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Error Analysis of an Explicit Finite Difference Approximation for the Space Fractional Wave Equations

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ABSTRACT: In this paper, the space fractional wave equation (SFWE) is numerically studied, where the fractional derivative is defined in the sense of Caputo. An explicit finite difference approximation (EFDA) for SFWE is presented. The stability and the error analysis of the EFDA are discussed. To demonstrate the effectiveness of the approximated method, some test examples are presented.

KEYWORDS: Fractional order wave equation, Caputo's derivative, Stability condition, Stability matrix analysis.

تحليل خطأ تقريب الفروق المحدودة الظاهرة عند حل المعادلات الموجية في فضاء الرتب الكسرية

ناصر حسن سويلم و تغريد عبدالرحمن عسيري

ملخص: لقد تم في هذا البحث دراسة عددية لمعادلات موجية في فضاء الرتب الكسرية، حيث تم تعريف المشتقة من الرتب الكسرية باستخدام تعريف كابوتو. وتم استخدام الفروق المحدودة الظاهرة لإيجاد حلول تقريبية للمعادلة الموجية. كما تم مناقشة الاستقرار وتحليل الخطأ لتلك الفروق. لتوضيح فعالية هذه الطريقة التقريبية، فقد تم عرض بعض الأمثلة الاختبارية.

1. Introduction

Fractional derivatives in mathematics are natural extension of integer-order derivatives, where the order is non integer. Fractional order differential equations have been the focus of many studies due to their frequent appearance in various applications especially in the fields of fluid mechanics, viscoelasticity, biology, physics and engineering (Bagley and Torvik, 1984; Mainardi, 1995; Mainardi and Paradisi, 1997; Podlubny, 1999; 2002). Consequently, considerable attention has been given to the solutions of fractional ordinary/partial differential equations (Sweilam *et al*., 2011). Numerical approximations are the main tool to simulate and study the behaviour of the solutions of such model problems (Fix and Roop, 2004; Meerschaert and Tadjeran, 2004; Sweilam *et al*., 2007; Sweilam and Khader, 2010; Tenreiro Machado, 2003; Yuste, 2011; Yuste and Acedo, 2005). Difference methods and, in particular, explicit finite difference methods, are an important class of numerical methods for solving fractional differential equations (Morton and Mayers, 1994; West and Seshadri, 1982; Xu *et al*., 2001). The usefulness of the explicit method and the reason why they are widely employed is based on their particularly attractive features (Yuste, 2011; Yuste and Acedo, 2005).

In this paper, an EFDA scheme is designed for solving a fractional order wave equation where the fractional derivative is in the Caputo sense. Moreover, since the explicit methods may be unstable, then, it is crucial to determine under which conditions, if any, these methods are stable. We will use here a kind of fractional von Neumann stability analysis to derive the stability conditions. We consider in this paper the following SFWE model:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = d(x,t)\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}, \quad 0 < x < L, \quad t \geq 0, \quad 1 < \alpha \leq 2, \tag{1}$$

$$u(x,0)=f_1(x), \quad u_t(x,0)=f_2(x), \quad 0 \leq x \leq L, \tag{2}$$

$$u(0,t)=\phi(t), \quad u(L,t)=\psi(t), \tag{3}$$

where the variable coefficient $d(x, t) > 0.$ The parameter α refers to the fractional order of spatial derivatives, and the Caputo's fractional derivative $D_x^\alpha u(x),$ is defined as follows (Podlubny, 1999).

$$\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = D_x^\alpha u(x,t) = \begin{cases} \dfrac{d^m u(x,t)}{dx^m}, & \alpha = m \in N, \\ \dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^x (x-\xi)^{m-\alpha-1}\frac{\partial^m u(\xi,t)}{\partial \xi^m}d\xi, & m-1 < \alpha < m, \end{cases} \tag{4}$$

where $\Gamma(\,.\,)$ is the gamma function.

