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Stochastic multiplicative processes with reset events

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We study a stochastic multiplicative process with reset events. It is shown that the model develops a stationary power-law probability distribution for the relevant variable, whose exponent depends on the model parameters. Two qualitatively different regimes are observed, corresponding to intermittent and regular behavior. In the boundary between them, the mean value of the relevant variable is time independent, and the exponent of the stationary distribution equals -2. The addition of diffusion to the system modifies in a nontrivial way the profile of the stationary distribution. Numerical and analytical results are presented. [S1063-651X(99)05305-2]

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The occurrence of power-law distributions (PLDs) is a common feature in the description of natural phenomena. These distributions appear in a wide class of nonequilibrium systems, ranging from physical processes such as dielectric breakdown, percolation, and rupture [1], to biological processes such as dendritic growth and large-scale evolution [2], to sociological phenomena such as urban development [3]. Power laws have been associated with the effect of the complex driving mechanisms inherent to these systems and with their intricate dynamical structure. Criticality, fractals, and chaotic dynamics are known to be intimately related to PLDs [4].

In view of the ubiquity of PLDs in the mathematical description of Nature, much work has been recently devoted to detecting universal mechanisms able to give rise to such distributions. There is a class of systems where PLDs are a mathematical artifact originating from standard distributions through a mere change of variables [5]. On the other hand, many other instances are known where PLDs arise as a genuine and characteristic feature of the involved phenomena. In the frame of equilibrium processes, for instance, power laws have been shown to derive from generalized maximum-entropy formulations [6]. For nonequilibrium phenomena, self-organized criticality (SOC) and stochastic multiplicative processes (SMPs) have been identified as sources of PLDs. According to the SOC conjecture [7], some nonequilibrium systems are continuously driven by their own internal dynamics to a critical state where, as for equilibrium phase transitions, power laws are omnipresent. On the other hand, SMPs [8] provide a (more flexible) mechanism for generating PLDs, based in the presence of underlying replication events.

It is, however, well known that a pure SMP,

$$n(t+1)=\mu(t)n(t), \tag{1}$$

with μ a random variable, does not generate a stationary PLD for $n(t)$. Rather, it gives rise to a time-dependent log-normal distribution. To model the abovementioned phenomena, therefore, SMPs have to be combined with additional mechanisms. It has been shown that transport processes [9], sources [10], and boundary constraints [11] are able to induce a SMP to generate power laws. The aim of the present paper is to discuss an alternative additional mechanism, namely, randomly reseting of the relevant variable to a given reference value. In a real system, this would represent catastrophic annihilation or death events, seemingly originated outside the system.

We consider a discrete-time stochastic multiplicative process $n(t)$, added with reset events in the following way. At each time step, n is reset with probability q to a new value n_0, drawn from a probability distribution $P_0(n_0)$. If the reset event does not occur, n is multiplied by a random positive factor μ with probability distribution $P(\mu)$. Namely,

$$n(t+1)=\begin{cases} n_0(t+1) & \text{with probability } q, \\ \mu(t)n(t) & \text{with probability } 1-q. \end{cases} \tag{2}$$

Between two consecutive reset events, $n(t)$ thus behaves as a pure multiplicative process. When one of such events occurs, the multiplicative sequence starts again.

In order to gain insight into the dynamics of process (2) we first consider the simplest case where $n_0(t)$ and $\mu(t)$ are constant for all t. Since an arbitrary factor in the initial value of n is irrelevant to its subsequent evolution, we take $n_0=1$ without loss of generality. We have thus

$$n(t+1)=\begin{cases} 1 & \text{with probability } q, \\ \mu n(t) & \text{with probability } 1-q. \end{cases} \tag{3}$$

This stochastic recursive equation can be readily solved to give

$$n(t)=\begin{cases} \mu^k & \text{with probability } p_k=q(1-q)^k \ (0\leqslant k\leqslant t-1), \\ \mu^t & \text{with probability } p_t=(1-q)^t. \end{cases} \tag{4}$$

Note that the possible values of $n(t)$, $n_k=\mu^k$ ($k=0,1,\ldots,t$), lie in the interval $[\mu^t,1]$ for $\mu<1$ and in $[1,\mu^t]$ for $\mu>1$. Except for the extreme value $n_t=\mu^t$, the associated probabilities are time independent. As time elapses, the probability of each possible value of $n(t)$ is therefore quenched for $n\neq\mu^t$, and the corresponding probability distribution evolves at this extreme value only. Thus, the distribution sequentially builds up in zones that lie increasingly further from $n=1$.

