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On the $K(1)$-local homotopy of tmf $\wedge$ tmf

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Abstract

As a step towards understanding the tmf-based Adams spectral sequence, we compute the $K(1)$-local homotopy of $\mathrm{tmf} \wedge \mathrm{tmf}$, using a small presentation of $L_{K(1)}\mathrm{tmf}$ due to Hopkins. We also describe the $K(1)$-local tmf-based Adams spectral sequence.

Keywords Topological modular forms · Chromatic homotopy theory · Hopf algebroid · Bousfield localization

Contents

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1 Introduction

This paper calculates the $K(1)$-local homotopy of $\mathrm{tmf} \wedge \mathrm{tmf}$. The motivation behind this traces back to Mahowald's work on *bo*-resolutions. In his seminal papers on the subject [22,24], Mahowald was able to use the *bo*-based Adams spectral sequence

(1) to prove the height 1 telescope conjecture at the prime $p = 2$,
(2) and, with Wolfgang Lellmann, to exhibit the *bo*-based Adams spectral sequence as a viable tool for computations.

An initial difficulty with this spectral sequence is the fact that bo_*bo does not satisfy Adams' flatness assumption, resulting in the E_2-term not having a description in terms of Ext. One can still work with the spectral sequence, but one has to understand both the algebra bo_*bo and the homotopy theory of *bo*-modules extremely well, and Mahowald's breakthrough decomposition of $bo \wedge bo$ in terms of Brown–Gitler spectra satisfied both goals.

Mahowald later initiated the study of resolutions over tmf, first known as eo_2. Early work on this was done by Mahowald and Rezk in [25], and then developed further in the work of Behrens–Ormsby–Stapleton–Stojanoska in [4]. Again, to work with the tmf-based Adams spectral sequence, one first needs to understand of the homotopy groups $\pi_*(\mathrm{tmf} \wedge \mathrm{tmf})$. This computation was seriously studied in [4] at the prime 2, and at the prime 3 is ongoing work of the first author and Vesna Stojanoska.

Behrens–Ormsby–Stapleton–Stojanoska take a number of approaches to $\mathrm{tmf}_*\mathrm{tmf}$:

(1) The *rational homotopy* $\mathrm{tmf}_*\mathrm{tmf} \otimes \mathbb{Q}$, can be described as a ring of rational, 2-variable modular forms.
(2) The $K(2)$*-local homotopy* $\pi_* L_{K(2)}(\mathrm{tmf} \wedge \mathrm{tmf})$ can be described in terms of Morava E-theory using the methods of [11]. To be precise, one has

$$L_{K(2)}(\mathrm{tmf} \wedge \mathrm{tmf}) \simeq \left(\mathrm{Map}^c(\mathbb{S}_2/G_{24}, \overline{E_2})^{hG_{24}}\right)^{hGal}.$$

(3) Using a change of rings isomorphism, one can write the *classical Adams spectral sequence* as

$$E_2^{*,*} = \mathrm{Ext}_{A_*}^{*,*}(H_*\mathrm{tmf} \wedge \mathrm{tmf}, \mathbb{F}_2) \cong \mathrm{Ext}_{A(2)_*}^{*,*}(A /\!\!/ A(2)_*, \mathbb{F}_2) \implies \pi_*\mathrm{tmf} \wedge \mathrm{tmf}.$$

However, the E_2-term is rather difficult to calculate since the algebra $A /\!\!/ A(2)_*$ is very complicated. Indeed, a full computation of the Adams E_2-term has yet to be done. The approach via the Adams spectral sequence is further complicated by the presence of differentials. Such differentials were first discovered in [25], and even more were found in [4].

Chromatic homotopy theory in principle allows the reassembly of tmf $\wedge$ tmf from its rationalization, $K(1)$-localizations at all primes, and $K(2)$-localizations at all primes. In this paper, we approach the as-yet-unstudied chromatic layer, giving a complete description of $L_{K(1)}(\mathrm{tmf} \wedge \mathrm{tmf})$. Our main tool is a construction due to Hopkins of $K(1)$-local tmf as a small cell complex in $K(1)$-local E_∞-rings [14].

Let us briefly mention some intuition and notation before stating the main result. First, the ring $\pi_* L_{K(1)}\mathrm{tmf}$ is essentially a graded version of the ring of functions on the p-complete moduli stack $\mathscr{M}_{\mathrm{ell}}^{\mathrm{ord}}$ of ordinary, generalized elliptic curves [21]. At small primes $p \leq 5$, we have

$$\pi_0 L_{K(1)}\mathrm{tmf} = \mathbb{Z}_p[j^{-1}]_p^\wedge,$$

where j^{-1} is the inverse of the modular j-invariant. (Note that, at these primes, $\mathscr{M}_{\mathrm{ell}}^{\mathrm{ord}}$ includes the point $j = \infty$, corresponding to the nodal cubic, but not the point $j = 0$, which is supersingular for $p \leq 5$.) If one writes KO for 2-complete real K-theory if $p = 2$, or the p-complete Adams summand for $p > 2$, the formula in all degrees (still for $p \leq 5$) becomes

$$\pi_* L_{K(1)}\mathrm{tmf} = (KO_*[j^{-1}])_p^\wedge.$$

This has p-torsion just at $p = 2$.

Second, the 0th homotopy group of a $K(1)$-local E_∞-ring is naturally a θ-algebra, bearing an algebraic structure studied extensively by Bousfield [7] and described briefly in our Appendix A.1. We write $\mathbb{T}(x)$ for the free θ-algebra on a generator x; by a theorem of Bousfield, as a ring, $\mathbb{T}(x)$ is polynomial on x, $\theta(x)$, $\theta^2(x)$, and so on.

We can now state the main result.

Theorem A *At primes $p \leq 5$,*

$$\pi_* L_{K(1)}(\mathrm{tmf} \wedge \mathrm{tmf}) \cong \left(KO_*[j^{-1}, \overline{j^{-1}}] \otimes \mathbb{T}(\lambda)/(\psi^p(\lambda) - \lambda - j^{-1} + \overline{j^{-1}})\right)_p^\wedge.$$

Given this, the last remaining obstacle to a chromatic understanding of $\mathrm{tmf}_*\mathrm{tmf}$ is a calculation of the transchromatic map

$$L_{K(1)}(\mathrm{tmf} \wedge \mathrm{tmf}) \to L_{K(1)} L_{K(2)}(\mathrm{tmf} \wedge \mathrm{tmf}).$$

We hope to study this in future work.

Let us describe a few consequences of this result. One is a computation of the $K(1)$-local Adams spectral sequence based on tmf.

Theorem B *For any spectrum X, there is a conditionally convergent spectral sequence*

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\pi_* L_{K(1)}(\mathrm{tmf} \wedge \mathrm{tmf})}(\pi_* L_{K(1)}\mathrm{tmf}, \pi_* L_{K(1)}(\mathrm{tmf} \wedge X)) \Rightarrow \pi_{t-s} L_{K(1)} X.$$

If tmf $\wedge X$ *is* $K(1)$*-locally pro-free over* tmf*, then the* E_2 *page of this spectral sequence is isomorphic to*

$$\begin{aligned}&\mathrm{Ext}_{\pi_* L_{K(1)}(\mathrm{tmf}\wedge\mathrm{tmf})}(\pi_* L_{K(1)}\mathrm{tmf}, \pi_* L_{K(1)}(\mathrm{tmf}\wedge X)\\ &\quad\cong \mathrm{Ext}_{\pi_* L_{K(1)}(KO\wedge KO)}(KO_*, \pi_* L_{K(1)}(KO\wedge X))\\ &\quad\cong H^*_{cts}(\mathbb{Z}_p^\times/\mu, \pi_* L_{K(1)}(KO\wedge X)),\end{aligned}$$

where μ *is the maximal finite subgroup of* $\mathbb{Z}_p^\times$*.*

In particular, the spectral sequence for the sphere vanishes at E_2 above cohomological degree 1, and so collapses immediately. While the $K(1)$-local tmf-based Adams spectral sequence is thus uninteresting, one obtains some nontrivial information about the global tmf-based Adams spectral sequence, namely that its v_1-periodic classes occur only on the 0 and 1 lines.

To put these results into perspective, it helps to return to bo. $K(1)$-locally, bo is the same as KO, and its $K(1)$-local co-operations algebra is simply:

$$\pi_* L_{K(1)}(bo\wedge bo) = \pi_* L_{K(1)}(KO\wedge KO) = KO_*\otimes \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p^\times/\mu, \mathbb{Z}_p).$$

As $bo\wedge bo$ is E_∞, this ring has an alternative θ-algebraic description, namely

$$\pi_* L_{K(1)}(bo\wedge bo) = KO_*\otimes \mathbb{T}(b)/(\psi^p(b)-b).$$

Here b is an explicit choice of group isomorphism $\mathbb{Z}_p^\times/\mu \xrightarrow{\cong} \mathbb{Z}_p$, and the single relation expands to

$$p\theta(b) = b - b^p,$$

a relation between b and $\theta(b)$. In the formula of Theorem A, the modular forms j^{-1}, $\overline{j^{-1}}$ also satisfy θ-algebra relations forced on them by number theory, and one obtains a relation between λ, $\theta(\lambda)$, and $\theta^2(\lambda)$, a sort of second-order version of the bo calculation.

It is also worth noting that, for the sake of calculating Adams spectral sequences, one is interested in the coalgebra of bo_*bo as much as its algebra – and the original, non-θ-algebraic calculation

$$\pi_* L_{K(1)}(bo\wedge bo) = KO_*\otimes \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p^\times/\mu, \mathbb{Z}_p)$$

is actually better suited for this purpose. It is this realization, and a search for an analogue for tmf, that eventually led to the proof of Theorem B.

As a final remark, our calculation also doubles as a calculation of a purely number-theoretic object. Namely, consider the moduli problem $\mathcal{M}_{\mathrm{pair}}$ over Spf $\mathbb{Z}_p$ that sends a p-complete ring R to the groupoid of data

$$(E, E', \phi : E \xrightarrow{\sim} E'),$$

where E and E' are ordinary generalized elliptic curves over R and ϕ is an isomorphism of their formal groups. Just as the structure sheaf of the moduli of generalized elliptic curves extends to a locally even periodic sheaf of E_∞ ring spectra whose global sections are (the nonconnective) Tmf [5,13], there is such a sheaf on $\mathcal{M}_{\text{pair}}$ whose global sections are $L_{K(1)}(\text{tmf} \wedge \text{tmf})$. Moreover, $\mathcal{M}_{\text{pair}}$ is an affine scheme in the case $p > 2$, and has a double cover by an affine scheme in the case $p = 2$. In both cases, its ring of global functions R_{pair} is exactly $\pi_0 L_{K(1)}(\text{tmf} \wedge \text{tmf})$. We can think of this ring as a ring of “ordinary 2-variable p-adic modular functions”. As examples of ordinary 2-variable p-adic modular functions, we have the functions

$$j^{-1} : (E, E', \phi) \mapsto j^{-1}(E), \quad \overline{j^{-1}} : (E, E', \phi) \mapsto j^{-1}(E').$$

Of course, these examples are somewhat trivial because they are really 1-variable modular functions. The results of this paper tell us that, *as a θ-algebra*, R_{pair} is generated over these 1-variable functions by a single other generator. This generator is explicitly given as the generator λ described in Remark 6.2.

In fact, the θ-algebra structure on $\pi_0 L_{K(1)}(\text{tmf} \wedge \text{tmf})$ has an equivalent definition in terms of number theory, and the generators we give can be identified in terms of modular forms. While the following is essentially a restatement of the original calculation, it is of independent enough interest to deserve explicit mention:

Theorem C *At the primes 2 and 3, the ring of ordinary 2-variable p-adic modular forms is generated as a θ-algebra by j^{-1}, $\overline{j^{-1}}$, and a single other generator.*

1.1 Outline of the paper

This paper is almost entirely set inside the $K(1)$-local category. This leads to some unusual choices about notation, for the sake of which we encourage even the expert reader to take a look at Sect. 1.2 below. In Sect. 2, we give some background information about $K(1)$-local homotopy theory, in particular reviewing the relevant notion of completeness and associated issues of homological algebra. Building on [1,17,19], and [3], we set up some fundamental tools, such as a relative Künneth formula, a change of rings theorem, and the theory of $K(1)$-local Adams spectral sequences, that we will use later on.

In Sect. 3, we study the E_∞ cone on the class $\zeta \in \pi_{-1} L_{K(1)} S$, called T_ζ by Hopkins. This object was used in [14] and [21] as a partial version of tmf, and the results in this section can mostly be found in those papers. However, in the process of reading those papers, the authors found some problems with the calculation of $\pi_* T_\zeta$ (see Remark 3.29). Part of our motivation in writing down this calculation in detail is to fill these gaps.

In Sect. 4, we compute the cooperations algebra $\pi_* L_{K(1)}(T_\zeta \wedge T_\zeta)$, which is an approximation to $\pi_* L_{K(1)}(\text{tmf} \wedge \text{tmf})$.

In Sect. 5, we return to the work of Hopkins and Laures to review their construction of $L_{K(1)}\text{tmf}$. Again, the material in this section can be found in [14] or [21], but we include for the reader’s convenience.

In Sect. 6, we compute the $K(1)$-local co-operations algebra for tmf, and prove Theorems A and B.

In Sect. 7, we discuss the relationship between our results and the theory of p-adic modular forms, and prove Theorem C.

We have also included an appendix containing technical information about θ-algebras and λ-rings.

1.2 Notation and conventions

The rest of this paper takes place inside the $K(1)$-local category, at a fixed prime $p \leq 5$. To avoid notational clutter, we adopt a blanket convention that all objects are implicitly $K(1)$-localized and/or p-completed, unless it is explicitly stated otherwise. To be precise, this includes the following conventions for algebra:

- All rings are implicitly L-completed with respect to the prime p (see Sect. 2.1, and note that the L-completion agrees with the ordinary p-completion when the ring is torsion-free). For example, by $\mathbb{Z}_p[j^{-1}]$ we really mean the completed polynomial algebra

$$\mathbb{Z}_p[j^{-1}]_p^\wedge = \left\{ \sum_{n \geq 0} a_n j^{-n} : |a_n|_p \to 0 \text{ as } n \to \infty \right\}.$$

- By $\otimes$ we mean the L-completed tensor product (see Sect. 2.1).
- We write $\mathsf{Mod}_*^\wedge$ for the category of L-complete graded $\mathbb{Z}_p$-modules, and $\mathsf{CAlg}_*^\wedge$ for the category of L-complete commutative graded $\mathbb{Z}_p$-algebras.
- More generally, if R_* is an L-complete ring, then $\mathsf{Mod}_{R_*}^\wedge$ is the category of L-complete R_*-modules and $\mathsf{CAlg}_{R_*}^\wedge$ the category of L-complete commutative R_*-algebras. If (R_*, Γ_*) is an L-complete Hopf algebroid, then $\mathsf{Comod}_{\Gamma_*}^\wedge$ is its category of L-complete comodules (see Sect. 2.3).
- Ext_{Γ_*} is the relative Ext functor for comodules defined in Definition 2.16.
- $\mathbb{T}(x_1, \ldots, x_n)$ is the free p-complete θ-algebra on the generators $x_1, \ldots, x_n$ (see Theorem A.5).

It includes the following conventions for topology:

- All smash products are implicitly $K(1)$-localized.
- Sp is the category of $K(1)$-local spectra, and CAlg is the category of $K(1)$-local E_∞-algebras.
- $\mathbb{P}(X)$ is the free $K(1)$-local E_∞-algebra on a spectrum X.

We will also employ the following notation:

- μ is the maximal finite subgroup of $\mathbb{Z}_p^\times$, so $\mu \cong C_2$ if $p = 2$ or C_{p-1} if p is odd, and $\mathbb{Z}_p^\times/\mu \cong \mathbb{Z}_p$.
- ω is a fixed generator of μ (so $\omega = -1$ at $p = 2$).
- For $p > 2$, g is a fixed topological generator of $\mathbb{Z}_p^\times$ (for example, we can take $g = \omega(1+p)$). Note that g maps to a topological generator of $\mathbb{Z}_p^\times/\mu$. For $p = 2$, g

is a fixed element of $\mathbb{Z}_2^\times$ mapping to a topological generator of $\mathbb{Z}_2^\times/\mu$ (for example, we can take $g = 3$).

- K is p-completed complex K-theory, and tmf is $K(1)$-local tmf. KO is (2-complete) KO if $p = 2$, or the (p-complete) Adams summand if p is odd.

Remark 1.1 (Restrictions on p). Unless otherwise stated, the results of this paper are valid only at $p = 2$, 3, and 5. This is primarily a matter of convenience: at these primes, there is a unique supersingular j-invariant congruent to 0 mod p, which implies that $\pi_0 L_{K(1)}\mathrm{tmf}$ is a p-complete polynomial in the generator j^{-1}. At larger primes, $\pi_0 L_{K(1)}\mathrm{tmf}$ is the p-complete ring of functions on

$$\mathbb{P}^1_{\mathbb{Z}_p} - \{\text{supersingular } j\text{-invariants}\},$$

which grows more complicated as the number of supersingular j-invariants increases, though presumably not in an essential way.

Our restriction on p is also a matter of interest: it is only at $p = 2$ and 3 that the homotopy groups of the unlocalized spectrum tmf has torsion; at larger primes tmf_* is just the ring of level 1 modular forms.

The reader will also note that the $K(1)$-local category behaves differently at the prime 2 than at all other primes. For example, while $\pi_*\mathrm{tmf}$ has 2- and 3-torsion, $\pi_* L_{K(1)}\mathrm{tmf}$ only has torsion at the prime 2.

2 Complete Hopf algebroids and comodules

One often attempts to study a $K(1)$-local spectrum X through its completed K-homology or KO-homology,

$$K_*X = \pi_* L_{K(1)}(K \wedge X) \text{ and } KO_*X = \pi_* L_{K(1)}(KO \wedge X).$$

These are not just graded abelian groups, but satisfy a condition known since [19] as L-completeness. In Sect. 2.1, we review the definition of L-completeness and some basic properties of the L-complete category. Next, in Sect. 2.2, we review the important technical notion of pro-freeness, which is to be the appropriate replacement for flatness in the L-complete setting. As we have to deal with some relative tensor products of $K(1)$-local ring spectra, we need a relative definition of pro-freeness that is more general than that used by other authors, e.g. [17]. We use this definition to give a Künneth formula for relative tensor products in which one of the modules is pro-free. In Sect. 2.3, we discuss homological algebra over L-complete Hopf algebroids, a concept originally due to Baker [1], and conclude with an examination of the $K(1)$-local Adams spectral sequence. Finally, in Sect. 2.4, we give the classical examples of the Hopf algebroids for K and KO, and describe their categories of comodules.

The results of this section should be compared with Barthel–Heard's work on the $K(n)$-local E_n-based Adams spectral sequence [3]. While we ultimately want to write down $K(1)$-local Adams spectral sequences over more general bases than K itself, the work involved is substantially simplified by certain convenient features of height

1, mostly boiling down to the fact that direct sums of L-complete $\mathbb{Z}_p$-modules are exact—the analogue of which is not true at higher heights [17, Sect. 1.3]. The reader who wishes to do similar work at higher heights should therefore proceed with caution.

2.1 Background on *L*-completeness

In the category Sp of $K(1)$-local spectra, there is a well-known equivalence [19, Proposition 7.10]

$$X \simeq \mathrm{holim}_i\, X \wedge S/p^i$$

Replacing X by the $K(1)$-local smash product $K \wedge X$, we have an equivalence

$$K \wedge X \simeq \mathrm{holim}_i\, K \wedge X \wedge S/p^i.$$

This shows that K_*X is derived complete, in a sense we now make precise.