2. Explicit finite difference approximation for SFWE

Let us consider $h = L/K,$ where K is a positive integer, by using a second order difference approximation and (4), we get for $m = 2$ that

$$\begin{aligned} D_x^\alpha u(x,t) &= \frac{1}{\Gamma(2-\alpha)}\int_0^x (x-\xi)^{1-\alpha}\frac{\partial^2 u(\xi,t)}{\partial \xi^2}d\xi \\ &= \frac{1}{\Gamma(2-\alpha)}\sum_{j=0}^{k-1}\int_{jh}^{(j+1)h} z^{1-\alpha}\frac{\partial^2 u(x-z,t)}{\partial z^2}dz \\ &\approx \frac{1}{\Gamma(2-\alpha)}\sum_{j=0}^{k-1}\frac{u(x-(j-1)h,t)-2u(x-jh,t)+u(x-(j+1)h,t)}{h^2}\times\int_{jh}^{(j+1)h} z^{1-\alpha}dz \\ &\approx \frac{h^{-\alpha}}{\Gamma(3-\alpha)}\sum_{j=0}^{k-1}\left[u(x-(j-1)h,t)-2u(x-jh,t)+u(x-(j+1)h,t)\right]\times\left[(j+1)^{2-\alpha}-j^{2-\alpha}\right]. \end{aligned}$$

Let $\Delta t = \tau > 0$ be the grid step in time, $t_n = n\tau,\ 0 \leq t_n \leq T,\ n = 0,\cdots,N-1,\ N = T/\tau$ and $\Delta x = h > 0$ be the grid step in space, $x_k = kh,\ 0 \leq x_k \leq L,\ k = 1,\cdots,K-1,$ so that $u_k^n = u(kh, n\tau)$ and $d_k = d(x_k, 0).$ Applying the forward finite difference formula to the initial conditions (2), we obtain

$$u_k^0 = f_1(x_k), \quad u_k^1 = f_1(x_k)\tau + f_2(x_k).$$

Now the discrete form of (1) using the explicit finite difference scheme can be written as

$$\frac{u_k^{n+1} - 2u_k^n + u_k^{n-1}}{\tau^2} = \frac{d_k h^{-\alpha}}{\Gamma(3-\alpha)} \sum_{j=0}^{k-1} [u_{k-j+1}^n - 2u_{k-j}^n + u_{k-j-1}^n][(j+1)^{2-\alpha} - j^{2-\alpha}],$$

and

$$u_k^{n+1} = 2u_k^n - u_k^{n-1} + \bar{s}[u_{k-1}^n - 2u_k^n + u_{k+1}^n] + \bar{s}\sum_{j=1}^{k-1} g_j [u_{k-j+1}^n - 2u_{k-j}^n + u_{k-j-1}^n], \tag{5}$$

where

$$\bar{s} = \frac{d_k s}{\Gamma(3-\alpha)}, \; s = \frac{\tau^2}{h^\alpha}, \; g_j = (j+1)^{2-\alpha} - j^{2-\alpha}.$$

The general form of (5) with initial conditions, can take the following form

$$U^1 = \tau U^0 + f_2(x_k), \qquad U^{n+1} = AU^n - U^{n-1}, \tag{6}$$

where $U^n = \left(u_1^n, u_2^n, \cdots, u_{k-1}^n\right)^T$ and A is the coefficients matrix with elements a_{ij} obtained from (5).

3. Stability analysis of EFDA

It is well known that the explicit difference schemes are not always stable for integer order differential equations. Then, for any α, there are always choices of Δt and Δx for which the numerical schemes may become unstable. Therefore, it is important to determine under which conditions, if any, the explicit method presented here is stable. To analyze the stability of the numerical scheme (6), we will use here a kind of fractional von Neumann stability analysis.

