# Gravity-induced segregation of cohesionless granular mixtures

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Working with the context of a theory proposed recently by Fried et al. (2001), we consider a one-dimensional problem involving granular mixture of $K > 2$ discrete sizes bounded below by an impermeable base, above by an evolving free surface, and subject to gravity. We demonstrate the existence of a solution in which the medium segregates by particle size. For a mixture of small and large particles ($K = 2$), we use methods of Smoller (1994) to show that the segregated solution is unique. Further, for a mixture of small, medium, and large particles ($K = 3$), we use LeVeque's (1994) CLAWPACK to construct numerical solutions and find that these compare favorably with analytical predictions.

## 1. Introduction

A granular material is a collection of solid particles together with an interstitial fluid such as air or water. Generally, granular materials consist not of identical particles, but, rather, of various particle types that may differ in size, shape, density, resilience, and roughness. Of interest in this article is segregation by particle size. Although it is relatively easy to achieve homogenous mixing with miscible fluids, it is very difficult to do so in a granular mixture involving particles of different sizes. For example, stirring a container of freeze-dried coffee causes large grains to rise to the surface.

Applications in which size-based segregation occur and are of importance are abundant. While mineral processing technologies exploit the tendency for granular materials to segregate, industrial mixing technologies must counteract this tendency. Though vital to the chemical, pharmaceutical, powder metallurgy, glass, ceramic, paint, food, and construction industries, separation and mixing technologies are presently limited by a reliance upon empirically-based heuristics. It therefore seems reasonable to expect that an enhanced understanding of the mechanisms underlying segregation will lead to advances in a broad spectrum of industrial enterprizes.

Theoretical studies of granular flows have been undertaken using molecular dynamics, statistical physics, and continuum mechanics. The approach based on molecular dynamics involves modelling the granular material as an assembly of rigid bodies, taking into account translational and rotational degrees of freedom and allowing for hysteretic interactions. Euler's equations of motion are formulated and solved numerically. Numerical results obtained by molecular dynamic simulations exhibit mixed agreement with observations. For example, in a simulation of plane Couette flow, Thompson & Grest (1991) employed a Hookean-type elastic force model with 750 soft particles of equal radii. A plug-like motion of the core and a boundary shear layer of 6–12 particle diameters in thickness was found, showing good agreement with experimental results of Hanes & Inman (1985). However, the shear stress did not show a quadratic dependence on the mean shear rate as observed in the experiments of Bagnold (1954). Furthermore, even with the most advanced computing resources currently available, the inherent memory demands of this method make it infeasible for systems of more than $10^6$ particles. Applications of molecular dynamics to size-based segregation can be found in Dolgunin & Ukolov 1995; Gallas et al. 1996; Herrmann & Ludig 1998; Hirshfled & Rapaport 1997; Hong et al. 2001; Moaker et al. 2000; Ohstuki et al. 1993, Poschel et al. 2000; Ristow 2000; Shinbrot et al. 2001; Shoichi 1998; Smith et al. 2001.

In the limit as the number of particles becomes infinitely large, this method is replaced by a statistical approach in which moments of a Boltzmann-type equation are used. Interactions between the individual bodies are expressed by the collision operator, taking into account energy losses during collisions. The number of moments taken determines the complexity of the theory, which is now continuous for fields that are statistical averages of quantities that exhibit large fluctuations on the microscale. The important results are the evolution equations for the density, the velocity, and the granular temperature (Jenkins & Savage 1983; Lun et al. 1984; Makse & Kurchan 2002). Basic to these kinetic models are the assumptions that momentum transfer occurs via collisions and that only binary collisions occur. This implies that the granular phase must be sufficiently dispersed, which for dry granular flows subject to gravity, seems plausible for only very rapid motions. However, many flows of practical interest fall into the intermediate regime where both frictional contacts and particle-particle collisions are significant (Ancey et al. 1996; Johnson et al. 1990). Applications of this statistical approach in the study of segregation problem can be found in Arnarson & Jenkins (2000), Arnarson & Willits (1998), Jenkins (1998), Jenkins & Mancini (1989), among others.

Continuum mechanical models are purely phenomenological descriptions and are restricted to macroscopic length scales that extend over many particle diameters. Closure conditions are based on standard invariance requirements and thermodynamical restrictions and may account for microstructural effects. Insofar as monodisperse granular materials are concerned, continuum-level theories have been exploited to considerable advantage (Hutter & Rajagopal 1994; Savage 1984; Wang & Hutter 2001).

Here, we use a recent continuum model developed by Fried et al. (2001) to study gravity-driven segregation and compaction of polydisperse granular material. Based on a kinematical treatment of voids, the basic physical laws allows for segregation via diffusion of different particle types and voids. This theory ignores inertial effects and is thus restricted to situations involving relatively slow flow. For a cohesionless granular material consisting of particles that differ only by size, this theory yields evolution equations for the volume-weighted mixture-velocity $\boldsymbol{v}$, the pressure $p$, and $K \geq 1$ particulate volume fractions $\varphi_k$ constrained to obey

$$0 \leq \varphi_k \leq 1 \qquad \text{and} \qquad \varphi^{\mathrm{p}} = \sum_{k=1}^{K} \varphi_k \leq \overset{*}{\varphi}, \tag{1.1}$$

with $\varphi^{\mathrm{p}}$ the total particulate volume fraction and $\overset{*}{\varphi}$ the volume fraction at random close-packing. Writing $\boldsymbol{f}$ for the external body force per unit mass, these equations consist of the constraint

$$\operatorname{div} \boldsymbol{v} = 0, \tag{1.2}$$

which enforces the requirement that all volume be accounted for (locally) by particles and voids, the force balance

$$\operatorname{div} \boldsymbol{S} + \sum_{k=1}^{K} m_k \varphi_k \boldsymbol{f} = \operatorname{grad} p, \tag{1.3}$$

with $\boldsymbol{S}$ the extra stress (given constitutively as a function of the volume fractions $(\varphi_1, \varphi_2, \ldots, \varphi_K)$ and the strain-rate $\boldsymbol{D} = \frac{1}{2}(\operatorname{grad}\boldsymbol{v} + (\operatorname{grad}\boldsymbol{v})^\top)$, which, by (1.2), must obey $\operatorname{tr}\boldsymbol{D} = 0$) and $m_k$ the density of a particle of type $k$, and the particulate volume balances

$$\frac{\mathrm{D}\varphi_k}{\mathrm{D}t} = -\operatorname{div}\big(\varphi_k \alpha_k(\imath_{\boldsymbol{D}}) h(\varphi^{\mathrm{p}})\boldsymbol{f}\big), \qquad k = 1, 2, \ldots, K, \tag{1.4}$$

in which $\alpha_k > 0$ is the effective mobility of particles of type $k$, depending on the $\imath_{\boldsymbol{D}} = (|\boldsymbol{D}|, \det \boldsymbol{D})$, and $h$ is the compaction function. The mobilities are stipulated to be positive (even when $\boldsymbol{D} = \boldsymbol{0}$). Thus, if we assume a direct correspondence between particle size and particle mobility, with smaller particles being less mobile than large particles, no generality is lost by taking

$$\alpha_1(\imath_{\boldsymbol{D}}) > \alpha_2(\imath_{\boldsymbol{D}}) > \cdots > \alpha_K(\imath_{\boldsymbol{D}}) > 0 \tag{1.5}$$

for all $\boldsymbol{D}$. Further, it is assumed that

$$h(\varphi^{\mathrm{p}}) > 0 \ \text{ for } \ 0 < \varphi^{\mathrm{p}} < \overset{*}{\varphi} \qquad \text{and} \qquad h(\varphi^{\mathrm{p}}) = 0 \ \text{ for } \ \varphi^{\mathrm{p}} \geq \overset{*}{\varphi}. \tag{1.6}$$

Due to the hyperbolicity of (1.4), the evolution equations (1.2)–(1.4) must be supplemented by relations which hold across shock surfaces across which the particulate volume fraction, the strain-rate, and the extra stress suffer jump discontinuities. Considering such a shock and writing $\boldsymbol{n}_{\mathrm{shock}}$ for its unit normal field and $V_{\mathrm{shock}}$ for the associated (scalar normal) velocity relative to the mixture, the relations associated with the particulate volume balances are

$$[\![\varphi_k]\!] V_{\mathrm{shock}} = [\![\varphi_k \alpha_k(\imath_{\boldsymbol{D}}) h(\varphi^{\mathrm{p}})]\!] \boldsymbol{f} \cdot \boldsymbol{n}_{\mathrm{shock}}, \qquad k = 1, 2, \ldots, K, \tag{1.7}$$

where, given a field $g$, $[\![g]\!] = g^+ - g^-$, with $g^+$ the limit of $g$, on the shock surface, taken from the region into which $\boldsymbol{n}_{\mathrm{shock}}$ points and $g^-$ the corresponding limit from the other side of the shock surface.

In addition to the evolution equations and shock relations, the theory delivers boundary conditions at free surfaces and impermeable solid boundaries. Focusing only on the conditions associated with the particulate volume balances, the conditions that hold at a free surface with unit normal $\boldsymbol{n}_{\mathrm{free}}$ directing into the region of pure voids and associated (scalar normal) velocity $V_{\mathrm{free}}$ relative to the mixture are

$$\alpha_k(\imath_{\boldsymbol{D}}) h(\varphi^{\mathrm{p}}) = \frac{V_{\mathrm{free}}}{\boldsymbol{f} \cdot \boldsymbol{n}_{\mathrm{free}}}, \tag{1.8}$$

for any particle type $k$ present at the surface, while the conditions

$$\varphi_k = 0 \qquad \text{or} \qquad \alpha_k(\imath_{\boldsymbol{D}}) h(\varphi^{\mathrm{p}}) = 0, \qquad k = 1, 2, \ldots, K, \tag{1.9}$$

must hold at an impermeable solid boundary.

In what follows, we consider a one-dimensional specialization of the equations listed above. Under this specialization, which we discuss in Section 2, the velocity field is constant and the governing equations reduce to a hyperbolic system. Our analysis is based on a particular packing function, which we introduce in Section 3. A nondimensionalization, which renders the problem in terms of packing fractions as opposed to volume fractions, is performed in Section 4. In Section 5, we pose a problem for a granular mixture bounded above by the free surface and below by an impermeable base. For a system with $K$ particle sizes, we obtain a solution to this problem which, at steady state, involves $K$ different layers—with the presence of particles of a given size in a given layer determined by their size. For the case $K = 3$, we use LeVeque's (1994) CLAWPACK to construct numerical solutions. This robust code requires a user-supplied Riemann solver. Our Riemann solver is based on the flux splitting scheme of Roe (1981), incorporating proper entropy corrections and flux limiters. In Section 6, we use an approach due to Smoller (1994) to establish the uniqueness of the solution presented in Section 5 for the case $K = 2$.

## 2. A class of one-dimensional problems

We suppose that the body force is purely gravitational, viz.,

$$\boldsymbol{f} = \boldsymbol{g}, \tag{2.1}$$

with $\boldsymbol{g}$ the gravitational acceleration. We let $x$ denote the Cartesian coordinate in the direction $\boldsymbol{e} = -\boldsymbol{g}/|\boldsymbol{g}|$ and seek solutions of the evolution equations for which $\boldsymbol{v}$ and $p$ are functions of $x$ and $t$, with

$$\boldsymbol{v} = v\boldsymbol{e}. \tag{2.2}$$

Then, since $\operatorname{div}\boldsymbol{v} = 0$, $v$ must be constant and we may, without loss in generality, assume that $v \equiv 0$. Thus, $\boldsymbol{D} = \boldsymbol{0}$ and the extra stress $\boldsymbol{S}$ either vanishes or is indeterminate (Fried et al. (2001)). For the indeterminate case, we take $\boldsymbol{S} = S\boldsymbol{e} \otimes \boldsymbol{e}$. Force balance requires that

$$\frac{\partial(p - S)}{\partial x} = -\sum_{k=1}^{K} m_k \varphi_k |\boldsymbol{g}|, \tag{2.3}$$

which, given the volume fractions $\varphi_k$, determines the difference $p - S$ between the pressure and the extra stress. Equation (2.3) holds with $S = 0$ when the constitutive relation for $\boldsymbol{S}$ is well-defined at $\boldsymbol{D} = \boldsymbol{0}$.

In view of the foregoing, the particulate volume balances (1.4) take the form

$$\frac{\partial \varphi_k}{\partial t} = \alpha_k g \frac{\partial\big(\varphi_k h(\varphi^{\mathrm{p}})\big)}{\partial x}, \qquad k = 1, 2, \ldots, K, \tag{2.4}$$

where, for simplicity, we use $\alpha_k$ to denote the values of the effective mobilities for $\boldsymbol{D} = \boldsymbol{0}$.

