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Theorems on Fixed Points for Asymptotically Regular Sequences and Maps in b-Metric Space

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ABSTRACT: In this paper, we present some fixed point theorems for asymptotically regular sequences and asymptotically regular maps in complete b- metric spaces. Our results extend and generalize the well-known fixed point theorems of Hardy- Roger [1] and Reich [2].

Keywords: b-Metric space; Fixed point; Asymptotically regular sequence; Asymptotically regular maps.

نظريات حول نقاط ثابتة لمتتالية عادية متقاربة وإسقاط في فضاء متري

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الملخص: نقدم في هذه الورقة بعض نظريات النقطة الثابتة لمتتاليات عادية متقاربة واسقاط عادي متقارب في فضاءات مترية متكاملة. تعتبر نتائجنا توسيع وتعميم لنظريات النقطة الثابتة المعروفة لكل من هاردي - روجر[2] وريخ [1].

الكلمات المفتاحية: الفضاء المتري، النقطة الثابتة، متتالية عادية متقاربة، إسقاط عادي متقارب.

1. Introduction

Metric fixed point theory was born with the well-known Banach contraction principle that was initially published in 1922. This principle states that on a complete metric space(X, d) , a self mapping T for which $d(Tx, Ty) \leq kd(x, y)$, for all $x, y \in X, 0 < k < 1$, has a unique fixed point. Several generalizations and extensions of this celebrated result have been appeared in the last few decades. The fixed point theorem in metric spaces plays a significant role to construct methods to solve the problems in mathematics and sciences. Metric fixed point theory is a vast field of study and is capable of solving many equations. To overcome the problem of measurable functions with respect to a measure and their convergence, [3] needed an extension of metric space. Using this idea, he introduced the concept of b-metric space and presented the contraction mapping in b-metric spaces that is generalization of the Banach contraction principle in metric spaces [4-7]. After that, several papers have dealt with fixed point theory or the variational principle for single-valued and multi-valued operators in b-metric spaces [8-14]. In this paper our aim is to show the validity of some fixed point theorems for asymptotically regular sequences. We also present results on fixed points of asymptotically regular mappings.

2. Preliminaries:

Consistent with [3] and [4,15], we recall some definitions and properties for b-metric space.

Definition 2.1. Let M be a nonempty set and the mapping $\rho: M \times M \to \mathbb{R}^+$ ($\mathbb{R}^+$ stands for nonnegative reals) satisfies:
(i) $\rho(x, y) = 0$ if and only if $x = y$ for all $x, y \in M$;
(ii) $\rho(x, y) = \rho(y, x)$ for all $x, y \in M$;
(iii) there exists a real number $s \geq 1$ such that $\rho(x, y) \leq s[\rho(x, z) + \rho(z, y)]$ for all $x, y, z \in M$.
Then ρ is called a b-metric on M and the pair (M, ρ) is called a b- metric space with coefficient s.

Remark 2.1. The class of b-metric spaces is larger than the class of metric spaces since any metric space is a b-metric space $s = 1$. Therefore, it is obvious that b-metric spaces generalizes metric spaces. We present an example which shows that introducing a b-metric space instead of a metric space is meaningful since there exists b-metric space instead of a metric space which are not metric spaces.

Example 2.1. Let $M = [0,\infty)$ and $\rho: M \times M \to \mathbb{R}^+$ defined by $\rho(x,y) = |x-y|^p$, where p is a real number such that $p > 1$. Let $x,y,z \in M$. By taking $u = x - z$ and $v = z - y$ we have

$$\begin{aligned}|x-y|^p = |u+v|^p &\leq (|u|+|v|)^p \\ &\leq (2\max\{|u|,|v|\})^p \\ &\leq 2^p(|x-z|^p + |z-y|^p),\end{aligned}$$

which implies that $\rho(x,y) \leq 2^p[\rho(x,z)+\rho(z,y)]$. Therefore (M,ρ) is a b-metric space with coefficient 2^p. On the other hand, for $x > z > y$, we have

$|x-y|^p = |u+v|^p = (u+v)^p > u^p + v^p = (x-z)^p + (z-y)^p = |x-z|^p + |z-y|^p$,

which implies that $\rho(x,y) > \rho(x,z) + \rho(z,y)$. Therefore (M,ρ) is not a metric space.