Theorem 1 The explicit finite-difference scheme (6) for SFWE is conditionally stable if

$$s \le s_x \equiv \frac{2^{1-\alpha}\left(2^\alpha - 2^3\right)\zeta(\alpha-2)}{\left(\Gamma(3-\alpha)\right)^{-1}}.$$

Proof. Let us analyze the stability of (5) by substituting in a separated solution $U_j^n = \zeta_n e^{iqj\Delta x}$ where q is a real spatial wave-number. Inserting this expression we get

$$\zeta_{n+1} = 2\zeta_n - \zeta_{n-1} + \bar{s}\,[e^{iq\Delta x} - 2 + e^{-iq\Delta x}]\zeta_{n+1} + \bar{s}\sum_{j=1}^{k} g_j [\zeta_{n+1-j} - 2\zeta_{n-j} + \zeta_{n-1-j}],$$

where $\zeta(x)$ means the Riemann zeta function. The stability will be determined by the behaviour of ζ_n. If we write $\zeta_{n+1} = \xi\zeta_n$ and assume that $\xi \equiv \xi(q)$ is independent of time, then we can obtain

$$\xi - 2 + \xi^{-1} - \bar{s}\sum_{j=1}^{k} g_j (\xi^{1-j} - 2\xi^{-j} + \xi^{-1-j}) = \bar{s}\,(e^{iq\Delta x} - 2 + e^{-iq\Delta x}).$$

Inserting the extrema value $\xi = -1$ into this equation, we obtain the following stability bound on s:

$$\bar{s}\sin^2\left(\frac{q\Delta x}{2}\right) \le \bar{s}_x^{n} = 1 + \sum_{j=1}^{n} (-1)^j \left[(j+1)^{2-\alpha} - j^{2-\alpha}\right],$$

with

$$\bar{s}_x = \sum_{j=1}^{\infty} (-1)^j \left[(j+1)^{2-\alpha} - j^{2-\alpha}\right],$$

or, in terms of the Riemann zeta function

$$\bar{s}_x = 2\left(1-2^{3-\alpha}\right)\zeta(\alpha-2).$$

Then, the method is stable when

$$s \le s_x \equiv \frac{2^{1-\alpha}\left(2^{\alpha}-2^{3}\right)\zeta(\alpha-2)}{\left(\Gamma(3-\alpha)\right)^{-1}}. \qquad \square$$

Table 1. The exact and EFDA solutions at $t = 0.05$ when $h = 0.005$, $\tau = 0.0025$.

x_i	$\alpha = 2$	$\alpha = 1.8$
0.0000	0.00000000	0.00000000
0.0500	0.43696211	0.49019405
0.1000	0.83115133	0.92270888
0.1500	1.14398166	1.25716135
0.2000	1.34483109	1.46300554
0.2500	1.41403909	1.52125246
0.3000	1.34483109	1.42692326
0.3500	1.14398166	1.18975019
0.4000	0.83115133	0.83331667
0.4500	0.46696211	0.39279598
0.5000	0.00000000	-0.08846503
0.5500	-0.43696211	-0.56317267
0.6000	-0.83115133	-0.98470522
0.6500	-1.14398166	-1.31166940
0.7000	-1.34483109	-1.51194717
0.7500	-1.41403909	-1.56583600
0.8000	-1.34483109	-1.46797473
0.8500	-1.14398166	-1.22786629
0.9000	-0.83115133	-0.86894593
0.9500	-0.43696211	-0.43396401
1.000	-0.00000000	0.00000000

Theorem 2 The truncation error of SFWE is $T(x,t) = O(\Delta t)^2 + O(\Delta x)$.

