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MARKOV TYPE EQUATIONS WITH SOLUTIONS IN LUCAS SEQUENCES

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Abstract. Here we look at the Markov equations $ax^2 + by^2 + cz^2 = dxyz$ with integer solutions (x, y, z) which are all members of a Lucas sequence whose characteristic equation has roots which are quadratic units.

1. Introduction

The Markov equation is the equation

$$x^2 + y^2 + z^2 = 3xyz.$$

It is known that it has infinitely many positive integer solutions (x, y, z). Letting $\{F_n\}_{n\ge0}$ be the Fibonacci sequence $F_0 = 0,\ F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \ge 0$, the identity

$$1 + F_{2n-1}^2 + F_{2n+1}^2 = 3F_{2n-1}F_{2n+1}$$

implies that $(1, F_{2n-1}, F_{2n+1})$ is a solution of the Markov equation for all positive integers $n \ge 2$. Up to identifying F_2 with F_1, these ones are the only solutions of the Markov equation whose components are Fibonacci numbers as shown by Luca and Srinivasan in [3]. Tengely [6] studied triples of positive integers (x, y, z) whose components are Fibonacci numbers such that

$$ax^2 + by^2 + cz^2 = dxyz \tag{1}$$

for a few other specific choices of coefficients (a, b, c, d) such as

$$(1, 1, 1, 1),\ (1, 1, 2, 2),\ (1, 1, 2, 4),\ (1, 1, 5, 5),\ (1, 2, 3, 6). \tag{2}$$

He proved that there are only sporadic solutions (finitely many) and he found them all. The largest component of any solution in Fibonacci numbers is at most 5. His method involved finding integer points on several elliptic curves. The choices of coefficients shown at (2) were first studied by Rosenberger in [4]. He showed that these are the only choices of coefficients (a, b, c, d) which are positive integers and satisfy $\gcd(a, b) = \gcd(a, c) = \gcd(b, c) = 1$ and all of $a,\ b,\ c$ divide d such that furthermore equation (1) has infinitely many positive integer solutions (x, y, z).

The Pell sequence $\{P_n\}_{n\ge0}$ is given by $P_0 = 0,\ P_1 = 1$ and $P_{n+2} = 2P_{n+1} + P_n$ for all $n \ge 0$. The identity

$$P_2^2 + P_{2n-1}^2 + P_{2n+1}^2 = 3P_2P_{2n-1}P_{2n+1}$$

shows that there exist infinitely many solutions of the Markov equation whose components are Pell numbers. It turns out that, like in the case of the Fibonacci numbers, the triples $(P_2, P_{2n-1}, P_{2n+1})$ for positive integers n are the only solutions of the Markov equation whose components are Pell numbers. This has been proved

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independently in [1] and [5]. The proofs from [6] rely heavily on calculations of integer points on elliptic curves, while the proofs from [1], [3] and [5] rely on identities with Fibonacci and Pell numbers. Here, we look at the general equation (1), where a, b, c, d are given positive integers. We put

$$A := \max\{a, b, c, d, 2\}.$$

We let $s \in \{\pm 1\}$, $r \geq 1$ be an integer and $\{U_n(r)\}_{n\geq 0}$ be the Lucas sequence given by $U_0(r) = 0,\ U_1(r) = 1$ and $U_{n+2}(r) = rU_{n+1}(r) + sU_n(r)$ (when $r = 1,\ 2$, we only allow $s = 1$ and not $s = -1$). Unless we want to emphasise r, we will drop the dependence on r in what follows in order to ease the notation. We also put $\{V_n\}_{n\geq 0}$ for the companion Lucas sequence of $\{U_n\}_{n\geq 0}$ which satisfies $V_0 = 2,\ V_1 = r$ and $V_{n+2} = rV_{n+1} + sV_n$ for all $n \geq 0$. We assume that $(x, y, z) = (U_k, U_m, U_n)$ is a solution of equation (1). We also assume that $k \leq m \leq n$. Note that we do not assume any particular ordering on a, b, c, so we can always permute the coefficients a, b, c in the left–hand side of equation (1) so that to assume that $k \leq n \leq m$. Put $i := n-m$. Let α be the largest solution of the characteristic equation $x^2-rx-s = 0$. We put $\phi := (1 + \sqrt{5})/2$ and $\alpha \geq \phi$. Note that $x = y = z = 1$ (so $k = m = n = 1$) is always a solution provided that $a + b + c = d$ regardless of r. We call this *the trivial solution*. Now we have the following result.