For large times, when the number of possible values of $n(t)$ becomes also large, it is convenient to define a probability distribution $f(n)$ for $n\in(\mu^t,1]$ for $\mu<1$ and $n\in[1,\mu^t)$ for $\mu>1$ as

$$f(n)=\frac{p_k}{|\Delta n|}=\frac{q}{|\ln\mu|}n^{-\alpha}, \tag{5}$$

where Δn is the variation in n when k is increased by one unit, and $\alpha=1-\ln(1-q)/\ln\mu$. In order to account for the contribution at $n=\mu^t$, $f(n)$ should be added with a δ-like term $f_0(t)\delta(n-\mu^t)$, where the factor f_0 can be obtained from the normalization of $f(n)$.

According to Eq. (5), the stochastic process (3) gives rise to a *stationary power-law* distribution $f(n)$ in an increasingly large interval of values of n. For $t\to\infty$, $f(n)$ is a stationary power-law distribution in $(0,1]$ for $\mu<1$, and in $[1,\infty)$ for $\mu>1$. In contrast with multiplicative processes with boundary constraints [11], there are no conditions on the parameters to obtain a stationary power-law distribution. For $1-q<\mu<1$, the exponent of this distribution is positive ($\alpha<0$), and $f(n)$ grows with n. In this situation, however, the distribution is defined for $0<n\leqslant 1$ and exhibits a cutoff at $n=1$. On the other hand, for $\mu<1-q$ or $\mu>1$ the exponent is negative ($\alpha>0$). For $\mu>1$, i.e., when $n(t)\in[1,\infty)$, the moments $m_i=\int f(n)n^i dn$ diverge for $i>\alpha-1$, indicating the presence of intermittent amplifications [9,12]. For $\mu<1-q$, m_i diverges for $i<\alpha-1$.

It is interesting to relate the exponent of the power-law distribution with the evolution of the mean value $\langle n(t)\rangle$. From Eq. (4), this mean value can be written as

$$\langle n(t)\rangle=\frac{q}{1-(1-q)\mu}+\frac{(1-q)(1-\mu)}{1-(1-q)\mu}\mu^t(1-q)^t. \tag{6}$$

For $\mu(1-q)<1$, the mean value of $n(t)$ converges to a finite value $\langle n\rangle=q/[1-(1-q)\mu]$, whereas for $\mu(1-q)>1$ it ''explodes.'' In the boundary between both regimes, where $\mu=1/(1-q)$, the exponent of the distribution is $\alpha=2$ and $f(n)\sim n^{-2}$. This exponent is therefore to be associated with the explosion threshold.

The power-law distribution in Eq. (5) can also be inferred from a description of the evolution of $f(n)$. In fact, since at each time step where no reset occurs the probability contribution to $f(n)$ comes from $n'=n/\mu$, we can write

$$f_{t+1}(n)\Delta n=(1-q)f_t(n/\mu)\Delta n/\mu. \tag{7}$$

Assuming now that this distribution is stationary, $f_{t+1}\equiv f_t$, a solution to Eq. (7) is given by $f(n)=An^{-\alpha}$, with

$$(1-q)\mu^{\alpha-1}=1, \tag{8}$$

which produces the same value of α as in Eq. (5) [13]. Note that Eq. (7) does not hold for $n=1$, where the contributions to the probability come from reset events.

The above argument provides a method for dealing with the general multiplicative process with reset events, Eq. (2), when both μ and n_0 are drawn from prescribed probability distributions $P(\mu)$ and $P_0(n_0)$. We assume that $P_0(n_0)$ is appreciably different from zero in a bounded region, where the contributions from reset events are relevant. Outside this region the evolution of $f(n)$ can be written as

$$f_{t+1}(n)\Delta n=(1-q)\int_0^\infty d\mu\, P(\mu)f_t(n/\mu)\Delta n/\mu, \tag{9}$$

which generalizes Eq. (7). Under the assumption of stationarity, this equation is solved by $f(n)=An^{-\alpha}$, where the exponent α must verify

$$(1-q)\int_0^\infty d\mu\, \mu^{\alpha-1}P(\mu)=1. \tag{10}$$

For regular forms of $P(\mu)$ this equation has at least one solution for α. When the probability is mainly concentrated in values of μ larger than unity the solution is expected to be positive ($\alpha>0$) and vice versa.

As in the case of constant μ and n_0, a close relation exists here between the evolution of the average $\langle n(t)\rangle$ and the exponent of the power-law distribution. In particular, $\langle n(t)\rangle$ is found to remain stationary along the whole process when $\alpha=2$. Again, thus, the exponent $\alpha=2$ is associated with the explosion threshold, and marks the boundary between regular and intermittent evolution. This can be seen, for instance, from Eq. (9). Multiplication of this equation by n and integration over n yields

$$\langle n(t+1)\rangle=(1-q)\left[\int_0^\infty d\mu\, \mu P(\mu)\right]\langle n(t)\rangle. \tag{11}$$

Comparing with Eq. (10), we readily note that the multiplicative constant $(1-q)\int d\mu\, \mu P(\mu)$ that governs the evolution of $\langle n(t)\rangle$ in Eq. (11) equals unity for $\alpha=2$.

