On Internal Constraints in Continuum Mechanics

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Abstract. When a body is subject to simple internal constraints, the deformation gradient must belong to a certain manifold. This is in contrast to the situation in the unconstrained case, where the deformation gradient is an element of the open subset of second-order tensors with positive determinant. Commonly, following Truesdell and Noll [1], modern treatments of constrained theories start with an *a priori* additive decomposition of the stress into reactive and active components with the reactive component *assumed* to be powerless in all motions that satisfy the constraints and the active component given by a constitutive equation. Here, we obtain this same decomposition automatically by making a purely geometrical and general direct sum decomposition of the space of all second-order tensors in terms of the normal and tangent spaces of the constraint manifold. As an example, our approach is used to recover the familiar theory of constrained hyperelasticity.

Keywords: continuum mechanics, internal constraints, constitutive theory, hyperelasticity.

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Dedicated to the memory of Clifford A. Truesdell

1. Introduction

Most contemporary works in constrained theories of continuum mechanics follow the approach of Truesdell and Noll [1],[1] wherein the stress is decomposed *a priori* into reactive and active terms with the reactive stress *assumed* to be powerless in all motions consistent with the constraints and the active stress given by a constitutive equation. The approach of Truesdell and Noll was motivated by the Ericksen and Rivlin [2] treatment of constrained hyperelasticity, which is based on the requirement that the constitutive equations for the stress and internal energy satisfy balance of energy in all motions consistent with the constraints. The main feature of the Ericksen–Rivlin hyperelastic

[1] See Carlson and Tortorelli [3] for a fuller account of other work in this area.

development is that the stress is automatically decomposed into the sum of two terms. One term has zero power in any motion meeting the constraints and is determined by the constraints to within scalar multipliers; it is natural to think of this term as being present to maintain the constraints and to call it the *reactive stress*. The other term is, roughly speaking, the gradient of the internal-energy density with respect to the strain, and it is called the *active stress*. Carlson and Tortorelli [3] replaced the Lagrange multiplier formalism of the Ericksen–Rivlin approach with an elementary geometrical argument—essentially, the assertion that, if a vector $\boldsymbol{a}$ is orthogonal to every vector $\boldsymbol{b}$ that is orthogonal to some vector $\boldsymbol{c}$, then $\boldsymbol{a}$ is parallel to $\boldsymbol{c}$—used in the Truesdell–Noll method for determining the form of the reactive stress.

It is widely accepted that many of the advances in modern continuum mechanics rest in large part on the clear separation of kinematics, basic laws of balance and growth, and constitutive equations that characterizes the subject. Where do internal constraints fit into this hierarchy? While internal constraints do delimit aspects of material response, they apply to broad classes of materials; for instance, the constraint of incompressibility applies equally well to both hyperelastic solids and viscous fluids. Hence, we view internal constraints as being more fundamental than constitutive equations. It is natural then to attempt to ascertain the implications of the kinematical nature of internal constraints. Motivated by this point of view, Anderson, Carlson, and Fried [4] used a modified version of the geometrical argument of Carlson and Tortorelli [3] to deal with the constraints of incompressibility and microstructural inextensibility present in their theory of nematic elastomers. They started with a purely geometrical direct sum decomposition of the relevant fields based on the normal and tangent spaces of the constraint manifold to obtain automatically the familiar decomposition into active and reactive components—*without the use of any balance laws or constitutive assumptions*. It is the purpose of the present paper to return to the simpler context of classical continuum mechanics for presentation of this improved approach and to emphasize its generality. We also take this opportunity to treat multiple constraints.

In Section 2, we consider the case where the deformation gradient is restricted by n independent constraints. Thus, the deformation gradient is constrained to belong to a certain manifold in contrast to being an arbitrary element of the open subset of second-order tensors with positive determinant as in the unconstrained case. Next, we use the projection theorem to effect a unique orthogonal decomposition of the space of all second-order tensors in terms of the normal and tangent spaces of the constraint manifold.

In the absence of thermal contributions, the general thermomechanical principles of energy balance and entropy growth combine to yield a free-energy inequality, which may be simplified by means of the power-identity theorem. These considerations are developed in Section 3.