Regarding the shock relations, we choose $\boldsymbol{n}_{\text{shock}} = \boldsymbol{e}$, so that $\boldsymbol{f} \cdot \boldsymbol{n}_{\text{shock}} = -|\boldsymbol{g}|$. Thus, (1.7) becomes

$$[\![\varphi_k]\!] V_{\text{shock}} = -\alpha_k |\boldsymbol{g}| [\![\varphi_k h(\varphi^{\mathrm{p}})]\!], \qquad k = 1, 2, \ldots, K. \tag{2.5}$$

Similarly, the condition (1.8) that hold for any particle of type $k$ at a free surface becomes

$$\alpha_k h(\varphi^{\mathrm{p}}) = -\frac{V_{\text{free}}}{|\boldsymbol{g}|}. \tag{2.6}$$

## 3. Constitutive specialization of the packing function

For simplicity, we choose the packing function $h$ to be of the particular form

$$h(\varphi^{\mathrm{p}}) = \varphi^* - \sum_{k=1}^{K} \varphi_k, \tag{3.1}$$

which obeys the requirements expressed in (1.6).

## 4. Nondimensionalization

On introducing a characteristic length $L$ and a characteristic time $T$, we define dimensionless quantities

$$\tilde{x} = \frac{x}{L}, \quad \tilde{t} = \frac{t}{T}, \quad \tilde{\varphi}_k = \frac{\varphi_k}{\varphi^*}, \quad \tilde{\varphi}^{\mathrm{p}} = \frac{\varphi^{\mathrm{p}}}{\varphi^*}, \quad \tilde{V} = \frac{VT}{L}, \tag{4.1}$$

and

$$\tilde{\alpha}_k = \frac{\alpha_k |\boldsymbol{g}| T \varphi^*}{L}, \tag{4.2}$$

where $V$ is used to denote the velocity of a generic shock or a generic free surface. We refer to $\tilde{\varphi}_k$ as the *packing fraction*.

Bearing in mind the constitutive specialization (3.1) and dropping tildes, the hyperbolic system (2.4) becomes

$$\frac{\partial \varphi_k}{\partial t} = \alpha_k \frac{\partial}{\partial x}\Big(\varphi_k(1-\varphi^{\mathrm{p}})\Big), \qquad k = 1, 2, \ldots, K, \tag{4.3}$$

and the shock relations (2.5) become

$$[\![\varphi_k]\!] V_{\mathrm{shock}} = -\alpha_k [\![\varphi_k(1-\varphi^{\mathrm{p}})]\!], \qquad k = 1, 2, \ldots, K. \tag{4.4}$$

Further, the boundary condition (2.6) for a particle of type $k$ at a free surface becomes

$$\alpha_k(1-\varphi^{\mathrm{p}}) = -V_{\mathrm{free}}, \tag{4.5}$$

while the boundary condition (4.6) at an impermeable solid surface becomes

$$\varphi_k = 0 \qquad \text{or} \qquad \alpha_k(1-\varphi^{\mathrm{p}}) = 0, \qquad k = 1, 2, \ldots, K. \tag{4.6}$$

By virtue of (1.1) and $(4.1)_{3,4}$, the packing fractions must obey

$$0 \le \varphi_k \le 1 \qquad \text{and} \qquad \varphi^{\mathrm{p}} = \sum_{k=1}^{K} \le 1. \tag{4.7}$$

## 5. Solution to a particular problem

We now consider an open container that occupies the interval $0 \le x \le 1$, with $x = 0$ an impermeable base. We assume that, initially, each particulate packing fraction $\varphi_k$ has a prescribed constant value, viz.,

$$\varphi_k(x, 0) = \mathring{\varphi}_k, \qquad 0 \le x \le 1. \tag{5.1}$$

Further, we require that

$$0 < \mathring{\varphi}^{\mathrm{p}} = \sum_{k=1}^{K} \mathring{\varphi}_k < 1, \tag{5.2}$$

so that, initially, the mixture is loosely packed and has a free surface at the top $x = 1$ of the container.

Granted the foregoing initial conditions, the boundary conditions (4.6) at the base $x = 0$, and the free surface condition (4.5) for any particle type $k$ at the free surface, we seek solutions of the hyperbolic system (4.3) that consist of uniform states separated by shocks across which the jump conditions (4.4) hold.

As a candidate for such a solution, we consider a generalization of the solution obtained

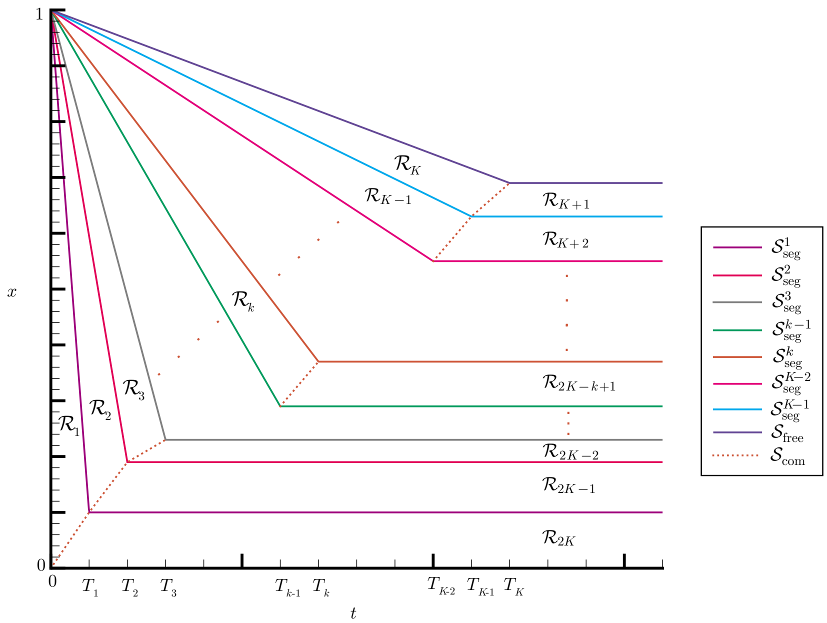


FIGURE 1. Compaction and segregation, by gravity, of a granular aggregate involving particles of $K$ sizes in a fixed container. At any time $t$, the solution consists of uniform states separated by shocks

for a mixture of small and large particles ($K = 2$) by Fried et al. (2001). This candidate can be depicted (Fig. 1) as the union of $2K$ regions, $\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_{2K}$, containing particles of various types and in various proportions. For $k = 1, 2, \dots, K$, regions $\mathcal{R}_k$ and $\mathcal{R}_{2K-k+1}$ contain particles of types $k, k+1, \dots, K$. In particular, region $\mathcal{R}_1$ corresponds to the initial state, so that the mixture is given by (5.1). While the mixtures in regions $\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_K$ are loosely packed, those in regions $\mathcal{R}_K, \mathcal{R}_{K+1}, \dots, \mathcal{R}_{2K}$ are closely packed. Region $\mathcal{R}_1$ is bounded by a segregation shock $\mathcal{S}^1_{\rm seg}$ emanating from the free surface and a compaction shock $\mathcal{S}_{\rm com}$ emanating from the base. These shocks meet at time $T_1$, at which point they are both deflected. Subsequent to $T_1$, $\mathcal{S}^1_{\rm seg}$ becomes horizontal while $\mathcal{S}_{\rm com}$ continues to travel upwards. In the region $\mathcal{R}_{2K}$ lying between the base, the portion of $\mathcal{S}_{\rm com}$ adjacent to $\mathcal{R}_1$, and the horizontal portion of $\mathcal{S}^1_{\rm seg}$, the packing fraction $\varphi_i^{2K}$ of particles of type $i$ given by

$$\varphi_i^{2K} = \mathring{\varphi}_i + \frac{\alpha_i(1 - \sum_{l=1}^K \mathring{\varphi}_l)\mathring{\varphi}_i}{\sum_{l=1}^K \alpha_l \mathring{\varphi}_l} \tag{5.3}$$

for $i = 1, 2, \dots, K$. In addition, until time $T_1$, the velocities of $\mathcal{S}^1_{\rm seg}$ and $\mathcal{S}_{\rm com}$ are given by

$$V_{\mathcal{S}^1_{\rm seg}} = \alpha_1(1 - \sum_{l=1}^K \mathring{\varphi}_l) \qquad \text{and} \qquad V_{\mathcal{S}_{\rm com}} = -\sum_{l=1}^K \alpha_l \mathring{\varphi}_l. \tag{5.4}$$

At time $T_2$, $\mathcal{S}_{\rm com}$ meets another segregation shock $\mathcal{S}^2_{\rm seg}$ emanating from the free surface. At this time, $\mathcal{S}_{\rm com}$ and $\mathcal{S}^2_{\rm seg}$ are deflected. Subsequent to $T_2$, $\mathcal{S}_{\rm com}$ continues to travel upward and $\mathcal{S}^2_{\rm seg}$ becomes horizontal. This process continues until time $T_{K-1}$, when $\mathcal{S}_{\rm com}$ is deflected upward and $\mathcal{S}^{K-1}_{\rm seg}$ becomes horizontal. For $k = 2, \dots, K$, the packing fraction $\varphi_i^k$ of particles of type $i$ in region $\mathcal{R}_k$ is given by the positive square root of the quadratic equation

$$\alpha_i(\varphi_i^k)^2 - \Big(\alpha_i(1-\sum_{j=k,\neq i}^{K}\varphi_j^k) - \alpha_{k-1}(1-\sum_{j=k-1}^{K}\varphi_j^{k-1})\Big)\varphi_i^k$$
$$- \varphi_i^{k-1}\Big(1-\sum_{j=k-1}^{K}\varphi_j^{k-1}\Big)(\alpha_{k-1}-\alpha_i) = 0, \quad (5.5)$$

for $i = 1, 2, \ldots, K$. In addition, the velocity of $\mathcal{S}_{\rm seg}^k$ until $T_k$ and the velocity of $\mathcal{S}_{\rm com}$ between $T_{k-1}$ and $T_k$ are given by

$$V_{\mathcal{S}_{\rm seg}^k} = \alpha_k(1-\sum_{l=k}^{K}\varphi_l^k) \qquad \text{and} \qquad V_{\mathcal{S}_{\rm com}} = -\sum_{l=k}^{K}\alpha_l\varphi_l^k. \quad (5.6)$$

At time $T_K$, $\mathcal{S}_{\rm com}$ and $\mathcal{S}_{\rm free}$ meet and $\mathcal{S}_{\rm free}$ becomes horizontal. As mentioned above, the region $\mathcal{R}_{K+1}$ contains only the largest particles—those of type $K$. For $k = 2, \ldots, K$, the particulate packing fraction $\varphi_i^{2K-k+1}$ in region $\mathcal{R}_{2K-k+1}$ is given by

$$\varphi_i^{2K-k+1} = \varphi_i^k + \frac{\alpha_i(1-\sum_{l=k}^{K}\varphi_l^k)\varphi_i^k}{\sum_{l=1}^{K}\alpha_l\varphi_l^k} \quad (5.7)$$

where $i = k, \ldots, K$.

To verify that the candidate described above is a solution, we have only to show that the following conditions are satisfied:

($i$) the solid-boundary condition (cf. $(4.6)_2$)

$$\alpha_l\Big(1-\sum_{i=1}^{K}\varphi_i^{2K}\Big) = 0, \qquad l = 1, 2, \ldots, K, \quad (5.8)$$

for $t > 0$;

($ii$) for each $k = 2, 3, \ldots, K$, the jump conditions (cf. (4.4))

$$(\varphi_l^k - \varphi_l^{2K-k+1})V_{\mathcal{S}_{\rm com}} = \alpha_l\varphi_l^k\Big(1-\sum_{i=k}^{K}\varphi_i^k\Big) \qquad \text{on} \qquad \mathcal{S}_{\rm com}, \quad (5.9)$$

where $l = k, k+1, \ldots, K$ and $T_{k-1} < t < T_k$, and, for $0 < t < T_k$

$$[\![\varphi_l^k]\!]V_{\mathcal{S}_{\rm seg}^{k-1}} = \alpha_l[\![\varphi_l^k(1-\sum_{i=k}^{K}\varphi_i^k)]\!] \qquad \text{on} \qquad \mathcal{S}_{\rm seg}^{k-1}, \quad (5.10)$$

where $l = k, k+1, \ldots, K$;

($iii$) the interface condition (cf. (4.6))

$$V_{\rm seg}^k = 0 \qquad \text{on} \qquad \mathcal{S}_{\rm seg}^k, \quad (5.11)$$

for $t \geq T_k$;

($v$) the free-surface conditions (cf. (4.5))

$$V_{\rm free} = \alpha_K(1-\varphi_K^K) \qquad \text{on} \qquad \mathcal{S}_{\rm free}, \quad (5.12)$$

for $0 < t < T_K$, and

$$V_{\rm free} = 0 \qquad \text{on} \qquad \mathcal{S}_{\rm free}, \quad (5.13)$$

for $t \geq T_K$.