In summary, depending on q and $P(\mu)$ the system can be in a regular regime where $\langle n(t)\rangle$ converges to a finite value, or in an intermittence regime, where $\langle n(t)\rangle$ diverges. At the boundary, i.e., at the explosion threshold, $\langle n(t)\rangle$ remains constant and, independently of the specific value of q and of the particular form of $P(\mu)$, the probability distribution $f(n)$ exhibits a power-law tail with a characteristic exponent $f(n)\sim n^{-2}$.

We have numerically checked that the exponent of the stationary profile of $f(n)$ does not depend on the particular form of the distribution of reset values $P_0(n_0)$. In Fig. 1 we present the function $f(n)$ obtained with constant μ and q, for three different choices of $P_0(n_0)$: A uniform distribution between $n_0=0$ and $n_0=1$ (circles), an exponential distribution, $P_0(n_0)=\langle n_0\rangle^{-1}\exp(-n_0/\langle n_0\rangle)$, with $\langle n_0\rangle=100$ (squares), and a discrete distribution $P_0(n_0)=[\delta(n_0-1)+\delta(n_0-100)]/2$. The particular form of P_0 sets a lower boundary for the region where $f(n)$ behaves as a power law, but does not affect the corresponding exponent. Solid lines in the log-

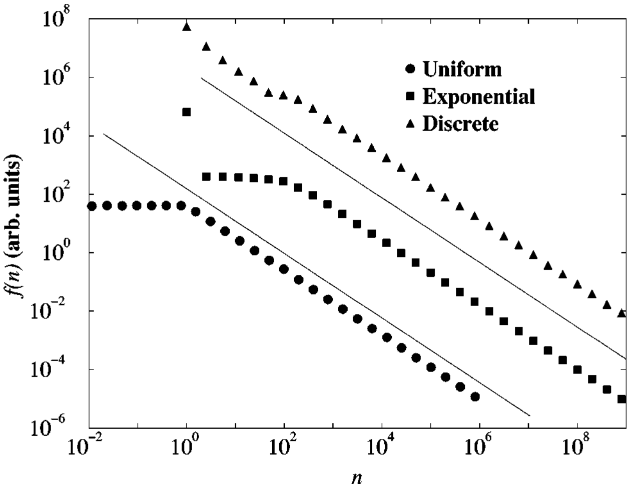

FIG. 1. Stationary distribution $f(n)$ for $\mu = 1.1$ and $q = 0.01$, and for different distributions of reset values $P_0(n_0)$ (see text). Straight lines have the theoretical slope $\alpha = 1.2195\cdots$.

log plot of Fig. 1 have the theoretical slope $\alpha = 1.1054\cdots$.

Figure 2 shows our simulation results for three different forms of $P(\mu)$: An exponential distribution $P(\mu) = \langle\mu\rangle^{-1}\exp(-\mu/\langle\mu\rangle)$ with $\langle\mu\rangle = 2$, a uniform distribution $P(\mu) = 5/2$ with $\mu \in [9/10,13/10]$, and a discrete distribution $P(\mu) = \Sigma_{k=1}^{3}\delta(\mu-\mu_k)/3$ with $\mu_1 = 1$, $\mu_2 = 6/5$ and $\mu_3 = 7/5$. The slope of the solid lines has been obtained numerically for various values of q from Eq. (10). This yields $\alpha = 1.4965\cdots$ for the exponential distribution with $q = 0.2$, $\alpha = 1.2195\cdots$ for the uniform distribution with $q = 0.02$, and $\alpha = 1.8965\cdots$ for the discrete distribution with $q = 0.15$. In all cases, our numerical and analytical results are in full agreement within six to nine decades in the power-law region.

