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Analysis of a Thin-Walled Curved Rectangular Beam with Five Degrees of Freedom

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ANALYSIS OF A THIN-WALLED CURVED RECTANGULAR

BEAM WITH FIVE DEGREES OF FREEDOM

By

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A study of a thin-walled curved rectangular box beam under torsion and out-of-plane bending is documented in this thesis. A new one-dimensional theory that takes into account warping and distortion in the beam cross-sections is the main focus. Existing available theories for thin-walled curved beams lack rigorous theoretical development, and most have ignored the effects of warping and distortion.

A higher order theory including two additional degrees of freedom corresponding to warping and distortion was derived. The conventional three degrees of freedom model was compared with the new five degrees of freedom model. The variation of beam thickness to control and decrease the high distortion variable is investigated.

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CHAPTER I

INTRODUCTION

Thin-walled beams are extensively used in engineering and other fields because of their high stiffness to mass ratio. In engineering, the automotive industry has used thin-walled beams in areas such as joints in automotive bodies, rocker joints and T joints. Instead of plate and shell elements, simple beam elements are sometimes used for analysis of such areas to reduce the difficulty of the problem. In this study, the automotive curved joints are the main focus. Initial design and analysis of a thin-walled automotive joint is usually based on a one-dimensional beam model. For this reason, the beam element is chosen over plate and shell elements even though plate/shell finite element analysis would accurately predict the dynamics of thin-walled joints.

There are various finite beam elements, which are based on conventional beam theory [11], but most of these available theories lack rigorous theoretical development. These elements typically have three degrees of freedom from shear, bending, and rotation, but effects of warping and distortion have been ignored. A few models do have warping deformation effects present along with the three degrees of freedom from the conventional beam theory, but distortion has been ignored. There is only one model that has included effects of distortion and warping along with three degrees of freedom from conventional beam theory. Only recently has both warping and distortion variable been included in analysis of curved beam out-of-plane bending problems. Yoon Young Kim and Youngkyu Kim of Seoul National University (South Korea) were the first to consider both of these variables along with the three conventional variables of lift, bending, and torsion. Timoshenko solved the case of a thin-walled, rectangular, curved beam with in-plane loading in 1923. This solution by S. P. Timoshenko gave designers guidelines on the magnitude of the effect and how to design them. Kim and Kim presented the comparable out-of-plane solution as a 1-D finite element in 2002. A new five degree-of-freedom one-dimensional theory of Kim and Kim [10] is used in the analysis for curved box beam with out-of-plane load. The objective of this research is to first duplicate the Kim and Kim numerical results and derivation of the theory and then use the 1-D FEA model to develop design guidelines.

In addition, the problems with the new 1-D theory have been discussed, and the proposed solutions to solve these problems are mentioned as well. Tailor welded patches are proposed to be used for the present model. Tailor welded patches suggest that the thickness in the beam's cross section may be varied to obtain better solution accuracy. These patches are used for this model to see the behavior in results for the conventional three DOF model and new five DOF model.

CHAPTER II

BACKGROUND: HISTORY OF IN-PLANE BENDING OF CURVED BEAMS (LITERATURE REVIEW)

Bending of curved beams is an important topic in strength of materials because such beams have many practical applications, such as in the automotive industry. Many well known researchers have contributed to this field. In this chapter, contributions to the topic of bending of curved beams made by two important researchers, Saint Venant and S. P. Timoshenko, will be discussed. Saint Venant considered only the in-plane bending of curved beams and did not consider out-of-plane bending. Timoshenko investigated cases of both in-plane and out-of-plane bending. For the out-of-plane bending case for curved rectangular beams, he considered only the three kinematics variables of shear, bend, and rotation and excluded the effects of warping and distortion variables even though he knew the effects of these variables were present in his analysis.

The main work of Saint Venant deals with mathematical theory of elasticity, but he also contributed to the theory of bending of beams. He was the first to test the accuracy of the fundamental assumptions of bending [11], that the cross section of a beam stays plane during deformation and that the longitudinal fibers of a beam do not press upon each other during bending and are in a state of simple tension or compression.

He also showed that these two assumptions worked only in uniform bending when the beam has two equal and opposite couples applied at the ends. For a rectangular box beam subjected to bending (Figure 1), Saint Venant showed that the changes in length of fibers and the corresponding lateral deformations satisfy both the fundamental assumptions of bending mentioned above and the condition of continuity of deformation.

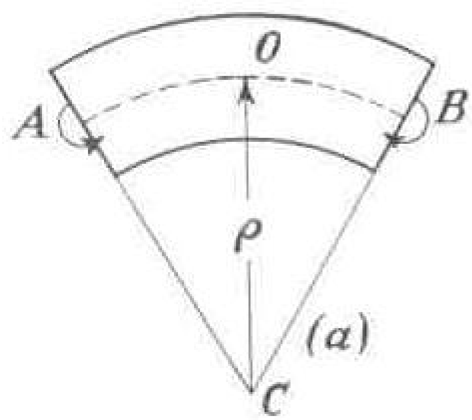

Figure 1. Pure Bending of Rectangular Beam

In addition, he showed that the initially rectangular cross section of the beam changes its shape (Figure 2) because of the lateral contraction of the fibers on the convex side and expansion on the concave side.

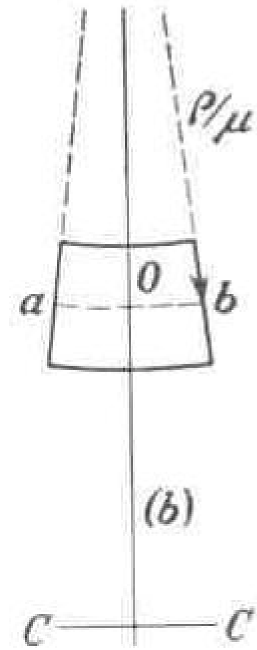

Figure 2. Changed Cross Section of Rectangular Beam

In figure 2, the initially straight line "ab" becomes slightly bent and the corresponding radius of curvature becomes $\frac{\rho}{\mu}$, where ρ is the radius of curvature and μ is the Poisson's ratio. Because of this lateral displacement, the distances of the neutral fibers "a" and "b" from the upper and lower surfaces of the beam are changed. This behavior found by Saint Venant was the first time in history of the strength of materials that the distortion of the shape of the box beam in bending was investigated [11].

Saint Venant also found that there is warping present in beam bending, which was ignored in the classical beam theory. He showed that shearing stresses act in the planes of cross section, and due to the presence of these stresses, the cross sections do not remain plane during bending (figure 3).

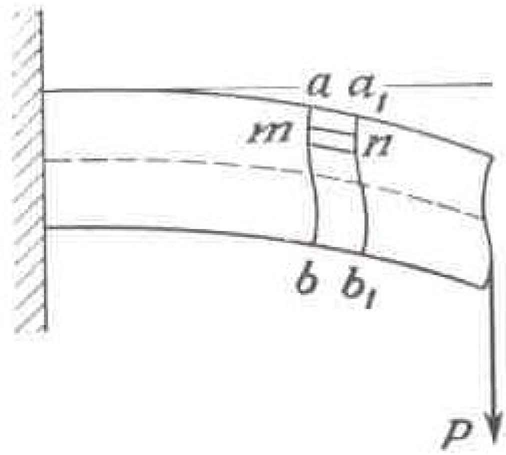

Figure 3. Cantilever Loaded at the Free End

In fact, the cross sections are warped as shown in figure 3. Since warping is the same for any two cross sections, it produces no changes in length of fibers, unlike distortion. Thus, warping will not affect the bending stresses, which are calculated using the assumption that the cross sections remain plane during bending.

The other researcher who has contributed significantly to the theory of bending in beams is S.P. Timoshenko. He not only worked with the more common cases of bending of curved bars in the plane of their initial curvature, but he also worked with the cases in which the forces acting on a curved bar do not lie in the plane of the center line of the bar.

To find the deflection in curved bars in plane bending case, Timoshenko used Castigliano's theorem [10]. To find the bending stresses in curved bar for the in plane case, Timoshenko assumed that the shape of the cross section remains unchanged. He further clarified that such an assumption is justifiable as long as the bar is solid because the very small displacements in the plane of the cross section, which are due to lateral contraction and expansion, have no substantial affect on the stress distribution.

For the out-of-plane bending case, Timoshenko found it necessary to consider the deflection of the bar in two perpendicular planes and the twist of the bar. Figure 4 (a) shows a simple case, which Timoshenko used in his analysis, where a portion of a horizontal circular ring built in at A is loaded by a vertical out-of-plane load P, and is applied at the end B.

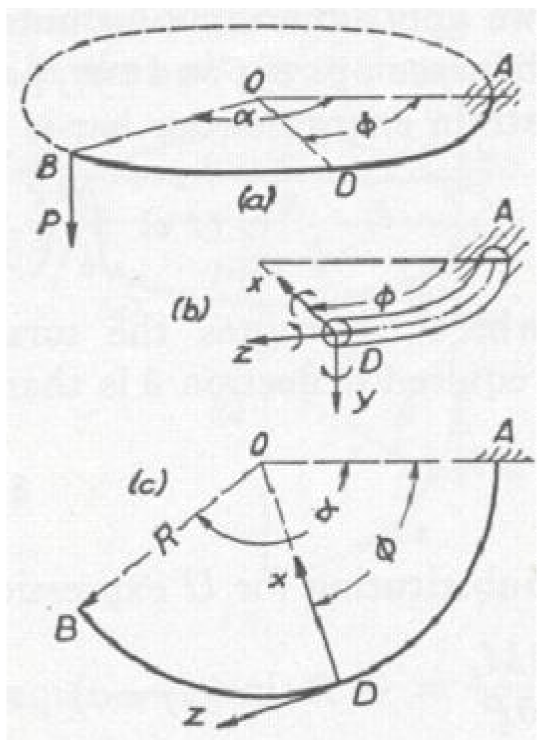

Figure 4. Bending of Curved Bar Out of its Plane of Initial Curvature

Timoshenko used the coordinate system as displayed in figure 4 (b) and 4 (c) to find the moments

$$M_x = -PR\sin(\alpha - \phi) \tag{1}$$

$$M_y = 0 \tag{2}$$

$$M_z = PR[1-\cos(\alpha-\phi)] \tag{3}$$

of external load P with respect to these axes. By using these equations, Timoshenko found the bending and torsion stresses in any cross section of the curved bar.

In calculating deflection at the end B, Timoshenko first used Castigliano's theorem to find the total strain energy of the bar. He assumed that the cross sectional dimensions of the bar are small compared with the radius R of the bar. He found the total strain energy of the bar to be

$$U = \int_0^{\alpha}\left(\frac{M_x{}^2}{2EI_x} + \frac{M_z{}^2}{2C}\right)Rd\phi\ , \tag{4}$$

where C is the torsional rigidity of the bar. Further, Timoshenko used the following equation to find the deflection of the bar:

$$\delta = \frac{\partial U}{\partial P}\ . \tag{5}$$

Substituting for U and finding partial derivatives of P with respect to moment,

$$\frac{\partial M_x}{\partial P} = -R\sin(\alpha-\phi) \tag{6}$$

$$\frac{\partial M_z}{\partial P} = R[1-\cos(\alpha-\phi)]\ . \tag{7}$$

Substituting the following in equation 5 gives the following expression for deflection:

$$\delta = \frac{PR^3}{EI_x}\int_0^{\alpha}\{\sin^2(\alpha-\phi) + \frac{EI_x}{C}[1-\cos(\alpha-\phi)]^2\}d\phi\ . \tag{8}$$

For the case when $\alpha = \dfrac{\pi}{2}$, equation 8 simplifies to

$$\delta = \frac{PR^3}{EI_x}\left[\frac{\pi}{4} + \frac{EI_x}{C}\left(\frac{3\pi}{4} - 2\right)\right]. \tag{9}$$

Equation 9 [10] gives the deflection at any point of the curved bar knowing the given variables in the equation. This equation is useful for any curved bar subjected to out-of-plane load at one end with the other end being fixed in all degrees of freedom. Timoshenko used this equation for only three degrees of freedom: lift, bending, and torsion. Effects of warping and distortion variables are not included in Timoshenko's analysis. Although he was familiar with these variables, he ignored their effects in his derivation for the deflection equation.

Only recently has both warping and distortion variables been included in analysis of curved beam out-of-plane bending problem. There has not been a model present that included both warping and distortion effects within it. Kim and Kim were the first to consider both of these variables along with the three conventional beam model variables of lift, bending, and torsion.

CHAPTER III

BASIC THEORY AND ASSUMPTIONS IN KIM AND KIM MODEL

The theory of bending and torsion is the foundation on which the procedures of engineering design and analysis are based. In the design of buildings, bridges, ships, or automobiles, beam and torsion theory is the most important tool available to engineers. The theory is well developed for thin-walled straight beam structures, where many simplifying assumptions can safely be made. However, there is no present theory for thin-walled curved beam structures. In this chapter, the present Kim and Kim model will be discussed in detail, and than how the one-dimensional curved box beam theory was derived using the theory for straight box beams will be shown.

The straight box beam theory is based on two well-accepted assumptions in the theory of thin-walled members, classical bending theory and thin-walled torsion theory. First, these two theories will be discussed, and than it will be shown how by extending these theories the one-dimensional curved box beam theory was developed.

Classical bending theory, commonly referred to as Euler-Bernoulli beam theory, assumes that a one-dimensional stress state exists for a beam under pure bending. In other words, plane sections remain plane during bending, usually referred to as Navier assumption [11]. This assumption also works for a beam loaded by a shear load if the beam is sufficiently long relative to the other dimensions. These and other assumptions about the beam deformations lead to the "flexure formula" for bending stress [11],

$$\sigma_y = \frac{M_x z}{I_{xx}}, \tag{10}$$

where M_x is the bending moment, z is the distance from the neutral axis, and I_{xx} is the moment of inertia. Since beam theory assumes a one-dimensional stress state, the transverse and shear stresses are zero.

Torsion theory is considered to be completely determined by the theory of St. Venant [11], which was developed in 1850s. The crucial point in St. Venant's torsion theory is that the cross section is free to deform out-of-its plane during torsion. The basic assumption in thin wall torsion theory is that the shear stress is tangential to the cross-section and constant throughout the thickness. All other stresses are neglected; therefore, this is also a one-dimensional theory.

Kim and Kim [5] used the bending and torsion theories and the straight box beam theory to find different deformation functions that are important for the derivation of the one-dimensional curved box beam theory. These section deformation functions will be discussed later in this chapter. First, two of the more important variables of warping and distortion will be discussed.

Discussion of Warping Variable

As mentioned in chapter two, Saint Venant was the first researcher to find and effectively include the warping effect in his thin-walled beam theory. He learned from the mistake of Navier who attempted to apply his reasoning of plane sections remaining plane to noncircular cross sections. Navier was unsuccessful to include the effects of warping in his theory [6]. Saint Venant, a student of Navier, proposed a different assumption than Navier and assumed that plane sections do not remain plane but warp during torsion for non-circular cross sections. He used the displacement functions to derive the warping. Saint Venant assumed that the warping could be described by a function that does not vary with z-axis (out-of-plane). He found the displacements as presented in the following equations [6]:

$$u_x = -\alpha z y \tag{11}$$

$$u_y = \alpha z x \tag{12}$$

$$u_z = \alpha \psi(x, y). \tag{13}$$

The warping function is $\psi(x, y)$, and α is the rate of twist. It is shown that the warping function comes from the Saint Venant torsion theory for non-circular cross sections. For a circular cross section, the warping function is zero as Navier [6] showed in his theory, but for non-circular cross sections the warping function is not zero as was presented by Saint Venant. Now, some discussion of the distortion variable will be presented, and it will be shown where the distortion motion and function comes from.

Discussion of Distortion Variable

Before proceeding further, Kim and Kim model will be discussed more in detail to obtain a theoretical feel for the concepts in the model. Distortion is present in the model and this variable is the cause of problems in the model, as will be seen from the analysis results in the next chapter. First, it is important to explain why the model undergoes distortion.

This section will discuss some ideas explaining why the model distorts. To simplify the explanation, one element is chosen from the current thin-walled curved beam model. Figure 5 presents the top view of an element from the beam model.

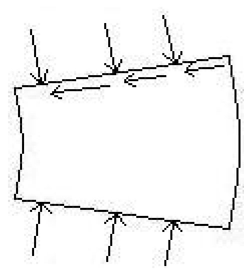

Figure 5. Curved Beam Element (Top View)

The figure presents the bending load present in the beam. From the figure, it is apparent that some stresses are present in the element. The vertical forces are the normal forces that are acting against each other at the top and bottom of the element. These opposing normal forces cause the element to be in compression. Further, another force is present that acts along the normal forces. The presence of normal forces requires that this particular force be present as well. These forces are the shear loads, which are shown in figure 6.

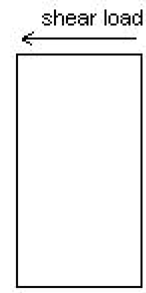

Figure 6. Curved Beam Element (Front View)

Figure 6 shows the front view of the beam element. The topside in figure 6 is in compression. There must be an opposing force present that has to exert the shear load. This reaction force is depicted in figure 7.

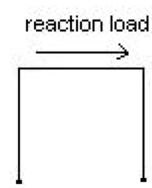

Figure 7. Half Beam Element Model

Figure 7 is a half model of figure 6 sectioned in exact middle of the element. For the shear load, there is a reaction load exerted against it as presented in figure 3. This reaction load for the shear load causes the element to appear as in figure 8.

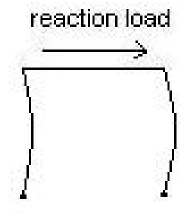

Figure 8. Half Beam Element Model with Distortion

Figure 8 depicts the distortion behavior caused by the reaction force for shear load. Similar behavior is observed from Kim and Kim analysis and the present research through FEA ALGOR, as will be seen later in this chapter. The behavior presented in figure 8 appears similar to the wall distortions present in the current analysis.

Now that the whereabouts of the distortion effects are made clear and known, it is important that these effects should not be ignored in the current thin-walled beam theory. This distortion effect causes significant problems in the analysis. Some of the problems are discussed now. Figure 9 depicts the beam element as seen from the top view, taking the distortion effects into account.

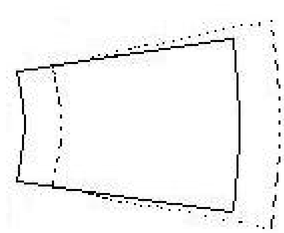

Figure 9. Original and Distorted Model

Figure 9 presents both the original model and the distorted model. The beam, which is in compression, is shifted outwards. As a result of this shift, the forces in the beam's cross section drop. Figure 10 indicates why the forces drop in the distorted model.

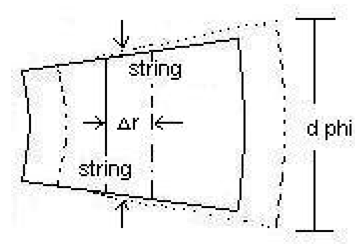

Figure 10. Original and Distorted Model Labeled

Assuming that there is a piece of string present in the beam model, the solid vertical lines and the dotted vertical lines in the figure represent pieces of strings present in both the original (solid line) and the distorted (dotted line) model. The model is in compression, and $d\phi$ presents the angle associated with the beam curvature. The piece of string moves slightly to the right and it is also wider when the distortion effects are present. The string has more width (ΔL) to it because it is farther out. The stress function is shown in equation 14

$$\sigma = E\varepsilon, \tag{14}$$

where E is modulus of elasticity and epsilon is

$$\varepsilon = \frac{\Delta L}{L}. \tag{15}$$

The change in length (ΔL) is less in the distorted model (dotted line) in figure 10. Since ΔL is less in the distorted model, the forces are also less than in the original model.

As presented in figure 10, since the distorted model is shifted out, the model has to be squeezed in more in order for the whole system to work. This can be addressed through the thickness of the model as presented in figure 11.

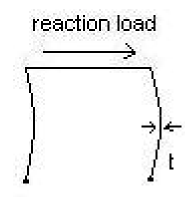

Figure 11. Half Beam Element Model with Distortion Revisited

Figure 11 presents the distorted front view of the beam element. The symbol t represents the thickness of the model. The moment of inertia of the beam is shown by the following equation:

$$I = \frac{1}{12} h t^3 . \tag{16}$$

In equation 16, the t^3 term comes from the distortion effect in the beam model. As seen in figure 11, the thin-walled beam is restrained to move in the direction of the force. The greater restraint present, the better the results that are obtained since the distortion effects disappear with increased restraint.

Revisiting Figure 10, which presents Δr term, taking this term into account, the epsilon (equation 15) changes to

$$\varepsilon = \varepsilon_o + \Delta r d\phi , \tag{17}$$

where ε_o is the original epsilon. The addition of $\Delta r d\phi$ (distortion effect) is the key term that causes troubling high stresses. Consequently, adjustments must be made to maintain these troublesome stresses. In order to maintain these stresses, the forces cancel out because the beam must carry the load. The epsilon is fixed, and the load must be carried. The original epsilon, ε_o, must increase in order to compensate. As a result, the new epsilon is likely to be much bigger than before, which means that the beam compressed more at the top and expanded more at the bottom. This indicates that more curvature is present in the beam to maintain a bigger displacement in the distorted model. As a result, the overall beam system is likely to be much weaker than has been assumed.

Explanation of Beam as a Boundary Value Problem

The distortion variable will be discussed further in this section, and the derivation of the distortion sectional deformation function will be shown, which is clearly important in developing the one-dimensional thin-walled beam theory. The present beam problem will be discussed in this section as a boundary value problem. Beams deflect more than do axially loaded members. Bending is the most common problem in all the loading problems in design. For a simple beam, the flexure equations [8] are written as:

$$\frac{V}{EI} = \frac{d^3 s}{dx^3} \tag{18}$$

$$\frac{M}{EI} = \frac{d^2 s}{dx^2} \tag{19}$$

$$\theta = \frac{ds}{dx} \tag{20}$$

$$s = s(x) \ . \tag{21}$$

The nomenclature and conventions are presented by the following simple beam:

Figure 12. Simple Beam

The beam is fixed at one end and has a moment applied at the other end. Some assumptions are made for this boundary value problem. First, the moment at the midpoint of the beam is zero. Second, the slope at half the length (L/2) is also zero. The beam is presented as a half model in the following figure.

