

The totality of soft-states in a neo-classical nematic elastomer

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The remarkable ability of nematic elastomers to exist in multiple stable equilibrium configurations in the absence of force and energy cost is known as soft elasticity. Here, we determine all of the soft states that are possible in two particular classes of nematic elastomers: the neo-classical nematic elastomers, which have received considerable recent attention; and a natural theoretical generalization, which we call the Mooney nematic elastomers. For both of these classes, the soft states are minimizers of the free energy. Our starting point is to develop a nonlinear continuum theory for nematic elastomers, the constitutive framework of which includes the special classes noted above. This theory is based on two distinct force balances—one arising from the macroscopic deformation and the other from the nematicity-induced distortion of the polymer molecules—and allows for coexistent isotropic, uniaxial, and biaxial phases. These balances are supplemented by a principle of dissipation imbalance, a principle that in conjunction with invariance requirements is used to obtain restrictions on constitutive equations.

Keywords: nematic elastomers; soft states; energy minimizers.

1. Introduction

Of the many interesting phenomena exhibited by nematic elastomers,[†] one of the most fascinating is *soft elasticity*—the ability of the material to exist in multiple stable equilibrium configurations in the absence of force and energy cost. Soft elasticity was predicted by Golubovic & Lubensky (1989) and it has been experimentally observed by Kundler & Finkelmann (1995). The terminology ‘soft elasticity’ was coined by Warner, Bladon & Terentjev (1994). Other references that we have found to be especially germane to our investigation of soft elasticity are Olmsted (1994), Warner (1999), and DeSimone & Dolzmann (2000).

Here, we determine the totality of soft states that exist when the nematic elastomer is described by the so-called *neo-classical free-energy density function* derived on a molecular-statistical basis by Warner, Gelling & Vilgis (1988) and studied by Warner and his collaborators (Bladon, Terentjev & Warner 1993; Bladon, Terentjev & Warner 1994; Warner, Bladon & Terentjev 1994; Terentjev, Warner & Bladon 1994).

[†] See Warner & Terentjev (1996) for a rather complete review.

To define meaningfully the notion of a soft state, we begin, in Section 2, by outlining a general theory for nematic elastomers that allows for isotropy, uniaxiality, and biaxiality of the conformation tensor. This theory leads to a coupled system of evolution equations involving the deformation and the conformation. Under conditions of equilibrium, these equations specialize to equations that would arise by considering variations of a relevant free-energy functional. Our notion of a soft state contains information concerning not only force but also energetics. In Section 3, we briefly define the neo-classical free-energy density function and examine some of its properties. In Sections 4 and 5, we determine, for the first time, all of the soft states for neo-classical nematic elastomers. Such states are given by simple formulas in terms of an arbitrary rotation tensor and an arbitrary proper unimodular tensor. In addition, we show that, for a neo-classical nematic elastomer, the free-energy density is minimized in all soft states. This suggests an analogy between soft states in nematic elastomers and stress-free energy minimizing variants in crystalline solids (Ball & James 1987). Related to this analogy is the connection between crystal twinning and nematic-elastomeric striping, a connection discussed by DeSimone & Dolzmann (2000). Hence, whereas the spectrum of stress-free energy minimizing states available to a typical crystalline solid is discrete, the neo-classical model predicts that the spectrum of soft states available to a nematic elastomer is “doubly” continuous in the sense that it is parameterized by a pair of tensors. Granted the validity of the neo-classical model, one would therefore expect nematic elastomers to exhibit a rich variety of striped microstructures. We conclude Section 5 by noting a few particular examples of soft states. Finally, in Section 6, we extend our work to a class of free-energy density functions more likely to be realistic. In particular, we consider the case of the nematic elastomeric counterpart of Mooney’s (1940) free-energy density function. Surprisingly, we find that the set of soft states available to such a material is identical to that available for a neo-classical nematic elastomer and that the free-energy density is minimized in all of these states.

2. General theory

(a) Kinematics

We consider an incompressible nematic elastomer which, in a reference state, occupies a region \mathcal{B} in a three-dimensional point space \mathcal{E} . At each point \mathbf{x} of \mathcal{B} , we envision a cross-linked network of polymeric molecules which include nematic mesogens as main-chain elements or as pendant side-groups. Generally, the kinematic state of a nematic elastomer is described by two fields:

- the *deformation* \mathbf{y} , a time-dependent mapping of \mathcal{B} into \mathcal{E} , that serves as a macroscopic measure of the distortion of the network cross-links; and
- the *conformation* \mathbf{A} , a time-dependent mapping of \mathcal{B} into the set Psym of symmetric and positive-definite second-order tensors, that serves as a macroscopic measure of the nematically-induced distortion of the polymer molecules in the current state.†

† The conformation is a dimensionless version of the *step-length tensor* introduced by Warner, Gelling & Vilgis (1988).

To account for the possibility of nematically-induced distortion of polymer molecules in the reference state, we introduce a referential conformation field $\mathbf{A}_* : \mathcal{B} \rightarrow \text{Psym}$. Whereas \mathbf{A}_* is a material property which describes nematically-induced anisotropy in the reference state and may vary from point to point in the region \mathcal{B} , \mathbf{A} is a kinematical field which describes nematically-induced anisotropy in the deformed state and, like \mathbf{y} , varies not only with position in \mathcal{B} but also in time. As in the theory of nonlinear elasticity (Ogden 1997), the distinction between the reference configuration and the current configuration is of primary importance in our theory.

We use Grad and Div to denote differentiation on \mathcal{B} and, in particular, write

$$\mathbf{F} = \text{Grad} \mathbf{y} \quad (2.1)$$

for the deformation gradient. Since the medium is assumed to be incompressible, we require that \mathbf{F} belong to the set Unim^+ of second-order tensors with unit determinant.

(b) *Basic laws*

We formulate the basic laws in the referential setting, using \mathcal{P} to designate an arbitrary regular part of \mathcal{B} with boundary $\partial\mathcal{P}$ oriented by a unit normal field $\boldsymbol{\nu}$ directed outward from \mathcal{P} .

For simplicity, inertial effects are incorporated in the external body force densities.

(i) *Balance of deformational forces*

We introduce a deformational force system consisting of a *stress* \mathbf{S} belonging to the space Lin of second-order tensors and an *external deformational body force density* \mathbf{b} belonging to the translation space \mathcal{V} of \mathcal{E} . The body force density \mathbf{b} is assumed to include inertial forces. We impose balance of deformational forces by requiring that

$$\int_{\partial\mathcal{P}} \mathbf{S} \boldsymbol{\nu} da + \int_{\mathcal{P}} \mathbf{b} dv = \mathbf{0} \quad (2.2)$$

for all \mathcal{P} and all time, or, equivalently, that

$$\text{Div} \mathbf{S} + \mathbf{b} = \mathbf{0} \quad (2.3)$$

on \mathcal{B} for all time.

