HU-EP-21/15-RTG

Gravitational Bremsstrahlung and Hidden Supersymmetry of Spinning Bodies

Gustav Uhre Jakobsen,[1, 2, *] Gustav Mogull,[1, 2, †] Jan Plefka,[1, ‡] and Jan Steinhoff[2, §]

[1] *Institut für Physik und IRIS Adlershof, Humboldt-Universität zu Berlin, Zum Großen Windkanal 2, 12489 Berlin, Germany*

[2] *Max Planck Institute for Gravitational Physics (Albert Einstein Institute), Am Mühlenberg 1, 14476 Potsdam, Germany*

The recently established formalism of a worldline quantum field theory, which describes the classical scattering of massive bodies in Einstein gravity, is generalized up to quadratic order in spin — for a pair of Kerr black holes revealing a hidden $\mathcal{N}=2$ supersymmetry. The far-field time-domain waveform of the gravitational waves produced in such a spinning encounter is computed at leading order in the post-Minkowskian (weak field, but generic velocity) expansion, and exhibits this supersymmetry. From the waveform we extract the leading-order total radiated angular momentum in a generic reference frame, and the total radiated energy in the center-of-mass frame to leading order in a low-velocity approximation.

arXiv:2106.10256v1 [hep-th] 18 Jun 2021

The rise of gravitational wave (GW) astronomy [1] offers new paths to explore our universe, including black hole (BH) population and formation studies [2], tests of gravity in the strong-field regime [3], measurements of the Hubble constant [4], and investigations of strongly interacting matter inside neutron stars [5]. This form of astronomy relies heavily on Bayesian methods to infer probability distributions for theoretical GW predictions (templates), depending on a source's parameters, to match the measured strain on detectors. With the network of GW observatories steadily increasing in sensitivity [6], theoretical GW predictions need to keep pace with the accuracy requirements placed on templates [7]. For the inspiral and merger phases of a binary an important strategy is to synergistically combine approximate and numerical relativity predictions [8], each applicable only to a corner of the parameter space [9].

In this Letter we calculate gravitational waveforms — the primary observables of GW detectors — produced in the parameter-space region of highly eccentric (scattering) spinning BHs and neutron stars (NSs), to leading order in the weak-field, or post-Minkowskian (PM), approximation. Following the above strategy, this is a valuable input for future eccentric waveform models. Indeed, the extension of contemporary quasi-circular (non-eccentric) waveform models for spinning binaries to eccentric orbits (including scattering) is under active investigation [10]. This is motivated, for instance, by the potential insight gained on the formation channels or astrophysical environments of binary BHs (BBHs) through measurements of eccentricity [11] and spins [12], or the search for scattering BHs [13] in our universe.

Accurate predictions for GWs from BBHs should crucially also account for the BHs' spins [14], and this is an important aspect of the present work. The gravitational waveforms presented here are valid up to quadratic order in angular momenta (spins) of the compact stars; that is, we extend Crowley, Kovacs and Thorne's seminal non-spinning result [15]. We also improve on our earlier reproduction of the non-spinning result [16] by presenting results in a compact Lorentz-covariant form, using an improved integration strategy.

To obtain these results we generalize the recently introduced worldline quantum field theory (WQFT) formalism [16, 17] to include spinning particles on the worldline. This is achieved by including anticommuting worldline fields carrying the spin degrees of freedom, building upon Refs. [18–20]. Our formalism manifests a hidden $\mathcal{N}=2$ extended worldline supersymmetry (SUSY), to linear order in spin for NSs and quadratic order for Kerr-BHs. In fact, SUSY leaves an imprint on the spinning waveform, which we show to be invariant under flat-space $\mathcal{N}=2$ SUSY transformations to the relevant orders. This realization of SUSY hints towards a bootstrap of BH interactions from symmetry principles.

The spinning WQFT innovates over previous approaches to classical spin based on corotating-frame variables [21, 22] in the effective field theory (EFT) of compact objects [23, 24] — see Refs. [25] for the construction of PM integrands and Refs. [26, 27] for worldline and spin deflections (in agreement with scattering amplitude results [28, 29]). The worldline EFT was applied to radiation also in the weak-field and slow-motion, i.e. post-Newtonian (PN), approximation [30]— see Refs. [31] for more traditional methods. Other approaches to PM spin effects can be found in Refs. [32].

Spinning Worldline Quantum Field Theory. — It has been known since the 1980s [18] that the relativistic wave equation for a massless or massive spin-$\mathcal{N}/2$ field in flat spacetime (generalizing the Klein-Gordon, Dirac and Maxwell or Proca equations) may be obtained by quantization of an extended supersymmetric particle model where one augments the bosonic trajectory $x^\mu(\tau)$ by $\mathcal{N}$ anticommuting, real worldline fields. Generalizing this to a curved background spacetime comes with consistency problems beyond $\mathcal{N}=2$. Yet the situation for spins up to one is well understood [20], and sufficient for our purposes of describing two-body scattering up to quadratic

* gustav.uhre.jakobsen@physik.hu-berlin.de
† gustav.mogull@aei.mpg.de
‡ jan.plefka@hu-berlin.de
§ jan.steinhoff@aei.mpg.de

order in spin.

We therefore augment the worldline trajectories $x_i^\mu(\tau_i)$ $(i = 1, 2)$ of our two massive bodies by anticommuting *complex* Grassmann fields $\psi_i^a(\tau_i)$. These are vectors in the flat tangent Minkowski spacetime connected to the curved spacetime via the vierbein $e_\mu^a(x)$. The worldline action in the massive case for each body takes the form (suppressing the i subscripts) [20, 33]

$$S = -m\int \mathrm{d}\tau \Big[\tfrac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + i\bar{\psi}_a \tfrac{D\psi^a}{D\tau} + \tfrac{1}{2} R_{abcd}\bar{\psi}^a\psi^b\bar{\psi}^c\psi^d\Big]\,, \tag{1}$$

where $g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}$ is the metric in mostly minus signature, $\frac{D\psi^a}{D\tau} = \dot{\psi}^a + \dot{x}^\mu \omega_\mu{}^a{}_b \psi^b$ includes the spin connection $\omega_{\mu ab}$ and the Riemann tensor is $R_{\mu\nu ab} = e_\mu^c e_\nu^d R_{abcd} = 2(\partial_{[\mu}\omega_{\nu]ab} + \omega_{[\mu\,a}{}^c\omega_{\nu]cb})$. This theory enjoys a global $\mathcal{N} = 2$ SUSY: it is invariant under

$$\delta x^\mu = i\bar{\epsilon}\psi^\mu + i\epsilon\bar{\psi}^\mu\,, \quad \delta\psi^a = -\epsilon e_\mu^a \dot{x}^\mu - \delta x^\mu\,\omega_\mu{}^a{}_b\psi^b\,, \tag{2}$$

with constant SUSY parameters ϵ and $\bar{\epsilon} = \epsilon^\dagger$.

The connection to a traditional description of spinning bodies in general relativity, using the spin field $S^{\mu\nu}$ and the Lorentz body-fixed frame Λ_μ^A [21, 22, 24, 34, 35], comes about upon identifying the spin field $S^{\mu\nu}(\tau)$ with the Grassmann bilinear:

$$S^{\mu\nu} = -2ie_a^\mu e_b^\nu\,\bar{\psi}^{[a}\psi^{b]}\,. \tag{3}$$

One can easily show that S^{ab} obeys the Lorentz algebra under Poisson brackets $\{\psi^a, \bar{\psi}^b\}_{\text{P.B.}} = -i\eta^{ab}$. In fact, the spin-supplementary condition (SSC) and preservation of spin length may be related to $\mathcal{N} = 2$ SUSY-related constraints [33]. Finally, by deriving the classical equations of motion from the action these can be shown to match the Mathisson-Papapetrou equations [36] at quadratic spin order. This fascinatingly points to a hidden $\mathcal{N} = 2$ SUSY in the actions of Refs. [22, 34, 35] for the Kerr-BH.

The actions of Refs. [22, 34, 35] also carry a first spin-induced *multipole moment term* at quadratic order in spins with an undertermined Wilson coefficient C_E, where here $C_E = 0$ for a Kerr BH. Translating it to our formalism this term reads

$$S_{ES^2} := -m\int \mathrm{d}\tau\, C_E E_{ab}\bar{\psi}^a\psi^b\,\bar{\psi}\cdot\psi\,, \tag{4}$$

where $E_{ab} := R_{a\mu b\nu}\dot{x}^\mu\dot{x}^\nu$ is the "electric" part of the Riemann tensor. This term breaks the $\mathcal{N} = 2$ SUSY.

In order to describe a scattering scenario we expand the worldline fields about solutions of the equations of motion along straight-line trajectories:

$$\begin{aligned} x_i^\mu(\tau_i) &= b_i^\mu + v_i^\mu\tau_i + z_i^\mu(\tau_i)\,,\\ \psi_i^a(\tau_i) &= \Psi_i^a + \psi_i'^a(\tau_i)\,, \end{aligned} \tag{5}$$

where $\mathcal{S}_i^{\mu\nu} := -2i\bar{\Psi}_i^{[\mu}\Psi_i^{\nu]}$ captures the initial spin of the two massive objects. The weak gravity expansion of the vierbein reads

$$e_\mu^a = \eta^{a\nu}\left(\eta_{\mu\nu} + \frac{\kappa}{2}h_{\mu\nu} - \frac{\kappa^2}{8}h_{\mu\rho}h^\rho{}_\nu + \mathcal{O}(\kappa^3)\right)\,, \tag{6}$$

introducing the graviton field $h_{\mu\nu}(x)$ and the gravitational coupling $\kappa^2 = 32\pi G$. Note that in this perturbative framework the distinction between curved $\mu, \nu, \ldots$ and tangent $a, b, \ldots$ indices necessarily drops.

The spinning WQFT has the partition function [17, 33]

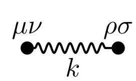
$$\begin{aligned} \mathcal{Z}_{\text{WQFT}} := \text{const} \times \int D[h_{\mu\nu}]\, e^{i(S_{\text{EH}}+S_{\text{gf}})} \\ \times \int \prod_{i=1}^{2} D[z_i^\mu] D[\psi_i'^\mu] \exp\Big[i\sum_{i=1}^{2} S^{(i)} + S_{ES^2}^{(i)}\Big], \end{aligned} \tag{7}$$

where S_{EH} is the Einstein-Hilbert action and the gauge-fixing term S_{gf} enforces de Donder gauge. The SUSY variations (2) leave an imprint on the free energy (or eikonal) $F_{\text{WQFT}}(b_i, v_i, \mathcal{S}_i) := -i\log\mathcal{Z}_{\text{WQFT}}$: after integrating out the fluctuations z^μ and ψ'^μ in the path integral (7), the SUSY variations of the background trajectories (5) remain intact in an asymptotically flat spacetime. That is, the transformations

$$\begin{aligned} &\delta b_i^\mu = i\bar{\epsilon}\Psi_i^\mu + i\epsilon\bar{\Psi}_i^\mu\,, \quad \delta v_i^\mu = 0\,, \quad \delta\Psi_i^\mu = -\epsilon v_i^\mu \\ &\quad\Rightarrow \quad \delta\mathcal{S}_i^{\mu\nu} = v_i^\mu\,\delta b_i^\nu - v_i^\nu\,\delta b_i^\mu \end{aligned} \tag{8}$$

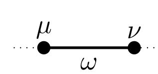
are a symmetry of $F_{\text{WQFT}}(b_i, v_i, \mathcal{S}_i)$ up to the SUSY-breaking C_E terms. As we shall see, this is also a symmetry of the waveform. In general we choose $b\cdot v_i = 0$ (which can be achieved with a suitable shift of the proper times τ_i); however, the variables $b_i, v_i, \mathcal{S}_i$ need to be left unconstrained under the SUSY variation (8).