To establish (5.8), we substitute for $\varphi_i^{2K}$ from (5.3) and sum over the index $i$. Similarly, (5.9) follows directly if we substitute for $\varphi_l^{2K-k+1}$ from (5.7) and for $V_{\mathcal{S}_{\rm com}}$ from (5.6). Also, on substituting for $V_{\mathcal{S}_{\rm seg}^{k-1}}$ from (5.6), it can be shown that the packing fraction $\varphi_l^k$ satisfying (5.10) is given by (5.5). Hence, (5.10) follows if we can show that the packing fractions in question are determined uniquely by the quadratic equation (5.5). Since $\alpha_{k-1} > \alpha_k$ for $k = 2, \ldots, K$ and the packing in the region is not close, it follows that (5.5) has only one real positive root, $\varphi_l^k$, and that $0 < \varphi_l^k < 1$. Hence, the only positive root of (5.5) is indeed the packing fraction. For $t \geq T_k$, the shock $\mathcal{S}_{\rm seg}^k$ separates two compacted zones. From (4.6), it follows directly that $\alpha_i(1 - \sum_{j=k}^{K} \varphi_j^{2K-k+1}) = 0$. Hence, $V_{\rm seg}^k = 0$ (5.11). Since the shock $\mathcal{S}_{\rm free}$ separates the region $\mathcal{R}_K$ (which contains only particles of type $K$, with a packing fraction $\varphi_K^K$) from the free surface (where $\varphi_k = 0$, $k = 1, \ldots, K$), it follows directly as a consequence of (4.5) that $V_{\rm free} = \alpha_K(1 - \varphi_K^K)$ (for $t < T_K$). However, since, for $t \geq T_K$, the shock $\mathcal{S}_{\rm free}$ now separates the compacted zone $\mathcal{R}_{K+1}$ (where $\varphi_K^{K+1} = 1$) from the free surface, it follows that $1 - \varphi_K^{K+1} = 0$ and, hence, by (4.6), we have $V_{\rm free} = 0$.

Consistent with intuitive expectations, the solution exhibits both compaction and segregation. Segregation is by particle size, with the region closest to the free surface consisting only of the largest particles. As an illustration, we consider the particular case of a mixture containing particles of three sizes, with effective mobilities

$$\alpha_1 = 0.3, \qquad \alpha_2 = 0.2, \qquad \text{and} \qquad \alpha_3 = 0.1, \tag{5.14}$$

and use CLAWPACK to generate a solution to the problem described above, with

$$\mathring{\varphi}_k = 0.2, \qquad k = 1, 2, 3. \tag{5.15}$$

Figs. 2–4 show characteristic plots for each of the particle types and Figs. 5–6 show profiles of the packing fractions as functions of position at various times. These plots are a result of the numerical solutions obtained using CLAWPACK. In keeping with the general results presented above, these plots exhibit compaction shocks and segregation shocks that separate states in which the particulate packing fractions are uniform. At each time, the domain consists of four distinct regions. Prior to time $t = 4.16$ ($\equiv T_1$), these regions are demarcated by the free surface, which moves with velocity $V_{\rm free} = 0.03$, segregation shocks $\mathcal{S}_{\rm seg}^1$ and $\mathcal{S}_{\rm seg}^2$ with velocities $V_{\mathcal{S}_{\rm seg}^1} = 0.16$ and $V_{\mathcal{S}_{\rm seg}^2} = 0.09$, and a compaction shock $\mathcal{S}_{\rm com}$ with velocity $V_{\mathcal{S}_{\rm com}} = -0.12$. The region closest to the free surface ($\mathcal{R}_3$) consists only of the largest particles with $\varphi_3 = 0.672680$. In the region between $\mathcal{S}_{\rm seg}^2$ and $\mathcal{S}_{\rm seg}^1$ ($\mathcal{R}_2$) , $\varphi_1 = 0$, $\varphi_2 = 0.309717$, and $\varphi_3 = 0.219433$. In the region between $\mathcal{S}_{\rm seg}^1$ and $\mathcal{S}_{\rm com}^2$ ($\mathcal{R}_1$) , the mixture is in its initial state. Initially at the base , the medium is closely packed with $\varphi_1 = 0.40$, $\varphi_2 = 0.35$, and $\varphi_3 = 0.25$. Between time $t = 4.16$ ($\equiv T_1$) and time $t = 4.77$ ($\equiv T_2$) , the velocity of $\mathcal{S}_{\rm com}$ is $V_{\mathcal{S}_{\rm com}} = -0.08$. At $t = 4.16$ ($\equiv T_1$) , a horizontal layer ($\mathcal{R}_6$) develops adjacent to the basal region and the packing fractions within this layer are given by $\varphi_1 = 0.399996$, $\varphi_2 = 0.333329$, and $\varphi_3 = 0.266674$. Subsequent to $t = 4.77$ ($\equiv T_2$) , the velocity of $\mathcal{S}_{\rm com}$ becomes $V_{\mathcal{S}_{\rm com}} = -0.06$. Time $t = 4.77$ ($\equiv T_2$) signals the development of a layer ($\mathcal{R}_5$) of closely packed large particles, where $\varphi_1 = 0$, $\varphi_2 = 0.657399$, and $\varphi_3 = 0.342599$. At time $t = 10.91$ ($\equiv T_3$), $\mathcal{S}_{\rm com}$ intersects the free surface and a steady state is achieved.

Table 1 compares the analytical and the numerical values of the distribution of all the three types of particles at the steady state in the zones where close packing is achieved. The maximum relative error is less than $2.6 \times 10^{-5}$.

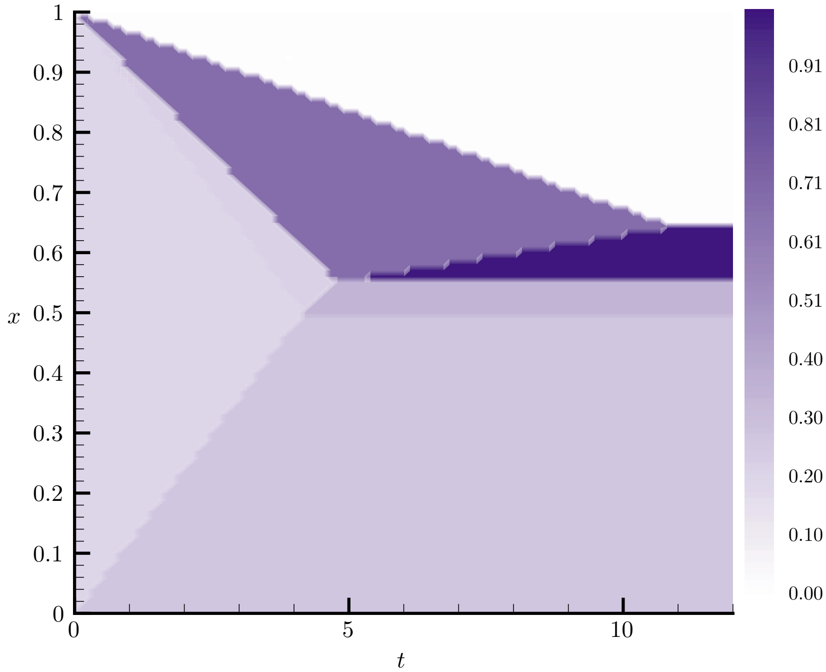


FIGURE 2. Characteristic plot of species type $k = 3$. Here, $(\alpha_1, \alpha_2, \alpha_3) = (0.3, 0.2, 0.1)$ and $(\varphi_1, \varphi_2, \varphi_3) = (0.2, 0.2, 0.2)$. To make this plot, the spatial and time steps were lowered from 0.001 to 0.01, resulting in a jagged shock profile

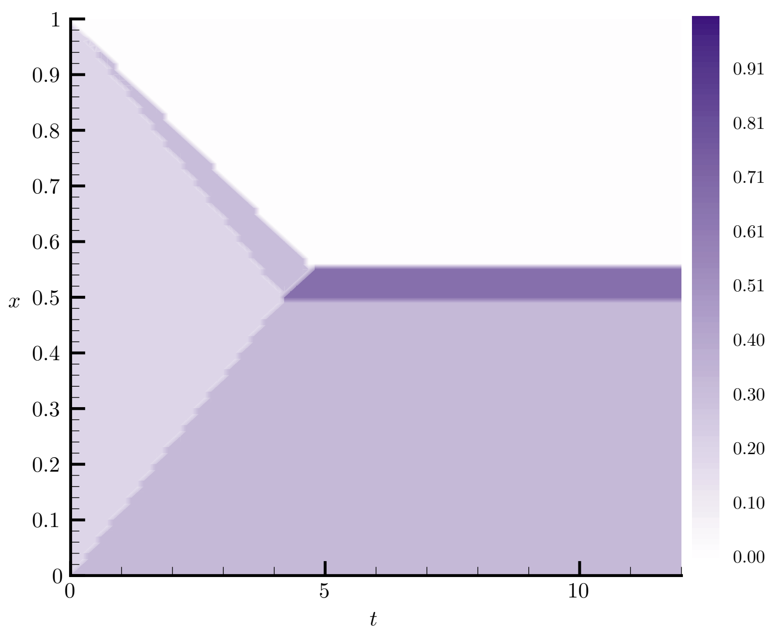


FIGURE 3. Characteristic plot of species type $k = 2$. Here, $(\alpha_1, \alpha_2, \alpha_3) = (0.3, 0.2, 0.1)$ and $(\varphi_1, \varphi_2, \varphi_3) = (0.2, 0.2, 0.2)$

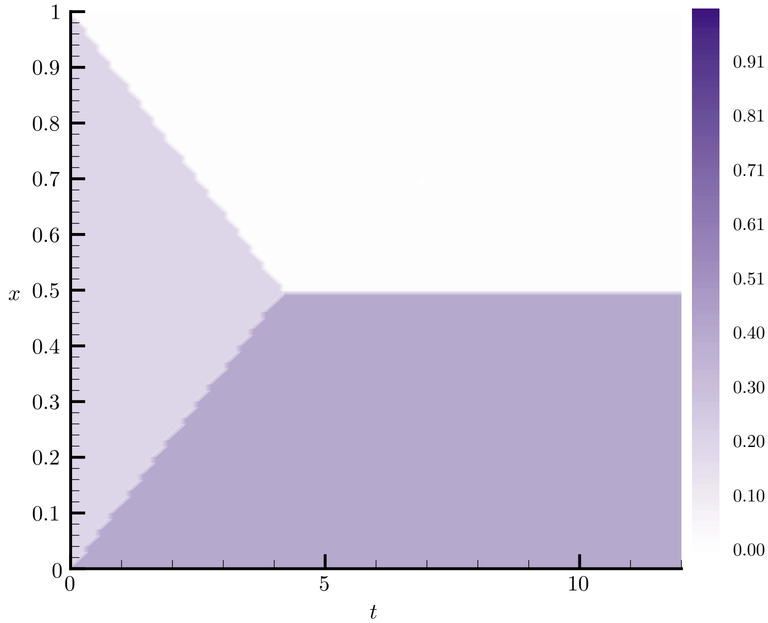


FIGURE 4. Characteristic Plot of species type $k = 1$. Here, $(\alpha_1, \alpha_2, \alpha_3) = (0.3, 0.2, 0.1)$ and $(\varphi_1, \varphi_2, \varphi_3) = (0.2, 0.2, 0.2)$

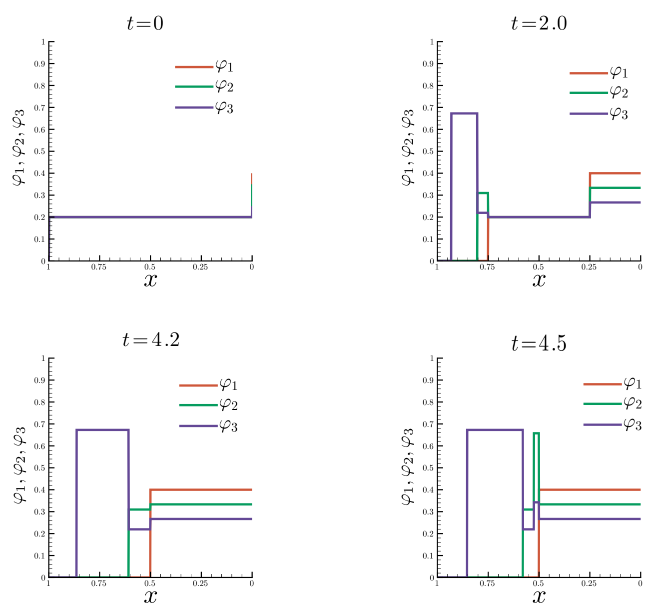


FIGURE 5. The distribution of particles in space at a given time. Here, $(\alpha_1, \alpha_2, \alpha_3) = (0.3, 0.2, 0.1)$ and $(\varphi_1, \varphi_2, \varphi_3) = (0.2, 0.2, 0.2)$

| Compaction zone | Packing fraction | Analytical value | Numerical value | Relative error |
|---|---|---|---|---|
| $\mathcal{R}_6$ | $\varphi_1$ | 0.400000 | 0.399996 | 0.000010 |
| | $\varphi_2$ | 0.333333 | 0.333329 | 0.000012 |
| | $\varphi_3$ | 0.266667 | 0.266674 | 0.000026 |
| $\mathcal{R}_5$ | $\varphi_2$ | 0.657400 | 0.657399 | 0.000002 |
| | $\varphi_3$ | 0.342600 | 0.342599 | 0.000002 |
| $\mathcal{R}_4$ | $\varphi_3$ | 1.000000 | 0.999998 | 0.000002 |

TABLE 1. Comparison of the analytical (Anal. ) and numerical (Num. ) values of the distribution of particles of all three types in various Compaction Zones

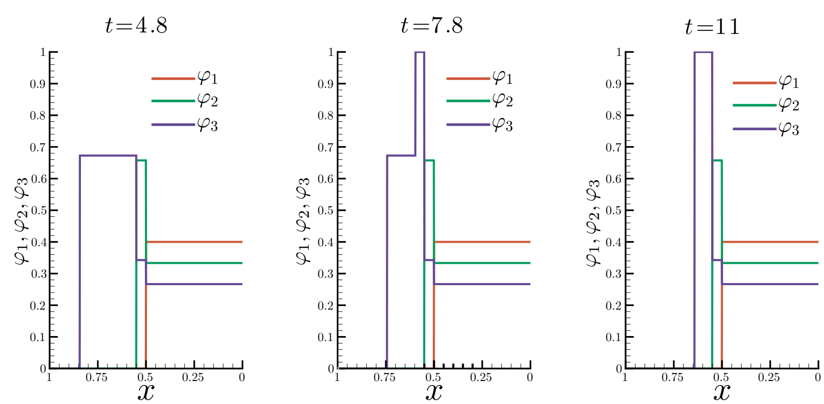


FIGURE 6. The distribution of particles in space at a given time. Here, $(\alpha_1, \alpha_2, \alpha_3) = (0.3, 0.2, 0.1)$ and $(\varphi_1, \varphi_2, \varphi_3) = (0.2, 0.2, 0.2)$

## 6. Uniqueness of the segregated solution for a mixture of small and large particles

In general, it would desirable to understand whether solution presented in the Section 5 is unique. Here, as a step toward this goal, we use methods developed by Smoller (1994) to resolve this issue for the case of a mixture of small and large particles.