(ii) *Balance of conformational forces*

We introduce a conformational force system consisting of *internal* and *external body force densities* $\mathbf{\Pi}$ and $\mathbf{\Delta}$ belonging to the space Sym of symmetric second-order tensors. The body force density $\mathbf{\Delta}$ is assumed to include inertial forces. We

impose balance of conformational forces by requiring that[†]

$$\int_{\mathcal{P}} (\mathbf{\Pi} + \mathbf{\Delta}) dv = \mathbf{0} \quad (2.4)$$

for all \mathcal{P} and all time, or, equivalently, that

$$\mathbf{\Pi} + \mathbf{\Delta} = \mathbf{0} \quad (2.5)$$

on \mathcal{B} for all time.

(iii) *Balance of moments*

We ignore external couples and impose moment balance by requiring that[‡]

$$\int_{\partial\mathcal{P}} \mathbf{y} \times \mathbf{S}\boldsymbol{\nu} da + \int_{\mathcal{P}} (\mathbf{y} \times \mathbf{b} + 2\text{dual}(\mathbf{A}\mathbf{\Delta})) da = \mathbf{0} \quad (2.6)$$

for all \mathcal{P} and all time, or, equivalently, in view of (2.3) and (2.5), that

$$\mathbf{S}\mathbf{F}^\top + \mathbf{A}\mathbf{\Pi} = (\mathbf{S}\mathbf{F}^\top + \mathbf{A}\mathbf{\Pi})^\top \quad (2.7)$$

on \mathcal{B} for all time.

(iv) *Imbalance of dissipation*

We consider a purely mechanical situation where the first and second laws combine to yield a principle of dissipation imbalance and, introducing the *free-energy density* ψ , impose that imbalance by requiring that[¶]

$$\overline{\int_{\mathcal{P}} \dot{\psi} dv} \leq \int_{\partial\mathcal{P}} \mathbf{S}\boldsymbol{\nu} \cdot \dot{\mathbf{y}} da + \int_{\mathcal{P}} (\mathbf{b} \cdot \dot{\mathbf{y}} + \mathbf{\Delta} \cdot \dot{\mathbf{A}}) dv \quad (2.8)$$

for all \mathcal{P} and all time, or, equivalently, in view of (2.3) and (2.5), that

$$\dot{\psi} - \mathbf{S} \cdot \dot{\mathbf{F}} + \mathbf{\Pi} \cdot \dot{\mathbf{A}} \leq 0 \quad (2.9)$$

on \mathcal{B} for all time.

(c) *Constitutive theory*

For simplicity, we restrict our attention to a constitutive theory in which the deformation gradient \mathbf{F} and conformation \mathbf{A} are the independent variables.

[†] More generally, we might introduce a conformational stress $\boldsymbol{\Sigma}$, a third-order tensor that, for each point \mathbf{x} in \mathcal{B} and each time t , maps elements of \mathcal{V} into elements of Sym . Under these circumstances, (2.5) would be replaced by $\int_{\partial\mathcal{P}} \boldsymbol{\Sigma}\boldsymbol{\nu} da + \int_{\mathcal{P}} (\mathbf{\Pi} + \mathbf{\Delta}) dv = \mathbf{0}$.

[‡] In keeping with Anderson, Carlson & Fried (1999), the torque $\int_{\mathcal{P}} 2\text{dual}(\mathbf{A}\mathbf{\Pi}) dv$ induced by $\mathbf{\Pi}$ is not accounted for in the moment balance (2.6) since this torque acts internally. Here, $\text{dual}\mathbf{M}$ is the element of \mathcal{V} defined so that $\frac{1}{2}(\mathbf{M} - \mathbf{M}^\top)\mathbf{v} = (\text{dual}\mathbf{M}) \times \mathbf{v}$ for all elements \mathbf{v} of \mathcal{V} . In the presence of conformational stress, it would be necessary to include the associated moment in (2.6).

[¶] By reasoning identical to that underlying the omission of internal torque from (2.6), the power $\int_{\mathcal{P}} \mathbf{\Pi} \cdot \dot{\mathbf{A}} dv$ expended by the internal conformational body force density $\mathbf{\Pi}$ is not included in (2.8). In the presence of conformational stress, it would be necessary to include the associated power in (2.8).

(i) *Thermodynamics*

Employing the approach taken by Anderson, Carlson & Fried (1999), we find that the imbalance (2.9) holds in all processes consistent with the incompressibility constraint and the force and moment balances if and only if

$$\psi = \hat{\psi}(\mathbf{F}, \mathbf{A}; \mathbf{A}_*), \quad \mathbf{S} = \frac{\partial \hat{\psi}(\mathbf{F}, \mathbf{A}; \mathbf{A}_*)}{\partial \mathbf{F}} - p\mathbf{F}^{-\top}, \quad \text{and} \quad \mathbf{\Pi} = -\frac{\partial \hat{\psi}(\mathbf{F}, \mathbf{A}; \mathbf{A}_*)}{\partial \mathbf{A}}, \quad (2.10)$$

where the parametric dependence on \mathbf{A}_* has been indicated explicitly. The partial derivative on the left side of (2.10)₂ indicates differentiation on the manifold determined by the constraint of incompressibility requiring that $\mathbf{F} \in \text{Unim}^+$, i.e.,

$$\frac{\partial \hat{\psi}(\mathbf{F}, \mathbf{A}; \mathbf{A}_*)}{\partial \mathbf{F}} = \left(\mathbf{I} - \frac{1}{|\mathbf{F}^{-1}|^2} \mathbf{F}^{-\top} \otimes \mathbf{F}^{-\top} \right) \frac{\partial \bar{\psi}(\mathbf{F}, \mathbf{A}; \mathbf{A}_*)}{\partial \mathbf{F}}, \quad (2.11)$$

where \mathbf{I} denotes the identity mapping from Lin into itself, $\mathbf{L} \otimes \mathbf{M}$ is the linear operator from Lin to Lin defined by $(\mathbf{L} \otimes \mathbf{M})\mathbf{N} = (\mathbf{M} \cdot \mathbf{N})\mathbf{L}$, $\bar{\psi}$ is a smooth extension of $\hat{\psi}$ to $\text{Lin}^+ \times \text{Psym}$, Lin^+ is the set of second-order tensors with strictly positive determinant, and the second factor on the right side of (2.11) is the standard partial derivative evaluated on the constraint manifold. In (2.10), \mathbf{S} is decomposed into an active component \mathbf{S}_a determined constitutively by

$$\mathbf{S}_a = \frac{\partial \hat{\psi}(\mathbf{F}, \mathbf{A}; \mathbf{A}_*)}{\partial \mathbf{F}} \quad (2.12)$$

and a constitutively indeterminate reactive contribution

$$\mathbf{S}_r = -p\mathbf{F}^{-\top}, \quad (2.13)$$

with p the *pressure* required to maintain the constraint of incompressibility. From (2.10)₃, the internal conformational body force density $\mathbf{\Pi}$ must belong to Sym .