We can regard p-completion as an endofunctor of the category of abelian groups. This functor is neither left nor right exact. However, it still has left derived functors, which we write as L_0 and L_1 (the higher left derived functors vanish in this case). Since p-completion is not right exact, it is generally *not* the case that $M_p^\wedge = L_0M$. There is, however, a canonical factorization of the completion map $M \to M_p^\wedge$:

$$M \longrightarrow L_0M \xrightarrow{\varepsilon_M} M_p^\wedge.$$

The second map is surjective, and in fact, there is a short exact sequence [19, Theorem A.2(b)]

$$0 \to \lim_n{}^1 \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/p^n, M) \to L_0M \to M_p^\wedge \to 0. \tag{2.1}$$

We also have [19, Theorem A.2(d)]

$$L_0M = \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p^\infty, M), \quad L_1M = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^\infty, M).$$

Definition 2.2 An abelian group A is L-complete if the natural map $A \to L_0A$ is an isomorphism. A graded abelian group A_* is L-complete if it is L-complete in each degree.

Being L-complete is quite close to being p-complete: for example, p-complete modules are L-complete, and if M is finitely generated, then $L_0M \cong M_p^\wedge$. In particular, K_* and KO_* are L-complete. More generally, for any $K(1)$-local spectrum X, π_*X is L-complete as a graded abelian group [17, Lemma 3.2].

Write $\mathsf{Mod}_*^\wedge$ for the category of L-complete graded $\mathbb{Z}_p$-modules. This is an abelian subcategory of the category of graded $\mathbb{Z}_p$-modules which is closed under extensions. It is also closed symmetric monoidal [2, Sect. A.2] under the L-completed tensor product

$$M_* \overline{\otimes} N_* = L_0(M_* \otimes N_*).$$

Following our general conventions (see Sect. 1.2), we will simply write $\otimes$ for this tensor product, where this does not cause confusion.

Write $\mathsf{CAlg}_*^\wedge$ for the category of commutative algebra objects in $\mathsf{Mod}_*^\wedge$. If $R_* \in \mathsf{CAlg}_*^\wedge$ (in particular, if $R_* = K_*$ or KO_*), there is an obvious abelian category of L-complete R_*-modules, which we denote $\mathsf{Mod}_{R_*}^\wedge$.

2.2 Pro-freeness

Definition 2.3 Let $R_* \in \mathsf{CAlg}_*^\wedge$, and let $M_* \in \mathsf{Mod}_{R_*}^\wedge$. Say that M_* is pro-free if it is of the form

$$M_* \cong L_0 F_*,$$

where F_* is a free graded R_*-module. Say that a map $R_* \to S_*$ of commutative rings in $\mathsf{Mod}_*^\wedge$ is pro-free if S_* is a pro-free R_*-module.

Pro-free modules are projective in the category $\mathsf{Mod}_{R_*}^\wedge$. In this height 1 case, they are also flat in this category. As is shown below, this follows from the fact that direct sums in $\mathsf{Mod}_*^\wedge$ are exact, which is, surprisingly, not true at higher heights.

Lemma 2.4 *Let $R_* \in \mathsf{CAlg}_*^\wedge$, and let M_* be a non-zero pro-free R_*-module. Then M_* is faithfully flat in $\mathsf{Mod}_{R_*}^\wedge$, that is, the functor $M_* \otimes_{R_*} \cdot$ is exact and conservative.*

Proof If M_* is a pro-free R_*-module, it is a coproduct of (possibly shifted) copies of R_* in the category $\mathsf{Mod}_{R_*}^\wedge$. Correspondingly, $M_* \otimes_{R_*} N_*$ is a coproduct of possibly shifted copies of N_*, which can be taken in $\mathsf{Mod}_*^\wedge$. This functor is exact because coproducts in $\mathsf{Mod}_*^\wedge$ are exact [17, Proposition 1.4]. Clearly, a coproduct of copies of N_* is zero iff N_* is zero, which together with exactness implies conservativity. $\square$

Lemma 2.5 *Pro-freeness is preserved by base change: if M_* is pro-free over R_* and $R_* \to S_*$ is a map of rings in $\mathsf{Mod}_*^\wedge$, then $M_* \otimes_{R_*} S_*$ is pro-free over S_*.*

Proof Again, M_* is a coproduct of copies of R_* in the category $\mathsf{Mod}_{R_*}^\wedge$. The tensor product is a left adjoint, so distributes over this coproduct. $\square$

Lemma 2.6 *Suppose that $R_* \in \mathsf{CAlg}_*^\wedge$ and $M_* \in \mathsf{Mod}_{R_*}^\wedge$. Suppose also that R_* is p-torsion-free. Then M_* is pro-free over R_* iff M_* is p-torsion-free and M_*/p is free over R_*/p.*

Proof Suppose that M_* is pro-free over R_*, and write $M_* = L_0\left(\bigoplus_\alpha \Sigma^{n_\alpha} R_*\right)$. By the exact sequence (2.1), M_* is the same as the p-completion of $\bigoplus_\alpha \Sigma^{n_\alpha} R_*$, and is, in particular, p-torsion-free. By [19, Proposition A.4],

$$L_0\left(\bigoplus_\alpha \Sigma^{n_\alpha} R_*\right)/p = \left(\bigoplus_\alpha \Sigma^{n_\alpha} R_*\right)/p = \bigoplus_\alpha \Sigma^{n_\alpha}(R_*/p),$$

which is clearly free over R_*/p (and flat, in particular).

For the converse, suppose that M_* is L-complete and p-torsion-free and M_*/p is free over R_*/p. Again using (2.1), we see that the natural surjection $M_* \to (M_*)^\wedge_p$ is an isomorphism, so that M_* is honestly p-complete. Choose generators for M_*/p as an R_*/p-module, and lift them to a map

$$\phi : F_* \to M_*$$

from a free graded R_*-module, which is an isomorphism mod p. Again, we observe that $L_0(F_*) = (F_*)^\wedge_p$, that it is p-torsion-free, and that $L_0(F_*)/p = F_*/p$. Applying the snake lemma to the diagram of graded $\mathbb{Z}_p$-modules

$$\begin{tikzcd} 0 \arrow[r] & L_0(F_*) \arrow[r, "p"] \arrow[d, "\phi^\wedge"'] & L_0(F_*) \arrow[r] \arrow[d, "\phi^\wedge"] & F_*/p \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M_* \arrow[r, "p"] & M_* \arrow[r] & M_*/p \arrow[r] & 0, \end{tikzcd}$$

we see that multiplication by p is an isomorphism on $\ker(\phi^\wedge)$ and $\operatorname{coker}(\phi^\wedge)$. Both of these are L-complete graded $\mathbb{Z}_p$-modules, and this implies that they are zero, by [19, Theorem A.6(d,e)]. □

Lemma 2.7 *Let R be a homotopy commutative $K(1)$-local ring spectrum, and let M be a $K(1)$-local R-module. Then M_* is pro-free over R_* if and only if there is an equivalence of $K(1)$-local R-modules,*

$$M \simeq \bigvee \Sigma^{n_\alpha} R.$$

(Here, as always, the coproduct is taken in the $K(1)$-local category).

Proof Suppose that M_* is pro-free over R_*. Choose generators $x_\alpha \in M_{n_\alpha}$ such that the natural map

$$R_*\{x_\alpha\} \to M_*$$

becomes an isomorphism after L-completion. Each x_α corresponds to a map of spectra $S^{n_\alpha} \to M$, and they assemble to a map of $K(1)$-local R-modules

$$\bigvee \Sigma^{n_\alpha} R \to M.$$

This is an equivalence by a result of Hovey [17, Theorem 3.3], which states that the functor π_* sends ($K(1)$-local) coproducts to (L-complete) direct sums. The converse also follows from Hovey's result. □

Note that Hovey's proof uses the same, height-1-specific fact that direct sums are exact in $\mathsf{Mod}^\wedge_{R_*}$.

Proposition 2.8 *Suppose that R is a $K(1)$-local homotopy commutative ring spectrum and M and N are R-modules, such that M_* is pro-free over R_*. Then the natural map of L-complete modules,*

$$M_* \otimes_{R_*} N_* \to \pi_*(M \wedge_R N),$$

is an isomorphism.

Proof By the previous lemma, we can write M as a wedge of suspensions of R,

$$M \simeq \bigvee \Sigma^{n_\alpha} R \simeq R \wedge \bigvee S^{n_\alpha}$$

(using the fact that the $K(1)$-local smash product is a left adjoint, so distributes over the $K(1)$-local coproduct). Thus,

$$M \wedge_R N \simeq N \wedge \bigvee S^{n_\alpha} \simeq \bigvee \Sigma^{n_\alpha} N.$$

Using Hovey's theorem again [17, Theorem 3.3], we obtain

$$\pi_*(M \wedge_R N) \cong L_0 \left(\bigoplus \Sigma^{n_\alpha} N_* \right) \cong L_0(F_* \otimes_{R_*} N_*),$$

where F_* is the free graded R_*-module on generators in the degrees n_α. By [19, A.7],

$$\pi_*(M \wedge_R N) \cong L_0(L_0(F_*) \otimes_{R_*} N_*) \cong M_* \otimes_{R_*} N_*.$$

It is clear that this isomorphism is induced by the natural map. □

2.3 Homological algebra of L-complete Hopf algebroids

We now turn to the problem of homological algebra over an L-complete Hopf algebroid. We begin with some definitions generalizing those of [1].

Definition 2.9 A L-complete Hopf algebroid is a cogroupoid object (R_*, Γ_*) in $\mathsf{CAlg}_*^{\wedge}$, such that Γ_* is pro-free as a left R_*-module. As usual, we write

$$\begin{aligned} \eta_L, \eta_R : R_* &\to \Gamma_* && \text{for the left and right units,} \\ \Delta : \Gamma_* &\to \Gamma_* \otimes_{R_*} \Gamma_* && \text{for the comultiplication,} \\ \epsilon : \Gamma_* &\to R_* && \text{for the counit, and} \\ \chi : \Gamma_* &\to \Gamma_* && \text{for the antipode.} \end{aligned}$$

Note that χ gives an isomorphism between Γ_* as a left R_*-module and Γ_* as a right R_*-module, so that Γ_* is also pro-free as a right R_*-module.

Remark 2.10 We should point out that in this $K(1)$-local setting, we impose the condition that Γ_* is pro-free over R_*, as opposed to the more common condition that Γ_* is flat over R_* in the unlocalized situation. This is required to produce an appropriate L-complete version of Ext (cf. [1,3]). In light of this, we often require a pro-freeness condition rather than a flatness condition (e.g. proposition 2.20).

Remark 2.11 At heights higher than 1, one has to deal with the fact that the left and right units generally do not act in the same way on the generators $(p, u_1, \ldots, u_{n-1})$ with respect to which L-completeness is defined. Thus, Baker's definition has the additional condition that the ideal $(p, u_1, \ldots, u_{n-1})$ is invariant. At height 1, this condition is trivial.

Definition 2.12 Let (R_*, Γ_*) be an L-complete Hopf algebroid. A **left comodule** over (R_*, Γ_*) (a **left Γ_*-comodule** for short) is $M_* \in \mathsf{Mod}^{\wedge}_{R_*}$ together with a coaction map

$$\psi : M_* \to \Gamma_* \otimes_{R_*} M_*$$

such that the diagrams

$$\begin{tikzcd} M_* \arrow[r, "\psi"] \arrow[d, "\psi"'] & \Gamma_* \otimes_{R_*} M_* \arrow[d, "\psi\otimes 1"] \\ \Gamma_* \otimes_{R_*} M_* \arrow[r, "\Delta\otimes 1"'] & \Gamma_* \otimes_{R_*} \Gamma_* \otimes_{R_*} M_* \end{tikzcd} \qquad \begin{tikzcd} M_* \arrow[r, "\psi"] \arrow[dr, equal] & \Gamma_* \otimes_{R_*} M_* \arrow[d, "\epsilon\otimes 1"] \\ & M_* \end{tikzcd}$$

commute. Write $\mathsf{Comod}^{\wedge}_{\Gamma_*}$ for the category of left Γ_*-comodules.

Lemma 2.13 *The category of left Γ_*-comodules is abelian, and the forgetful functor $\mathsf{Comod}^{\wedge}_{\Gamma_*} \to \mathsf{Mod}^{\wedge}_{R_*}$ is exact.*

Proof Suppose that

$$0 \to K_* \to M_* \xrightarrow{f} N_* \to 0$$

is an exact sequence of R_*-modules, and f is a map of Γ_*-comodules. A coaction map can then be defined on K_* via the diagram

$$\begin{tikzcd} 0 \arrow[r] & K_* \arrow[r] \arrow[d, dashed] & M_* \arrow[r, "f"] \arrow[d] & N_* \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Gamma_* \otimes_{R_*} K_* \arrow[r] & \Gamma_* \otimes_{R_*} M_* \arrow[r] & \Gamma_* \otimes_{R_*} N_* \arrow[r] & 0. \end{tikzcd}$$

The bottom sequence is exact because Γ_* is flat in $\mathsf{Mod}^{\wedge}_{R_*}$, by Lemma 2.4. One checks that this structure makes K_* a comodule by the usual diagram chase. A similar proof works for cokernels. $\square$

Again, this is generally not true at heights higher than 1, because Γ_* may not be flat—see [3, Sect. 2.2].

Definition 2.14 An extended comodule is one of the form

$$M_* = \Gamma_* \otimes_{R_*} N_*,$$

where $N_* \in \mathsf{Mod}^{\wedge}_{R_*}$, with coaction $\Delta \otimes 1_{N_*}$.

When working with uncompleted Hopf algebroids, one next constructs enough injectives in the comodule category by showing that a comodule extended from an injective R_*-module is injective [27, A1.2.2]. One cannot do this in this case, because $\mathsf{Mod}^{\wedge}_*$ does not have enough injectives [2, Section A.2]. For example, if I is an injective L-complete $\mathbb{Z}_p$-module containing a copy of $\mathbb{Z}/p$, then one can inductively construct extensions $\mathbb{Z}/p^n \to I$ and thus a nonzero map $\mathbb{Z}/p^\infty \to I$—but this means that I is not L-complete. Thus, one instead has to use relative homological algebra. We take the following definitions from [3, Sect. 2].

Definition 2.15 A relative injective comodule is a retract of an extended comodule. A relative monomorphism of comodules is a comodule map $M_* \to N_*$ which is a split injection as a map of R_*-modules. A relative short exact sequence is a sequence

$$M_* \xrightarrow{f} N_* \xrightarrow{g} P_*$$

where the image of f is the kernel of g, and f is a relative monomorphism. A relative injective resolution of a comodule M_* is a sequence

$$M_* = J_*^{-1} \to J_*^0 \to J_*^1 \to \cdots$$

where

- each J_*^s is relative injective for $s \geq 0$,
- each composition $J_*^{s-1} \to J_*^s \to J_*^{s+1}$ is zero,
- and if C_*^s is the cokernel of $J_*^{s-1} \to J_*^s$, the sequences

$$C_*^{s-1} \to J_*^s \to C_*^s$$

are relatively short exact.

Definition 2.16 Let M_* and N_* be two comodules over (R_*, Γ_*). Let $J_*^\bullet$ be a relative injective resolution of N_*. Define

$$\widehat{\mathrm{Ext}}^*_{\Gamma_*}(M_*, N_*)$$

to be the cohomology of the complex $\mathrm{Hom}_{\mathsf{Comod}^{\wedge}_{\Gamma_*}}(M_*, J_*^\bullet)$.

Following our general conventions, we will simply write $\mathrm{Ext}_{\Gamma_*}(M_*, N_*)$ for this functor, where this does not cause confusion.

Proposition 2.17

(a) *Every comodule has a relative injective resolution.*
(b) *The definition of* $\widehat{\mathrm{Ext}}$ *above is independent of the choice of resolution.*
(c) *We have*

$$\mathrm{Ext}^0_{\Gamma_*}(M_*, N_*) = \mathrm{Hom}_{\mathit{Comod}^{\wedge}_{\Gamma_*}}(M_*, N_*).$$

(d) *If* N_* *is relatively injective, then* $\mathrm{Ext}^s_{\Gamma_*}(M_*, N_*)$ *vanishes for* $s > 0$.
(e) *If* N_* *is an extended comodule* $\Gamma_* \otimes_{R_*} K_*$ *for an* R_**-module* K_**, then*

$$\mathrm{Ext}^0_{\Gamma_*}(M_*, N_*) = \mathrm{Hom}_{\mathit{Mod}^{\wedge}_{R_*}}(M_*, K_*).$$

Proof The first three statements follow from identical arguments to those in [3, 2.11, 2.12, 2.15]. (One should note, in particular, that if M_* is a comodule, the coaction

$$M_* \to \Gamma_* \otimes_{R_*} M_*$$

is a relative monomorphism into a relative injective.) Statement (d) is then trivial, as we can take N_* to be its own relative injective resolution. For (e), we use (c) and the adjunction

$$\mathrm{Hom}_{\mathsf{Comod}^{\wedge}_{\Gamma_*}}(M_*, \Gamma_* \otimes_{R_*} K_*) \cong \mathrm{Hom}_{\mathsf{Mod}^{\wedge}_{R_*}}(M_*, K_*).$$

□

Definition 2.18 The primitives of a comodule M_* are the R_*-module

$$\mathrm{Ext}^0_{\Gamma_*}(R_*, M_*),$$

which are naturally identified with a sub-R_*-module of M_*. If M_* is extended, $M_* = \Gamma_* \otimes_{R_*} K_*$, then the primitives of M_* are the submodule $1 \otimes K_*$.

In the following lemma and proof, all tensor products are over R_*.

Lemma 2.19 *A tensor product of an extended comodule with an arbitrary comodule is extended. More precisely, if* $M_* \in \mathit{Mod}^{\wedge}_{R_*}$ *and* $C_* \in \mathit{Comod}^{\wedge}_{\Gamma_*}$*, there is a natural isomorphism*

$$(\Gamma_* \otimes M_*) \otimes C_* \to \Gamma_* \otimes (M_* \otimes C_*)$$

where the source has diagonal coaction and the target is extended. The map

$$M_* \otimes C_* \xrightarrow{1\otimes\psi} M_* \otimes \Gamma_* \otimes C_* \xrightarrow{\mathrm{swap}\otimes 1} \Gamma_* \otimes M_* \otimes C_* \xrightarrow{\chi\otimes 1\otimes 1} \Gamma_* \otimes M_* \otimes C_*$$

induces an isomorphism

$$M_* \otimes C_* \cong \mathrm{Ext}^0_{\Gamma_*}(R_*, (\Gamma_* \otimes M_*) \otimes C_*).$$

Proof This is an L-complete version of [15, Lemma 1.1.5], and the same proof works here. The formula for the primitives follows from the following observations. Define

$$g : \Gamma_* \otimes (M_* \otimes C_*) \xrightarrow{1\otimes 1\otimes \psi} \Gamma_* \otimes (M_* \otimes \Gamma_* \otimes C_*)$$
$$\xrightarrow{1\otimes \mathrm{swap}\otimes 1} \Gamma_* \otimes \Gamma_* \otimes M_* \otimes C_* \xrightarrow{\mathrm{mult}\circ(1\otimes\chi)\otimes 1\otimes 1} (\Gamma_* \otimes M_*) \otimes C_* = T_*.$$

(Note that the map

$$\mathrm{mult} \circ (1 \otimes \chi) : \Gamma_* \otimes_{R_*} \Gamma_* \to \Gamma_*$$

is part of the structure of the Hopf algebroid (R_*, Γ_*), though the multiplication on Γ_* itself may not factor through the R_*-module tensor product $\Gamma_* \otimes \Gamma_*$.)