In Section 4, the orthogonal decomposition of Section 2 is applied to the stress tensor. We find that, for motions consistent with the constraints, the normal component is automatically powerless and only the tangential component enters into the free-energy inequality. Consequently, the tangential component is called the active stress, and one would expect to write a constitutive equation for it. On the other hand, the normal component, termed the reactive stress, is determined by the constraints to within scalar multipliers that we take to be constitutively indeterminate. Thus, our approach to internal constraints has the same level of generality as that of Truesdell and Noll [1] and provides exactly the same results. However, our decomposition of the stress, rather than being *a priori*, is dictated by the geometry of the constraint manifold.

In Section 5, as an application of the general theory, we make elastic constitutive assumptions for the free energy and the stress and require that the free-energy inequality be satisfied for all motions consistent with the constraints to recover the theory of constrained hyperelasticity; and, in this sense, the present paper replaces the paper of Carlson and Tortorelli [3]. Finally, in Section 6, we show that when the principle of material frame-indifference is invoked in constrained hyperelasticity, the active and reactive stresses individually satisfy local balance of moment of momentum.

Throughout, we use the notations of modern continuum mechanics; see, e.g., the text of Gurtin [5].

2. The Geometry of the Constraint Manifold

We use a referential formulation. Accordingly, the body is identified with the region of space $\mathcal{B}$ that it occupies in a fixed reference configuration. We write $\boldsymbol{y}$ for the motion of the body and

$$\boldsymbol{F} = \operatorname{Grad} \boldsymbol{y}, \tag{2.1}$$

with $\det \boldsymbol{F} > 0$, for the deformation gradient.

We consider the case where the motion of the body is restricted by n *simple constraints*; i.e., the deformation gradient is required to meet[2]

$$\hat{\gamma}_i(\boldsymbol{F}) = 0, \qquad i = 1, \dots, n, \tag{2.2}$$

where the constraint functions $\hat{\gamma}_i : \mathrm{Lin}^+ \to \mathbb{R}$ are suitably smooth and independent in the sense that the set $\{\mathrm{Grad}\,\hat{\gamma}_i(\boldsymbol{F}), i = 1, \dots, n\}$ is linearly independent at each $\boldsymbol{F}$ belonging to Lin^+. In other words, the deformation gradient must belong to the *constraint manifold*

$$\mathrm{Con} := \{\boldsymbol{F} \in \mathrm{Lin}^+ : \hat{\gamma}_i(\boldsymbol{F}) = 0, i = 1, \dots, n\}. \tag{2.3}$$

Of great use to us will be the *normal space* to Con at $\boldsymbol{F}$,

$$\mathrm{Norm}(\boldsymbol{F}) := \mathrm{Lsp}\{\mathrm{Grad}\,\hat{\gamma}_i(\boldsymbol{F}), i = 1, \dots, n\}, \tag{2.4}$$

and its orthorgonal complement in Lin,

$$\begin{aligned}(\mathrm{Norm}(\boldsymbol{F}))^{\perp} &= \{\boldsymbol{A} \in \mathrm{Lin} : \boldsymbol{A}\cdot\boldsymbol{B} = 0, \forall \boldsymbol{B} \in \mathrm{Norm}(\boldsymbol{F})\} \\ &= \{\boldsymbol{A} \in \mathrm{Lin} : \boldsymbol{A}\cdot\mathrm{Grad}\,\hat{\gamma}_i(\boldsymbol{F}) = 0, i = 1, \dots, n\} \\ &=: \mathrm{Tan}(\boldsymbol{F}),\end{aligned} \tag{2.5}$$

which is the *tangent space* to Con at $\boldsymbol{F}$.

Of course, the constraint equations (2.2) must hold for all time, and time differentiation yields

$$\mathrm{Grad}\,\hat{\gamma}_i(\boldsymbol{F})\cdot\dot{\boldsymbol{F}} = 0, i = 1, \dots, n, \tag{2.6}$$

which, in view of (2.5), is equivalent to

$$\dot{\boldsymbol{F}} \in \mathrm{Tan}(\boldsymbol{F}). \tag{2.7}$$

If the body actually occupies the reference configuration at some reference time, then (2.6) implies (2.2) (see Carlson and Tortorelli [3]); hence, in this case, (2.7) is equivalent to (2.2).