### 6.1. *Riemann problem*

Toward addressing the uniqueness of the segregated solution for a mixture of small and large particles, we consider the system

$$\left.\begin{aligned} \frac{\partial \varphi_1}{\partial t} &= \alpha_1 \frac{\partial}{\partial x}\Big(\varphi_1(1-\varphi_1-\varphi_2)\Big), \\ \frac{\partial \varphi_2}{\partial t} &= \alpha_2 \frac{\partial}{\partial x}\Big(\varphi_2(1-\varphi_1-\varphi_2)\Big), \end{aligned}\right\} \tag{6.1}$$

which arises upon specializing (4.3) to the case $k=2$, for $(x,t)$ belonging to $(-\infty,\infty)\times(0,\infty)$ and subject to the initial conditions

$$\varphi_k(x,0)=\begin{cases}\varphi_k^< & \text{if } x<0,\\ \\ \varphi_k^> & \text{if } x>0,\end{cases} \tag{6.2}$$

with $\varphi_k^{\lessgtr}$ constant. For the moment, we interpret (6.1) in the weak sense, so that the discontinuity conditions that must hold across shocks, free surfaces, and solid boundaries are implicitly satisfied. In view of (1.5),

$$\alpha_1>\alpha_2>0. \tag{6.3}$$

Further, in view of (4.7), $(\varphi_1,\varphi_2)$ must belong to

$$\mathcal{A}=\{\varphi_1,\varphi_2):0\le\varphi_1\le 1, 0\le\varphi_2\le 1,\varphi_1+\varphi_2\le 1\}, \tag{6.4}$$

which we refer to as the set of admissible packing fractions.

Generally, we require only that $(\varphi_1^{\lessgtr},\varphi_2^{\lessgtr})$ belong to $\mathcal{A}$. This includes special problems corresponding to less than closely packed mixtures of semi-infinite extent that are bounded above by free surfaces ($\varphi_1^<=\varphi_2^<=0$ and $\varphi_1^>+\varphi_2^><1$) or below by solid bases ($\varphi_1^<+\varphi_2^<<1$ and $\varphi_1^>+\varphi_2^>=1$). However, our primary interest is in problems where $\varphi_1^{\lessgtr}$ and $\varphi_2^{\lessgtr}$ obey $0<\varphi_1^{\lessgtr}+\varphi_2^{\lessgtr}<1$.

The Jacobian of the system (6.1) is simply

$$\begin{bmatrix}\alpha_1(1-2\varphi_1-\varphi_2) & -\alpha_1\varphi_1\\ -\alpha_2\varphi_2 & \alpha_2(1-\varphi_1-2\varphi_2)\end{bmatrix}, \tag{6.5}$$

which has eigenvalues

$$\begin{aligned}\lambda_\pm(\varphi_1,\varphi_2)=\frac{1}{2}\Big(&\alpha_1(1-2\varphi_1-\varphi_2)+\alpha_2(1-2\varphi_2-\varphi_1)\Big)\\ &\pm\frac{1}{2}\sqrt{\big(\alpha_1(1-2\varphi_1-\varphi_2)-\alpha_2(1-2\varphi_2-\varphi_1)\big)^2+4\alpha_1\alpha_2\varphi_1\varphi_2}\,.\end{aligned} \tag{6.6}$$

By inspection,

$$0\le\lambda_-(\varphi_1,\varphi_2)<\lambda_+(\varphi_1,\varphi_2). \tag{6.7}$$

Hence, (6.1) is hyperbolic and we may conclude that solutions to the Riemann problem (6.1)–(6.2) must be uniform states separated by either ($i$) discontinuities (which may be compaction or segregation shocks, free surfaces, or solid boundaries) or ($ii$) rarefactions. Since one of the eigenvalues defined in (6.6) may vanish, (6.1) admits solutions involving static discontinuities.

#### 6.1.1. *Prelimary results concerning discontinuities*

Here, we consider the problem of determining all uniform states $(\varphi_1,\varphi_2)$ in $\mathcal{A}$ that may be connected to $(\varphi_1^<,\varphi_2^<)$ by a discontinuity emanating from the origin of the $(x,t)$-plane and moving with velocity $V$. Such a state must satisfy not only the jump conditions

$$\left.\begin{aligned}\alpha_1\varphi_1(1-\varphi_1-\varphi_2)-\alpha_1\varphi_1^<(1-\varphi_1^<-\varphi_2^<)=V(\varphi_1-\varphi_1^<),\\ \alpha_2\varphi_2(1-\varphi_1-\varphi_2)-\alpha_2\varphi_2^<(1-\varphi_1^<-\varphi_2^<)=V(\varphi_2-\varphi_2^<),\end{aligned}\right\} \tag{6.8}$$

but also the Lax (1957) entropy condition, which in the present context requires that either

$$\lambda_-(\varphi_1,\varphi_2) < V < \lambda_+(\varphi_1,\varphi_2) \qquad \text{and} \qquad V < \lambda_-(\varphi_1^<,\varphi_2^<) \tag{6.9}$$

or

$$\lambda_-(\varphi_1^<,\varphi_2^<) < V < \lambda_+(\varphi_1^<,\varphi_2^<) \qquad \text{and} \qquad \lambda_+(\varphi_1,\varphi_2) < V. \tag{6.10}$$

Eliminating $V$ between $(6.8)_1$ and $(6.8)_2$ yields a quadratic equation

$$A(\varphi_1,\varphi_1^<,\varphi_2^<)\varphi_2^2 + B(\varphi_1,\varphi_1^<,\varphi_2^<)\varphi_2 + C(\varphi_1,\varphi_1^<,\varphi_2^<) = 0 \tag{6.11}$$

for $\varphi_2$ in terms of $\varphi_1$, $\varphi_1^<$, and $\varphi_2^<$. Here,

$$\left.\begin{aligned} A(\varphi_1,\varphi_1^<,\varphi_2^<) &= \alpha_2(\varphi_1-\varphi_1^<) - \alpha_1\varphi_1, \\ B(\varphi_1,\varphi_1^<,\varphi_2^<) &= \alpha_1\varphi_1\varphi_2^< + \alpha_1\varphi_1 - \alpha_1\varphi_1^<(1-\varphi_1^<-\varphi_2^<) \\ &\quad - \alpha_1\varphi_1^2 + \alpha_2\varphi_1^< - \alpha_2\varphi_1 + \alpha_2\varphi_1^2 - \alpha_2\varphi_1\varphi_1^<, \\ C(\varphi_1,\varphi_1^<,\varphi_2^<) &= \alpha_1\varphi_1^<\varphi_2^<(1-\varphi_1^<-\varphi_2^<) - \alpha_1\varphi_1\varphi_2^< + \alpha_1\varphi_1^2\varphi_2^< \\ &\quad - \alpha_2\varphi_1^<\varphi_2^<(1-\varphi_1^<-\varphi_2^<) + \alpha_1\varphi_1\varphi_2^<(1-\varphi_1^<-\varphi_2^<). \end{aligned}\right\} \tag{6.12}$$

The roots

$$\begin{aligned} d_\pm(\varphi_1,\varphi_1^<,\varphi_2^<) = \frac{-B(\varphi_1,\varphi_1^<,\varphi_2^<)}{2} \\ \mp \frac{\sqrt{B^2(\varphi_1,\varphi_1^<,\varphi_2^<) - 4A(\varphi_1,\varphi_1^<,\varphi_2^<)C(\varphi_1,\varphi_1^<,\varphi_2^<)}}{2A(\varphi_1,\varphi_1^<,\varphi_2^<)} \end{aligned} \tag{6.13}$$

of the quadratic (6.11) describe curves within $\mathcal{A}$. A straightforward calculation too lengthy to display here shows that the curve

$$\mathcal{D}_-(\varphi_1,\varphi_1^<,\varphi_2^<) = \{(\varphi_1,\varphi_2) : \varphi_1^< < \varphi_1 < 1, \varphi_2 = d_-(\varphi_1,\varphi_1^<,\varphi_2^<)\} \tag{6.14}$$

describes all pairs $(\varphi_1,\varphi_2)$ consistent with the jump conditions (6.8) and the inequalities (6.9). Similarly, the curve

$$\mathcal{D}_+(\varphi_1,\varphi_1^<,\varphi_2^<) = \{(\varphi_1,\varphi_2) : \varphi_1^< < \varphi_1 < 1, \varphi_2 = d_+(\varphi_1,\varphi_1^<,\varphi_2^<)\} \tag{6.15}$$

describes all pairs $(\varphi_1,\varphi_2)$ consistent with the jump conditions (6.8) and the inequalities (6.10). A straightforward argument shows that $\mathcal{D}_+(\cdot,\varphi_1^<,\varphi_2^<)$ decreases monotonically over its domain. Thus, given $\varphi_2 = d_+(\cdot,\varphi_1^<,\varphi_2^<)$, we may write $\varphi_1 = \mathcal{D}_+^{-1}(\varphi_2,\varphi_1^<,\varphi_2^<)$. Fig. 7 depicts $\mathcal{D}_+$ and $\mathcal{D}_-$ for an arbitrary choice of $(\varphi_1^<,\varphi_2^<)$.

6.1.2. *Preliminary results concerning rarefactions*

Here, we consider the problem of determining all rarefactions $(\varphi_1,\varphi_2)$ taking values in $\mathcal{A}$, with

$$\varphi_k(x,t) = \phi_k(\xi), \qquad \xi = \frac{x}{t}, \tag{6.16}$$

that may be continuously connected to $(\varphi_1^<,\varphi_2^<)$ across a line emanating from the origin of the $(x,t)$-plane. Such a state must satisfy not only the system

$$\left.\begin{aligned} \xi\phi_1'(\xi) + \alpha_1\big(\phi_1(\xi)(1-\phi_1(\xi)-\phi_2(\xi))\big)' = 0, \\ \xi\phi_2'(\xi) + \alpha_2\big(\phi_2(\xi)(1-\phi_1(\xi)-\phi_2(\xi))\big)' = 0, \end{aligned}\right\} \tag{6.17}$$

which arises on inserting (6.16) in (6.1), but also, since the (non-negative) eigenvalues $\lambda_{\pm}(\phi_1(\xi),\phi_2(\xi))$ must increase with $\xi$, either

$$\lambda_{-}(\varphi_1^{<},\varphi_2^{<}) < \lambda_{-}(\phi_1,\phi_2) \tag{6.18}$$

or

$$\lambda_{+}(\varphi_1^{<},\varphi_2^{<}) < \lambda_{+}(\phi_1,\phi_2). \tag{6.19}$$

On writing (6.17) in the form

$$\begin{bmatrix} \alpha_1(1-2\phi_1(\xi)-\phi_2(\xi))+\xi & -\alpha_1\phi_1(\xi) \\ -\alpha_2\phi_2(\xi) & \alpha_2(1-\phi_1(\xi)-2\phi_2(\xi))+\xi \end{bmatrix} \begin{bmatrix} \phi_1'(\xi) \\ \phi_2'(\xi) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{6.20}$$

and recalling that the eigenvalues of the Jacobian (6.5) are unique, it follows that $\xi = -\lambda_{\pm}(\phi_1,\phi_2)$.