For fixed C_*, g is a natural transformation of functors of M_* valued in $\mathsf{Mod}^{\wedge}_{R_*}$. In the case $M_* = R_*$, it is an isomorphism (and is precisely the inverse given in Hovey's proof). Thus, g is an isomorphism for all pro-free modules M_*, using exactness of the direct sum, and an isomorphism for all M_* using the right exactness of the tensor product. □

Proposition 2.20 *Let R be a $K(1)$-local homotopy commutative ring spectrum such that R_*R is pro-free over R_*.Then for any $K(1)$-local spectrum X, the $K(1)$-local R-based Adams spectral sequence for X has E_2 page*

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{R_*R}(R_*, R_*X).$$

Proof This spectral sequence is the same as the Bousfield-Kan homotopy spectral sequence of the cosimplicial object

$$C^\bullet := R^{\wedge \bullet+1} \wedge X.$$

This is of the form

$$E_1^{*,*} = \pi_*(R^{\wedge *+1} \wedge X) \Rightarrow \pi_* \mathrm{Tot}(C^\bullet).$$

By Proposition 2.8, we have

$$\pi_*(R^{\wedge s+1} \wedge X) = R_*R^{\otimes_{R_*} s} \otimes_{R_*} R_*X,$$

which is a resolution of R_*X by extended comodules, so that the E_2 page is precisely $\mathrm{Ext}^{*,*}_{R_*R}(R_*, R_*X)$. □

We next discuss convergence of the spectral sequence. The Bousfield-Kan spectral sequence converges conditionally to the homotopy of its totalization, so this spectral sequence converges conditionally to π_*X if and only if the map

$$X \to \operatorname{holim} R^{\wedge \bullet+1} \wedge X$$

is an equivalence. Questions of this type were first studied by Bousfield [6], and in the local case by Devinatz-Hopkins [10]. We recall their definitions here:

Definition 2.21 [10, Appendix I] Let R be a $K(1)$-local homotopy commutative ring spectrum. The class $K(1)$-local R-nilpotent spectra is the smallest class $\mathcal{C}$ of $K(1)$-local spectra such that:

(1) $R \in \mathcal{C}$,
(2) $\mathcal{C}$ is closed under retracts and cofibers,
(3) and if $X \in \mathcal{C}$ and Y is an arbitrary $K(1)$-local spectrum, then $X \wedge Y \in \mathcal{C}$.

Proposition 2.22 [10, Appendix I] *Assume that X is $K(1)$-local R-nilpotent. Then the $K(1)$-local R-based Adams spectral sequence converges conditionally to $\pi_* X$.*

Finally, we write down a change of rings theorem, generalizing [18, Theorem 3.3].

Proposition 2.23 *Suppose that $f : (A, \Gamma_A) \to (B, \Gamma_B)$ is a morphism of L-complete Hopf algebroids such that the natural map*

$$B \otimes_A \Gamma_A \otimes_A B \to \Gamma_B$$

is an isomorphism, and such that there exists a map $g : B \otimes_A \Gamma \to A$ such that the composition

$$A \overset{1\otimes\eta_R}{\to} B \otimes_A \Gamma \overset{g}{\to} A$$

is the identity. Then for any Γ_A-comodule M, the induced map

$$\mathrm{Ext}^*_{\Gamma_A}(A, M) \to \mathrm{Ext}^*_{\Gamma_B}(B, B \otimes_A M)$$

is an isomorphism.

This statement can probably be obtained via the method of [16], but rather than taking a further detour into L-complete stacks, we have instead followed [9] (where this theorem is proved in the very similar setting of complete Hopf algebroids). We begin with some definitions and lemmas.

In the standard fashion, an L-complete Hopf algebroid (A, Γ) defines a functor

$$h_{(A,\Gamma)} : \mathsf{CAlg}^{\wedge}_* \to \mathsf{Grpd},$$

in which the objects of $h_{(A,\Gamma)}(R)$ are the ring homomorphisms $A \to R$, and the morphisms of $h_{(A,\Gamma)}(R)$ are the ring homomorphisms $\Gamma \to R$. Moreover, a morphism of L-complete Hopf algebroids, $f : (A, \Gamma_A) \to (B, \Gamma_B)$, induces a natural transformation $f^* : h_{(B,\Gamma_B)} \to h_{(A,\Gamma_A)}$.

Lemma 2.24 *Let $\phi : h_{(B,\Gamma_B)} \to h_{(A,\Gamma_A)}$ be a natural transformation of functors $\mathsf{CAlg}^{\wedge}_* \to \mathsf{Grpd}$. Then there is a morphism $f : (A, \Gamma_A) \to (B, \Gamma_B)$ such that $\phi = f^*$.*

Proof This is a variant of the Yoneda lemma. One can find the ring map $A \to B$ by evaluating ϕ on the object of $h_{(B,\Gamma_B)}(B)$ corresponding to id_B, and likewise one can find the map $\Gamma_A \to \Gamma_B$ by evaluating ϕ on the morphism of $h_{(B,\Gamma_B)}(\Gamma_B)$ corresponding to id_{Γ_B}. That these define an actual morphism of Hopf algebroids requires checking the commutativity of various diagrams of L-complete rings, which can be done in a similar fashion. □

As Grpd is really a (2,1)-category, the functor category $\mathsf{Fun}(\mathsf{CAlg}_*^\wedge, \mathsf{Grpd})$ is as well. We say that a morphism of L-complete Hopf algebroids is an equivalence if it is an equivalence in this functor category. In other words, $f : (A, \Gamma_A) \to (B, \Gamma_B)$ is an equivalence iff there is a morphism $g : (B, \Gamma_B) \to (A, \Gamma_A)$ and natural 2-equivalences

$$\mathrm{id}_{h_{(A,\Gamma_A)}} \Rightarrow f^* g^*, \quad g^* f^* \Rightarrow \mathrm{id}_{h_{(B,\Gamma_B)}}.$$

Lemma 2.25 *Suppose that $f, g : (A, \Gamma_A) \to (B, \Gamma_B)$ are morphisms, and $\tau : f^* \to g^*$ is a natural 2-equivalence. Then τ induces a natural equivalence of base change functors $\mathsf{Comod}^\wedge_{\Gamma_A} \to \mathsf{Comod}^\wedge_{\Gamma_B}$,*

$$\tau^* : B\,{}^g\!\otimes_A M \xrightarrow{\cong} B\,{}^f\!\otimes_A M.$$

Moreover, $\tau^* g^*$ and f^* induce the same map on cohomology,

$$\mathrm{Ext}^*_{\Gamma_A}(A, M) \to \mathrm{Ext}^*_{\Gamma_B}(B, B\,{}^f\!\otimes_A M).$$

Proof This is the L-complete verison of [9, 1.15, 1.17], and has the same proof. □

Lemma 2.26 *Suppose that $f : (A, \Gamma_A) \to (B, \Gamma_B)$ is an equivalence of L-complete Hopf algebroids. Then the base change functor*

$$B \otimes_A \cdot : \mathsf{Comod}^\wedge_{\Gamma_A} \to \mathsf{Comod}^\wedge_{\Gamma_B}$$

is an equivalence of categories. Moreover, the induced map

$$\mathrm{Ext}^*_{\Gamma_A}(A, M) \to \mathrm{Ext}^*_{\Gamma_B}(B, B \otimes_A M)$$

is an isomorphism.

Proof This follows immediately from the previous lemma. □

Lemma 2.27 *Let $\mathcal{C}$ be a small category. Suppose that $f : F \to G$ is a natural transformation of functors $\mathcal{C} \to$ Grpd such that:*

(i) *$f_c : F(c) \to G(c)$ is fully faithful for each $c \in \mathcal{C}$;*
(ii) *for each $c \in \mathcal{C}$ and $x \in G(c)$, there is an essential lift of x – in other words, a pair*

$$(\widetilde{x} \in F(c), \alpha_x : f_c(\widetilde{x}) \xrightarrow{\sim} x);$$

(iii) *and these lifts can be chosen functorially in $c \in \mathcal{C}$. In other words, there is a choice of essential lift for every c and x such that, given $h : c \to d$ in $\mathcal{C}$ and $x \in F(c)$, the essential lift of $h(x) \in F(d)$ is $(F(h)(\widetilde{x}), G(h)(\alpha_x))$.*

Then f is an equivalence in the functor 2-category $\mathsf{Fun}(\mathcal{C}, \mathsf{Grpd})$.

Remark 2.28 If f is only assumed objectwise fully faithful and essentially surjective, then an attempt to construct an inverse will, in general, only produce a pseudonatural transformation $g : G \to F$. Thus, some additional hypothesis like (iii) above is required.

Proof of Lemma 2.27 Let $(\widetilde{x}, \alpha_x)$ be the lifts functorial in $\mathcal{C}$ described in (ii) and (iii). Define $g : G \to F$ as follows:

- For each $x \in G(c)$, $g_c(x) = \widetilde{x}$.
- For each morphism $\phi : x \to y$ in $G(c)$, $g_c(\phi) : \widetilde{x} \to \widetilde{y}$ is the unique morphism such that $f_c(g_c(\phi))$ is the composite

$$f_c(\widetilde{x}) \overset{\alpha_x}{\to} x \overset{\phi}{\to} y \overset{\alpha_y}{\leftarrow} f_c(\widetilde{y}).$$

  This exists by (i).

One has to check that g is a natural transformation, or in other words that, given $h : c \to d$ in $\mathcal{C}$, $g_d \circ G(\phi) = F(\phi) \circ g_c$. This is an immediate consequence of (iii).

It remains to show that f and g are inverse equivalences, or in other words that there are natural transformations

$$\epsilon : f_c g_c \Rightarrow \mathrm{id}_{G(c)}, \qquad \eta : \mathrm{id}_{F(c)} \Rightarrow g_c f_c$$

that are natural in c. For $x \in G(c)$, define

$$\epsilon_x = \alpha_x : f_c g_c(x) \to x.$$

For $y \in F(c)$, let $\eta_y : y \to g_c f_c(y)$ be the unique lift of the identity morphism $f_c(y) \to f_c(y) = f_c g_c f_c(y)$. It is easy to check that these are natural transformations for each c, and natural in c. □

Proof of Proposition 2.23 It suffices to prove that f is an equivalence of Hopf algebroids. We do this by checking the conditions of Lemma 2.27, meaning that for each R, the functor of groupoids

$$f^* : h_{(B,\Gamma_B)}(R) \to h_{(A,\Gamma)}(R)$$

is fully faithful and essentially surjective, and the essential lifts can be chosen functorially in R.

Given $x, y \in h_{(B,\Gamma_B)}(R)$, we can identify Maps$(x, y)$ with the set of maps $\phi : \Gamma_B \to R$ such that $\phi\eta_L = x$, $\phi\eta_R = y$. Since $\Gamma_B = B \otimes_A \Gamma \otimes_A B$, such a map is equivalent to a map $\Gamma \to R$ such that $\phi\eta_L = f^*(x)$, $\phi\eta_R = f^*(y)$. This proves that f^* is fully faithful.

An object $x \in h_{(A,\Gamma)}(R)$ is given by a ring map $x : A \to R$. Precomposing with $g : B \otimes_A \Gamma \to A$ gives $xg : B \otimes_A \Gamma \to R$, which corresponds to an object $y \in h_{(B,\Gamma_B)}(R)$ and an isomorphism of $h_{(A,\Gamma)}(R)$ with source $f^*(y)$. Since $g(f \otimes_R \eta_R) = \mathrm{id}_A$, the target of this isomorphism is x. Thus, f^* is essentially surjective. Moreover, as the essential lifts are given by precomposing with a morphism of rings, they are clearly functorial in R. □

2.4 The Hopf algebroids for *K* and *KO*

The K-theory spectrum has a group action by $\mathbb{Z}_p^\times$ via E_∞ ring maps. For $k \in \mathbb{Z}_p^\times$, we write ψ^k for the corresponding endomorphism of K, called the kth Adams operation. On homotopy, writing u for the Bott element, we have

$$\psi^k : K_* \to K_* : \quad u^n \mapsto k^n u^n. \tag{2.29}$$

The group $\mathbb{Z}_p^\times$ has a maximal finite subgroup μ of order $p-1$ (if p is odd) or 2 (if $p = 2$), and we write $KO = K^{h\mu}$. (This agrees with the p-completion of the real K-theory spectrum at $p = 2$ and 3). Then KO inherits an action by the topologically cyclic group $\mathbb{Z}_p^\times/\mu$, which we also refer to as an action by Adams operations.

The Adams operations give us a way to analyze the completed cooperations algebras K_*K and KO_*KO. Define

$$\Phi_K : K_*K \to \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p^\times, K_*)$$

as adjoint to the map

$$K_*K \times \mathbb{Z}_p^\times \to K_*$$

defined by taking an element $x : S^t \to K \wedge K$ and p-adic unit $k \in \mathbb{Z}_p^\times$ to the composite

$$S^t \xrightarrow{x} K \wedge K \xrightarrow{K \wedge \psi^k} K \wedge K \xrightarrow{m} K$$

Likewise, there is a map

$$\Phi_{KO} : KO_*KO \to \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p^\times/\mu, KO_*).$$

Theorem 2.30 (cf. [16]) *The map*

$$\Phi_K : K_*K \to \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, K_*)$$

is an isomorphism. It induces an isomorphism of Hopf algebroids

$$(K_*, K_*K) \cong (K_*, \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p^\times, K_*)),$$

where the latter has the following Hopf algebroid structure:

- *The left unit* $\eta_L : K_* \to \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p^\times, K_*)$ *is the inclusion of constant functions.*
- *The right unit* $\eta_R : K_* \to \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p^\times, K_*)$ *sends* x *to the function* $a \mapsto \psi^a(x)$.
- *The coproduct,*

$$\begin{aligned}\Delta : \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p^\times, K_*) &\to \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p^\times, K_*) \otimes \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p^\times, K_*)\\ &\cong \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times, K_*),\end{aligned}$$

  is given by sending a function f *to the function* $(a, b) \mapsto f(ab)$.
- *The antipode* $\mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p^\times, K_*) \to \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p^\times, K_*)$ *sends a function* f *to* $a \mapsto f(a^{-1})$.
- *The augmentation map* $\mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p^\times, K_*) \to K_*$ *is given by evaluation at* 1.

Analogous statements hold for KO.

Note that $\eta_L = \eta_R$ in degree zero, so that K_0K and KO_0KO are Hopf algebras.

Remark 2.31 The reader should note that it follows from Mahler's theorem that K_*K and KO_*KO are pro-free over K_* and KO_* respectively.

Remark 2.32 The cooperations algebra K_*K carries *two* actions by Adams operations, coming from the two copies of K. Given $f \in K_0K$, we can represent f both as a map $f : S^0 \to K \wedge K$ and as an element of $\mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p^\times, K_0)$. Then, for $a, b \in \mathbb{Z}_p^\times$, we have

$$((\psi^a \wedge K) \circ f)(b) = f(ab) \tag{2.33}$$

and

$$((K \wedge \psi^a) \circ f)(b) = \psi^a(f(a^{-1}b)). \tag{2.34}$$

Now suppose that M_* is an L-complete K_*K-comodule with coaction ψ_{M_*}. Then there is a map

$$\begin{aligned}M_* &\overset{\psi_{M_*}}{\to} K_*K \otimes_{K_*} M_*\\ &\cong \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p^\times, K_*) \otimes_{K_*} M_*\\ &\cong \mathrm{Hom}_{\mathsf{Mod}_*^\wedge}(\mathbb{Z}_p[[\mathbb{Z}_p^\times]], K_*) \otimes_{K_*} M_*\\ &\to \mathrm{Hom}_{\mathsf{Mod}_*^\wedge}(\mathbb{Z}_p[[\mathbb{Z}_p^\times]], M_*).\end{aligned}$$

Here $\mathrm{Hom}_{\mathsf{Mod}_*^\wedge}$ is the ordinary space of maps between $\mathbb{Z}_p$-modules, which is automatically L-complete when the modules are L-complete [2, Sect. A.2]. As $\mathsf{Mod}_*^\wedge$ is closed symmetric monoidal, this map is adjoint to one of the form

$$M_* \otimes \mathbb{Z}_p[[\mathbb{Z}_p^\times]] \to M_*. \tag{2.35}$$

In the case where M_* is p-complete, this defines a continuous group action by $\mathbb{Z}_p^\times$ on M_*. If M_* is merely L-complete, then one still gets a group action by $\mathbb{Z}_p^\times$ on M_*, and the only reasonable definition of "continuous group action" appears to be that it extends to a map of L-complete modules of the form (2.35). In either case, we call this the action by Adams operations on M_*. Of course, if M_* is the completed K-theory of a spectrum X, $M_* = \pi_* L_{K(1)}(K \wedge X)$, then this action is induced by the Adams operations on K.

If M_* is p-complete then the standard relative injective resolution of M_*,

$$M_* \to K_*K \otimes_{K_*} M_* \to K_*K \otimes_{K_*} K_*K \otimes_{K_*} M_* \to \cdots,$$

is isomorphic to the complex of continuous $\mathbb{Z}_p^\times$-cochains,

$$M_* \to \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p^\times, M_*) \to \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times, M_*).$$

Thus, we can identify the relative Ext of Definition 2.16 with continuous group cohomology:

$$\mathrm{Ext}^{*,*}_{K_*K}(K_*, M_*) = H^*_{cts}(\mathbb{Z}_p^\times, M_*).$$

Similar remarks apply to KO: a KO_*KO-comodule M_* has a continuous group action by $\mathbb{Z}_p^\times/\mu$, and if M_* is p-complete, we have

$$\mathrm{Ext}^{*,*}_{KO_*KO}(KO_*, M_*) = H^*_{cts}(\mathbb{Z}_p^\times/\mu, M_*).$$

(Again, if M_* is merely L-complete, then one should instead take these Ext groups as a definition of continuous group cohomology with coefficients in M_*!)

One recovers the familiar $K(1)$-local Adams spectral sequences based on KO as an immediate consequence.

Proposition 2.36 *Let X be a $K(1)$-local spectrum. Then there is a strongly convergent Adams spectral sequence*

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{KO_*KO}(KO_*, KO_*X) = H^s_{cts}(\mathbb{Z}_p^\times/\mu, KO_tX) \Rightarrow \pi_{t-s}X.$$

The spectral sequence always collapses at the E_2 page.

Proof The calculation of the E_2 pages follows from the above discussion and Proposition 2.20. Since $\mathbb{Z}_p^\times/\mu$ has cohomological dimension 1, the spectral sequence collapses at E_2, and in particular, converges strongly. To establish that the limit is π_*X, we must show that every $K(1)$-local X is $K(1)$-local KO-nilpotent (see Proposition 2.22). But the sphere is a fiber of copies of KO (see (3.1) below), so S is $K(1)$-local KO-nilpotent, so the same is true for arbitrary X. $\square$

3 Cones on ζ

There is a class ζ in $\pi_{-1}L_{K(1)}S$ which vanishes in the homotopy of $K(1)$-local tmf (as well as $K(1)$-local K and KO), simply because these spectra have no nontrivial homotopy in degree -1. As a result, the cone $C(\zeta)$ and the E_∞-cone T_ζ on ζ mediate between the sphere and tmf. We describe these spectra in this section, which is mostly an exposition of material found in [14].

3.1 The spectrum cone on ζ

Recall from Sect. 1.2 that g is a fixed topological generator of $\mathbb{Z}_p^\times$ (or, when $p = 2$, a fixed element of $\mathbb{Z}_2^\times$ mapping to a topological generator of $\mathbb{Z}_2^\times/\mu$), and that ω is a fixed generator of μ. The fiber sequence

$$S \longrightarrow KO \xrightarrow{\psi^g - 1} KO \tag{3.1}$$

gives a long exact sequence on homotopy groups

$$\cdots \longrightarrow \pi_n S \longrightarrow \pi_n KO \xrightarrow{\psi^g - 1} \pi_n KO \xrightarrow{\partial} \pi_{n-1} S \longrightarrow \cdots .$$

Recall that the action of ψ^g on $\pi_0 KO$ is trivial, so the connecting homomorphism gives an isomorphism

$$\mathbb{Z}_p = \pi_0 KO \cong \pi_{-1} S.$$

This isomorphism does depend on the choice of topological generator g. We let $\zeta := \partial(1)$, and we define $C(\zeta)$ to be the cone on ζ, i.e. the cofibre

$$S^{-1} \xrightarrow{\zeta} S \longrightarrow C(\zeta).$$

Since $\pi_{-1}KO = 0$, we get a morphism of cofibre sequences

$$\begin{tikzcd}
S^{-1} \arrow[r, "\zeta"] & S^0 \arrow[r] \arrow[d, "="] & C(\zeta) \arrow[r, "\delta"] \arrow[d, "\iota"] & S^0 \arrow[r, "\zeta"] \arrow[d, "\eta"] & S^1 \arrow[d, "="] \\
 & S^0 \arrow[r, "\eta"] & KO \arrow[r, "\psi^g - 1"] & KO \arrow[r] & S^1
\end{tikzcd} \tag{3.2}$$

The morphism ι is a nullhomotopy of $\eta \circ \zeta$.

