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Lossless Linear Integer signal Resampling

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Abstract-This paper describes about signal resampling based on polynomial interpolation is reversible for all types of signals, i.e., the original signal can be reconstructed losslessly from the resampled data. This paper also discusses Matrix factorization method for reversible uniform shifted resampling and uniform scaled and shifted resampling. Generally, signal resampling is considered to be irreversible process except in some special cases because of strong attenuation of high frequency components. The matrix factorization method is actually a new way to compute linear transform. The factorization yields three elementary integer-reversible matrices. This method is actually a lossless integer-reversible implementation of linear transform. Some examples of lower order resampling solutions are also presented in this paper.

Keywords- ***polynomial interpolation, resampling, TERM matrix factorization, Stirling numbers.***

I.INTRODUCTION

Resampling is a statistical method that approximates a discrete set of points of a signal using another discrete set of known points of the signal. Signal resampling is used for different purposes in signal processing, like audio resampling and especially in image processing [3], [6], and [7]. Resampling is employed in image processing and computer graphics for better display, finer registration, less-distortion rotation, and more realistic texture mapping.

Signal resampling is implemented by two steps: interpolation of given discrete signal into continuous signal and then sampling the interpolated signal at new coordinates points. Signal format conversion, sample rate conversion, decimation, alignment, and warping are all basic operations that are involve in resampling.

Reversible resampling is process of estimating original discrete samples losslessly from the resampled data. In signal interpolation, a signal can be exactly reconstructed from its samples if the signal is band limited and the sampling rate is above the nyquist rate. But in general these conditions cannot always be satisfied for digitized signal and are not true for digital images.

This paper is organized as follows. The general resampling as a linear transform is discussed in Section II, a reversible shifted resampling is formulated in Section III, and a uniform scaled and shifted resampling is described in Section IV, error analysis is presented in Section V, Some examples in the form of experimental results are presented in Section VI, and finally paper concluded in section VII.

II. LOSSLESS RESAMPLING METHOD

Let the given discrete samples of a signal at n coordinate points $t_1, t_2, t_3, \ldots, t_n$ are $p_1, p_2, p_3, \ldots, p_n$, a resampling method is to find new samples at m coordinate points $x_1, x_2, x_3, \ldots, x_m$ be $p(x_1), p(x_2), p(x_3), \ldots p(x_m)$. The original samples are represented in the form of Vectors as $p_n=[p_1, p_2, p_3, \ldots, p_n,]^T$.

If a polynomial interpolation is employed for resampling, then the following function gives expression of the interpolated continuous signal from the n samples as shown below

$$p(t) = [1, t, t^2, t^3, \ldots\ldots, t^{n-1}].A.p_n \quad (1)$$

The interpolated signal is resampled at new coordinates and the new sampled data is obtained with the following expression

$$p(x_k) = [1, x_k, x_k^2, \ldots\ldots x_k^{n-1}].A.p_n \quad (2)$$

Where, $k=1,2,3,\ldots.m$.

$$\begin{bmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ \vdots \\ p(x_m) \end{bmatrix} = \begin{bmatrix} x_1^0 & x_1^1 & x_1^2 \ldots\ldots x_1^{n-1} \\ x_2^0 & x_2^1 & x_2^2 \ldots\ldots x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ x_m^0 & x_m^1 & x_m^2 \ldots\ldots x_m^{n-1} \end{bmatrix} . A. \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_n \end{bmatrix} \quad (3)$$

Let $V(x_1, x_2, x_3 \ldots., x_m)$ denote the *m*-by-*n* vandermonde matrix and is given by:

$$V(x_1, x_2, x_3 \ldots., x_m) = \begin{bmatrix} x_1^0 & x_1^1 & x_1^2 \ldots\ldots x_1^{n-1} \\ x_2^0 & x_2^1 & x_2^2 \ldots\ldots x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ x_m^0 & x_m^1 & x_m^2 \ldots\ldots x_m^{n-1} \end{bmatrix} \quad (4)$$