Feynman rules. — As the Feynman rules for the Einstein-Hilbert action are conventional we will not dwell on them; the only subtlety is our use of a *retarded* graviton propagator:

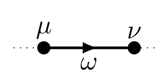


$$= i\frac{P_{\mu\nu;\rho\sigma}}{(k^0 + i\epsilon)^2 - \mathbf{k}^2}\,, \tag{9}$$

with $P_{\mu\nu;\rho\sigma} := \eta_{\mu(\rho}\eta_{\sigma)\nu} - \frac{1}{2}\eta_{\mu\nu}\eta_{\rho\sigma}$. On the worldline we work in one-dimensional energy (frequency) space: the propagators for the fluctuations $z^\mu(\omega)$ and anticommuting vectors $\psi'^\mu(\omega)$ are respectively



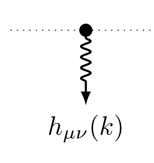
$$= -i\frac{\eta^{\mu\nu}}{m\,(\omega + i\epsilon)^2}\,, \tag{10a}$$



$$= -i\frac{\eta^{\mu\nu}}{m\,(\omega + i\epsilon)}\,, \tag{10b}$$

which also both involve a retarded $i\epsilon$ prescription. The former was already used in Refs. [16, 17].

Next we consider the worldline vertices. The simplest of these is the single-graviton emission vertex:



$$\begin{aligned} &= -i\frac{m\kappa}{2}e^{ik\cdot b}\delta\!\!\!{}^-(k\cdot v)\Big(v^\mu v^\nu + ik_\rho\mathcal{S}^{\rho(\mu}v^{\nu)} \\ &\quad + \frac{1}{2}k_\rho k_\sigma\mathcal{S}^{\rho\mu}\mathcal{S}^{\nu\sigma} + \frac{C_E}{2}v^\mu v^\nu(k\cdot\mathcal{S}\cdot\mathcal{S}\cdot k)\Big)\,, \end{aligned} \tag{11}$$

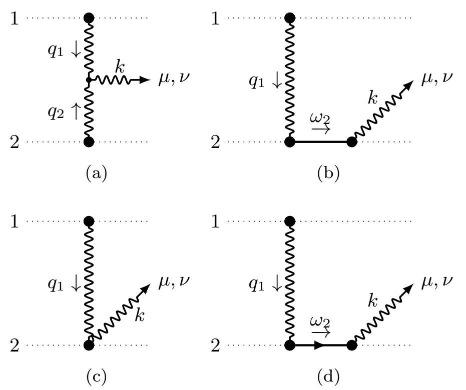


FIG. 1. The four diagram topologies contributing to the 2PM Bremsstrahlung up to $\mathcal{O}(\mathcal{S}^2)$, where $\omega_i = k\cdot v_i$ by energy conservation at the worldline vertices. For diagrams (b)–(d) we also include the corresponding flipped topologies with massive bodies 1↔2; for diagram (d) (which includes the propagating fermion $\psi_2'^{\mu}$) we also include the graph with the arrow reversed.

where $\delta\!\!\!{}^-\!(\omega) := (2\pi)\delta(\omega)$ and we have used $\mathcal{S}^{\mu\nu} = -2i\bar{\Psi}^{[\mu}\Psi^{\nu]}$. The other worldline-based vertices required for the 2PM Bremsstrahlung all appear in Fig. 1: the two-point interaction between a graviton and a single z^μ mode in (b), the two-graviton emission vertex in (c), and the two-point interaction between a graviton and ψ'^{μ} in (d). Full expressions for these vertices are provided in the Supplementary Material.

Waveform from WQFT. — To describe the Bremsstrahlung at 2PM order including spin effects we compute the expectation value $k^2\langle h_{\mu\nu}(k)\rangle_{\rm WQFT}$. This requires us to compute four kinds of Feynman graphs, illustrated in Fig. 1. Explicit expressions for the first two graphs (a) and (b) were given in the non-spinning case [16]; these are now modified by terms up to $\mathcal{O}(\mathcal{S}^2)$. Graphs (c) and (d) are unique to the spinning case — for the latter we sum over both routings of the fermion line.

From this result we seek to obtain the waveform in spacetime in the *wave zone*, where the distance to the observer $|\mathbf{x}| = r$ is large compared to all other lengths. Following Ref. [16] the gauge-invariant *frequency-domain waveform* $4G\,\epsilon^{\mu\nu}S_{\mu\nu}(k^\mu = \Omega\,(1,\hat{\mathbf{x}}))$ is extracted from the WQFT via

$$S_{\mu\nu}(k) = \frac{2}{\kappa}k^2\langle h_{\mu\nu}(k)\rangle_{\rm WQFT}\,, \tag{12}$$

where Ω is the GW frequency and $\hat{\mathbf{x}} = \mathbf{x}/r$ points towards the observer. However, it is advantageous to study the *time-domain waveform* $f(u,\hat{\mathbf{x}})$ which is given by a Fourier transform:

$$\kappa\epsilon^{\mu\nu}h_{\mu\nu} = \frac{f(u,\hat{\mathbf{x}})}{r} = \frac{4G}{r}\int_\Omega e^{-ik\cdot x}\,\epsilon^{\mu\nu}\,S_{\mu\nu}(k)\Big|_{k^\mu=\Omega\,\rho^\mu}\,. \tag{13}$$

We have contracted with a polarization tensor $\epsilon^{\mu\nu} = \frac{1}{2}\epsilon^\mu\epsilon^\nu$, $\int_\Omega := \int_{-\infty}^{\infty}\frac{\mathrm{d}\Omega}{2\pi}$, and $\rho^\mu = (1,\hat{\mathbf{x}})$; in a PM decomposition $f = \sum_n G^n f^{(n)}$ we seek the 2PM component $f^{(2)}$. Note that $k\cdot x = \Omega(t-r)$ yields the retarded time $u = t-r$, and $\epsilon\cdot\epsilon = \epsilon\cdot\rho = 0$.

Integration. — Our integration procedure follows closely that used for the non-spinning calculation in Ref. [16], the main difference being that we maintain four-dimensional Lorentz covariance. Each diagram contributing to $k^2\langle h_{\mu\nu}(k)\rangle_{\rm WQFT}$ carries the overall factor

$$\mu_{1,2}(k) = e^{i(q_1\cdot b_1 + q_2\cdot b_2)}\delta\!\!\!{}^-\!(q_1\cdot v_1)\delta\!\!\!{}^-\!(q_2\cdot v_2)\delta\!\!\!{}^-\!(k-q_1-q_2)\,. \tag{14}$$

We integrate over q_i, the momentum emitted from each worldline (see Fig. 1). When we also integrate over Ω — as in Eq. (13) — the full integration measure becomes

$$\int_{\Omega,q_1,q_2}\mu_{1,2}(k)e^{-ik\cdot x} = \frac{1}{\rho\cdot v_2}\int_{q_1}\delta\!\!\!{}^-\!(q_1\cdot v_1)e^{-iq_1\cdot\tilde{b}}\,, \tag{15}$$

where $\int_{q_i} := \int\frac{\mathrm{d}^4 q_i}{(2\pi)^4}$; the delta function constraints give $\Omega = \frac{q_1\cdot v_2}{\rho\cdot v_2}$ and $q_2 = k - q_1$. The shifted impact parameter,

$$\tilde{b}^\mu = \tilde{b}_2^\mu - \tilde{b}_1^\mu\,, \qquad \tilde{b}_i^\mu = b_i^\mu + u_i v_i^\mu\,, \tag{16}$$

extends the original impact parameter $b^\mu = b_2^\mu - b_1^\mu$ along the undeflected trajectories of the two bodies. Finally, u_i is the retarded time in the i'th rest frame:

$$u_i = \frac{\rho\cdot(x-b_i)}{\rho\cdot v_i}\,, \tag{17}$$

This implies $\rho\cdot\tilde{b}_i = \rho\cdot x = u$, so $\rho\cdot\tilde{b} = 0$.

Rewriting the integral measure as in Eq. (15) is convenient for performing the integrals of diagrams (b)–(d), in the rest frame of body 1. The mirrored counterparts to these diagrams are easily recovered after integration using the $1 \leftrightarrow 2$ symmetry of the waveform. To integrate diagram (a) we insert the partial-fraction identity $q_1^{-2}q_2^{-2} = -q_1^{-2}(2k\cdot q_1)^{-1} - q_2^{-2}(2k\cdot q_2)^{-1}$ (which is valid for k on-shell) and focus on the first term.

The full 2PM waveform is then written schematically as (dropping the subscript on q_1)

$$\frac{f^{(2)}}{m_1m_2} = 4\pi\int_q\delta\!\!\!{}^-\!(q\cdot v_1)\frac{e^{-iq\cdot\tilde{b}}}{q^2}\left(\frac{\mathcal{N}(q)}{q\cdot v_2 + i\epsilon} + \frac{\mathcal{M}(q)}{(q\cdot v_2)(q\cdot\rho)}\right)\,, \\ + (1\leftrightarrow 2)\,, \tag{18}$$

the $\mathcal{N}$- and $\mathcal{M}$-contributions corresponding to diagrams (b)–(d) and (a) in Fig. 1 respectively. The numerators $\mathcal{N}(q)$ and $\mathcal{M}(q)$ have a uniform power counting in q for each spin order:

$$\begin{aligned}\mathcal{N}(q) &= \mathcal{N}_\mu q^\mu + \mathcal{N}_{\mu\nu}q^\mu q^\nu + \mathcal{N}_{\mu\nu\rho}q^\mu q^\nu q^\rho\,,\\ \mathcal{M}(q) &= \mathcal{M}_{\mu\nu}q^\mu q^\nu + \mathcal{M}_{\mu\nu\rho}q^\mu q^\nu q^\rho + \mathcal{M}_{\mu\nu\rho\sigma}q^\mu q^\nu q^\rho q^\sigma\,,\end{aligned} \tag{19}$$

and the non-spinning result involves only $\mathcal{N}_\mu$ and $\mathcal{M}_{\mu\nu}$. We present full expressions for $\mathcal{N}$ and $\mathcal{M}$ in the ancillary file attached to the `arXiv` submission of this Letter.