By the projection theorem, Lin admits the direct sum decomposition

$$\mathrm{Lin} = \mathrm{Norm}(\boldsymbol{F}) \oplus \mathrm{Tan}(\boldsymbol{F}); \tag{2.8}$$

i.e., each $\boldsymbol{A} \in \mathrm{Lin}$ can be written uniquely as[3]

$$\boldsymbol{A} = \boldsymbol{A}_{\perp} + \boldsymbol{A}_{\parallel}, \quad \boldsymbol{A}_{\perp} \in \mathrm{Norm}(\boldsymbol{F}), \quad \boldsymbol{A}_{\parallel} \in \mathrm{Tan}(\boldsymbol{F}). \tag{2.9}$$

[2] At this level of generality, it must be required that $n < 9$. However, once the principle of material frame-indifference is imposed (cf. the developments of Section 6), the constraint functions $\hat{\gamma}_i$ are seen to depend on $\boldsymbol{F}$ only through the symmetric tensor $\boldsymbol{F}^{\top}\boldsymbol{F}$. Consequently, we must, in fact, have $n < 6$.

[3] Our usage of the subscripts $\perp$ and $\|$ here is exactly opposite to that used by Anderson, Carlson, and Fried [4].

In view of (2.5), (2.7), and (2.9),

$$\boldsymbol{A}_{\perp}\cdot\dot{\boldsymbol{F}} = 0, \qquad \boldsymbol{A}\cdot\dot{\boldsymbol{F}} = \boldsymbol{A}_{\parallel}\cdot\dot{\boldsymbol{F}}. \tag{2.10}$$

3. Free-energy Inequality

We restrict attention to processes in which the temperature is independent of position and time; in this case, the principles of energy balance and entropy growth, or the first and second laws of thermodynamics, combine to yield a free-energy inequality. On using $\mathcal{P}$ to denote an arbitrary regular part of $\mathcal{B}$ with boundary $\partial\mathcal{P}$ and unit outward normal field $\boldsymbol{n}$, this free-energy inequality requires that

$$\overline{\int_{\mathcal{P}} \rho(\psi + \tfrac{1}{2}|\boldsymbol{v}|^2)\,dv}^{\,\cdot} \leq \int_{\partial\mathcal{P}} \boldsymbol{S}\boldsymbol{n}\cdot\boldsymbol{v}\,da + \int_{\mathcal{P}} \rho\boldsymbol{b}\cdot\boldsymbol{v}\,dv \tag{3.1}$$

for each instant and for all parts. Here, ρ is the referential mass density, $\boldsymbol{v}$ is the velocity field, ψ is the free energy per unit mass in the reference configuration, $\boldsymbol{S}$ is the first Piola–Kirchhoff stress tensor, $\boldsymbol{b}$ is the body force per unit mass in the reference configuration, and the superposed dot indicates time differentiation.

Next, we recall that an easy consequence of the principles of mass balance and momentum balance is the power-identity theorem, which asserts that

$$\int_{\partial\mathcal{P}} \boldsymbol{S}\boldsymbol{n}\cdot\boldsymbol{v}\,da + \int_{\mathcal{P}} \rho\boldsymbol{b}\cdot\boldsymbol{v}\,dv = \int_{\mathcal{P}} \boldsymbol{S}\cdot\dot{\boldsymbol{F}}\,dv + \overline{\int_{\mathcal{P}} \tfrac{1}{2}\rho|\boldsymbol{v}|^2\,dv}^{\,\cdot} \tag{3.2}$$

for each instant and all parts. Equations (3.1) and (3.2) imply that

$$\overline{\int_{\mathcal{P}} \rho\psi\,dv}^{\,\cdot} \leq \int_{\mathcal{P}} \boldsymbol{S}\cdot\dot{\boldsymbol{F}}\,dv \tag{3.3}$$

for each instant and all parts. The local equivalent of (3.3) is

$$\rho\dot{\psi} \leq \boldsymbol{S}\cdot\dot{\boldsymbol{F}}, \tag{3.4}$$

and it is this inequality on which our subsequent considerations of hyperelasticity are based.