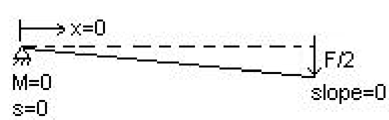

Figure 13. Simple Beam (Half Model)

The assumptions are depicted through this half model. The slope at the hard corner (at the right end) is zero. The moment at the left end is zero at the pin joint. As for all pin joints, the moment and the displacement is zero. Now, the general equation for the beam is written as follows:

$$F(s) = \frac{a_1}{6}s^3 + \frac{a_2}{2}s^2 + a_3 s + a_o = 0 \tag{22}$$

where a_1 represents the applied load, a_2 represents the applied moment, and a_3 represents the slope. Kim and Kim used this simple beam model to find the distortion function. As mentioned earlier, there are three boundary conditions for this problem. First, for the pin joint, the moment and the displacement is zero. The other boundary condition is that the slope is zero at the hard corner of the beam, as presented earlier by figure 13. Now, since the boundary conditions are known the problem is ready to be solved.

According to general equation of the beam (equation 22), there are four terms that have to be taken into consideration. Since, because of the pin joint, s is zero at zero, than a_o is zero. Since, also at the pin joint, the moment is zero, than a_2 is zero. Setting the known quantity equal to zero and simplifying the general equation gives:

$$F(s) = \frac{a_1}{6}s^3 + a_3 s = 0\,. \tag{23}$$

Now, the third boundary condition is used and the slope (the derivative of above equation) is set to zero at the distance L/2

$$F'(s) = \frac{a_1}{2}s^2 + a_3 = 0\,. \tag{24}$$

Now, substituting for s equal L/2 where the derivative is zero and solving for a_3, gives

$$a_3 = -a_1\frac{L^2}{8}\,. \tag{25}$$

Substituting this value for a_3 into simplified general equation for the beam gives

$$F(s) = \frac{a_1}{6}s^3 - \frac{a_1 L^2 s}{8}. \tag{26}$$

It is known that a_1 is the applied load, which in turn makes a_1

$$a_1 = \frac{V}{EI}. \tag{27}$$

Simplifying this equation further gives

$$F(s) = \frac{a_1 L^2}{12}\left[\frac{2s^3}{L^2} - \frac{3s}{2}\right]. \tag{28}$$

The expression inside the bracket in equation 28 is exactly what Kim and Kim found and used for their distortion function. The same function is also used in the current research and analysis as will be seen in later chapters.

With the help of this boundary value problem and earlier theoretical explanation, the distortion motion and function, as well as the warping function, are explained in detail and seen where each of them came from. It will be shown later in the results section that if the warping and distortion variables are treated separately along with the three conventional variables of lift, bend, and torsion, than the resulting displacement for the model will not be as high. When the warping and the distortion variables are treated together along with the three conventional variables, than the resulting displacements will be high. The cause of this increase in displacement is the interaction between the distortion and warping variables. The distortion can be predicted if no warping is present and the warping can be predicted when no distortion is present. When both variables are present, it is almost impossible to predict the behavior of the beam analytically. The interaction between the two variables results is a complicated problem. The conventional beam model with three degrees of freedom of lift, bend, and torsion does not have any interactions present since there is no warping and distortion terms. This is why Timoshenko was able to solve the conventional beam model analytically [10]. When there is interaction between warping and distortion present, then the problem becomes difficult to solve analytically, and finite element method is required to solve it. The finite element model and the results are presented later and it will be seen how the results vary with and without the presence of both warping and distortion variables. First, the Kim and Kim model will be discussed in more detail and the derivation of the one-dimensional thin-walled curved beam theory will be presented.

Derivation of Kim and Kim Thin-Walled Curved Beam Theory

The main idea behind the Kim and Kim model is to take into consideration the warping and distortion functions that are ignored in existing curved beam theories. The importance of these functions was discussed earlier in this chapter. The Kim and Kim model [10] employs five degrees of freedom corresponding to flexure, torsion, out-of-plane bending, warping, and distortion. This model is reduced from a three-dimensional variation expression, and careful considerations have been made to coupling due to curvature.

In order to analyze the beam problem using one-dimensional beam theory, correct sectional deformations for the problem had to be used. These sectional deformations were very important for the development of one-dimensional theory. Before analyzing the sectional deformations for this problem, the given problem along with its geometry will be presented. A thin-walled rectangular box beam, which was used for the analysis, is depicted in Figure 14 [10].

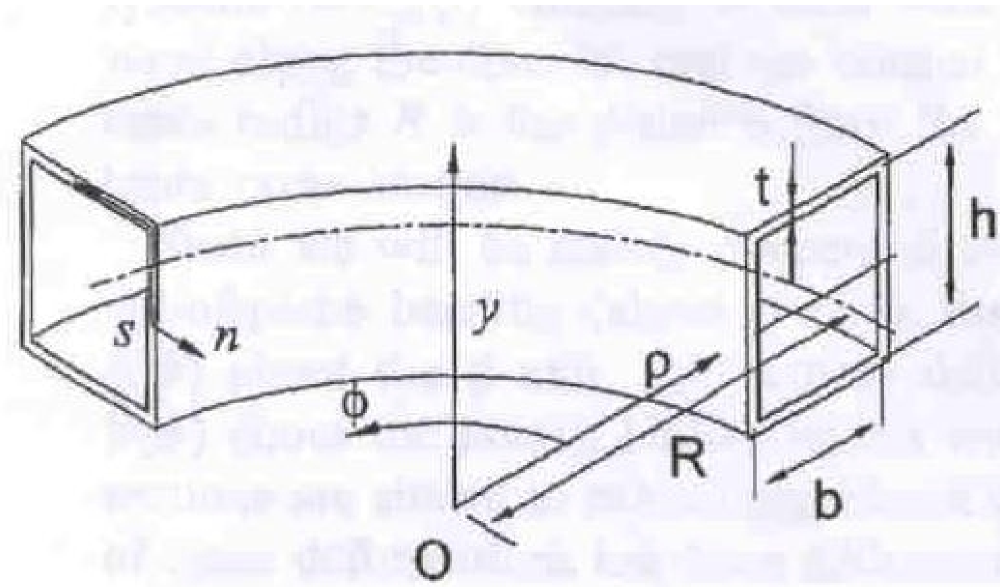

Figure 14. Geometry of a Thin-Walled Rectangular Curved Beam

The width and height of the beam are presented by b=50mm and h=100mm. The thickness t=2mm of the beam wall is assumed to be uniform and smaller than other dimensions. Other dimensions and material properties include radius R=1m, modulus of Elasticity E=200GPa, Poisson's ratio $\nu = 0.3$, the central angle $\phi = 90°$, and the vertical force V=100N. Also, the global coordinate system (ρ , ϕ , y) is shown in figure 14. The mean radius R is the distance from the center O of the curvature to the centroid line of the beam cross section.

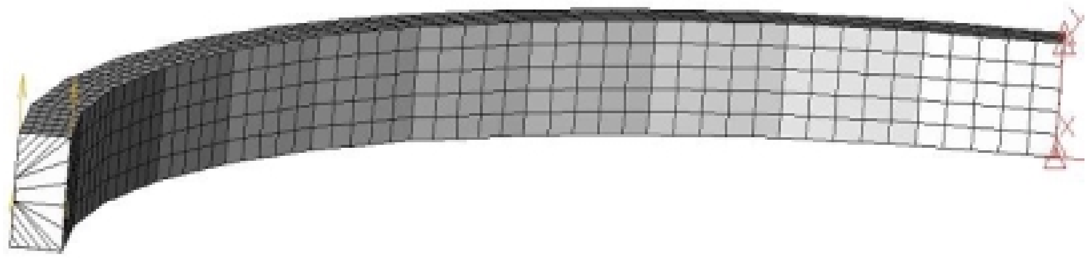

Figure 15. ALGOR Representation of Thin-Walled Curved Beam

Finite Element Analysis representation of the given problem is depicted in figure 15. The vertical out-of-plane force, along with the boundary conditions of fully constrained on one end and rigidly constrained on the loaded end, are presented in figure 15.

The geometry and the coordinate system used for the problem are shown in Figure 16.

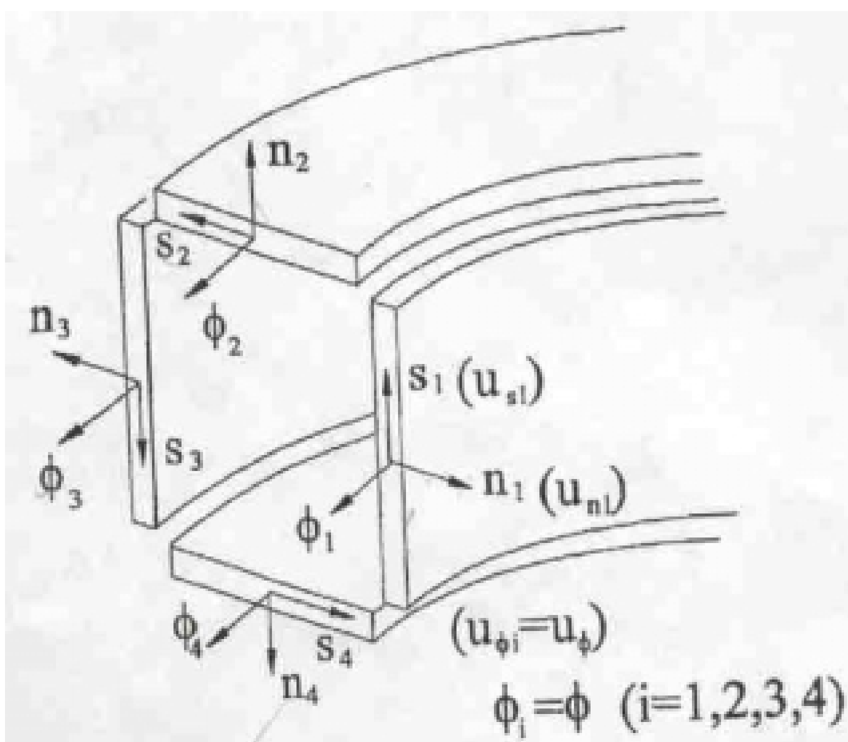

Figure 16. Coordinate System for the Thin-Walled Curved Box Beam

Figure 16 presents the local coordinate system (η_i, s_i, ϕ) that was used for this problem. The normal coordinate η_i directs outward from the surface, and the tangential coordinate s_i is measured along the contour. Bending and axial deformations are analyzed using existing beam elements based on Euler or Timoshenko beam theories [2]. Using figure 16, the displacements are described in terms of normal $u_{ni}(s_i,\phi)$, tangential $u_{si}(s_i,\phi)$, and axial $u_{\phi i}(s_i,\phi)$ components. These displacements are often called shell displacements since their dependence on the tangential coordinate s_i seems explicit. If the beam deformation at given ϕ is denoted by the amounts of axial rotation $\theta(\phi)$, warping $U(\phi)$, and distortion $\chi(\phi)$, the shell displacements becomes

$$u_{ni}(s_i,\phi) = \psi_{ni}^{\theta}(s_i)\theta(\phi) + \psi_{ni}^{\eta}(s_i)\eta(\phi) + \psi_{ni}^{\beta}(s_i)\beta(\phi) + \psi_{ni}^{U}(s_i)U(\phi) + \psi_{ni}^{\chi}(s_i)\chi(\phi) \quad (29a)$$

$$u_{si}(s_i,\phi) = \psi_{si}^{\theta}(s_i)\theta(\phi) + \psi_{si}^{\eta}(s_i)\eta(\phi) + \psi_{si}^{\beta}(s_i)\beta(\phi) + \psi_{si}^{U}(s_i)U(\phi) + \psi_{si}^{\chi}(s_i)\chi(\phi) \quad (29b)$$

$$u_{\phi i}(s_i,\phi) = \psi_{\phi i}^{\theta}(s_i)\theta(\phi) + \psi_{\phi i}^{\eta}(s_i)\eta(\phi) + \psi_{\phi i}^{\beta}(s_i)\beta(\phi) + \psi_{\phi i}^{U}(s_i)U(\phi) + \psi_{\phi i}^{\chi}(s_i)\chi(\phi), \quad (29c)$$

where $\psi(s)'s$ represent the functions describing the contour deformations of the cross sections. The functions $\psi^{\theta}(s)$, $\psi^{\eta}(s)$, and $\psi^{\beta}(s)$ are found using Timoshenko and St. Venant beam theories [5].

The sectional deformations are now presented for all five variables of bending, vertical displacement, torsion, warping, and distortion. For rotation $\theta(\phi)$ about the ϕ axis, the correct section deformation functions are found using Timoshenko and St. Venant theories and are presented in the following equations [5]:

$$\psi_{ni}^{\theta}(s_i) = -s_i \quad (30a)$$

$$\psi_{si}^{\theta}(s_i) = \frac{b}{2} \qquad [i = odd] \tag{30b}$$

$$\psi_{si}^{\theta}(s_i) = \frac{h}{2} \qquad [i = even] \tag{30c}$$

$$\psi_{\phi i}^{\theta}(s_i) = 0\,. \tag{30d}$$

For vertical displacement $\eta(\phi)$ in the y-axis, the section deformation functions become

$$\psi_{ni}^{\eta}(s_i) = 0 \qquad [i = odd] \tag{31a}$$

$$\psi_{ni}^{\eta}(s_i) = (-1)^{(i-2)/2} \qquad [i = even] \tag{31b}$$

$$\psi_{si}^{\eta}(s_i) = (-1)^{(i-2)/2} \qquad [i = odd] \tag{31c}$$

$$\psi_{si}^{\eta}(s_i) = 0 \qquad [i = even]\,. \tag{31d}$$

For rotation $\beta(\phi)$ about ρ axis, the sectional deformations are

$$\psi_{ni}^{\beta}(s_i) = 0 \tag{32a}$$

$$\psi_{si}^{\beta}(s_i) = 0 \tag{32b}$$

$$\psi_{\phi i}^{\beta}(s_i) = (-1)^{(i+1)/2} s_i \quad [i = odd]\,. \tag{32c}$$

$$\psi_{\phi i}^{\beta}(s_i) = (-1)^{i/2} \frac{h}{2} \qquad [i = even] \tag{32d}$$

For warping and distortion, the sectional deformation functions were taken from Kim and Kim [2] and presented in equation 33 and 34. Figure 17 [5] shows the warping and distortion deformations in the beam as found by Kim and Kim in their analysis.

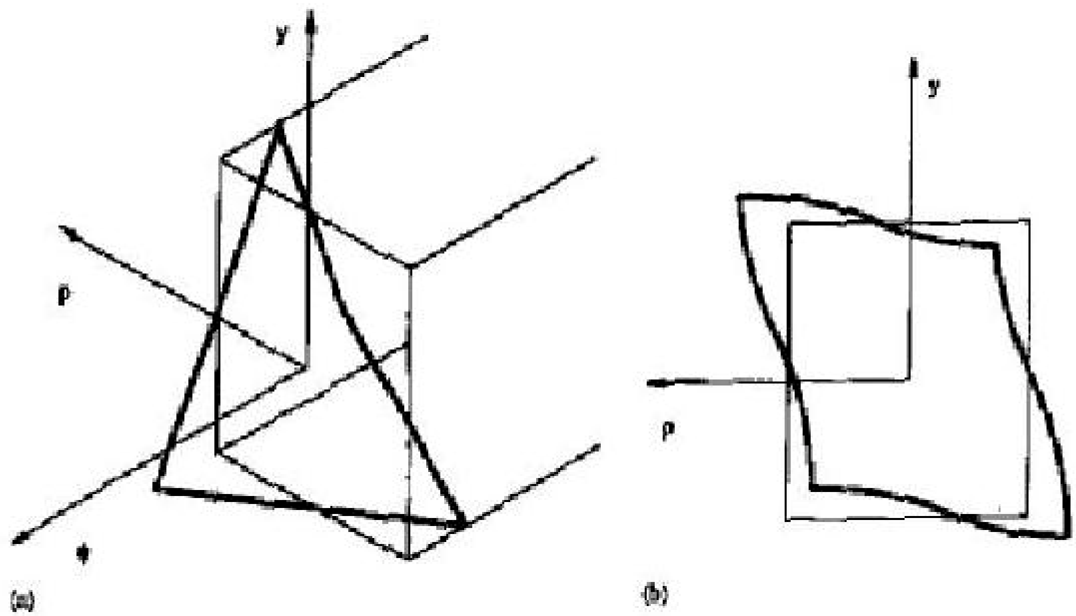

Figure 17. Kim and Kim Warping and Distortion Section Deformation Shapes

Figure 17a presents the out-of-plane warping, and figure 17b presents the in-plane distortion section deformation shapes. Earlier in figure 8, the whereabouts of the distortion variable were discussed and shown.

For warping deformation $U(\phi)$, the sectional deformation functions are

$$\psi_{ni}^{U}(s_i) = 0 \tag{33a}$$

$$\psi_{ni}^{U}(s_i) = 0 \tag{33b}$$

$$\psi_{\phi i}^{U}(s_i) = \frac{b}{2}s_i \quad [i = odd] \tag{33c}$$

$$\psi_{\phi i}^{U}(s_i) = -\frac{h}{2}s_i \qquad [i = even]. \tag{33d}$$

For distortion, section deformation functions are presented in equation 34 [5]:

$$\psi_{ni}^{\chi}(s_i) = \frac{2h}{b+h}\left(\frac{2s_i^{\ 3}}{h^2} - \frac{3s_i}{2}\right) \quad [i = odd] \tag{34a}$$

$$\psi_{ni}^{\chi}(s_i) = -\frac{2b}{b+h}\left(\frac{2s_i^{\ 3}}{b^2} - \frac{3s_i}{2}\right) \qquad [i = even] \tag{34b}$$

$$\psi_{si}^{\chi}(s_i) = -\frac{b}{2} \quad [i = odd] \tag{34c}$$

$$\psi_{si}^{\chi}(s_i) = \frac{h}{2} \quad [i = even] \tag{34d}$$

$$\psi_{\phi i}^{\chi}(s_i) = 0\,. \tag{34e}$$

All the sectional deformations presented in this chapter were used to derive the one-dimensional thin-walled curved box beam theory. In addition to understanding the sectional deformations, determining the three-dimensional displacements, strains, and stresses for the curved beam was also very important in deriving the one-dimensional thin-walled theory. The three-dimensional displacements $\tilde{u}_n$, $\tilde{u}_s$, and $\tilde{u}_\phi$ of a generic point in the beam were derived and shown in following equations [5].

$$\tilde{u}_{ni}(n_i, s_i, \phi) = \psi_{ni}^{\theta}(s_i)\theta(\phi) + \psi_{ni}^{\eta}(s_i)\eta(\phi) + \psi_{ni}^{\chi}(s_i)\chi(\phi) \tag{35a}$$

$$\tilde{u}_{si}(n_i, s_i, \phi) = \psi_{si}^{\theta}(s_i)\theta(\phi) + \psi_{si}^{\eta}(s_i)\eta(\phi) + \psi_{si}^{\chi}(s_i)\chi(\phi) - n_i\frac{d\psi_{ni}^{\chi}(s_i)}{ds_i}\chi(\phi) \tag{35b}$$

$$\tilde{u}_{\phi i}(n_i, s_i, \phi) = \psi_{\phi i}^{\beta}(s_i)\beta(\phi) + \psi_{\phi i}^{U}(s_i)U(\phi) \tag{35c}$$

The three-dimensional strains were found by Kim and Kim [2] and are presented in following equations [5]:

$$\varepsilon_{zzi}(n_i,s_i,z)=\xi_i^{\,2}\left(1+\xi_i\frac{b}{2R}\right)\left\{\psi_{zi}^{\beta}(s_i)\frac{d\beta(z)}{dz}+\psi_{zi}^{U}(s_i)\frac{dU(z)}{dz}\right\}$$
$$+\zeta_i^{\,2}\left(1+\zeta_i\frac{s_i}{2R}\right)\left\{\psi_{zi}^{\beta}(s_i)\frac{d\beta(z)}{dz}+\psi_{zi}^{U}(s_i)\frac{dU(z)}{dz}\right\}$$
$$-\frac{\xi_i}{R}\left(1+\xi_i\frac{b}{2R}\right)\left\{\psi_{ni}^{\theta}(s_i)\theta(z)+\psi_{ni}^{\chi}(s_i)\chi(z)\right\}$$
$$+\frac{\zeta_i}{R}\left(1+\zeta_i\frac{s_i}{2R}\right)\left\{\psi_{si}^{\theta}(s_i)\theta(z)+\psi_{si}^{\chi}(s_i)\chi(z)-n_i\frac{d\psi_{ni}^{\chi}(s_i)}{ds_i}\chi(z)\right\}$$

(36a)

$$\varepsilon_{ss}(n_i,s_i,z)=-n_i\frac{d^2\psi_{ni}^{\chi}(s_i)}{ds_i^{\,2}}\chi(z) \tag{36b}$$

$$\varepsilon_{szi}(n_i,s_i,z)=\frac{1}{2}\begin{bmatrix}\frac{d\psi_{zi}^{U}(s_i)}{ds_i}U(z)+\xi_i^{2}\frac{d\psi_{zi}^{\beta}(s_i)}{ds_i}\beta(z)\\ +\xi_i^{2}\left(1+\xi_i\frac{b}{2R}\right)\left\{\psi_{si}^{\theta}(s_i)\frac{d\theta(z)}{dz}+\psi_{si}^{\eta}(s_i)\frac{d\eta(z)}{dz}+\psi_{si}^{\chi}(s_i)\frac{d\chi(z)}{dz}-n_i\frac{d\psi_{ni}^{\chi}(s_i)}{ds_i}\frac{d\chi(z)}{dz}\right\}\\ +\frac{\zeta_i}{R}\left(1+\zeta_i\frac{s_i}{2R}\right)\left\{\psi_{zi}^{U}(s_i)U(z)+\psi_{zi}^{\beta}(s_i)\beta(z)\right\}\\ +\zeta_i^{2}\left(1+\zeta_i\frac{s_i}{2R}\right)\left\{\psi_{si}^{\theta}(s_i)\frac{d\theta(z)}{dz}+\psi_{si}^{\chi}(s_i)\frac{d\chi(z)}{dz}-n_i\frac{d\psi_{ni}^{\chi}(s_i)}{ds_i}\frac{d\chi(z)}{dz}\right\}\end{bmatrix}.$$

(36c)

In equation 36, the terms, ξ_i and ζ_i, are defined in equation 37 and 38 [5].

$$\xi_i=(-1)^{(i-1)/2} \qquad i=odd \tag{37a}$$

$$\xi_i=0 \qquad i=even \tag{37b}$$

$$\zeta_i=0 \qquad i=odd \tag{38a}$$

$$\zeta_i=(-1)^{i/2} \qquad i=even \tag{38b}$$

Using the three-dimensional strain components found in equation 36, the three-dimensional stresses were found and presented in equation 39 [5]

$$\sigma_{zzi}(n_i, s_i, z) = E_1(\varepsilon_{zzi} + \nu\varepsilon_{ssi}) \tag{39a}$$

$$\sigma_{ssi}(n_i, s_i, z) = E_1(\varepsilon_{ssi} + \nu\varepsilon_{zzi}) \tag{39b}$$

$$\sigma_{szi}(n_i, s_i, z) = 2G\varepsilon_{szi}\,. \tag{39c}$$

In equation 39, G represents the shear modulus, and ν is the Poisson's ratio. In addition, E_1 is depicted by equation 40

$$E_1 = \frac{E}{1-\nu^2}\,, \tag{40}$$

where E is the modulus of elasticity.