Since ζ is nullhomotopic in KO, the top cofibre sequence in (3.2) splits after smashing with KO, giving $KO \wedge C(\zeta) \simeq KO \wedge (S^0 \vee S^0)$. In fact, there is a canonical splitting, coming from the diagram

$$\begin{tikzcd}
KO \wedge S^0 \arrow[r] \arrow[d, "="] & KO \wedge C(\zeta) \arrow[r, "KO\wedge\delta"] \arrow[d, "KO\wedge\iota"] & KO \wedge S^0 \arrow[d, "KO\wedge\eta"] \\
KO \wedge S^0 \arrow[r, "KO\wedge\eta"] \arrow[dr, "="'] & KO \wedge KO \arrow[r, "KO\wedge(\psi^g-1)"] \arrow[d, "m"] & KO \wedge KO \\
& KO &
\end{tikzcd} \tag{3.3}$$

We see that

$$m \circ (KO \wedge \iota) : KO \wedge C(\zeta) \to KO$$

splits the inclusion $KO \to KO \wedge C(\zeta)$. Thus, we can choose classes $a, b \in KO_0 C(\zeta)$ by

$$\begin{aligned} m(KO \wedge \iota)(a) &= 1, & (KO \wedge \delta)(a) &= 0, \\ m(KO \wedge \iota)(b) &= 0, & (KO \wedge \delta)(b) &= -1, \end{aligned}$$

and $\{a, b\}$ is a KO_*-module basis for $KO_* C(\zeta)$.

Proposition 3.4 *Under the morphism*

$$KO \wedge \iota : KO_0 C(\zeta) \to KO_0 KO = \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p^\times/\mu, \mathbb{Z}_p),$$

the element a is mapped to the constant function 1 and b is mapped to the unique group homomorphism sending g to 1.

Proof We use the formulas from Theorem 2.30 and (2.34). For the sake of brevity, let $\overline{a}$ and $\overline{b}$ be the images of a and b under $KO \wedge \iota$, which we think of as continuous functions from the topologically cyclic group $\mathbb{Z}_p^\times/\mu$ to $\mathbb{Z}_p$. Since $m(\overline{a}) = 1$, by Theorem 2.30, the function $\overline{a}$ satisfies $\overline{a}(1) = 1$. We also have

$$(KO \wedge (\psi^g - 1))(\overline{a}) = (KO \wedge \eta)(KO \wedge \delta)(a) = 0,$$

and by (2.34), together with the fact that ψ^g acts trivially on KO_0,

$$\overline{a}(g^{-1}n) - \overline{a}(n) = 0$$

for any $n \in \mathbb{Z}_p^\times/\mu$. Together with continuity of $\overline{a}$, this implies that $\overline{a}$ is constant.

Applying the same arguments to $\overline{b}$, we obtain

$$\overline{b}(1) = 0, \quad \overline{b}(g^{-1}n) - \overline{b}(n) = -1.$$

It follows that

$$\overline{b}(g^k) = k$$

for any $k \in \mathbb{Z}$, and by continuity, for any $k \in \mathbb{Z}_p$. $\square$

Corollary 3.5 *The map*

$$\iota_* : KO_0C(\zeta) \to KO_0KO$$

is injective.

Proof One just has to observe that the functions $\overline{a}$, $\overline{b}$ are linearly independent in KO_0KO. □

Corollary 3.6 *In $KO_*C(\zeta)$, the Adams operations fix a and $\psi^g(b) = b + a$.*

Proof By the previous corollary, the Adams operations can be calculated in KO_0KO, where they are given by (2.33). □

Corollary 3.7 *We have*

$$K_*C(\zeta) \cong K_*\{a, b\},$$

where the Adams operations fix a and satisfy

$$\psi^g(b) = b + a, \quad \psi^\omega(b) = b.$$

Proof The KO-module $KO \wedge C(\zeta)$ is free on the generators $\{a, b\}$, so $K \wedge C(\zeta)$ is free on the same generators as a K-module. Since the generators of $K_*C(\zeta)$ are in the image of $KO_*C(\zeta)$, they are fixed by ψ^ω. □

3.2 The E_∞-cone on ζ

The previous subsection allows us to start the analysis of the E_∞-cone on ζ. Recall from Sect. 1.2 that we write CAlg for the category of $K(1)$-local E_∞-algebras, and $\mathbb{P}(X)$ for the free E_∞-algebra on X.

Definition 3.8 The spectrum T_ζ is defined by the following homotopy pushout square in the category CAlg.

$$\begin{tikzcd} \mathbb{P}(S^{-1}) \arrow[r, "0"] \arrow[d, "\zeta"'] & S^0 \arrow[d] \\ S^0 \arrow[r] & T_\zeta \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{3.9}$$

Just as $C(\zeta)$ classifies nullhomotopies of ζ in spectra equipped with a map from S^0, T_ζ classifies nullhomotopies of ζ in E_∞-algebras. That is, there is a natural equivalence of mapping spaces

$$\mathsf{CAlg}(T_\zeta, R) \simeq \mathsf{Sp}_{S^0/}(C(\zeta), R),$$

where $\mathsf{Sp}_{S^0/}$ is the category of spectra equipped with a map from S^0. In particular, there is a canonical morphism $C(\zeta) \to T_\zeta$, and a canonical factorization

$$\begin{tikzcd} C(\zeta) \arrow[r] \arrow[rr, bend right, "\iota"'] & T_\zeta \arrow[r, "\pi"] & KO, \end{tikzcd} \tag{3.10}$$

making KO a commutative T_ζ-algebra. We also have the following.

Proposition 3.11 *Let R be any E_∞-algebra such that $\pi_{-1}R = 0$. Then there is an equivalence in CAlg_R:*

$$R \wedge T_\zeta \simeq R \wedge \mathbb{P}(S^0).$$

Proof Smashing R with the pushout diagram for T_ζ produces a pushout diagram

$$\begin{tikzcd} R \wedge \mathbb{P}(S^{-1}) \arrow[r, "*"] \arrow[d, "R\wedge\zeta"] & R \wedge S^0 \arrow[d] \\ R \wedge S^0 \arrow[r] & R \wedge T_\zeta \end{tikzcd}$$

Observe the equivalence $\mathbb{P}_R(R \wedge S^0) \simeq R \wedge \mathbb{P}(S^0)$. Note that $R \wedge \zeta$ is adjoint to the map

$$R \wedge \zeta : R \wedge S^{-1} \to R \wedge S^0$$

in R-modules. This morphism is itself adjoint to the map

$$\zeta : S^{-1} \to S^0 \to R \wedge S^0$$

in S-modules. As $\pi_{-1}R = 0$, this map is null, which implies $R \wedge \zeta$ is null in R-modules. Thus the morphism $R \wedge \zeta$ in CAlg_R is adjoint to the null morphism. So the pushout diagram is in fact the pushout of the following

$$\begin{tikzcd} \mathbb{P}_R(R \wedge S^{-1}) \arrow[r, "0"] \arrow[d, "0"] & R \\ R & \end{tikzcd}$$

which gives $R \wedge \mathbb{P}(S^0)$. □

Corollary 3.12 *There is an equivalence of KO-algebras*

$$KO \wedge T_\zeta \simeq KO \wedge \mathbb{P}(S^0).$$

More explicitly, we can choose this equivalence so that the following diagram commutes:

$$\begin{array}{ccc} KO \wedge C(\zeta) & \xrightarrow{a\vee b} & KO \wedge (S^0 \vee S^0) \\ \downarrow & & \downarrow \\ KO \wedge T_\zeta & \longrightarrow & KO \wedge \mathbb{P}(S^0) \end{array}$$

Here, the map

$$S^0 \vee S^0 \to \mathbb{P}(S^0)$$

is the unit on the left summand, and the inclusion of the generator on the right one. This allows us to calculate the KO-homology of T_ζ completely.

Corollary 3.13 *As a θ-algebra over KO_*,*

$$KO_* T_\zeta \cong KO_* \otimes \mathbb{T}(b), \ \textit{with}\ \psi^g(b) = b + 1,$$

where b is the image of the element of $KO_0C(\zeta)$ described in Proposition 3.4. Likewise,

$$K_* T_\zeta \cong K_* \otimes \mathbb{T}(b), \ \textit{with}\ \psi^g(b) = b + 1, \ \ \psi^\omega(b) = b.$$

Proof This is a consequence of the E_∞ equivalence $KO \wedge T_\zeta \simeq KO \wedge \mathbb{P}(S^0)$, McClure's theorem A.6, and the commutativity of (3.2). Since b is in the image of $KO_0C(\zeta)$, its Adams operations follow from Corollary 3.6. As the Adams operations on KO_* are known and ψ^g commutes with θ, the calculation of $\psi^g(b)$ determines the Adams operations on all of $KO_*T_\zeta = KO_* \otimes \mathbb{T}(b)$. Tensoring up to K, one also gets the formula for K_*T_ζ. □

3.3 The homotopy groups of T_ζ

In this subsection we compute the homotopy groups of T_ζ. This has been done before in [14] and [21]. As this calculation is important for the work on co-operations to follow, we review it here in detail.

We may approach the homotopy groups of T_ζ using the KO-based Adams spectral sequence, which we saw in Proposition 2.36 takes the form

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{KO_*KO}(KO_*, KO_*T_\zeta) = H^s_{cts}(\mathbb{Z}_p^\times/\mu; KO_t T_\zeta) \implies \pi_{t-s}T_\zeta. \quad (3.14)$$

The key point of Hopkins' calculation in [14] is as follows:

Theorem 3.15 [14,21]. *The KO-homology of T_ζ is an extended KO_*KO-comodule. More specifically, there is an isomorphism of KO_*KO-comodules*

$$KO_*T_\zeta \cong KO_*KO \otimes \mathbb{T}(f) \cong \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p^\times/\mu, \mathbb{Z}_p) \otimes KO_* \otimes \mathbb{T}(f),$$

where $f = \psi^p(b) - b$, and $\mathbb{T}(f)$ has trivial coaction.

This allows an immediate derivation of $\pi_* T_\zeta$.

Corollary 3.16 *The homotopy groups of T_ζ are*

$$\pi_* T_\zeta \cong KO_* \otimes \mathbb{T}(f).$$

Proof By Proposition 2.17, the cohomology of an extended comodule is concentrated in degree zero, and

$$\begin{aligned}&\mathrm{Ext}^0_{KO_*KO}(KO_*, KO_*KO \otimes \mathbb{T}(f))\\ &\quad = \mathrm{Hom}_{KO_*}(KO_*, KO_* \otimes \mathbb{T}(f)) = KO_* \otimes \mathbb{T}(f).\end{aligned}$$

□

The proof of Theorem 3.15 will take up the remainder of this section. As it is somewhat involved, let us give an outline first. The map $\pi : T_\zeta \to KO$ induces a map

$$KO_0 T_\zeta \to KO_0 KO = \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p^\times/\mu, \mathbb{Z}_p),$$

which we also denote by π. This is a map of θ-algebras and of KO_0KO-comodules, and there are also natural Hopf algebra structures on both objects making it a Hopf algebra map. We also consider the leaky λ-ring structures of Definition A.10. Using all this structure, we prove that the Hopf algebra kernel is just $\mathbb{T}(f)$, and construct a coalgebra splitting. This implies that $KO_0 T_\zeta$ is an induced KO_0KO-comodule by a general theorem about Hopf algebras. Finally, one can explicitly construct $\mathbb{Z}_p^\times/\mu$-invariant elements in nonzero degrees of $KO_* T_\zeta$, multiplication by which allows us to transport the result in degree zero to nonzero degrees.

Lemma 1 *The map of θ-algebras $i : \mathbb{T}(f) \to \mathbb{T}(b)$ sending f to $\psi^p(b) - b$ is injective and pro-free.*

Proof Let $b_0 = b$ and $b_i = \theta_i(b)$, and likewise with f_i, where the operations θ_i are as defined in Theorem A.5. Then

$$\mathbb{T}(b) = \mathbb{Z}_p[b_0, b_1, \ldots] \text{ and } \mathbb{T}(f) = \mathbb{Z}_p[f_0, f_1, \ldots].$$

We claim that

$$f_i \equiv b_i^p - b_i \mod (p, b_0, \ldots, b_{i-1}). \tag{3.18}$$

This is true for $i = 0$. Suppose it has been proven for $i = 0, \ldots, n-1$. Then

$$\psi^{p^n}(f) = \psi^{p^{n+1}}(b) - \psi^{p^n}(b),$$

or in other words,

$$f_0^{p^n} + pf_1^{p^{n-1}} + \cdots + p^n f_n = b_0^{p^{n+1}}$$
$$- b_0^{p^n} + p(b_1^{p^n} - b_1^{p^{n-1}}) + \cdots + p^n(b_n^p - b_n) + p^{n+1}b_{n+1}. \tag{3.19}$$

Since $f_i \equiv b_i^p - b_i \bmod (p, b_0, \ldots, b_{i-1})$, we have

$$f_i \equiv 0 \mod (p, b_0, \ldots, b_i),$$

and thus

$$p^i f_i^{p^{n-i}} \equiv 0 \mod (p^{i+n-i+1}, b_0, \ldots, b_i).$$

Thus, (3.19) reduces mod $(p^{n+1}, b_0, \ldots, b_{n-1})$ to

$$p^n f_n \equiv p^n(b_n^p - b_n) \mod (p^{n+1}, b_0, \ldots, b_{n-1})$$

or just

$$f_n \equiv b_n^p - b_n \mod (p, b_0, \ldots, b_{n-1}),$$

which is (3.18) for $i = n$.

Thus, $\mathbb{T}(b)/p = \mathbb{F}_p[b_0, b_1, \ldots]$ is freely generated over $\mathbb{T}(f)/p = \mathbb{F}_p[f_0, f_1, \ldots]$ by the monomials $b_0^{n_0} b_1^{n_1} \cdots$ in which all $n_i < p$ and all but finitely many of the n_i are zero. By Lemma 2.6, $\mathbb{T}(b)$ is pro-free over $\mathbb{T}(f)$. In particular, the unit map is an injection. $\square$

It will be helpful to make the identification

$$KO_0KO = \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p^\times/\mu, \mathbb{Z}_p) \cong \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p)$$

using the continuous group isomorphism

$$\mathbb{Z}_p^\times/\mu \xrightarrow{\cong} \mathbb{Z}_p, \quad g \mapsto 1.$$

By proposition 3.4, $b \in KO_0KO$ goes to the identity under this identification.

The map $T_\zeta \to KO$ induces a map

$$\pi : \mathbb{T}(b) = KO_0T_\zeta \to KO_0KO \cong \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p),$$

This is a θ-algebra map, determined by the fact that $\pi(b) = \mathrm{id}$. By Proposition A.4, $\psi^p(\pi(b)) = \pi(b)$. Thus, there is an induced map

$$\overline{\pi} : \mathbb{T}(b) \otimes_{\mathbb{T}(f)} \mathbb{Z}_p \to \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p)$$

where $\mathbb{T}(f) \to \mathbb{Z}_p$ sends all $\theta^k(f)$ to 0.

Definition 3.20 We give $\mathbb{T}(b)$ and $\mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p)$ the leaky λ-ring structures $\mathcal{L}(\mathbb{T}(b))$, $\mathcal{L}(\mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p))$ of Definition A.10. In each of these λ-rings, the Adams operations ψ^k associated to the λ-ring structure are the identity for k prime to p, while ψ^p is equal to the operation ψ^p associated to the θ-algebra structure.

By Example A.11, the λ-operations on $\phi \in \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p)$ are given by

$$\lambda^n(\phi)(x) = \binom{\phi(x)}{n}.$$

Lemma 3.21 *The map*

$$\pi : \mathbb{T}(b) \to \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p)$$

is a map of λ-rings.

Proof This map is obtained by applying the functor $\mathcal{L}$ to a map of ψ-θ-algebras. □

Proposition 3.22 *The map*

$$\overline{\pi} : \mathbb{T}(b) \otimes_{\mathbb{T}(f)} \mathbb{Z}_p \to \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p)$$

is an isomorphism.

Proof First, let's show the map is surjective. Since the map $\mathbb{T}(b) \to \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p)$ is a map of λ-rings with $\mathrm{id}_{\mathbb{Z}_p}$ in its image, $\lambda^k(\mathrm{id})$ is also in its image for all $k \in \mathbb{N}$. Observe that $\lambda^k(\mathrm{id})$ is precisely the binomial coefficient function $\beta_k : x \mapsto \binom{x}{k}$. It is a theorem of Mahler [29, 4.2.4] that $\mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p)$ is generated (as a complete $\mathbb{Z}_p$-module) by the binomial functions β_k for $k \in \mathbb{N}$. Thus the map $\pi : \mathbb{T}(b) \to \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p)$ is surjective.

We now introduce an alternative description of $\mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p)$. Any element of $\mathbb{Z}_p$ has a unique description

$$a = \sum_{i \geq 0} a_i p^i$$

where each a_i is a Teichmüller lift, i.e., either zero or a $(p-1)$th root of unity. Define

$$\alpha_i(a) = a_i.$$

A continuous map $\mathbb{Z}_p \to \mathbb{Z}/p^n$ can be described in terms of a finite number of the α_i, so we have

$$\mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}/p^n) = \mathbb{Z}/p^n[\alpha_0, \alpha_1, \ldots]/(\alpha_i^p - \alpha_i).$$

Taking the limit gives

$$\mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p) = \mathbb{Z}_p[\alpha_0, \alpha_1, \ldots]/(\alpha_i^p - \alpha_i).$$

Let $b_n := \theta_n b$ in $\mathbb{T}(b)$, so $\mathbb{T}(b) = \mathbb{Z}_p[b_0, b_1, \ldots]$. Recall the identities

$$\psi^{p^n} b = b_0^{p^n} + p b_1^{p^{n-1}} + \cdots + p^n b_n. \tag{3.23}$$

We claim that

$$\pi(b_n) \equiv \alpha_n \mod p$$

for all n. We proceed by induction: first, $\pi(b_0) = \mathrm{id}$ is congruent to α_0 mod p. Suppose we have shown that

$$\pi(b_i) \equiv \alpha_i \mod p$$

for each $i < n$. It follows that

$$\pi(b_i^{p^{n-i}}) \equiv \alpha_i^{p^{n-i}} = \alpha_i \mod p^{n-i+1}$$

and so

$$\pi(p^i b_i^{p^i}) \equiv p^i \alpha_i \mod p^{n+1}.$$

Thus, applying π to (3.23) and using the fact that ψ^p is the identity on $\mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p)$, we get

$$\mathrm{id} \equiv \alpha_0 + p\alpha_1 + \cdots + p^{n-1}\alpha_{n-1} + p^n \pi(b_n) \mod p^{n+1}.$$

But of course $\mathrm{id} = \sum p^i \alpha_i$ on the nose, so solving for $\pi(b_n)$ gives

$$\pi(b_n) \equiv \alpha_n \mod p.$$

We can now compute the kernel of π. First note that it contains each

$$\theta_n(f) = \psi^p(b_n) - b_n.$$

This is just because it's a θ-algebra map whose kernel contains f, and was needed to define the map $\overline{\pi}$ in the first place. We want to show that the $\theta_n(f)$ generate the kernel of π. But we know that

$$\pi/p : \mathbb{F}_p[b_0, b_1, \ldots] \to \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{F}_p) = \mathbb{F}_p[\alpha_0, \alpha_1, \ldots]/(\alpha_i^p - \alpha_i)$$

sends b_i to α_i, so that $\ker(\pi/p)$ is generated by the elements $b_n^p - b_n \equiv \psi^p(b_n) - b_n$ (mod p).