4. Active and Reactive Stresses

On employing the decomposition (2.10) in the particular case when $\boldsymbol{A}$ is identified with the first Piola–Kirchhoff stress $\boldsymbol{S}$, it follows from the power-identity theorem (3.2) that only the component $\boldsymbol{S}_\parallel$ expends nonzero power over a constrained motion, and we refer to $\boldsymbol{S}_\parallel$ as the *active component of the stress* and write

$$\boldsymbol{S}_\parallel = \boldsymbol{S}_a. \tag{4.1}$$

On the other hand, the component $\boldsymbol{S}_\perp$ is powerless in a constrained motion, and we refer to $\boldsymbol{S}_\perp$ as the *reactive component of the stress* and write

$$\boldsymbol{S}_\perp = \boldsymbol{S}_r. \tag{4.2}$$

Finally, since $\boldsymbol{S}_r$ belongs to $\mathrm{Norm}(\boldsymbol{F})$, it follows from (2.4) that there exist scalar fields $\lambda_1, \ldots, \lambda_n$ such that

$$\boldsymbol{S}_r = \sum_{i=1}^{n} \lambda_i \mathrm{Grad}\,\hat{\gamma}_i(\boldsymbol{F}). \tag{4.3}$$

Thus, we have shown that, when a body is internally constrained by simple constraints of the form (2.2), the geometry of the constraint manifold dictates that the stress is automatically decomposed into the sum of two components: a powerless component $\boldsymbol{S}_r$ that is determined to within scalar multipliers by (4.3); and a component $\boldsymbol{S}_a$ that does expend power and consequently appears in the free-energy inequality. We emphasize that this result is independent of any constitutive considerations other than the "simple" nature of the constraints; in particular, the body need not be elastic.

A noteworthy feature of our approach is that, in view of (4.1), (4.2), and (2.9),

$$\boldsymbol{S}_a \cdot \boldsymbol{S}_r = 0. \tag{4.4}$$

This automatic normalization is important, because the presence of the constitutively indeterminate multipliers in $\boldsymbol{S}_r$ (see (4.3)) means that the response function for any component of $\boldsymbol{S}_a$ not orthogonal to $\boldsymbol{S}_r$ could not be measured.

5. Constrained Hyperelasticity

In the constrained case, it follows from (2.10) and (4.1) that the local free-energy inequality (3.4) reduces to

$$\rho\dot{\psi} \leq \boldsymbol{S}_a \cdot \dot{\boldsymbol{F}}. \tag{5.1}$$

For *hyperelasticity*, we make the constitutive assumptions that

$$\psi = \hat{\psi}(\boldsymbol{F}), \qquad \hat{\psi} : \mathrm{Con} \to \mathbb{R}, \tag{5.2}$$

and

$$\boldsymbol{S}_a = \hat{\boldsymbol{S}}_a(\boldsymbol{F}), \qquad \hat{\boldsymbol{S}}_a : \mathrm{Con} \to \mathrm{Tan}(\boldsymbol{F}). \tag{5.3}$$

Now, with $\hat{\psi}$ assumed to be smooth,

$$\dot{\psi} = \mathrm{Grad}_{\parallel}\hat{\psi}(\boldsymbol{F}) \cdot \dot{\boldsymbol{F}}; \tag{5.4}$$

so the local free-energy inequality becomes

$$(\hat{\boldsymbol{S}}_a(\boldsymbol{F}) - \rho \mathrm{Grad}_{\parallel}\hat{\psi}(\boldsymbol{F})) \cdot \dot{\boldsymbol{F}} \leq 0. \tag{5.5}$$