The energy approach was used to derive the one-dimensional thin-walled theory for curved box beams. Knowing the three-dimensional displacements, stresses, and strains, the one-dimensional theory is derived by integrating over the beam's cross sections using the system potential energy $\prod$. The potential energy of the system is presented in equation 41 [3]:

$$\Pi = \frac{1}{2}\int_{z1}^{z2}\int_{A}\sigma_{kl}\varepsilon_{kl}\,dAdz - \int\int(p\tilde{u}_z + q\tilde{u}_s)dAdz - \int\Big[\sigma_{zz}\tilde{u}_z + \sigma_{sz}\tilde{u}_s\Big]_{z=z_1}^{z=z_2} dA\,, \tag{41}$$

where A is the total cross-sectional area of the thin-walled curved beam. The terms, p and q, represent the external loads acting in axial and transverse directions. The coordinates z_1 and z_2 represent the location of two ends of the beam, and $\tilde{(\ )}$ represents the quantities at the ends of the beam.

Substituting equations for displacements (equation 35), strains (equation 36), and stresses (equation 39) into potential energy equation 41 gives

$$\Pi = \frac{1}{2}\int_{z_1}^{z_2}\begin{bmatrix} E_1\left(a_1\beta'^2 + a_2\theta^2 + 2a_3\theta\beta'\right) \\ +G\left(b_1\theta'^2 + b_2\eta'^2 + b_4\beta^2 + 2b_6\theta'\eta' + 2b_{12}\eta'\beta + 2b_{13}\theta'\beta\right) \\ +E_1\left\{c_1U'^2 + (c_2 + c_3 + 2\nu c_4)\chi^2 + 2c_5U'\theta + 2c_6\beta'U' + 2c_7\theta\chi + 2c_9\beta'\chi + c_{11}U'\chi\right\} \\ +G(d_1U^2 + d_2\chi'^2 + 2d_3\theta'U + 2d_4\eta'U + 2d_5\beta U + 2d_6\theta'\chi' + 2d_7\eta'\chi' + 2d_8\beta\chi' + 2d_9U\chi') \end{bmatrix} dz$$
$$- \int_{z_1}^{z_2}(q_1\theta + q_2\eta + p_1\beta + p_2U + q_3\chi)dz - \left[\bar{H}\,\theta + \bar{V}\,\eta + \bar{M}\,\beta + \bar{B}U + \bar{Q}\,\chi\right]_{z=z_1}^{z=z_2}, \tag{42}$$

where $(\ )'$ represents the differentiation with respect to z. The coefficients $a_1, b_1, and\ c_1$ are sectional coefficients, which are found by integrating the deformation shape functions ψ. These coefficients are presented in Appendix A. The generalized stress resultants $\bar{H}, \bar{V}, \bar{M}, \bar{B}, and\ \bar{Q}$ [4] in equation 42 [5] are described in the following equation

$$\bar{H} = \sum_{i=1}^{4}\int_{A_i}\sigma_{szi}\psi_{si}^{\theta}dA_i = G(b_1\theta' + b_6\eta' + b_{13}\beta + d_3U + d_6\chi') \tag{43a}$$

$$\bar{V} = \sum_{i=1}^{4}\int_{A_i}\sigma_{szi}\psi_{si}^{\eta}dA_i = G(b_6\theta' + b_2\eta' + b_{12}\beta + d_4U + d_7\chi') \tag{43b}$$

$$\bar{M} = \sum_{i=1}^{4}\int_{A_i}\sigma_{zzi}\psi_{zi}^{\beta}dA_i = E_1(a_3\theta + a_1\beta' + c_6U' + c_9\chi) \tag{43c}$$

$$\bar{B} = \sum_{i=1}^{4}\int_{A_i}\sigma_{zzi}\psi_{zi}^{U}dA_i = E_1(c_5\theta + c_6\beta' + c_1U'' + c_{11}\chi') \tag{43d}$$

$$\bar{Q} = \sum_{i=1}^{4}\int_{A_i}\sigma_{szi}\psi_{si}^{\chi}dA_i = G(d_6\theta' + d_7\eta' + d_8\beta + d_9U + d_2\chi'), \tag{43e}$$

where $dA_i = ds_in_i$. In addition, from equation 42, terms p and q are described in the following equations:

$$q_1 = \sum_{i=1}^{4} \int_A q \psi_{si}^{\theta} dA_i \tag{44a}$$

$$q_2 = \sum_{i=1}^{4} \int_A q \psi_{si}^{\eta} dA_i \tag{44b}$$

$$q_3 = \sum_{i=1}^{4} \int_A q \psi_{si}^{\chi} dA_i \tag{44c}$$

$$p_1 = \sum_{i=1}^{4} \int_A p \psi_{zi}^{\beta} dA_i \tag{44d}$$

$$p_2 = \sum_{i=1}^{4} \int_A p \psi_{zi}^{U} dA_i \ . \tag{44e}$$

Now, the principle of minimum potential energy is used to find the governing equations for each of the five degrees of freedom of $\theta, \eta, \beta, U, and\ \chi$. These governing equations are found with the help of one-dimensional form of potential energy equation 42 as presented by Kim and Kim [5] in the following equations:

$$E_1(a_2\theta + a_3\beta^{'} + c_5U^{'} + c_7\chi) - G(b_1\theta^{''} + b_6\eta^{''} + b_{13}\beta^{'} + d_3U^{'} + d_6\chi^{''}) = q_1 \tag{45a}$$

$$-G(b_6\theta^{''} + b_2\eta^{''} + b_{12}\beta^{'} + d_4U^{'} + d_7\chi^{''}) = q_2 \tag{45b}$$

$$-E_1(a_3\theta^{'} + a_1\beta^{''} + c_6U^{''} + c_9\chi^{'}) + G(b_{13}\theta^{'} + b_{12}\eta^{'} + b_4\beta + d_5U + d_8\chi^{'}) = p_1 \tag{45c}$$

$$-E_1(c_5\theta^{'} + c_6\beta^{''} + c_1U^{''} + c_{11}\chi^{'}) + G(d_3\theta^{'} + d_4\eta^{'} + d_5\beta + d_1U + d_9\chi^{'}) = p_2 \tag{45d}$$

$$E_1\{c_7\theta + c_9\beta^{'} + c_{11}U^{'} + (c_2 + c_3 + 2\nu c_4)\chi\} - G(d_6\theta^{''} + d_7\eta^{''} + d_8\beta^{'} + d_9U^{'} + d_2\chi^{''}) = q_3 \, . \tag{45e}$$

Once the governing equations for each of the five variables are found, the stiffness matrix is found using the Finite Element Method. The simplest two-node continuous finite element model based on the energy form (equation 42) is used in the numerical analysis.

With the help of the one-dimensional form of potential energy equation 42 and governing equation 45, the stiffness matrix is found to be as presented in appendix A using the Finite Element Method [12]. The explicit components of the stiffness matrix can be obtained through either numerical or exact integration.

As mentioned before, the present theory is the only case that accounts for χ (distortion) and U (warping) variables with the addition of three conventional variables of $\theta(torsion), \eta(shear), and\ \beta(bending)$. As it will be shown through the analysis and results, these two additional variables are the key to obtaining accurate results. In the following analysis, the displacement results from Kim and Kim and conventional beam theory model are compared with the analytical Finite Element Results. The following analysis also shows the results for the case where either warping or distortion is added to the conventional beam model.

CHAPTER IV

VERIFICATION PROCESS AND RESULTS

The verification process for the Kim and Kim model included several steps. First, the sectional deformation functions and coefficients had to be verified and derived. The calculations for the sectional deformation coefficients are presented in Appendix A. After the sectional deformation coefficients were calculated with the help of Mathcad, the stiffness matrix was found using the finite element method and subsequently verified with Kim and Kim findings [5]. Using the stiffness matrix, the desired stresses and displacements were found with the help of Matlab (as presented in Appendix B) and compared with Kim and Kim results.

Another step that was taken to verify the Kim and Kim model was to vary the number of variables used in the model. Four different models were used for this analysis, and each model consisted of different variables from the three kinematics variables to warping and distortion variables. For each of the four cases, the effects of varying the degrees of freedom on displacement results are presented in this section. The Kim and Kim model is flexible in the sense that once the correct model is found, it can be manipulated to be used for any number of variables, as will be seen in the following four cases.

Three Degrees of Freedom (DOF) (Conventional Beam Model)

The first case is the conventional beam model with three degrees of freedom of out-of-plane bending, rotation, and flexure. Displacement results at the loaded end were calculated using Finite Element Analysis software (ALGOR) and were compared to the results obtained from Timoshenko (using Castigliano's Theorem) and the current Kim and Kim thin-walled beam theory. For each model, displacements were calculated for both original (1T) and double (2T) thickness cases. Using the displacements for 1T and 2T model, the lambda for each element was also calculated. Before proceeding further, some explanation of what lambda is will be provided.

Lambda is a weight reduction factor that expresses the response of the design to a change in thickness. When lambda=1, it means that the weight of an equivalent stiffness structure will depend only on the specific modulus. When lambda>1, it means that the weight of an equivalent stiffness structure will depend on both the specific modulus and the density of the material; lighter weight materials will weigh less when lambda is greater than 1. The case of lambda<1 is not practical since it rarely occurs in any application; therefore, it will not be discussed. For the Kim and Kim model, lambda was calculated using equation 46

$$\lambda = -\frac{\log\left(\frac{\theta_2}{\theta_1}\right)}{\log\left(\frac{t_2}{t_1}\right)}, \tag{46}$$

where $t_2=2*t_1$, θ_2 is the element displacement for the double thickness model, and θ_1 is the element displacement for the original thickness model. Equation 44 [7] describes the relationship between stiffness and lambda and tells why consideration of lambda is important. The stiffness, K of a part under some applied load is expressed as:

$$K = C*E*t^{\lambda}, \tag{47}$$

where C is geometry constant, E is Young's modulus, and t is material thickness. The exponent λ is important because it expresses the response of the design to a change in thickness, as mentioned earlier. Equation 47 is an approximation. For automotive structures, λ usually varies between 1 and 2, although in few cases it can be below 1 or as high as 3. Now that lambda has been defined, the results of the conventional beam model will be presented.

Table 1 shows the displacement results at the loaded end for the three degrees of freedom (DOF) model along with the calculated lambdas. All the displacements presented in the following tables are in units of meters, and all the lambdas were dimensionless.

Table 1. Displacement Data at the Loaded End (Meters) for 3 DOF Case

Vertical Displacements	**1T**	**2T**	**Lambda**
3D FEA plate model [theoretical]	1.780E-03	6.700E-04	1.410E+00
1D FEA Kim & Kim, 3 DOF	1.040E-03	5.202E-04	1.000E+00
Timoshenko [Bending and Torsion] Analytical	1.270E-03	6.939E-04	1.000E+00
Displacements for Other Variables			
1D FEA Kim & Kim, 3 DOF [Max Twist]	5.948E-04	2.974E-04	1.000E+00
1D FEA Kim & Kim, 3 DOF [Bending]	1.126E-03	5.629E-04	1.000E+00

Three-dimensional (3D) Finite Element Analysis (FEA) plate results (theoretical results) for displacement were obtained from ALGOR FEA. One-dimensional (1D) displacement results with three degrees of freedom (DOF) were calculated using the current Kim and Kim beam theory with three degrees of freedom with the help of Matlab and Mathcad. Bending and torsion analytical results were obtained using Timoshenko's beam theory, which uses Castigliano's Theorem approach. All three of these displacement results were calculated at the loaded end for a 100 N vertical load. As seen from table 1, lambdas obtained for current one-dimensional Kim and Kim 3 DOF displacement and Timoshenko's beam theory displacement were one. Only the Finite Element Analysis results from ALGOR gave lambda greater than one.

In addition, maximum twist and bend angles were found for the current Kim and Kim 3 DOF model. For both cases, lambda was one, as presented in table 1. The maximum twist and bending term could not be presented using FEA software since it is limited to showing only the vertical displacements and stresses. Only the current Kim and Kim model presents these additional results with the help of Matlab.

For this case of 3 DOF, the displacement results obtained from 1D Kim and Kim model and Timoshenko theory do not match well with theoretical results obtained from FEA software. The displacement results seem to be significantly different compared to the theoretical FEA plate model results.

Four Degrees of Freedom (DOF) with Warping

The second case is one with four degrees of freedom. Along with three kinematics variables associated with conventional beam model, this model has one additional variable. The degrees of freedom associated with this model are rotation, flexure, out-of-plane bending, and warping. Similar to 3 DOF model, displacement results at the loaded end were calculated using Finite Element Analysis software (ALGOR). In addition, the displacement results were compared to the results obtained from Timoshenko theory (using Castigliano's Theorem) and the current Kim and Kim thin-walled beam theory. For each model, displacements were calculated for both original (1T) and double (2T) thickness cases. Using the displacements for 1T and 2T model, the lambda for each element was also calculated.

Table 2. Displacement Data at the Loaded End (Meters) for 4 DOF Case with Warping

Vertical Displacements	**1T**	**2T**	**Lambda**
3D FEA plate model [theoretical]	1.780E-03	6.700E-04	1.410E+00
1D FEA Kim & Kim, 4 DOF	1.108E-03	5.541E-04	1.000E+00
Timoshenko [Bending and Torsion] Analytical	1.270E-03	6.939E-04	1.000E+00
Displacements for Other Variables			
1D FEA Kim & Kim, 4 DOF [Max Twist]	6.774E-04	3.387E-04	1.000E+00
1D FEA Kim & Kim, 4 DOF [Bending]	1.224E-03	6.118E-04	1.000E+00
1D FEA Kim & Kim, 4 DOF [Max Warping]	5.431E-04	2.715E-04	1.000E+00

Similar to the results of 3 DOF model, lambdas obtained for current one-dimensional Kim and Kim 4 DOF displacement and Timoshenko's beam theory displacement were 1 as shown in table 3. Only the Finite Element Analysis results from ALGOR gave lambda greater than 1. In addition, maximum twist, warping, and bend were found for the current Kim and Kim 4 DOF beam, and for all the cases, lambda was 1. The maximum twist, maximum warping, and bending term could not be presented using FEA software since it is limited to showing only the vertical displacements and stresses. Only the current Kim and Kim model presents these additional results with the help of Matlab.

The addition of the warping term did not have much effect on the overall results. The displacement results obtained from 1D Kim and Kim model and Timoshenko theory still did not match well with theoretical results obtained from FEA software. However, there was a slight improvement from 3 DOF results. The addition of warping term slightly increased the displacement toward the correct theoretical result. From table 1, the 1D Kim and Kim 3 DOF displacement was 1.04E-3, but with addition of warping, the displacement was increased to 1.10E-3, as seen in table 2 for 1D Kim and Kim 4 DOF result.

Four Degrees of Freedom (DOF) with Distortion

The third case is with four degrees of freedom. Along with three kinematics variables associated with conventional beam model, this model has one additional variable. The degrees of freedom associated with this model are rotation, flexure, out-of-plane bending, and distortion. The additional distortion term is ignored in almost all of the classical beam theories. Displacement results at the loaded end were calculated using

Finite Element Analysis software (ALGOR). The results were compared to the results obtained from Timoshenko theory (using Castigliano's Theorem) and the current Kim and Kim thin-walled beam theory. For each model, displacements were calculated for both original (1T) and double (2T) thickness cases. Using the displacements for the 1T and 2T models, the lambda for each element was also calculated.

Table 3. Displacement Data at the Loaded End (Meters) for 4 DOF Case with Distortion

Vertical Displacements	**1T**	**2T**	**Lambda**
3D FEA plate model [theoretical]	1.780E-03	6.700E-04	1.410E+00
1D FEA Kim & Kim, 4 DOF	1.194E-03	5.624E-04	1.086E+00
Timoshenko [Bending and Torsion] Analytical	1.270E-03	6.939E-04	1.000E+00
Displacements for Other Variables			
1D FEA Kim & Kim, 4 DOF [Max Twist]	4.886E-04	2.682E-04	8.653E-01
1D FEA Kim & Kim, 4 DOF [Bending]	1.262E-03	5.984E-04	1.077E+00
1D FEA Kim & Kim, 4 DOF [Max Distortion]	2.388E-04	6.417E-05	1.896E+00

Unlike the 3 DOF model and 4 DOF model with warping, in this model some variations in lambdas are seen. The addition of the distortion term has changed the lambda behavior for 1D Kim and Kim theory. The same variation in lambdas is seen through the additional 1D Kim and Kim results for maximum twist, distortion, and bending. For maximum distortion, bending, and flexure, lambda has increased in table 3. Only maximum twist lambda drops below one.

The addition of distortion term has affects the lambdas for all cases, as seen in table 3. However, the addition of the distortion term did not have much affect on the overall displacement results. The displacement results for the 1D Kim and Kim model, are still different compared to the ALGOR finite element results. However, a slight improvement is seen from 3 DOF results and 4 DOF results with warping. The addition of the distortion term slightly increased the displacement toward the correct theoretical result. From table 1 and 2, the 1D Kim and Kim 3 DOF displacement was 1.04E-3, and for 4 DOF with the warping added result was 1.10E-3, but with the addition of distortion, the displacement was increased to 1.19E-3, as seen in table 3 for 1D Kim and Kim 4 DOF result. The addition of the distortion term has given the closest results that have been seen so far compared to all other variations of degrees of freedom.

Five Degrees of Freedom (DOF) (Kim and Kim Model)

The fourth and final case is with five degrees of freedom. Along with the three kinematics variables associated with conventional beam model, this model has two additional variables. The degrees of freedom associated with this model are rotation, flexure, out-of-plane bending, warping, and distortion. As mentioned previously, this is the first and only model in the history of strength of materials that has considered both warping and distortion variables in the analysis of a thin-walled curved rectangular beam. The importance of both warping and distortion variables is presented by the following figure 18. Kim and Kim found the presence of both warping and distortion variables in their analysis as presented in figure 17. Similar affects were found in the present analysis as depicted by the following figures.

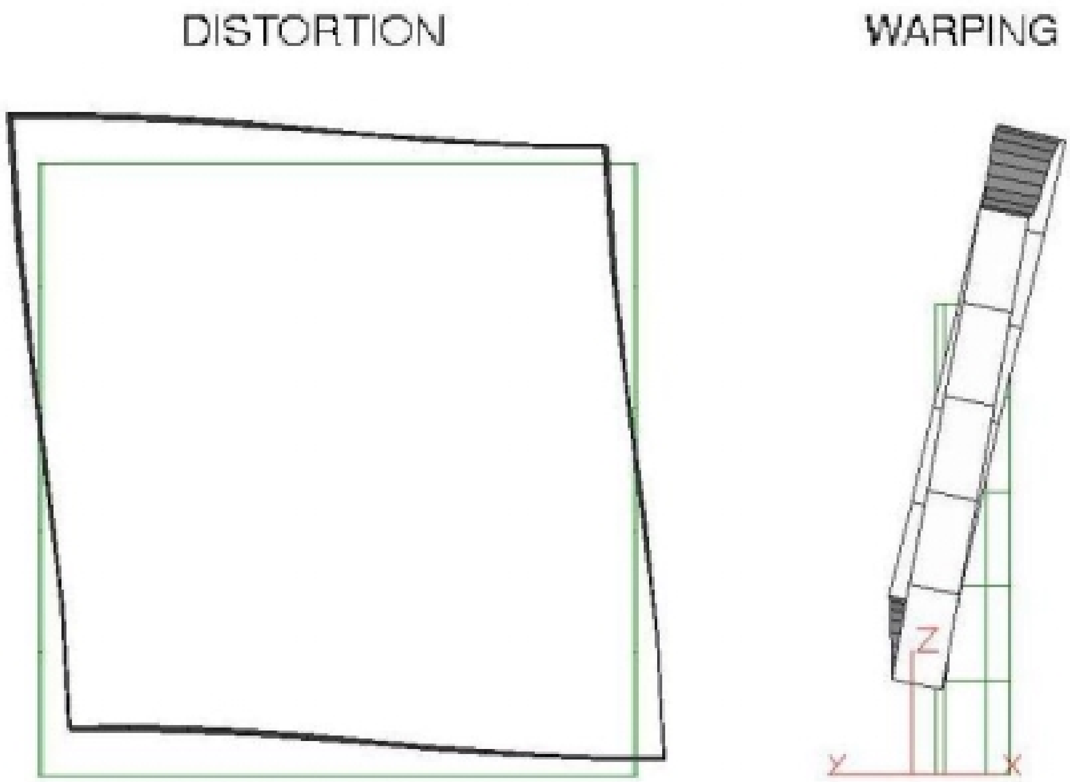

Figure 18. Section Deformation Shapes of Warping and Distortion (ALGOR FEA.)

These variables of warping and distortion were shown by ALGOR FEA. The presence of these effects emphasizes that these variables cannot be ignored from the analysis as has happened in the past by most thin-walled curved beam theories. Similar effects were observed by Cook [1] and Timoshenko [9] in their analysis of a curved thin-walled rectangular beam.

Similar to earlier cases, displacement results at the loaded end for this case were calculated using Finite Element Analysis software (ALGOR). In addition, the ALGOR displacement results, which were taken to be theoretical, were compared to the results obtained from Timoshenko theory (using Castigliano's Theorem) and the current Kim and Kim thin-walled beam theory. For each model, displacements were calculated for both original (1T) and double (2T) thickness cases. Using the displacements for 1T and 2T model, the lambda for each element was also calculated.