To lowest order in q^μ, the first integral in eq. (18) is

$$4\pi \int_q \hat{\delta}(q\cdot v_1)\frac{e^{-iq\cdot\tilde{b}}}{q^2}\frac{q^\mu}{q\cdot v_2 + i\epsilon} = \frac{P_1^{\mu\nu}v_{2,\nu}}{(\gamma^2-1)|\tilde{\mathbf{b}}|_1} - \frac{b^\mu}{b^2}\left(\frac{1}{\sqrt{\gamma^2-1}} + \frac{u_2}{|\tilde{\mathbf{b}}|_1}\right), \tag{20}$$

where $P_i^{\mu\nu} := \eta^{\mu\nu} - v_i^\mu v_i^\nu$ is a projector into the rest frame of the i'th body; $b^2 = |b|^2 = -b^\mu b_\mu$ (the impact parameter is spacelike) and

$$|\tilde{\mathbf{b}}|_{1,2} := \sqrt{-\tilde{b}_\mu P_{1,2}^{\mu\nu}\tilde{b}_\nu} = \sqrt{b^2 + (\gamma^2-1)u_{2,1}^2} \tag{21}$$

are the lengths of the shifted impact parameter $\tilde{b}^\mu$ (16) in the two rest frames. The second integral in eq. (18) is

$$4\pi \int_q \hat{\delta}(q\cdot v_1)\frac{e^{-iq\cdot\tilde{b}}}{q^2}\frac{q^\mu q^\nu}{q\cdot v_2\; q\cdot\rho} = \frac{K_1^{\mu\nu}\; v_2\cdot K_1\cdot\rho - 2(v_2\cdot K_1)^{(\mu}(\rho\cdot K_1)^{\nu)}}{(\gamma^2-1)\;(\rho\cdot v_1)^2\; b^2\; |\tilde{b}|^2\; |\tilde{\mathbf{b}}|_1}, \tag{22}$$

where we have introduced the symmetric tensor

$$K_i^{\mu\nu} := P_i^{\mu\nu}|\tilde{\mathbf{b}}|_i^2 + (P_i\cdot\tilde{b})^\mu(P_i\cdot\tilde{b})^\nu\,, \tag{23}$$

with the property that $K_i^{\mu\nu}v_{i,\nu} = K_i^{\mu\nu}\tilde{b}_\nu = 0$. Both integrals are derived in the Supplementary Material; one generalizes to higher powers of q^μ in the numerators by taking derivatives with respect to $\tilde{b}^\mu$.

Results. — The 2PM waveform takes the schematic form

$$\frac{f^{(2)}}{m_1 m_2} = \sum_{s=0}^{2}\frac{1}{|\tilde{\mathbf{b}}|_1^{2s+1}}\left[\alpha_1^{(s)} + \frac{\beta_1^{(s)}}{|\tilde{b}|^{2s+2}}\right] + (1\leftrightarrow 2)\,, \tag{24}$$

where the coefficients $\alpha_i^{(s)}$, $\beta_i^{(s)}$, provided in the ancillary file with the covariant SSC $v_{i,\mu}\mathcal{S}_i^{\mu\nu} = 0$, are associated with the $\mathcal{N}$- and $\mathcal{M}$-type contributions in Eq. (18) respectively; they are functions of u_i, b^μ, v_i^μ, ρ^μ, and $\mathcal{S}_i^{\mu\nu}$ and bi-linear in ϵ^μ. In the Kerr-BH case ($C_{E,i}=0$) the waveform f is invariant under the SUSY transformations in Eq. (8). To see this we expand the waveform at all PM orders in powers of spin:

$$f = f_0 + \sum_{i=1}^{2}\mathcal{S}_{i,\mu\nu}f_i^{\mu\nu} + \sum_{i,j=1}^{2}\mathcal{S}_{i,\mu\nu}\mathcal{S}_{j,\rho\sigma} + \mathcal{O}(\mathcal{S}^3)\,, \tag{25}$$

where SUSY links higher-spin to lower-spin terms,

$$\frac{1}{2}\frac{\partial f_0}{\partial b_{i,\mu}} = v_{i,\nu}\, f_i^{[\mu\nu]}\,, \qquad \frac{1}{4}\frac{\partial f_i^{\mu\nu}}{\partial b_{j,\rho}} = v_{j,\sigma}\, f_{ij}^{\mu\nu;[\rho\sigma]}\,. \tag{26}$$

These identities are satisfied by the waveform in Eq. (24).

To illustrate the waveform we consider the *gravitational wave memory* $\Delta f(\hat{\mathbf{x}}) := f(+\infty,\hat{\mathbf{x}}) - f(-\infty,\hat{\mathbf{x}})$. The constant spin tensors are decomposed in terms of the Pauli-Lubanski vectors a_i^μ as $\mathcal{S}_i^{\mu\nu} = \epsilon^{\mu\nu}{}_{\rho\sigma}v_i^\rho a_i^\sigma$, the latter satisfying $a_i\cdot v_i = 0$. In the aligned-spin case $a_i\cdot b = a_i\cdot v_j = 0$, i.e. the spin vectors are orthogonal to the plane of scattering. Writing $|a_i| = \sqrt{-a_i^2}$ the wave memory is then proportional to the non-spinning result:

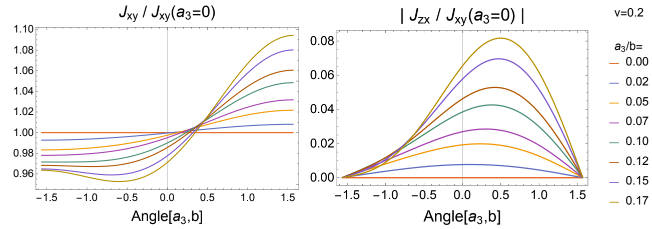


FIG. 2. Total radiated angular momenta for the scattering of two Kerr-BHs with $v = 0.2$ as a function of the angle between the total initial spins $\mathbf{a_3} = \mathbf{a_1} + \mathbf{a_2}$ and $\mathbf{b}$ (with $\mathbf{a_i}\cdot\mathbf{v_i} = 0$) for a range of ratios $|\mathbf{a_3}|/|\mathbf{b}|$. We show the normalized ratio of angular momenta emitted orthogonal to the $\mathbf{b}, \mathbf{v}$ plane (left plot) and in the $\mathbf{b}$ direction (right plot), normalization is w.r.t. angular momentum emitted in the spinless case.

$$\Delta f^{(2)} = \left(1 + \frac{2v|a_3|}{b(1+v^2)} + \frac{|a_3|^2}{b^2} - \sum_{i=1}^{2}\frac{C_{E,i}|a_i|^2}{b^2}\right)\Delta f^{(2)}_{\mathcal{S}=0},$$
$$\frac{\Delta f^{(2)}_{\mathcal{S}=0}}{m_1 m_2} = \frac{4(2\gamma^2-1)\epsilon\cdot v_1(2b\cdot\epsilon\,\rho\cdot v_1 - b\cdot\rho\,\epsilon\cdot v_1)}{b^2\sqrt{\gamma^2-1}(\rho\cdot v_1)^2} + (1\leftrightarrow 2)\,, \tag{27}$$

where $a_3^\mu = a_1^\mu + a_2^\mu$. For two Kerr black holes ($C_{E,i}=0$) with equal-and-opposite spins ($a_1^\mu = -a_2^\mu$) we see that $\Delta f^{(2)} = \Delta f^{(2)}_{\mathcal{S}=0}$, which we observe also when the spins are mis-aligned to the plane of scattering.

There is also a 1PM (non-radiating) contribution to the waveform consisting of single-graviton emission from either massive body:

$$f^{(1)}(\hat{\mathbf{x}}) = \frac{2m_1}{\rho\cdot v_1}(\epsilon\cdot v_1)^2 + \frac{2m_2}{\rho\cdot v_2}(\epsilon\cdot v_2)^2\,. \tag{28}$$

At 1PM order there is manifestly no dependence on either the spins $\mathcal{S}_i^{\mu\nu}$ or impact parameters b_i^μ, so the SUSY identities in Eq. (26) are trivially satisfied.

Finally, the wave memory and 1PM part of the waveform contribute to the total radiated angular momentum $J_{ij}^{\rm rad}$. Using three-dimensional Cartesian basis vectors $\hat{\mathbf{e}}_i$, we choose a frame of reference with the initial velocities v_i^μ restricted to the t–x plane; $\mathbf{b} = b\,\hat{\mathbf{e}}_2$ is orthogonal to these. Then we find two non-zero components of $J_{ij}^{\rm rad}$: $J_{xy}^{\rm rad}$ and $J_{zx}^{\rm rad}$, which are conveniently arranged into

$$\frac{J_{xy}^{\rm rad} + iJ_{zx}^{\rm rad}}{J_{xy}^{\rm init}\big|_{\mathcal{S}=0}} = \frac{4G^2m_1m_2}{b^2}\frac{(2\gamma^2-1)}{\sqrt{\gamma^2-1}}\mathcal{I}(v) \times\left(1 - \frac{2iv\,\mathbf{a}_3\cdot\mathbf{l}}{b(1+v^2)} - \frac{(\mathbf{a}_3\cdot\mathbf{l})^2}{b^2} + \sum_{i=1}^{2}\frac{C_{E,i}}{b^2}(\mathbf{a}_i\cdot\mathbf{l})^2\right) + \mathcal{O}(G^3)\,. \tag{29}$$

We normalize with respect to $J_{xy}^{\rm init}\big|_{\mathcal{S}=0}$, the initial angular momentum in the non-spinning case. The spin vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are taken in the rest frame of each massive body; $\mathbf{a}_3=\mathbf{a}_1+\mathbf{a}_2$, $\mathbf{l}=\hat{\mathbf{e}}_2+i\hat{\mathbf{e}}_3$, and

$$\mathcal{I}(v)=-\frac{8}{3}+\frac{1}{v^2}+\frac{(3v^2-1)}{v^3}\text{arctanh}(v) \tag{30}$$

is a universal prefactor. Eq. (29) holds in the rest frame of either body or the center-of-mass (c.o.m.) frame; see Fig. 2 for plots. For a derivation we refer the reader to the Supplementary Material. There we also compute the total radiated energy in the c.o.m. frame. Due to the multi-scale nature of the waveform it is difficult to perform the necessary time and solid angle-integrals, so we performed a low velocity expansion. For terms up to $\mathcal{O}(v^2)$ we find

$$\begin{aligned}E_{\rm CoM}^{\rm rad,LO}=\frac{vG^3m_1^2m_2^2\pi}{b^3}\bigg[&\frac{37}{15}+\frac{v(65m_1+69m_2)(\mathbf{a}_1\cdot\hat{\mathbf{e}}_3)}{10b(m_1+m_2)}+\frac{1503(\mathbf{a}_1\cdot\hat{\mathbf{e}}_1)(\mathbf{a}_2\cdot\hat{\mathbf{e}}_1)-3559(\mathbf{a}_1\cdot\hat{\mathbf{e}}_2)(\mathbf{a}_2\cdot\hat{\mathbf{e}}_2)+1816(\mathbf{a}_1\cdot\hat{\mathbf{e}}_3)(\mathbf{a}_2\cdot\hat{\mathbf{e}}_3)}{320b^2}\\&+\frac{9(185-176C_{E,1})(\mathbf{a}_1\cdot\hat{\mathbf{e}}_1)^2-(3385-3472C_{E,1})(\mathbf{a}_1\cdot\hat{\mathbf{e}}_2)^2+8(245-236C_{E,1})(\mathbf{a}_1\cdot\hat{\mathbf{e}}_3)^2}{320b^2}+(1\leftrightarrow2)+\mathcal{O}\left(v^2\right)\bigg],\end{aligned} \tag{31}$$

where the swap $(1\leftrightarrow2)$ does not affect the basis vectors $\hat{\mathbf{e}}_\mathbf{i}$ or the constant term $\frac{37}{15}$. It is straighforward to extend this result to higher orders in v.