(ii) *Invariance*

We require that $\hat{\psi}$ be invariant under superposed rigid motions, i.e.,

$$\hat{\psi}(\mathbf{Q}\mathbf{F}, \mathbf{Q}\mathbf{A}\mathbf{Q}^\top; \mathbf{A}_*) = \hat{\psi}(\mathbf{F}, \mathbf{A}; \mathbf{A}_*) \quad (2.14)$$

for all elements \mathbf{Q} of the set Orth^+ of rotations. In the usual way, we see that (2.14) is met if and only if $\hat{\psi}$ can be expressed in the reduced form

$$\hat{\psi}(\mathbf{F}, \mathbf{A}; \mathbf{A}_*) = \tilde{\psi}(\mathbf{C}, \mathbf{F}^\top \mathbf{A} \mathbf{F}; \mathbf{A}_*), \quad (2.15)$$

where

$$\mathbf{C} = \mathbf{F}^\top \mathbf{F} \quad (2.16)$$

is the right Cauchy–Green strain tensor.

(iii) *Symmetry*

The *material symmetry group* Matsym of a nematic elastomeric point defined by the free-energy density function $\hat{\psi}$ corresponds to the set of rotations of the reference configuration that leave the subsequent response of the elastomer unchanged, i.e.,

$$\text{Matsym} = \{\mathbf{Q} \in \text{Orth}^+ : \hat{\psi}(\mathbf{F}\mathbf{Q}, \mathbf{A}; \mathbf{Q}^\top \mathbf{A}_* \mathbf{Q}) = \hat{\psi}(\mathbf{F}, \mathbf{A}; \mathbf{A}_*)\}. \quad (2.17)$$

Combining the notion of symmetry with the requirement of invariance yields

$$\text{Matsym} = \{\mathbf{Q} \in \text{Orth}^+ : \tilde{\psi}(\mathbf{Q}\mathbf{C}\mathbf{Q}^\top, \mathbf{Q}\mathbf{F}^\top \mathbf{A}\mathbf{F}\mathbf{Q}^\top; \mathbf{Q}^\top \mathbf{A}_* \mathbf{Q}) = \tilde{\psi}(\mathbf{C}, \mathbf{F}^\top \mathbf{A}\mathbf{F}; \mathbf{A}_*)\}. \quad (2.18)$$

(d) *Normalization*

Without loss of generality, we assume that the free-energy function is normalized so that it vanishes in the reference state, viz.,

$$\hat{\psi}(\mathbf{I}, \mathbf{A}_*; \mathbf{A}_*) = 0. \quad (2.19)$$

(e) *Governing equations*

Combining the deformational force balance (2.3) and the relation (2.10)₂ for the deformational stress \mathbf{S} , we obtain the partial differential equation

$$\text{Div} \left(\frac{\partial \hat{\psi}(\mathbf{F}, \mathbf{A}; \mathbf{A}_*)}{\partial \mathbf{F}} \right) + \mathbf{b} = \mathbf{F}^{-\top} \text{Grad} p. \quad (2.20)$$

Similarly, combining the conformational force balance (2.5) and the relation (2.10)₃ for the internal conformational force density \mathbf{II} , we obtain the partial differential equation

$$\frac{\partial \hat{\psi}(\mathbf{F}, \mathbf{A}; \mathbf{A}_*)}{\partial \mathbf{A}} = \mathbf{\Delta}. \quad (2.21)$$

A direct calculation shows that, granted the invariance postulate (2.14), the moment balance (2.7) holds.

The governing equations (2.20) and (2.21) can be viewed as dynamical generalizations of the Euler–Lagrange equations that would arise from variations of the energy functional

$$\mathcal{E}(\mathbf{y}, \mathbf{A}) = \int_{\mathcal{B}} \hat{\psi}(\mathbf{F}, \mathbf{A}; \mathbf{A}_*) dv + \phi(\mathbf{y}, \mathbf{A}), \quad (2.22)$$

with ϕ being a joint potential that generates the equilibrium contributions to the external force densities \mathbf{b} and $\mathbf{\Delta}$ and any deformational traction distributed over $\partial\mathcal{B}$. While varying \mathbf{y} subject to the constraint $\det \mathbf{F} = 1$ yields the equilibrium counterpart of (2.20), varying \mathbf{A} yields the equilibrium counterpart of (2.21).

Our dynamical theory allows for the description of important processes, such as stripe formation and evolution, which may occur away from equilibrium. Although our constitutive framework does not account for dissipation, it could be easily incorporated (Fried & Gurtin 1994; Fried 1996).

(f) *Soft states*

Of course, there needs to be agreement as to what constitutes soft elasticity, and we address this next.

Definition 2.1. For a given nematic elastomeric material point \mathbf{x} in \mathcal{B} (defined by prescription of the free-energy density function $\hat{\psi}$ at \mathbf{x}), a deformation gradient and conformation pair

$$(\mathbf{F}, \mathbf{A}) \text{ in } \text{Unim}^+ \times \text{Psym} \tag{2.23}$$

is a local soft state at \mathbf{x} if the corresponding free-energy density, active deformational stress, and internal conformational body force density (as defined by (2.10)₁, (2.12), and (2.10)₃) vanish at \mathbf{x} , i.e.,

$$\psi = 0, \quad \mathbf{S}_a = \mathbf{0}, \quad \text{and} \quad \mathbf{\Pi} = \mathbf{0} \quad \text{at} \quad \mathbf{x}. \tag{2.24}$$

Of course, the external force system that supports a soft state is not arbitrary, and, in view of (2.24)_{2,3}, an immediate consequence of (2.10)_{2,3}, (2.20), and (2.21) is

Theorem 2.1. For (\mathbf{F}, \mathbf{A}) in $\text{Unim}^+ \times \text{Psym}$ to be a local soft state at \mathbf{x} , it is necessary that, at \mathbf{x} , the external body force density be suitably related to the pressure gradient and the external conformational body force density vanish, i.e.,

$$\text{Grad} p = \mathbf{F}^\top \mathbf{b} \quad \text{and} \quad \mathbf{\Delta} = \mathbf{0} \quad \text{at} \quad \mathbf{x}. \tag{2.25}$$

The notion of a local soft state is readily extended to the whole body.