Table 4. Displacement Data at the Loaded End for 5 DOF Case

Vertical Displacements	**1T**	**2T**	**Lambda**
3D FEA plate model [theoretical]	1.780E-03	6.700E-04	1.410E+00
1D FEA Kim & Kim, 5 DOF	1.830E-03	6.674E-04	1.455E+00
Timoshenko [Bending and Torsion] Analytical	1.270E-03	6.939E-04	1.000E+00
Displacements for Other Variables			
1D FEA Kim & Kim, 5 DOF [Max Twist]	5.568E-04	3.138E-04	8.273E-01
1D FEA Kim & Kim, 5 DOF [Bending]	1.808E-03	6.991E-04	1.371E+00
1D FEA Kim & Kim, 5 DOF [Max Warping]	3.275E-03	7.977E-04	2.038E+00
1D FEA Kim & Kim, 5 DOF [Max Distortion]	9.326E-04	1.436E-04	2.699E+00

As seen from the results of 4 DOF with distortion added (table 3), the lambdas deviated from one because of the additional distortion term. The same behavior is seen in this model. Lambdas for this case do not follow the same behavior as 3 DOF conventional beam model. Similar variations in lambdas were present for maximum distortion, bending, warping, and twist that were observed for the four DOF case with distortion. The maximum warping and distortion terms has deviated the most in this case from one.

This five DOF case is the closest that the displacement results have gotten to the theoretical value. The addition of both warping and distortion terms along with the three kinematics variables seem to solve the problem. Table 5 presents the summarized results for all four cases with varying degrees of freedom along with the theoretical FEA plate model and Timoshenko (Castigliano's Theorem) theory.

Table 5. Displacement Results Summarized for All Four Cases

Vertical Displacements	**1T**	**2T**	**Lambda**
3D FEA Plate Model [theoretical]	1.780E-03	6.700E-04	1.410E+00
Timoshenko (bending and Torsion) Analytical	1.270E-03	6.939E-04	1.000E+00
1D FEA Kim & Kim, 3 DOF	1.040E-03	5.202E-04	1.000E+00
1D FEA Kim & Kim, 4 DOF with warping	1.108E-03	5.541E-04	1.000E+00
1D FEA Kim & Kim, 4 DOF with distortion	1.194E-03	5.624E-04	1.086E+00
1D FEA Kim & Kim, 5 DOF [both warping and distortion]	1.830E-03	6.674E-04	1.455E+00

As seen in table 5, the 1D Kim and Kim 5 DOF model shows the most accuracy in displacement and lambda results compared to the 3D FEA plate model. The conclusion can be drawn that the addition of distortion or warping terms alone will not give accurate results. The results will be accurate only when the effects of both distortion and warping terms are taken into account. This interaction between warping and distortion terms is the key in obtaining accurate displacements for a thin-walled curved rectangular beam.

Furthermore, as seen from the results of the 4 DOF with distortion and 5 DOF cases, the lambda began to deviate from one when distortion was present. This leads to the conclusion that the stiffness of the thin-walled rectangular beam is linear with thickness except for the distortion variable. Hence, distortion is the only non-linear variable that creates problems and causes lambda to deviate from normal. Suggested solutions to this problem with the distortion variables will be discussed in chapter 6. First, additional observation from Kim and Kim 5 DOF model will be discussed in more detail in the next chapter.

CHAPTER V

ADDITIONAL OBSERVATION FROM KIM AND KIM MODEL

The new 5 degrees of freedom (DOF) Kim and Kim model is a very useful tool in design engineering because the behavior of many different variables throughout the beam can be observed. These observations were made from the Kim and Kim 5 DOF model for the original and double thickness cases. This comparison between the thicknesses is presented in the following graphs of vertical displacement, twist, bend, warping, and distortion. Figure 19 shows the displacement results from original and double thickness of the model.

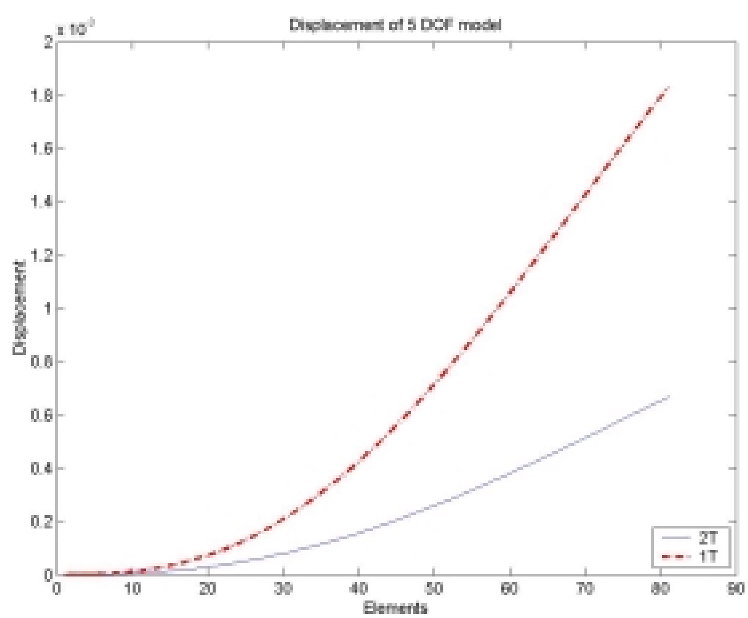

Figure 19. Vertical Displacement Plot for 5 DOF Model

The current beam model consists of 80 elements. The behavior of displacement is presented in all 80 elements of the beam. The displacement is highest at element 80 where the beam is loaded with 100 N vertical force. Also, the displacement of double thickness (2T) is much lower than original thickness, which shows that doubling the thickness stiffens the beam. The displacement value at the loaded end for original thickness (1T) is three times as much as the value for the double thickness (2T) case. Figure 20 presents the behavior of twist displacements throughout the beam.

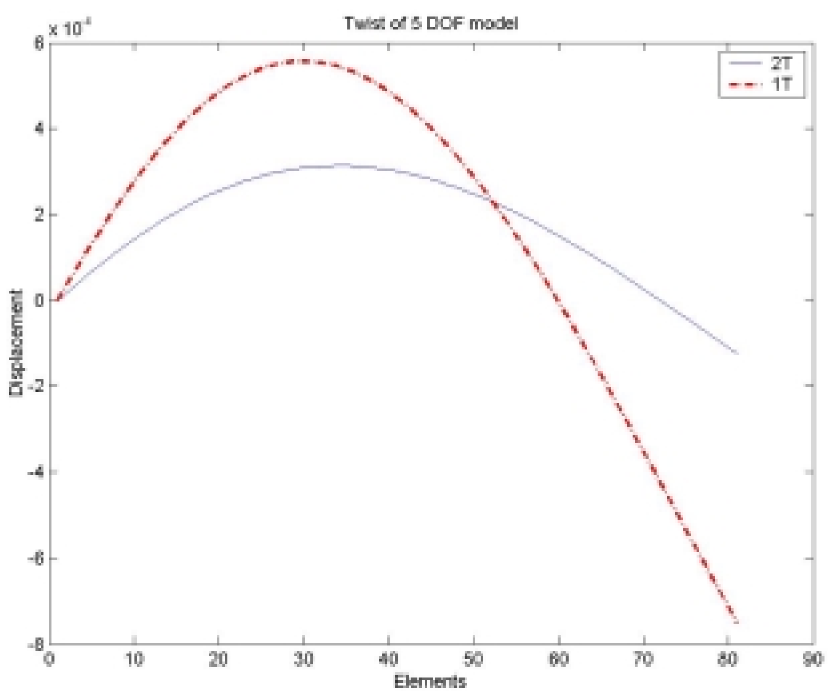

Figure 20. Twist Plot for 5 DOF Model

From looking at the twist plot for both original and double thickness cases, it is apparent that the maximum twist value for original thickness (1T) is almost twice as much as the double thickness case (2T). Figure 21 presents the behavior of the bending variable throughout the beam.

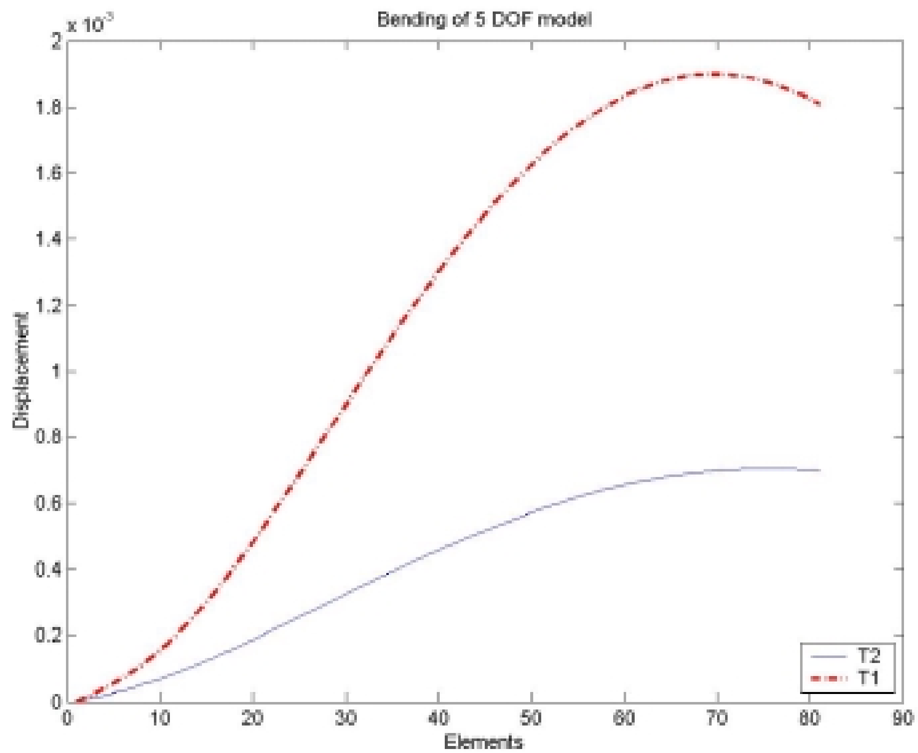

Figure 21. Bending Plot for 5 DOF Model

Similar to the vertical displacement and twist plot, the bending plot shows the stiffer double thickness (2T) case with lesser displacement than the original thickness (1T) case. The maximum bending value for original thickness is again almost three times as much as the value for the double thickness case. Figure 22 presents the warping behavior of the beam.

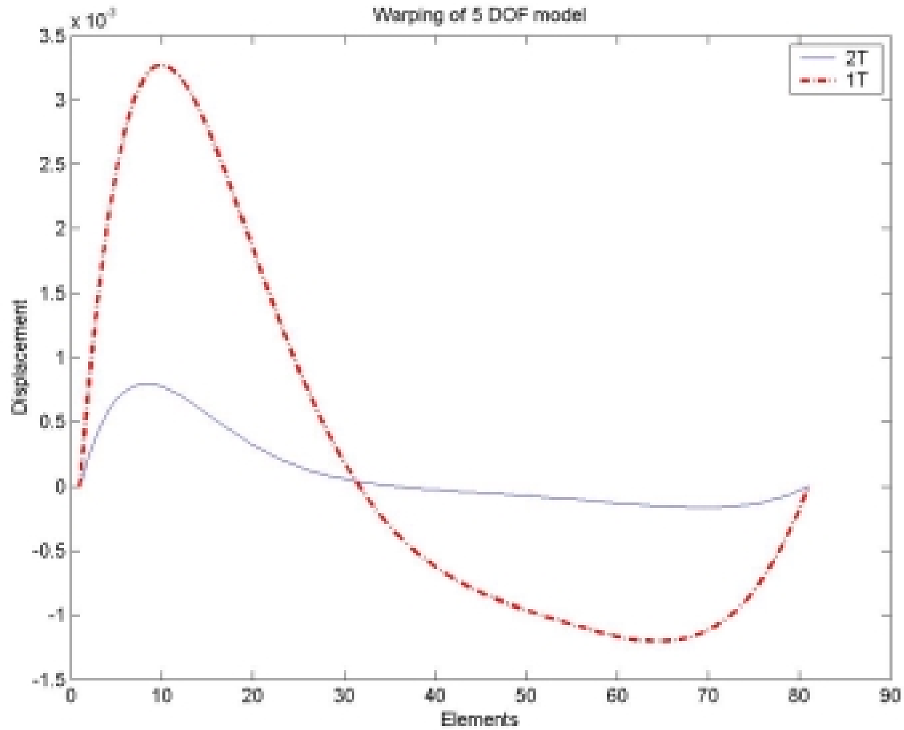

Figure 22. Warping Plot for 5 DOF Model

The maximum warping term for original thickness (1T) is again approximately four times as much as the value for double thickness (2T) case. The warping value stays positive until element 38, almost at the center of the beam, and than stays negative after the midpoint of the beam until the loaded end. Figure 14 shows the distortion variable behavior throughout the beam. It is nearly impossible to measure the distortion or the warping in a plate FEA model or a curved beam. However, it is possible to see warping and distortion in the cross-sections, but it is very difficult to quantify them.

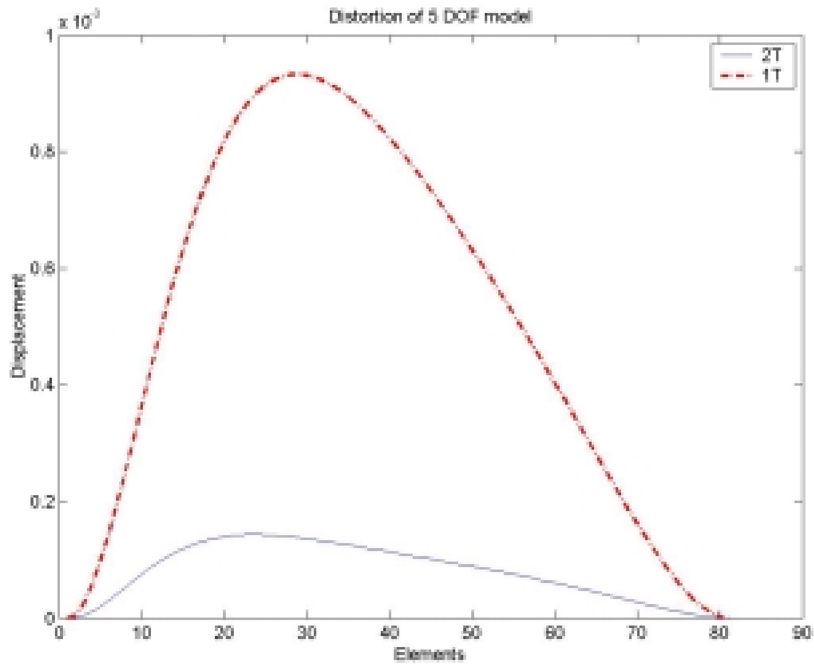

Figure 23. Distortion Plot for 5 DOF Model

Similar to the other variables, the original thickness (1T) case has higher displacements than the double thickness (2T) case. The maximum distortion value for the original thickness case is almost seven times as much as the value for the double thickness case. The overall results for the behavior of all five variables for both original and double thickness cases are summarized in the following table.

Table 6. Maximum Displacement Results for All Five Variables

	1T	2T	1T/2T
Vertical Displacement	1.830E-03	6.674E-04	2.7
Max Twist	5.568E-04	3.138E-04	1.8
Bending	1.808E-03	6.991E-04	2.6
Max Warping	3.275E-03	7.977E-04	4.1
Max Distortion	9.326E-04	1.436E-04	6.5

From table 6, it is clear that the difference between the original (1T) and double (2T) thickness variables is the highest in distortion variable, where the difference is almost seven times. This further clarifies that the distortion variable is the most sensitive variable to thickness. In other words, stiffness of the thin-walled rectangular beam is linear with thickness, except for the distortion variable. Hence, distortion is the only non-linear variable that gives thin-walled problems, and in order to reduce these effects, the design should concentrate on the distortion variable.

CHAPTER VI

CONTROLLING THE DISTORTION VARIABLE

As seen from previous chapters, the addition of the distortion variable increases the displacements in the beam. This increase in displacement was proven to be the accurate and correct result because it compared well with the results obtained from the FEA plate model. However, the displacement must be decreased in order to make a stiffer beam. This can be achieved by controlling the distortion variable, which is the cause of this increase in displacement result.

Distortion can be reduced locally by altering the thickness in a small area of a joint, thus stiffening the joint with minimal weight change. Thickness alterations can be achieved through tailor-welded patches (area of increased thickness), as will be seen in this section. The model that was used for the thickness alteration was the same model as the one used for Kim and Kim thin-walled beam theory. The beam model is shown in the following figure, which also shows where the beam was thickened.

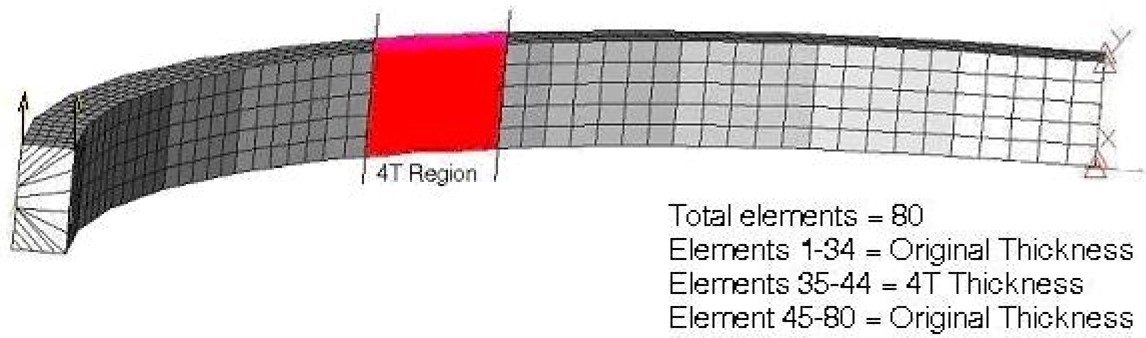

Figure 24. Thin-Walled Curved Beam with Altered Thickness (ALGOR FEA)

The thickness is altered to four times the original thickness at the approximate center of the beam. As presented in figure 24, the total element size for this model was 80 elements, and the thickness was quadrupled from element 35 through 44.

The Kim and Kim 5 DOF model was very useful in this case because it enables the study of each variable throughout the beam. The following figures present the behavior for all five degrees of freedom used in this model. Both the regular five DOF Kim and Kim results and the five DOF Kim and Kim with altered thickness results are presented in these figures. The original model is presented in the figures as 1T, and the model with altered thickness is referred to as 1T/4T. Figure 16 shows a better representation for vertical displacement behavior throughout the beam for this case. The region where the beam thickness was quadrupled (Tailor-welded patch) is also presented in the following figure.

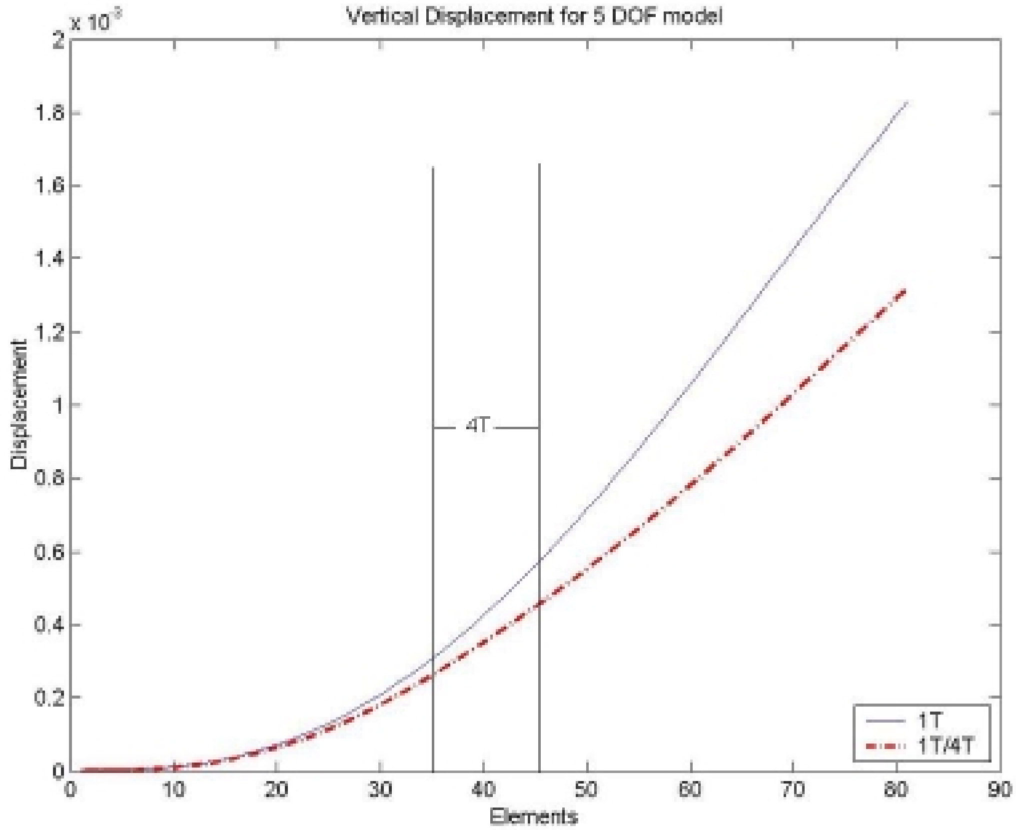

Figure 25. Vertical Displacement Plot for 5 DOF Model with Altered Thickness

As seen from this figure, the Kim and Kim model with quadrupled thickness (1T/4T) present in the middle of beam had lower displacement values at each element than the normal Kim and Kim 5 DOF case (1T). Therefore, quadrupling the thickness seems to affect the displacements in this case. Figure 26 shows the behavior of the rotation (twist) variable.