Theorem 1.1. *All quadruples (r, s, k, m, n), $s \in \{\pm 1\}$ with the property that $(x, y, z) = (U_k, U_m, U_n)$ is a nontrivial solution of equation (1) with $k \leq m \leq n$ satisfy the following conditions:*

(1)

$$\begin{aligned} k &\leq \max\{11, 8 + \log(70A^4)/\log\alpha\}, \\ i &\leq \max\{15 + \log A/\log\alpha, 12 + \log(70A^5)/\log\alpha. \end{aligned}$$

(2) *In addition, one of the following holds:*

(2.i) *All of the following conditions hold:*

$$b = c,\ s = 1,\ i \equiv 0 \pmod 2,\ aU_k^2 = cU_i^2,\ dU_k/b = V_i,\ n \equiv 1 \pmod 2,$$

in which case also $r < 2A$. Furthermore, if this is the case, then for any $n > i$ odd, the triple $(x, y, z) = (U_k, U_{n-i}, U_n)$ satisfies (1), or

(2.ii) *Not all conditions from (2.i) above hold, in which case*

$$n \leq 47 + \log(10^6 A^{15})/\log\alpha.$$

Furthermore, in this case we also have $r \leq \max\{10^6 A^{15}, 2^{105} A\}$.

Note that part (2.i) of Theorem 1.1 gives us that for any sequence $\{U_n(r)\}_{n\geq 0}$ with $s = 1$ there are coefficients (a, b, c, d) such that relation (1) holds for infinitely many $(x, y, z) = (U_k, U_m, U_n)$. Indeed, take i even, $(a, b, c, d) := (U_i, U_i, U_i, V_i)$, then $(k, m, n) = (i, n-i, n)$ gives a valid solution for any $n > i$ odd.

Our proof is elementary that is, it only uses the Binet formula for $\{U_n\}_{n\geq 0}$ and some calculations.

2. The proof of Theorem 1.1

Put $\Delta := r^2 + 4s$, then

$$(\alpha, \beta) := \left(\frac{r + \sqrt{\Delta}}{2}, \frac{r - \sqrt{\Delta}}{2}\right)$$

are the two roots of the characteristic equation $x^2-rx-s=0$ of our Lucas sequence. Clearly, $\alpha>1$ and $\beta=-s/\alpha=\pm\alpha^{-1}$. The Binet formula for U_n is

$$U_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}\qquad \text{for all}\qquad n\geq 0. \tag{3}$$

The inequality

$$\alpha^{n-2}\leq U_n<\alpha^n \tag{4}$$

holds for all $n\geq 1$ and can be proved easily by induction. Assume now that $(x,y,z)=(U_k,U_m,U_n)$ is a solution of equation (1).

Lemma 2.1. *We have*

$$-6-\frac{\log(3A)}{\log\alpha}\leq n-(m+k)\leq 4+\frac{\log A}{\log\alpha}.$$

Proof. This is immediate from inequality (4). Indeed, we have

$$\alpha^{2(n-2)}\leq U_n^2=z^2\leq ax^2+by^2+cz^2=dxyz\leq A\alpha^{n+m+k};$$
$$3A\alpha^{2n}\geq 3AU_n^2=3Az^2=ax^2+by^2+cz^2=dxyz\geq \alpha^{(n-2)+(m-2)+(k-2)},$$

and the desired inequalities follow by manipulating the above inequalities and taking logarithms. □

We next bound k. The argument only involves the elementary estimates given by Lemma 2.1 as well as a norm calculation in the quadratic field $\mathbb{Q}[\sqrt{\Delta}]$.