In the spirit of Green [6, 7], Ericksen and Rivlin [2], and Coleman and Noll [8], we require that our constitutive equations be restricted such that the local free-energy inequality (5.5) is always satisfied. To make this precise, we say that a *constrained hyperelastic process* consists of:

i. a motion $\boldsymbol{y}$ consistent with the constraint equations (2.2),

ii. scalar fields $\lambda_1 \ldots, \lambda_n$,

iii. a free-energy field ψ given in terms of ψ by constitutive equation (5.2),

iv. an active stress field $\boldsymbol{S}_a$ given in terms of $\boldsymbol{y}$ by constitutive equation (5.3),

v. a reactive stress field $\boldsymbol{S}_r$ given in terms of $\boldsymbol{y}$ and $\lambda_1 \ldots, \lambda_n$ through (4.3), and

vi. a body force field $\boldsymbol{b}$ determined in terms of the above fields through local balance of momentum.

Then, we insist that the local free-energy inequality (5.5) be satisfied for every constrained hyperelastic process. At least locally, it is possible to choose a constrained hyperelastic process such that, at any given position and time, $\boldsymbol{F}$ and $\dot{\boldsymbol{F}}$ take on arbitrary values in Con and $\mathrm{Tan}(\boldsymbol{F})$, respectively. Since both $\boldsymbol{S}_a(\boldsymbol{F})$ and $\mathrm{Grad}_{\|}\hat{\psi}(\boldsymbol{F})$ belong to $\mathrm{Tan}(\boldsymbol{F})$, we conclude that

$$\hat{\boldsymbol{S}}_a(\boldsymbol{F}) = \rho\mathrm{Grad}_{\|}\hat{\psi}(\boldsymbol{F}). \tag{5.6}$$

In (5.4)–(5.6), $\mathrm{Grad}_{\|}\hat{\psi}(\boldsymbol{F})$ represents the tangential gradient of $\hat{\psi}$ at $\boldsymbol{F}$. When the response function $\hat{\psi}$ admits a smooth extension off of the constraint manifold to an open subset of Lin^+, then

$$\mathrm{Grad}_{\|}\hat{\psi}(\boldsymbol{F}) = (\mathsf{I} - \sum_{i=1}^{n} \boldsymbol{N}_i \otimes \boldsymbol{N}_i)\mathrm{Grad}\,\hat{\psi}(\boldsymbol{F}), \tag{5.7}$$

where the fourth-order tensor I is the identity operator on Lin, $\{\boldsymbol{N}_i, i = 1, \ldots, n\}$ is an orthonormal basis for the linear subspace Norm $(\boldsymbol{F})$, and $\boldsymbol{A}\otimes\boldsymbol{B}$ is the fourth-order tensor defined such that $(\boldsymbol{A}\otimes\boldsymbol{B})\boldsymbol{C} = (\boldsymbol{B}\cdot\boldsymbol{C})\boldsymbol{A}$ for any second-order tensor $\boldsymbol{C}$.

6. Material Frame-indifference and Moment-of-momentum Balance

An interesting feature of hyperelasticity in the unconstrained case is that the principle of balance of moment-of-momentum need not be taken as an axiom; rather it appears as a theorem in the theory primarily as a consequence of the principle of material frame-indifference. In this section, we show that this is the case also in the constrained theory as developed above.

As noted in the introduction, internal constraints do delimit aspects of material response. Thus, the kinematical restrictions embodied in (2.2) are subject to the principle of material frame-indifference:

$$\hat{\gamma}_i(\boldsymbol{QF}) = \hat{\gamma}_i(\boldsymbol{F}), \quad i = 1, \ldots, n, \quad \forall(\boldsymbol{Q}, \boldsymbol{F}) \in \mathrm{Orth}^+ \times \mathrm{Lin}^+. \tag{6.1}$$

A standard consequence of (6.1) is that for each i

$$\hat{\gamma}_i(\boldsymbol{F}) = \bar{\gamma}_i(\boldsymbol{C}), \qquad \bar{\gamma}_i : \mathrm{Psym} \to \mathbb{R}, \tag{6.2}$$

where

$$\boldsymbol{C} = \boldsymbol{F}^{\top}\boldsymbol{F} \tag{6.3}$$

is the right Cauchy–Green deformation tensor.