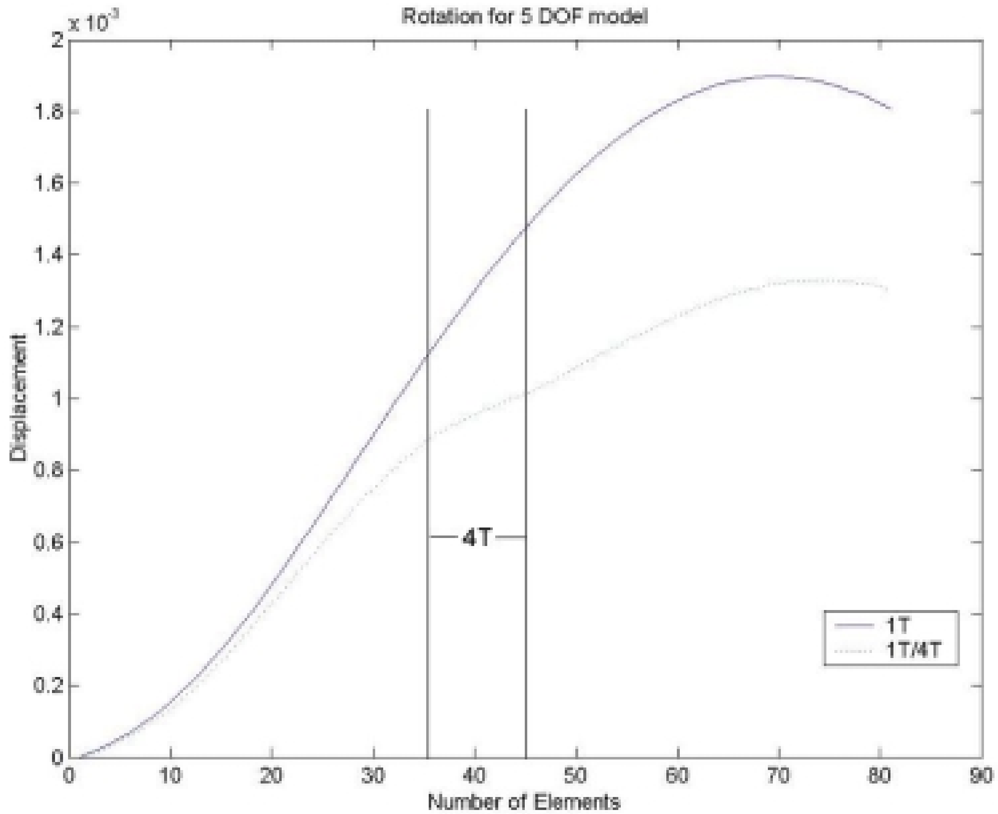

Figure 26. Bending Rotation Plot for 5 DOF Model with Altered Thickness

Again, the 1T/4T (quadruple thickness case) shows lesser displacements in twist variable than regular the 1T Kim and Kim model. Figure 27 presents the behavior of the warping variable.

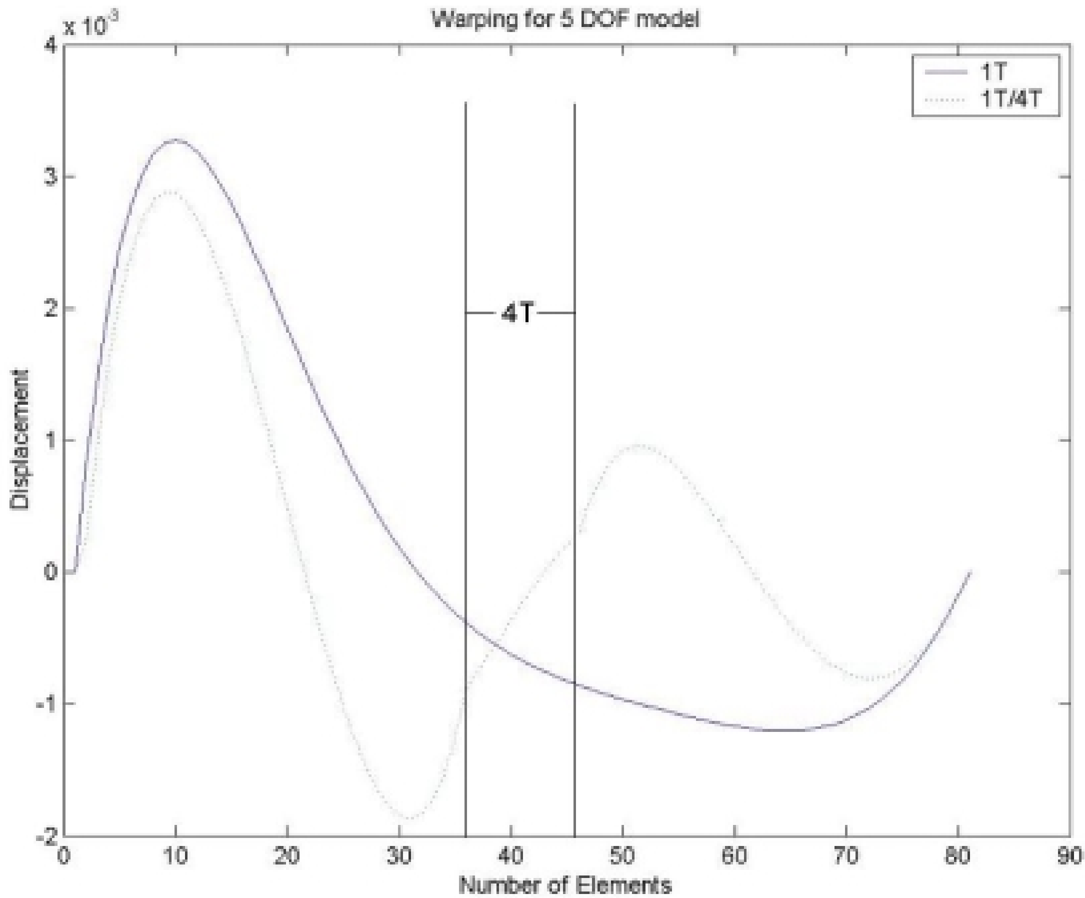

Figure 27. Warping Plot for 5 DOF Model with Altered Thickness

Similar behavior is shown in this figure in that the 1T/4T case shows lesser displacement. In the Tailor welded patch region, the displacement starts to decrease and when the thickness is changed back to original than the displacements starts to return to its normal behavior. Figure 28 shows the behavior of the distortion variable.

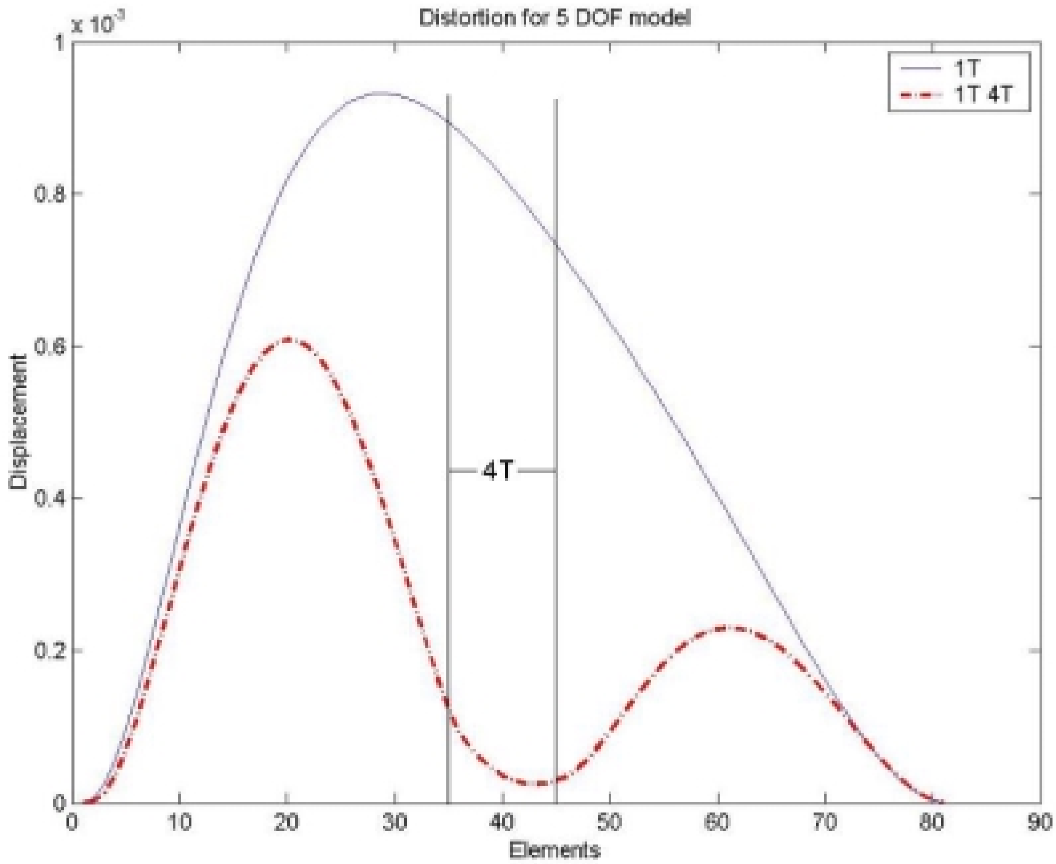

Figure 28. Distortion Plot for 5 DOF Model with Altered Thickness

The distortion plot shows similar behavior compared to the warping plot. The overall displacements in the distortion variable are smaller than those in the original 1T case. The 4T region shows a bump in the displacement. This is because when the thickness is increased in the region, the beam stiffens up and displacements are decreased.

As previously mentioned in earlier discussions and as shown in these graphs, the distortion and other displacements in the beam drop considerably when thickness is increased. Tailor-welded patches (area of increased thickness) reduce the distortion, which is the only non-linear variable with regards to thickness. This is further proven by the following summarized data of the preceding figures. The results of displacement of each variable are presented in table 7.

Table 7. Comparison of Displacement Values for Each Variable for 5 DOF (Original Model) and 5 DOF (Altered Thickness Model)

Five Variables	5 DOF 1T	5 DOF 1T/4T
Vertical Displacement	1.83E-03	1.31E-03
Max Twist	5.568E-04	5.672E-04
Bending	1.808E-03	1.305E-03
Max Warping	3.275E-03	2.885E-03
Max Distortion	9.326E-04	6.085E-04

The displacement values are compared for Kim and Kim 5 DOF original model with the model with Tailor welded patch applied. For each variable, the 1T/4T case has a lower displacement value than the original model (1T).

Furthermore, the results obtained from this case of Kim and Kim 5 DOF model with thickness altered in the middle of the beam were compared to the results of the regular Kim and Kim 5 DOF case results, the 3D FEA plate model results, and the Kim and Kim 3 DOF model results. For each model, displacements were calculated for both original (1T) and double (2T) thickness cases. Using the displacements for the 1T and 2T models, the lambda for each element was also calculated. The results are presented in table 8.

Table 8. Displacement Data at the Loaded End (Meters) for 3 DOF, 5 DOF (Original Model), FEA Plate, and 5 DOF (Altered Thickness Model)

Vertical Displacements	**1T**	**2T**	**Lambda**
3D FEA Plate Model [theoretical]	1.78E-03	6.70E-04	1.41E+00
1D FEA Kim & Kim, 3 DOF	1.04E-03	5.20E-04	1.00E+00
1D FEA Kim & Kim, 5 DOF	1.83E-03	6.67E-04	1.46E+00
1D FEA Kim & Kim, 5 DOF [4T thickness in middle]	1.31E-03	2.77E-04	1.12E+00

The results in table 8 display that thickening the beam in the middle causes substantial decreases in vertical displacement. In addition, the lambda for the 5 DOF model with quadrupled thickness in the middle of the beam was reduced from 1.45 (original Kim and Kim 5 DOF model) to 1.12. This was achieved through thickening the beam in the center. As a result of this, the beam was stiffened up to 36 percent.

CHAPTER VII

CONCLUSIONS

A new five DOF model is proposed by Kim and Kim and used in this research. This is the first model of its kind that takes into account both warping and distortion variables along with the three kinematics variables of lift, bend, and twist. The new model works very well for the out-of-plane bending case for the thin-walled curved rectangular beams and gives accurate displacement results compared to the 3D Finite Element Analysis plate theory results presented by ALGOR FEA.

Unlike the straight thin-walled beams where the effects of warping and distortion are local, these two variables significantly affect the overall stiffness of the curved beam. However it is shown that the additional consideration of warping or distortion alone does not improve the solution accuracy. The interaction between warping and distortion variables gives accurate results.

The conventional beam model (3 DOF model) does not compare well with the 3D FEA plate model displacement results neither does the models with just either warping or distortion added or the Timoshenko (Castigliano's Theorem approach) model. The complexity resulting from the introduction of additional variables is well compensated by the great improvement in solution accuracy. This was seen as the displacement results neared to the theoretical 3D FEA displacement value with the addition of both warping and distortion variables.

Along with three DOF of vertical lift, bending, and torsion, warping and distortion variables must be present to get accurate results for a curved beam subjected to out-of-plane load. Only the Kim and Kim model with 5 DOF model gave accurate displacement results. This model showed that distortion is the only non-linear variable that gives problems and makes lambda to be non-linear as well. In other words, the stiffness is linear with thickness except for the distortion variable. .

The use of Tailor-welded patches is proposed as a solution for controlling the troublesome distortion variable that increases the displacements in the beam. The results show that the tailor-welded patches decrease not only the distortion but also decrease all other displacements in the beam. Therefore, distortion can be controlled and reduced locally by altering the thickness in a small area of the beam. This in turn pushes lambda toward 1.0, thus stiffening the beam with minimal weight change.

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APPENDIX A

MATHCAD SAMPLE WORKSHEET FOR SOLVING SECTION DEFORMATIONS

Mathcad Sample Worksheets for Solving Sectional Deformations

Dimension and boundary condition Information: All Units are Metric meter, Newton $\quad$ ORIGIN:= 1

$b := 0.05$ meter — base of the rectangular beam

$h := 0.1$ meter — height of the rectangular beam

$t := 0.002$ meter — thickness of the rectangular beam

$R := 1$ meter — distance from the center of the curvature to the centroid line of the beam cross section

$E := 200 \cdot 10^{9}$ Pa — modulus of elasticity

$\nu := .3$ — poisson's ratio

$G := \dfrac{E}{2 \cdot (1+\nu)} \qquad G = 7.692 \times 10^{10}$ Pa — modulus of rigidity

$V := 100$ N — applied vertical force

$\phi := 90 \cdot deg$ — central angle

$E_1 := \dfrac{E}{1-\nu^2} \qquad E_1 = 2.198 \times 10^{11}$ Pa — material property

Calculation of element length (L)

$side_1 := 1 \qquad side_2 := 1 \qquad angle_3 := \dfrac{9}{8} \cdot deg$

$side_3 := \sqrt{side_1{}^2 + side_2{}^2 - 2 \cdot side_1 \cdot side_2 \cdot \cos(angle_3)} \qquad side_3 = 0.02$

$L := side_3 \qquad L = 0.02$ meter — element length

Defining range $\quad i := 1, 2 .. 4$

Defining section deformation functions for rotation theta(phi) about phi axis

$\psi_{\theta\eta i}(s_i) := -s_i \qquad \psi_{\theta s i_i} := \begin{cases} \dfrac{b}{2} & \text{if } i = 1 \\ \dfrac{b}{2} & \text{if } i = 3 \\ \dfrac{h}{2} & \text{otherwise} \end{cases}$

Defining section deformation functions for vertical displacement eta(phi) about y-axis

$$\psi_{\eta\eta i_i} := \begin{cases} 1 & \text{if } i = 2 \\ -1 & \text{if } i = 4 \\ 0 & \text{otherwise} \end{cases} \qquad \psi_{\eta si_i} := \begin{cases} 1 & \text{if } i = 1 \\ -1 & \text{if } i = 3 \\ 0 & \text{otherwise} \end{cases}$$

Defining section deformation functions for the rotation beta(phi) about p-axis

$$\psi_{\beta\phi1}(s_i) := -s_i \qquad \psi_{\beta\phi2}(s_i) := \frac{-h}{2} \qquad \psi_{\beta\phi3}(s_i) := s_i \qquad \psi_{\beta\phi4}(s_i) := \frac{h}{2}$$

Defining section deformation functions for warping U(phi)

$$\psi_{U\phi1}(s_i) := \frac{b}{2}\cdot s_i \qquad \psi_{U\phi2}(s_i) := \frac{-h}{2}\cdot s_i \qquad \psi_{U\phi3}(s_i) := \frac{b}{2}\cdot s_i \qquad \psi_{U\phi4}(s_i) := \frac{-h}{2}\cdot s_i$$

Defining section deformation functions for distortion chi(phi)

$$\psi_{\chi\eta1}(s_i) := \frac{2\cdot h}{b+h}\cdot\left(\frac{2\cdot s_i^{\,3}}{h^2} - \frac{3\cdot s_i}{2}\right) \qquad \psi_{\chi\eta3}(s_i) := \frac{2\cdot h}{b+h}\cdot\left(\frac{2\cdot s_i^{\,3}}{h^2} - \frac{3\cdot s_i}{2}\right)$$

$$\psi_{\chi\eta2}(s_i) := \frac{-2\cdot b}{b+h}\cdot\left(\frac{2\cdot s_i^{\,3}}{b^2} - \frac{3\cdot s_i}{2}\right) \qquad \psi_{\chi\eta4}(s_i) := \frac{-2\cdot b}{b+h}\cdot\left(\frac{2\cdot s_i^{\,3}}{b^2} - \frac{3\cdot s_i}{2}\right)$$

$$\psi_{\chi si_i} := \begin{cases} \frac{-b}{2} & \text{if } i = 1 \\ \frac{-b}{2} & \text{if } i = 3 \\ \frac{h}{2} & \text{otherwise} \end{cases}$$

Defining additional functions that are necessary for section deformation coefficients calculations

$$\xi_i := \begin{cases} 1 & \text{if } i = 1 \\ (-1) & \text{if } i = 3 \\ 0 & \text{otherwise} \end{cases} \qquad \zeta_i := \begin{cases} -1 & \text{if } i = 2 \\ 1 & \text{if } i = 4 \\ 0 & \text{otherwise} \end{cases}$$

Calculating sectional deformation coefficients

$$a_1 := \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_1)^2 \cdot \left(1 + \xi_1 \cdot \frac{b}{2 \cdot R}\right) + (\zeta_1)^2 \cdot \left(1 + \zeta_1 \cdot \frac{s_i}{R}\right) \right]^2 \cdot \psi_{\beta\phi 1}(s_i)^2 \, d\eta_i \, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_2)^2 \cdot \left(1 + \xi_2 \cdot \frac{b}{2 \cdot R}\right) + (\zeta_2)^2 \cdot \left(1 + \zeta_2 \cdot \frac{s_i}{R}\right) \right]^2 \cdot \psi_{\beta\phi 2}(s_i)^2 \, d\eta_i \, ds_i \ldots$$

$$+ \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_3)^2 \cdot \left(1 + \xi_3 \cdot \frac{b}{2 \cdot R}\right) + (\zeta_3)^2 \cdot \left(1 + \zeta_3 \cdot \frac{s_i}{R}\right) \right]^2 \cdot \psi_{\beta\phi 3}(s_i)^2 \, d\eta_i \, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_4)^2 \cdot \left(1 + \xi_4 \cdot \frac{b}{2 \cdot R}\right) + (\zeta_4)^2 \cdot \left(1 + \zeta_4 \cdot \frac{s_i}{R}\right) \right]^2 \cdot \psi_{\beta\phi 4}(s_i)^2 \, d\eta_i \, ds_i$$

$$\boxed{a_1 = 8.336 \times 10^{-7}}$$

$$a_2 := \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\frac{-\xi_1}{R} \cdot \left(1 + \xi_1 \cdot \frac{b}{2 \cdot R}\right) \cdot \psi_{\theta\eta\, i}(s_i) - \left(\frac{\zeta_1}{R}\right) \cdot \left(1 + \zeta_1 \cdot \frac{s_i}{R}\right) \cdot \psi_{\theta si_1} \right]^2 d\eta_i \, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\frac{-\xi_2}{R} \cdot \left(1 + \xi_2 \cdot \frac{b}{2 \cdot R}\right) \cdot \psi_{\theta\eta\, i}(s_i) - \left(\frac{\zeta_2}{R}\right) \cdot \left(1 + \zeta_2 \cdot \frac{s_i}{R}\right) \cdot \psi_{\theta si_2} \right]^2 d\eta_i \, ds_i \ldots$$

$$+ \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\frac{-\xi_3}{R} \cdot \left(1 + \xi_3 \cdot \frac{b}{2 \cdot R}\right) \cdot \psi_{\theta\eta\, i}(s_i) - \left(\frac{\zeta_3}{R}\right) \cdot \left(1 + \zeta_3 \cdot \frac{s_i}{R}\right) \cdot \psi_{\theta si_3} \right]^2 d\eta_i \, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\frac{-\xi_4}{R} \cdot \left(1 + \xi_4 \cdot \frac{b}{2 \cdot R}\right) \cdot \psi_{\theta\eta\, i}(s_i) - \left(\frac{\zeta_4}{R}\right) \cdot \left(1 + \zeta_4 \cdot \frac{s_i}{R}\right) \cdot \psi_{\theta si_4} \right]^2 d\eta_i \, ds_i$$