In short the (*i*,*j*)th element of the vandermonde matrix is given by

$$V_{i,j}(x, x_2, x_3 \ldots., x_m) = x_i^{j-1} \quad (5)$$

Where $i=1,2,3,\ldots,m$ and $j=1,2,3\ldots,n$

Matrix can be denoted as V(x)=V(1+*x*, 2+*x*, 3+*x*,…*m*+*x*) and $\bar{V} = V(0)$ for *m*=*n*. Subscriptions also used to represent a matrix e.g., $V_{m\times n}$

With (1) and $\bar{V}$, a matrix relation can be obtained as $\bar{V}$.A=I or A = $\bar{V}^{-1}$.

Form (3), (4) and (5) rewriting the resampling model as:

$$\begin{bmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ \vdots \\ p(x_m) \end{bmatrix} = V(x_1, x_2, x_3 \ldots., x_m) . \bar{V}^{-1}. p_n \quad (6)$$

$$R(x_1, x_2, x_3, \ldots, x_m) = V(x_1, x_2, x_3, \ldots, x_m). \bar{V}^{-1} \quad (7)$$

Where $R(x_1, x_2, x_3, \ldots, x_m)$ is the $m \times n$ transform matrix for the *m*-point resampling from *n*-point discrete signal. The resampling method based on the above assumptions is actually a linear transform. Based on *m* and *n* the following conditions may exist:

If *m*<*n*, the equation of the system is under-determined, and append *n*-*m* more rows to the matrix to make it invertible, which implies that resampled data is not sufficient for the recovery of the original samples.

If *m*=*n*, the resampling matrix is R is a square matrix. It is invertible if all the sampling points are different from each other, and it is integer reversible if $|\det(R)| = 1$ [2]. A lossless shifted resampling method for this case is presented in the next section.

If *m*>*n*, the resampling relation is over determined. Because the resampling matrix is a product of vandermonde matrix and inverse of another vandermonde matrix, if all the resampling points are different then only *n*-samples are used to recover the original signal. For this case the scaled and shifted resampling is presented in section IV.

III.LOSSLESS SHIFTED RESAMPLING MODEL

A simple shifted resampling model is to shift a digitized signal by a fractional delay and resample the signal at the same rate as the original signal, i.e., *m*=*n*, $x_k=k+x$, where *x* is a real number. Such a shifted resampling is formulated as to find *p(k+x)* from p_k for *k*=1, 2, 3, …,*n*, and reversibility can be achieved by recovering all the original samples p_k from *p(k+x)* by resampling with a shift of ' –*x*'.

Here the polynomial interpolation can be done as

$$[1,t+x,(t+x)^2,(t+x),^3\ldots\ldots(t+x)^{n-1}] = [1,t,t^2,t^3,\ldots\ldots t^{n-1}].T(x) \quad (8)$$

Where T(*x*) is given as:

$$T_{i,j}(x) = \begin{cases} 0 & (i > j) \\ \binom{j-1}{i-1} x^{j-i} & (i \le j) \end{cases} \quad \text{for } i, j = 1,2,3\ldots n \quad (9)$$

Here $\binom{m}{n}$ denote the binomial coefficient " *m* choose *n* " and T(*x*) is an upper triangular matrix as a function of *x*.

Based on (5) the shifted resampling problem can be formulated as

$$P(x) = V(x).\bar{V}^{-1}.p_n = R(x).p_n \qquad (10)$$

Where, $p(x)=[p(1+x),p(2+x),p(3+x),\ldots\ldots p(n+x)]^T$ and $R(s) = V(x).\bar{V}^{-1}$

But from (7) it is clear that

$$V(x)=\bar{V}.T(x) \qquad (11)$$

From (9) and (10) transform matrix is written as

$$R(x) = V(x).\bar{V}^{-1}=\bar{V}.T(x).\bar{V}^{-1} \qquad (12)$$

And the determinant of transform matrix is

$$\det(R(x)) = \det(\bar{V}).\det(T(x)).\det(\bar{V}^{-1})=1 \qquad (13)$$

A matrix can be factorized into three TERMs if the determinant of the matrix is absolutely 1. A TERM can directly transform an integer vector into another integer vector, and original vector can be recovered losslessly.