By (6.2) and (6.3),

$$\mathrm{Grad}\,\hat{\gamma}_i(\boldsymbol{F}) = 2\boldsymbol{F}\mathrm{Grad}\,\bar{\gamma}_i(\boldsymbol{C}), \tag{6.4}$$

and (4.3) becomes

$$\boldsymbol{S}_r = \sum_{i=1}^{n} \lambda_i \boldsymbol{F} \operatorname{Grad} \bar{\gamma}_i(\boldsymbol{C}) \tag{6.5}$$

in terms of the reduced constraint functions $\bar{\gamma}_i$, where the factor of 2 has been absorbed into the constitutively indeterminate multipliers. An immediate consequence of (6.5) is that

$$\boldsymbol{S}_r \boldsymbol{F}^\top = \boldsymbol{F} \boldsymbol{S}_r^\top, \tag{6.6}$$

which is the local form of balance of moment-of-momentum for the reactive stress.

Similarly, material frame-indifference requires that the constitutive equation (5.2) for the free-energy density reduce to

$$\psi = \bar{\psi}(\boldsymbol{C}). \tag{6.7}$$

Here, of course, the domain of $\bar{\psi}$ is the reduced constraint manifold

$$\overline{\mathrm{Con}}(\boldsymbol{C}) := \{\boldsymbol{C} \in \mathrm{Psym} : \bar{\gamma}_i(\boldsymbol{C}) = 0, i = 1, \ldots, n\}. \tag{6.8}$$

In terms of $\bar{\psi}$, (5.6) becomes

$$\boldsymbol{S}_a = \bar{\boldsymbol{S}}_a(\boldsymbol{C}) = 2\rho \boldsymbol{F} \operatorname{Grad}_{\|} \bar{\psi}(\boldsymbol{C}), \tag{6.9}$$

where $\operatorname{Grad}_{\|}$ now denotes the tangential gradient with respect to the manifold $\overline{\mathrm{Con}}$. Furthermore, it follows from (6.9) that

$$\boldsymbol{S}_a \boldsymbol{F}^\top = \boldsymbol{F} \boldsymbol{S}_a^\top, \tag{6.10}$$

which is the local form of moment-of-momentum balance for the active stress.