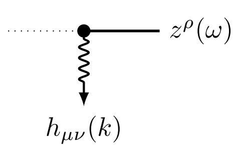
Conclusions. — In this Letter we extended the WQFT to describe spinning compact bodies to quadratic order in spin, and calculated the leading-PM order waveform for highly eccentric (scattering) orbits. Our upcoming work [33] will present an application to further observables such as the spin kick and deflection [26, 29] at 2PM order. The radiated energy (31) should also be particularly useful for future studies. In Refs. [37, 38] the $\mathcal{O}(G^3)$ energy loss from a scattering of non-spinning black holes was recently computed to all orders in velocity using the KMOC formalism [39] (see also Ref. [40]); a similar result could conceivably be obtained at $\mathcal{O}(\mathcal{S}^2)$, and then checked against Eq. (31) in the low-velocity limit. Similarly, the remarkably simple result for radiated angular momentum (29) at 2PM order is intriguing; it may be important for understanding the high-energy limit, see Ref. [41, 42] for the non-spinning case.

The application of modern on-shell and integration techniques to compute scattering amplitudes [37, 43–47] holds great promise for pushing calculations to higher PM orders. This is demonstrated by the impressive calculation of the 4PM conservative dynamics in the potential region [47, 48] — see also Refs. [41, 42, 45, 49–53]. The connection between amplitudes and classical physics was studied in Refs. [39, 40, 54], and Refs. [27, 54] discussed the connection to bound orbits. Our WQFT framework [16, 17] provides another, rather intuitive way to connect amplitude and (classical) worldline EFT calculations. It may therefore benefit from modern amplitude techniques at higher PM orders in future work, building on the compact Lorentz-covariant master integrals provided here.

Acknowledgments. — We would like to thank F. Bautista, R. Bonezzi, A. Buonanno, P. Pichini and J. Vines for very helpful discussions. We are also grateful for use of G. Kälin's C++ graph library. GUJ's and GM's research is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) Projektnummer 417533893/GRK2575 "Rethinking Quantum Field Theory".

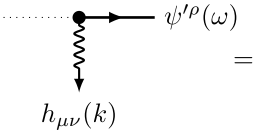
SUPPLEMENTARY MATERIAL

Feynman rules. — Here we give explicit expressions for the worldline Feynman rules used in the main calculation, the single-graviton emission vertex having already been given in Eq. (11). Adding an outgoing z^μ line we have:



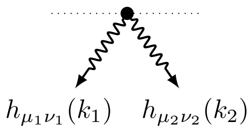
$$\begin{aligned}&=\frac{m\kappa}{2}e^{ik\cdot b}\hat{\delta}(k\cdot v+\omega)\bigg(2\omega v^{(\mu}\delta_\rho^{\nu)}+v^\mu v^\nu k_\rho+i(k\cdot\mathcal{S})^{(\mu}(k_\rho v^{\nu)}+\omega\delta_\rho^{\nu)})+\frac{1}{2}k_\rho(k\cdot\mathcal{S})^\mu(\mathcal{S}\cdot k)^\nu\\&\qquad+\frac{C_E}{2}\left(\left(2\omega v^{(\mu}\delta_\rho^{\nu)}+v^\mu v^\nu k_\rho\right)(k\cdot\mathcal{S}\cdot\mathcal{S}\cdot k)-\omega^2k_\rho(\mathcal{S}\cdot\mathcal{S})^{\mu\nu}+2\omega^2(k\cdot\mathcal{S}\cdot\mathcal{S})^{(\mu}\delta_\rho^{\nu)}\right)\bigg),\end{aligned} \tag{32}$$

where we have adopted the shorthands $(k\cdot\mathcal{S})^\mu=k_\nu\mathcal{S}^{\nu\mu}$, $(\mathcal{S}\cdot\mathcal{S})^{\mu\nu}=\mathcal{S}^{\mu\rho}\mathcal{S}_\rho{}^\nu$ and $(\mathcal{S}\cdot k)^\mu=\mathcal{S}^{\mu\nu}k_\nu$. Both this and the single-graviton emission vertex appear in the non-spinning case, and by setting $\mathcal{S}^{\mu\nu}=0$ we recover the corresponding expressions from Refs. [16, 17]. New to the spinning case is the coupling with ψ'^{μ}:



$$= -im\kappa e^{ik\cdot b}\hat{\delta}(k\cdot v+\omega)\Big(k_{[\rho}\delta^{(\mu}_{\sigma]}\big(v^{\nu)}-i(\mathcal{S}\cdot k)^{\nu)}\big)+iC_E\big(v^{(\mu}k_\lambda+\omega\delta^{(\mu}_{\lambda}\big)\big(v^{\nu)}k_{[\rho}+\omega\delta^{\nu)}_{[\rho}\big)\mathcal{S}^{\lambda}{}_{\sigma]}\Big)\bar{\Psi}^{\sigma}. \tag{33}$$

The vertex with $\bar{\psi}'^{\mu}(\omega)$ on an outgoing line is identical, except with $\bar{\Psi}^{\mu}\to\Psi^{\mu}$. Finally, starting at linear order in spin there is also the two-graviton emission vertex:



$$\begin{aligned}
&= -\frac{m\kappa^2}{4}e^{i(k_1+k_2)\cdot b}\hat{\delta}((k_1+k_2)\cdot v)\Big((k_1\cdot\mathcal{S})^{\mu_2}v^{\mu_1}\eta^{\nu_1\nu_2}-\mathcal{S}^{\mu_1\mu_2}\big(v^{\nu_1}k_1^{\nu_2}-\tfrac{1}{2}k_1\cdot v\eta^{\nu_1\nu_2}\big)\\
&+i\left((\mathcal{S}\cdot k_1)^{\mu_1}(\mathcal{S}\cdot k_1)^{\mu_2}+\tfrac{1}{2}(\mathcal{S}\cdot k_2)^{\mu_1}(\mathcal{S}\cdot k_1)^{\mu_2}-\tfrac{1}{2}\mathcal{S}^{\mu_1\mu_2}(k_1\cdot\mathcal{S}\cdot k_2)\right)\eta^{\nu_1\nu_2}\\
&+\tfrac{i}{4}k_1\cdot k_2\mathcal{S}^{\mu_1\nu_2}\mathcal{S}^{\mu_2\nu_1}-ik_1^{\nu_2}(\mathcal{S}\cdot(k_1+k_2))^{\mu_1}\mathcal{S}^{\mu_2\nu_1}\\
&+i\,C_E\left(2k_1\cdot v(\mathcal{S}\cdot\mathcal{S}\cdot(k_1+k_2))^{\mu_2}v^{\mu_1}-\tfrac{1}{2}(k_1\cdot v)^2(\mathcal{S}\cdot\mathcal{S})^{\mu_1\mu_2}-\tfrac{1}{2}(k_1\cdot\mathcal{S}\cdot\mathcal{S}\cdot k_2)v^{\mu_1}v^{\mu_2}\right)\eta^{\nu_1\nu_2}\\
&+i\,C_E\big(-\tfrac{1}{2}k_1\cdot k_2(\mathcal{S}\cdot\mathcal{S})^{\nu_1\nu_2}v^{\mu_1}v^{\mu_2}+k_1^{\nu_2}(\mathcal{S}\cdot\mathcal{S}\cdot k_2)^{\nu_1}v^{\mu_1}v^{\mu_2}-k_1^{\nu_2}(\mathcal{S}\cdot\mathcal{S}\cdot k_1)^{\mu_2}v^{\mu_1}v^{\nu_1}\\
&-k_1^{\nu_2}(\mathcal{S}\cdot\mathcal{S}\cdot k_2)^{\mu_2}v^{\mu_1}v^{\nu_1}-(\mathcal{S}\cdot\mathcal{S})^{\mu_2\nu_2}\left(k_1\cdot vk_2^{\nu_1}-\tfrac{1}{2}k_1\cdot k_2v^{\nu_1}\right)v^{\mu_1}\big)\Big)+(1\leftrightarrow 2)\,,
\end{aligned} \tag{34}$$

with implicit symmetrization on (μ_1,ν_1) and (μ_2,ν_2).

Integration. — To compute the 2PM waveform we require explicit results for the following integrals:

$$\mathcal{J}^{\mu_1\mu_2\dots\mu_n}=4\pi\int_q\hat{\delta}(q\cdot v_1)\frac{e^{-iq\cdot\tilde{b}}}{q^2}\frac{q^{\mu_1}q^{\mu_2}\dots q^{\mu_n}}{q\cdot v_2+i\epsilon}\,, \tag{35}$$

$$\mathcal{I}^{\mu_1\mu_2\dots\mu_n}=4\pi\int_q\hat{\delta}(q\cdot v_1)\frac{e^{-iq\cdot\tilde{b}}}{q^2}\frac{q^{\mu_1}q^{\mu_2}\dots q^{\mu_n}}{q\cdot v_2\;q\cdot\rho}\,, \tag{36}$$

with $n=1,2,3$ for the $\mathcal{J}$-integrals and $n=2,3,4$ for the $\mathcal{I}$-integrals. Expressions for $\mathcal{J}^{\mu}$ and $\mathcal{I}^{\mu\nu}$ were presented in Eqs. (20) and (22) of the main text respectively — we derive these first, then generalize to higher-orders in q^μ by taking derivatives with respect to the shifted impact parameter $\tilde{b}^\mu$.