The simple factorization of three TERM as follows:

The LDU factorization of vandermonde matrix $\bar{V}$ is $\bar{V}$ = L.D.$\bar{U}$ and U(x) denote the combination of three upper triangular matrices and two diagonal matrices as

$$U(x)=D.\bar{U}.T(x).\bar{U}^{-1}.D^{-1} \qquad (14)$$

Then, the factorization of three TERM for (12) becomes

$$R(x) = L.D.\bar{U}.T(x).\bar{U}^{-1}.D^{-1}.L^{-1}$$

$$= L.U(x).L^{-1} \qquad (15)$$

Where L is unit lower triangular matrix and the matrices are presented by elements as

$$L_{i,j} = \begin{cases} 0 & (i < j) \\ \binom{i-1}{j-1} & (i \geq j) \end{cases} \qquad (16)$$

$$D_{i,j} = (i-1)!\,\delta_{i,j} \qquad (17)$$

$$U_{i,j} = \begin{cases} 0 & (i > j) \\ \frac{x^{\underline{j-i}}}{(j-i)!} & (i \leq j) \end{cases} \qquad (18)$$

Where $a^{\underline{m}} = (a-1).(a-2).\ldots(a-m+1)$ (19) represents falling polynomial factorial.

The upper triangular matrix $\bar{U}$ is made of the Stirling numbers of the second kind, " m subset n" $\left\{ {m \atop n} \right\}$, and its inverse $\bar{U}^{-1}$ is of the stirling numbers of the first kind, , " m cycle n" $\left[{m \atop n} \right]$. The ($i$, j)th elements are

$$\bar{U}_{i,j} = \begin{cases} 0 & (i > j) \\ \left\{ {j \atop i} \right\} & (i \leq j) \end{cases} \qquad (20)$$

$$\bar{U}^{-1}{}_{i,j} = \begin{cases} 0 & (i > j) \\ (-1)^{i+j} \left[{j \atop i} \right] & (i \leq j) \end{cases} \qquad (21)$$

Similar to the falling factorial polynomials, the rising factorials are defined as

$$a^{\overline{m}} = (a+1).(a+2).\ldots(a+m-1) \qquad (22)$$

Which is related to the falling factorials by a difference in sign as :

$$(-a)^{\overline{m}} = (-1)^m (a)^{\overline{m}} \qquad (23)$$

By using (20) and (21) a relation is obtained as

$$U^{-1}(x) = U(-x) \qquad (24)$$

Hence,

$$R^{-1}(x) = L.U^{-1}(x).L^{-1}$$
$$= L.U(-x).L^{-1} = R(-x) \qquad (25)$$

Which implies that the resampling with a shift of 'x'and re-resampling with a backward shift of '-x' restores the original signal losslessly. For integer reversible mapping, the transform with L^{-1} can be implemented by the reverse transform of L, so that the combined transform with $R(x)$ and $R(-x)$ can give the result exactly the same as the original if the integer-reversible implementation is employed [4]. The shifted resampling has applications in registration of images with only subpixel shift, rotation of images by shears [7].