Definition 2.2. Let (M,ρ) be a b-metric space. Then $\{x_n\}$ in M is called

1. a Cauchy sequence if and only if for all $\epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$ such that for each $n.m \geq n(\epsilon)$, we have $\rho(x_n, x_m) < \epsilon$.
2. a convergent sequence if and only if there exists $x \in M$ such that for all $\epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$ such that for every $n \geq n(\epsilon)$, we have $\rho(x_n, x) < \epsilon$.

Definition 2.3 If (M,ρ) is a b-metric space then a subset $L \subset M$ is called

(i) compact if and only if for every sequence of elements of L there exists a subsequence that converges to an element of L.
(ii) closed if and only if for each sequence $\{x_n\}$ in L which converges to an element x, we have $x \in L$.
(iii) The b-metric space is complete if every Cauchy sequence in M converges in M.

Definition 2.4 [13] Let (M,ρ) be a b-metric space. A sequence $\{x_n\}$ in M is said to be asymptotically T-regular if

$$\lim_{n\to\infty} \rho(x_n, Tx_n) = 0.$$

Example 2.2. Let $M = [0,\infty)$ and $\rho: X \times X \to \mathbb{R}^+$ defined by $\rho(x,y) = |x-y|^p, p \geq 1$ then clearly (M,ρ) is a b-metric space with coefficient 2^p. Now let T be a self map of M such that $Tx = \frac{x}{2}$ and choose a sequence $\{x_n\}$, $x_n \neq 0$ for any positive integer n, which converges to zero in metric in M . We deduce that

$$\begin{aligned}\lim_{n\to\infty} \rho(x_n, Tx_n) &= \lim_{n\to\infty} |x_n - Tx_n|^p \\ &= \lim_{n\to\infty}\left|x_n - \frac{x_n}{2}\right|^p = \lim_{n\to\infty}\left|\frac{x_n}{2}\right|^p = 0.\end{aligned}$$

Hence $\{x_n\}$ is an asymptotically T-regular sequence in(M,ρ).

Definition 2.5 [17] Let (M,ρ) be a b-metric space. A mappings T of M into itself is said to be asymptotically regular at a point x in M if

$$\lim_{n\to\infty} \rho(T^n x, T^{n+1}x) = 0.$$

Example 2.3. Let (M,ρ) be a b-metric space as defined in Example 2.2 and let $T: M \to M$ be such that $Tx = \frac{x}{4}$ where $x \in M$. Then we have

$$\begin{aligned}\lim_{n\to\infty} \rho(T^n x, T^{n+1}x) &= \lim_{n\to\infty} |T^n x - T^{n+1}x|^p \\ &= \lim_{n\to\infty}\left|\frac{x}{4^n} - \frac{x}{4^{n+1}}\right|^p \\ &= \lim_{n\to\infty}\left|\frac{3x}{4^{n+1}}\right|^p = 0.\end{aligned}$$

Hence T is an asymptotically regular map at all points in M.

3. Main Results

Theorems 3.1. Let (M,ρ) be a complete b-metric space with the coefficient $s \geq 1$ and T be a self mapping of M satisfying the following inequality

$$\rho(Tx,Ty) \leq a_1\rho(x,Tx) + a_2\rho(y,Ty) + a_3\rho(x,Ty) + a_4\rho(y,Tx) + a_5\rho(x,y),$$

for all $x,y \in M$, where a_i ($i = 1,2,3,4,5$) are non-negative real numbers with $max\{(a_1 s + a_4 s^2), (a_3 s^3 + a_4 s^2 + a_5)\} < 1$ for $s \geq 1$. If there exists an asymptotically T-regular sequence in M, then T has a unique fixed point.