We ignore the case where $\phi_1$ and $\phi_2$ are constant, which yields the trivial solution $\phi_k = \varphi_k^{<}$ and $\phi_2 = \varphi_2^{<}$, and assume that neither $\phi_1'$ nor $\phi_2'$ vanishes. Then, from (6.20) we obtain a pair,

$$\frac{\mathrm{d}\phi_2}{\mathrm{d}\phi_1} = \frac{\alpha_1(1-2\phi_1-\phi_2)-\lambda_{\pm}(\phi_1,\phi_2)}{\alpha_2\phi_1}, \tag{6.21}$$

of first order differential equations which must be solved subject to the inequalities (6.18) and (6.19) and the initial condition

$$\phi_2 = \varphi_2^{<} \qquad \text{when} \qquad \phi_1 = \varphi_1^{<}. \tag{6.22}$$

As solutions we find

$$\phi_2 = r_{\pm}(\phi_1,\varphi_1^{<},\varphi_2^{<}) \tag{6.23}$$

with

$$\begin{aligned} r_{\pm}(\phi_1,\varphi_1^{<},\varphi_2^{<}) = {} & \frac{-B(\phi_1,\varphi_1^{<},\varphi_2^{<})}{2} \\ & \pm \frac{\sqrt{B^2(\phi_1,\varphi_1^{<},\varphi_2^{<}) - 4A(\phi_1,\varphi_1^{<},\varphi_2^{<})C(\phi_1,\varphi_1^{<},\varphi_2^{<})}}{2A(\phi_1,\varphi_1^{<},\varphi_2^{<})} \end{aligned} \tag{6.24}$$

and $A$, $B$, and $C$ as defined in (6.12). A direct calculation too lengthy to display here shows that the curve

$$\mathcal{R}_{-}(\phi_1,\varphi_1^{<},\varphi_2^{<}) = \{(\phi_1,\varphi_2) : \varphi_1 < \varphi_1^{<}, \varphi_2 = r_{-}(\phi_1,\varphi_1^{<},\varphi_2^{<})\} \tag{6.25}$$

describes all rarefactions $(\phi_1,\phi_2)$ satisfying the constraint (6.18). Similarly, the curve

$$\mathcal{R}_{+}(\phi_1,\varphi_1^{<},\varphi_2^{<}) = \{(\phi_1,\varphi_2) : \varphi_1 < \varphi_1^{<}, \varphi_2 = r_{+}(\phi_1,\varphi_1^{<},\varphi_2^{<})\} \tag{6.26}$$

describes all rarefactions $(\phi_1,\phi_2)$ satisfying the constraint (6.19). A straightforward argument shows that $\mathcal{R}_{-}(\cdot,\varphi_1^{<},\varphi_2^{<})$ decreases monotonically over its domain. Thus, given $\varphi_2 = r_{-}(\cdot,\varphi_1^{<},\varphi_2^{<})$, we may write $\varphi_1 = \mathcal{R}_{-}^{-1}(\varphi_2,\varphi_1^{<},\varphi_2^{<})$. Fig. 7 depicts $\mathcal{R}_{+}$ and $\mathcal{R}_{-}$ for an arbitrary choice of $(\varphi_1^{<},\varphi_2^{<})$.

### 6.1.3. *Solutions of the Riemann problem*

Given packing fractions $\varphi_1^{<}$ and $\varphi_2^{<}$ consistent with $0 < \varphi_1^{<} + \varphi_2^{<} < 1$, the curves

$$\mathcal{C}_{-}(\varphi_1,\varphi_1^{<},\varphi_2^{<}) = \mathcal{D}_{-}(\varphi_1,\varphi_1^{<},\varphi_2^{<}) \cup \mathcal{R}_{-}(\varphi_1,\varphi_1^{<},\varphi_2^{<}) \tag{6.27}$$

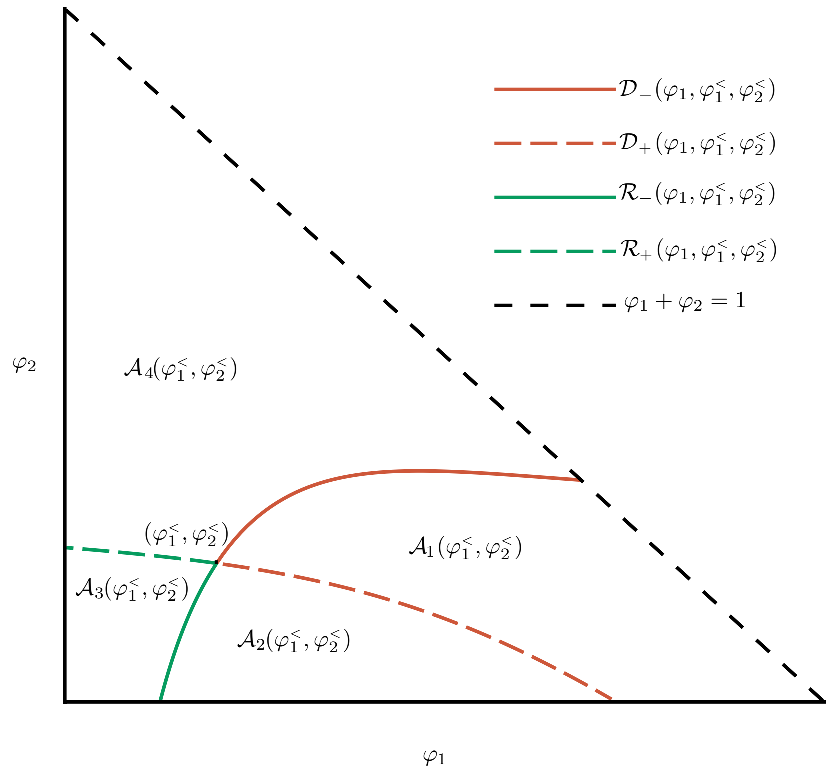


FIGURE 7. The phase portrait of states $(\varphi_1,\varphi_2)$ connecting to the right of $(\varphi_1^<,\varphi_2^<)$

and

$$\mathcal{C}_+(\varphi_1,\varphi_1^<,\varphi_2^<)=\mathcal{D}_+(\varphi_1,\varphi_1^<,\varphi_2^<)\cup\mathcal{R}_+(\varphi_1,\varphi_1^<,\varphi_2^<) \tag{6.28}$$

divide the region $\mathcal{A}$ into four subregions $\mathcal{A}_1(\varphi_1^<,\varphi_2^<)$, $\mathcal{A}_2(\varphi_1^<,\varphi_2^<)$, $\mathcal{A}_3(\varphi_1^<,\varphi_2^<)$, and $\mathcal{A}_4(\varphi_1^<,\varphi_2^<)$ (see Fig. 7) where

$$\left.\begin{aligned}
\mathcal{A}_1(\varphi_1^<,\varphi_2^<) &= \{(\varphi_1,\varphi_2)\in\mathcal{A}:\varphi_1^<<\varphi_1,\\
&\qquad \mathcal{D}_-(\varphi_1,\varphi_1^<,\varphi_2^<)\le\varphi_2\le\mathcal{D}_+(\varphi_1,\varphi_1^<,\varphi_2^<)\}\\
\mathcal{A}_2(\varphi_1^<,\varphi_2^<) &= \{(\varphi_1,\varphi_2)\in\mathcal{A}:\varphi_2^<>\varphi_2,\\
&\qquad \mathcal{R}_-^{-1}(\varphi_2,\varphi_1^<,\varphi_2^<)\le\varphi_1\le\mathcal{D}_+^{-1}(\varphi_2,\varphi_1^<,\varphi_2^<)\},\\
\mathcal{A}_3(\varphi_1^<,\varphi_2^<) &= \{(\varphi_1,\varphi_2)\in\mathcal{A}:\varphi_1^<>\varphi_1,\\
&\qquad \mathcal{R}_+(\varphi_1,\varphi_1^<,\varphi_2^<)\le\varphi_2\le\mathcal{R}_-(\varphi_1,\varphi_1^<,\varphi_2^<)\},\\
\mathcal{A}_4(\varphi_1^<,\varphi_2^<) &= \mathcal{A}\setminus(\mathcal{A}_1(\varphi_1^<,\varphi_2^<)\cup\mathcal{A}_2(\varphi_1^<,\varphi_2^<)\cup\mathcal{A}_3(\varphi_1^<,\varphi_2^<)).
\end{aligned}\right\} \tag{6.29}$$

Hence, if $(\varphi_1^>,\varphi_2^>)$ lies in any of the four regions $\mathcal{A}_i(\varphi_1^<,\varphi_2^<)$ $(i=1,2,3,4)$, we can find an admissible state $(\varphi_1,\varphi_2)$ lying in $\mathcal{C}_-(\varphi_1,\varphi_1^<,\varphi_2^<)$ and also the nature of the solution (i.e. discontinuities or rarefactions) as follows: Consider the family of curves (Fig. 8), $\Im=\big\{\mathcal{C}_+(\varphi_1,\bar\varphi_1,\bar\varphi_2):(\bar\varphi_1,\bar\varphi_2)\in\mathcal{C}_-(\varphi_1,\varphi_1^<,\varphi_2^<)\big\}$. Since the region $\mathcal{A}$ is closed and bounded, the $(\varphi_1,\varphi_2)$-plane is covered univalently by the family of curves $\Im$; that is, through each point $(\varphi_1^>,\varphi_2^>)$, there passes exactly one curve $\mathcal{C}_+(\varphi_1,\bar\varphi_1,\bar\varphi_2)$ belonging to $\Im$.

• Let $(\varphi_1^>, \varphi_2^>)$ lie in $\mathcal{A}_1(\varphi_1^<, \varphi_2^<)$. As shown in Fig. 9a, for each $(\varphi_1^>, \varphi_2^>)$, there is a unique point $(\bar{\varphi}_1, \bar{\varphi}_2) \in \mathcal{C}_-(\varphi_1, \varphi_1^<, \varphi_2^<)$ for which the curve $\mathcal{C}_+(\varphi_1, \bar{\varphi}_1, \bar{\varphi}_2)$ belongs to $\Im$ and passes through $(\varphi_1^>, \varphi_2^>)$. However, in $\mathcal{A}_1(\varphi_1^<, \varphi_2^<)$,

$$\mathcal{C}_-(\varphi_1, \varphi_1^<, \varphi_2^<) = \mathcal{D}_-(\varphi_1, \varphi_1^<, \varphi_2^<)$$

and

$$\mathcal{C}_+(\varphi_1, \bar{\varphi}_1, \bar{\varphi}_2) = \mathcal{D}_+(\varphi_1, \bar{\varphi}_1, \bar{\varphi}_2).$$

Hence, $(\bar{\varphi}_1, \bar{\varphi}_2)$ is connected to $(\varphi_1^<, \varphi_2^<)$ on the right by a discontinuity that satisfies the constraint (6.9). Similarly, $(\varphi_1^>, \varphi_2^>)$ is connected to $(\bar{\varphi}_1, \bar{\varphi}_2)$ on the right by a discontinuity that satisfies the constraint (6.10). The problem is completely solved once $(\bar{\varphi}_1, \bar{\varphi}_2)$ is determined. To this purpose, we solve $\varphi_2^> = d_+(\varphi_1^>, \bar{\varphi}_1, d_-(\bar{\varphi}_1, \varphi_1^<, \varphi_2^<))$ for $\bar{\varphi}_1$. Given $\bar{\varphi}_1$, we have $\bar{\varphi}_2 = d_-(\bar{\varphi}_1, \varphi_1^<, \varphi_2^<)$.

• Let $(\varphi_1^>, \varphi_2^>)$ lie in $\mathcal{A}_2(\varphi_1^<, \varphi_2^<)$. As shown in Fig. 9b, for each $(\varphi_1^>, \varphi_2^>)$, there is a unique point $(\bar{\varphi}_1, \bar{\varphi}_2) \in \mathcal{C}_-(\varphi_1, \varphi_1^<, \varphi_2^<)$ for which the curve $\mathcal{C}_+(\varphi_1, \bar{\varphi}_1, \bar{\varphi}_2)$ belongs to $\Im$ and passes through $(\varphi_1^>, \varphi_2^>)$. However, in $\mathcal{A}_2(\varphi_1^<, \varphi_2^<)$,

$$\mathcal{C}_-(\varphi_1, \varphi_1^<, \varphi_2^<) = \mathcal{R}_-(\varphi_1, \varphi_1^<, \varphi_2^<)$$

and

$$\mathcal{C}_+(\varphi_1, \bar{\varphi}_1, \bar{\varphi}_2) = \mathcal{D}_+(\varphi_1, \bar{\varphi}_1, \bar{\varphi}_2).$$

Hence, $(\bar{\varphi}_1, \bar{\varphi}_2)$ is connected to $(\varphi_1^<, \varphi_2^<)$ on the right by a rarefaction that satisfies the constraint (6.18). Similarly, $(\varphi_1^>, \varphi_2^>)$ is connected to $(\bar{\varphi}_1, \bar{\varphi}_2)$ on the right by a discontinuity that satisfies the constraint (6.10). The problem is completely solved once $(\bar{\varphi}_1, \bar{\varphi}_2)$ is determined. To this purpose, we solve $\varphi_2^> = d_+(\varphi_1^>, \bar{\varphi}_1, r_-(\bar{\varphi}_1, \varphi_1^<, \varphi_2^<))$ for $\bar{\varphi}_1$. Given $\bar{\varphi}_1$, we have $\bar{\varphi}_2 = r_-(\bar{\varphi}_1, \varphi_1^<, \varphi_2^<)$.