$$\boxed{a_2 = 8.336 \times 10^{-7}}$$

$$a_3 := \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\frac{-\xi_1}{R}\cdot\left(1+\xi_1\cdot\frac{b}{2\cdot R}\right)\cdot\psi_{\theta\eta i}(s_i) - \left(\frac{\zeta_1}{R}\right)\cdot\left(1+\zeta_1\cdot\frac{s_i}{R}\right)\cdot\psi_{\theta si_1}\right]\cdot\left[(\xi_1)^2\cdot\left(1+\xi_1\cdot\frac{b}{2\cdot R}\right)+(\zeta_1)^2\cdot\left(1+\zeta_1\cdot\frac{s_i}{R}\right)\right]\cdot\psi_{\beta\phi 1}(s_i)\, d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\frac{-\xi_2}{R}\cdot\left(1+\xi_2\cdot\frac{b}{2\cdot R}\right)\cdot\psi_{\theta\eta i}(s_i) - \left(\frac{\zeta_2}{R}\right)\cdot\left(1+\zeta_2\cdot\frac{s_i}{R}\right)\cdot\psi_{\theta si_2}\right]\cdot\left[(\xi_2)^2\cdot\left(1+\xi_2\cdot\frac{b}{2\cdot R}\right)+(\zeta_2)^2\cdot\left(1+\zeta_2\cdot\frac{s_i}{R}\right)\right]\cdot\psi_{\beta\phi 2}(s_i)\, d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\frac{-\xi_3}{R}\cdot\left(1+\xi_3\cdot\frac{b}{2\cdot R}\right)\cdot\psi_{\theta\eta i}(s_i) - \left(\frac{\zeta_3}{R}\right)\cdot\left(1+\zeta_3\cdot\frac{s_i}{R}\right)\cdot\psi_{\theta si_3}\right]\cdot\left[(\xi_3)^2\cdot\left(1+\xi_3\cdot\frac{b}{2\cdot R}\right)+(\zeta_3)^2\cdot\left(1+\zeta_3\cdot\frac{s_i}{R}\right)\right]\cdot\psi_{\beta\phi 3}(s_i)\, d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\frac{-\xi_4}{R}\cdot\left(1+\xi_4\cdot\frac{b}{2\cdot R}\right)\cdot\psi_{\theta\eta i}(s_i) - \left(\frac{\zeta_4}{R}\right)\cdot\left(1+\zeta_4\cdot\frac{s_i}{R}\right)\cdot\psi_{\theta si_4}\right]\cdot\left[(\xi_4)^2\cdot\left(1+\xi_4\cdot\frac{b}{2\cdot R}\right)+(\zeta_4)^2\cdot\left(1+\zeta_4\cdot\frac{s_i}{R}\right)\right]\cdot\psi_{\beta\phi 4}(s_i)\, d\eta_i\, ds_i$$

$$\boxed{a_3 = -8.336\times 10^{-7}}$$

$$b_1 := \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_1)^2\cdot\left(1+\xi_1\cdot\frac{b}{2\cdot R}\right)+(\zeta_1)^2\cdot\left(1+\zeta_1\cdot\frac{s_i}{R}\right)\right]^2\cdot\left(\psi_{\theta si_1}\right)^2 d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_2)^2\cdot\left(1+\xi_2\cdot\frac{b}{2\cdot R}\right)+(\zeta_2)^2\cdot\left(1+\zeta_2\cdot\frac{s_i}{R}\right)\right]^2\cdot\left(\psi_{\theta si_2}\right)^2 d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_3)^2\cdot\left(1+\xi_3\cdot\frac{b}{2\cdot R}\right)+(\zeta_3)^2\cdot\left(1+\zeta_3\cdot\frac{s_i}{R}\right)\right]^2\cdot\left(\psi_{\theta si_3}\right)^2 d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_4)^2\cdot\left(1+\xi_4\cdot\frac{b}{2\cdot R}\right)+(\zeta_4)^2\cdot\left(1+\zeta_4\cdot\frac{s_i}{R}\right)\right]^2\cdot\left(\psi_{\theta si_4}\right)^2 d\eta_i\, ds_i$$

$$\boxed{b_1 = 7.503\times 10^{-7}}$$

$$b_2 := \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} (\xi_1)^2 \cdot \left(1 + \xi_1 \cdot \frac{b}{2 \cdot R}\right)^2 \cdot \left(\psi_{\eta si_1}\right)^2 \, d\eta_i \, ds_i \; ...$$
$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} (\xi_2)^2 \cdot \left(1 + \xi_2 \cdot \frac{b}{2 \cdot R}\right)^2 \cdot \left(\psi_{\eta si_2}\right)^2 \, d\eta_i \, ds_i \; ...$$
$$+ \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} (\xi_3)^2 \cdot \left(1 + \xi_3 \cdot \frac{b}{2 \cdot R}\right)^2 \cdot \left(\psi_{\eta si_3}\right)^2 \, d\eta_i \, ds_i \; ...$$
$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} (\xi_4)^2 \cdot \left(1 + \xi_4 \cdot \frac{b}{2 \cdot R}\right)^2 \cdot \left(\psi_{\eta si_4}\right)^2 \, d\eta_i \, ds_i$$

$$\boxed{b_2 = 4.002 \times 10^{-4}}$$

$$b_4 := \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_1)^2 \cdot \frac{d}{ds_i}\psi_{\beta\phi 1}(s_i) + \left(\frac{\zeta_1}{R}\right) \cdot \left(1 + \zeta_1 \cdot \frac{s_i}{R}\right) \cdot \psi_{\beta\phi 1}(s_i)\right]^2 d\eta_i \, ds_i \; ...$$
$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_2)^2 \cdot \frac{d}{ds_i}\psi_{\beta\phi 2}(s_i) + \left(\frac{\zeta_2}{R}\right) \cdot \left(1 + \zeta_2 \cdot \frac{s_i}{R}\right) \cdot \psi_{\beta\phi 2}(s_i)\right]^2 d\eta_i \, ds_i \; ..$$
$$+ \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_3)^2 \cdot \frac{d}{ds_i}\psi_{\beta\phi 3}(s_i) + \left(\frac{\zeta_3}{R}\right) \cdot \left(1 + \zeta_3 \cdot \frac{s_i}{R}\right) \cdot \psi_{\beta\phi 3}(s_i)\right]^2 d\eta_i \, ds_i \; ..$$
$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_4)^2 \cdot \frac{d}{ds_i}\psi_{\beta\phi 4}(s_i) + \left(\frac{\zeta_4}{R}\right) \cdot \left(1 + \zeta_4 \cdot \frac{s_i}{R}\right) \cdot \psi_{\beta\phi 4}(s_i)\right]^2 d\eta_i \, ds_i$$

$$\boxed{b_4 = 4.005 \times 10^{-4}}$$

$$b_6 := \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_1)^2 \cdot \left(1 + \xi_1 \cdot \frac{b}{2 \cdot R}\right) + (\zeta_1)^2 \cdot \left(1 + \zeta_1 \cdot \frac{s_i}{R}\right)\right] \cdot (\xi_1)^2 \cdot \left(1 + \xi_1 \cdot \frac{b}{2 \cdot R}\right) \cdot \left(\Psi_{\theta si_1}\right) \cdot \left(\Psi_{\eta si_1}\right) d\eta_i \, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_2)^2 \cdot \left(1 + \xi_2 \cdot \frac{b}{2 \cdot R}\right) + (\zeta_2)^2 \cdot \left(1 + \zeta_2 \cdot \frac{s_i}{R}\right)\right] \cdot (\xi_2)^2 \cdot \left(1 + \xi_2 \cdot \frac{b}{2 \cdot R}\right) \cdot \left(\Psi_{\theta si_2}\right) \cdot \left(\Psi_{\eta si_2}\right) d\eta_i \, ds_i \ldots$$

$$+ \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_3)^2 \cdot \left(1 + \xi_3 \cdot \frac{b}{2 \cdot R}\right) + (\zeta_3)^2 \cdot \left(1 + \zeta_3 \cdot \frac{s_i}{R}\right)\right] \cdot (\xi_3)^2 \cdot \left(1 + \xi_3 \cdot \frac{b}{2 \cdot R}\right) \cdot \left(\Psi_{\theta si_3}\right) \cdot \left(\Psi_{\eta si_3}\right) d\eta_i \, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_4)^2 \cdot \left(1 + \xi_4 \cdot \frac{b}{2 \cdot R}\right) + (\zeta_4)^2 \cdot \left(1 + \zeta_4 \cdot \frac{s_i}{R}\right)\right] \cdot (\xi_4)^2 \cdot \left(1 + \xi_4 \cdot \frac{b}{2 \cdot R}\right) \cdot \left(\Psi_{\theta si_4}\right) \cdot \left(\Psi_{\eta si_4}\right) d\eta_i \, ds_i$$

$\boxed{b_6 = 5 \times 10^{-7}}$

$$b_{12} := \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_1)^2 \cdot \frac{d}{ds_i} \Psi_{\beta\phi 1}(s_i) + \left(\frac{\zeta_1}{R}\right) \cdot \left(1 + \zeta_1 \cdot \frac{s_i}{R}\right) \cdot \Psi_{\beta\phi 1}(s_i)\right] \cdot (\xi_1)^2 \cdot \left(1 + \xi_1 \cdot \frac{b}{2 \cdot R}\right) \cdot \Psi_{\eta si_1} \, d\eta_i \, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_2)^2 \cdot \frac{d}{ds_i} \Psi_{\beta\phi 2}(s_i) + \left(\frac{\zeta_2}{R}\right) \cdot \left(1 + \zeta_2 \cdot \frac{s_i}{R}\right) \cdot \Psi_{\beta\phi 2}(s_i)\right] \cdot (\xi_2)^2 \cdot \left(1 + \xi_2 \cdot \frac{b}{2 \cdot R}\right) \cdot \Psi_{\eta si_2} \, d\eta_i \, ds_i \ldots$$

$$+ \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_3)^2 \cdot \frac{d}{ds_i} \Psi_{\beta\phi 3}(s_i) + \left(\frac{\zeta_3}{R}\right) \cdot \left(1 + \zeta_3 \cdot \frac{s_i}{R}\right) \cdot \Psi_{\beta\phi 3}(s_i)\right] \cdot (\xi_3)^2 \cdot \left(1 + \xi_3 \cdot \frac{b}{2 \cdot R}\right) \cdot \Psi_{\eta si_3} \, d\eta_i \, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_4)^2 \cdot \frac{d}{ds_i} \Psi_{\beta\phi 4}(s_i) + \left(\frac{\zeta_4}{R}\right) \cdot \left(1 + \zeta_4 \cdot \frac{s_i}{R}\right) \cdot \Psi_{\beta\phi 4}(s_i)\right] \cdot (\xi_4)^2 \cdot \left(1 + \xi_4 \cdot \frac{b}{2 \cdot R}\right) \cdot \Psi_{\eta si_4} \, d\eta_i \, ds_i$$

$\boxed{b_{12} = -4 \times 10^{-4}}$

$$b_{13} := \int_{\frac{-h}{2}}^{\frac{h}{2}}\int_{\frac{-t}{2}}^{\frac{t}{2}}\left[(\xi_1)^2\cdot\left(1+\xi_1\cdot\frac{b}{2\cdot R}\right)+(\zeta_1)^2\cdot\left(1+\zeta_1\cdot\frac{s_i}{R}\right)\right]\cdot\psi_{\theta si_1}\cdot\left[(\xi_1)^2\cdot\frac{d}{ds_i}\psi_{\beta\phi1}(s_i)+\left(\frac{\zeta_1}{R}\right)\cdot\left(1+\zeta_1\cdot\frac{s_i}{R}\right)\cdot\psi_{\beta\phi1}(s_i)\right]d\eta_i\,ds_i\,\ldots$$

$$+\int_{\frac{-b}{2}}^{\frac{b}{2}}\int_{\frac{-t}{2}}^{\frac{t}{2}}\left[(\xi_2)^2\cdot\left(1+\xi_2\cdot\frac{b}{2\cdot R}\right)+(\zeta_2)^2\cdot\left(1+\zeta_2\cdot\frac{s_i}{R}\right)\right]\cdot\psi_{\theta si_2}\cdot\left[(\xi_2)^2\cdot\frac{d}{ds_i}\psi_{\beta\phi2}(s_i)+\left(\frac{\zeta_2}{R}\right)\cdot\left(1+\zeta_2\cdot\frac{s_i}{R}\right)\cdot\psi_{\beta\phi2}(s_i)\right]d\eta_i\,ds_i\,\ldots$$

$$+\int_{\frac{-h}{2}}^{\frac{h}{2}}\int_{\frac{-t}{2}}^{\frac{t}{2}}\left[(\xi_3)^2\cdot\left(1+\xi_3\cdot\frac{b}{2\cdot R}\right)+(\zeta_3)^2\cdot\left(1+\zeta_3\cdot\frac{s_i}{R}\right)\right]\cdot\psi_{\theta si_3}\cdot\left[(\xi_3)^2\cdot\frac{d}{ds_i}\psi_{\beta\phi3}(s_i)+\left(\frac{\zeta_3}{R}\right)\cdot\left(1+\zeta_3\cdot\frac{s_i}{R}\right)\cdot\psi_{\beta\phi3}(s_i)\right]d\eta_i\,ds_i\,\ldots$$

$$+\int_{\frac{-b}{2}}^{\frac{b}{2}}\int_{\frac{-t}{2}}^{\frac{t}{2}}\left[(\xi_4)^2\cdot\left(1+\xi_4\cdot\frac{b}{2\cdot R}\right)+(\zeta_4)^2\cdot\left(1+\zeta_4\cdot\frac{s_i}{R}\right)\right]\cdot\psi_{\theta si_4}\cdot\left[(\xi_4)^2\cdot\frac{d}{ds_i}\psi_{\beta\phi4}(s_i)+\left(\frac{\zeta_4}{R}\right)\cdot\left(1+\zeta_4\cdot\frac{s_i}{R}\right)\cdot\psi_{\beta\phi4}(s_i)\right]d\eta_i\,ds_i$$

$$b_{13} = 2.501\times 10^{-7}$$

$$c_1 := \int_{\frac{-h}{2}}^{\frac{h}{2}}\int_{\frac{-t}{2}}^{\frac{t}{2}}\left[(\xi_1)^2\cdot\left(1+\xi_1\cdot\frac{b}{2\cdot R}\right)+(\zeta_1)^2\cdot\left(1+\zeta_1\cdot\frac{s_i}{R}\right)\right]^2\cdot\psi_{U\phi1}(s_i)^2\,d\eta_i\,ds_i\,\ldots$$

$$+\int_{\frac{-b}{2}}^{\frac{b}{2}}\int_{\frac{-t}{2}}^{\frac{t}{2}}\left[(\xi_2)^2\cdot\left(1+\xi_2\cdot\frac{b}{2\cdot R}\right)+(\zeta_2)^2\cdot\left(1+\zeta_2\cdot\frac{s_i}{R}\right)\right]^2\cdot\psi_{U\phi2}(s_i)^2\,d\eta_i\,ds_i\,\ldots$$

$$+\int_{\frac{-h}{2}}^{\frac{h}{2}}\int_{\frac{-t}{2}}^{\frac{t}{2}}\left[(\xi_3)^2\cdot\left(1+\xi_3\cdot\frac{b}{2\cdot R}\right)+(\zeta_3)^2\cdot\left(1+\zeta_3\cdot\frac{s_i}{R}\right)\right]^2\cdot\psi_{U\phi3}(s_i)^2\,d\eta_i\,ds_i\,\ldots$$

$$+\int_{\frac{-b}{2}}^{\frac{b}{2}}\int_{\frac{-t}{2}}^{\frac{t}{2}}\left[(\xi_4)^2\cdot\left(1+\xi_4\cdot\frac{b}{2\cdot R}\right)+(\zeta_4)^2\cdot\left(1+\zeta_4\cdot\frac{s_i}{R}\right)\right]^2\cdot\psi_{U\phi4}(s_i)^2\,d\eta_i\,ds_i$$

$$c_1 = 3.127\times 10^{-10}$$

$$c_2 := \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\left(\frac{-\xi_1}{R}\right)\cdot\left(1+\xi_1\cdot\frac{b}{2\cdot R}\right)\cdot\psi_{\chi\eta1}(s_i) - \left(\frac{\zeta_1}{R}\right)\cdot\left(1+\zeta_1\cdot\frac{s_i}{R}\right)\cdot\left(\psi_{\chi si_1} - \eta_i\cdot\frac{d}{ds_i}\psi_{\chi\eta1}(s_i)\right)\right]^2 d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\left(\frac{-\xi_2}{R}\right)\cdot\left(1+\xi_2\cdot\frac{b}{2\cdot R}\right)\cdot\psi_{\chi\eta2}(s_i) - \left(\frac{\zeta_2}{R}\right)\cdot\left(1+\zeta_2\cdot\frac{s_i}{R}\right)\cdot\left(\psi_{\chi si_2} - \eta_i\cdot\frac{d}{ds_i}\psi_{\chi\eta2}(s_i)\right)\right]^2 d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\left(\frac{-\xi_3}{R}\right)\cdot\left(1+\xi_3\cdot\frac{b}{2\cdot R}\right)\cdot\psi_{\chi\eta3}(s_i) - \left(\frac{\zeta_3}{R}\right)\cdot\left(1+\zeta_3\cdot\frac{s_i}{R}\right)\cdot\left(\psi_{\chi si_3} - \eta_i\cdot\frac{d}{ds_i}\psi_{\chi\eta3}(s_i)\right)\right]^2 d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\left(\frac{-\xi_4}{R}\right)\cdot\left(1+\xi_4\cdot\frac{b}{2\cdot R}\right)\cdot\psi_{\chi\eta4}(s_i) - \left(\frac{\zeta_4}{R}\right)\cdot\left(1+\zeta_4\cdot\frac{s_i}{R}\right)\cdot\left(\psi_{\chi si_4} - \eta_i\cdot\frac{d}{ds_i}\psi_{\chi\eta4}(s_i)\right)\right]^2 d\eta_i\, ds_i$$

$$\boxed{c_2 = 1.364\times 10^{-6}}$$

$$c_3 := \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left(-\eta_i\cdot\frac{d^2}{ds_i^2}\psi_{\chi\eta1}(s_i)\right)^2 d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[-\eta_i\cdot\left(\frac{d^2}{ds_i^2}\psi_{\chi\eta2}(s_i)\right)\right]^2 d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[-\eta_i\cdot\left(\frac{d^2}{ds_i^2}\psi_{\chi\eta3}(s_i)\right)\right]^2 d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left(-\eta_i\cdot\frac{d^2}{ds_i^2}\psi_{\chi\eta4}(s_i)\right)^2 d\eta_i\, ds_i$$

$$\boxed{c_3 = 4.267\times 10^{-7}}$$

$$c_4 := \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\left(\frac{-\xi_1}{R}\right)\cdot\left(1+\xi_1\cdot\frac{b}{2\cdot R}\right)\cdot\psi_{\chi\eta1}(s_i) - \left(\frac{\zeta_1}{R}\right)\cdot\left(1+\zeta_1\cdot\frac{s_i}{R}\right)\cdot\left(\psi_{\chi si_1} - \eta_i\cdot\frac{d}{ds_i}\psi_{\chi\eta1}(s_i)\right)\right]\cdot\left(-\nu\cdot\eta_i\cdot\frac{d^2}{ds_i^2}\psi_{\chi\eta1}(s_i)\right) d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\left(\frac{-\xi_2}{R}\right)\cdot\left(1+\xi_2\cdot\frac{b}{2\cdot R}\right)\cdot\psi_{\chi\eta2}(s_i) - \left(\frac{\zeta_2}{R}\right)\cdot\left(1+\zeta_2\cdot\frac{s_i}{R}\right)\cdot\left(\psi_{\chi si_2} - \eta_i\cdot\frac{d}{ds_i}\psi_{\chi\eta2}(s_i)\right)\right]\cdot\left(-\nu\cdot\eta_i\cdot\frac{d^2}{ds_i^2}\psi_{\chi\eta2}(s_i)\right) d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\left(\frac{-\xi_3}{R}\right)\cdot\left(1+\xi_3\cdot\frac{b}{2\cdot R}\right)\cdot\psi_{\chi\eta3}(s_i) - \left(\frac{\zeta_3}{R}\right)\cdot\left(1+\zeta_3\cdot\frac{s_i}{R}\right)\cdot\left(\psi_{\chi si_3} - \eta_i\cdot\frac{d}{ds_i}\psi_{\chi\eta3}(s_i)\right)\right]\cdot\left(-\nu\cdot\eta_i\cdot\frac{d^2}{ds_i^2}\psi_{\chi\eta3}(s_i)\right) d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\left(\frac{-\xi_4}{R}\right)\cdot\left(1+\xi_4\cdot\frac{b}{2\cdot R}\right)\cdot\psi_{\chi\eta4}(s_i) - \left(\frac{\zeta_4}{R}\right)\cdot\left(1+\zeta_4\cdot\frac{s_i}{R}\right)\cdot\left(\psi_{\chi si_4} - \eta_i\cdot\frac{d}{ds_i}\psi_{\chi\eta4}(s_i)\right)\right]\cdot\left(-\nu\cdot\eta_i\cdot\frac{d^2}{ds_i^2}\psi_{\chi\eta4}(s_i)\right) d\eta_i\, ds_i$$

$$\boxed{c_4 = 5.333\times 10^{-12}}$$

$$c_5 := \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\left(\frac{-\xi_1}{R}\right)\cdot\left(1+\xi_1\cdot\frac{b}{2\cdot R}\right)\cdot\psi_{\theta\eta i}(s_i) - \left(\frac{\zeta_1}{R}\right)\cdot\left(1+\zeta_1\cdot\frac{s_i}{R}\right)\cdot\psi_{\theta si_1}\right]\cdot\left[(\xi_1)^2\cdot\left(1+\xi_1\cdot\frac{b}{2\cdot R}\right) + (\zeta_1)^2\cdot\left(1+\zeta_1\cdot\frac{s_i}{R}\right)\right]\cdot\psi_{U\phi1}(s_i)\, d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\left(\frac{-\xi_2}{R}\right)\cdot\left(1+\xi_2\cdot\frac{b}{2\cdot R}\right)\cdot\psi_{\theta\eta i}(s_i) - \left(\frac{\zeta_2}{R}\right)\cdot\left(1+\zeta_2\cdot\frac{s_i}{R}\right)\cdot\psi_{\theta si_2}\right]\cdot\left[(\xi_2)^2\cdot\left(1+\xi_2\cdot\frac{b}{2\cdot R}\right) + (\zeta_2)^2\cdot\left(1+\zeta_2\cdot\frac{s_i}{R}\right)\right]\cdot\psi_{U\phi2}(s_i)\, d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\left(\frac{-\xi_3}{R}\right)\cdot\left(1+\xi_3\cdot\frac{b}{2\cdot R}\right)\cdot\psi_{\theta\eta i}(s_i) - \left(\frac{\zeta_3}{R}\right)\cdot\left(1+\zeta_3\cdot\frac{s_i}{R}\right)\cdot\psi_{\theta si_3}\right]\cdot\left[(\xi_3)^2\cdot\left(1+\xi_3\cdot\frac{b}{2\cdot R}\right) + (\zeta_3)^2\cdot\left(1+\zeta_3\cdot\frac{s_i}{R}\right)\right]\cdot\psi_{U\phi3}(s_i)\, d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\left(\frac{-\xi_4}{R}\right)\cdot\left(1+\xi_4\cdot\frac{b}{2\cdot R}\right)\cdot\psi_{\theta\eta i}(s_i) - \left(\frac{\zeta_4}{R}\right)\cdot\left(1+\zeta_4\cdot\frac{s_i}{R}\right)\cdot\psi_{\theta si_4}\right]\cdot\left[(\xi_4)^2\cdot\left(1+\xi_4\cdot\frac{b}{2\cdot R}\right) + (\zeta_4)^2\cdot\left(1+\zeta_4\cdot\frac{s_i}{R}\right)\right]\cdot\psi_{U\phi4}(s_i)\, d\eta_i\, ds_i$$