Definition 2.2. For a given nematic elastomeric body \mathcal{B} (defined by prescriptions of a free-energy density function $\hat{\psi}$ over \mathcal{B}), a pair of deformation gradient and conformation fields

$$\mathbf{F} : \mathcal{B} \rightarrow \text{Comp}, \quad \mathbf{A} : \mathcal{B} \rightarrow \text{Psym}, \tag{2.26}$$

with

$$\text{Comp} = \{ \mathbf{F} = \text{Grad} \mathbf{y} : \mathbf{y} \text{ an isochoric deformation of } \mathcal{B} \} \tag{2.27}$$

the set of compatible deformation gradients on \mathcal{B} , is a global soft state on \mathcal{B} if $(\mathbf{F}(\mathbf{x}), \mathbf{A}(\mathbf{x}))$ is a local soft state at each \mathbf{x} in \mathcal{B} .

Of course, it follows from Theorem 2.1 that, for (\mathbf{F}, \mathbf{A}) to be a global soft state on \mathcal{B} , it is necessary that (2.25) hold at each \mathbf{x} in \mathcal{B} .

3. The neo-classical free-energy density function

Following Warner & Terentjev (1996), we refer to a nematic elastomeric material as *neo-classical* if

$$\psi = \hat{\psi}(\mathbf{F}, \mathbf{A}; \mathbf{A}_*) = \frac{1}{2} \mu (|\mathbf{A}^{-\frac{1}{2}} \mathbf{F} \mathbf{A}_*^{\frac{1}{2}}|^2 - \log \det(\mathbf{A}^{-1} \mathbf{A}_*) - 3). \tag{3.1}$$

The expression on the right side of (3.1) was obtained by Warner, Gelling & Vilgis (1988) in their generalization of the molecular-statistical theory of rubber elasticity to account for the nematically-induced distortion of polymer molecules. It is, therefore, not surprising that, for $\mathbf{A} = \mathbf{A}_* = \mathbf{I}$, the right side of (3.1) specializes to the neo-Hookean free-energy function $\frac{1}{2}\mu(|\mathbf{F}|^2 - 3)$.

Direct calculations show that the neo-classical free-energy density function defined in (3.1) satisfies the invariance property (2.14) and that $\text{Matsym} = \text{Orth}^+$.

Two important properties of the neo-classical free-energy density function are expressed in

Theorem 3.1. *The neo-classical free-energy density function defined in (3.1) is bounded below by zero,*

$$0 \leq \hat{\psi}(\mathbf{F}, \mathbf{A}; \mathbf{A}_*) \quad \text{for all } (\mathbf{F}, \mathbf{A}) \quad \text{in } \text{Unim}^+ \times \text{Psym}, \quad (3.2)$$

and attains that bound for all (\mathbf{F}, \mathbf{A}) of the form[†]

$$(\mathbf{F}, \mathbf{A}) = (\mathbf{Q}, \mathbf{Q}\mathbf{A}_*\mathbf{Q}^\top) \quad (3.3)$$

with \mathbf{Q} an arbitrary element of Orth^+ .

Proof. Observe that (3.1) can be written alternatively in the form

$$\hat{\psi}(\mathbf{F}, \mathbf{A}; \mathbf{A}_*) = \frac{1}{2}\mu(\xi^{\frac{1}{3}}|\boldsymbol{\Phi}|^2 - \log \xi - 3), \quad (3.4)$$

with $\boldsymbol{\Phi} = (\det(\mathbf{A}^{-1}\mathbf{A}_*))^{-\frac{1}{6}}\mathbf{A}^{-\frac{1}{2}}\mathbf{F}\mathbf{A}_*^{\frac{1}{2}}$ and $\xi = \det(\mathbf{A}^{-1}\mathbf{A}_*)$. Since $\det \mathbf{F} = 1$, it follows that $\det \boldsymbol{\Phi} = 1$ and (Moriarty 1971) that $|\tilde{\mathbf{F}}|^2 \geq 3$. Thus, defining $f : (0, \infty) \rightarrow [0, \infty)$ by $f(\xi) = \frac{1}{2}\mu(3(\xi^{\frac{1}{3}} - 1) - \log \xi)$ and observing that $f(\xi) \geq f(0) = 0$ for all $\xi \in (0, \infty)$, it follows that

$$\hat{\psi}(\mathbf{F}, \mathbf{A}; \mathbf{A}_*) \geq f(\det(\mathbf{A}^{-1}\mathbf{A}_*)) \geq 0. \quad (3.5)$$

Further, a direct calculation shows that $\hat{\psi}(\mathbf{Q}, \mathbf{Q}\mathbf{A}_*\mathbf{Q}^\top; \mathbf{A}_*) = 0$ for any $\mathbf{Q} \in \text{Orth}^+$. \square

Thus, in view of Definition 2.1, the free-energy density is minimized at a point \mathbf{x} at which a neo-classical nematic elastomer attains a soft state.

4. Local soft states for neo-classical nematic elastomers

To the neo-classical free-energy function (3.1), there corresponds the active deformational stress

$$\mathbf{S}_a = \frac{\partial \hat{\psi}(\mathbf{F}, \mathbf{A}; \mathbf{A}_*)}{\partial \mathbf{F}} = \mu \left(\mathbf{A}^{-1} \mathbf{F} \mathbf{A}_* - \frac{(\mathbf{A}^{-1} \mathbf{F} \mathbf{A}_*) \cdot \mathbf{F}^{-\top}}{|\mathbf{F}^{-1}|^2} \mathbf{F}^{-\top} \right) \quad (4.1)$$

and the internal conformational body force density

$$\boldsymbol{\Pi} = -\frac{\partial \hat{\psi}(\mathbf{F}, \mathbf{A}; \mathbf{A}_*)}{\partial \mathbf{A}} = \frac{1}{2}\mu(\mathbf{A}^{-1} \mathbf{F} \mathbf{A}_* \mathbf{F}^\top - \mathbf{I}) \mathbf{A}^{-1}. \quad (4.2)$$

As a key step in the determination of all soft states in this case, we have

[†] Additional minimizers will be given in the next section.