IV.LOSSLESS SCALED AND SHIFTED RESAMPLING

Scaled resampling is nothing but the sampling rate conversion. Sample rate conversion has application in image registration, texture mapping and some other applications. For this type of applications uniform scaled and shifted resampling points must be used. For this case the resampling points are at $x_k = k.{}^{n}/_{m} + x$, where $m>n$. In such case resampling rate is higher than that of original input signal, and resampling matrix is of $m \times n$.

$$R_{m\times n}(x) = R_{m\times n}\left(\frac{n}{m}+x, \frac{2n}{m}+x, \frac{3n}{m}+s, \ldots \frac{mn}{m}+x\right)$$

$$= V_{m\times n}\left(\frac{n}{m}+x, \frac{2n}{m}+x, \frac{3n}{m}+x, \ldots . \frac{mn}{m}+x\right). \bar{V}^{-1}_{n\times n} \quad (26)$$

Using (10), the following relation exists,

$$V_{m\times n}\left(\frac{n}{m}+x, \frac{2n}{m}+x, \frac{3n}{m}+x, \ldots\ldots \frac{mn}{m}+x\right) = V_{m\times n}\left(\frac{n}{m}, \frac{2n}{m}, \frac{3n}{m}, \ldots\ldots \frac{mn}{m}\right). T_{n\times n}(x) \quad (27)$$

With the LDU factorization, $\bar{V}_{n\times n} = L_{n\times n}.D_{n\times n}.\bar{U}_{n\times n}$, and let denoting $M_{m\times n} = V_{m\times n}\left(\frac{n}{m}, \frac{2n}{m}, \frac{3n}{m}, \ldots\ldots \frac{mn}{m}\right). D^{-1}_{n\times n}$ and $Q_{n\times n}(x) = D_{n\times n}.T_{n\times n}(x).\bar{U}^{-1}_{n\times n}.D^{-1}_{n\times n}$, then the factorization of transform matrix is given as

$$R_{m\times n}\ (x) = M_{m\times n}.Q_{n\times n}(x).L^{-1}_{n\times n} \quad (28)$$

Where $M_{m\times n}$ is a constant matrix of full column rank, $Q_{n\times n}(x)$ is a unit upper triangular matrix with x as its independent variable and $L_{n\times n}$ is a unit lower triangular matrix as defined in (16).

The reverse can be done by using n samples for the inverse of $M_{m\times n}$ and then using n X-reversed samples for the inverse of $Q_{n\times n}(x)$ and $L^{-1}_{n\times n}$.

Equation (28) tells that uniform scaled and shifted resampling can also be integer reversible. Note that the factorization in (28) is not unique if $m>n+1$.

V. ERROR ANALYSIS

The difference between the original signal and resampled signal is defined as resampling error. It gives the information about how much the resampled signal drifts away from the original signal.

The approximated error for the shifted resampling case is defined as follows:

$$E_A = R(x).p_n - p_n = [R(x) - I].p_n$$

$$= L.[U((x) - I].L^{-1}.p_n \quad (29)$$

Equation (27) gives the absolute error.

The shifted resampling is formulated as a linear transform and transform matrix can be factorized into three matrices as given in (15). From the formulas of L and L^{-1}, but here all the components of both matrices are integers. Multiplication and addition of integers will produce another integer, but the rounding error of integer signal resampling only results from the rounding operations with done with$U(-x)$. For low degree cases, the error due to rounding operation is small.

The basic types of errors that are considering here are: i) Mean absolute error (Mean); ii) Root mean square error (RMSE); iii) maximum absolute error (Max). The computational formulas are

$$Mean = \frac{1}{n}\sum_k |p(k+x) - p_n| \quad (30)$$

$$RMSE = \sqrt{\frac{1}{n}\sum_k |p(k+x)| - p_k} \quad (31)$$

$$Max = \max_k |p(k+x) - p_k| \quad (32)$$

Here the errors such as mean square error, mean absolute error, and the maximum error can be measured between the input and reconstructed output. These results are compared with the traditional irreversible resampling method.

VI. EXPERIMENTAL RESULTS

In this section some low degree resampling examples are presented.

For **m=n=1**,$\boldsymbol{R(x) = L = U(x) = 1}$, here the shifted resampling involves nearest neighbor interpolation method.