Lemma 2.2. *One of the following holds:*

(i) $\alpha^{2k-16}\leq 40A^3$;
(ii) $\alpha<4A$ *and* $\alpha^{2k-16}<4900A^8$;
(iii) $\alpha\geq 4A$, $n-m-k=-1$ *and* $k\leq 11$.

Proof. We write the right–hand side of equation (1) as

$$dxyz=dU_kU_mU_n=\frac{d\alpha^{k+m+n}}{\Delta^{3/2}}(1+\zeta_k)(1+\zeta_m)(1+\zeta_n),\quad \zeta_\ell:=-\left(\frac{\beta}{\alpha}\right)^\ell,\quad \ell=k,m,n.$$

Since $|\zeta_\ell|=1/\alpha^{2\ell}\leq 1/\alpha^{2k}$ for $\ell\in\{k,m,n\}$, it follows that

$$(1+\zeta_k)(1+\zeta_m)(1+\zeta_n)=:1+\zeta,\quad \text{where}\quad |\zeta|\leq\frac{7}{\alpha^{2k}}. \tag{5}$$

We write the left–hand side as

$$ax^2+by^2+cz^2=cz^2\left(1+\frac{ax^2+by^2}{cz^2}\right)=\frac{c\alpha^{2n}}{\Delta}(1+\zeta_n)^2(1+\zeta'),\quad \zeta':=\frac{ax^2+by^2}{cz^2}.$$

Now by Lemma 2.1, we have

$$\zeta'\leq\frac{2AU_m^2}{U_n^2}<\frac{2A\alpha^{2m}}{\alpha^{2n-4}}=\frac{2A}{\alpha^{2(n-m)-4}}<\frac{18A^3}{\alpha^{2k-16}},$$

where we used the fact that $\alpha^{n-m}>\alpha^{k-6}/(3A)$, which is given by Lemma 2.1. Thus,

$$(1+\zeta_n)^2(1+\zeta')=1+\zeta'',$$

where

$$|\zeta''| < \frac{3}{\alpha^{2k}} + \frac{18A^3}{\alpha^{2k-16}} + \left(\frac{3}{\alpha^{2k}}\right)\left(\frac{18A^3}{\alpha^{2k-16}}\right)$$

$$= \frac{1}{\alpha^{2k-16}}\left(\frac{3}{\alpha^{16}} + 18A^3 + \left(\frac{3}{\alpha^{16}}\right)(18A^3)\right) < \frac{20A^3}{\alpha^{2k-16}}, \quad (6)$$

where we used the fact that $\alpha^{16} \geq ((1+\sqrt{5})/2)^{16} > 100$. We assume that (1) of the lemma does not hold so

$$\alpha^{2k-16} > 40A^3. \quad (7)$$

Thus, $|\zeta''| < 1/2$. Since in this case $k > 8$, it follows that $|\zeta| < 7/\alpha^{2k} < 1/2$. In the equation

$$\frac{c\alpha^{2n}}{\Delta}(1+\zeta'') = \frac{d\alpha^{n+m+k}}{\Delta^{3/2}}(1+\zeta),$$

we have $1+\zeta'',\ 1+\zeta \in (1/2, 3/2)$. Simplifying the above relation as

$$c\Delta^{1/2}\alpha^{n-(m+k)}(1+\zeta'') = d(1+\zeta),$$

we get that

$$c\Delta^{1/2}\alpha^{n-(m+k)}/d \in [1/3, 3]. \quad (8)$$

In particular,

$$\begin{aligned} |c\Delta^{1/2}\alpha^{n-(m+k)} - d| &\leq d|\zeta| + c\Delta^{1/2}\alpha^{n-(m+k)}|\zeta''| \\ &< A|\zeta| + 3A|\zeta''| \\ &< \frac{7A}{\alpha^{2k}} + \frac{60A^4}{\alpha^{2k-16}} \\ &< \frac{61A^4}{\alpha^{2k-16}}. \quad (9) \end{aligned}$$

From here we distinguish three cases.