Proof. Evaluating (1) at the point (x_k, t_n) gives $\left[\frac{\partial^2 u}{\partial t^2} - d\frac{\partial^\alpha u}{\partial x^\alpha}\right]_{(x_k,\, t_n)} = 0$,

by the difference equation

$$\Delta_t^2 u_k^n - d\,\Delta_x u_{k+1}^n = T(x_k, t_n). \qquad (7)$$

Neglecting the truncation error term $T(x_k, t_n)$, we get the explicit difference scheme (5). From (1) and (7), we get

$$\left[\frac{\partial^2 u}{\partial t^2}\Big|_{(x_k,t_n)} - \Delta_t^2 u_k^n\right] - d\left[\frac{\partial u^\alpha}{\partial x^\alpha}\Big|_{(x_k,t_n)} - \Delta_x u_{k+1}^n\right] = T(x_k, t_n),$$

$$\frac{\partial^2}{\partial t^2} u(x_k, t_n) = \Delta_t^2 u(x_k, t_n) + O(\Delta t)^2, \tag{8}$$

$$\Delta_x u_{k+1}^n = \frac{\partial^\alpha u}{\partial x^\alpha}\Big|_{(x_k, t_n)} + O(\Delta x)^2,$$

$$\frac{\partial^\alpha u}{\partial x^\alpha}\Big|_{(x_{k+1}, t_n)} = \frac{\partial^\alpha u}{\partial x^\alpha}\Big|_{(x_k, t_n)} + \Delta_x \frac{d}{dx}\frac{\partial^\alpha u}{\partial x^\alpha}\Big|_{(x_k, t_n)} + O(\Delta x)^2.$$

So that

$$\Delta_x u_{k+1}^n = \Delta_x u_k^n + O(\Delta x) + O(\Delta x)^2.$$

From this result and from (8), we claim that $T(x, t) = O(\Delta t)^2 + O(\Delta x)$. □

4. Numerical results

Example 1. Consider the space fractional wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^{1.8} u(x,t)}{\partial x^{1.8}}, \quad 0 < x < 1, \quad 0 < t \le 1, \tag{9}$$

$$u(x,0) = \sin(2\pi x), \quad u_t(x,0) = 2\pi \sin(2\pi x), \quad u(0,t) = u(1,t) = 0.$$

When $\alpha = 2$ (instead of 1.8 in (9)), the exact solution is

$$u(x,t) = \sin 2\pi x(\cos 2\pi t + \sin 2\pi t). \tag{10}$$

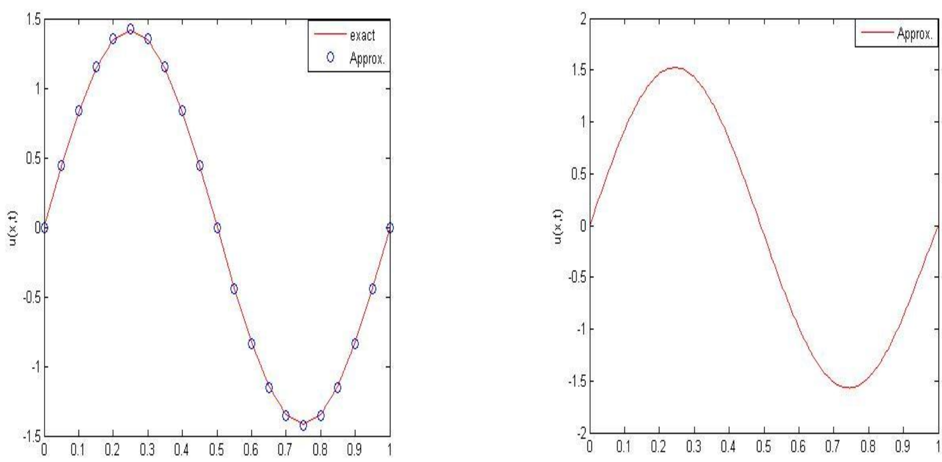


Figure 1. EFDA solutions when $h = 0.005$ and $\tau = 0.0025$: (left) comparison with the exact solution for $\alpha = 2$ at $t = 0.05$, (right) for $\alpha = 1.8$ at $t = 0.125$.