Lemma 4.1. *Let a neo-classical nematic elastomer be given. If the internal conformational body force $\mathbf{\Pi}$ density vanishes at a particular material point \mathbf{x} , then there exist tensors $\mathbf{\Omega}$ in Orth^+ and $\mathbf{\Gamma}$ in Unim^+ such that at \mathbf{x} the deformation gradient \mathbf{F} and the conformation \mathbf{A} have the forms*

$$\mathbf{F} = \mathbf{A}^{\frac{1}{2}} \mathbf{\Omega} \mathbf{A}_*^{-\frac{1}{2}} = \mathbf{\Gamma} \mathbf{A}_*^{\frac{1}{2}} \mathbf{\Omega} \mathbf{A}_*^{-\frac{1}{2}} \quad (4.3)$$

and

$$\mathbf{A} = \mathbf{\Gamma} \mathbf{A}_* \mathbf{\Gamma}^\top = \mathbf{F} \mathbf{A}_* \mathbf{F}^\top, \quad (4.4)$$

respectively. Conversely, if, for any $\mathbf{\Omega}$ in Orth^+ and $\mathbf{\Gamma}$ in Unim^+ , \mathbf{F} and \mathbf{A} are given at \mathbf{x} by (4.3) and (4.4), then \mathbf{F} in Unim^+ , \mathbf{A} in Psym , and $\mathbf{\Pi}$ vanishes at \mathbf{x} ; furthermore, the active deformational stress \mathbf{S}_a and the free-energy density ψ also vanish at \mathbf{x} .

Proof. Suppose that $\mathbf{\Pi} = \mathbf{0}$. Then, it follows from (4.2) that $\mathbf{A}^{-1} \mathbf{F} \mathbf{A}_* \mathbf{F}^\top = \mathbf{I}$, which, on successive multiplications, implies that $\mathbf{A}^{-\frac{1}{2}} \mathbf{F} \mathbf{A}_*^{\frac{1}{2}} = (\mathbf{A}^{-\frac{1}{2}} \mathbf{F} \mathbf{A}_*^{\frac{1}{2}})^\top$. Thus,

$$\mathbf{A}^{-\frac{1}{2}} \mathbf{F} \mathbf{A}_*^{\frac{1}{2}} = \mathbf{\Omega} \in \text{Orth}^+. \quad (4.5)$$

By (4.5),

$$\mathbf{F} = \mathbf{A}^{\frac{1}{2}} \mathbf{\Omega} \mathbf{A}_*^{-\frac{1}{2}}, \quad (4.6)$$

which implies that $\det(\mathbf{A}^{\frac{1}{2}} \mathbf{A}_*^{-\frac{1}{2}}) = 1$, i.e.,

$$\mathbf{A}^{\frac{1}{2}} \mathbf{A}_*^{-\frac{1}{2}} = \mathbf{\Gamma} \in \text{Unim}^+. \quad (4.7)$$

By (4.7) and the symmetry of \mathbf{A} and \mathbf{A}_* ,

$$\mathbf{A}^{\frac{1}{2}} = \mathbf{A}_*^{\frac{1}{2}} \mathbf{\Gamma}^\top = \mathbf{\Gamma} \mathbf{A}_*^{\frac{1}{2}}. \quad (4.8)$$

On using both of (4.8) in $\mathbf{A} = \mathbf{A}^{\frac{1}{2}} \mathbf{A}_*^{\frac{1}{2}}$, we obtain

$$\mathbf{A} = \mathbf{\Gamma} \mathbf{A}_* \mathbf{\Gamma}^\top. \quad (4.9)$$

By (4.6) and (4.8)₂,

$$\mathbf{F} = \mathbf{\Gamma} \mathbf{A}_*^{\frac{1}{2}} \mathbf{\Omega} \mathbf{A}_*^{-\frac{1}{2}}. \quad (4.10)$$

Further, by (4.10), $\mathbf{\Gamma} = \mathbf{F} \mathbf{A}_*^{\frac{1}{2}} \mathbf{\Omega}^\top \mathbf{A}_*^{-\frac{1}{2}}$, which, on substitution into (4.9), yields

$$\mathbf{A} = \mathbf{F} \mathbf{A}_* \mathbf{F}^\top. \quad (4.11)$$

Equations (4.6), (4.10), (4.10), and (4.11) comprise the statements claimed in (4.3) and (4.4).

Turning next to the converse assertion, we consider the various possibilities for the pair (\mathbf{F}, \mathbf{A}) supplied by (4.3)–(4.4) for given $\mathbf{\Omega} \in \text{Orth}^+$ and $\mathbf{\Gamma} \in \text{Unim}^+$.

- (i) $\mathbf{F} = \mathbf{A}^{\frac{1}{2}} \boldsymbol{\Omega} \mathbf{A}_*^{-\frac{1}{2}}$ and $\mathbf{A} = \boldsymbol{\Gamma} \mathbf{A}_* \boldsymbol{\Gamma}^\top$: Easily, $\mathbf{A} \in \text{Psym}$. To see that $\mathbf{F} \in \text{Unim}^+$, we first note that $\det \mathbf{F} = (\det \mathbf{A}^{\frac{1}{2}})(\det \mathbf{A}_*^{\frac{1}{2}})^{-1}$ and then that $(\det \mathbf{A}^{\frac{1}{2}})^2 = \det \mathbf{A} = (\det \boldsymbol{\Gamma})(\det \mathbf{A}_*)(\det \boldsymbol{\Gamma}^\top) = \det \mathbf{A}_* = (\det \mathbf{A}_*^{\frac{1}{2}})^2$. Consequently, $\det \mathbf{A}^{\frac{1}{2}} = \det \mathbf{A}_*^{\frac{1}{2}}$, and $\det \mathbf{F} = 1$. Also, it follows from $\mathbf{F} = \mathbf{A}^{\frac{1}{2}} \boldsymbol{\Omega} \mathbf{A}_*^{-\frac{1}{2}}$ that $\mathbf{A}^{-1} \mathbf{F} \mathbf{A}_* = \mathbf{A}^{-\frac{1}{2}} \boldsymbol{\Omega} \mathbf{A}_*^{\frac{1}{2}}$ and, therefore,