Our starting point for $\mathcal{J}^\mu$ is

$$4\pi\int_q\hat{\delta}(q\cdot v_1)e^{-iq\cdot\tilde{b}}\frac{q^\mu}{q^2}=-i\frac{P_1^{\mu\nu}\tilde{b}_\nu}{|\tilde{\mathbf{b}}|_1^3}\,, \tag{37}$$

which is easily derived by specializing to the rest frame of massive body 1 — $P_1^{\mu\nu}\tilde{b}_\nu$ and $|\tilde{\mathbf{b}}|_1$ (21) are the covariant "uplifts" of $\tilde{\mathbf{b}}^i$ and $|\tilde{\mathbf{b}}|$ from this frame. Using

$$\int_\omega e^{-i\omega\tau}\frac{f(\omega)}{\omega+i\epsilon}=-i\int_{-\infty}^{\tau}\mathrm{d}\tau'\int_\omega e^{-i\omega\tau'}f(\omega) \tag{38}$$

the $\mathcal{J}^\mu$ integral can be re-written as

$$\mathcal{J}^\mu=-4\pi i\int_{-\infty}^{u_2}\mathrm{d}u_2'\int_q\hat{\delta}(q\cdot v_1)e^{-iq\cdot\tilde{b}'}\frac{q^\mu}{q^2}\,, \tag{39}$$

where $\tilde{b}'^\mu=b^\mu+u_2'v_2^\mu-u_1v_1^\mu$. Inserting (37) and performing the one-dimensional u_2' integration produces Eq. (20).

In addition to v_1^μ the $\mathcal{I}^{\mu\nu}$ integral is also orthogonal to $\tilde{b}^\mu$, i.e. $\tilde{b}_\mu\mathcal{I}^{\mu\nu}=v_{1,\mu}\mathcal{I}^{\mu\nu}=0$. This follows from

$$\tilde{b}_\mu\mathcal{I}^{\mu\nu}=i|\tilde{b}|\frac{\partial}{\partial|\tilde{b}|}\mathcal{I}^{\nu}=0\,, \tag{40}$$

where the first equality is derived from the $\mathcal{I}$-type integrals definition (36). The integrand of $\mathcal{I}^\mu$ is dimensionless in q^μ, so its integrated form depends only on dimensionless combinations of $\tilde{b}^\mu$ — hence the second equality. $\mathcal{I}^{\mu\nu}$ therefore lives in a two-dimensional subspace orthogonal to v_1^μ and $\tilde{b}^\mu$, and we make an ansatz:

$$\mathcal{I}^{\mu\nu}=c_1K_1^{\mu\nu}+c_2(v_2\cdot K_1)^{(\mu}(\rho\cdot K_1)^{\nu)}\,. \tag{41}$$

$K_1^{\mu\nu}$ was defined in Eq. (23) as the four-dimensional projector into this subspace. We solve for the coefficients by contracting $\mathcal{I}^{\mu\nu}$ with v_2^μ and/or ρ^μ, evaluating the resulting scalar integrals to obtain

$$\rho_\mu v_{2\nu}\mathcal{I}^{\mu\nu}=-\frac{1}{|\tilde{\mathbf{b}}|_1}\,,\quad \rho_\mu\rho_\nu\mathcal{I}^{\mu\nu}=\frac{v_2\cdot K_1\cdot\rho}{(\gamma^2-1)b^2|\tilde{\mathbf{b}}|_1}\,. \tag{42}$$

These allow us to fix c_1 and c_2, and we recover Eq. (22).

By differentiating these integrals with respect to $\tilde{b}^\mu$ one can pull down additional factors of q^μ. For the $\mathcal{J}$-type integrals this procedure is unambiguous; special care should be taken for the $\mathcal{I}$-type integrals as $\tilde{b}^\mu$ is constrained by $\rho\cdot\tilde{b}=0$. However, provided one always works in the three-dimensional subspace defined by $P_1^{\mu\nu}$ then one overcomes this problem, as all contractions involve $P_1^{\mu\nu}$ and $\rho\cdot P_1\cdot\tilde{b}\neq 0$.

Radiated energy and angular momentum. — In Ref. [16] we used the spin-less Bremsstrahlung waveform to compute expressions for the radiated energy and angular momentum, so here we extend these to include spin. The relevant starting points are the same [42, 55]:

$$P^\mu_{\rm rad} = \frac{1}{32\pi G}\int \mathrm{d}u\mathrm{d}\sigma[\dot f_{ij}]^2\rho^\mu\,,\quad \text{where } f = f_{ij}\epsilon^{ij} \tag{43}$$

$$J^{\rm rad}_{ij} = \frac{1}{8\pi G}\int \mathrm{d}u\mathrm{d}\sigma\left(f_{k[i}\dot f_{j]k} - \frac{1}{2}x_{[i}\partial_{j]}f_{kl}\dot f_{kl}\right)\,, \tag{44}$$

with $\dot f_{ij} := \partial_u f_{ij}$ and $\mathrm{d}\sigma = \sin\theta\mathrm{d}\theta\mathrm{d}\phi$ is the unit sphere measure. Here we have introduced a spherical polar coordinate system via

$$\hat{\mathbf{x}} = \hat{\mathbf{e}}_1\cos\theta + \sin\theta\big(\hat{\mathbf{e}}_2\cos\phi + \hat{\mathbf{e}}_3\sin\phi\big)\,, \tag{45}$$

which defines the angles θ and ϕ towards the observer; $\hat{\mathbf{e}}_i$ are Cartesian spatial unit vectors (with Latin indices $i,j,\ldots$). Without loss of generality we assume that $\hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_3$ are orthogonal to the initial velocities v_1^μ and v_2^ν, and $\mathbf{b} = b\,\hat{\mathbf{e}}_2$ where $b^\mu = (0,\mathbf{b})$. The waveform $f_{ij}(u,\theta,\phi)$ is conveniently decomposed on a basis of transverse-traceless polarization tensors:

$$f_{ij} = f_+(e_+)_{ij} + f_\times(e_\times)_{ij}\,, \tag{46}$$

where $f_{+,\times} = \frac{1}{2}(e_{+,\times})_{ij}f_{ij}$ and the polarization tensors are explicitly given as

$$e_+^{ij} = \hat{\boldsymbol{\theta}}^i\hat{\boldsymbol{\theta}}^j - \hat{\boldsymbol{\phi}}^i\hat{\boldsymbol{\phi}}^j\,,\quad e_\times^{ij} = \hat{\boldsymbol{\theta}}^i\hat{\boldsymbol{\phi}}^j + \hat{\boldsymbol{\phi}}^i\hat{\boldsymbol{\theta}}^j\,. \tag{47}$$

The two angular vectors orthogonal to $\hat{\mathbf{x}}$ are $\hat{\boldsymbol{\theta}} := \partial_\theta\hat{\mathbf{x}}$ and $\hat{\boldsymbol{\phi}} := (\sin\theta)^{-1}\partial_\phi\hat{\mathbf{x}}$.

Given our starting point of a fully Lorentz-covariant expression for the waveform f_{ij}, we can make different choices of inertial frame for intermediate expressions. There are two of particular interest to us: the rest frame of the first massive body, and the center-of-mass (c.o.m.) frame. In either case, we decompose the velocities v_i^μ and Pauli-Lubanski spin vectors a_i^μ; defined via $\mathcal{S}_i^{\mu\nu} = \epsilon^{\mu\nu}{}_{\rho\sigma}v_i^\rho a_i^\sigma$; as

$$v_i^\mu = \begin{pmatrix}\gamma_i\\ \gamma_i\mathbf{v}_i\end{pmatrix}\,,\quad a_i^\mu = \begin{pmatrix}\gamma_i(\mathbf{v}_i\cdot\mathbf{a}_i)\\ \mathbf{a}_i + \frac{\gamma_i^2}{1+\gamma_i}(\mathbf{v}_i\cdot\mathbf{a}_i)\mathbf{v}_i\end{pmatrix}\,, \tag{48}$$

where $\mathbf{v}_i \parallel \hat{\mathbf{e}}_2$. These choices manifestly ensure that $a_i\cdot v_i = 0$, $v_i^2 = 1$, and $a_i^2 = -\mathbf{a}_i^2$. Note that $\mathbf{a_i}$ always denotes the spin vector of the ith body in its restframe.

In the first rest frame $\mathbf{v}_1 = \mathbf{0} \implies \gamma_1 = 1$, $\mathbf{v}_2 = \mathbf{v} \implies \gamma_2 = \gamma$; in the c.o.m. frame $\mathbf{v}_1 = v_1\hat{\mathbf{e}}_1$ and $\mathbf{v}_2 = -v_2\hat{\mathbf{e}}_1$, where

$$v_i = \frac{p_\infty}{E_i}\,,\qquad \gamma_i = \frac{E_i}{m_i}\,, \tag{49}$$

and $E_i = \sqrt{m_i^2 + p_\infty^2}$. The initial momenta are $p_1^\mu = m_1 v_1^\mu = (E_1, p_\infty, 0, 0)$ and $p_2^\mu = m_2 v_2^\mu = (E_2, -p_\infty, 0, 0)$. The c.o.m. momentum p_∞ is

$$p_\infty = \frac{m_1 m_2\sqrt{\gamma^2-1}}{\sqrt{m_1^2+m_2^2+2\gamma m_1 m_2}}\,. \tag{50}$$

When working in the c.o.m. frame we prefer to express intermediate results in terms of γ_i and v_i, then use

$$v = \frac{v_1+v_2}{1+v_1 v_2} \tag{51}$$

to reassemble final expressions in terms of γ and v.

We begin with the radiated angular momentum $J^{\rm rad}_{ij}$, which contributes at leading PM order G^2. There are two non-zero components: $J^{\rm rad}_{zx}$ and $J^{\rm rad}_{xy}$. As $f^{(1)}$ (28) is static the u-integration is trivially performed by expressing $J^{\rm rad}_{zx}$ and $J^{\rm rad}_{xy}$ in terms of the wave memories $\Delta f_{+,\times} := f_{+,\times}|_{u=\infty} - f_{+,\times}|_{u=-\infty}$:

$$J^{\rm rad}_{xy} + iJ^{\rm rad}_{zx} = \frac{1}{8\pi}\int\mathrm{d}\sigma\, e^{-i\phi}\Big[i\frac{f_+^{(1)}\Delta f_\times}{\sin\theta} - \partial_\theta f_+^{(1)}\frac{\Delta f_+}{2}\Big] + \mathcal{O}(G^3)\,. \tag{52}$$

The result after integration is Eq. (29). It holds in both the rest frame of the first body and the c.o.m. frame: in the former case $J^{\rm init}_{xy}\big|_{\mathcal{S}=0} = m_2\sqrt{\gamma^2-1}\,b$; in the latter $J^{\rm init}_{xy}\big|_{\mathcal{S}=0} = p_\infty b$.