Proof. Let $\{x_n\}$ be an asymptotically T-regular sequence in M. Then for $n,m \in \mathbb{N}\ with\ m \geq n$, we have

$\rho(x_n, x_m) \leq s[\rho(x_n, Tx_n) + \rho(Tx_n, x_m)]$

$$\begin{aligned}
&\leq s[\rho(x_n, Tx_n) + s\rho(Tx_n, Tx_m) + s\rho(Tx_m, x_m)]\\
&\leq s\,\rho(x_n, Tx_n) + s^2\rho(Tx_n, Tx_m) + s^2\rho(Tx_m, x_m)\\
&\leq s\,\rho(x_n, Tx_n) + s^2\rho(Tx_m, x_m) + s^2[a_1\rho(x_n, Tx_n) + a_2\rho(x_m, Tx_m)\\
&\quad +a_3\,\rho(x_n, Tx_m) + a_4\rho(x_m, Tx_n) + a_5\rho(x_n, x_m)]\\
&\leq (s + a_1s^2 + a_4s^3)\rho(x_n, Tx_n) + (s^2 + a_2s^2 + a_3s^3)\rho(x_m, Tx_m)\\
&\quad +(a_3s^3 + a_4s^3 + a_5)\rho(x_n, x_m).
\end{aligned}$$

Thus, we get

$$\rho(x_n, x_m) \leq \left\{\frac{(s+a_1s^2+a_4s^3)}{1-(a_3s^3+a_4s^3+a_5)}\right\} \rho(x_n, Tx_n) + \left\{\frac{(s^2+a_2s^2+a_3s^3)}{1-(a_3s^3+a_4s^3+a_5)}\right\} \rho(x_m, Tx_m).$$

Taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \rho(x_n, x_m) = 0,$$

which shows that $\{x_n\}$ is a Cauchy sequence. Since M is a complete b-metric space, the sequence $\{x_n\}$ converges in M. So let $\lim_{n\to\infty} x_n = z$ for some $z \in M$.
Now we show that z is a fixed point of T.
Consider,

$$\begin{aligned}
\rho(Tz, z) &\leq s[\rho(Tz, Tx_n) + \rho(Tx_n, z)]\\
&\leq s\rho(Tz, Tx_n) + s^2\rho(Tx_n, x_n) + s^2\rho(x_n, z).\\
&\leq s[a_1\rho(z, Tz) + a_2\rho(x_n, Tx_n) + a_3\rho(z, Tx_n) + a_4\rho(x_n, Tz) + a_5\rho(z, x_n)]\\
&\quad + s^2\rho(Tx_n, x_n) + s^2\rho(x_n, z)\\
&\leq a_1s\rho(z, Tz) + a_2s\rho(x_n, Tx_n) + a_3s^2\rho(z, x_n)\\
&\quad +a_3s^2\rho(x_n, Tx_n) + s^2a_4\rho(x_n, z) + s^2a_4\rho(z, Tz)\\
&\quad +a_5s\rho(z, x_n) + s^2\rho(Tx_n, x_n) + s^2\rho(x_n, z).
\end{aligned}$$

Therefore,

$$\begin{aligned}
(1 - a_1s - s^2a_4)\rho(Tz, z) \leq\ &(s^2 + a_2s + a_3s^2)\rho(x_n, Tx_n)\\
&+(s^2 + a_3s^2 + a_4s^2 + a_5s)\rho(x_n, z),
\end{aligned}$$

which implies that

$$\rho(Tz, z) \leq \left\{\frac{(s^2+a_2s+a_3s^2)}{(1-a_1s-s^2a_4)}\right\} \rho(x_n, Tx_n) + \left\{\frac{(s^2+a_3s^2+a_4s^2+a_5s)}{(1-a_1s-s^2a_4)}\right\}\rho(x_n, z).$$