Since $\mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p)$ is a free complete $\mathbb{Z}_p$-module, we have that

$$\mathrm{Tor}^1_{\mathbb{Z}_p}(\mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p), \mathbb{F}_p) = 0,$$

and so

$$\ker(\pi/p) \cong \ker(\pi) \otimes \mathbb{F}_p.$$

Since $\mathbb{T}(b)$ is p-adically complete and torsion free, it follows that the elements $\psi^p(b_n) - b_n$ also generate $\ker(\pi)$, concluding the proof. □

Lemma 3.24 *The map $i : \mathbb{T}(f) \to \mathbb{T}(b)$ is a map of Hopf algebras, where $\mathbb{T}(f)$ and $\mathbb{T}(b)$ both have the Hopf algebra structure of Example A.7. The induced Hopf algebra structure on $\mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p) = \mathbb{T}(b) /\!\!/ \mathbb{T}(f)$ is the same as that induced by addition on the source $\mathbb{Z}_p$.*

Proof The first statement follows from the fact that the functor $\mathbb{T}$ naturally takes values in Hopf algebras. In particular, the diagonal map $\Delta : \mathbb{T}(M) \to \mathbb{T}(M) \otimes \mathbb{T}(M)$ is functorial in M. Thus i is a map of Hopf algebras.

For the second statement, it suffices to show that the given map $\pi : \mathbb{T}(b) \to \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p)$ is a Hopf algebra map. This can be checked after tensoring with $\mathbb{Q}$, in which case it suffices to check that $\pi(\psi^{p^n}(b))$ is still primitive. However, we have seen that each $\psi^{p^n}(b)$ goes to the identity of $\mathbb{Z}_p$, which is primitive in $\mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p)$. □

Lemma 3.25 *The map $\pi : \mathbb{T}(b) \to \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p)$ admits a coalgebra section $s : \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p) \to \mathbb{T}(b)$.*

Proof By Mahler's theorem cited above, $\mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p)$ is a free complete $\mathbb{Z}_p$-module on the binomial functions $\beta_k : x \mapsto \binom{x}{k}$, for $k \in \mathbb{N}$. In the proof of Proposition 3.22, we saw that, in terms of the λ-ring structure on $\mathbb{T}(b)$, $\pi(\lambda^k(b)) = \beta_k$. We can define a continuous $\mathbb{Z}_p$-module section by

$$s(\beta_k) = \lambda^k(b).$$

It remains to see that this is also a coalgebra section. It follows from Lemma A.13 that the coproduct Δ is a morphism of λ-algebras.

The binomial functions have comultiplication

$$\Delta(\beta_n) = \sum_{i=0}^{n} \beta_i \otimes \beta_{n-i}.$$

Therefore,

$$(s \otimes s)\Delta(\beta_n) = \sum_{i=0}^{n} \lambda^i(b) \otimes \lambda^{n-i}(b) = \Delta s(\beta_n).$$

So s is a coalgebra map. □

Equipped with the above lemmas, we can finally prove Theorem 3.15. We begin by proving the degree zero part.

Proposition 3.26 *There is an isomorphism of $\mathbb{T}(f)$-modules and KO_0KO-comodules*

$$\mathbb{T}(f) \otimes KO_0KO \cong \mathbb{T}(b).$$

Proof **Note:** *For the duration of this proof, we will make all completions explicit.*
We wish to show that

$$KO_*T_\zeta = KO_* \otimes \mathbb{T}(f) \cong KO_*KO \otimes \mathbb{T}(b).$$

At this point, we have maps of complete Hopf algebras

$$\mathbb{T}(f) \xrightarrow{i} \mathbb{T}(b) \xrightarrow{\pi} KO_0KO, \tag{3.27}$$

such that $KO_0KO = \mathbb{T}(b)\overline{\otimes}_{\mathbb{T}(f)}\mathbb{Z}_p$, together with a coalgebra section s of π. We claim that

$$\widehat{\phi} : \mathbb{T}(f) \,\overline{\otimes}\, KO_0KO \xrightarrow{i\otimes s} \mathbb{T}(b) \,\overline{\otimes}\, \mathbb{T}(b) \xrightarrow{m} \mathbb{T}(b)$$

is the desired isomorphism. This uses a variant of the arguments in [26, Sect. 1]. The situation is slightly complicated by the omnipresence of completion, as well as the fact that the objects involved are not graded in any manageable way.

First, we handle the completions. Let A be the uncompleted polynomial ring

$$A := \mathbb{Z}_p[f, \theta(f), \theta_2(f), \dots],$$

and likewise let B be the uncompleted polynomial ring on the $\theta_n(b)$. Let C be the sub-$\mathbb{Z}_p$-algebra of $\mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p)$ consisting of those functions which can be written as polynomials with $\mathbb{Q}_p$ coefficients. As an uncompleted $\mathbb{Z}_p$-module, C is free on the β_n. The sequence (3.27) restricts to a sequence of maps of Hopf algebras

$$A \xrightarrow{i} B \xrightarrow{\pi} C, \tag{3.28}$$

such that $C = B \otimes_A \mathbb{Z}_p$, together with a coalgebra section s of π. Write ϕ for the map

$$A \otimes C \xrightarrow{i\otimes s} B \otimes B \xrightarrow{mult} B.$$

Since $\widehat{\phi}$ is the completion of ϕ, it suffices to prove that ϕ is an isomorphism of A-modules and C-comodules.

Now, ϕ is clearly an A-module homomorphism, and it is also a map of C-comodules since s is a map of coalgebras. We will show that ϕ is injective by the method of [26, Proposition 1.7]. Note that C has a coalgebra grading in which the degree of β_n is n. This induces filtrations on $A \otimes C$ and $B \otimes C$, in which

$$F_{\leq n}(A \otimes C) = \sum_{q \leq n} A \otimes C_q,$$

and likewise for $B \otimes C$. Consider the map

$$\nu : A \otimes C \xrightarrow{\phi} B \xrightarrow{\Delta} B \otimes B \xrightarrow{1 \otimes \pi} B \otimes C.$$

Using the comultiplicativity of s, we see that

$$\nu(1 \otimes \beta_n) = \sum_{i=0}^{n} s(\beta_i) \otimes \beta_{n-i}.$$

Furthermore, since ν is a left A-module map, it preserves the filtration. Thus, there is an induced map $\overline{\nu}$ on associated graded objects. However, as C is the direct sum of the C_q, the associated graded objects are simply $A \otimes C$ and $B \otimes C$. Once again, one computes that

$$\overline{\nu}(1 \otimes \beta_n) = 1 \otimes \beta_n.$$

As $\overline{\nu}$ is a left A-module map, we can identify it with

$$i \otimes 1 : A \otimes C \to B \otimes C.$$

Since C is flat over $\mathbb{Z}_p$, this map is injective. Thus ν is injective, so ϕ is injective, as desired.

For surjectivity, we use a version of [26, Proposition 1.6]. Filter A as follows: the elements of filtration $\geq s$ are the polynomials in $f, \theta(f), \theta_2(f), \ldots$ all of whose terms are of degree $\geq s$. Giving B the analogous filtration, the map $i : A \to B$ is a filtered A-module map, and the counit $\epsilon : A \to \mathbb{Z}_p$ kills the ideal of positively filtered elements. The A-module structure on C factors through ϵ, and we give C the trivial filtration $C = C_{\geq 0} = C_{\geq 1} = \cdots$. Then $\pi : B \to C$ is also filtered.

Claim 1 Let M be a nonnegatively filtered A-module. Then $M = 0$ iff $\mathbb{Z}_p \otimes_A M = 0$.

Indeed, if M is nonzero, then it has a nonzero element x of lowest possible filtration, say s. But the kernel of $M \to \mathbb{Z}_p \otimes_A M$ is precisely $A_{>0} \cdot M$, so if $\mathbb{Z}_p \otimes_A M = 0$, then x is an A-multiple of an element of lower filtration.

Claim 2 Let $g : M_1 \to M_2$ be a filtered A-module map, where M_1 and M_2 are nonnegatively filtered. Then g is surjective iff

$$\mathbb{Z}_p \otimes_A g : \mathbb{Z}_p \otimes_A M_1 \to \mathbb{Z}_p \otimes_A M_2$$

is surjective.

The direction ($\Rightarrow$) holds because the tensor product is right exact. For the direction ($\Leftarrow$), let $N = \mathrm{coker}(g)$. The A-module N receives a filtration in an evident way. Again using right exactness of the tensor product, we have that

$$\mathbb{Z}_p \otimes_A N = \mathrm{coker}(\mathbb{Z}_p \otimes_A g).$$

If $\mathbb{Z}_p \otimes_A g$ is surjective, then $\mathbb{Z}_p \otimes_A N = 0$, so $N = 0$ by Claim 1. (Since completion is neither left nor right exact in general, we need to work with the uncompleted tensor product here.)

Finally, $\phi : A \otimes C \to B$ is a filtered A-module map whose source and target are nonnegatively filtered. We have

$$\mathbb{Z}_p \otimes_A \phi = \mathrm{id} : C \to C = \mathbb{Z}_p \otimes_A B.$$

By Claim 2, ϕ is surjective. □

Proof of Theorem 3.15 We have already constructed an isomorphism of KO_0KO-comodules

$$\mathbb{T}(f) \otimes KO_0KO \to KO_0T_\zeta.$$

To extend this to a map

$$\mathbb{T}(f) \otimes KO_*KO = KO_* \otimes \mathbb{T}(f) \otimes KO_0KO \to KO_*T_\zeta,$$

one has identify the image of KO_* in KO_*T_ζ, which will consist of elements which are invariant under the Adams operations. First, suppose that $p > 2$. Since $g \in \mathbb{Z}_p^\times$ maps to a topological generator of $\mathbb{Z}_p^\times/\mu$, we have $g^{p-1} \in 1 + p\mathbb{Z}_p$. Write $g^{p-1} = 1 + h$ where $h \in p\mathbb{Z}_p$. Then the series

$$g^{-b(p-1)} := (1+h)^{-b} = \sum_{n \geq 0} \binom{-b}{n} h^n$$

converges in $\mathbb{T}(b)$. Indeed, each term has p-adic valuation at least $n - v_p(n!)$, and these converge to ∞ with n. In $KO_*T_\zeta = KO_* \otimes \mathbb{T}(b)$,

$$\psi^g(g^{-b(p-1)}v_1) = (1+h)^{-(b+1)} \cdot g^{p-1}v_1 = g^{-b(p-1)}v_1.$$

Thus, writing $\widetilde{v_1} = g^{-b(p-1)}v_1 \in KO_{2(p-1)}T_\zeta$, we see that multiplication by $\widetilde{v_1}^k$ induces an isomorphism of KO_0KO-comodules

$$KO_0T_\zeta \xrightarrow{\sim} KO_{2(p-1)k}T_\zeta.$$

As KO_0T_ζ is an extended comodule, the same follows for KO_*T_ζ, and we obtain

$$\pi_*T_\zeta = (KO_*T_\zeta)^{\mathbb{Z}_p^\times/\mu} = \mathbb{Z}_p[\widetilde{v_1}^{\pm 1}] \otimes \mathbb{T}(f).$$

The isomorphism with $KO_* \otimes \mathbb{T}(f)$ is given by mapping $\widetilde{v_1}$ to v_1.

Now suppose that $p = 2$, in which case KO_* is generated by $\eta \in KO_1$, $v = 2u^2 \in KO_4$, and $w = u^4 \in KO_8$, where $u \in K_2$ is the Bott element. We have that $g^2 = 1+h$ where $h \in 4\mathbb{Z}_2$. Again, this means that the series $g^{-2b} = (1+h)^{-b}$ converges, and we can define $\widetilde{v} = g^{-2b}v$, $\widetilde{w} = g^{-4b}w$. By the same arguments, $KO_{4*}T_\zeta$ is an etended comodule. To deal with the rest, we note that

$$KO_{8k+1}T_\zeta \cong KO_{8k+2}T_\zeta \cong KO_0T_\zeta \otimes_{\mathbb{Z}_2} \mathbb{F}_2$$

as KO_0KO-comodules. Tensoring the exact sequence

$$0 \to \pi_0 T_\zeta \to KO_0T_\zeta \xrightarrow{\psi^g - 1} KO_0T_\zeta \to 0$$

with $\mathbb{F}_2$ and noting that KO_0T_ζ is flat over $\mathbb{Z}_2$, we obtain the desired result. □

Remark 3.29 As we mentioned earlier, Hopkins' argument from [14] has errors. In particular, he correctly claims that the map

$$\mathbb{T}(f) \otimes \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p) \xrightarrow{i \otimes s} \mathbb{T}(b) \otimes \mathbb{T}(b) \xrightarrow{\text{mult}} \mathbb{T}(b)$$

is an isomorphism. However, he argues this by asserting that the inverse to this map is given by

$$\mathbb{T}(b) \xrightarrow{\Delta} \mathbb{T}(b) \otimes \mathbb{T}(b) \xrightarrow{(1 - s \circ \pi) \otimes \pi} \mathbb{T}(f) \otimes \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p).$$

But this map simply cannot be the inverse, indeed it is not even injective. To see this, let β_n denote the nth binomial coefficient function. The section s is a map of coalgebras and the diagonal on the β_n satisfy the Cartan formula. Thus

$$\Delta(s\beta_n) = \sum_{i+j=n} s(\beta_i) \otimes s(\beta_j).$$

Thus, under the above map, one computes that

$$s(\beta_n) \mapsto \sum_{i+j=n} \pi(s(\beta_i)) \otimes (1 - s\pi)(s\beta_j).$$

Since s is a section, $\pi s = 1$. Note that

$$(1 - s\pi)(s(\beta_j)) = s(\beta_j) - s\pi s(\beta_j) = s(\beta_j) - s(\beta_j) = 0.$$

Note that this includes the case when $j = 0$, in which case $\beta_j = \beta_0 = 1$. Thus the above map has a nontrivial kernel, and so is not injective.

4 Co-operations for T_ζ

We saw in Proposition 3.11 that $KO_*T_\zeta \cong KO_* \otimes \mathbb{T}(b)$. As $\mathbb{T}(b)$ is a completion of a polynomial ring, KO_*T_ζ is pro-free over KO_*. Moreover, we have an equivalence of KO-modules in Sp,

$$KO \wedge T_\zeta \wedge T_\zeta \simeq (KO \wedge T_\zeta) \wedge_{KO} (KO \wedge T_\zeta).$$

So it follows from Proposition 2.8 that,

$$KO_*(T_\zeta \wedge T_\zeta) \cong KO_*T_\zeta \otimes_{KO_*} KO_*T_\zeta \cong KO_* \otimes \mathbb{T}(b, b'). \tag{4.1}$$

Recall that the KO_*KO-comodule structure is given by an action of the group $\mathbb{Z}_p^\times/\mu$. In this case, the action comes from the diagonal action on the two tensor factors, so that

$$\psi^g(b) = b + 1, \quad \psi^g(b') = b' + 1.$$

As we saw in the previous section, the computation of π_*T_ζ followed from knowing that KO_*T_ζ was an extended comodule. The same strategy allows us to compute the co-operations algebra $\pi_*(T_\zeta \wedge T_\zeta)$.

Lemma 4.2 *A tensor product of extended KO_0KO-comodules is extended. More precisely, if M and N are $\mathbb{Z}_p$-modules, then*

$$(KO_0KO \otimes M) \otimes (KO_0KO \otimes N) \cong \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p^\times/\mu \times \mathbb{Z}_p^\times/\mu, \mathbb{Z}_p) \otimes M \otimes N$$

is extended on the primitive submodule

$$\{\phi \otimes m \otimes n \in \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p^\times/\mu \times\mathbb{Z}_p^\times/\mu, \mathbb{Z}_p) \otimes M \otimes N : \phi(a, b) \textit{ only depends on } a^{-1}b\}.$$

Proof This is the special case of Lemma 2.19 in which both comodules are extended. The formula for the primitives follows from the formula there, using Theorem 2.30 to describe the maps. □

In the following, we will frequently use x and $\overline{x}$ to denote the image of x along respectively the left and right units of a Hopf algebroid.

Theorem 4.3 *There is an isomorphism of θ-algebras*

$$\pi_*(T_\zeta \wedge T_\zeta) \cong KO_* \otimes \mathbb{T}(f, \overline{f}, \ell)/(\psi^p(\ell) - \ell - f + \overline{f}).$$

Proof As KO_*T_ζ is KO_*-pro-free, we have

$$KO_*(T_\zeta \wedge T_\zeta) \cong KO_*(T_\zeta) \otimes KO_*(T_\zeta)$$

as KO_*KO-comodules. We saw in the proof of Theorem 3.15 that KO_*T_ζ is an extended comodule. The lemma then implies that $KO_*(T_\zeta \wedge T_\zeta)$ is extended. Using Proposition 2.36, there is an additive isomorphism

$$\begin{aligned}\pi_*(T_\zeta \wedge T_\zeta) &= \mathrm{Hom}_{\mathsf{Comod}^\wedge_{KO_*}}(KO_*, KO_*(T_\zeta \wedge T_\zeta))\\ &\cong \pi_*T_\zeta \otimes_{KO_*} \pi_*T_\zeta \otimes \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p^\times/\mu, \mathbb{Z}_p).\end{aligned}$$

By Corollary 3.16, this is isomorphic to

$$KO_* \otimes \mathbb{T}(f, \overline{f}) \otimes \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p^\times/\mu, \mathbb{Z}_p),$$

where f and $\overline{f}$ come from the left and right copies of $\pi_0 T_\zeta$ respectively.

Note that, as the isomorphism $KO_*T_\zeta \cong \mathbb{T}(f) \otimes KO_*KO$ of Theorem 3.15 is an isomorphism of comodules but not of comodule algebras; the above isomorphism is only additive. We can nevertheless identify the multiplicative structure on $\pi_*(T_\zeta \wedge T_\zeta)$ by locating the primitive elements identified above inside the ring

$$KO_*(T_\zeta \wedge T_\zeta) = KO_* \otimes \mathbb{T}(b, \overline{b}).$$

In fact, the θ-algebra $\mathbb{T}(f, \overline{f})$ is just that generated by $f = \psi^p(b) - b$ and $\overline{f} = \psi^p(\overline{b}) - b$ inside $KO_*(T_\zeta \wedge T_\zeta)$. Likewise, there is a primitive copy of KO_* inside $KO_*(T_\zeta \wedge T_\zeta)$, namely that generated by the left unit on $\widetilde{v_1}$ (or by the left unit on η, $\widetilde{v}$, and $\widetilde{w}$ if $p = 2$).

