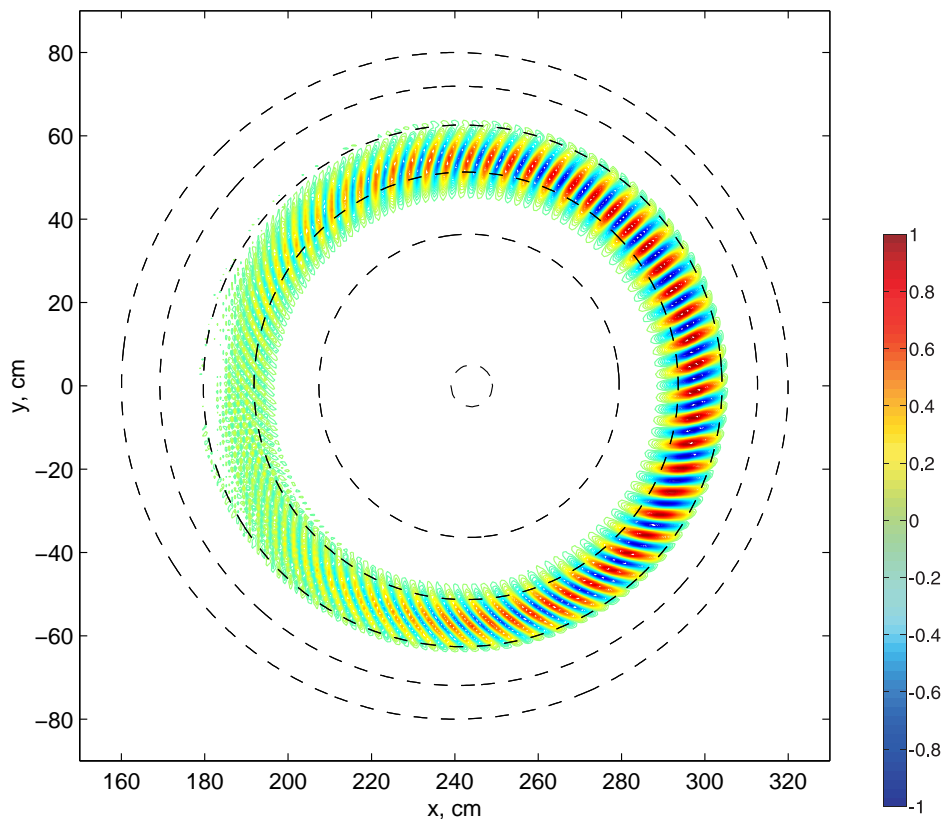


Figure 14.7: Linear ITG mode in a tokamak. $Re(\phi)$ (color coded) in the poloidal plane (Courtesy V. Kornilov).

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