$$\mathbf{A}^{-1} \mathbf{F} \mathbf{A}_* = \mathbf{F}^{-\top}. \quad (4.12)$$

By (4.12), $\mathbf{A}^{-1} \mathbf{F} \mathbf{A}_* \mathbf{F}^\top = \mathbf{I}$; and we see from (4.2) that $\mathbf{II} = \mathbf{0}$. By (4.12), $(\mathbf{A}^{-1} \mathbf{F} \mathbf{A}_*) \cdot \mathbf{F}^{-\top} = \mathbf{F}^{-\top} \cdot \mathbf{F}^{-\top} = |\mathbf{F}^{-1}|^2$. Using this result and (4.12) in (4.1), we find that $\mathbf{S}_a = \mathbf{0}$. From $\mathbf{F} = \mathbf{A}^{\frac{1}{2}} \boldsymbol{\Omega} \mathbf{A}_*^{-\frac{1}{2}}$, $\mathbf{A}^{-\frac{1}{2}} \mathbf{F} \mathbf{A}_*^{\frac{1}{2}} = \boldsymbol{\Omega}$; and, since $\boldsymbol{\Omega} \in \text{Orth}^+$, $|\mathbf{A}^{-\frac{1}{2}} \mathbf{F} \mathbf{A}_*^{\frac{1}{2}}|^2 = 3$. From $\mathbf{A} = \boldsymbol{\Gamma} \mathbf{A}_* \boldsymbol{\Gamma}^\top$, $\mathbf{A}^{-1} \mathbf{A}_* = \boldsymbol{\Gamma}^{-\top} \mathbf{A}_*^{-1} \boldsymbol{\Gamma}^{-1} \mathbf{A}_*$, which implies that $\det(\mathbf{A}^{-1} \mathbf{A}_*) = 1$. By these results and (3.1), $\psi = 0$.

- (ii) $\mathbf{F} = \mathbf{A}^{\frac{1}{2}} \boldsymbol{\Omega} \mathbf{A}_*^{-\frac{1}{2}}$ and $\mathbf{A} = \mathbf{F} \mathbf{A}_* \mathbf{F}^\top$: By essentially the same arguments as in Case (i), $\mathbf{F} \in \text{Unim}^+$, $\mathbf{A} \in \text{Psym}$, and $\mathbf{II} = \mathbf{0}$, $\mathbf{S}_a = \mathbf{0}$, and $\psi = 0$.
- (iii) $\mathbf{F} = \boldsymbol{\Gamma} \mathbf{A}_*^{\frac{1}{2}} \boldsymbol{\Omega} \mathbf{A}_*^{-\frac{1}{2}}$ and $\mathbf{A} = \boldsymbol{\Gamma} \mathbf{A}_* \boldsymbol{\Gamma}^\top$: As in the previous two cases, $\mathbf{A} \in \text{Psym}$; and, by direct calculation, $\mathbf{F} \in \text{Unim}^+$. Also it follows directly from these representations that $\mathbf{A}^{-1} \mathbf{F} \mathbf{A}_* \mathbf{F}^\top = \mathbf{I}$ and $\mathbf{A}^{-1} \mathbf{A} \mathbf{A}_* = \mathbf{F}^{-\top}$. Consequently, $\mathbf{II} = \mathbf{0}$ and $\mathbf{S}_a = \mathbf{0}$ by (4.2) and (4.1), respectively. From $\mathbf{F} = \boldsymbol{\Gamma} \mathbf{A}_*^{\frac{1}{2}} \boldsymbol{\Omega} \mathbf{A}_*^{-\frac{1}{2}}$, $|\mathbf{A}^{-\frac{1}{2}} \mathbf{F} \mathbf{A}_*^{\frac{1}{2}}|^2 = \text{tr}(\mathbf{A}^{-\frac{1}{2}} \boldsymbol{\Gamma} \mathbf{A}_* \boldsymbol{\Gamma}^\top \mathbf{A}^{-\frac{1}{2}}) = 3$ in view of $\mathbf{A} = \boldsymbol{\Gamma} \mathbf{A}_* \boldsymbol{\Gamma}^\top$. Also, exactly as in Case (i), it follows from $\mathbf{A} = \boldsymbol{\Gamma} \mathbf{A}_* \boldsymbol{\Gamma}^\top$ that $\det(\mathbf{A}^{-1} \mathbf{A}_*) = 1$; and we conclude from (3.1) that $\psi = 0$.
- (iv) $\mathbf{F} = \boldsymbol{\Gamma} \mathbf{A}_*^{\frac{1}{2}} \boldsymbol{\Omega} \mathbf{A}_*^{-\frac{1}{2}}$ and $\mathbf{A} = \mathbf{F} \mathbf{A}_* \mathbf{F}^\top$: By essentially the same arguments as in Case (iii), $\mathbf{F} \in \text{Unim}^+$, $\mathbf{A} \in \text{Psym}$, and $\mathbf{II} = \mathbf{0}$, $\mathbf{S}_a = \mathbf{0}$, and $\psi = 0$.

□

As an immediate consequence of Lemma 4.1, we have

Theorem 4.1. *Necessary and sufficient that a deformation gradient and conformation pair (\mathbf{F}, \mathbf{A}) belonging to $\text{Unim}^+ \times \text{Psym}$ be a local soft state at \mathbf{x} in \mathcal{B} for a given neo-classical nematic elastomer is that there exist tensors $\boldsymbol{\Omega}$ in Orth^+ and $\boldsymbol{\Gamma}$ in Unim^+ such that \mathbf{F} and \mathbf{A} are given by any pair of the equations in (4.3)–(4.4) with the referential conformation \mathbf{A}_* evaluated at \mathbf{x} .*

Thus, there is a “doubly” continuous spectrum (parametrized by the two tensors $\boldsymbol{\Omega}$ in Orth^+ and $\boldsymbol{\Gamma}$ in Unim^+) of local soft states possible in every neo-classical nematic elastomer. The nearest result to Theorem 4.1 that we have been able to find in the literature is Olmsted’s (1994) equation (3.6), a special case of our (4.3)₂ (obtained by restricting $\boldsymbol{\Gamma}$ to be in Orth^+). Olmsted asserts that it delivers a continuum of deformation gradients that leave the free-energy density of a neo-classical nematic elastomer invariant. Theorem 4.1 shows that this is indeed the case *provided* that these deformation gradients are paired with appropriate conformation tensors. Theorem 4.1 also makes it clear that Olmsted’s (1994) continuum of deformation gradients does not include all of those corresponding to soft states of a neo-classical nematic elastomer.

Some of the simpler possibilities contained in Theorem 4.1 are noted in the following three items, where it is understood that at a given material point \mathbf{x} a particular neo-classical nematic elastomeric material is defined by specification of $\mu(\mathbf{x}) > 0$ and $\mathbf{A}_*(\mathbf{x})$ in Psym in (3.1).