For $m=n=2$, the shifted resampling matrix with its factorization is

$$\boldsymbol{R(x) = [R_1(x), R_2(x)] = V(x).\bar{V}^{-1}}$$

$$\boldsymbol{= L.U(x).L^{-1}}$$

Which said to be linear interpolated resampling, the equivalent basis function is shown in Fig. 1(a)

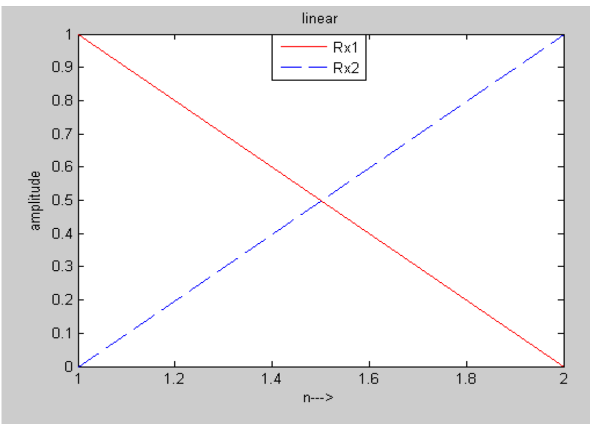
For ***m*=3** and ***n*=2**, here resampling rate is higher and the transformation matrix is

$$=\begin{bmatrix}1 & 1+x\\ 1 & 2+x\end{bmatrix}.\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}^{-1}$$

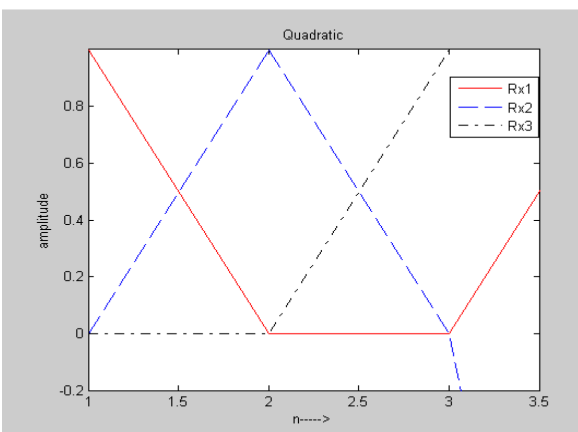
$$=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}.\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix}.\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}^{-1}$$

$$\boldsymbol{R}_{3\times 2}(\boldsymbol{x}) = \boldsymbol{V}_{3\times 2}(\boldsymbol{x}).\overline{\boldsymbol{V}}_{2\times 2}^{-1}$$

$$=\begin{bmatrix}1 & \frac{2}{3}+x\\ 1 & \frac{4}{3}+x\\ 1 & 2+x\end{bmatrix}.\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}^{-1}$$



(a). **Linear**



(b).Quadratic

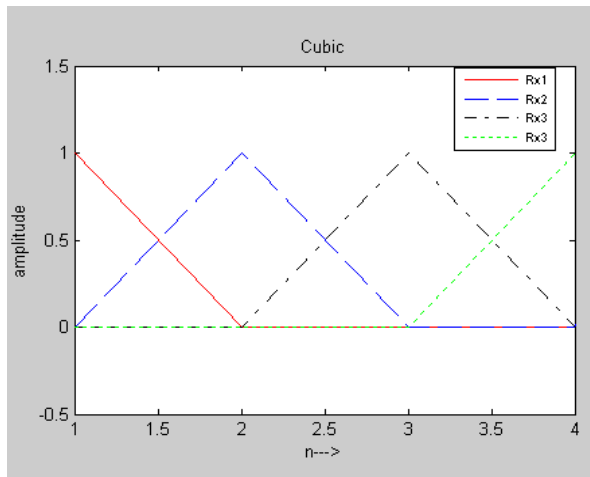
$$=\begin{bmatrix}1 & \frac{-1}{3}\\ 1 & \frac{1}{3}\\ 1 & 1\end{bmatrix}.\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix}.\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}^{-1}$$

$$= \boldsymbol{M}_{3\times 2}.\boldsymbol{Q}_{2\times 2}(\boldsymbol{x}).\boldsymbol{L}_{2\times 2}^{-1}$$

The resampling is done from two samples to three samples, and the original input can be reconstructed from the three samples by reverse computation.