The numerical studies are given as follows: the exact solutions for $\alpha = 2$ (as given by (10)) and the EFDA solution for $\alpha = 1.8$ at $t = 0.05$ when $h = 0.005$ and $\tau = 0.0025$ are given in Table 1. In order to test the numerical scheme, we also plot in Figure 1 the exact and approximate solutions for integer case $\alpha = 2$. Moreover, the approximate solution for $\alpha = 1.8$ at $t = 0.125$ when $h = 0.005$ and $\tau = 0.0025$ is also shown in Figure 1. To study the behaviour of these solutions, Figure 2 is plotted to show the 3D-EFDA solutions for $\alpha = 2$ and $\alpha = 1.8$ respectively. Figure 3 shows the unstable solutions behaviour when $h = 0.157$ and , $\tau = 0.001$ where the value of s is larger than the stability bound s_x. For more details see Theorem 1.

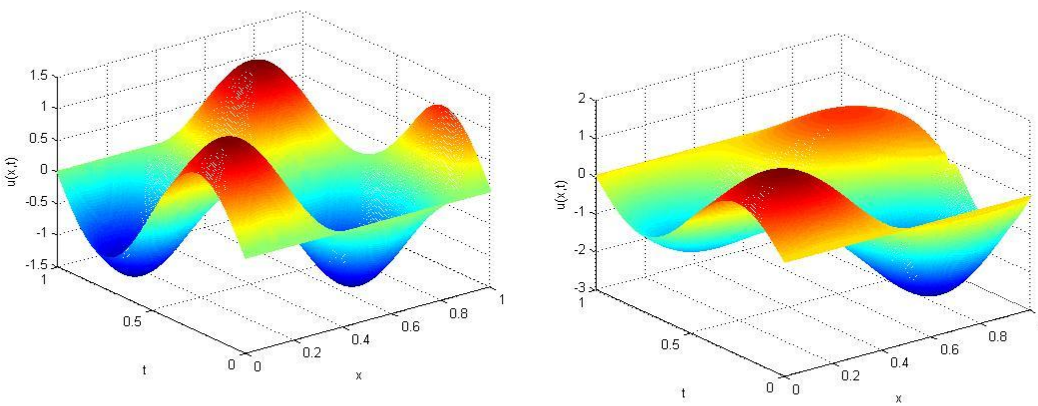


Figure 2. 3D-EFDA solutions for: (left) $\alpha = 2$, (right) $\alpha = 1.8$.

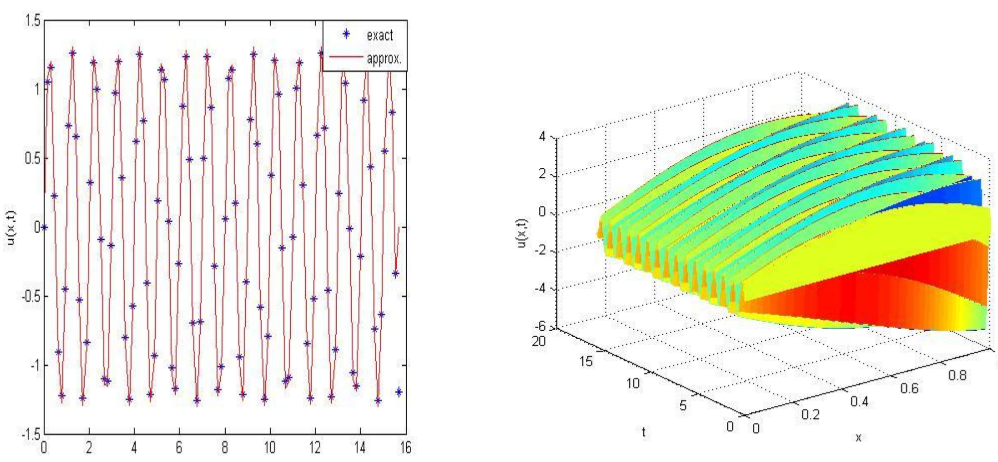


Figure 3. Unstable EFDA solutions when $h = 0.157$ and $\tau = 0.001$: (left) comparison with the exact solution, (right) 3D-EFDA solutions.