- With the choice $\mathbf{\Omega} = \mathbf{\Gamma} = \mathbf{I}$, (4.3)–(4.4) yield $\mathbf{F} = \mathbf{I}$ and $\mathbf{A} = \mathbf{A}_*$ and we thereby recover the fact that the reference state is locally soft at each material point.
- With the choice $\mathbf{\Omega}$ in Orth^+ and $\mathbf{\Gamma} = \mathbf{I}$, (4.3)–(4.4) imply $\mathbf{F} = \mathbf{A}_*^{\frac{1}{2}} \mathbf{\Omega} \mathbf{A}_*^{-\frac{1}{2}}$ and $\mathbf{A} = \mathbf{A}_*$. Thus, in the context of locally soft states, there exists a family of deformation gradients that induce no change in the conformation.
- With the choice $\mathbf{\Omega} = \mathbf{I}$ and $\mathbf{\Gamma}$ in Unim^+ , (4.3)–(4.4) imply $\mathbf{F} = \mathbf{\Gamma}$ and $\mathbf{A} = \mathbf{F} \mathbf{A}_* \mathbf{F}^\top$. Thus, locally, to any deformation gradient \mathbf{F} there corresponds a soft state with associated conformation $\mathbf{A} = \mathbf{F} \mathbf{A}_* \mathbf{F}^\top$. On the other hand, each equation in (4.4) implies that $\det \mathbf{A} = \det \mathbf{A}_*$. Thus, in the context of locally soft states, it is not possible to arbitrarily specify the current conformation \mathbf{A} in Psym .

In view of Theorem 3.1 and Theorem 4.1, we have

Theorem 4.2. *For a given neo-classical nematic elastomer, the free-energy density ψ at a material point \mathbf{x} in \mathcal{B} is minimized in a local soft state at \mathbf{x} .*

5. Global soft states for neo-classical nematic elastomers

The counterpart of Theorem 4.1 for global soft states is readily established.

Theorem 5.1. *Necessary and sufficient that a deformation gradient and conformation pair (\mathbf{F}, \mathbf{A}) in $\text{Comp} \times \text{Psym}$ be a global soft state for a given nematic elastomeric body \mathcal{B} is that there exist tensor fields*

$$\mathbf{\Omega} : \mathcal{B} \rightarrow \text{Orth}^+ \quad \text{and} \quad \mathbf{\Gamma} : \mathcal{B} \rightarrow \text{Psym} \quad (5.1)$$

such that \mathbf{F} and \mathbf{A} are given on \mathcal{B} by any pair of the equations in (4.3)–(4.4) with \mathbf{F} in Comp .

Proof. Suppose that (\mathbf{F}, \mathbf{A}) is a global soft state on \mathcal{B} . Then, necessarily $(\mathbf{F}(\mathbf{x}), \mathbf{A}(\mathbf{x}))$ is a local soft state at each $\mathbf{x} \in \mathcal{B}$, and, by Theorem 4.1, there exist tensor fields $\mathbf{\Omega} : \mathcal{B} \rightarrow \text{Orth}^+$ and $\mathbf{\Gamma} : \mathcal{B} \rightarrow \text{Psym}$ such that \mathbf{F} and \mathbf{A} are given on \mathcal{B} by any pair of the equations in (4.3)–(4.4). Moreover since \mathbf{F} is a deformation gradient, $\mathbf{\Omega}$ and $\mathbf{\Gamma}$ must be such that $\mathbf{F} \in \text{Comp}$.

Conversely, suppose that the fields \mathbf{F} and \mathbf{A} are given by any pair of the equations in (4.3)–(4.4) with $\mathbf{\Omega}$ and $\mathbf{\Gamma}$ such that $\mathbf{F} \in \text{Comp}$. Then, by Theorem 4.1, (\mathbf{F}, \mathbf{A}) is a global soft state on \mathcal{B} . \square

It follows from Theorem 5.1 that, to each of the local soft states noted after Theorem 4.1, there corresponds an analogous global soft state. We discuss these in the subsequent items.

- Setting $\boldsymbol{\Omega} = \boldsymbol{\Gamma} = \mathbf{I}$ on \mathcal{B} in (4.3)–(4.4) implies $\mathbf{F} = \mathbf{I}$, $\mathbf{A} = \mathbf{A}_*$ on \mathcal{B} . Thus, as expected, the reference state is globally soft.
- The choice $\boldsymbol{\Omega} : \mathcal{B} \rightarrow \text{Orth}^+$, $\boldsymbol{\Gamma} = \mathbf{I}$ on \mathcal{B} in (4.3)–(4.4) implies $\mathbf{F} = \mathbf{A}_*^{\frac{1}{2}} \boldsymbol{\Omega} \mathbf{A}_*^{-\frac{1}{2}}$ on \mathcal{B} . (Of course, $\boldsymbol{\Omega}$ must be such that \mathbf{F} is an element of Comp), $\mathbf{A} = \mathbf{A}_*$ on \mathcal{B} . Thus, for this class of global soft states, the conformation is unchanged.
- The choice $\boldsymbol{\Omega} = \mathbf{I}$, $\boldsymbol{\Gamma} \in \text{Comp}$ in (4.3)–(4.4) implies $\mathbf{F} = \boldsymbol{\Gamma}$, $\mathbf{A} = \mathbf{F} \mathbf{A}_* \mathbf{F}^\top$ on \mathcal{B} . Accordingly, to any isochoric deformation \mathbf{y} of \mathcal{B} whatsoever (and, hence, compatible deformation gradient $\mathbf{F} = \text{Grad} \mathbf{y}$) there corresponds the conformation $\mathbf{A} = \mathbf{F} \mathbf{A}_* \mathbf{F}^\top$ on \mathcal{B} such that the resulting state is globally soft. In other words, any deformation may be effected without force by inducing the appropriate conformation.
- Finally, we observe that, if the referential conformation \mathbf{A}_* is uniform, then the choice of uniform $\boldsymbol{\Omega}$ in Orth^+ and uniform $\boldsymbol{\Gamma}$ in Unim^+ leads via (4.3)–(4.4) to uniform (and, hence, compatible) $\mathbf{F} = \mathbf{A}_*^{\frac{1}{2}} \boldsymbol{\Omega} \mathbf{A}_*^{-\frac{1}{2}}$, and this state is globally soft.