• Let $(\varphi_1^>, \varphi_2^>)$ lie in $\mathcal{A}_3(\varphi_1^<, \varphi_2^<)$. As shown in Fig. 9c, for each $(\varphi_1^>, \varphi_2^>)$, there is a unique point $(\bar{\varphi}_1, \bar{\varphi}_2) \in \mathcal{C}_-(\varphi_1, \varphi_1^<, \varphi_2^<)$ for which the curve $\mathcal{C}_+(\varphi_1, \bar{\varphi}_1, \bar{\varphi}_2)$ belongs to $\Im$ and passes through $(\varphi_1^>, \varphi_2^>)$. However, in $\mathcal{A}_3(\varphi_1^<, \varphi_2^<)$,

$$\mathcal{C}_-(\varphi_1, \varphi_1^<, \varphi_2^<) = \mathcal{R}_-(\varphi_1, \varphi_1^<, \varphi_2^<)$$

and

$$\mathcal{C}_+(\varphi_1, \bar{\varphi}_1, \bar{\varphi}_2) = \mathcal{R}_+(\varphi_1, \bar{\varphi}_1, \bar{\varphi}_2).$$

Hence, $(\bar{\varphi}_1, \bar{\varphi}_2)$ is connected to $(\varphi_1^<, \varphi_2^<)$ on the right by a rarefaction that satisfies the constraint (6.18). Similarly, $(\varphi_1^>, \varphi_2^>)$ is connected to $(\bar{\varphi}_1, \bar{\varphi}_2)$ on the right by a rarefaction that satisfies the constraint (6.19). The problem is completely solved once $(\bar{\varphi}_1, \bar{\varphi}_2)$ is determined. To this purpose, we solve $\varphi_2^> = r_+(\varphi_1^>, \bar{\varphi}_1, r_-(\bar{\varphi}_1, \varphi_1^<, \varphi_2^<))$ for $\bar{\varphi}_1$. Given $\bar{\varphi}_1$, we have $\bar{\varphi}_2 = r_-(\bar{\varphi}_1, \varphi_1^<, \varphi_2^<)$.

• Let $(\varphi_1^>, \varphi_2^>)$ lie in $\mathcal{A}_4(\varphi_1^<, \varphi_2^<)$. As shown in Fig. 9d , for each $(\varphi_1^>, \varphi_2^>)$, there is a unique point $(\bar{\varphi}_1, \bar{\varphi}_2) \in \mathcal{C}_-(\varphi_1, \varphi_1^<, \varphi_2^<)$ for which the curve $\mathcal{C}_+(\varphi_1, \bar{\varphi}_1, \bar{\varphi}_2)$ belongs to $\Im$ and passes through $(\varphi_1^>, \varphi_2^>)$. However, in $\mathcal{A}_4(\varphi_1^<, \varphi_2^<)$,

$$\mathcal{C}_-(\varphi_1, \varphi_1^<, \varphi_2^<) = \mathcal{D}_-(\varphi_1, \varphi_1^<, \varphi_2^<)$$

and

$$\mathcal{C}_+(\varphi_1, \bar{\varphi}_1, \bar{\varphi}_2) = \mathcal{R}_+(\varphi_1, \bar{\varphi}_1, \bar{\varphi}_2).$$

Hence, $(\bar{\varphi}_1, \bar{\varphi}_2)$ is connected to $(\varphi_1^<, \varphi_2^<)$ on the right by a discontinuity that satisfies the constraint (6.9). Similarly, $(\varphi_1^>, \varphi_2^>)$ is connected to $(\bar{\varphi}_1, \bar{\varphi}_2)$ on the right by a rarefaction that satisfies the constraint (6.19). The problem is completely solved once $(\bar{\varphi}_1, \bar{\varphi}_2)$ is determined. To this purpose, we solve $\varphi_2^> = r_+(\varphi_1^>, \bar{\varphi}_1, d_-(\bar{\varphi}_1, \varphi_1^<, \varphi_2^<))$ for $\bar{\varphi}_1$. Once $\bar{\varphi}_1$ is determined, we use $\bar{\varphi}_2 = d_-(\bar{\varphi}_1, \varphi_1^<, \varphi_2^<)$ to find $\bar{\varphi}_2$.

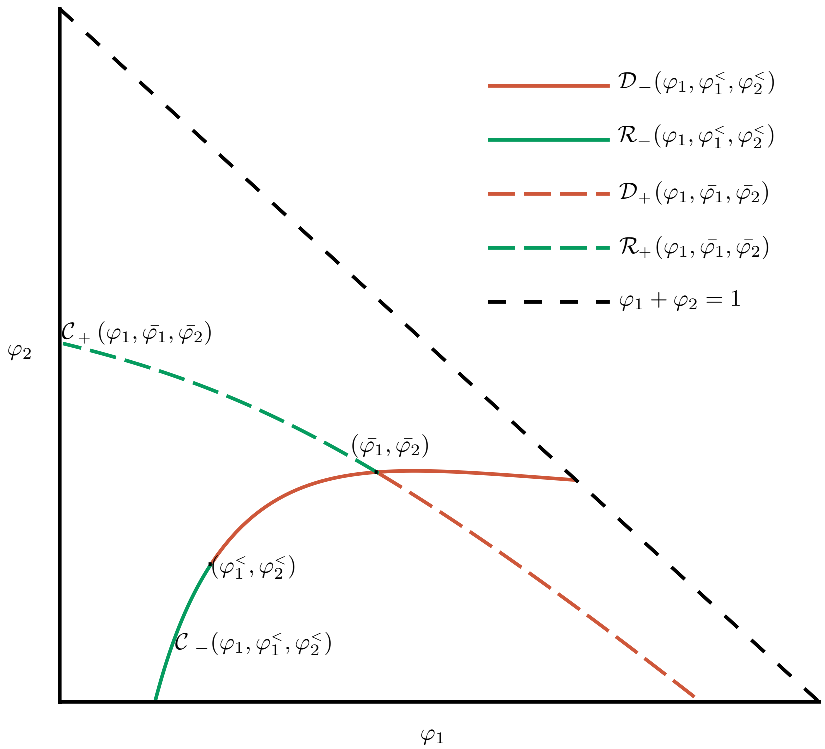


FIGURE 8. Definition of the family of states $\Im = \left\{\mathcal{C}_+(\varphi_1, \bar\varphi_1, \bar\varphi_2) : (\bar\varphi_1, \bar\varphi_2) \in \mathcal{C}_-(\varphi_1, \varphi_1^<, \varphi_2^<)\right\}$

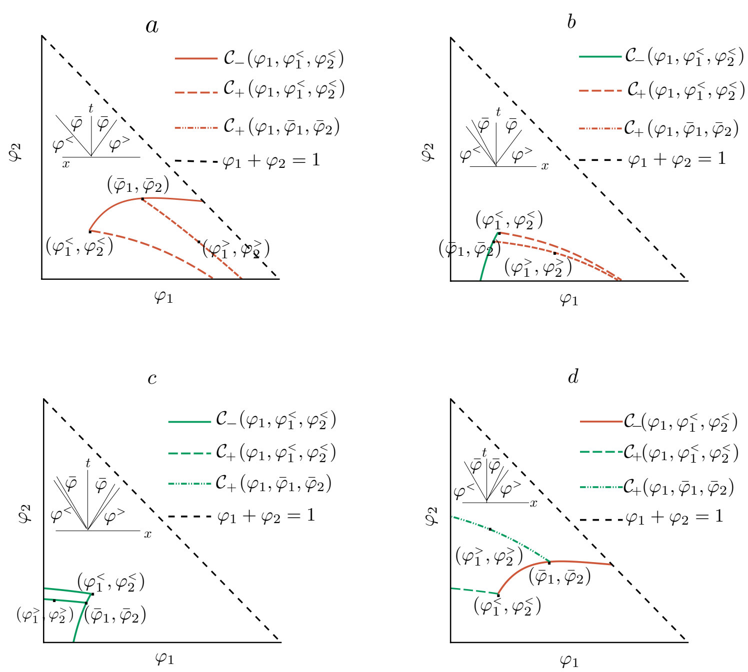


FIGURE 9. Solution of the Riemann problem when $(\varphi_1^>, \varphi_2^>)$ lies in (a) $\mathcal{A}_1(\varphi_1^<, \varphi_2^<)$, (b) $\mathcal{A}_2(\varphi_1^<, \varphi_2^<)$, (c) $\mathcal{A}_3(\varphi_1^<, \varphi_2^<)$, and (d) $\mathcal{A}_4(\varphi_1^<, \varphi_2^<)$

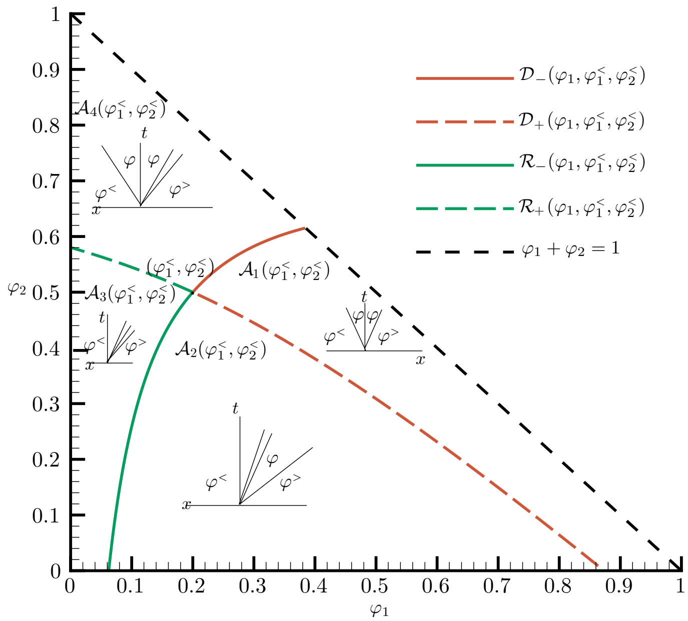


FIGURE 10. The states $(\varphi_1, \varphi_2)$ connecting to the right of $(\varphi_1^<, \varphi_2^<) = (0.2, 0.5)$. Here, $(\alpha_1, \alpha_2) = (0.4, 0.1)$

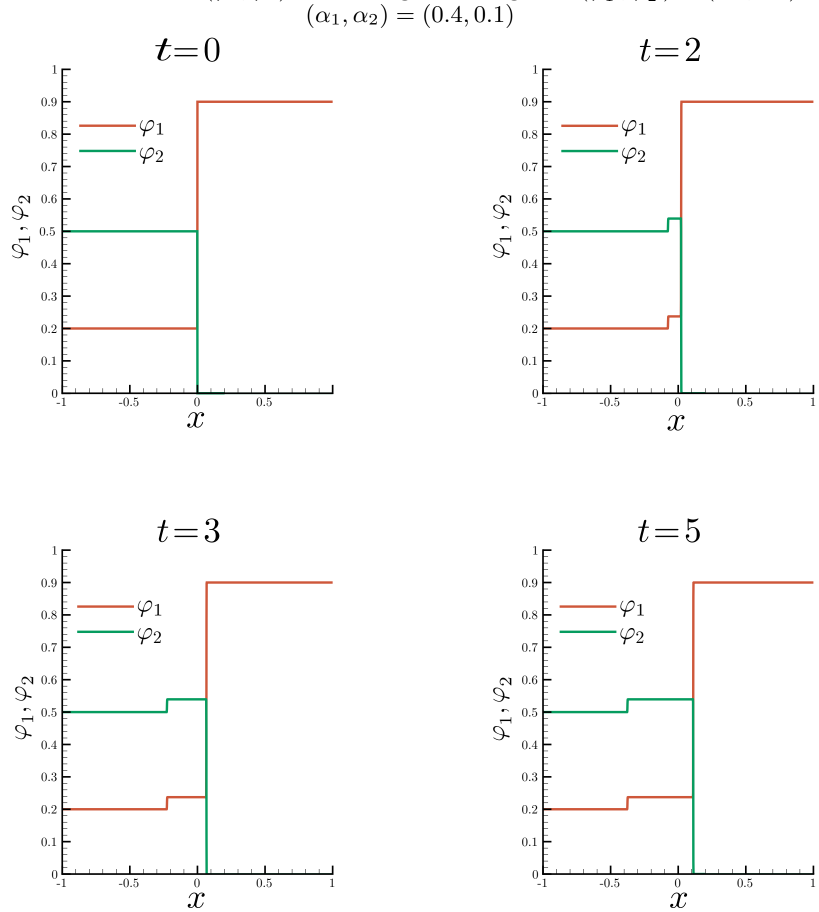


FIGURE 11. Solution of the Riemann problem when $(\varphi_1^>, \varphi_2^>) = (0.9, 0.0)$ lies in $\mathcal{A}_1(\varphi_1^<, \varphi_2^<)$