Example 2. Consider the space fractional wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^{1.6} u(x,t)}{\partial x^{1.6}}, \quad 0 \le x \le 5, \quad 0 < t \le 1, \tag{11}$$

$$u(x,0) = \sin x, \quad u_t(x,0) = 0, \quad u(0,t) = 0, \quad u(5,t) = \sin(5)\cos(t).$$

When $\alpha = 2$ (instead of 1.6 in (11)) the exact solution is

$$u(x,t) = \sin x \cos t \ . \tag{12}$$

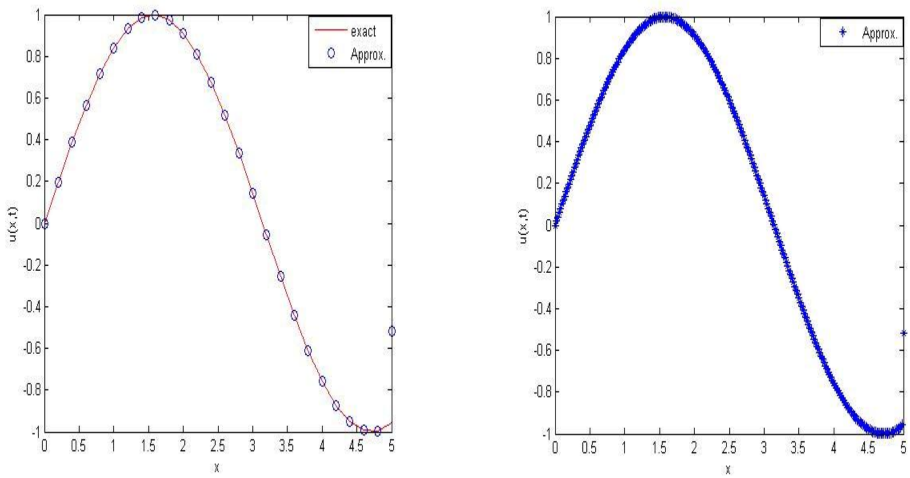


Figure 4. EFDA solutions when $h = 0.002$ and $\tau = 0.001$: (left) comparison with the exact solution for $\alpha = 2$ at $t = 0.01$, (right) for $\alpha = 1.6$ at $t = 0.01$.

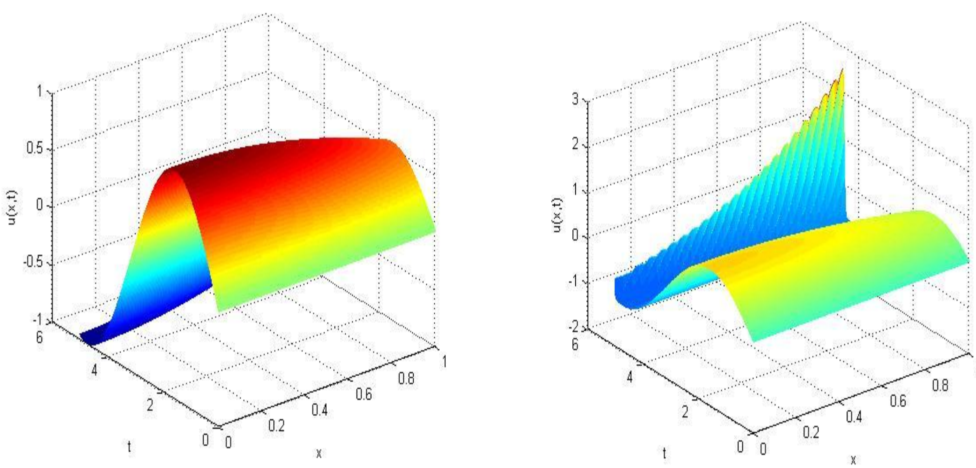
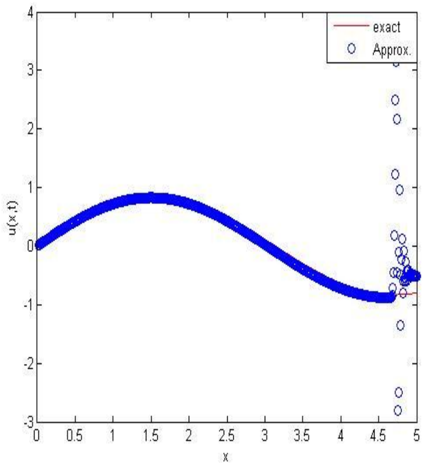