We still have to identify the $\mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p)$ factor. Lemma 4.2 tells us that, under the isomorphism

$$KO_0(T_\zeta \wedge T_\zeta) \cong \mathbb{T}(f, \overline{f}) \otimes \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_p),$$

this factor is precisely

$$\begin{aligned}&\{1 \otimes \phi \in \mathbb{T}(f, \overline{f}) \otimes \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_p) : \phi(x, y) \text{ only depends on } y - x\}\\ &\quad = \{1 \otimes \phi : \phi(x, y) = \phi(0, y - x)\}.\end{aligned} \tag{4.4}$$

The submodule $1 \otimes \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_p)$ is the image of

$$s \otimes \overline{s} : \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p) \otimes \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p) \to KO_0(T_\zeta \wedge T_\zeta),$$

where s is as defined in Lemma 3.25. That is,

$$(s \otimes \overline{s})(\beta_n \otimes \beta_m) = \lambda^n(b)\lambda^m(\overline{b}).$$

By Mahler's theorem, the submodule of $f \in \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_p)$ satisfying the condition of (4.4) is spanned by

$$(x, y) \mapsto \binom{y - x}{n}.$$

Thus, the invariant $\mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p)$ factor in $KO_0(T_\zeta \wedge T_\zeta)$ is spanned by

$$\lambda^n(b - \overline{b}).$$

In particular, the sub-λ-algebra of $KO_0(T_\zeta \wedge T_\zeta)$ generated by $b - \overline{b}$ contains this $\mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p)$. But this is the same as the sub-θ-algebra generated by $b - \overline{b}$. Let

$$\ell = b - \overline{b}.$$

The formula $\psi^p(b) - b = f$, and the analogous one for $\overline{f}$, show that

$$f - \overline{f} = \psi^p(\ell) - \ell.$$

Thus, there is an epimorphism

$$\mathbb{T}(f, \overline{f}, \ell)/(\psi^p(\ell) - \ell - f + \overline{f}) \twoheadrightarrow \pi_0(T_\zeta \wedge T_\zeta). \tag{4.5}$$

To see that this is an isomorphism, note that Proposition 3.11 implies that

$$\pi_0(T_\zeta \wedge T_\zeta) \cong \pi_0 T_\zeta \otimes \mathbb{T}(x)$$

as $\pi_0(T_\zeta)$-modules. That is, it is a free θ-algebra on two generators. But the left-hand side of Eq. (4.5) is free on the generators f and ℓ, and any nontrivial quotient of it would not be free on two generators. Thus, we have

$$\pi_0(T_\zeta \wedge T_\zeta) = \mathbb{T}(f, \overline{f}, \ell)/(\psi^p(\ell) - \ell - f + \overline{f}).$$

This concludes the proof. □

5 $K(1)$-local tmf

We continue to work $K(1)$-locally, and fix $p = 2$ or 3, so that $j = 0$ is the unique supersingular j-invariant. It is simple to extend this story to larger primes with a single supersingular j-invariant; slightly more complicated to extend it to other primes; but in neither case is it quite as interesting. As in section 4, the statements in this section are due to [14].

Proposition 5.1 *For any $x \in KO_0\mathrm{tmf}$ such that $\psi^g(x) = x + 1$, there is a unique homotopy class of E_∞ maps $T_\zeta \to \mathrm{tmf}$ sending $b \in KO_0 T_\zeta$ to x.*

Proof Clearly, any map $T_\zeta \to \mathrm{tmf}$ acts this way on KO-homology. Conversely, since $\pi_{-1}\mathrm{tmf} = 0$, the set of homotopy classes of E_∞ maps $T_\zeta \to \mathrm{tmf}$ is parametrized by

$$\pi_0\mathrm{tmf} = \mathrm{Maps}_{\theta\text{-}\mathsf{Alg}}(\mathbb{T}(f), \pi_0\mathrm{tmf}).$$

Since KO_0T_ζ is the induced KO_0KO-comodule on π_0T_ζ, any such θ-algebra map extends uniquely to a ψ-θ-algebra map

$$KO_0T_\zeta \to KO_0\mathrm{tmf}$$

and thus to

$$KO_*T_\zeta \to KO_*\mathrm{tmf}.$$

$\square$

Remark 5.2 In particular, we can pick

$$g = 3 \text{ and } x = -\frac{\log c_4/w}{\log 3^4} \text{ at } p = 2,$$
$$g = 2 \text{ and } x = -\frac{\log c_6/v_1^3}{\log 2^6} \text{ at } p = 3.$$

Proposition 5.3 [14, 7.1]. *Let b be as above and let $f = \psi^p(b) - b$. Then $f \equiv j^{-1}$ mod p, and as an element of $\mathbb{Z}_p[j^{-1}]$, f has constant term zero. Thus, the map $\mathbb{Z}_p[f] \to \mathbb{Z}_p[j^{-1}]$ is an isomorphism.*

Proof This is a calculation using q-expansions. See [14, 7.1]. $\square$

It follows that the map $q : T_\zeta \to \mathrm{tmf}$ induces a surjective map on π_0,

$$q : \mathbb{T}(f) \twoheadrightarrow \mathbb{Z}_p[j^{-1}].$$

Thus, $\theta(f)$ maps to some completed polynomial in j^{-1}. Since $f \equiv j^{-1}$ mod p, this can also be written as a completed polynomial in f, say $h(f)$. It follows that the kernel of q is the θ-ideal generated by $\theta(f) - h(f)$.

Lemma 5.4 *The map of θ-algebras $F : \mathbb{T}(x) \to \mathbb{T}(b)$ sending x to $\theta(f) - h(f)$ makes $\mathbb{T}(b)$ into a pro-free $\mathbb{T}(x)$-module.*

Proof This is similar to Lemma 1. Again, let us write

$$x_i = \theta_i(x), \quad b_i = \theta_i(b), \quad i \geq 0.$$

(See Theorem A.5 for θ_i.) We will prove by induction that

$$F(x_i) = b_{i+1}^p - b_{i+1} \mod (p, b_0, \ldots, b_i). \tag{5.5}$$

When $i = 0$,

$$\begin{aligned} F(x) &= \theta(f) - h(f) \\ &= \theta(\psi^p(b) - b) - h(\psi^p(b) - b) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{p}(\psi^{p^2}(b) - \psi^p(b) - (\psi^p(b) - b)^p) - h(\psi^p(b) - b) \\
&= \frac{1}{p}(b_0^{p^2} - b_0^p + p(b_1^p - b_1) + p^2 b_2 - (b_0^p - b_0 + pb_1)^p) \\
&\quad - h(b_0^p - b_0 + pb_1) \\
&\equiv b_1^p - b_1 + pb_2 - p^{p-1}b_1^p - h(pb_1) \pmod{b_0} \\
&\equiv b_1^p - b_1 \pmod{(p, b_0)}.
\end{aligned}
$$

Suppose that we have proved (5.5) for $i = 0, \ldots, n-1$. Then for these values of i,

$$
F(x_i) \equiv 0 \mod (p, b_0, \ldots, b_{i+1}),
$$

and so

$$
p^i F(x_{n-i})^{p^{n-i}} \equiv 0 \mod (p^{n+1}, b_0, \ldots, b_{i+1}). \tag{5.6}
$$

We also have

$$
\begin{aligned}
\theta(\psi^{p^{n+1}}(b) - \psi^{p^n}(b)) &= \frac{1}{p}(\psi^{p^{n+2}}(b) - \psi^{p^{n+1}}(b) - (\psi^{p^{n+1}}(b) - \psi^{p^n}(b))^p) \\
&\equiv \frac{1}{p}(p^{n+1}(b_{n+1}^p - b_{n+1}) + p^{n+2}b_{n+2} \\
&\quad - (p^{n+1}b_{n+1})^p) \mod (b_0, \ldots, b_n) \\
&\equiv p^n(b_{n+1}^p - b_{n+1}) \mod (p^{n+1}, b_0, \ldots, b_n).
\end{aligned}
$$

Finally,

$$
h(\psi^{p^{n+1}}(b) - \psi^{p^n}(b)) \equiv 0 \mod (p^{n+1}, b_0, \ldots, b_n)
$$

because h is a completed polynomial over $\mathbb{Z}_p$. Putting this all together,

$$
\begin{aligned}
\psi^{p^n}(F(x)) &= \psi^{p^n}(\theta(\psi^p(b) - b) - h(\psi^p(b) - b)) \\
&= \theta(\psi^{p^{n+1}}(b) - \psi^{p^n}(b)) - h(\psi^{p^{n+1}}(b) - \psi^{p^n}(b)) \\
&\equiv p^n(b_{n+1}^p - b_{n+1}) \mod (p^{n+1}, b_0, \ldots, b_n).
\end{aligned}
$$

The left-hand side is congruent to $p^n F(x_n)$ modulo this ideal by (5.6), which proves (5.5).

It follows that the map

$$
\mathbb{F}_p[b_0, x_0, x_1, \ldots] \to \mathbb{F}_p[b_0, b_1, b_2, \ldots]
$$

makes the target into a free module over the source, by the same argument as in Lemma 1. But $\mathbb{F}_p[b_0, x_0, x_1, \ldots]$ is clearly free over $\mathbb{F}_p[x_0, x_1, \ldots]$. By Lemma 2.6, $\mathbb{T}(b)$ is pro-free over $\mathbb{T}(x)$. This finishes the proof of the lemma. □

Theorem 5.7 [14, 7.2] *There is a homotopy pushout square of $K(1)$-local E_∞ rings,*

$$\begin{tikzcd} \mathbb{P}(S^0) \arrow[r, "0"] \arrow[d, "\theta(f)-h(f)"'] & S^0 \arrow[d] \\ T_\zeta \arrow[r, "q"'] & \mathrm{tmf}. \end{tikzcd}$$

Proof Let Y be the homotopy pushout of the above square, so

$$Y \simeq T_\zeta \wedge_{\mathbb{P}(S^0)} S^0.$$

Since $\theta(f) = h(f)$ in $\pi_0\mathrm{tmf}$, there is a map $Y \to \mathrm{tmf}$, which we will show is an isomorphism on homotopy groups.

We note that $KO_*\mathbb{P}(S^0) \to KO_*T_\zeta$ is precisely the map of the previous lemma, tensored by KO_*. By Lemma 2.5, KO_*T_ζ is pro-free over $KO_*\mathbb{P}(S^0)$. Then by Proposition 2.8 and the previous lemma, we have the Künneth formula,

$$KO_*Y = KO_*T_\zeta \otimes_{KO_*\mathbb{P}(S^0)} KO_* \cong KO_*T_\zeta \otimes_{KO_0\mathbb{P}(S^0)} \mathbb{Z}_p.$$

By Proposition 3.26 and the proof of Theorem 3.15, we have an isomorphism

$$KO_*T_\zeta \cong \pi_*T_\zeta \otimes KO_0KO$$

as π_*T_ζ-modules and KO_0KO-comodules. Since $KO_0\mathbb{P}(S^0) \to KO_0T_\zeta$ factors through π_0T_ζ, we likewise have

$$KO_*Y = KO_*T_\zeta \otimes_{KO_0\mathbb{P}(S^0)} \mathbb{Z}_p \cong (\pi_*T_\zeta \otimes_{KO_0\mathbb{P}(S^0)} \mathbb{Z}_p) \otimes KO_0KO$$

as KO_0KO-comodules. That is, KO_*Y is an induced comodule, and

$$\pi_*Y = \pi_*T_\zeta \otimes_{KO_0\mathbb{P}(S^0)} \mathbb{Z}_p = KO_* \otimes \mathbb{T}(f)/(\theta(f) - h(f)) = \mathbb{Z}_p[f] = \pi_*\mathrm{tmf}.$$

(Here the quotient is by the θ-ideal generated by $\theta(f) - h(f)$.) $\square$

Corollary 5.8 *There is an E_∞ map $r : \mathrm{tmf} \to KO$.*

Proof One has an E_∞ map $T_\zeta \to KO$, which by arguments similar to the ones above fits into a pushout square of E_∞ rings

$$\begin{tikzcd} \mathbb{P}(S^0) \arrow[r, "0"] \arrow[d, "f"'] & S^0 \arrow[d] \\ T_\zeta \arrow[r] & KO. \end{tikzcd}$$

The left-hand vertical map sends the θ-algebra generator x of $KO_0\mathbb{P}(S^0)$ to $f = \psi^p(b) - b \in KO_0 T_\zeta$. There is an E_∞ factorization

$$\begin{tikzcd} \mathbb{P}(S^0) \arrow[r, "\theta(x)-h(x)"] \arrow[rr, bend right, "\theta(f)-h(f)"'] & \mathbb{P}(S^0) \arrow[r, "f"] & T_\zeta . \end{tikzcd}$$

This induces a map from the E_∞ cofiber of the composite, namely tmf, to the E_∞ cofiber of the right-hand map, namely KO. ◻

On coefficients, the map r is just

$$KO_*[j^{-1}] \to KO_* : \quad j^{-1} \mapsto 0.$$

Despite the obvious splitting of r at the level of coefficients, it is not clear whether or not there exists an E_∞ map from KO to tmf.

6 Co-operations for $K(1)$-local tmf

The preceding Theorem 5.7 gave a presentation of $K(1)$-local tmf in terms of finitely many E_∞ cells. We can now use this presentation to describe the $K(1)$-localization of tmf $\wedge$ tmf.

Theorem 6.1 *The homotopy groups of* tmf $\wedge$ tmf *are given by*

$$\pi_*(\mathrm{tmf} \wedge \mathrm{tmf}) = KO_* \otimes \mathbb{Z}_p[f, \overline{f}] \otimes \mathbb{T}(\ell)/(\psi^p(\ell) - \ell - f + \overline{f}).$$

Proof Write $F : \mathbb{P}(S^0) \to T_\zeta$ for the map sending the generator $x \in KO_0\mathbb{P}(S^0) = \mathbb{T}(x)$ to $\theta(f) - h(f)$. We saw in the previous section that F induces a pro-free map on KO-homology, and that

$$\mathrm{tmf} = S^0 \wedge^F_{\mathbb{P}(S^0)} T_\zeta .$$

Therefore,

$$\mathrm{tmf} \wedge \mathrm{tmf} = (S^0 \wedge^F_{\mathbb{P}(S^0)} T_\zeta) \wedge (T_\zeta \,{}^F\!\wedge_{\mathbb{P}(S^0)} S^0) \simeq (S^0 \wedge S^0) \wedge^{F\wedge F}_{\mathbb{P}(S^0)\wedge\mathbb{P}(S^0)} (T_\zeta \wedge T_\zeta).$$

Since $F : KO_*\mathbb{P}(S^0) \to KO_* T_\zeta$ is flat, this has KO-homology

$$\begin{aligned} KO_*(\mathrm{tmf} \wedge \mathrm{tmf}) &= (KO_*(T_\zeta) \otimes_{KO_*} KO_*(T_\zeta)) \otimes^{F\otimes F}_{KO_*(\mathbb{P}S^0 \wedge \mathbb{P}S^0)} KO_* \\ &= KO_* \otimes \mathbb{T}(b, \overline{b}) \otimes_{F\otimes F, \mathbb{T}(x,\overline{x})} \mathbb{Z}_p . \end{aligned}$$

But $KO_* \otimes \mathbb{T}(b, \overline{b})$ is an extended KO_*KO-comodule, and $(F \otimes F)$ factors through its fixed points, which are

$$\pi_*(T_\zeta \wedge T_\zeta) = \mathbb{T}(f, \overline{f}, \ell)/(\psi^p(\ell) - \ell - f + \overline{f}).$$

By the arguments of Theorem 5.7, $KO_*(\mathrm{tmf} \wedge \mathrm{tmf})$ is also extended, and

$$\begin{aligned}\pi_*(\mathrm{tmf} \wedge \mathrm{tmf}) &= KO_* \otimes \mathbb{T}(f, \overline{f}, \ell)/(\theta(f) - h(f), \theta(\overline{f}) - h(\overline{f}), \psi^p(\ell) \\ &\quad - \ell - f + \overline{f}) \\ &\cong KO_*[f, \overline{f}] \otimes \mathbb{T}(\ell)/(\psi^p(\ell) - \ell - f + \overline{f}).\end{aligned}$$

□

Remark 6.2 For a more modular presentation of this ring, recall from Proposition 5.3 that $f = \alpha(j^{-1})$ for some invertible power series $\alpha \in \mathbb{Z}_p[j^{-1}]$. Thus, $KO_*[f, \overline{f}] = KO_*[j^{-1}, \overline{j^{-1}}]$. Letting

$$\lambda = \alpha(\ell),$$

we can equivalently write

$$\mathrm{tmf}_*\mathrm{tmf} = KO_* \otimes \mathbb{Z}_p[j^{-1}, \overline{j^{-1}}] \otimes \mathbb{T}(\lambda)/(\psi^p(\lambda) - \lambda - j^{-1} + \overline{j^{-1}}).$$

We now consider the Hopf algebroid for $K(1)$-local tmf.

To obtain $\mathrm{tmf}_*\mathrm{tmf}$ from $T_{\zeta,*}T_\zeta$, we take the θ-algebra quotient induced by the relation $\theta(f) = h(f)$, and the same relation for $\overline{f}$. We obtain

$$\mathrm{tmf}_*\mathrm{tmf} = \mathrm{tmf}_* \otimes \mathbb{T}(\ell)/(\theta(f + \ell - \psi^p(\ell)) - h(f + \ell - \psi^p(\ell))), \tag{6.3}$$

where again the quotient is by a θ-ideal.

This formula should be compared to the analogous one for $K(1)$-local KO-cooperations: as a θ-algebra, we have

$$KO_*KO \cong KO_* \otimes \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p) \cong KO_* \otimes \mathbb{T}(b)/(\psi^p(b) - b),$$

where the last isomorphism follows from Proposition 3.22. That is, KO_*KO is generated as a θ-algebra over KO_* by a single generator b, with an algebraic relation between b and $p\theta(b)$. Likewise, $\mathrm{tmf}_*\mathrm{tmf}$ is generated over tmf_* by a single generator ℓ, with an algebraic relation over the coefficient ring $\mathbb{Z}_p[f]$ that relates ℓ, $\theta(\ell)$, and $p\theta_2(\ell)$. One can think of this as a second-order version of the θ-algebraic structure underlying KO_*KO.

Now, the coalgebra presentation $KO_*KO = KO_* \otimes \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p)$ is in fact more useful than the algebra presentation $KO_*KO = KO_* \otimes \mathbb{T}(b)/(\psi^p(b) - b)$. The former allows the simple computation of the KO-based Adams spectral sequence for arbitrary X: its E_2 page is just the group cohomology

$$E_2^{*,*} = H^*_{cts}(\mathbb{Z}_p^\times/\mu, KO_*X),$$

and as this is concentrated on two lines, the spectral sequence collapses at E_2 and always converges. As it turns out, very similar statements are true for tmf.

Proposition 6.4 *The left unit* $\mathrm{tmf}_* \to \mathrm{tmf}_*\mathrm{tmf}$ *is pro-free.*

Proof While one can prove pro-freeness algebraically by applying Lemmas 2.5 and 2.6 to the formula (6.3), it is easier to use Laures's [21, Corollary 3], which gives an additive equivalence of homology theories

$$\mathrm{tmf}_*X \cong KO_*X[j^{-1}],$$

and correspondingly an additive equivalence of $K(1)$-local spectra

$$\mathrm{tmf} \simeq KO[j^{-1}] = \bigvee_{n=1}^{\infty} KO.$$

Thus, to show that $\mathrm{tmf}_*\mathrm{tmf}$ is pro-free over tmf_*, it suffices to show that $KO_*\mathrm{tmf}$ is pro-free over KO_*. From Lemma 2.7, one observes that the property of KO_*X being pro-free over KO_* is closed under coproducts. As KO_*KO is pro-free over KO_*, and tmf is a coproduct of copies of KO, $KO_*\mathrm{tmf}$ is also pro-free. □

Corollary 6.5 *There is an L-complete Hopf algebroid* $(\mathrm{tmf}_*, \mathrm{tmf}_*\mathrm{tmf})$. *For any* $K(1)$*-local spectrum* X*, the* $K(1)$*-local Adams spectral sequence based on* tmf *is conditionally convergent and takes the form*

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathrm{tmf}_*\mathrm{tmf}}(\mathrm{tmf}_*, \mathrm{tmf}_*X) \Rightarrow \pi_{t-s}X.$$

Proof Since $\mathrm{tmf}_* \to \mathrm{tmf}_*\mathrm{tmf}$ is pro-free, one has an L-complete Hopf algebroid by Definition 2.9. Then by Proposition 2.20, the E_2 page of the Adams spectral sequence has the form described.