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968	Riahi, D. N.	Effects of rotation on convection in a porous layer during alloy solidification—Chapter 12 in *Transport Phenomena in Porous Media* (D. B. Ingham and I. Pop, eds.), 316–340 (2002)	Apr. 2001
969	Damljanovic, V., and R. L. Weaver	Elastic waves in cylindrical waveguides of arbitrary cross section—*Journal of Sound and Vibration* (submitted)	May 2001
970	Gioia, G., and A. M. Cuitiño	Two-phase densification of cohesive granular aggregates—*Physical Review Letters* **88**, 204302 (2002) (in extended form and with added co-authors S. Zheng and T. Uribe)	May 2001
971	Subramanian, S. J., and P. Sofronis	Calculation of a constitutive potential for isostatic powder compaction—*International Journal of Mechanical Sciences* (submitted)	June 2001
972	Sofronis, P., and I. M. Robertson	Atomistic scale experimental observations and micromechanical/ continuum models for the effect of hydrogen on the mechanical behavior of metals—*Philosophical Magazine* (submitted)	June 2001
973	Pushkin, D. O., and H. Aref	Self-similarity theory of stationary coagulation—*Physics of Fluids* **14**, 694–703 (2002)	July 2001
974	Lian, L., and N. R. Sottos	Stress effects in ferroelectric thin films—*Journal of the Mechanics and Physics of Solids* (submitted)	Aug. 2001
975	Fried, E., and R. E. Todres	Prediction of disclinations in nematic elastomers—*Proceedings of the National Academy of Sciences* **98**, 14773–14777 (2001)	Aug. 2001
976	Fried, E., and V. A. Korchagin	Striping of nematic elastomers—*International Journal of Solids and Structures* **39**, 3451–3467 (2002)	Aug. 2001
977	Riahi, D. N.	On nonlinear convection in mushy layers: Part I. Oscillatory modes of convection—*Journal of Fluid Mechanics* **467**, 331–359 (2002)	Sept. 2001
978	Sofronis, P., I. M. Robertson, Y. Liang, D. F. Teter, and N. Aravas	Recent advances in the study of hydrogen embrittlement at the University of Illinois—Invited paper, Hydrogen–Corrosion Deformation Interactions (Sept. 16–21, 2001, Jackson Lake Lodge, Wyo.)	Sept. 2001
979	Fried, E., M. E. Gurtin, and K. Hutter	A void-based description of compaction and segregation in flowing granular materials—*Proceedings of the Royal Society of London A* (submitted)	Sept. 2001
980	Adrian, R. J., S. Balachandar, and Z.-C. Liu	Spanwise growth of vortex structure in wall turbulence—*Korean Society of Mechanical Engineers International Journal* **15**, 1741–1749 (2001)	Sept. 2001
981	Adrian, R. J.	Information and the study of turbulence and complex flow—*Japanese Society of Mechanical Engineers Journal B*, in press (2002)	Oct. 2001
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985	Kasimov, A. R., and Stewart, D. S.	Spinning instability of gaseous detonations—*Journal of Fluid Mechanics* (submitted)	Oct. 2001
986	Brown, E. N., N. R. Sottos, and S. R. White	Fracture testing of a self-healing polymer composite—*Experimental Mechanics* (submitted)	Nov. 2001
987	Phillips, W. R. C.	Langmuir circulations—*Surface Waves* (J. C. R. Hunt and S. Sajjadi, eds.), in press (2002)	Nov. 2001
988	Gioia, G., and F. A. Bombardelli	Scaling and similarity in rough channel flows—*Physical Review Letters* **88**, 014501 (2002)	Nov. 2001
989	Riahi, D. N.	On stationary and oscillatory modes of flow instabilities in a rotating porous layer during alloy solidification—*Journal of Porous Media*, in press (2002)	Nov. 2001
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No.	Authors	Title	Date
991	Christensen, K. T., and R. J. Adrian	Measurement of instantaneous Eulerian acceleration fields by particle-image accelerometry: Method and accuracy—*Experimental Fluids* (submitted)	Dec. 2001
992	Liu, M., and K. J. Hsia	Interfacial cracks between piezoelectric and elastic materials under in-plane electric loading—*Journal of the Mechanics and Physics of Solids*, in press (2002)	Dec. 2001
993	Panat, R. P., S. Zhang, and K. J. Hsia	Bond coat surface rumpling in thermal barrier coatings—*Acta Materialia*, in press (2002)	Jan. 2002
994	Aref, H.	A transformation of the point vortex equations—*Physics of Fluids* **14**, 2395–2401 (2002)	Jan. 2002
995	Saif, M. T. A, S. Zhang, A. Haque, and K. J. Hsia	Effect of native Al_2O_3 on the elastic response of nanoscale aluminum films—*Acta Materialia* **50**, 2779–2786 (2002)	Jan. 2002
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998	Christensen, K. T., and R. J. Adrian	The velocity and acceleration signatures of small-scale vortices in turbulent channel flow—*Journal of Turbulence*, in press (2002)	Jan. 2002
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1001	Dolbow, J. E., E. Fried, and A. Q. Shen	Point defects in nematic gels: The case for hedgehogs—*Proceedings of the National Academy of Sciences* (submitted)	Feb. 2002
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1003	Carlson, D. E., E. Fried, and S. Sellers	The totality of soft-states in a neo-classical nematic elastomer—*Proceedings of the Royal Society A* (submitted)	Mar. 2002
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1010	Zhang, S., R. Panat, and K. J. Hsia	Influence of surface morphology on the adhesive strength of aluminum/epoxy interfaces—*Journal of Adhesion Science and Technology* (submitted)	Oct. 2002
1011	Carlson, D. E., E. Fried, and D. A. Tortorelli	On internal constraints in continuum mechanics—*Journal of Elasticity* (submitted)	Oct. 2002