Figure 5. 3D-EFDA solutions for: (left) $\alpha = 2$, (right) $\alpha = 1.6$.

The numerical studies for Example 2 can be presented as follows: the exact solutions for $\alpha = 2$ (as given by (12)) and the EFDA solution for $\alpha = 1.6$ at $t = 0.01$ where $h = 0.002$ and $\tau = 0.001$ are given in Table 2. In order to test the numerical scheme, we also plot in Figure 4 the exact and approximate solutions for integer case $\alpha = 2$. Moreover, the approximate solution for $\alpha = 1.6$ at $t = 0.01$ when $h = 0.002$, $\tau = 0.001$ is also shown in Figure 4. To study the behaviour of these solutions, Figure 5 is plotted to show the 3D-EFDA solutions for $\alpha = 2$ and $\alpha = 1.6$ respectively. Figure 6 shows the unstable solutions' behaviour when $h = 0.008$ and $\tau = 0.001$, where the value of s is larger than the stability bound s_x. For more details see Theorem 1.



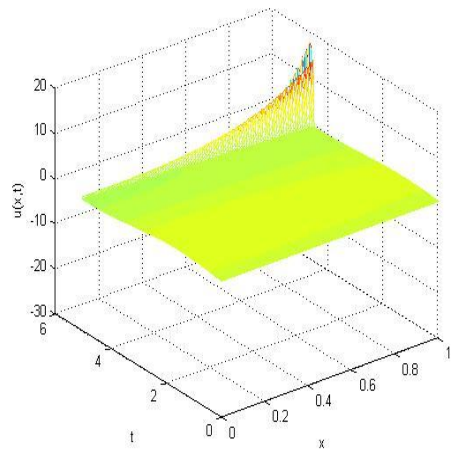


Figure 6. Unstable EFDA solutions when $h = 0.008$ and $\tau = 0.001$: (left) comparison with the exact solution, (right) 3D-EFDA solutions.

Table 2. The exact and EFDA solutions at $t = 0.01$ when $h = 0.002$, $\tau = 0.001$.

x_i	$\alpha = 2$	$\alpha = 1.6$
0.0000	0.00000000	0.00000000
0.2000	0.19866128	0.19866128
0.4000	0.38940257	0.38940432
0.6000	0.56461961	0.56462215
0.8000	0.71732704	0.71733027
1.0000	0.84143691	0.84144069
1.2000	0.93200134	0.93200553
1.4000	0.98540982	0.98541426
1.6000	0.99953312	0.99953762
1.8000	0.97380819	0.97381257
2.0000	0.90926060	0.90926469
2.2000	0.80846366	0.80846730
2.4000	0.67543582	0.67543886
2.6000	0.51548049	0.51548281
2.8000	0.33497458	0.33497609
3.0000	0.14111429	0.14111493
3.2000	-0.05837178	-0.05837204
3.4000	-0.25553075	-0.25553190
3.6000	-0.44250252	-0.44250451
3.8000	-0.61183311	-0.61183586
4.0000	-0.75677158	-0.75677525

5. Conclusions

An explicit finite difference approximation for SFWE has been explored, where the fractional derivative was in the Caputo sense. Error analysis and stability of the explicit numerical method for SFWE were discussed by means of a fractional version of the von Neumann stability analysis. Finally, some numerical results of EFDA were presented. These numerical results demonstrate that the EFDA is a computationally simple and efficient method for SFWE.

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