The radiated four-momentum $P^\mu_{\rm rad}$ (43) contributes to leading PM order G^3. In the center-of-mass frame the radiated energy is $E_{\rm rad,CoM} = v^\mu_{\rm CoM}P^{\rm rad}_\mu$, where

$$v_{\rm CoM} = \frac{m_1 v_1 + m_2 v_2}{\sqrt{m_1^2+m_2^2+2\gamma m_1 m_2}}\,. \tag{53}$$

Due to the multi-scale nature of the waveform f_{ij} it is difficult to perform the time and solid-angle integrations in Eq. (43) directly; however, in a low velocity expansion we suceeded and the result is stated in Eq. (31).

[1] B.P. Abbott *et al.* (LIGO Scientific, Virgo), "Observation of Gravitational Waves from a Binary Black Hole Merger," Phys. Rev. Lett. **116**, 061102 (2016), arXiv:1602.03837 [gr-qc]; "GWTC-1: A Gravitational-Wave Transient Catalog of Compact Binary Mergers Observed by LIGO and Virgo during the First and Second Observing Runs," Phys. Rev. X **9**, 031040 (2019), arXiv:1811.12907 [astro-ph.HE]; R. Abbott *et al.* (LIGO Scientific, Virgo), "GWTC-2: Compact Binary Coalescences Observed by LIGO and Virgo During the First Half of the Third Observing Run," (2020), arXiv:2010.14527 [gr-qc].

[2] R. Abbott *et al.* (LIGO Scientific, Virgo), "Population Properties of Compact Objects from the Second LIGO-Virgo Gravitational-Wave Transient Catalog," (2020), arXiv:2010.14533 [astro-ph.HE].

[3] R. Abbott *et al.* (LIGO Scientific, Virgo), "Tests of General Relativity with Binary Black Holes from the second LIGO-Virgo Gravitational-Wave Transient Catalog," (2020), arXiv:2010.14529 [gr-qc].

[4] B. P. Abbott *et al.* (LIGO Scientific, Virgo), "A Gravitational-wave Measurement of the Hubble Constant Following the Second Observing Run of Advanced LIGO and Virgo," Astrophys. J. **909**, 218 (2021), arXiv:1908.06060 [astro-ph.CO].

[5] B. P. Abbott *et al.* (LIGO Scientific, Virgo), "GW170817: Measurements of neutron star radii and equation of state," Phys. Rev. Lett. **121**, 161101 (2018), arXiv:1805.11581 [gr-qc].

[6] J. Aasi *et al.* (LIGO Scientific), "Advanced LIGO," Class. Quant. Grav. **32**, 074001 (2015), arXiv:1411.4547 [gr-qc]; F. Acernese *et al.* (VIRGO), "Advanced Virgo: a second-generation interferometric gravitational wave detector," Class. Quant. Grav. **32**, 024001 (2015), arXiv:1408.3978 [gr-qc]; Yoichi Aso, Yuta Michimura, Kentaro Somiya, Masaki Ando, Osamu Miyakawa, Takanori Sekiguchi, Daisuke Tatsumi, and Hiroaki Yamamoto (KAGRA), "Interferometer design of the KAGRA gravitational wave detector," Phys. Rev. D **88**, 043007 (2013), arXiv:1306.6747 [gr-qc].

[7] Pürrer, Michael and Haster, Carl-Johan, "Gravitational waveform accuracy requirements for future ground-based detectors," Phys. Rev. Res. **2**, 023151 (2020), arXiv:1912.10055 [gr-qc].

[8] A. Buonanno and T. Damour, "Effective one-body approach to general relativistic two-body dynamics," Phys. Rev. D **59**, 084006 (1999), arXiv:gr-qc/9811091; Parameswaran Ajith *et al.*, "Phenomenological template family for black-hole coalescence waveforms," Class. Quant. Grav. **24**, S689–S700 (2007), arXiv:0704.3764 [gr-qc].

[9] Maarten van de Meent and Harald P. Pfeiffer, "Intermediate mass-ratio black hole binaries: Applicability of small mass-ratio perturbation theory," Phys. Rev. Lett. **125**, 181101 (2020), arXiv:2006.12036 [gr-qc].

[10] Antoni Ramos-Buades, Sascha Husa, Geraint Pratten, Héctor Estellés, Cecilio García-Quirós, Maite Mateu-Lucena, Marta Colleoni, and Rafel Jaume, "First survey of spinning eccentric black hole mergers: Numerical relativity simulations, hybrid waveforms, and parameter estimation," Phys. Rev. D **101**, 083015 (2020), arXiv:1909.11011 [gr-qc]; Danilo Chiaramello and Alessandro Nagar, "Faithful analytical effective-one-body waveform model for spin-aligned, moderately eccentric, coalescing black hole binaries," Phys. Rev. D **101**, 101501 (2020), arXiv:2001.11736 [gr-qc]; Alessandro Nagar, Alice Bonino, and Piero Rettegno, "Effective one-body multipolar waveform model for spin-aligned, quasicircular, eccentric, hyperbolic black hole binaries," Phys. Rev. D **103**, 104021 (2021), arXiv:2101.08624 [gr-qc]; Xiaolin Liu, Zhoujian Cao, and Zong-Hong Zhu, "A higher-multipole gravitational waveform model for an eccentric binary black holes based on the effective-one-body-numerical-relativity formalism," (2021), arXiv:2102.08614 [gr-qc]; Mohammed Khalil, Alessandra Buonanno, Jan Steinhoff, and Justin Vines, "Radiation-reaction force and multipolar waveforms for eccentric, spin-aligned binaries in the effective-one-body formalism," (2021), arXiv:2104.11705 [gr-qc]; Tanja Hinderer and Stanislav Babak, "Foundations of an effective-one-body model for coalescing binaries on eccentric orbits," Phys. Rev. D **96**, 104048 (2017), arXiv:1707.08426 [gr-qc]; Tousif Islam, Vijay Varma, Jackie Lodman, Scott E. Field, Gaurav Khanna, Mark A. Scheel, Harald P. Pfeiffer, Davide Gerosa, and Lawrence E. Kidder, "Eccentric binary black hole surrogate models for the gravitational waveform and remnant properties: comparable mass, nonspinning case," Phys. Rev. D **103**, 064022 (2021), arXiv:2101.11798 [gr-qc].

[11] Johan Samsing, "Eccentric Black Hole Mergers Forming in Globular Clusters," Phys. Rev. D **97**, 103014 (2018), arXiv:1711.07452 [astro-ph.HE]; Carl L. Rodriguez, Pau Amaro-Seoane, Sourav Chatterjee, and Frederic A. Rasio, "Post-Newtonian Dynamics in Dense Star Clusters: Highly-Eccentric, Highly-Spinning, and Repeated Binary Black Hole Mergers," Phys. Rev. Lett. **120**, 151101 (2018), arXiv:1712.04937 [astro-ph.HE]; László Gondán and Bence Kocsis, "High Eccentricities and High Masses Characterize Gravitational-wave Captures in Galactic Nuclei as Seen by Earth-based Detectors," (2020), arXiv:2011.02507 [astro-ph.HE].

[12] B. P. Abbott *et al.* (LIGO Scientific, Virgo), "Binary Black Hole Population Properties Inferred from the First and Second Observing Runs of Advanced LIGO and Advanced Virgo," Astrophys. J. Lett. **882**, L24 (2019), arXiv:1811.12940 [astro-ph.HE].

[13] Bence Kocsis, Merse E. Gaspar, and Szabolcs Marka, "Detection rate estimates of gravity-waves emitted during parabolic encounters of stellar black holes in globular clusters," Astrophys. J. **648**, 411–429 (2006), arXiv:astro-ph/0603441; Sajal Mukherjee, Sanjit Mitra, and Sourav Chatterjee, "Detectability of hyperbolic encounters of compact stars with ground-based gravitational waves detectors," (2020), arXiv:2010.00916 [gr-qc]; Michael Zevin, Johan Samsing, Carl Rodriguez, Carl-Johan Haster, and Enrico Ramirez-Ruiz, "Eccentric Black Hole Mergers in Dense Star Clusters: The Role of Binary–Binary Encounters," Astrophys. J. **871**, 91 (2019), arXiv:1810.00901 [astro-ph.HE]; Rossella Gamba, Matteo Breschi, Gregorio Carullo, Piero Rettegno, Simone Albanesi, Sebastiano Bernuzzi, and Alessandro Nagar, "GW190521: A dynamical capture of two black holes," (2021), arXiv:2106.05575 [gr-qc].

[14] Barak Zackay, Tejaswi Venumadhav, Liang Dai, Javier Roulet, and Matias Zaldarriaga, "Highly spinning and aligned binary black hole merger in the Advanced LIGO first observing run," Phys. Rev. D **100**, 023007 (2019), arXiv:1902.10331 [astro-ph.HE]; Yiwen Huang, Carl-Johan Haster, Salvatore Vitale, Aaron Zimmerman, Javier Roulet, Tejaswi Venumadhav, Barak Zackay, Liang Dai, and Matias Zaldarriaga, "Source properties of the lowest signal-to-noise-ratio binary black hole detections," Phys. Rev. D **102**, 103024 (2020), arXiv:2003.04513 [gr-qc].

[15] K. S. Thorne and S. J. Kovacs, "The generation of gravitational waves. I. Weak-field sources." Astrophys. J. **200**, 245–262 (1975); R. J. Crowley and K. S. Thorne, "The generation of gravitational waves. II. The postlinear formation revisited." Astrophys. J. **215**, 624–635 (1977); S.J. Kovacs and K.S. Thorne, "The Generation of Gravitational Waves. 3. Derivation of Bremsstrahlung Formulas," Astrophys. J. **217**, 252–280 (1977); "The Generation of Gravitational Waves. 4. Bremsstrahlung," Astrophys. J. **224**, 62–85 (1978).

[16] Gustav Uhre Jakobsen, Gustav Mogull, Jan Plefka, and Jan Steinhoff, "Classical Gravitational Bremsstrahlung from a Worldline Quantum Field Theory," Phys. Rev. Lett. **126**, 201103 (2021), arXiv:2101.12688 [gr-qc].

[17] Gustav Mogull, Jan Plefka, and Jan Steinhoff, "Classical black hole scattering from a worldline quantum field theory," JHEP **02**, 048 (2021), arXiv:2010.02865 [hep-th].

[18] Paul S. Howe, Silvia Penati, Mario Pernici, and Paul K. Townsend, "Wave Equations for Arbitrary Spin From Quantization of the Extended Supersymmetric Spinning Particle," Phys. Lett. B **215**, 555–558 (1988).

[19] G. W. Gibbons, R. H. Rietdijk, and J. W. van Holten, "SUSY in the sky," Nucl. Phys. B **404**, 42–64 (1993), arXiv:hep-th/9303112.