For ***m*=*n*=3**, the shifted resampling matrix

$$\boldsymbol{R}(\boldsymbol{x}) = [\boldsymbol{R}_1(\boldsymbol{x}), \boldsymbol{R}_2(\boldsymbol{x}), \boldsymbol{R}_3(\boldsymbol{x})]$$



(c).Cubic

$$=\begin{bmatrix}1 & 1+x & (1+x)^2\\ 1 & 2+x & (2+x)^2\\ 1 & 3+x & (3+x)^2\end{bmatrix}.\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 2^2\\ 1 & 3 & 3^2\end{bmatrix}^{-1}$$

$$=\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 2 & 1\end{bmatrix}.\begin{bmatrix}1 & x & \frac{1}{2}x(x-1)\\ 0 & 1 & x\\ 0 & 0 & 1\end{bmatrix}.\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 2 & 1\end{bmatrix}^{-1}$$

Which is the quadratic polynomial and the basis functions are given in Fig. 1(b).

For higher order scaled and shifted resampling, if ***m*=4, *n*=3** resampling matrix is shown in next page.

For ***m*=*n*=4**, the shifted resampling matrix factors are determined as

$$\boldsymbol{L}=\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&2&1&0\\1&3&3&1\end{bmatrix}$$

$$\boldsymbol{U(x)}=\begin{bmatrix}1&x&\frac{1}{2}x(x-1)&\frac{1}{6}x(x-1)(x-2)\\0&1&x&\frac{1}{2}x(x-1)\\0&0&1&x\\0&0&0&1\end{bmatrix}$$

Which is the cubic polynomial and is shown in Fig.1(c).

It should be mentioned that interpolation basis functions for scaled and shifted resampling are same for the scaled resampling at same order, except that the scaling and the function segment length are different.

Fig.1 Basis functions for lossless resampling method

For experiments to test the method above basis functions are used to resample a randomly generated signal of 12 integers, {258, 81, 171, 240, 187, 216, 43, 101, 89, 56, 99, 9}, by a shift of -1/4. The resampling results with traditional B-spline interpolation methods are given in Fig.(2)a, and reconstructed signal by b-spline method is shown in Fig.(2)b, and the results with the lossless resampling method are illustrated in Fig.(3)a, and the reconstructed signal is shown in Fig(3)b for comparison. The input and resampled signal, resampled error and re-resampled error for the traditional B-spline interpolation and lossless resampling method results are listed in table-I for comparison.

TABLE-I

Shifted resampling and Re-Resampling errors

Interpolation method		**Resampled signal**												**Resampling Error**			**Re-Resampling Error**		
Input signal		258	81	171	240	187	216	43	101	89	56	99	9	Mean	RMSE	Max	Mean	RMSE	Max
B-spline interpolation method	Linear	191	135	198	207	174	140	67	92	74	60	52	4	24	39	76	23	27	29
	Quadratic	289	98	154	250	137	238	38	92	102	29	100	10	17	22	50	29	35	72
	Cubic	344	37	194	257	174	223	-1	116	86	48	110	-14	25	33	86	10	12	27
Lossless Resampling	Linear	317	140	148	217	177	206	24	82	100	68	129	39	25	30	59	0	0	0
	Quadratic	376	110	111	276	196	197	8	89	101	12	99	54	34	46	118	0	0	0
	Cubic	426	96	129	237	58	250	96	29	153	48	89	67	54	72	168	0	0	0

Mean: Mean absolute error; RMSE: root mean squared error; Max: Maximum absolute error