$$\boxed{c_5 = 6.25\times 10^{-10}}$$

$$c_6 := \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_1)^2 \cdot \left(1+\xi_1 \cdot \frac{b}{2 \cdot R}\right) + (\zeta_1)^2 \cdot \left(1+\zeta_1 \cdot \frac{s_i}{R}\right)\right]^2 \cdot \psi_{U\phi 1}(s_i) \cdot \psi_{\beta\phi 1}(s_i)\, d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_2)^2 \cdot \left(1+\xi_2 \cdot \frac{b}{2 \cdot R}\right) + (\zeta_2)^2 \cdot \left(1+\zeta_2 \cdot \frac{s_i}{R}\right)\right]^2 \cdot \psi_{U\phi 2}(s_i) \cdot \psi_{\beta\phi 2}(s_i)\, d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_3)^2 \cdot \left(1+\xi_3 \cdot \frac{b}{2 \cdot R}\right) + (\zeta_3)^2 \cdot \left(1+\zeta_3 \cdot \frac{s_i}{R}\right)\right]^2 \cdot \psi_{U\phi 3}(s_i) \cdot \psi_{\beta\phi 3}(s_i)\, d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_4)^2 \cdot \left(1+\xi_4 \cdot \frac{b}{2 \cdot R}\right) + (\zeta_4)^2 \cdot \left(1+\zeta_4 \cdot \frac{s_i}{R}\right)\right]^2 \cdot \psi_{U\phi 4}(s_i) \cdot \psi_{\beta\phi 4}(s_i)\, d\eta_i\, ds_i$$

$$\boxed{c_6 = -6.25 \times 10^{-10}}$$

$$c_7 := \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\left(\frac{-\xi_1}{R}\right) \cdot \left(1+\xi_1 \cdot \frac{b}{2 \cdot R}\right) \cdot \psi_{\chi\eta 1}(s_i) - \left(\frac{\zeta_1}{R}\right) \cdot \left(1+\zeta_1 \cdot \frac{s_i}{R}\right) \cdot \left(\psi_{\chi si_1} - \eta_i \cdot \frac{d}{ds_i} \psi_{\chi\eta 1}(s_i)\right)\right] \cdot \left[\left(\frac{-\xi_1}{R}\right) \cdot \left(1+\xi_1 \cdot \frac{b}{2 \cdot R}\right) \cdot \psi_{\theta\eta 1}(s_i) - \left(\frac{\zeta_1}{R}\right) \cdot \left(1+\zeta_1 \cdot \frac{s_i}{R}\right) \cdot \psi_{\theta si_1}\right] d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\left(\frac{-\xi_2}{R}\right) \cdot \left(1+\xi_2 \cdot \frac{b}{2 \cdot R}\right) \cdot \psi_{\chi\eta 2}(s_i) - \left(\frac{\zeta_2}{R}\right) \cdot \left(1+\zeta_2 \cdot \frac{s_i}{R}\right) \cdot \left(\psi_{\chi si_2} - \eta_i \cdot \frac{d}{ds_i} \psi_{\chi\eta 2}(s_i)\right)\right] \cdot \left[\left(\frac{-\xi_2}{R}\right) \cdot \left(1+\xi_2 \cdot \frac{b}{2 \cdot R}\right) \cdot \psi_{\theta\eta 2}(s_i) - \left(\frac{\zeta_2}{R}\right) \cdot \left(1+\zeta_2 \cdot \frac{s_i}{R}\right) \cdot \psi_{\theta si_2}\right] d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\left(\frac{-\xi_3}{R}\right) \cdot \left(1+\xi_3 \cdot \frac{b}{2 \cdot R}\right) \cdot \psi_{\chi\eta 3}(s_i) - \left(\frac{\zeta_3}{R}\right) \cdot \left(1+\zeta_3 \cdot \frac{s_i}{R}\right) \cdot \left(\psi_{\chi si_3} - \eta_i \cdot \frac{d}{ds_i} \psi_{\chi\eta 3}(s_i)\right)\right] \cdot \left[\left(\frac{-\xi_3}{R}\right) \cdot \left(1+\xi_3 \cdot \frac{b}{2 \cdot R}\right) \cdot \psi_{\theta\eta 3}(s_i) - \left(\frac{\zeta_3}{R}\right) \cdot \left(1+\zeta_3 \cdot \frac{s_i}{R}\right) \cdot \psi_{\theta si_3}\right] d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\left(\frac{-\xi_4}{R}\right) \cdot \left(1+\xi_4 \cdot \frac{b}{2 \cdot R}\right) \cdot \psi_{\chi\eta 4}(s_i) - \left(\frac{\zeta_4}{R}\right) \cdot \left(1+\zeta_4 \cdot \frac{s_i}{R}\right) \cdot \left(\psi_{\chi si_4} - \eta_i \cdot \frac{d}{ds_i} \psi_{\chi\eta 4}(s_i)\right)\right] \cdot \left[\left(\frac{-\xi_4}{R}\right) \cdot \left(1+\xi_4 \cdot \frac{b}{2 \cdot R}\right) \cdot \psi_{\theta\eta 4}(s_i) - \left(\frac{\zeta_4}{R}\right) \cdot \left(1+\zeta_4 \cdot \frac{s_i}{R}\right) \cdot \psi_{\theta si_4}\right] d\eta_i\, ds_i$$

$$\boxed{c_7 = 1.034 \times 10^{-6}}$$

$$
\begin{aligned}
c_9 := & \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_1)^2 \cdot \left(1+\xi_1 \cdot \frac{b}{2 \cdot R}\right) + (\zeta_1)^2 \cdot \left(1+\zeta_1 \cdot \frac{s_i}{R}\right)\right] \cdot \psi_{\beta\phi 1}(s_i) \cdot \left[\left(\frac{-\xi_1}{R}\right) \cdot \left(1+\xi_1 \cdot \frac{b}{2 \cdot R}\right) \cdot \psi_{\chi\eta 1}(s_i) - \left(\frac{\zeta_1}{R}\right) \cdot \left(1+\zeta_1 \cdot \frac{s_i}{R}\right) \cdot \left(\psi_{\chi s i_1} - \eta_i \cdot \frac{d}{ds_i} \psi_{\chi\eta 1}(s_i)\right)\right] d\eta_i \, ds_i \ldots \\
& + \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_2)^2 \cdot \left(1+\xi_2 \cdot \frac{b}{2 \cdot R}\right) + (\zeta_2)^2 \cdot \left(1+\zeta_2 \cdot \frac{s_i}{R}\right)\right] \cdot \psi_{\beta\phi 2}(s_i) \cdot \left[\left(\frac{-\xi_2}{R}\right) \cdot \left(1+\xi_2 \cdot \frac{b}{2 \cdot R}\right) \cdot \psi_{\chi\eta 2}(s_i) - \left(\frac{\zeta_2}{R}\right) \cdot \left(1+\zeta_2 \cdot \frac{s_i}{R}\right) \cdot \left(\psi_{\chi s i_2} - \eta_i \cdot \frac{d}{ds_i} \psi_{\chi\eta 2}(s_i)\right)\right] d\eta_i \, ds_i \ldots \\
& + \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_3)^2 \cdot \left(1+\xi_3 \cdot \frac{b}{2 \cdot R}\right) + (\zeta_3)^2 \cdot \left(1+\zeta_3 \cdot \frac{s_i}{R}\right)\right] \cdot \psi_{\beta\phi 3}(s_i) \cdot \left[\left(\frac{-\xi_3}{R}\right) \cdot \left(1+\xi_3 \cdot \frac{b}{2 \cdot R}\right) \cdot \psi_{\chi\eta 3}(s_i) - \left(\frac{\zeta_3}{R}\right) \cdot \left(1+\zeta_3 \cdot \frac{s_i}{R}\right) \cdot \left(\psi_{\chi s i_3} - \eta_i \cdot \frac{d}{ds_i} \psi_{\chi\eta 3}(s_i)\right)\right] d\eta_i \, ds_i \ldots \\
& + \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_4)^2 \cdot \left(1+\xi_2 \cdot \frac{b}{2 \cdot R}\right) + (\zeta_4)^2 \cdot \left(1+\zeta_4 \cdot \frac{s_i}{R}\right)\right] \cdot \psi_{\beta\phi 4}(s_i) \cdot \left[\left(\frac{-\xi_4}{R}\right) \cdot \left(1+\xi_4 \cdot \frac{b}{2 \cdot R}\right) \cdot \psi_{\chi\eta 4}(s_i) - \left(\frac{\zeta_4}{R}\right) \cdot \left(1+\zeta_4 \cdot \frac{s_i}{R}\right) \cdot \left(\psi_{\chi s i_4} - \eta_i \cdot \frac{d}{ds_i} \psi_{\chi\eta 4}(s_i)\right)\right] d\eta_i \, ds_i
\end{aligned}
$$

$$c_9 = -1.034 \times 10^{-6}$$

$$
\begin{aligned}
c_{11} := & \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_1)^2 \cdot \left(1+\xi_1 \cdot \frac{b}{2 \cdot R}\right) + (\zeta_1)^2 \cdot \left(1+\zeta_1 \cdot \frac{s_i}{R}\right)\right] \cdot \left(\psi_{U\phi 1}(s_i)\right) \cdot \left[\left(\frac{-\xi_1}{R}\right) \cdot \left(1+\xi_1 \cdot \frac{b}{2 \cdot R}\right) \cdot \psi_{\chi\eta 1}(s_i) - \left(\frac{\zeta_1}{R}\right) \cdot \left(1+\zeta_1 \cdot \frac{s_i}{R}\right) \cdot \left(\psi_{\chi s i_1} - \eta_i \cdot \frac{d}{ds_i} \psi_{\chi\eta 1}(s_i)\right)\right] d\eta_i \, ds_i \ldots \\
& + \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_2)^2 \cdot \left(1+\xi_2 \cdot \frac{b}{2 \cdot R}\right) + (\zeta_2)^2 \cdot \left(1+\zeta_2 \cdot \frac{s_i}{R}\right)\right] \cdot \left(\psi_{U\phi 2}(s_i)\right) \cdot \left[\left(\frac{-\xi_2}{R}\right) \cdot \left(1+\xi_2 \cdot \frac{b}{2 \cdot R}\right) \cdot \psi_{\chi\eta 2}(s_i) - \left(\frac{\zeta_2}{R}\right) \cdot \left(1+\zeta_2 \cdot \frac{s_i}{R}\right) \cdot \left(\psi_{\chi s i_2} - \eta_i \cdot \frac{d}{ds_i} \psi_{\chi\eta 2}(s_i)\right)\right] d\eta_i \, ds_i \ldots \\
& + \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_3)^2 \cdot \left(1+\xi_3 \cdot \frac{b}{2 \cdot R}\right) + (\zeta_3)^2 \cdot \left(1+\zeta_3 \cdot \frac{s_i}{R}\right)\right] \cdot \left(\psi_{U\phi 3}(s_i)\right) \cdot \left[\left(\frac{-\xi_3}{R}\right) \cdot \left(1+\xi_3 \cdot \frac{b}{2 \cdot R}\right) \cdot \psi_{\chi\eta 3}(s_i) - \left(\frac{\zeta_3}{R}\right) \cdot \left(1+\zeta_3 \cdot \frac{s_i}{R}\right) \cdot \left(\psi_{\chi s i_3} - \eta_i \cdot \frac{d}{ds_i} \psi_{\chi\eta 3}(s_i)\right)\right] d\eta_i \, ds_i \ldots \\
& + \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_4)^2 \cdot \left(1+\xi_4 \cdot \frac{b}{2 \cdot R}\right) + (\zeta_4)^2 \cdot \left(1+\zeta_4 \cdot \frac{s_i}{R}\right)\right] \cdot \left(\psi_{U\phi 4}(s_i)\right) \cdot \left[\left(\frac{-\xi_4}{R}\right) \cdot \left(1+\xi_4 \cdot \frac{b}{2 \cdot R}\right) \cdot \psi_{\chi\eta 4}(s_i) - \left(\frac{\zeta_4}{R}\right) \cdot \left(1+\zeta_4 \cdot \frac{s_i}{R}\right) \cdot \left(\psi_{\chi s i_4} - \eta_i \cdot \frac{d}{ds_i} \psi_{\chi\eta 4}(s_i)\right)\right] d\eta_i \, ds_i
\end{aligned}
$$

$$c_{11} = 8.75 \times 10^{-10}$$

$$d_1 := \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\frac{d}{ds_i}\psi_{U\phi1}(s_i) + \left(\frac{\zeta_1}{R}\right)\cdot\left(1+\zeta_1\cdot\frac{s_i}{R}\right)\cdot\psi_{U\phi1}(s_i)\right]^2 d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\frac{d}{ds_i}\psi_{U\phi2}(s_i) + \left(\frac{\zeta_2}{R}\right)\cdot\left(1+\zeta_2\cdot\frac{s_i}{R}\right)\cdot\psi_{U\phi2}(s_i)\right]^2 d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\frac{d}{ds_i}\psi_{U\phi3}(s_i) + \left(\frac{\zeta_3}{R}\right)\cdot\left(1+\zeta_3\cdot\frac{s_i}{R}\right)\cdot\psi_{U\phi3}(s_i)\right]^2 d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\frac{d}{ds_i}\psi_{U\phi4}(s_i) + \left(\frac{\zeta_4}{R}\right)\cdot\left(1+\zeta_4\cdot\frac{s_i}{R}\right)\cdot\psi_{U\phi4}(s_i)\right]^2 d\eta_i\, ds_i$$

$$\boxed{d_1 = 7.503\times 10^{-7}}$$

$$d_2 := \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_1)^2\cdot\left(1+\xi_1\cdot\frac{b}{2\cdot R}\right) + (\zeta_1)^2\cdot\left(1+\zeta_1\cdot\frac{s_i}{R}\right)\right]^2\cdot\left(\psi_{\chi si_1} - \eta_i\cdot\frac{d}{ds_i}\psi_{\chi\eta1}(s_i)\right)^2 d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_2)^2\cdot\left(1+\xi_2\cdot\frac{b}{2\cdot R}\right) + (\zeta_2)^2\cdot\left(1+\zeta_2\cdot\frac{s_i}{R}\right)\right]^2\cdot\left(\psi_{\chi si_2} - \eta_i\cdot\frac{d}{ds_i}\psi_{\chi\eta2}(s_i)\right)^2 d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_3)^2\cdot\left(1+\xi_3\cdot\frac{b}{2\cdot R}\right) + (\zeta_3)^2\cdot\left(1+\zeta_3\cdot\frac{s_i}{R}\right)\right]^2\cdot\left(\psi_{\chi si_3} - \eta_i\cdot\frac{d}{ds_i}\psi_{\chi\eta3}(s_i)\right)^2 d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_4)^2\cdot\left(1+\xi_4\cdot\frac{b}{2\cdot R}\right) + (\zeta_4)^2\cdot\left(1+\zeta_4\cdot\frac{s_i}{R}\right)\right]^2\cdot\left(\psi_{\chi si_4} - \eta_i\cdot\frac{d}{ds_i}\psi_{\chi\eta4}(s_i)\right)^2 d\eta_i\, ds_i$$

$$\boxed{d_2 = 7.506\times 10^{-7}}$$

$$d_3 := \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_1)^2 \cdot \left(1 + \xi_1 \cdot \frac{b}{2 \cdot R}\right) + (\zeta_1)^2 \cdot \left(1 + \zeta_1 \cdot \frac{s_i}{R}\right)\right] \cdot \psi_{\theta si_1} \cdot \left[\frac{d}{ds_i} \psi_{U\phi 1}(s_i) + \left(\frac{\zeta_1}{R}\right) \cdot \left(1 + \zeta_1 \cdot \frac{s_i}{R}\right) \cdot \psi_{U\phi 1}(s_i)\right] d\eta_i \, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_2)^2 \cdot \left(1 + \xi_2 \cdot \frac{b}{2 \cdot R}\right) + (\zeta_2)^2 \cdot \left(1 + \zeta_2 \cdot \frac{s_i}{R}\right)\right] \cdot \psi_{\theta si_2} \cdot \left[\frac{d}{ds_i} \psi_{U\phi 2}(s_i) + \left(\frac{\zeta_2}{R}\right) \cdot \left(1 + \zeta_2 \cdot \frac{s_i}{R}\right) \cdot \psi_{U\phi 2}(s_i)\right] d\eta_i \, ds_i \ldots$$

$$+ \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_3)^2 \cdot \left(1 + \xi_3 \cdot \frac{b}{2 \cdot R}\right) + (\zeta_3)^2 \cdot \left(1 + \zeta_3 \cdot \frac{s_i}{R}\right)\right] \cdot \psi_{\theta si_3} \cdot \left[\frac{d}{ds_i} \psi_{U\phi 3}(s_i) + \left(\frac{\zeta_3}{R}\right) \cdot \left(1 + \zeta_3 \cdot \frac{s_i}{R}\right) \cdot \psi_{U\phi 3}(s_i)\right] d\eta_i \, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_4)^2 \cdot \left(1 + \xi_4 \cdot \frac{b}{2 \cdot R}\right) + (\zeta_4)^2 \cdot \left(1 + \zeta_4 \cdot \frac{s_i}{R}\right)\right] \cdot \psi_{\theta si_4} \cdot \left[\frac{d}{ds_i} \psi_{U\phi 4}(s_i) + \left(\frac{\zeta_4}{R}\right) \cdot \left(1 + \zeta_4 \cdot \frac{s_i}{R}\right) \cdot \psi_{U\phi 4}(s_i)\right] d\eta_i \, ds_i$$

$$d_3 = -2.502 \times 10^{-7}$$

$$d_4 := \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\frac{d}{ds_i} \psi_{U\phi 1}(s_i) + \left(\frac{\zeta_1}{R}\right) \cdot \left(1 + \zeta_1 \cdot \frac{s_i}{R}\right) \cdot \psi_{U\phi 1}(s_i)\right] \cdot (\xi_1)^2 \cdot \left(1 + \xi_1 \cdot \frac{b}{2 \cdot R}\right) \cdot \psi_{\eta si_1} \, d\eta_i \, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\frac{d}{ds_i} \psi_{U\phi 2}(s_i) + \left(\frac{\zeta_2}{R}\right) \cdot \left(1 + \zeta_2 \cdot \frac{s_i}{R}\right) \cdot \psi_{U\phi 2}(s_i)\right] \cdot (\xi_2)^2 \cdot \left(1 + \xi_2 \cdot \frac{b}{2 \cdot R}\right) \cdot \psi_{\eta si_2} \, d\eta_i \, ds_i \ldots$$

$$+ \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\frac{d}{ds_i} \psi_{U\phi 3}(s_i) + \left(\frac{\zeta_3}{R}\right) \cdot \left(1 + \zeta_3 \cdot \frac{s_i}{R}\right) \cdot \psi_{U\phi 3}(s_i)\right] \cdot (\xi_3)^2 \cdot \left(1 + \xi_3 \cdot \frac{b}{2 \cdot R}\right) \cdot \psi_{\eta si_3} \, d\eta_i \, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\frac{d}{ds_i} \psi_{U\phi 4}(s_i) + \left(\frac{\zeta_4}{R}\right) \cdot \left(1 + \zeta_4 \cdot \frac{s_i}{R}\right) \cdot \psi_{U\phi 4}(s_i)\right] \cdot (\xi_4)^2 \cdot \left(1 + \xi_4 \cdot \frac{b}{2 \cdot R}\right) \cdot \psi_{\eta si_4} \, d\eta_i \, ds_i$$

$$d_4 = 2.5 \times 10^{-7}$$

$$d_5 := \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\left(\frac{d}{ds_i}\psi_{\beta\phi1}(s_i)\right)\cdot(\xi_1)^2 + \left(\frac{\zeta_1}{R}\right)\cdot\left(1+\zeta_1\cdot\frac{s_i}{R}\right)\cdot\psi_{\beta\phi1}(s_i)\right]\cdot\left[\frac{d}{ds_i}\psi_{U\phi1}(s_i) + \left(\frac{\zeta_1}{R}\right)\cdot\left(1+\zeta_1\cdot\frac{s_i}{R}\right)\cdot\psi_{U\phi1}(s_i)\right] d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\left(\frac{d}{ds_i}\psi_{\beta\phi2}(s_i)\right)\cdot(\xi_2)^2 + \left(\frac{\zeta_2}{R}\right)\cdot\left(1+\zeta_2\cdot\frac{s_i}{R}\right)\cdot\psi_{\beta\phi2}(s_i)\right]\cdot\left[\frac{d}{ds_i}\psi_{U\phi2}(s_i) + \left(\frac{\zeta_2}{R}\right)\cdot\left(1+\zeta_2\cdot\frac{s_i}{R}\right)\cdot\psi_{U\phi2}(s_i)\right] d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\left(\frac{d}{ds_i}\psi_{\beta\phi3}(s_i)\right)\cdot(\xi_3)^2 + \left(\frac{\zeta_3}{R}\right)\cdot\left(1+\zeta_3\cdot\frac{s_i}{R}\right)\cdot\psi_{\beta\phi3}(s_i)\right]\cdot\left[\frac{d}{ds_i}\psi_{U\phi3}(s_i) + \left(\frac{\zeta_3}{R}\right)\cdot\left(1+\zeta_3\cdot\frac{s_i}{R}\right)\cdot\psi_{U\phi3}(s_i)\right] d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\left(\frac{d}{ds_i}\psi_{\beta\phi4}(s_i)\right)\cdot(\xi_4)^2 + \left(\frac{\zeta_4}{R}\right)\cdot\left(1+\zeta_4\cdot\frac{s_i}{R}\right)\cdot\psi_{\beta\phi4}(s_i)\right]\cdot\left[\frac{d}{ds_i}\psi_{U\phi4}(s_i) + \left(\frac{\zeta_4}{R}\right)\cdot\left(1+\zeta_4\cdot\frac{s_i}{R}\right)\cdot\psi_{U\phi4}(s_i)\right] d\eta_i\, ds_i$$