Case 1. *The case when $n \geq m+k$.*

From (8), we deduce that $\Delta^{1/2} \leq \Delta^{1/2}\alpha^{n-(m+k)} \leq d/c \leq 3A$. From the fact that $\Delta^{1/2} = \alpha - \beta$, we get that

$$\alpha < 4A. \quad (10)$$

Further, we note that $c\Delta^{1/2}\alpha^{n-(m+k)} - d \neq 0$, since either $n = m+k$ and $\Delta^{1/2}$ is not rational or $n > m+k$ and $\alpha^{2(n-(m+k))}$ is not rational. Thus,

$$|-c\Delta^{1/2}\beta^{n-(m+k)} - d| \leq c\Delta^{1/2} + d \leq 4d \leq 4A. \quad (11)$$

Multiplying (9) and (11), we get

$$1 \leq |(c\Delta^{1/2}\alpha^{n-(m+k)} - d)(-c\Delta^{1/2}\beta^{n-(m+k)} - d)| \leq \frac{244A^5}{\alpha^{2k-16}}.$$

The left-most inequality above follows from the fact that the number inside the absolute value is the norm of the non-zero algebraic integer $c\Delta^{1/2}\alpha^{n-(m+k)} - d$ in the quadratic field $\mathbb{K} := \mathbb{Q}[\sqrt{\Delta}]$ and as such it is at least as large as 1. So,

$$\alpha^{2k-16} \leq 244A^5. \quad (12)$$

Case 2. *The case $n-(m+k) \leq -2$.*

In this case, inequality (8) gives $\alpha^2 \le \alpha^{(m+k)-n} \le (3c/d)\Delta^{1/2} \le 3A(\alpha+1)$, which gives

$$\alpha < 4A, \tag{13}$$

and so $\Delta^{1/2} < \alpha + 1 < 5A$. Thus, $\alpha^{(m+k)-n} \le (3c/d)\Delta^{1/2} < 15A^2$. Now

$$|-c\Delta^{1/2}\beta^{n-(m+k)} - d| \le c\Delta^{1/2}\alpha^{(m+k)-n} + d < A(5A)(15A^2) + A < 76A^4. \tag{14}$$

Multiplying (9) with (14), we get

$$1 \le |(c\Delta^{1/2}\alpha^{n-(m+k)} - d)(-c\Delta^{1/2}\beta^{n-(m+k)} - d)| < \frac{4636A^8}{\alpha^{2k-16}},$$

so

$$\alpha^{2k-16} < 4636A^8. \tag{15}$$

Case 3. *The case when $n-(m+k) = -1$.*

If

$$\alpha < 4A, \tag{16}$$

then $\Delta^{1/2} < \alpha + 1 < 5A$. In this case,

$$|-c\Delta^{1/2}\beta^{n-(m+k)} - d| = |-c\Delta^{1/2}\beta^{-1} - d| \le c\Delta^{1/2}\alpha + d \le A(4A)(5A) + A < 21A^3,$$

therefore

$$1 \le |(c\Delta^{1/2}\alpha^{n-(m+k)} - d)(-c\Delta^{1/2}\beta^{n-(m+k)} - d)| < \frac{1281A^7}{\alpha^{2k-16}}.$$

This last inequality leads to

$$\alpha^{2k-16} < 1281A^7. \tag{17}$$

Assume next that $\alpha > 4A$. Then $\Delta^{1/2} = \alpha - \beta < \alpha + 1 < 2\alpha$. In this last case, we have

$$|-c\Delta^{1/2}\beta^{n-(m+k)} - d| = |-c\Delta^{1/2}\beta^{-1} - d| \le c\Delta^{1/2}\alpha + d < A(2\alpha^2) + A < 3A\alpha^2.$$