Since T is asymptotically T-regular, letting limit $n \to \infty$ we get $\rho(Tz, z) = 0$ $i.e. Tz = z$. Hence z is a fixed point of T.
Uniqueness: Let u be another fixed point such that $z \neq u$. Then,

$$\begin{aligned}
\rho(z, u) &= \rho(Tz, Tu)\\
&\leq a_1\rho(z, Tz) + a_2\rho(u, Tu) + a_3\rho(z, Tu) + a_4\rho(u, Tz) + a_5\rho(z, u).
\end{aligned}$$

From the last inequality, we have $(1 - a_3 - a_4 - a_5)\,\rho(z, u) = 0$. Since $a_3 + a_4 + a_5 < 1$, therefore $z = u$,.
Next, we discuss the problem of the existence of a fixed point of an operator without using any contractive condition. We shall first consider the situation in a metric space.

Theorem 3.2. Let (M, ρ) be a metric space, and T be a continuous self-mapping on M. If there exists an asymptotically T-regular sequence $\{x_n\}$ such that $\lim_{n\to\infty} x_n = z$, then z is a fixed point of T.

Proof. Let us consider the inequality

$$\begin{aligned}
\rho(Tz, z) &\leq [\rho(Tz, Tx_n) + \rho(Tx_n, z)]\\
&\leq \rho(Tz, Tx_n) + \rho(Tx_n, x_n) + \rho(x_n, z).
\end{aligned}$$

Taking the limit as $n \to \infty$, we obtain $\rho(Tz, z) \leq 0$ giving thereby $Tz = z$.
Now, let us see the similar situation in a b-metric space.

Theorem 3.3. Let (M, ρ) be a b-metric space with the coefficient $s \geq 1$, and T be a self-mapping of M. If there exists an asymptotically T-regular sequence $\{x_n\}$ with $\lim_{n\to\infty} x_n = z$, then z is not necessarily a fixed point.

Proof. Using the triangle inequality twice in a b-metric space, we obtain

$$\begin{aligned}
\rho(Tz, z) &\leq s[\rho(Tz, Tx_n) + \rho(Tx_n, z)]\\
&\leq s\,\rho(Tz, Tx_n) + s^2\rho(Tx_n, x_n) + s^2\rho(x_n, z).
\end{aligned}$$

Taking the limit as $n \to \infty$, we obtain

$$(1 - s^2)\rho(Tz, z) \leq 0\ ,$$

which is equivalent to the inequality

$$(s^2 - 1)\rho(Tz, z) \geq 0,$$

which provides no definite information. Thus z may or may not be a fixed point of T.

Remarks 3.1.

1. Even if T is continuous in Theorem 3.3, we cannot get any information from it about the existence of a/ the fixed point of T.
2. From Theorem 3.2 and Theorem 3.3, we notice yet another difference between the behavior of the two types of metrics defined on a set.

Theorem 3.4. Let (M,ρ) be a complete b-metric space with the coefficient $s \geq 1$ and T be a self mapping of M satisfying the inequality

$$\rho(Tx,Ty) \leq a_1\rho(x,Tx) + a_2\rho(y,Ty) + a_3\rho(x,Ty) + a_4\rho(y,Tx) + a_5\rho(x,y),$$

for all $x,y \in M$, where a_i ($i = 1,2,3,4,5$) are non-negative real numbers with $max\{(a_1 s + a_4 s^2),(a_3 + a_4 + a_5)\} < 1$ for $s \geq 1$. If T is asymptotically regular at some point $x_0 \in M$, then there exists a unique fixed point of T.