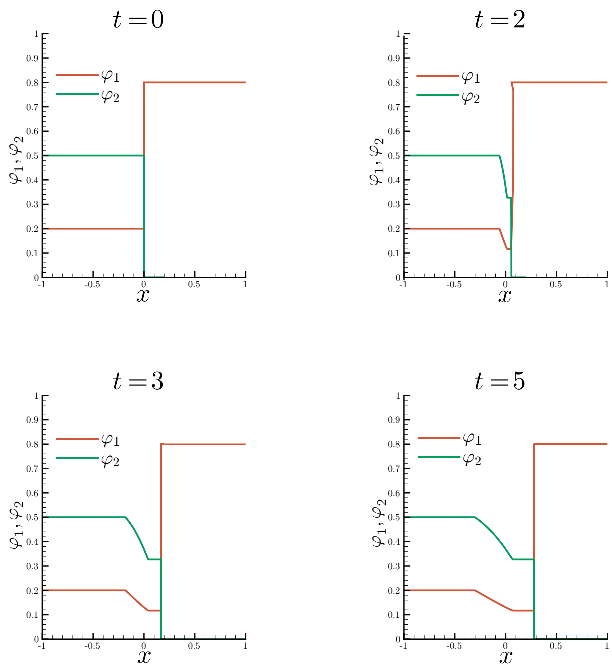


FIGURE 12. Solution of the Riemann problem when $(\varphi_1^>, \varphi_2^>) = (0.8, 0.0)$ lies in $\mathcal{A}_2(\varphi_1^<, \varphi_2^<)$

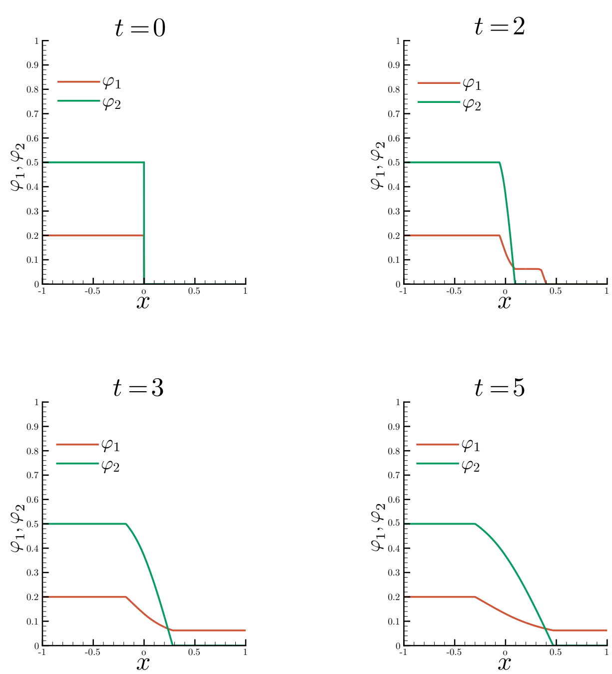


FIGURE 13. Solution of the Riemann problem when $(\varphi_1^>, \varphi_2^>) = (0.0, 0.0)$ lies in $\mathcal{A}_3(\varphi_1^<, \varphi_2^<)$

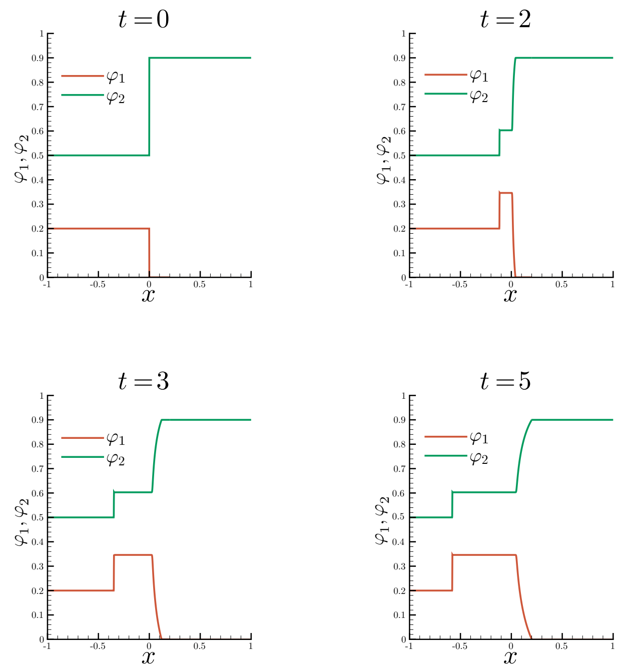


FIGURE 14. Solution of the Riemann problem when $(\varphi_1^>, \varphi_2^>) = (0.0, 0.9)$ lies in $\mathcal{A}_4(\varphi_1^<, \varphi_2^<)$

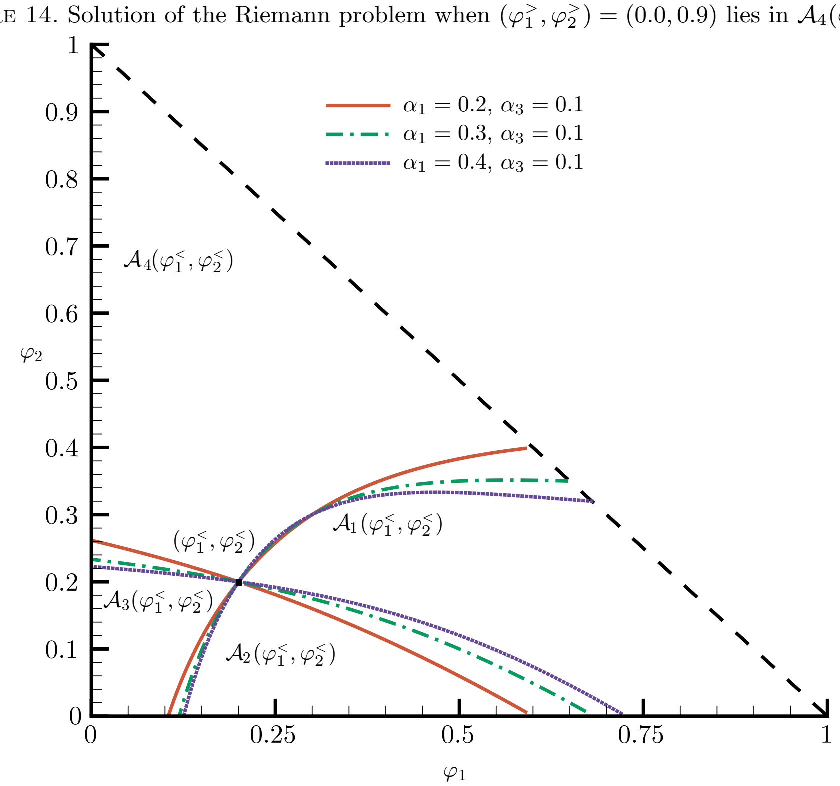


FIGURE 15. The effect of varying $\alpha_1$ on the states $(\varphi_1, \varphi_2)$ connecting to the right of $(\varphi_1^<, \varphi_2^<) = (0.2, 0.2)$

The presence of discontinuities or rarefactions depends on the rate in which voids are generated and the mobilities of particles of type 1 and type 2. In the region $\mathcal{A}_1(\varphi_1^<, \varphi_2^<)$, the rates at which particles of type 1 and type 2 diffuse to fill voids exceeds the rate in which voids are generated, resulting in a solution involving discontinuities. However, in $\mathcal{A}_2(\varphi_1^<, \varphi_2^<)$, $\mathcal{A}_3(\varphi_1^<, \varphi_2^<)$ or $\mathcal{A}_4(\varphi_1^<, \varphi_2^<)$, the rates at which voids are generated exceeds the rates at which particles of either type diffuse. This results in solutions involving rarefactions only (as in $\mathcal{A}_3(\varphi_1^<, \varphi_2^<)$) or combinations of rarefactions and discontinuities (as in $\mathcal{A}_2(\varphi_1^<, \varphi_2^<)$ or $\mathcal{A}_4(\varphi_1^<, \varphi_2^<)$).

As an illustration, we present numerical results generated by CLAWPACK for the case $(\alpha_1, \alpha_2) = (0.4, 0.1)$ and $(\varphi_1^<, \varphi_2^<) = (0.2, 0.5)$. Fig. 10 describes all possible states connecting to the right of $(0.2, 0.5)$. The solution of the Riemann problem (6.1) for the cases when $(\varphi_1^>, \varphi_2^>)$ lies in $\mathcal{A}_1(\varphi_1^<, \varphi_2^<)$, $\mathcal{A}_2(\varphi_1^<, \varphi_2^<)$, $\mathcal{A}_3(\varphi_1^<, \varphi_2^<)$ or $\mathcal{A}_4(\varphi_1^<, \varphi_2^<)$ is shown in Fig. 11, 12, 13, and 14, respectively.

Fig. 15 depict the influence of the dimensionless effective mobility $\alpha_1$, in the case $(\varphi_1^<, \varphi_2^<) = (0.2, 0.2)$. As we increase $\alpha_1$, the area of the region $\mathcal{A}_1(\varphi_1^<, \varphi_2^<)$, where the solution is described only by discontinuities (or where the rate at which the particles diffuse exceeds the rate at which the volume fraction of the voids increases) diminishes. Thus, as the size of particles of type 1 decreases, solutions involving rarefactions become more likely.

### 6.2. *Application*

Here, we use the results concerning the Riemann problem to establish the uniqueness of the solution (in Section 5) in the case of a mixture of small and large particles. In this case, the solution is the union of:

- a uniform state bounded by a free surface $\mathcal{S}_{\text{free}}$ and a segregation shock $\mathcal{S}_{\text{seg}}$ and involving only large particles with packing fraction

$$\varphi_2 = -\frac{1}{2}\Big(\frac{\alpha_1(1-\mathring{\varphi}_1-\mathring{\varphi}_2)}{\alpha_2} - 1\Big) + \frac{1}{2}\sqrt{\Big(\frac{\alpha_1(1-\mathring{\varphi}_1-\mathring{\varphi}_2)}{\alpha_2} - 1\Big)^2 + 4\Big(\frac{\alpha_1}{\alpha_2} - 1\Big)\mathring{\varphi}_2(1-\mathring{\varphi}_1-\mathring{\varphi}_2)}; \quad (6.30)$$

- a mixed uniform state bounded by the segregation shock $\mathcal{S}_{\text{seg}}$ and a compaction shock $\mathcal{S}_{\text{com}}$ and involving small and large particles with packing fractions $\varphi_1 = \mathring{\varphi}_1$ and $\varphi_2 = \mathring{\varphi}_2$;
- a mixed uniform state closely-packed bounded by the container base and the compaction shock $\mathcal{S}_{\text{com}}$ until the time $T_1$, by the container base and $\mathcal{S}_{\text{seg}}$ after $T_1$, and involving small and large particles with packing fractions

$$\varphi_1^* = \mathring{\varphi}_1 + \frac{(1-\mathring{\varphi}_1-\mathring{\varphi}_2)\alpha_1\mathring{\varphi}_1}{\alpha_1\mathring{\varphi}_1+\alpha_2\mathring{\varphi}_2}, \qquad \varphi_2^* = \mathring{\varphi}_2 + \frac{(1-\mathring{\varphi}_1-\mathring{\varphi}_2)\alpha_2\mathring{\varphi}_2}{\alpha_1\mathring{\varphi}_1+\alpha_2\mathring{\varphi}_2}; \quad (6.31)$$

- a uniform closely-packed state bounded by a compaction shock $\mathcal{S}_{\text{com}}$ and the segregation shock $\mathcal{S}_{\text{seg}}$ until the time $T_2$, by the free surface and $\mathcal{S}_{\text{seg}}$ after $T_2$ and involving only large particles with packing fraction $\varphi_1 = 1$.

Prior to the instant $T_1 = \frac{1}{\alpha_1(1-\mathring{\varphi}_2)+\alpha_2\mathring{\varphi}_2}$, $V_{\text{seg}} = \alpha_1(1-\mathring{\varphi}_1-\mathring{\varphi}_2)$, $V_{\text{com}} = -(\alpha_1\mathring{\varphi}_1+\alpha_2\mathring{\varphi}_2)$ and $V_{\text{free}} = \alpha_1(1-\varphi_2)$, with $\varphi_2$ given by (6.30). Between the instants $T_1$ and $T_2 =$

$\dfrac{\frac{\alpha_1}{\alpha_2}(1-\mathring{\varphi}_1-\mathring{\varphi}_2)}{\alpha_1(1-\mathring{\varphi}_2)+\alpha_2\mathring{\varphi}_2}$, $V_{\rm com} = -\alpha_2\varphi_2$, with $\varphi_2$ given once again by (6.30). Subsequent to $T_2$, $V_{\rm seg} = 0$.