Multiplying the above inequality with (9), we get

$$1 \le |(c\Delta^{1/2}\alpha^{n-(m+k)} - d)(-c\Delta^{1/2}\beta^{n-(m+k)} - d)| \le \frac{183A^5}{\alpha^{2k-18}},$$

which yields

$$\alpha^{2k-18} < 183A^5.$$

Since $\alpha > 4A$, it follows that $2k-18 < 6$, so $k \le 11$. □

From Lemma 2.2, we get that k is bounded. That is, either $k \le 11$ (in case (iii)) or $k \le 8 + \log(70A^4)/\log\alpha$ (in cases (i) and (ii)). This gives the upper bound for k from estimate (1) of Theorem 1.1. Note that

$$k \le 11 + \log(70A^4)/\log\alpha \tag{18}$$

holds in all cases. By Lemma 2.1, recalling that $i := n - m$, we get that

$$i \le 4 + k + \log A/\log\alpha,$$

which together with the bound on k gives the upper bound from (1) of Theorem 1.1. Note that

$$i \le 15 + \log(70A^5)/\log\alpha \tag{19}$$

always holds. This finishes the proof of part (1) of Theorem 1.1. We next move to the proofs of part (2) of Theorem 1.1.

The equation has become

$$aU_k^2 + bU_{n-i}^2 + cU_n^2 = dU_kU_{n-i}U_n, \tag{20}$$

where k and i are bounded. If $\alpha \le 4A$, then also r is bounded. If $\alpha > 4A$, we don't know yet that A is bounded. To continue, we recall the following known property.

Lemma 2.3. *We have*

$$U_n^2 - U_{n+i}U_{n-i} = (-s)^{n-i}U_i^2. \tag{21}$$

Proof. Easy calculation using the Binet formula (3). □

Identity (21) implies that $U_{n-i} \mid cU_n^2 - c(-s)^{n-i}U_i^2$. From equation (20), we also have that $U_{n-i} \mid cU_n^2 + aU_k^2$. Thus,

$$U_{n-i} \mid aU_k^2 + c(-s)^{n-i}U_i^2. \tag{22}$$

Since k and i are bounded and $U_{n-i} \ge \alpha^{n-i-2} \ge \phi^{n-i-2}$, it follows that also n is bounded unless the expression from the right–hand side of divisibility relation (22) above is 0. Thus, if the expression from the right–hand side of (22) above is not zero, we should get the conclusions from (2.ii) of Theorem 1.1. Let us see this argument. If the above expression is not zero, then

$$\alpha^{n-i-2} \le U_{n-i} \le 2A\alpha^{2\max\{i,k\}} \le 10^4A^{10}\alpha^{30}$$

(see estimates (18), (19)), which gives

$$\alpha^n \le 10^4A^{10}\alpha^{32+i} \le (10^4A^{10})(70A^5)\alpha^{47} = 10^6A^{15}\alpha^{47}$$

(again by (19)), which gives the bound on n from (2.ii) of Theorem 1.1. If

$$r \le 10^6A^{15} \tag{23}$$

we then get the second part of (2.ii). Assume next that estimate (23) does not hold. Then $\alpha > r-1 \ge 10^6A^{15}$, so $n \le 47$, $k \le 11$ and $i \le 15$. In this last case, all three k, n, m are bounded. Fixing these three data, the equation

$$aU_n^2(r) + bU_m^2(r) + cU_n^2(r) - dU_k(r)U_m(r)U_n(r) = 0$$

is one of two (according to whether $s = \pm 1$) polynomial equations in r. It remains to show that this is not the zero polynomial. This is the content of the next lemma.

Lemma 2.4. *If a, b, c, d and m, n, k are fixed positive integers, then for $s \in \{\pm 1\}$ the polynomial*

$$aU_k(r)^2 + bU_m(r)^2 + cU_n(r)^2 - dU_k(r)U_m(r)U_n(r) \tag{24}$$

is not the zero polynomial in r except for $a+b+c=d$ and $k=m=n=1$.