Proof. Let T be an asymptotically regular mapping at $x_0 \in M$. Considering the sequence $\{T^n x_0\}$ and $m,n \in \mathbb{N}$, we have

$$\begin{aligned}\rho(T^m x_0,T^n x_0) &\leq a_1\rho(T^{m-1}x_0,T^m x_0) + a_2\rho(T^{n-1}x_0,T^n x_0) + a_3\rho(T^{m-1}x_0,T^n x_0)\\ &\quad +a_4\rho(T^{n-1}x_0,T^m x_0) + a_5\rho(T^{m-1}x_0,T^{n-1}x_0)\\ &\leq a_1\rho(T^{m-1}x_0,T^m x_0) + a_2\rho(T^{n-1}x_0,T^n x_0)\\ &\quad +a_3 s\rho(T^{m-1}x_0,T^m x_0) + a_3 s\rho(T^m x_0,T^n x_0)\\ &\quad +a_4 s\rho(T^{n-1}x_0,T^n x_0) + +a_4\rho s(T^n x_0,T^m x_0)\\ &\quad +a_5 s\rho(T^{m-1}x_0,T^m x_0) + a_5 s^2\rho(T^m x_0,T^n x_0) + a_5 s^2\rho(T^n x_0,T^{n-1}x_0),\end{aligned}$$

which implies that

$$\rho(T^m x_0,T^n x_0) \leq \left\{\frac{a_1+a_3 s+a_5 s}{1-(a_3 s+a_4 s+a_5 s^2)}\right\}\rho(T^{m-1}x_0,T^m x_0) + \left\{\frac{a_2+a_4 s+a_5 s^2}{1-(a_3 s+a_4 s+a_5 s^2)}\right\}\rho(T^{n-1}x_0,T^n x_0)\ .$$

Since T is asymptotically regular, as $m,n \to \infty$, the above inequality yields $\lim_{m,n\to\infty}\rho(T^m x_0,T^n x_0) = 0$. This shows that $\{T^n x_0\}$ is a Cauchy sequence. Since M is complete, $\lim_{n\to\infty} T^n x_0 = 0 = z$ for some $z \in M$.

Next we will show that z is a fixed point of T.

Consider,

$$\begin{aligned}\rho(Tz,z) &\leq s\rho(Tz,T^n x_0) + s\rho(T^n x_0,z)\\ &\leq a_1 s\rho(z,Tz) + a_2 s\rho(T^{n-1}x_0,T^n x_0) + a_3 s\rho(z,T^n x_0)\\ &\quad +a_4 s\rho(T^{n-1}x_0,Tz) + a_5 s\rho(z,T^{n-1}x_0) + s\rho(T^n x_0,z)\\ &\leq a_1 s\rho(z,Tz) + a_2 s\rho(T^{n-1}x_0,T^n x_0) + a_3 s\rho(z,T^n x_0)\\ &\quad +a_4 s^2\rho(T^{n-1}x_0,T^n x_0) + a_4 s^2\rho(T^n x_0,Tz)\\ &\quad +a_5 s^2\rho(z,T^n x_0) + a_5 s^2\rho(T^n x_0,T^{n-1}x_0) + s\rho(T^n x_0,z).\end{aligned}$$

Letting the limit be $n \to \infty$ and since $\{T^{n-1}x_0\}$ is a subsequence of $\{T^n x_0\}$, we obtain

$$\rho(Tz,z) \leq a_1 s\rho(z,Tz) + a_4 s^2\rho(z,Tz)$$

which gives $\rho(Tz,z) = 0$ i.e. $Tz = z$. Hence z is a fixed point of T. The uniqueness of the fixed point follows from Theorem 3.1.