We have also investigated the effects of diffusive transport on the process (3). With this aim, we have considered a one-dimensional array of elements whose individual dynamics is given by Eq. (3) and, at each time step, we have incorporated an interaction mechanism that mimics diffusion. After the multiplicative process with reset events has been applied, the state of each element is further changed to

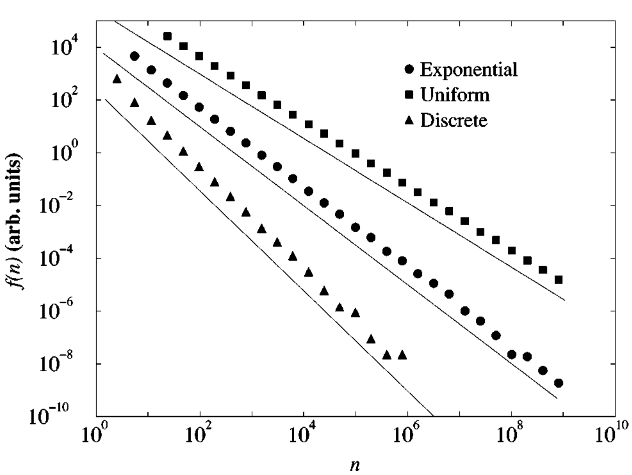

FIG. 2. Stationary distributions $f(n)$ for different forms of $P(\mu)$ and different values of q (see text). The slope of the straight lines has been obtained through numerical solution of Eq. (10).

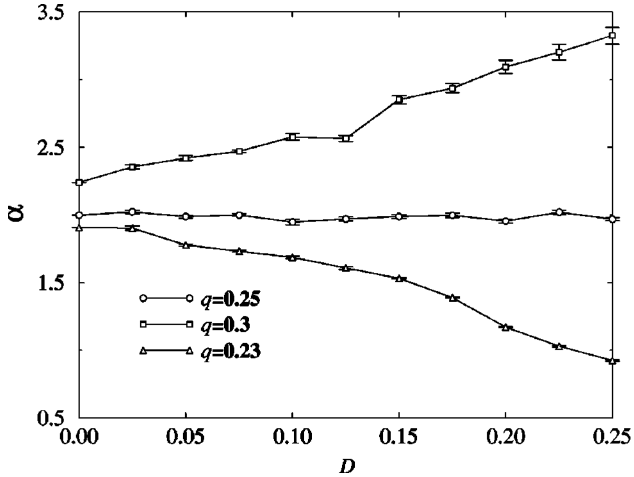

FIG. 3. Dependence of the exponent α on the diffusion coefficient D for $\mu = 4/3$ and three values of q corresponding to the intermittence regime ($q = 0.23$), the regular phase ($q = 0.3$), and the explosion threshold ($q = 0.25$). The error bars stand for the error of α in a least square fit to the numerical data.

$$n_i'(t) = (1-D)n_i(t) + \frac{D}{2}[n_{i+1}(t) + n_{i-1}(t)], \qquad (12)$$

where i labels the elements in the array, with periodic boundary conditions. Then, n_i' is used as the input state for the next step. In this deterministic, time-discrete version of diffusive transport, D plays the role of a diffusion constant.

Figure 3 summarizes our numerical results on the effect of diffusion on the SMP (3), displaying the dependence of the power-law exponent with the diffusion constant. We have chosen values of q and μ such that the different regimes of the process have been explored. The value of the multiplicative constant has been fixed in this case to $\mu = 4/3$. In the regular regime [i.e., $\mu(1-q) < 1$], diffusion produces a decrease of α in the power-law distribution. This can be understood if we consider that the role of diffusion is to deplete dense areas, transporting material to less occupied cells. The multiplicative process is not fast enough in this regime to balance the joint effect of reset events and diffusion. As a result, underpopulation occurs in the high-density region, and α decreases ($q = 0.3$ in Fig. 3). In the intermittent regime [$q = 0.23$, i.e., $\mu(1-q) > 1$], diffusion favors the opposite effect. Remarkably, diffusion does not have any effect on the value of α when the system is evolving at the explosion threshold. Within numerical errors, in fact, $\alpha = 2$ irrespectively of the value of D. It is also worth to point out that the qualitative behavior of the process depends on μ and q only. Changing D does not allow the system to switch between the intermittent and the regular regimes.

Summing up, in this paper we have studied a stochastic multiplicative process with reset events. The combination of this random reseting with the replication events driven by the stochastic process allows for the development of a stationary distribution in the system, both when the mean value of the relevant variable converges to a finite value (regular regime) and when it diverges (intermittent regime with persistence [14]). The regime at the boundary between regular and intermittent behavior is of particular interest. At this point, where the overall effects of the multiplicative process are exactly balanced by the random resets, the mean value of the relevant variable remains constant in time. We have shown that this property is closely related with the fact that the exponent of the power-law stationary distribution equals -2. This value is to be related with Zipf law, which predicts the same exponent of power-law distributions in a series of seemingly disparate natural systems [2,3]. Thus, the SMP with reset events offers an alternative explanation of this ubiquitous exponent. In fact, whereas a general trend of biological and social systems could be to improve their growth rates by increasing the parameter μ, it is on the other hand to be expected that external constrains are going to operate in order to avoid divergencies by increasing q. It is not unlikely that the competition between these two processes could lead real systems to this boundary between regular behavior and developed intermittency.

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