Proof. The polynomial $U_n(r)$ as a polynomial in r has degree $n-1$ and is monic. Further, $U_n(0) = 0$ for n even and $U_n(0) = \pm 1$ for n odd. These can be checked easily. Thus, if the relation

$$aU_k(r)^2 + bU_m(r)^2 + cU_n(r)^2 = dU_k(r)U_m(r)U_n(r) \tag{25}$$

holds as an equality of polynomials, then the polynomial on the left has degree $2n-2$ and the polynomial on the right has degree $n+m+k-3$. Thus, $2n-2 = n+m+k-3$, which gives $n = m+k-1$. Assume first that $n = m$. Then $k = 1$. Thus, $U_n(r)$ divides a, which is a constant non-zero polynomial, which gives $n = m = 1$. This is the trivial solution mentioned in the statement of the lemma. From now on, we assume that $n > m$. Comparing leading terms we get $c = d$. Evaluating in 0, we get

$$aU_k(0)^2 + bU_m(0)^2 + cU_n(0)^2 = \pm cU_k(0)U_m(0)U_n(0). \tag{26}$$

Since $n + m + k = 2(m + k) - 1$, it follows that either all n, m, k are odd, or two are even and the third is odd. If all of them are odd, the above relation (26) gives $a + b + c = \pm c$, a contradiction. If two of k, m, n are even and the third is odd, then two of the numbers $U_k(0)$, $U_m(0)$, $U_n(0)$ are zero and the third one is ± 1, and again relation (26) is impossible. □

One can prove, by induction using the formula

$$U_{n+2}(r) = rU_{n+1}(r) + sU_n(r), \qquad s \in \{\pm 1\},$$

that the sum of the absolute values of the coefficients of $U_n(r)$ is $\le 2^{n-1}$. Thus, for $k \le 11$, $m \le n \le 47$ and $(k, m, n) \ne (1, 1, 1)$, the polynomial

$$aU_k(r)^2 + bU_m(r)^2 + cU_n(r)^2 - dU_k(r)U_m(r)U_n(r)$$

is nonzero and has integer coefficients the sum of which in absolute values is at most

$$4 \cdot A \cdot (2^{47})^2 \times 2^{11} < 2^{105}A,$$

so its maximal positive integer root (which divides the coefficient of the monomial of the smallest degree participating in the polynomial) is also at most $2^{105}A$. Thus, $r \le 2^{105}A$. This completes the proof of the second part of (2.ii) of Theorem 1.1.

It remains to study the case when the expression in the right–hand side of (22) is zero. In this case, $s = 1$, $n - i$ is odd and $aU_k^2 = cU_i^2$. We evaluate (21) in $n - i$ and get

$$U_{n-i}^2 - U_nU_{n-2i} = (-1)^nU_i^2.$$

This formula holds also if $n < 2i$ because the sequence $\{U_\ell\}_{\ell\ge 0}$ can be extended to negative numbers either by recurrence or by allowing n to be negative in the formula (3). We then get $U_n \mid bU_{n-i}^2 - (-1)^nbU_i^2$. Since also $U_n \mid aU_k^2 + bU_{n-i}^2$, we get that $U_n \mid aU_k^2 + (-1)^nbU_i^2$. The previous argument shows again that n is bounded, and the bounds indicated at (2.ii) hold unless $aU_k^2 = (-1)^{n+1}bU_i^2$. In this last case, n is odd, i is even, and $b = c$. Inserting $U_n^2 - U_{n-i}U_{n+i} = -U_i^2$ into

$$aU_k^2 + bU_{n-i}^2 + bU_n^2 = dU_kU_{n-i}U_n,$$

we get

$$aU_k^2 + bU_{n-i}^2 + bU_{n-i}U_{n+i} - bU_i^2 = dU_kU_{n-i}U_{n+i}.$$

Since $aU_k^2 = bU_i^2$, we can simplify both sides of the above relation by U_{n-i} to get

$$U_{n+i} = (dU_k/b)U_n - U_{n-i}.$$

But it is well–known and easy to prove using the Binet formula that in fact the formula $U_{n+i} = V_iU_n - U_{n-i}$ holds for all $n > i$. Thus, we have $V_iU_n = (dU_k/b)U_n$, so $V_i = dU_k/b$. This gives all the conditions from part (2.i) of Theorem 1.1. Further if all the above conditions are met and $n > i$ is odd, then one can work backwards from the last relation

$$U_{n+i} = V_iU_n - U_{n-i} = (dU_k/b)U_n - U_{n-i},$$

through the algebraic manipulations we have just done, and conclude that it leads to