Theorem 3.5. Let (M,ρ) be a complete b-metric space with coefficient $s \geq 1$ and T be a self mapping of M satisfying the inequality

$$\rho(Tx,Ty) \leq a_1\rho(x,Tx) + a_2\rho(y,Ty) + a_3\rho(x,Ty) + a_4\rho(y,Tx) + a_5\rho(x,y),$$

for all $x,y \in M$, where a_i ($i = 1,2,3,4,5$) are non-negative real numbers with $max\{(a_2 + a_3)s^2,(a_3 + a_4 s + a_5 s)\} < 1$ for $s \geq 1$. If T is asymptotically regular at some point $x \in M$ and the sequence $\{T^n x\}$ of iterates has a subsequence $\{T^{n_k}x\}$ converging to a point z of M, then z is a unique fixed point of T and $\{T^n x\}$ also converges to z.

Proof. Let $\lim_{k\to\infty} T^{n_k}x = z$, then

$$\begin{aligned}\rho(z,Tz) &\leq s\rho(z,T^{n_k}x) + s^2\rho(T^{n_k}x,T^{n_k+1}x) + s^2\rho(T^{n_k+1}x,Tz)\\ &\leq s\rho(z,T^{n_k}x) + s^2\rho(T^{n_k}x,T^{n_k+1}x) + a_1 s^2\rho(T^{n_k}x,T^{n_k+1}x) + a_2 s^2\rho(z,Tz)\\ &\quad +a_3 s^2\rho(T^{n_k}x,Tz) + a_4 s^2\rho(z,T^{n_k+1}x) + a_5 s^2\rho(T^{n_k}x,z).\end{aligned}$$

Letting $k \to \infty$, we get

$$\rho(z,Tz) \leq (a_2 s^2 + a_3 s^2)\rho(z,Tz)$$

which gives z is a fixed point of T.

Now,

$$\begin{aligned}\rho(z,T^n x) &= \rho(Tz,T^n x)\\ &\leq a_1\rho(z,Tz) + a_2\rho(T^{n-1}x,T^n x) + a_3\rho(z,T^n x) + a_4\rho(T^{n-1}x,Tz) + a_5\rho(z,T^{n-1}x)\\ &\leq a_1\rho(z,Tz) + a_2\rho(T^{n-1}x,T^n x) + a_3\rho(z,T^n x)\\ &\quad +a_4 s\rho(T^{n-1}x,T^n x) + a_4 s\rho(T^n x,Tz)\\ &\quad +a_5 s\rho(z,T^n x) + a_5 s\rho(T^n x,T^{n-1}x)\end{aligned}$$

which implies that

$$(1 - a_3 - a_4 s - a_5 s)\rho(z,T^n x) \leq (a_2 + a_4 s + a_5 s)\rho(T^{n-1}x,T^n x).$$

Since T is asymptotically regular at $x \in M$ and using the fact that $max\{(a_2 + a_3)s^2, (a_3 + a_4 s + a_5 s)\} < 1$ for $s \geq 1$ implies that the sequence $\{T^n x\}$ converges to z in M. This completes the proof.

Example 3.1. Let $M = \mathbb{R}$ and $\rho: M \times M \to \mathbb{R}^+$ be defined by $\rho(x, y) = |x - y|^p$, where $p > 1$. Then (M, ρ) is a b-metric space. Define a self map T on M as follows $Tx = \frac{x}{2}$ for all $x \in M$. Clearly T is asymptotically regular for all $x \in M$. If we take $a_1 = a_2 = a_3 = a_4 = 0$ and $a_5 = \frac{1}{2^p}$, then the contractive condition used here holds and 0 is the unique fixed point of T.

Remark 3.2. The asymptotic regularity of the mapping T satisfies the Hardy-Roger's contraction condition. It is actually a consequence of $\sum_{i=1}^{5} a_i < 1$. Thus Theorem 3.3 and Theorem 3.4 extend results due to Hardy-Roger [1] in b-metric space. It is also worth mentioning that our condition on control constants says that $\sum_{i=1}^{5} a_i$ may exceed 1.

4. Conclusion

It has been demonstrated that one can use asymptotically regular sequences rather than sequences of iterates to obtain interesting results related to fixed point theorems in *b*-metric spaces.

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