[20] Fiorenzo Bastianelli, Paolo Benincasa, and Simone Giombi, "Worldline approach to vector and antisymmetric tensor fields," JHEP **04**, 010 (2005), arXiv:hep-th/0503155; "Worldline approach to vector and antisymmetric tensor fields. II." JHEP **10**, 114 (2005), arXiv:hep-th/0510010.

[21] Rafael A. Porto, "Post-Newtonian corrections to the motion of spinning bodies in NRGR," Phys. Rev. D **73**, 104031 (2006), arXiv:gr-qc/0511061.

[22] Michele Levi and Jan Steinhoff, "Spinning gravitating objects in the effective field theory in the post-Newtonian scheme," JHEP **09**, 219 (2015), arXiv:1501.04956 [gr-qc].

[23] Walter D. Goldberger and Ira Z. Rothstein, "An Effective field theory of gravity for extended objects," Phys. Rev. D **73**, 104029 (2006), arXiv:hep-th/0409156; "Towers of Gravitational Theories," Gen. Rel. Grav. **38**, 1537–1546 (2006), arXiv:hep-th/0605238; Walter D. Goldberger and Andreas Ross, "Gravitational radiative corrections from effective field theory," Phys. Rev. D **81**, 124015 (2010), arXiv:0912.4254 [gr-qc].

[24] Rafael A. Porto, "The effective field theorist's approach to gravitational dynamics," Phys. Rept. **633**, 1–104 (2016), arXiv:1601.04914 [hep-th]; Michèle Levi, "Effective Field Theories of Post-Newtonian Gravity: A comprehensive review," Rept. Prog. Phys. **83**, 075901 (2020), arXiv:1807.01699 [hep-th].

[25] Walter D. Goldberger, Jingping Li, and Siddharth G. Prabhu, "Spinning particles, axion radiation, and the classical double copy," Phys. Rev. D **97**, 105018 (2018), arXiv:1712.09250 [hep-th]; Walter D. Goldberger and Alexander K. Ridgway, "Radiation and the classical double copy for color charges," Phys. Rev. **D95**, 125010 (2017), arXiv:1611.03493 [hep-th]; Chia-Hsien Shen, "Gravitational Radiation from Color-Kinematics Duality," JHEP **11**, 162 (2018), arXiv:1806.07388 [hep-th].

[26] Zhengwen Liu, Rafael A. Porto, and Zixin Yang, "Spin Effects in the Effective Field Theory Approach to Post-Minkowskian Conservative Dynamics," (2021), arXiv:2102.10059 [hep-th].

[27] Gregor Kälin and Rafael A. Porto, "Post-Minkowskian Effective Field Theory for Conservative Binary Dynamics," JHEP **11**, 106 (2020), arXiv:2006.01184 [hep-th].

[28] Zvi Bern, Andres Luna, Radu Roiban, Chia-Hsien Shen, and Mao Zeng, "Spinning Black Hole Binary Dynamics, Scattering Amplitudes and Effective Field Theory," (2020), arXiv:2005.03071 [hep-th].

[29] Dimitrios Kosmopoulos and Andres Luna, "Quadratic-in-Spin Hamiltonian at $\mathcal{O}(G^2)$ from Scattering Amplitudes," (2021), arXiv:2102.10137 [hep-th].

[30] Rafael A. Porto, Andreas Ross, and Ira Z. Rothstein, "Spin induced multipole moments for the gravitational wave flux from binary inspirals to third Post-Newtonian order," JCAP **1103**, 009 (2011), arXiv:1007.1312 [gr-qc]; "Spin induced multipole moments for the gravitational wave amplitude from binary inspirals to 2.5 Post-Newtonian order," JCAP **1209**, 028 (2012), arXiv:1203.2962 [gr-qc]; Natalia T. Maia, Chad R. Galley, Adam K. Leibovich, and Rafael A. Porto, "Radiation reaction for spinning bodies in effective field theory I: Spin-orbit effects," Phys. Rev. **D96**, 084064 (2017), arXiv:1705.07934 [gr-qc]; "Radiation reaction for spinning bodies in effective field theory II: Spin-spin effects," Phys. Rev. **D96**, 084065 (2017), arXiv:1705.07938 [gr-qc]; Gihyuk Cho, Brian Pardo, and Rafael A. Porto, "Gravitational radiation from inspiralling compact objects: Spin-spin effects completed at the next-to-leading post-Newtonian order," (2021), arXiv:2103.14612 [gr-qc].

[31] Chandra Kant Mishra, Aditya Kela, K. G. Arun, and Guillaume Faye, "Ready-to-use post-Newtonian gravitational waveforms for binary black holes with nonprecessing spins: An update," Phys. Rev. **D93**, 084054 (2016), arXiv:1601.05588 [gr-qc]; Alessandra Buonanno, Guillaume Faye, and Tanja Hinderer, "Spin effects on gravitational waves from inspiraling compact binaries at second post-Newtonian order," Phys. Rev. **D87**, 044009 (2013), arXiv:1209.6349 [gr-qc].

[32] Justin Vines, "Scattering of two spinning black holes in post-Minkowskian gravity, to all orders in spin, and effective-one-body mappings," Class. Quant. Grav. **35**, 084002 (2018), arXiv:1709.06016 [gr-qc]; Donato Bini and Thibault Damour, "Gravitational spin-orbit coupling in binary systems, post-Minkowskian approximation and effective one-body theory," Phys. Rev. **D96**, 104038 (2017), arXiv:1709.00590 [gr-qc]; "Gravitational spin-orbit coupling in binary systems at the second post-Minkowskian approximation," Phys. Rev. **D98**, 044036 (2018), arXiv:1805.10809 [gr-qc]; Alfredo Guevara, "Holomorphic Classical Limit for Spin Effects in Gravitational and Electromagnetic Scattering," JHEP **04**, 033 (2019), arXiv:1706.02314 [hep-th]; Justin Vines, Jan Steinhoff, and Alessandra Buonanno, "Spinning-black-hole scattering and the test-black-hole limit at second post-Minkowskian order," Phys. Rev. D **99**, 064054 (2019), arXiv:1812.00956 [gr-qc]; Alfredo Guevara, Alexander Ochirov, and Justin Vines, "Scattering of Spinning Black Holes from Exponentiated Soft Factors," JHEP **09**, 056 (2019), arXiv:1812.06895 [hep-th]; Ming-Zhi Chung, Yu-Tin Huang, Jung-Wook Kim, and Sangmin Lee, "The simplest massive S-matrix: from minimal coupling to Black Holes," JHEP **04**, 156 (2019), arXiv:1812.08752 [hep-th]; Alfredo Guevara, Alexander Ochirov, and Justin Vines, "Black-hole scattering with general spin directions from minimal-coupling amplitudes," Phys. Rev. D **100**, 104024 (2019), arXiv:1906.10071 [hep-th]; Ming-Zhi Chung, Yu-Tin Huang, and Jung-Wook Kim, "Classical potential for general spinning bodies," JHEP **09**, 074 (2020), arXiv:1908.08463 [hep-th]; Poul H. Damgaard, Kays Haddad, and Andreas Helset, "Heavy Black Hole Effective Theory," JHEP **11**, 070 (2019), arXiv:1908.10308 [hep-ph]; Rafael Aoude, Kays Haddad, and Andreas Helset, "On-shell heavy particle effective theories," JHEP **05**, 051 (2020), arXiv:2001.09164 [hep-th]; Alfredo Gue-

vara, Ben Maybee, Alexander Ochirov, Donal O'Connell, and Justin Vines, "A worldsheet for Kerr," (2020), arXiv:2012.11570 [hep-th].

[33] Gustav Uhre Jakobsen, Gustav Mogull, Jan Plefka, and Jan Steinhoff, "Susy in the sky with gravitons," To appear.

[34] Justin Vines, Daniela Kunst, Jan Steinhoff, and Tanja Hinderer, "Canonical Hamiltonian for an extended test body in curved spacetime: To quadratic order in spin," Phys. Rev. D **93**, 103008 (2016), arXiv:1601.07529 [gr-qc].

[35] Rafael A Porto and Ira Z. Rothstein, "Next to Leading Order Spin(1)Spin(1) Effects in the Motion of Inspiralling Compact Binaries," Phys. Rev. D **78**, 044013 (2008), [Erratum: Phys.Rev.D 81, 029905 (2010)], arXiv:0804.0260 [gr-qc].

[36] Myron Mathisson, "Neue Mechanik materieller Systeme," Acta Phys. Polon. **6**, 163–2900 (1937); Achille Papapetrou, "Spinning test particles in general relativity. 1." Proc. Roy. Soc. Lond. A **209**, 248–258 (1951); W. G. Dixon, "Dynamics of extended bodies in general relativity. I. Momentum and angular momentum," Proc. Roy. Soc. Lond. A **314**, 499–527 (1970).

[37] Enrico Herrmann, Julio Parra-Martinez, Michael S. Ruf, and Mao Zeng, "Gravitational Bremsstrahlung from Reverse Unitarity," (2021), arXiv:2101.07255 [hep-th].

[38] Enrico Herrmann, Julio Parra-Martinez, Michael S. Ruf, and Mao Zeng, "Radiative Classical Gravitational Observables at $\mathcal{O}(G^3)$ from Scattering Amplitudes," (2021), arXiv:2104.03957 [hep-th].

[39] David A. Kosower, Ben Maybee, and Donal O'Connell, "Amplitudes, Observables, and Classical Scattering," JHEP **02**, 137 (2019), arXiv:1811.10950 [hep-th].

[40] Ben Maybee, Donal O'Connell, and Justin Vines, "Observables and amplitudes for spinning particles and black holes," JHEP **12**, 156 (2019), arXiv:1906.09260 [hep-th].

[41] Paolo Di Vecchia, Carlo Heissenberg, Rodolfo Russo, and Gabriele Veneziano, "Universality of ultra-relativistic gravitational scattering," Phys. Lett. B **811**, 135924 (2020), arXiv:2008.12743 [hep-th].

[42] Thibault Damour, "Radiative contribution to classical gravitational scattering at the third order in G," Phys. Rev. D **102**, 124008 (2020), arXiv:2010.01641 [gr-qc].

[43] Zvi Bern, Lance J. Dixon, David C. Dunbar, and David A. Kosower, "One loop n point gauge theory amplitudes, unitarity and collinear limits," Nucl. Phys. **B425**, 217–260 (1994), arXiv:hep-ph/9403226 [hep-ph]; "Fusing gauge theory tree amplitudes into loop amplitudes," Nucl. Phys. **B435**, 59–101 (1995), arXiv:hep-ph/9409265 [hep-ph]; Ruth Britto, Freddy Cachazo, and Bo Feng, "Generalized unitarity and one-loop amplitudes in N=4 super-Yang-Mills," Nucl. Phys. **B725**, 275–305 (2005), arXiv:hep-th/0412103 [hep-th].