$$aU_k^2 + bU_{n-i}^2 + cU_n^2 = dU_k U_{n-i} U_n,$$

which is what we wanted. Finally, it remains to deal with the statement about r. Since $aU_k^2 = bU_i^2$, if $a \neq b$, then it follows that $k \neq i$. We assume without loss of generality that $a > b$. Then $i > k$, so $A \geq a/b = (U_i/U_k)^2 \geq (U_i/U_{i-1})^2 \geq (r-1)^2$, where we used the fact that $U_i = rU_{i-2} + sU_{i-2} \geq (r-1)U_{i-1}$ (note that $i - 1 \geq 1$ since i is even). Thus, we get $r < 1 + \sqrt{A} < 2A$ in this case. Thus, it remains to treat the case when $a = b = c$ so $k = i$. Then $dU_i/b = V_i$. It is well-known and it follows from the formula

$$V_i^2 - \Delta U_i^2 = 4(-s)^i$$

that $\gcd(U_i, V_i) = 1, 2$. Thus, $V_i \mid 2d$, showing that $2A \geq 2d \geq \alpha^i + \alpha^{-i} > \alpha^2$, so $\alpha < \sqrt{2A}$, therefore $r < \alpha + 1 < 1 + \sqrt{2A} < 2A$. So, indeed $r < 2A$ in case (2.i).

This finishes the proof of the theorem.

3. Particular cases

Take $(a, b, c, d) = (1, 1, 1, 3)$. Then part (1) of Theorem 1.1 gives us the bounds $k \leq 25,\ i \leq 32$. For (2.i), we want $U_k = U_i$ and $3U_i = V_i$. Since $\gcd(U_i, V_i) = 1, 2$, it follows that either $(U_i, V_i) = (1, 3)$ or $(U_i, V_i) = (2, 6)$. Since i is even, $r \mid U_i$. Thus, we have $i = 2$, $U_i = r$, and we get that either $r = 1$, which is the Fibonacci case, or $r = 2$, which is the Pell case. Here, $U_k = 1,\ 2$, respectively, and m, n are consecutive odd integers. We have encountered the known parametric families and the above arguments show that there are no other parametric families. The sporadic solutions should have, by (2.ii), $n \leq 109$. We generated all the polynomials

$$U_k(r)^2 + U_m(r)^2 + U_n(r)^2 - 3U_k(r)U_m(r)U_n(r) \tag{27}$$

for all $1 \leq k \leq 25,\ k \leq m \leq 109,\ n \in [m, \min\{m+32, 109\}]$ for both cases $s \in \{\pm 1\}$ and computed candidates for their integer roots r. In Lemma 2.4 it was shown that the above polynomial was non-zero for $(k, m, n) \neq (1, 1, 1)$ by simultaneously looking at its degree, leading term and last coefficient. In order to bound r, we should understand better the last non-zero coefficient. Here is a partial result that does not do the job in all cases but it suffices for some applications. It is based on the formula

$$U_m(r) = \sum_{\substack{0 \leq k \leq m \\ k \not\equiv m \pmod 2}} \binom{(m+k-1)/2}{k} r^k, \quad \text{when} \quad s = 1$$

(see [2]). For $s = -1$, the coefficients of $U_m(r)$ are the same as above in absolute values but their signs are alternating (starting with the leading coefficient whose value is 1). The following lemma is then immediate.

Lemma 3.1. *The last coefficient (free term) of the polynomial shown at* (24) *is*

(i) $a + b + c \pm d$ *if* k, m, n *are all odd;*
(ii) *One of* $a, b, c, a + b, a + c, b + c$ *if at least one of* k, m, n *is even but not all.*
(iii) *In case all* k, m, n *are even, the coefficient of* r^2 *is*

$$\pm(a(k/2)^2 + b(m/2)^2 + c(n/2)^2).$$

In case k, m, n are all odd, and the free term is 0*, then the coefficient of r^2 is one of*

$$\pm\left((2a-d)\binom{(k+1)/2}{2}+(2b-d)\binom{(m+1)/2}{2}+(2c-d)\binom{(n+1)/2}{2}\right).$$

In particular, for the polynomial shown at (27), the last coefficient (free term) is a nonzero integer of absolute value at most 6 unless either all m, n, k are even or odd. If they are all even, then r divides $k^2+m^2+n^2$ and if they are all odd, then r divides $(k^2-1)+(m^2-1)+(n^2-1)$ which is nonzero for $(k,m,n)\neq(1,1,1)$. Thus, one can just loop over all the possibilities for (k,n,m) and then over all the possible divisors r of the above numbers and check whether (27) evaluates to 0. No nontrivial examples were found. So, we proved the following numerical corollary.