6. Soft states in Mooney nematic elastomers

The neo-classical free-energy function defined in (3.1) is the counterpart for nematic elastomers of the neo-Hookean free-energy function for conventional rubber. Despite the molecular-statistical basis of the neo-Hookean model, it is well-understood that this model is not without its drawbacks. We anticipate the same for the neo-classical model of nematic rubber elasticity. For conventional rubber, the need for models which more closely match experimental observations led Mooney (1940) and Rivlin (1948) to propose phenomenologically-based constitutive relations giving the free-energy density as a function of the first and second principle invariants

$$I_1(\mathbf{B}) = \text{tr} \mathbf{B} \quad \text{and} \quad I_2(\mathbf{B}) = \frac{1}{2}(I_1^2(\mathbf{B}) - I_1(\mathbf{B}^2)) \quad (6.1)$$

of the left Cauchy–Green tensor $\mathbf{B} = \mathbf{F} \mathbf{F}^\top$. In our present context, the natural generalization of \mathbf{B} is simply

$$\tilde{\mathbf{B}} = \mathbf{A}^{-\frac{1}{2}} \mathbf{F} \mathbf{A}_*^{\frac{1}{2}} (\mathbf{A}^{-\frac{1}{2}} \mathbf{F} \mathbf{A}_*^{\frac{1}{2}})^\top = \mathbf{A}^{-\frac{1}{2}} \mathbf{F} \mathbf{A}_* \mathbf{F}^\top \mathbf{A}^{-\frac{1}{2}}. \quad (6.2)$$

Following the developments of Mooney (1940) and Rivlin (1948), we might, therefore, consider phenomenological descriptions of nematic elastomers with the form

$$\hat{\psi}(\mathbf{F}, \mathbf{A}; \mathbf{A}_*) = \hat{\Psi}(I_1(\tilde{\mathbf{B}}), I_2(\tilde{\mathbf{B}}), I_3(\tilde{\mathbf{B}})), \quad (6.3)$$

where, since \mathbf{F} belongs to Unim^+ ,

$$I_3(\tilde{\mathbf{B}}) = \det((\mathbf{A}^{-\frac{1}{2}} \mathbf{F} \mathbf{A}_*^{\frac{1}{2}})^2) = \det(\mathbf{A}^{-1} \mathbf{A}_*). \quad (6.4)$$

For all such materials, $\text{Matsym} = \text{Orth}^+$.

The earliest attempts to improve upon the predictions of the neo-Hookean model of rubber elasticity were based upon Mooney’s (1940) free-energy density function

$$\frac{1}{2} \mu (\alpha I_1(\mathbf{B}) + (1 - \alpha) I_2(\mathbf{B}) - 3), \quad (6.5)$$

the nematic elastomeric counterpart of which is

$$\hat{\Psi}(\mathbf{F}, \mathbf{A}; \mathbf{A}_*) = \frac{1}{2}\mu(\alpha I_1(\tilde{\mathbf{B}}) + (1 - \alpha)I_2(\tilde{\mathbf{B}}) - (2 - \alpha)\log I_3(\tilde{\mathbf{B}}) - 3), \quad (6.6)$$

with $0 \leq \alpha \leq 1$. The molecular-statistical basis for (6.5), given recently by Fried (2002), is readily extended to account for the nematic-induced distortion of polymer molecules and, thus, to yield (6.6). We refer to (6.6) as the *Mooney free-energy density function* and to a nematic elastomeric material as a *Mooney nematic elastomer* if its free-energy density is given by (6.6).

Analogous to Theorem 3.1, we have

Theorem 6.1. *The Mooney free-energy density function defined in (6.6) is bounded below by zero,*

$$0 \leq \hat{\Psi}(\mathbf{F}, \mathbf{A}; \mathbf{A}_*) \quad \text{for all } (\mathbf{F}, \mathbf{A}) \text{ in } \text{Unim}^+ \times \text{Psym}, \quad (6.7)$$

and attains that bound for all (\mathbf{F}, \mathbf{A}) of the form

$$(\mathbf{F}, \mathbf{A}) = (\mathbf{Q}, \mathbf{Q}\mathbf{A}_*\mathbf{Q}^\top) \quad (6.8)$$

with \mathbf{Q} an arbitrary element of Orth^+ .

Proof. Observe that (3.1) can be written alternatively in the form

$$\hat{\Psi}(\mathbf{F}, \mathbf{A}; \mathbf{A}_*) = \frac{1}{2}\mu(\alpha\xi^{\frac{1}{3}}I_1(\boldsymbol{\Phi}\boldsymbol{\Phi}^\top) + (1 - \alpha)\xi^{\frac{2}{3}}I_2(\boldsymbol{\Phi}\boldsymbol{\Phi}^\top) - (2 - \alpha)\log\xi - 3), \quad (6.9)$$

with $\boldsymbol{\Phi}$ and ξ as defined in the proof of Theorem 3.1. As before, since $\det \mathbf{F} = 1$, $\det \boldsymbol{\Phi} = 1$ and it follows (Moriarty 1971) that $I_1(\boldsymbol{\Phi}\boldsymbol{\Phi}^\top) \geq 3$ and that $I_2(\boldsymbol{\Phi}\boldsymbol{\Phi}^\top) \geq 3$. Thus, defining $f : (0, \infty) \rightarrow [0, \infty)$ by $f(\xi) = \frac{1}{2}\mu(3\alpha\xi^{\frac{1}{3}} + 3(1 - \alpha)\xi^{\frac{2}{3}} - (2 - \alpha)\log\xi - 3)$ and observing that, bearing in mind that $0 \leq \alpha \leq 1$, $f(\xi) \geq f(0) = 0$ for all $\xi \in (0, \infty)$, it follows that

$$\hat{\Psi}(\mathbf{F}, \mathbf{A}; \mathbf{A}_*) \geq f(\det(\mathbf{A}^{-1}\mathbf{A}_*)) \geq 0. \quad (6.10)$$

Further, a direct calculation shows that $\hat{\Psi}(\mathbf{Q}, \mathbf{Q}\mathbf{A}_*\mathbf{Q}^\top; \mathbf{A}_*) = 0$ for any $\mathbf{Q} \in \text{Orth}^+$. \square

Thus, as is the case for a neo-classical nematic elastomer, the free-energy density is minimized at a point \mathbf{x} at which a Mooney nematic elastomer attains a soft state.

The following theorem, which can be established in the same manner as was Theorem 4.1, delineates the full set of local soft states attainable by Mooney nematic elastomers.

Theorem 6.2. *Necessary and sufficient that a deformation gradient and conformation pair (\mathbf{F}, \mathbf{A}) belonging to $\text{Unim}^+ \times \text{Psym}$ be a local soft state at \mathbf{x} in \mathcal{B} for a given Mooney nematic elastomer is that there exist tensors $\boldsymbol{\Omega}$ in Orth^+ and $\boldsymbol{\Gamma}$ in Unim^+ such that \mathbf{F} and \mathbf{A} are given by any pair of the equations in (4.3)–(4.4) with the referential conformation \mathbf{A}_* evaluated at \mathbf{x} .*

Hence, the local and global soft states for a Mooney nematic elastomer are identical to those for a neo-classical nematic elastomer.

It might be worthwhile to determine the subclass of free-energy functions of the form (6.3) which give rise to the same set of local and global soft states as those arising for neo-classical and Mooney nematic elastomers.

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