Concerning this solution, we have the following

THEOREM 1. *The solution delineated above solves uniquely the problem*

$$\left.\begin{aligned}\frac{\partial\varphi_1}{\partial t} &= \alpha_1\frac{\partial}{\partial x}\Big(\varphi_1(1-\varphi_1-\varphi_2)\Big),\\ \frac{\partial\varphi_2}{\partial t} &= \alpha_2\frac{\partial}{\partial x}\Big(\varphi_2(1-\varphi_1-\varphi_2)\Big),\end{aligned}\right\} \tag{6.32}$$

*subject to the initial conditions*

$$\varphi_k(x,0) = \begin{cases} 0 & \textit{if } x > 1,\\ \\ \mathring{\varphi}_k & \textit{if } 0 < x \le 1,\end{cases} \tag{6.33}$$

*the jump conditions*

$$\left.\begin{aligned}[\![\varphi_1]\!]V_{\rm shock} &= -\alpha_1[\![\varphi_1(1-\varphi_1-\varphi_2)]\!],\\ [\![\varphi_2]\!]V_{\rm shock} &= -\alpha_2[\![\varphi_2(1-\varphi_1-\varphi_2)]\!],\end{aligned}\right\} \tag{6.34}$$

*across any shock (moving with velocity $V_{\rm shock}$), the condition*

$$\alpha_k(1-\varphi_1-\varphi_2) = V_{\rm free}, \qquad k = 1,2 \tag{6.35}$$

*for particles of type $k$ at a free surface (moving with velocity $V_{\rm free}$), and condition*

$$\varphi_1 + \varphi_2 = 1 \tag{6.36}$$

*at the base.*

*Proof.* We first show that rarefactions may not issue from the points $(x,t) = (1,0)$ and $(x,t) = (0,0)$ corresponding to the initial free surface and to the base. Suppose that, at $t = 0$, a rarefaction occurs at $x = 1$. Then

$$\lambda_\pm(0,0) < \lambda_\pm(\phi_1(\xi),\phi_2(\xi)), \qquad \xi = \frac{x}{t}. \tag{6.37}$$

However, from (6.6), $\lambda_-(0,0) = \alpha_2$ and $\lambda_-(\phi_1,\phi_2) < \alpha_2(1-\phi_1-2\phi_2)$. Thus, by (6.37), $\alpha_2 < \alpha_2(1-\phi_1-2\phi_2)$ or $\phi_1 + 2\phi_2 < 0$, which cannot occur since $\phi_1$ and $\phi_2$ must be in $\mathcal{A}$. Thus, the first characteristic does not give rise to a rarefaction. Similarly, from (6.6), $\lambda_+(0,0) = \alpha_1$ and $\lambda_+(\phi_1,\phi_2) < \alpha_1$. Thus, by (6.37), $\alpha_1 = \lambda_2(\phi_1,\phi_2) < \alpha_1$, which cannot hold. Hence, by contradiction, no rarefactions may emanate from $(x,t) = (1,0)$. At the base ($x = 1$), the state $(\varphi_1,\varphi_2)$ connecting to the right of the state $(\mathring{\varphi}_1,\mathring{\varphi}_2)$ satisfies the relation $1-\varphi_1-\varphi_2 = 0$. Such a state must lie on the intersection of $\mathcal{D}_-(\varphi_1,\mathring{\varphi}_1,\mathring{\varphi}_2)$ with the line $\varphi_1+\varphi_2 = 1$. Hence, the characteristic at $x = 1$ satisfies only (6.9) and is not a rarefaction. Thus, by contradiction, no rarefactions may emanate from $(x,t) = (0,0)$.

At the point $(x,t) = (X_1,T_1)$, with $X_1 = \frac{\alpha_1(1-\mathring{\varphi}_1-\mathring{\varphi}_2)}{\alpha_1(1-\mathring{\varphi}_2)+\alpha_2\mathring{\varphi}_2}$, the segregation shock $S_{\rm seg}$ and the compaction shock $S_{\rm com}$ meet. To understand the nature of the solution at $t = T_1$, we solve for (6.32) subject to the condition $(\varphi_1(x < X_1, t = T_1), \varphi_2(x < X_1, t = T_1)) = (0,\varphi_2)$ and $(\varphi_1(x \ge X_1, t = T_1), \varphi_2(x \ge X_1, t = T_1)) = (\varphi_1^*,\varphi_2^*)$ where $\varphi_2$ is given as in (6.30) and $(\varphi_1^*,\varphi_2^*)$ as in (6.31). If $(\bar\varphi_1,\bar\varphi_2)$ denotes an admissible state connecting to the right of $(0,\varphi_2)$ by a rarefaction, then (6.24) together with constraints (6.18) and (6.19) imply that $\varphi_1 < 0$. This falls outside the domain $\mathcal{A}$. Hence, a rarefaction cannot emanate from $(x,t) = (X_1, T_1)$.

At the point $(x,t) = (X_2, T_2)$, with $X_2 = \frac{\alpha_2(1-\varphi_2)X_1}{\alpha_1}$ and $\varphi_2$ given by (6.30), the free surface $S_{\text{free}}$ and the compaction shock $S_{\text{com}}$ meet. To understand the nature of the solution at $t = T_2$, we solve for (6.32) subject to the condition $(\varphi_1(x < X_2, t = T_2), \varphi_2(x < X_2, t = T_2)) = (0,0)$ and $(\varphi_1(x \geq X_2, t = T_2), \varphi_2(x \geq X_2, t = T_2)) = (0,1)$. Following a proof identical to that used in the case of the point $(x,t) = (1,0)$, we conclude that a rarefaction cannot emanate from $(X_2, T_2)$.

Hence the solution cannot involve rarefactions. To complete the proof, we rely on the analysis of the Riemann problem detailed above.

- At the point $(x,t) = (1,0)$, we can regard $(\varphi_1^<, \varphi_2^<) = (0,0)$ and $(\varphi_1^>, \varphi_2^>) = (\mathring{\varphi}_1, \mathring{\varphi}_2)$. Since the admissible state $(\varphi_1, \varphi_2)$ lies in $D_-(\varphi_1, \varphi_1^< = 0, \varphi_2^< = 0)$, it follows that this state is connected to $(\varphi_1^<, \varphi_2^<) = (0,0)$ by a discontinuity (the free surface) that satisfies the constraint (6.9) together with the jump conditions

$$\left.\begin{aligned} \alpha_1\varphi_1(1-\varphi_1-\varphi_2) &= V_{\text{free}}\varphi_1, \\ \alpha_2\varphi_2(1-\varphi_1-\varphi_2) &= V_{\text{free}}\varphi_2. \end{aligned}\right\} \tag{6.38}$$

This implies that $\varphi_1 = 0$ and that $V_{\text{free}} = \alpha_2(1-\varphi_2)$. To determine $\varphi_2$, we note that $(\varphi_1^>, \varphi_2^>) = (\mathring{\varphi}_1, \mathring{\varphi}_2)$ lies in $D_+(\varphi_1^> = \mathring{\varphi}_1, 0, \varphi_2)$. Hence, the state $(0, \varphi_2)$ is connected to $(\varphi_1^>, \varphi_2^>) = (\mathring{\varphi}_1, \mathring{\varphi}_2)$ by a discontinuity (the segregation shock) satisfying the constraint (6.10) along with the jump conditions

$$\left.\begin{aligned} \alpha_1\varphi_1(1-\varphi_1-\varphi_2) - \alpha_1\mathring{\varphi}_1(1-\mathring{\varphi}_1-\mathring{\varphi}_2) &= V_{\text{seg}}(\varphi_1-\mathring{\varphi}_1), \\ \alpha_2\varphi_2(1-\varphi_1-\varphi_2) - \alpha_2\mathring{\varphi}_2(1-\mathring{\varphi}_1-\mathring{\varphi}_2) &= V_{\text{seg}}(\varphi_2-\mathring{\varphi}_2). \end{aligned}\right\} \tag{6.39}$$

The above conditions implies that $V_{\text{seg}} = \alpha_1(1-\mathring{\varphi}_1-\mathring{\varphi}_2)$ and that $\varphi_2$ given by (6.30). Since $\alpha_1 > \alpha_2$ and $1-\mathring{\varphi}_1-\mathring{\varphi}_2 > 0$, it follows that (6.30) has only one positive real root $\varphi_2$ and that $\varphi_2 < 1$. Hence, the root of (6.30) yields a packing fraction.

- At the point $(x,t) = (0,0)$, the state $(\varphi_1^*, \varphi_2^*)$ connecting to the right of $(\mathring{\varphi}_1, \mathring{\varphi}_2)$ satisfies the relation $\varphi_1^* + \varphi_2^* = 1$. However, from (6.6) it follows that $\lambda_-(\varphi_1^*, \varphi_2^*) = 0$ and $\lambda_+(\varphi_1^*, \varphi_2^*) = -(\alpha_1\varphi_1^* + \alpha_2\varphi_2^*)$. Since one of the eigenvalues is constant, a contact discontinuity (the compaction shock) emanates from the base. From the jump conditions

$$\left.\begin{aligned} \alpha_1\mathring{\varphi}_1(1-\mathring{\varphi}_1-\mathring{\varphi}_2) &= -V_{\text{com}}(\varphi_1^*-\mathring{\varphi}_1), \\ \alpha_2\mathring{\varphi}_2(1-\mathring{\varphi}_1-\mathring{\varphi}_2) &= -V_{\text{com}}(\varphi_2^*-\mathring{\varphi}_2), \end{aligned}\right\} \tag{6.40}$$

and the condition $\varphi_1^* + \varphi_2^* = 1$, it follows that $V_{\text{com}} = -(\alpha_1\mathring{\varphi}_1 + \alpha_2\mathring{\varphi}_2)$ and $(\varphi_1^*, \varphi_2^*)$ is given as in (6.31).

- At the point $(x,t) = (X_1, T_1)$, we can regard $(\varphi_1^<, \varphi_2^<) = (0, \varphi_2)$ and $(\varphi_1^>, \varphi_2^>) = (\varphi_1^*, \varphi_2^*)$, where $\varphi_1^* + \varphi_2^* = 1$. Since the admissible state $(\bar{\varphi}_1, \bar{\varphi}_2)$ lies in $\mathcal{D}_-(\bar{\varphi}_1, 0, \varphi_2^< = \varphi_2)$, this state must be connected to $(\varphi_1^<, \varphi_2^<) = (0, \varphi_2)$ by a discontinuity (a compaction shock) that satisfies the constraint (6.9) along with the jump conditions

$$\left.\begin{aligned} \alpha_1\bar{\varphi}_1(1-\bar{\varphi}_1-\bar{\varphi}_2) &= V_{\text{com}}\bar{\varphi}_1, \\ \alpha_2\bar{\varphi}_2(1-\bar{\varphi}_1-\bar{\varphi}_2) - \alpha_2\varphi_2(1-\varphi_2) &= V_{\text{com}}(\bar{\varphi}_2-\varphi_2). \end{aligned}\right\} \tag{6.41}$$

This implies that $\bar{\varphi}_1 = 0$ and that $V_{\text{com}} = -\alpha_2\varphi_2$. To determine $\bar{\varphi}_2$, we note that $(\varphi_1^>, \varphi_2^>) = (\varphi_1^*, \varphi_2^*)$ lies in $\mathcal{D}_+(\varphi_1^*, 0, \bar{\varphi}_2)$. Hence, the state $(0, \varphi_2)$ is connected to $(\varphi_1^>, \varphi_2^>) = (\varphi_1^*, \varphi_2^*)$ by a discontinuity (the segregation shock) that satisfies the constraint (6.10) along with the jump conditions

$$\left.\begin{aligned} \alpha_1\varphi_1^*(1-\varphi_1^*-\varphi_2^*) &= V_{\rm seg}(-\varphi_1^*), \\ \alpha_2\bar\varphi_2(1-\bar\varphi_1-\bar\varphi_2)-\alpha_2\varphi_2^*(1-\varphi_1^*-\varphi_2^*) &= V_{\rm seg}(\bar\varphi_2-\varphi_2^*). \end{aligned}\right\} \quad (6.42)$$

The above conditions imply that $V_{\rm seg}=0$ and $\varphi_2=1$.

- At the point $(x,t)=(X_2,T_2)$, we can regard $(\varphi_1^<,\varphi_2^<)=(0,0)$ and $(\varphi_1^>,\varphi_2^>)=(0,1)$. Proceeding exactly as in the case of the point $(x,t)=(0,0)$, it follows that the solution can be described by a contact discontinuity (the free surface) separating the free surface from the compacted layer where $(\varphi_1,\varphi_2)=(0,1)$.

□

## 7. Discussion

Using a model proposed by Fried et al. (2001), we have studied size-based segregation occuring under the action of gravity. For a flow with constant velocity, the model reduces to a system of one-dimensional conservation laws. We have presented a solution for a particular initial-value problem involving a mixture of particles of $K\geq 2$ sizes. This solution shows segregation and compaction by particle size. At steady state, this solution consists of layers of closely packed particles, with the upper-most layer consisting only of particles of the largest size. Numerical solutions for $K=3$ particles were computed using LeVeque's (1994) CLAWPACK. Relying on methods developed by Smoller (1994), we established the uniqueness of the solution for the case $K=2$ of a mixture of small and large particles. The issue of uniqueness in the case $K>2$ remains open.

The problem considered here is idealized in the sense that the flow field is trivial. Furthermore, we have ignored variations of the particulate mobilities with the strain-rate. As discussed by Fried et al. (2001), such variations should rule out particle diffusion in the absence of sufficient agitation. Under flow conditions more general than those considered here, strain-rate dependence of the mobilities would allow for the existence of regions in which particles would simply move with the mixture. Compaction and segregation would thus be confined to regions of sufficiently high agitation.

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