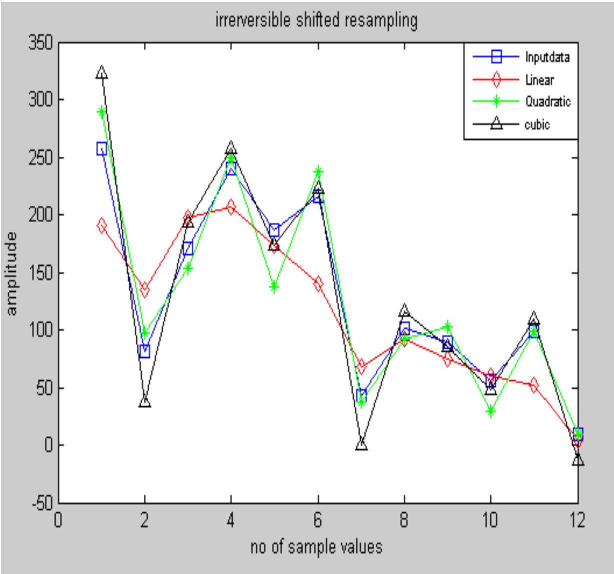


Fig.2.(a) Resampling by B-spline interpolation Method

the reconstructed signal is shown in Fig(3)b for comparison. The input and resampled signal, resampled error and re-resampled error for the traditional B-spline interpolation and lossless resampling method results are listed in table-I for comparison.

Similarly for scaled and shifted resampling, the i) Linear case is n=2, m=3 ii) Quadratic case is n=3, m=4 iii) Cubic case is n=4, m=6. The shift is considered is x=-1/4. The interpolation segment is given by $x_{km} = kn + 1 + x$.

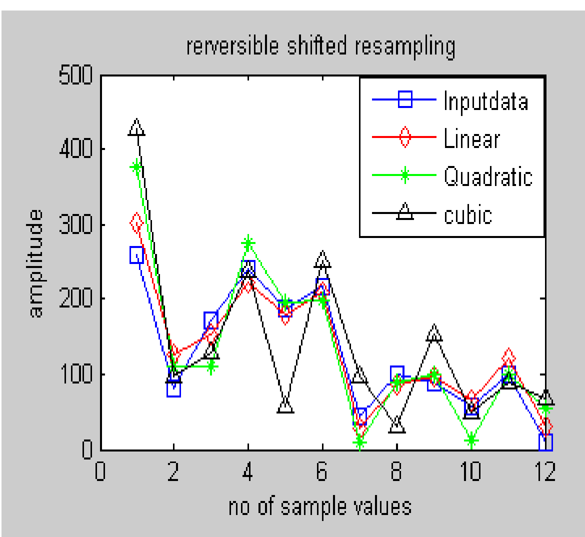


Fig.3(a). Shifted Resampling by lossless resampling method

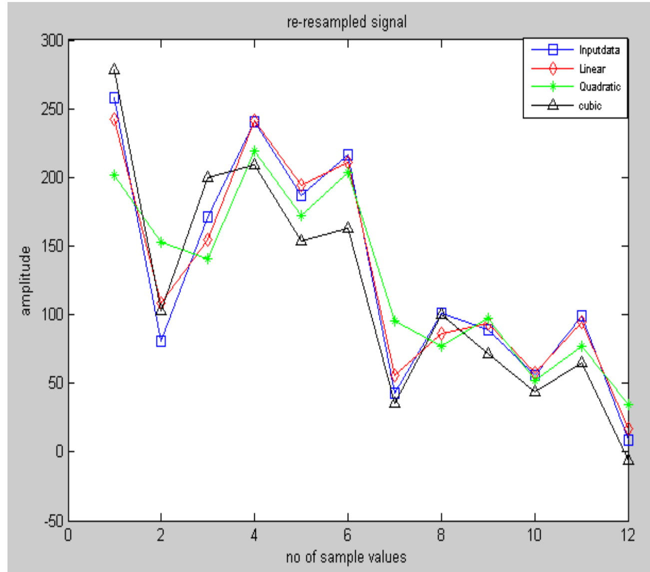


Fig.2.(b) Reconstructed signal by B-spline interpolation Method

The resampled signal is shown in Fig.4 (a), and Fig.4 (b) shows reconstructed signal which is same as the input signal. From the Fig.4 (b) it is clear that input signal and the reconstructed signal by the lossless resampling are identical i.e., the original is losslessly reconstructed from the resampled signal.