To establish convergence, one needs to show that X is $K(1)$-local tmf-nilpotent. Recall from [10, Appendix 1] and [6] that this is the largest class of $K(1)$-local spectra containing tmf and closed under retracts, cofibers, and $K(1)$-local smash products with arbitrary spectra. Now, multiplication by j^{-1} gives a cofiber sequence

$$\mathrm{tmf} \xrightarrow{j^{-1}} \mathrm{tmf} \to KO,$$

so that KO is $K(1)$-local tmf-nilpotent. The cofiber sequence

$$S \to KO \to KO$$

then shows that the sphere is $K(1)$-local tmf-nilpotent. This clearly implies that an arbitrary spectrum is $K(1)$-local tmf-nilpotent. □

We can now prove Theorem B.

Theorem 6.6 *Suppose that* tmf_*X *is pro-free over* tmf_*. *Then there is a natural isomorphism*

$$\mathrm{Ext}^{*,*}_{\mathrm{tmf}_*\mathrm{tmf}}(\mathrm{tmf}_*, \mathrm{tmf}_*X) \cong \mathrm{Ext}^{*,*}_{KO_*KO}(KO_*, KO_*X).$$

Proof The ring map tmf $\to KO$ induces a map of Hopf algebroids,

$$(\mathrm{tmf}_*, \mathrm{tmf}_*\mathrm{tmf}) \to (KO_*, KO_*KO).$$

The map $\mathrm{tmf}_* \to KO_*$ sends j^{-1} to zero, and thus sends $f = j^{-1} + O(pj^{-1}, j^{-2})$ to zero as well. We have

$$\begin{aligned}
&KO_* \otimes_{\mathrm{tmf}_*} \mathrm{tmf}_*\mathrm{tmf} \otimes_{\mathrm{tmf}_*} KO_* \\
&\quad = KO_* \otimes_{\mathrm{tmf}_*} (KO_* \otimes \mathbb{Z}_p[f, \overline{f}] \otimes \mathbb{T}(\ell)/(f - \overline{f} - \psi^p(\ell) + \ell)) \otimes_{\mathrm{tmf}_*} KO_* \\
&\quad \cong KO_* \otimes \mathbb{T}(\ell)/(\psi^p(\ell) - \ell).
\end{aligned}$$

We need to identify the image of ℓ in KO_*KO. Consider the commuting square

$$\begin{tikzcd}
\mathrm{tmf}_*\mathrm{tmf} \arrow[r] \arrow[d] & KO_*(\mathrm{tmf} \wedge \mathrm{tmf}) \arrow[d] \\
KO_*KO \arrow[r] & KO_*(KO \wedge KO).
\end{tikzcd}$$

The horizontal maps are both inclusions of KO_*KO-primitives, and, in particular, injective. Going from $\mathrm{tmf}_*\mathrm{tmf}$ to $KO_*(KO \wedge KO)$ around the top right corner sends ℓ to $b - \overline{b}$, where we recall that

$$b \in KO_0KO \cong \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p)$$

is the identity map on $\mathbb{Z}_p$. Using the Hopf algebroid formulas found in Theorem 2.30, together with the group isomorphism $\mathbb{Z}_p^\times/\mu \cong \mathbb{Z}_p$, we have

$$b - \overline{b} = \eta_L(b) - \eta_R(b) \in \mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_p) : (x, y) \mapsto x - y.$$

In the notation of Lemma 2.19, the primitives are included into $\mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p \times \mathbb{Z}_p, KO_*)$ via precomposition with

$$m : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_p : \ m(x, y) = x - y.$$

Thus, $b - \overline{b}$ is precisely $m^*(b)$. This proves that the map $\mathrm{tmf}_*\mathrm{tmf} \to KO_*KO$ sends ℓ to b. It follows that the map

$$KO_* \otimes_{\mathrm{tmf}_*} \mathrm{tmf}_*\mathrm{tmf} \otimes_{\mathrm{tmf}_*} KO_* \to KO_*KO$$

is an isomorphism.

Using the fiber sequence

$$\mathrm{tmf} \xrightarrow{j^{-1}} \mathrm{tmf} \to KO,$$

one obtains

$$\mathrm{tmf}_*KO = \mathrm{tmf}_*\mathrm{tmf}/(\overline{j^{-1}}) = \mathrm{tmf}_*\mathrm{tmf}/(\overline{f}) = KO_*[f] \otimes \mathbb{T}(\ell)/(f - \psi^p(\ell) + \ell).$$

There is a pushout square of L-complete rings

$$\begin{tikzcd}
\mathbb{T}(f) \arrow[r, "f \mapsto \psi^p(b) - b"] \arrow[d] & \mathbb{T}(b) \arrow[d, "b \mapsto \ell"] \\
KO_*[f] \arrow[r] & KO_*[f] \otimes \mathbb{T}(\ell)/(f - \psi^p(\ell) + \ell).
\end{tikzcd}$$

The top horizontal map is pro-free by Lemma 1. By Lemma 2.5, the bottom horizontal map is also pro-free.

Since tmf_*X is pro-free over tmf_*, we have

$$KO_*X \cong \pi_*(KO \wedge_{\mathrm{tmf}} (\mathrm{tmf} \wedge X)) \cong KO_* \otimes_{\mathrm{tmf}_*} \mathrm{tmf}_*X.$$

By the change of rings theorem, Proposition 2.23, the induced map on Ext is an equivalence. □

Corollary 6.7 *The $K(1)$-local* tmf*-based Adams spectral sequence for the sphere collapses at E_2, where it is concentrated on the 0 and 1 lines.*

Proof This follows immediately from the above theorem and Proposition 2.36. □

7 Connections to number theory

In this section, we recall the relationship between $K(1)$-local tmf and p-adic modular forms in the sense of Katz.

Definition 7.1 Let $\mathscr{M}_{\mathrm{ell}}^{\mathrm{ord}}$ be the p-complete moduli stack of generalized elliptic curves with ordinary reduction at p. Let ω be the line bundle of invariant differentials on $\mathscr{M}_{\mathrm{ell}}^{\mathrm{ord}}$. An ordinary modular form of weight k is a global section of $\omega^{\otimes k}$ over $\mathscr{M}_{\mathrm{ell}}^{\mathrm{ord}}$

A modular form of any weight over $\mathbb{Z}_p$ has a q-expansion in $\mathbb{Z}_p[[q]]$, given by evaluating it on the Tate curve. In [20], Katz defined the ring D of divided congruences to comprise those power series $f(q) \in \mathbb{Z}_p[[q]]$ such that, for some n, $p^n f(q)$ is a sum of q-expansions of modular forms (of possibly different weights). Equivalently, if R is the ring of modular forms over $\mathbb{Z}_p$, then $D \subseteq R[1/p]$ is the subring generated by those linear combinations of modular forms which have integral q-expansion.

As Katz shows, the ring of divided congruences also has a modular interpretation.

Definition 7.2 Let $\mathscr{M}_{\mathrm{fg}}^{\mathrm{mult}}$ be the p-complete moduli stack of one-dimensional formal groups with ordinary reduction at p.

As is well-known, any such formal group is étale-locally isomorphic to the multiplicative formal group $\widehat{\mathbb{G}_m}$. As

$$\underline{\mathrm{Aut}}(\widehat{\mathbb{G}_m}) \cong \mathbb{Z}_p^\times,$$

we have

$$\mathscr{M}_{\mathrm{fg}}^{\mathrm{mult}} \cong B\mathbb{Z}_p^\times.$$

Definition 7.3 Let $\mathscr{M}_{\mathrm{ell}}^{triv}$ be the moduli of trivialized elliptic curves over $\mathrm{Spf}\,\mathbb{Z}_p$, defined as the pullback in the diagram

$$\begin{tikzcd}
\mathscr{M}_{\mathrm{ell}}^{triv} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{M}_{\mathrm{ell}}^{\mathrm{ord}} \arrow[d] \\
\mathrm{Spf}\,\mathbb{Z}_p \arrow[r, "\widehat{\mathbb{G}_m}"'] & \mathscr{M}_{\mathrm{fg}}^{\leq 1} = B\mathbb{Z}_p^\times.
\end{tikzcd}$$

In other words, $\mathscr{M}_{\mathrm{ell}}^{triv}$ is a $\mathbb{Z}_p^\times$-Galois cover of $\mathscr{M}_{\mathrm{fg}}$, and represents the functor

$$R \mapsto \{(E, \alpha) : E \text{ an elliptic curve over } R, \quad \alpha : \widehat{\mathbb{G}_m} \xrightarrow{\sim} \widehat{E}\}.$$

This is a functor to groupoids, where an isomorphism $(E, \alpha) \to (E', \alpha')$ is a morphism of elliptic curves $f : E \to E'$ such that the square

$$\begin{tikzcd}
\widehat{\mathbb{G}_m} \arrow[r, equal] \arrow[d, "\alpha"'] & \widehat{\mathbb{G}_m} \arrow[d, "\alpha'"] \\
\widehat{E} \arrow[r, "f"'] & \widehat{E}'
\end{tikzcd}$$

commutes.

Theorem 7.4 [20] *The moduli problem $\mathscr{M}_{\mathrm{ell}}^{triv}$ is representable by a p-adic affine formal scheme* $\mathrm{Spf}\, V_\infty$*, and there is a canonical isomorphism $V_\infty \cong D_p^\wedge$.*

The modular definition of V_∞ as the ring of functions on $\mathscr{M}_{\mathrm{ell}}^{triv}$ also gives it additional algebraic structure. First, as it is the ring of functions on a $\mathbb{Z}_p^\times$-Galois cover of $\mathscr{M}_{\mathrm{ell}}^{\mathrm{ord}}$, it has a natural $\mathbb{Z}_p^\times$-action. This action can also be defined in terms of divided congruences [20, 2.4.1], by requiring that $[a] \in \mathbb{Z}_p^\times$ act on a modular form of weight k by

$$[a] f_k = a^k f_k,$$

and extending to V_∞ by linearity and continuity.

Second, let $(E, \alpha : \widehat{\mathbb{G}_m} \xrightarrow{\sim} \widehat{E})$ be a point of $\mathscr{M}_{\text{ell}}^{triv}$ over a p-complete ring R, and let E_0 be the mod p reduction of E. Since E is ordinary, the relative Frobenius on E_0,

$$F_0 : E_0 \to E_0^{(p)},$$

has kernel a connected group scheme of rank p, which the trivialization α identifies with $\mu_p \subseteq \mathbb{G}_m$. The isogeny F_0 deforms uniquely to one of the form

$$F : E \to E^{(p)},$$

and again α identifies the kernel of F with μ_p. Then there is an induced trivialization

$$\begin{tikzcd}
\mu_p \arrow[r] \arrow[d, "\sim"'] & \widehat{\mathbb{G}_m} \arrow[r, "{[p]}"] \arrow[d, "\alpha"] & \widehat{\mathbb{G}_m} \arrow[d, dashed, "\alpha^{(p)}"] \\
\widehat{E}[p] \arrow[r] & \widehat{E} \arrow[r, "F"'] & \widehat{E^{(p)}}.
\end{tikzcd}$$

The mapping

$$(E, \alpha) \mapsto (E^{(p)}, \alpha^{(p)})$$

defines a ring endomorphism $\psi^p : V_\infty \to V_\infty$. By [5, Lemma 5.4], the $\mathbb{Z}_p^\times$-action and the operator ψ^p define a ψ-θ-algebra structure on V_∞.

Now, since $\mathscr{M}_{\text{ell}}^{triv} = \operatorname{Spf} V_\infty$ is an affine formal scheme and a Galois cover of $\mathscr{M}_{\text{ell}}^{\text{ord}}$, one can compute the cohomology of $\mathscr{M}_{\text{ell}}^{\text{ord}}$, and in particular, the ring $\Gamma(\mathscr{M}_{\text{ell}}^{\text{ord}}, \mathcal{O}_{\mathscr{M}_{\text{ell}}^{\text{ord}}})$ of ordinary p-adic modular forms, as the group cohomology of $\mathbb{Z}_p^\times$ acting on $\operatorname{Spf} V_\infty$.

Theorem 7.5 *We have*

$$H^0(\mathbb{Z}_p^\times, V_\infty) = \mathbb{Z}_p[j^{-1}],$$

and $H^1(\mathbb{Z}_p^\times, V_\infty) = 0$. Thus, the ring of weight 0 ordinary p-adic modular forms is $\mathbb{Z}_p[j^{-1}]$.

There are several different ways to prove this, but below, we show how it can be recovered from known information about tmf.

First, we recall from [5] that the (uncompleted) moduli of generalized elliptic curves can be equipped with a sheaf $\mathscr{O}^{der}$ of locally even periodic E_∞ ring spectra, such that there is an isomorphism of sheafs of rings

$$\pi_* \mathscr{O}^{der} \cong \omega^{\otimes */2},$$

where ω is the line bundle of invariant differentials on $\mathscr{M}_{\text{ell}}$. The global sections of $\mathscr{O}^{der}$ over $\mathscr{M}_{\text{ell}}$ are the unlocalized spectrum Tmf. By standard results on E_∞-rings, this sheaf can be pulled back to $\mathscr{M}_{\mathrm{ell}}^{\mathrm{ord}}$, and

$$\Gamma(\mathscr{M}_{\mathrm{ell}}^{\mathrm{ord}}, \mathscr{O}^{der}) = L_{K(1)}\mathrm{Tmf} \simeq L_{K(1)}\mathrm{tmf}.$$

Likewise, it can be lifted along the pro-étale cover $\mathscr{M}_{\mathrm{ell}}^{triv} \to \mathscr{M}_{\mathrm{ell}}^{\mathrm{ord}}$.

Proposition 7.6 *The global sections of $\mathscr{O}^{der}$ along $\mathscr{M}_{\mathrm{ell}}^{triv}$ are*

$$\Gamma(\mathscr{M}_{\mathrm{ell}}^{triv}, \mathscr{O}^{der}) = L_{K(1)}(K \wedge \mathrm{tmf}).$$

Proof The group $\mathbb{Z}_p^\times$ acts on the p-complete K-theory spectrum K by E_∞ automorphisms. Therefore, the constant sheaf K on Spf $\mathbb{Z}_p$ descends to a locally even periodic spectral sheaf on $\mathscr{M}_{\mathrm{fg}}^{\mathrm{mult}}$. (This is special to height 1: for example, the moduli of p-complete, height ≤ 2 formal groups does not admit such a spectral enrichment.)

We have

$$\Gamma(\mathscr{M}_{\mathrm{fg}}^{\leq 1}, \mathscr{O}^{der}) = K^{h\mathbb{Z}_p^\times} = L_{K(1)}S.$$

Also, the line bundle $\pi_2\mathscr{O}^{der}$ on $\mathscr{M}_{\mathrm{fg}}^{\leq 1}$ is isomorphic to the line bundle that sends a formal group to its invariant differentials.

Thus, the pullback square (7.3) extends to a square of p-complete nonconnective spectral schemes. Taking global sections, which send pullbacks of nonconnective spectral schemes to pushouts of E_∞-rings [23, 1.1.5.6], we get an equivalence of p-complete E_∞-rings

$$K \wedge_{L_{K(1)}S} L_{K(1)}\mathrm{tmf} \simeq \Gamma(\mathscr{M}_{\mathrm{ell}}^{triv}, \mathscr{O}^{der}).$$

But $K(1)$-localization is smashing in the category of p-complete spectra (because $E(1)$-localization is smashing in the category of all spectra). So the left-hand side is precisely $L_{K(1)}(K \wedge \mathrm{tmf})$. □

Corollary 7.7 *We have*

$$K_*\mathrm{tmf} = V_\infty[u^{\pm 1}].$$

Proof The stack $\mathscr{M}_{\mathrm{ell}}^{triv} = \mathrm{Spf}\ V_\infty$ is an affine formal scheme, so the line bundles $\omega^{\otimes k}$ have no higher cohomology, and the descent spectral sequence

$$E_2^{s,2t} = H^s(\mathscr{M}_{\mathrm{ell}}^{triv}, \omega^{\otimes t}) \Rightarrow \pi_{2t-s}(K_*\mathrm{tmf})$$

collapses. Moreover, the line bundle ω is trivial. In fact, given a trivialized elliptic curve

$$(E, \alpha : \widehat{\mathbb{G}_m} \xrightarrow{\sim} \widehat{E}),$$

there is a canonical invariant differential on E, namely the pullback of the invariant differential $dT/T \in \omega_{\mathbb{G}_m}$ along α^{-1}. Letting u be a basis for ω over V_∞, the claim follows. □

Proof of Theorem 7.5 We will in fact show that $V_\infty^{\mathbb{Z}_p^\times} = \pi_0\mathrm{tmf}$ at all primes, which implies the above result at $p \leq 5$.

The homotopy fixed points spectral sequence computing the homotopy groups of $K(1)$-local tmf thus takes the form

$$E_2^{*,*} = H^*(\mathbb{Z}_p^\times, V_\infty[u^{\pm 1}]) \Rightarrow \pi_*\mathrm{tmf}.$$

If $p > 2$, then $\mathbb{Z}_p^\times$ has cohomological dimension 1, so the spectral sequence collapses at E_2. As we know, $\pi_* L_{K(1)}\mathrm{tmf}$ is concentrated in even degrees at these primes, which means that the group cohomology is concentrated in H^0. The statement is immediate at $p > 2$.

At $p = 2$, we need to do a little more work, first by understanding the spectral sequence

$$E_2^{*,*} = H^*(\{\pm 1\}, V_\infty[u^{\pm 1}]) \Rightarrow KO_*\mathrm{tmf}. \tag{7.8}$$

Given a trivialized elliptic curve (E, α) over R, the group acts on it by

$$[-1](E, \alpha) = (E, \widehat{\mathbb{G}_m} \xrightarrow{[-1]} \widehat{\mathbb{G}_m} \xrightarrow{\alpha} \widehat{E}).$$

But this is isomorphic in the groupoid $\mathscr{M}_{\mathrm{ell}}^{triv}(R)$ to (E, α), as is shown by the square

$$\begin{tikzcd} \widehat{G_m} \arrow[r, equal] \arrow[d, "\alpha"'] & \widehat{\mathbb{G}_m} \arrow[d, "\alpha\circ[-1]"] \\ \widehat{E} \arrow[r, "{[-1]}"'] & \widehat{E}. \end{tikzcd}$$

Thus, $\{\pm 1\}$ acts trivially on V_∞. Since the spectral sequence (7.8) is compatible with the spectral sequence

$$E_2^{*,*} = H^*(\{\pm 1\}, \mathbb{Z}_2[u^{\pm 1}]) \Rightarrow KO_*,$$

one easily obtains

$$KO_*\mathrm{tmf} \cong KO_* \otimes V_\infty.$$

The residual spectral sequence,

$$E_2^{*,*} = H^*(\mathbb{Z}_2^\times/\{\pm 1\}, V_\infty \otimes KO_*) \Rightarrow \pi_*\mathrm{tmf},$$

is again concentrated in cohomological degrees 0 and 1, so collapses at E_2. Since $\pi_{-1}\mathrm{tmf} = 0$, we have

$$H^1(\mathbb{Z}_2^\times/\{\pm 1\}, V_\infty) = 0.$$

This surjects onto $KO_1\mathrm{tmf} = V_\infty/(2)$, and we get

$$H^1(\mathbb{Z}_2^\times/\{\pm 1\}, V_\infty/(2)) = 0.$$

Thus,

$$\pi_0\mathrm{tmf} = H^0(\mathbb{Z}_2^\times/\{\pm 1\}, V_\infty) = V_\infty^{\mathbb{Z}_2^\times},$$

proving the claim about H^0. Finally,

$$H^1(\mathbb{Z}_2^\times, V_\infty) = H^1(\{\pm 1\}, \mathbb{Z}_2[j^{-1}]) = 0$$

as the coefficients are torsion-free. □

Combining this method with the results of Sect. 7 yields the following.