[44] Z. Bern, J. J. M. Carrasco, and Henrik Johansson, "New Relations for Gauge-Theory Amplitudes," Phys. Rev. **D78**, 085011 (2008), arXiv:0805.3993 [hep-ph]; Zvi Bern, John Joseph M. Carrasco, and Henrik Johansson, "Perturbative Quantum Gravity as a Double Copy of Gauge Theory," Phys. Rev. Lett. **105**, 061602 (2010), arXiv:1004.0476 [hep-th]; Z. Bern, J. J. M. Carrasco, L. J. Dixon, H. Johansson, and R. Roiban, "Simplifying Multiloop Integrands and Ultraviolet Divergences of Gauge Theory and Gravity Amplitudes," Phys. Rev. **D85**, 105014 (2012), arXiv:1201.5366 [hep-th]; Zvi Bern, John Joseph M. Carrasco, Wei-Ming Chen, Henrik Johansson, Radu Roiban, and Mao Zeng, "Five-loop four-point integrand of $N = 8$ supergravity as a generalized double copy," Phys. Rev. **D96**, 126012 (2017), arXiv:1708.06807 [hep-th]; Zvi Bern, John Joseph Carrasco, Wei-Ming Chen, Alex Edison, Henrik Johansson, Julio Parra-Martinez, Radu Roiban, and Mao Zeng, "Ultraviolet Properties of $\mathcal{N} = 8$ Supergravity at Five Loops," Phys. Rev. **D98**, 086021 (2018), arXiv:1804.09311 [hep-th]; Zvi Bern, John Joseph Carrasco, Marco Chiodaroli, Henrik Johansson, and Radu Roiban, "The Duality Between Color and Kinematics and its Applications," (2019), arXiv:1909.01358 [hep-th].

[45] Zvi Bern, Clifford Cheung, Radu Roiban, Chia-Hsien Shen, Mikhail P. Solon, and Mao Zeng, "Black Hole Binary Dynamics from the Double Copy and Effective Theory," JHEP **10**, 206 (2019), arXiv:1908.01493 [hep-th].

[46] Julio Parra-Martinez, Michael S. Ruf, and Mao Zeng, "Extremal black hole scattering at $\mathcal{O}(G^3)$: graviton dominance, eikonal exponentiation, and differential equations," JHEP **11**, 023 (2020), arXiv:2005.04236 [hep-th].

[47] Zvi Bern, Julio Parra-Martinez, Radu Roiban, Michael S. Ruf, Chia-Hsien Shen, Mikhail P. Solon, and Mao Zeng, "Scattering Amplitudes and Conservative Binary Dynamics at $\mathcal{O}(G^4)$," (2021), arXiv:2101.07254 [hep-th].

[48] Christoph Dlapa, Gregor Kälin, Zhengwen Liu, and Rafael A. Porto, "Dynamics of Binary Systems to Fourth Post-Minkowskian Order from the Effective Field Theory Approach," (2021), arXiv:2106.08276 [hep-th].

[49] Zvi Bern, Clifford Cheung, Radu Roiban, Chia-Hsien Shen, Mikhail P. Solon, and Mao Zeng, "Scattering Amplitudes and the Conservative Hamiltonian for Binary Systems at Third Post-Minkowskian Order," Phys. Rev. Lett. **122**, 201603 (2019), arXiv:1901.04424 [hep-th].

[50] Clifford Cheung and Mikhail P. Solon, "Classical gravitational scattering at $\mathcal{O}(\mathrm{G}^3)$ from Feynman diagrams," JHEP **06**, 144 (2020), arXiv:2003.08351 [hep-th]; Kälin, Gregor and Liu, Zhengwen and Porto, Rafael A., "Conservative Dynamics of Binary Systems to Third Post-Minkowskian Order from the Effective Field Theory Approach," Phys. Rev. Lett. **125**, 261103 (2020), arXiv:2007.04977 [hep-th].

[51] Paolo Di Vecchia, Carlo Heissenberg, Rodolfo Russo, and Gabriele Veneziano, "The Eikonal Approach to Gravitational Scattering and Radiation at $\mathcal{O}(G^3)$," (2021), arXiv:2104.03256 [hep-th]; N. E. J. Bjerrum-Bohr, P. H. Damgaard, L. Planté, and P. Vanhove, "The Amplitude for Classical Gravitational Scattering at Third Post-Minkowskian Order," (2021), arXiv:2105.05218 [hep-th].

[52] Donato Bini, Thibault Damour, and Andrea Geralico, "Scattering of tidally interacting bodies in post-Minkowskian gravity," Phys. Rev. **D101**, 044039 (2020), arXiv:2001.00352 [gr-qc]; Clifford Cheung and Mikhail P. Solon, "Tidal Effects in the Post-Minkowskian Expansion," Phys. Rev. Lett. **125**, 191601 (2020), arXiv:2006.06665 [hep-th]; Kays Haddad and Andreas Helset, "Tidal effects in quantum field theory," JHEP **12**, 024 (2020), arXiv:2008.04920 [hep-th]; Kälin, Gregor and Liu, Zhengwen and Porto, Rafael A., "Conservative Tidal Effects in Compact Binary Systems to Next-to-Leading Post-Minkowskian Order," Phys. Rev. **D102**, 124025 (2020), arXiv:2008.06047 [hep-th]; Andreas Brandhuber and Gabriele Travaglini, "On higher-derivative effects on

the gravitational potential and particle bending," JHEP **01**, 010 (2020), arXiv:1905.05657 [hep-th]; Manuel Accettulli Huber, Andreas Brandhuber, Stefano De Angelis, and Gabriele Travaglini, "Note on the absence of R^2 corrections to Newton's potential," Phys. Rev. **D101**, 046011 (2020), arXiv:1911.10108 [hep-th]; "Eikonal phase matrix, deflection angle and time delay in effective field theories of gravity," Phys. Rev. **D102**, 046014 (2020), arXiv:2006.02375 [hep-th]; Zvi Bern, Julio Parra-Martinez, Radu Roiban, Eric Sawyer, and Chia-Hsien Shen, "Leading Nonlinear Tidal Effects and Scattering Amplitudes," (2020), arXiv:2010.08559 [hep-th]; Clifford Cheung, Nabha Shah, and Mikhail P. Solon, "Mining the Geodesic Equation for Scattering Data," Phys. Rev. **D103**, 024030 (2021), arXiv:2010.08568 [hep-th]; Rafael Aoude, Kays Haddad, and Andreas Helset, "Tidal effects for spinning particles," (2020), arXiv:2012.05256 [hep-th].

[53] D. Amati, M. Ciafaloni, and G. Veneziano, "Higher Order Gravitational Deflection and Soft Bremsstrahlung in Planckian Energy Superstring Collisions," Nucl. Phys. **B347**, 550–580 (1990); Paolo Di Vecchia, Andrés Luna, Stephen G. Naculich, Rodolfo Russo, Gabriele Veneziano, and Chris D. White, "A tale of two exponentiations in $\mathcal{N} = 8$ supergravity," Phys. Lett. **B798**, 134927 (2019), arXiv:1908.05603 [hep-th]; Paolo Di Vecchia, Stephen G. Naculich, Rodolfo Russo, Gabriele Veneziano, and Chris D. White, "A tale of two exponentiations in $\mathcal{N} = 8$ supergravity at subleading level," JHEP **03**, 173 (2020), arXiv:1911.11716 [hep-th]; Zvi Bern, Harald Ita, Julio Parra-Martinez, and Michael S. Ruf, "Universality in the classical limit of massless gravitational scattering," Phys. Rev. Lett. **125**, 031601 (2020), arXiv:2002.02459 [hep-th]; Manuel Accettulli Huber, Andreas Brandhuber, Stefano De Angelis, and Gabriele Travaglini, "From amplitudes to gravitational radiation with cubic interactions and tidal effects," (2020), arXiv:2012.06548 [hep-th]; Paolo Di Vecchia, Carlo Heissenberg, Rodolfo Russo, and Gabriele Veneziano, "Radiation Reaction from Soft Theorems," (2021), arXiv:2101.05772 [hep-th]; Yilber Fabian Bautista and Alfredo Guevara, "From Scattering Amplitudes to Classical Physics: Universality, Double Copy and Soft Theorems," (2019), arXiv:1903.12419 [hep-th]; Alok Laddha and Ashoke Sen, "Gravity Waves from Soft Theorem in General Dimensions," JHEP **09**, 105 (2018), arXiv:1801.07719 [hep-th]; "Logarithmic Terms in the Soft Expansion in Four Dimensions," JHEP **10**, 056 (2018), arXiv:1804.09193 [hep-th]; Biswajit Sahoo and Ashoke Sen, "Classical and Quantum Results on Logarithmic Terms in the Soft Theorem in Four Dimensions," JHEP **02**, 086 (2019), arXiv:1808.03288 [hep-th]; Alok Laddha and Ashoke Sen, "Classical proof of the classical soft graviton theorem in $D > 4$," Phys. Rev. **D101**, 084011 (2020), arXiv:1906.08288 [gr-qc]; Arnab Priya Saha, Biswajit Sahoo, and Ashoke Sen, "Proof of the classical soft graviton theorem in $D = 4$," JHEP **06**, 153 (2020), arXiv:1912.06413 [hep-th]; Manu A, Debodirna Ghosh, Alok Laddha, and P. V. Athira, "Soft Radiation from Scattering Amplitudes Revisited," (2020), arXiv:2007.02077 [hep-th]; Biswajit Sahoo, "Classical Sub-subleading Soft Photon and Soft Graviton Theorems in Four Spacetime Dimensions," JHEP **12**, 070 (2020), arXiv:2008.04376 [hep-th].

[54] Thibault Damour, "Classical and quantum scattering in post-Minkowskian gravity," Phys. Rev. **D102**, 024060 (2020), arXiv:1912.02139 [gr-qc]; N. E. J. Bjerrum-Bohr, Andrea Cristofoli, and Poul H. Damgaard, "Post-Minkowskian Scattering Angle in Einstein Gravity," JHEP **08**, 038 (2020), arXiv:1910.09366 [hep-th]; N. Emil J. Bjerrum-Bohr, Poul H. Damgaard, Ludovic Planté, and Pierre Vanhove, "Classical Gravity from Loop Amplitudes," (2021), arXiv:2104.04510 [hep-th]; Kälin, Gregor and Porto, Rafael A., "From Boundary Data to Bound States," JHEP **01**, 072 (2020), arXiv:1910.03008 [hep-th]; "From boundary data to bound states. Part II. Scattering angle to dynamical invariants (with twist)," JHEP **02**, 120 (2020), arXiv:1911.09130 [hep-th].

[55] Béatrice Bonga and Eric Poisson, "Coulombic contribution to angular momentum flux in general relativity," Phys. Rev. D **99**, 064024 (2019), arXiv:1808.01288 [gr-qc].