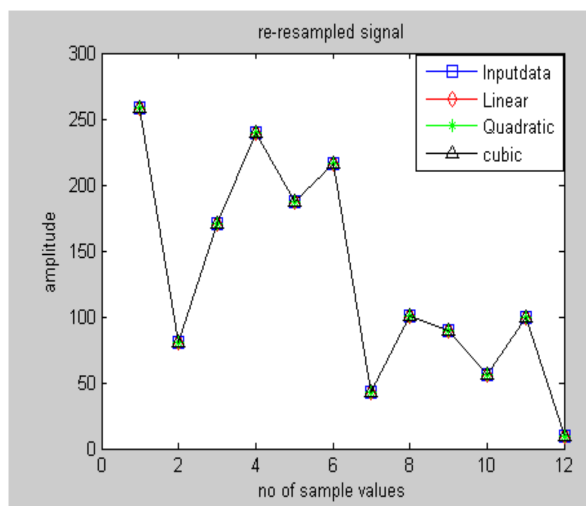


Fig3(b).Reconstructed signal from shifted resampled signal

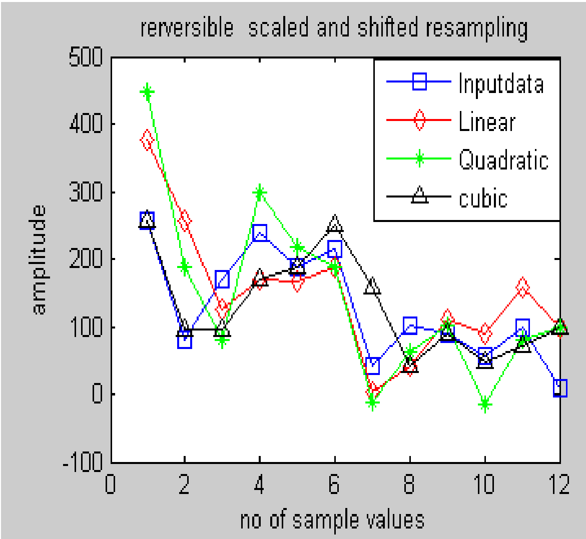


Fig.4.(a) Scaled and shifted resampling

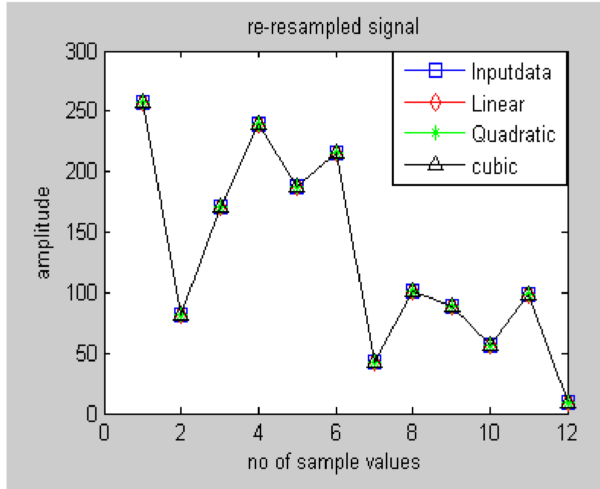


Fig.4 (b) Reconstructed signal from scaled and shifted resampling

VII. CONCLUSION

For this lossless resampling method, the error between original signal and reconstructed signal is negligible i.e., all the errors are zero hence, reconstructed signal is same as the input which is the significant characteristic of this method. This method is reversible yet there is need to accept some error which is caused by rounding operation. The interpolation method employed for resampling is reversible for any degree but there occurs Runge's phenomenon when using high degree polynomials instead of using piece-wise low degree interpolation for resampling.

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