Theorem 7.9 *Let $\mathbb{Z}_p^\times$ act diagonally on $V_\infty \otimes_{\mathbb{Z}_p} V_\infty$. Then*

$$\begin{aligned} H^0(\mathbb{Z}_p^\times, V_\infty \otimes V_\infty) &= \pi_0(\mathrm{tmf} \wedge \mathrm{tmf}) \\ &= \mathbb{Z}_p[j^{-1}, \overline{j^{-1}}] \otimes \mathbb{T}(\lambda)/(\psi^p(\lambda) - \lambda - j^{-1} + \overline{j^{-1}}), \\ H^1(\mathbb{Z}_p^\times, V_\infty \otimes V_\infty) &= 0. \end{aligned}$$

Proof Begin by considering the pullback diagram

$$\begin{tikzcd} \mathscr{M}_{\mathrm{ell}}^{triv} \times \mathscr{M}_{\mathrm{ell}}^{triv} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{M}_{\mathrm{ell}}^{triv} \arrow[d] \\ \mathscr{M}_{\mathrm{ell}}^{triv} \arrow[r] & \mathrm{Spf}\, \mathbb{Z}_p. \end{tikzcd}$$

All the maps in this diagram are pro-étale, so the spectral enrichment of $\mathscr{M}_{\mathrm{ell}}^{triv}$ pulls back to one on $\mathscr{M}_{\mathrm{ell}}^{triv} \times \mathscr{M}_{\mathrm{ell}}^{triv}$. Taking global sections, we see that

$$\Gamma(\mathscr{M}_{\mathrm{ell}}^{triv} \times \mathscr{M}_{\mathrm{ell}}^{triv}, \mathscr{O}^{der}) = (K \wedge \mathrm{tmf}) \wedge_K (K \wedge \mathrm{tmf}) \simeq K \wedge \mathrm{tmf} \wedge \mathrm{tmf}.$$

Thus, there is a homotopy fixed points spectral sequence

$$\begin{aligned} E_2^{*,*} &= H^*(\mathbb{Z}_p^\times, V_\infty[u^{\pm 1}] \otimes_{\mathbb{Z}_p[u^{\pm 1}]} V_\infty[u^{\pm 1}]) \\ &= H^*(\mathbb{Z}_p^\times, (V_\infty \otimes V_\infty)[u^{\pm 1}]) \Rightarrow \pi_*(\mathrm{tmf} \wedge \mathrm{tmf}). \end{aligned}$$

The group $\mathbb{Z}_p^\times$ acts diagonally on $V_\infty \otimes V_\infty$.

From this point, we can follow the arguments of Theorem 7.5. If $p > 2$, then $\mathbb{Z}_p^\times$ has cohomological dimension 1, and the fact that $\pi_{-1}(\mathrm{tmf}\wedge\mathrm{tmf}) = 0$ yields the result.

If $p = 2$, then we first take the fixed points of $\{\pm 1\} \subseteq \mathbb{Z}_2^\times$. Since this acts trivially on $V_\infty \otimes V_\infty$, we obtain

$$KO_*(\mathrm{tmf} \wedge \mathrm{tmf}) \cong KO_* \otimes V_\infty \otimes V_\infty.$$

This is zero in degree -1 and $\mathbb{Z}_2^\times/\{\pm 1\}$ has cohomological dimension 1, from which the result quickly follows. Again, we need the fact that $\pi_0(\mathrm{tmf} \wedge \mathrm{tmf})$ is torsion-free to prove the statement about H^1. $\square$

Definition 7.10 An ordinary 2-variable modular form of weight k is a section of $\omega^{\otimes k}$ on the moduli stack

$$\mathscr{M}_{\mathrm{ell}}^{\mathrm{ord}} \times_{\mathscr{M}_{\mathrm{fg}}^{\mathrm{mult}}} \mathscr{M}_{\mathrm{ell}}^{\mathrm{ord}}$$

which parametrizes pairs of ordinary elliptic curves together with an isomorphism of their formal groups.

Corollary 7.11 *The ring of weight 0 ordinary 2-variable modular forms is*

$$\mathbb{Z}_p[j^{-1}, \overline{j^{-1}}] \otimes \mathbb{T}(\lambda)/(\psi^p(\lambda) - \lambda - j^{-1} + \overline{j^{-1}}).$$

Proof There is a $\mathbb{Z}_p^\times$-Galois cover with affine source,

$$\mathscr{M}_{\mathrm{ell}}^{triv} \times \mathscr{M}_{\mathrm{ell}}^{triv} \to \mathscr{M}_{\mathrm{ell}}^{\mathrm{ord}} \times_{\mathscr{M}_{\mathrm{fg}}^{\mathrm{mult}}} \mathscr{M}_{\mathrm{ell}}^{\mathrm{ord}}.$$

So the cohomology of $\mathscr{M}_{\mathrm{ell}}^{\mathrm{ord}} \times_{\mathscr{M}_{\mathrm{fg}}^{\mathrm{mult}}} \mathscr{M}_{\mathrm{ell}}^{\mathrm{ord}}$ is the same as

$$H^*(\mathbb{Z}_p^\times, V_\infty \otimes V_\infty).$$

The result now follows from Theorem 7.9. $\square$

Remark 7.12 An ordinary 2-variable modular form of weight 0 may be thought of as a function on triples

$$(E, E', \alpha : \widehat{E} \xrightarrow{\sim} \widehat{E}').$$

Clearly, some of these come from ordinary 1-variable modular forms, by evaluating them on E or E' and forgetting the rest of the data. By Theorem 7.5, these generate a copy of $\mathbb{Z}_p[j^{-1}, \overline{j^{-1}}]$ inside the ring of weight 0 ordinary 2-variable modular forms.

The corollary then implies that the entire ring of weight 0 ordinary 2-variable modular forms is generated, *as a θ-algebra*, by a single other element. Naturally, one wonders what this element is. At the prime 2, one can take

$$b = -\frac{\log c_4/w}{\log 3^4} \in KO_0\mathrm{tmf}$$

(see Remark 5.2). Then

$$\ell = b - \overline{b} = \frac{\log \overline{c_4} - \log c_4}{\log 3^4} \in \mathrm{tmf}_0\mathrm{tmf},$$

where we have tacitly used periodicity to move 2-variable modular forms to weight 0. Thus, ℓ is a 2-adic unit times

$$\frac{\log \overline{c_4} - \log c_4}{16} = \frac{1}{16}\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n}((\overline{c_4} - 1)^n - (c_4 - 1)^n)\right).$$

The convergence and integrality of the expression can be checked using the q-expansion

$$c_4 = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n.$$

Finally, λ is related to ℓ by a (non-explicit) composition-invertible power series. Thus, the ring of ordinary 2-variable modular forms is generated as a θ-algebra by j^{-1}, $\overline{j^{-1}}$, and

$$\frac{\log \overline{c_4} - \log c_4}{16}.$$

At the prime 3, one likewise has the θ-algebra generator

$$\frac{\log \overline{c_6} - \log c_6}{9}.$$

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Appendix A: λ-Rings and Hopf algebras

This section collects useful algebra related to the multiplicative theory of $K(1)$-local spectra. As we discuss in Appendix A.1, any $K(1)$-local E_∞-ring has power operations on its π_0 making it into a θ-algebra (cf. [7,28]). We recall Bousfield's description of the free θ-algebra functor and note that it takes values in Hopf algebras. In Appendix A.2, we recall the definition of λ-rings, which are closely related to θ-algebras—see [7]. Unlike θ-algebras, which are a vital feature of $K(1)$-local homotopy theory, λ-rings will largely play a technical role in some of the proofs in this paper. For this reason, we take the opportunity to clarify some of the ways of passing between λ-rings and θ-algebras.

For the most part, we restrict to working with modules which are p-complete rather than merely L-complete. Let $\mathsf{Mod}^{\wedge}_{\mathbb{Z}_p}$ denote the category of L-complete $\mathbb{Z}_p$-modules, and recall from Sect. 1.2 that all algebraic statements carry a tacit completion.

A.1 E_∞-rings and θ-algebras

Definition A.1 A θ-algebra is an L-complete $\mathbb{Z}_p$-algebra R equipped with operations $\theta : R \to R$

$$\begin{aligned} \theta(x+y) &= \theta(x) + \theta(y) - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x^i y^{p-i}, \\ \theta(xy) &= x^p\theta(y) + y^p\theta(x) + p\theta(x)\theta(y) \qquad \text{for } x, y \in R_0, \\ \theta(1) &= 0. \end{aligned}$$

We will write $\psi^p(x) = x^p + p\theta(x)$ for x in degree zero. Note that the above formulas imply that ψ^p is a ring homomorphism in degree zero. Conversely, if R is p-torsion-free, then θ can be uniquely recovered from a ring homomorphism ψ^p satisfying $\psi^p(x) \equiv x^p \bmod p$.

Definition A.2 A ψ-θ-algebra is a p-complete θ-algebra R together with maps $\psi^k : R \to R$ for $k \in \mathbb{Z}_p^\times$ such that

(1) ψ^k is multiplicative on R,
(2) $k \mapsto \psi^k$ is a continuous endomorphism from $\mathbb{Z}_p^\times$ to the monoid of endomorphisms of R_*,
(3) and each ψ^k commutes with θ and ψ^p.

Proposition A.3 [8, Chapter IX], [12] *If X is a $K(1)$-local E_∞-ring spectrum such that K_*X is p-complete, then K_0X is naturally a ψ-θ-algebra, with ψ^k for $k \in \mathbb{Z}_p^\times$ given by the Adams operations.*

Since the Adams operations commute with the θ-algebra structure, the θ-algebra structure passes through the homotopy fixed points spectral sequence. Thus, if X is a $K(1)$-local E_∞-ring spectrum, $\pi_0 X$ is also a θ-algebra. In other words, the classes in $K_0 B\Sigma_p$ representing the power operations θ and ψ^p lift to $\pi_0 L_{K(1)} B\Sigma_p$—see [14].

Proposition A.4 *The θ-algebra structures on $\pi_0 K = \pi_0 KO = \mathbb{Z}_p$, on KO_0KO, and on K_0K are all given by $\psi^p = \mathrm{id}$.*

Proof There is a unique θ-algebra structure on $\mathbb{Z}_p$ satisfying the requirements of Definition A.2, and it is $\psi^p = \mathrm{id}$.

As for K_0K (the proof for KO_0KO is similar), the multiplication map $K \wedge K \to K$ is an E_∞-map. By Theorem 2.30, the map induced on π_0 is

$$\mathrm{Maps}_{\mathrm{cts}}(\mathbb{Z}_p^\times, \mathbb{Z}_p) \ni f \mapsto f(1) \in \mathbb{Z}_p.$$

Thus,

$$(\psi^p f)(1) = \psi^p(f(1)) = f(1).$$

Moreover, ψ^p commutes with the left action of the Adams operations, which act by

$$(\psi^k \wedge 1)(f)(x) = f(kx).$$

It follows that

$$(\psi^p f)(k) = f(k)$$

for every $k \in \mathbb{Z}_p^\times$. Thus, ψ^p acts by the identity. □

There is an adjunction

$$\mathbb{T} : \mathsf{Mod}_{\mathbb{Z}_p}^\wedge \rightleftarrows \mathsf{Alg}_\theta : U$$

where U is the forgetful functor, and $\mathbb{T}$ is the free θ-algebra functor. It is described explicitly as follows:

Theorem A.5 [7, 2.6, 2.9] *The free θ-algebra on a single generator x is a polynomial algebra: explicitly,*

$$\mathbb{T}(x) = \mathbb{Z}_p[x, \theta(x), \theta\theta(x), \ldots]_p^\wedge \cong \mathbb{Z}_p[x, \theta_1(x), \theta_2(x), \ldots]_p^\wedge,$$

where the elements $\theta_n(x)$ are inductively defined so that

$$\psi^{p^n}(x) = x^{p^n} + p\theta_1(x)^{p^{n-1}} + \cdots + p^n\theta_n(x).$$

Theta-algebras function as an algebraic approximation to $K(1)$-local E_∞-algebras, as was shown in the following form by [2,28] following work of [8]. Write $\mathbb{P} : \mathsf{Sp} \to \mathsf{CAlg}$ for the free $K(1)$-local E_∞-algebra functor. This is given by

$$\mathbb{P}(X) = L_{K(1)}\left(\bigvee_{i\geq 0} E\Sigma_{i+} \wedge_{\Sigma_i} X^{\wedge i}\right).$$

Theorem A.6 [2] *For a $K(1)$-local spectrum X, there is a natural map*

$$\mathbb{T}(K_*X) \to K_*(\mathbb{P}(X)),$$

which is an isomorphism if K_*X is flat as a K_*-module.

The category Alg_θ has tensor products, which are the coproducts in this category. The tensor product of R_1 and R_2 has underlying ring $R_1 \otimes R_2$, and θ-algebra structure

$$\theta(x \otimes y) = x^p \otimes \theta(y) + \theta(x) \otimes y^p + p\theta(x) \otimes \theta(y).$$

If R_1 and R_2 are ψ-θ-algebras, the tensor product has the same θ-algebra structure as above, and has Adams operations

$$\psi^k(x \otimes y) = \psi^k(x) \otimes \psi^k(y).$$

Recall the adjunction

$$\mathbb{T} : \mathsf{Mod}^{\wedge}_{\mathbb{Z}_p} \rightleftarrows \mathsf{Alg}_\theta : U.$$

If the underlying module carries Adams operations, then the free θ-algebra functor takes values in ψ-θ-algebras. This yields an adjunction

$$\mathbb{T} : \mathsf{Mod}^{\wedge}_{\mathbb{Z}_p[[\mathbb{Z}_p^\times]]} \rightleftarrows \mathsf{Alg}_{\psi,\theta} : U.$$

Since $\mathbb{T}$ is a left adjoint, it preserves coproducts. This results in the following natural isomorphism of θ-algebras (resp. ψ-θ-algebras),

$$\mathbb{T}(M \oplus N) \cong \mathbb{T}(M) \otimes \mathbb{T}(N).$$

In particular, this means that $\mathbb{T}(M)$ has a natural L-complete Hopf algebra structure, with comultiplication coming from the diagonal map

$$M \to M \oplus M.$$

If M is finitely generated and torsion-free, then $\mathbb{T}(M)$ is actually a p-complete Hopf algebra. Moreover, the structure maps are morphisms of θ-algebras (resp. ψ-θ-algebras).

Example A.7 Of particular interest to us is the free θ-algebra $\mathbb{T}(b)$ on a single generator b. Its underlying algebra structure is given by Theorem A.5 above. An elementary calculation shows that b is a Hopf algebra primitive, i.e.

$$\Delta(b) = b \otimes 1 + 1 \otimes b.$$

Since Δ is a morphism of ψ-θ algebras, we have

$$\psi^{p^n} \circ \Delta = \Delta \circ \psi^{p^n}.$$

Since ψ^{p^n} is a ring homomorphism for all n, we have

$$\Delta(\psi^{p^n}(b)) = \psi^{p^n}(b \otimes 1 + 1 \otimes b) = \psi^{p^n}(b) \otimes 1 + 1 \otimes \psi^{p^n}(b).$$

Thus $\psi^{p^n}(b)$ is a Hopf algebra primitive for all n. This uniquely determines the rest of the Hopf algebra structure.

This Hopf algebra is actually fairly classical. Recall that the additive group of p-typical Witt vectors of a p-complete ring R are classified by a Hopf algebra

$$\mathbb{W} = \mathbb{Z}_p[a_0, a_1, \dots].$$

The map that sends $\theta_n(b)$ to a_n is then an isomorphism $\mathbb{T}(b) \to \mathbb{W}$ of θ-algebras and Hopf algebras. The element $\psi^{p^n}(b)$ goes to the primitive element of $\mathbb{W}$,

$$w_n = a_0^{p^n} + p a_1^{p^{n-1}} + \cdots + p^n a_n,$$

which represents the nth ghost component.

A.2 λ-Rings

Definition A.8 [7,30] A λ-ring is a graded commutative p-complete $\mathbb{Z}_p$-algebra R equipped with operaitons $\lambda^n : R \to R$ for $n \geq 0$ such that

$$\begin{aligned}
\lambda^0(x) &= 1, \\
\lambda^1(x) &= x, \\
\lambda^n(1) &= 0 \text{ for } n \geq 1, \\
\lambda^n(x+y) &= \sum_{i+j=n} \lambda^i(x)\lambda^j(y) \\
\lambda^n(xy) &= P_n(\lambda^1(x), \dots, \lambda^n(x), \lambda^1(y), \dots, \lambda^n(y)), \text{ and} \\
\lambda^m(\lambda^n(x)) &= P_{m,n}(\lambda^1(x), \dots, \lambda^{mn}(x)).
\end{aligned}$$

where P_n and $P_{m,n}$ are certain universal polynomials with integral coefficients which can be recovered by taking λ^n to be the nth elementary symmetric polynomial in infinitely many variables.

The category Alg_λ of λ-rings is also symmetric monoidal. The tensor product is the ordinary p-complete tensor product with λ-operations defined by the Cartan formula,

$$\lambda^n = \sum_{i+j=n} \lambda^i \otimes \lambda^j.$$

The notions of a λ-ring and a ψ-θ-algebra are closely related. In particular, given a λ-ring we can associate to it Adams operations. Indeed, one defines

$$\psi^n(x) = \nu_n(\lambda^1(x), \dots, \lambda^n(x)).$$

Here, ν_n is the polynomial so that if σ_k denotes the kth elementary symmetric polynomial in infinitely many variables x_i and $p_k = \sum x_i^k$,

$$p_n(\underline{x}) = \nu_n(\sigma_1(\underline{x}), \dots \sigma_n(\underline{x})).$$

The operation ψ^p satisfies the Frobenius congruence $\psi^p(x) \equiv x^p \mod p$. Thus if R is a torsion-free p-complete λ-ring, then R is a ψ-θ-algebra. A partial converse also holds.

Theorem A.9 (Bousfield, [7, Theorem 3.6]) *Let R be a p-complete ψ-θ-algebra. Then R has a unique λ-ring structure in which the Adams operations are the given ψ^k and ψ^p.*

Definition A.10 As a result, there are not one but two functors from ψ-θ-algebras to λ-rings, both of which are the identity on underlying rings. The sealed functor,

$$\mathcal{S} : \mathsf{Alg}_{\psi,\theta} \to \mathsf{Alg}_{\lambda},$$

is the one given by Bousfield's theorem, and is an equivalence on the subcategories of torsion-free algebras. The leaky functor,

$$\mathcal{L} : \mathsf{Alg}_{\psi,\theta} \to \mathsf{Alg}_{\lambda},$$

first replaces all the ψ^k by the identity for k prime to p, and then applies $\mathcal{S}$ to the result.

We will also write $\mathcal{L}$ for the functor

$$\mathsf{Alg}_{\theta} \to \mathsf{Alg}_{\lambda}$$

which sets $\psi^k = 1$ for k prime to p and then applies $\mathcal{S}$ to the result.

Example A.11 Recall that $\mathbb{Z}_p$ has a unique θ-algebra structure, in which ψ^p is the identity. Thus $\mathcal{L}(\mathbb{Z}_p)$ is a λ-ring in which all Adams operations are the identity. The λ-operations are given by $\lambda^n(x) = \binom{x}{n}$ [7, Example 1.3].

Lemma A.12 *Both $\mathcal{S}$ and $\mathcal{L}$ are symmetric monoidal functors.*

Proof As the operation of replacing the prime-to-p Adams operations with the identity is clearly monoidal, it suffices to prove that $\mathcal{S}$ is monoidal. For this, it suffices to prove that the inverse operation, from λ-rings to rings with Adams operations ψ^n for $n \in \mathbb{Z}_p$, preserves the obvious tensor products, which is a simple calculation. □

Corollary A.13 *Let M be a torsion-free, p-complete $\mathbb{Z}_p$-module. Then the coproduct map*

$$\Delta : \mathcal{L}(\mathbb{T}(M)) \to \mathcal{L}(\mathbb{T}(M)) \otimes \mathcal{L}(\mathbb{T}(M))$$

is a morphism of λ-rings.

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