Corollary 3.1. *If $(a,b,c,d)=(1,1,1,3)$ then any nontrivial solution of equation* (1) *has $r=1,2$ so it is of the form $(1,F_{2t-1},F_{2t+1})$ or $(2,P_{2t-1},P_{2t+1})$ for some positive integer t.*

Let us now take a look at the quadruples (2) studied by Tengely. In this case, if there were infinitely many solutions, in a quadruple (a,b,c,d), we would need, up to permutations among a,b,c, that two of them coincide and that the product of the third one with any of these two is a square (because of the conditions $aU_k^2=bU_i^2$, which implies that ab is a square, and further $b=c$). Of the quadruples from (2) only $(a,b,c,d)=(1,1,1,1)$ has this property. For it, we get $U_k=U_i$ and $dU_i=V_i$. Thus, $U_i=V_i$ or $2U_i=V_i$. Since $\gcd(U_i,V_i)=1,2$, we get that $V_i=2,4$. However, since i is even, $V_i=\alpha^i+\alpha^{-i}>\alpha^2>4$ except for $\alpha=(1+\sqrt{5})/2$ for which $V_i\leq 4$ only for $i=2$, but then $V_2=3\notin\{U_2,2U_2\}=\{2,4\}$. So, these equations do not have parametric solutions. They might have sporadic solutions. For them, $A=6$, so Theorem 1.1 (i) and (iii) tell us that $k\leq 31,\ i\leq 39,\ n\leq 131$. As for the r, we look at the polynomials

$$aU_k(r)^2+bU_m(r)^2+cU_n(r)^2-dU_k(r)U_m(r)U_n(r).$$

By Lemma 3.1, if at least one of k,m,n is even but not all are, the last coefficient is non-zero and smaller than $a+b+c\leq 12$ in absolute value. If all k,m,n are even, then r divides $an^2+bm^2+ck^2$, over all permutations of a,b,c. If all k,m,n are odd, then the last coefficient is non-zero and smaller than 12 in absolute value except for $(a,b,c,d)=(1,1,2,4),\ (1,2,3,6)$ for which it might be 0. If this is the case, then the coefficient of r^2 is, up to sign, $(2a-d)(u^2-1)/8+(2b-d)(v^2-1)/8+(2c-d)(w^2-1)/8$, where (u,v,w) is a permutation of (k,m,n). Since in both cases $d=2c$, it follows that this is zero only if $u=v=1$. This entails $k=m=1$, so $x=y=1$, which leads to $z=1$, so $n=1$ as well, a trivial solution. So, r is a divisor of one of the above non-zero numbers as $\{u,v\}$ range over all subsets with two elements of $\{k,m,n\}$. Now we have everything we need to run a calculation. Only small solutions were found, namely for $(a,b,c,d)=(1,1,1,1),\ (1,1,2,2),\ (1,1,5,5)$, all solutions of the form $(x,y,z)=(U_k(r),U_m(r),U_n(r))$ have $z\leq 3$, for $(a,b,c,d)=(1,1,2,4),\ (1,2,3,6)$, they all have $z\leq 5$ as in Tengely's calculations where only the case $r=1$ (Fibonacci numbers) was allowed. So, we proved the following corollary.

Corollary 3.2. *Let $(x,y,z)=(U_k(r),U_m(r),U_n(r))$ be a solution of* (1)*, where $(a,b,c,d)\in\{(1,1,1,1),(1,1,2,2),(1,1,2,4),(1,1,5,5),(1,2,3,6)\}$. We then have $\max\{x,y,z\}\leq 5$.*

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