$$\boxed{d_5 = -5.002\times 10^{-7}}$$

$$d_6 := \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_1)^2\cdot\left(1+\xi_1\cdot\frac{b}{2\cdot R}\right) + (\zeta_1)^2\cdot\left(1+\zeta_1\cdot\frac{s_i}{R}\right)\right]\cdot\left(\psi_{\chi si_1} - \eta_i\cdot\frac{d}{ds_i}\psi_{\chi\eta1}(s_i)\right)\cdot\left[(\xi_1)^2\cdot\left(1+\xi_1\cdot\frac{b}{2\cdot R}\right) + (\zeta_1)^2\cdot\left(1+\zeta_1\cdot\frac{s_i}{R}\right)\right]\cdot\psi_{\theta si_1} d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_2)^2\cdot\left(1+\xi_2\cdot\frac{b}{2\cdot R}\right) + (\zeta_2)^2\cdot\left(1+\zeta_2\cdot\frac{s_i}{R}\right)\right]\cdot\left(\psi_{\chi si_2} - \eta_i\cdot\frac{d}{ds_i}\psi_{\chi\eta2}(s_i)\right)\cdot\left[(\xi_2)^2\cdot\left(1+\xi_2\cdot\frac{b}{2\cdot R}\right) + (\zeta_2)^2\cdot\left(1+\zeta_2\cdot\frac{s_i}{R}\right)\right]\cdot\psi_{\theta si_2} d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_3)^2\cdot\left(1+\xi_3\cdot\frac{b}{2\cdot R}\right) + (\zeta_3)^2\cdot\left(1+\zeta_3\cdot\frac{s_i}{R}\right)\right]\cdot\left(\psi_{\chi si_3} - \eta_i\cdot\frac{d}{ds_i}\psi_{\chi\eta3}(s_i)\right)\cdot\left[(\xi_3)^2\cdot\left(1+\xi_3\cdot\frac{b}{2\cdot R}\right) + (\zeta_3)^2\cdot\left(1+\zeta_3\cdot\frac{s_i}{R}\right)\right]\cdot\psi_{\theta si_3} d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_4)^2\cdot\left(1+\xi_4\cdot\frac{b}{2\cdot R}\right) + (\zeta_4)^2\cdot\left(1+\zeta_4\cdot\frac{s_i}{R}\right)\right]\cdot\left(\psi_{\chi si_4} - \eta_i\cdot\frac{d}{ds_i}\psi_{\chi\eta4}(s_i)\right)\cdot\left[(\xi_4)^2\cdot\left(1+\xi_4\cdot\frac{b}{2\cdot R}\right) + (\zeta_4)^2\cdot\left(1+\zeta_4\cdot\frac{s_i}{R}\right)\right]\cdot\psi_{\theta si_4} d\eta_i\, ds_i$$

$$\boxed{d_6 = 2.499\times 10^{-7}}$$

$$d_7 := \int_{\frac{-h}{2}}^{\frac{h}{2}}\int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_1)^2\cdot\left(1+\xi_1\cdot\frac{b}{2\cdot R}\right)+(\zeta_1)^2\cdot\left(1+\zeta_1\cdot\frac{s_i}{R}\right)\right]\cdot\left(\psi\chi si_1-\eta_i\cdot\frac{d}{ds_i}\psi\chi\eta 1(s_i)\right)\cdot(\xi_1)^2\cdot\left(1+\xi_1\cdot\frac{b}{2\cdot R}\right)\cdot\psi\eta si_1\,d\eta_i\,ds_i\ \ldots$$

$$+\int_{\frac{-b}{2}}^{\frac{b}{2}}\int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_2)^2\cdot\left(1+\xi_2\cdot\frac{b}{2\cdot R}\right)+(\zeta_2)^2\cdot\left(1+\zeta_2\cdot\frac{s_i}{R}\right)\right]\cdot\left(\psi\chi si_2-\eta_i\cdot\frac{d}{ds_i}\psi\chi\eta 2(s_i)\right)\cdot(\xi_2)^2\cdot\left(1+\xi_2\cdot\frac{b}{2\cdot R}\right)\cdot\psi\eta si_2\,d\eta_i\,ds_i\ \ldots$$

$$+\int_{\frac{-h}{2}}^{\frac{h}{2}}\int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_3)^2\cdot\left(1+\xi_3\cdot\frac{b}{2\cdot R}\right)+(\zeta_3)^2\cdot\left(1+\zeta_3\cdot\frac{s_i}{R}\right)\right]\cdot\left(\psi\chi si_3-\eta_i\cdot\frac{d}{ds_i}\psi\chi\eta 3(s_i)\right)\cdot(\xi_3)^2\cdot\left(1+\xi_3\cdot\frac{b}{2\cdot R}\right)\cdot\psi\eta si_3\,d\eta_i\,ds_i\ \ldots$$

$$+\int_{\frac{-b}{2}}^{\frac{b}{2}}\int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_4)^2\cdot\left(1+\xi_4\cdot\frac{b}{2\cdot R}\right)+(\zeta_4)^2\cdot\left(1+\zeta_4\cdot\frac{s_i}{R}\right)\right]\cdot\left(\psi\chi si_4-\eta_i\cdot\frac{d}{ds_i}\psi\chi\eta 4(s_i)\right)\cdot(\xi_4)^2\cdot\left(1+\xi_4\cdot\frac{b}{2\cdot R}\right)\cdot\psi\eta si_4\,d\eta_i\,ds_i$$

$$\boxed{d_7 = -5\times 10^{-7}}$$

$$d_8 := \int_{\frac{-h}{2}}^{\frac{h}{2}}\int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_1)^2\cdot\left(1+\xi_1\cdot\frac{b}{2\cdot R}\right)+(\zeta_1)^2\cdot\left(1+\zeta_1\cdot\frac{s_i}{R}\right)\right]\cdot\left(\psi\chi si_1-\eta_i\cdot\frac{d}{ds_i}\psi\chi\eta 1(s_i)\right)\cdot\left[(\xi_1)^2\cdot\frac{d}{ds_i}\psi\beta\phi 1(s_i)+\left(\frac{\zeta_1}{R}\right)\cdot\left(1+\zeta_1\cdot\frac{s_i}{R}\right)\cdot\psi\beta\phi 1(s_i)\right]d\eta_i\,ds_i\ \ldots$$

$$+\int_{\frac{-b}{2}}^{\frac{b}{2}}\int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_2)^2\cdot\left(1+\xi_2\cdot\frac{b}{2\cdot R}\right)+(\zeta_2)^2\cdot\left(1+\zeta_2\cdot\frac{s_i}{R}\right)\right]\cdot\left(\psi\chi si_2-\eta_i\cdot\frac{d}{ds_i}\psi\chi\eta 2(s_i)\right)\cdot\left[(\xi_2)^2\cdot\frac{d}{ds_i}\psi\beta\phi 2(s_i)+\left(\frac{\zeta_2}{R}\right)\cdot\left(1+\zeta_2\cdot\frac{s_i}{R}\right)\cdot\psi\beta\phi 2(s_i)\right]d\eta_i\,ds_i\ \ldots$$

$$+\int_{\frac{-h}{2}}^{\frac{h}{2}}\int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_3)^2\cdot\left(1+\xi_3\cdot\frac{b}{2\cdot R}\right)+(\zeta_3)^2\cdot\left(1+\zeta_3\cdot\frac{s_i}{R}\right)\right]\cdot\left(\psi\chi si_3-\eta_i\cdot\frac{d}{ds_i}\psi\chi\eta 3(s_i)\right)\cdot\left[(\xi_3)^2\cdot\frac{d}{ds_i}\psi\beta\phi 3(s_i)+\left(\frac{\zeta_3}{R}\right)\cdot\left(1+\zeta_3\cdot\frac{s_i}{R}\right)\cdot\psi\beta\phi 3(s_i)\right]d\eta_i\,ds_i\ \ldots$$

$$+\int_{\frac{-b}{2}}^{\frac{b}{2}}\int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_4)^2\cdot\left(1+\xi_2\cdot\frac{b}{2\cdot R}\right)+(\zeta_4)^2\cdot\left(1+\zeta_4\cdot\frac{s_i}{R}\right)\right]\cdot\left(\psi\chi si_4-\eta_i\cdot\frac{d}{ds_i}\psi\chi\eta 4(s_i)\right)\cdot\left[(\xi_4)^2\cdot\frac{d}{ds_i}\psi\beta\phi 4(s_i)+\left(\frac{\zeta_4}{R}\right)\cdot\left(1+\zeta_4\cdot\frac{s_i}{R}\right)\cdot\psi\beta\phi 4(s_i)\right]d\eta_i\,ds_i$$

$$\boxed{d_8 = 7.501\times 10^{-7}}$$

$$d_9 := \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_1)^2 \cdot \left(1 + \xi_1 \cdot \frac{b}{2 \cdot R}\right) + (\zeta_1)^2 \cdot \left(1 + \zeta_1 \cdot \frac{s_i}{R}\right) \right] \cdot \left(\psi_{\chi si_1} - \eta_i \cdot \frac{d}{ds_i} \psi_{\chi\eta 1}(s_i) \right) \cdot \left[\left(\frac{d}{ds_i} \psi_{U\phi 1}(s_i) \right) + \left(\frac{\zeta_1}{R} \right) \cdot \left(1 + \zeta_1 \cdot \frac{s_i}{R} \right) \cdot \psi_{U\phi 1}(s_i) \right] d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_2)^2 \cdot \left(1 + \xi_2 \cdot \frac{b}{2 \cdot R}\right) + (\zeta_2)^2 \cdot \left(1 + \zeta_2 \cdot \frac{s_i}{R}\right) \right] \cdot \left(\psi_{\chi si_2} - \eta_i \cdot \frac{d}{ds_i} \psi_{\chi\eta 2}(s_i) \right) \cdot \left[\left(\frac{d}{ds_i} \psi_{U\phi 2}(s_i) \right) + \left(\frac{\zeta_2}{R} \right) \cdot \left(1 + \zeta_2 \cdot \frac{s_i}{R} \right) \cdot \psi_{U\phi 2}(s_i) \right] d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_3)^2 \cdot \left(1 + \xi_3 \cdot \frac{b}{2 \cdot R}\right) + (\zeta_3)^2 \cdot \left(1 + \zeta_3 \cdot \frac{s_i}{R}\right) \right] \cdot \left(\psi_{\chi si_3} - \eta_i \cdot \frac{d}{ds_i} \psi_{\chi\eta 3}(s_i) \right) \cdot \left[\left(\frac{d}{ds_i} \psi_{U\phi 3}(s_i) \right) + \frac{\zeta_3}{R} \cdot \left(1 + \zeta_3 \cdot \frac{s_i}{R} \right) \cdot \psi_{U\phi 3}(s_i) \right] d\eta_i\, ds_i \ldots$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-t}{2}}^{\frac{t}{2}} \left[(\xi_4)^2 \cdot \left(1 + \xi_4 \cdot \frac{b}{2 \cdot R}\right) + (\zeta_4)^2 \cdot \left(1 + \zeta_4 \cdot \frac{s_i}{R}\right) \right] \cdot \left(\psi_{\chi si_4} - \eta_i \cdot \frac{d}{ds_i} \psi_{\chi\eta 4}(s_i) \right) \cdot \left[\left(\frac{d}{ds_i} \psi_{U\phi 4}(s_i) \right) + \left(\frac{\zeta_4}{R} \right) \cdot \left(1 + \zeta_4 \cdot \frac{s_i}{R} \right) \cdot \psi_{U\phi 4}(s_i) \right] d\eta_i\, ds_i$$

$$\boxed{d_9 = -7.502 \times 10^{-7}}$$

Calculating Stiffness Matrix

$$k_{11} := \begin{bmatrix} \frac{L}{3}\cdot E_1\cdot a_2 + \frac{1}{L}\cdot G\cdot b_1 & \frac{1}{L}\cdot G\cdot b_6 & \frac{-1}{2}\cdot E_1\cdot a_3 - \frac{1}{2}\cdot G\cdot b_{13} & \frac{-1}{2}\cdot E_1\cdot c_5 - \frac{1}{2}\cdot G\cdot d_3 & \frac{L}{3}\cdot E_1\cdot c_7 + \frac{1}{L}\cdot G\cdot d_6 \\ \frac{1}{L}\cdot G\cdot b_6 & \frac{1}{L}\cdot G\cdot b_2 & \frac{-1}{2}\cdot G\cdot b_{12} & \frac{-1}{2}\cdot G\cdot d_4 & \frac{1}{L}\cdot G\cdot d_7 \\ \frac{-1}{2}\cdot E_1\cdot a_3 - \frac{1}{2}\cdot G\cdot b_{13} & \frac{-1}{2}\cdot G\cdot b_{12} & \frac{1}{L}\cdot E_1\cdot a_1 + \frac{L}{3}\cdot G\cdot b_4 & \frac{1}{L}\cdot E_1\cdot c_6 + \frac{L}{3}\cdot G\cdot d_5 & \frac{-1}{2}\cdot E_1\cdot c_9 - \frac{1}{2}\cdot G\cdot d_8 \\ \frac{-1}{2}\cdot E_1\cdot c_5 - \frac{1}{2}\cdot G\cdot d_3 & \frac{-1}{2}\cdot G\cdot d_4 & \frac{1}{L}\cdot E_1\cdot c_6 + \frac{L}{3}\cdot G\cdot d_5 & \frac{1}{L}\cdot E_1\cdot c_1 + \frac{L}{3}\cdot G\cdot d_1 & \frac{-1}{2}\cdot E_1\cdot c_{11} - \frac{1}{2}\cdot G\cdot d_9 \\ \frac{L}{3}\cdot E_1\cdot c_7 + \frac{1}{L}\cdot G\cdot d_6 & \frac{1}{L}\cdot G\cdot d_7 & \frac{-1}{2}\cdot E_1\cdot c_9 - \frac{1}{2}\cdot G\cdot d_8 & \frac{-1}{2}\cdot E_1\cdot c_{11} - \frac{1}{2}\cdot G\cdot d_9 & \frac{L}{3}\cdot E_1\cdot\left(c_2 + c_3 + 2\cdot c_4\right) + \frac{1}{L}\cdot G\cdot d_2 \end{bmatrix}$$

$$k_{12} := \begin{bmatrix} \frac{L}{6}\cdot E_1\cdot a_2 - \frac{1}{L}\cdot G\cdot b_1 & \frac{-1}{L}\cdot G\cdot b_6 & \frac{1}{2}\cdot E_1\cdot a_3 - \frac{1}{2}\cdot G\cdot b_{13} & \frac{1}{2}\cdot E_1\cdot c_5 - \frac{1}{2}\cdot G\cdot d_3 & \frac{L}{6}\cdot E_1\cdot c_7 - \frac{1}{L}\cdot G\cdot d_6 \\ \frac{-1}{L}\cdot G\cdot b_6 & \frac{-1}{L}\cdot G\cdot b_2 & \frac{-1}{2}\cdot G\cdot b_{12} & \frac{-1}{2}\cdot G\cdot d_4 & \frac{-1}{L}\cdot G\cdot d_7 \\ \frac{-1}{2}\cdot E_1\cdot a_3 + \frac{1}{2}\cdot G\cdot b_{13} & \frac{1}{2}\cdot G\cdot b_{12} & \frac{-1}{L}\cdot E_1\cdot a_1 + \frac{L}{6}\cdot G\cdot b_4 & \frac{-1}{L}\cdot E_1\cdot c_6 + \frac{L}{6}\cdot G\cdot d_5 & \frac{-1}{2}\cdot E_1\cdot c_9 - \frac{1}{2}\cdot G\cdot d_8 \\ \frac{-1}{2}\cdot E_1\cdot c_5 + \frac{1}{2}\cdot G\cdot d_3 & \frac{1}{2}\cdot G\cdot d_4 & \frac{-1}{L}\cdot E_1\cdot c_6 + \frac{L}{6}\cdot G\cdot d_5 & \frac{-1}{L}\cdot E_1\cdot c_1 + \frac{L}{6}\cdot G\cdot d_1 & \frac{-1}{2}\cdot E_1\cdot c_{11} + \frac{1}{2}\cdot G\cdot d_9 \\ \frac{-L}{6}\cdot E_1\cdot c_7 - \frac{1}{L}\cdot G\cdot d_6 & \frac{-1}{L}\cdot G\cdot d_7 & \frac{1}{2}\cdot E_1\cdot c_9 - \frac{1}{2}\cdot G\cdot d_8 & \frac{1}{2}\cdot E_1\cdot c_{11} - \frac{1}{2}\cdot G\cdot d_9 & \frac{L}{6}\cdot E_1\cdot\left(c_2 + c_3 + 2\cdot c_4\right) - \frac{1}{L}\cdot G\cdot d_2 \end{bmatrix}$$

$$k_{21} := k_{12}^{T}$$

$$k_{22} := \begin{bmatrix} \frac{L}{3}\cdot E_1\cdot a_2 + \frac{1}{L}\cdot G\cdot b_1 & \frac{1}{L}\cdot G\cdot b_6 & \frac{1}{2}\cdot E_1\cdot a_3 + \frac{1}{2}\cdot G\cdot b_{13} & \frac{1}{2}\cdot E_1\cdot c_5 + \frac{1}{2}\cdot G\cdot d_3 & \frac{L}{3}\cdot E_1\cdot c_7 + \frac{1}{L}\cdot G\cdot d_6 \\ \frac{1}{L}\cdot G\cdot b_6 & \frac{1}{L}\cdot G\cdot b_2 & \frac{1}{2}\cdot G\cdot b_{12} & \frac{1}{2}\cdot G\cdot d_4 & \frac{1}{L}\cdot G\cdot d_7 \\ \frac{1}{2}\cdot E_1\cdot a_3 + \frac{1}{2}\cdot G\cdot b_{13} & \frac{1}{2}\cdot G\cdot b_{12} & \frac{1}{L}\cdot E_1\cdot a_1 + \frac{L}{3}\cdot G\cdot b_4 & \frac{1}{L}\cdot E_1\cdot c_6 + \frac{L}{3}\cdot G\cdot d_5 & \frac{1}{2}\cdot E_1\cdot c_9 + \frac{1}{2}\cdot G\cdot d_8 \\ \frac{1}{2}\cdot E_1\cdot c_5 + \frac{1}{2}\cdot G\cdot d_3 & \frac{1}{2}\cdot G\cdot d_4 & \frac{1}{L}\cdot E_1\cdot c_6 + \frac{L}{3}\cdot G\cdot d_5 & \frac{1}{L}\cdot E_1\cdot c_1 + \frac{L}{3}\cdot G\cdot d_1 & \frac{1}{2}\cdot E_1\cdot c_{11} + \frac{1}{2}\cdot G\cdot d_9 \\ \frac{L}{3}\cdot E_1\cdot c_7 + \frac{1}{L}\cdot G\cdot d_6 & \frac{1}{L}\cdot G\cdot d_7 & \frac{1}{2}\cdot E_1\cdot c_9 + \frac{1}{2}\cdot G\cdot d_8 & \frac{1}{2}\cdot E_1\cdot c_{11} + \frac{1}{2}\cdot G\cdot d_9 & \frac{L}{3}\cdot E_1\cdot\left(c_2 + c_3 + 2\cdot c_4\right) + \frac{1}{L}G\cdot d_2 \end{bmatrix}$$

$$k_{11} = \begin{pmatrix} 2.940511689\times 10^{6} & 1.9588615354\times 10^{6} & 8.1990041209\times 10^{4} & 9.5547161172\times 10^{3} & 9.8071373187\times 10^{5} \\ 1.9588615354\times 10^{6} & 1.5680686591\times 10^{9} & 1.5384615385\times 10^{7} & -9.6153846154\times 10^{3} & -1.9588615354\times 10^{6} \\ 8.1990041209\times 10^{4} & 1.5384615385\times 10^{7} & 9.5330427287\times 10^{6} & -7.2477650779\times 10^{3} & 8.475103022\times 10^{4} \\ 9.5547161172\times 10^{3} & -9.6153846154\times 10^{3} & -7.2477650779\times 10^{3} & 3.8776083135\times 10^{3} & 2.8758012821\times 10^{4} \\ 9.8071373187\times 10^{5} & -1.9588615354\times 10^{6} & 8.475103022\times 10^{4} & 2.8758012821\times 10^{4} & 2.9431429436\times 10^{6} \end{pmatrix}$$

$$k_{22} = \begin{pmatrix} 2.940511689\times 10^{6} & 1.9588615354\times 10^{6} & -8.1990041209\times 10^{4} & -9.5547161172\times 10^{3} & 9.8071373187\times 10^{5} \\ 1.9588615354\times 10^{6} & 1.5680686591\times 10^{9} & -1.5384615385\times 10^{7} & 9.6153846154\times 10^{3} & -1.9588615354\times 10^{6} \\ -8.1990041209\times 10^{4} & -1.5384615385\times 10^{7} & 9.5330427287\times 10^{6} & -7.2477650779\times 10^{3} & -8.475103022\times 10^{4} \\ -9.5547161172\times 10^{3} & 9.6153846154\times 10^{3} & -7.2477650779\times 10^{3} & 3.8776083135\times 10^{3} & -2.8758012821\times 10^{4} \\ 9.8071373187\times 10^{5} & -1.9588615354\times 10^{6} & -8.475103022\times 10^{4} & -2.8758012821\times 10^{4} & 2.9431429436\times 10^{6} \end{pmatrix}$$

$$k_{21} = \begin{pmatrix} -2.9387129709\times 10^{6} & -1.9588615354\times 10^{6} & 1.0122882326\times 10^{5} & -9.6920787546\times 10^{3} & -9.7997022576\times 10^{5} \\ -1.9588615354\times 10^{6} & -1.5680686591\times 10^{9} & -1.5384615385\times 10^{7} & 9.6153846154\times 10^{3} & 1.9588615354\times 10^{6} \\ -1.0122882326\times 10^{5} & 1.5384615385\times 10^{7} & -9.2305936966\times 10^{6} & 6.8700185438\times 10^{3} & -1.4245135073\times 10^{5} \\ 9.6920787546\times 10^{3} & -9.6153846154\times 10^{3} & 6.8700185438\times 10^{3} & -3.3109884828\times 10^{3} & 2.8950320513\times 10^{4} \\ -9.7848321354\times 10^{5} & 1.9588615354\times 10^{6} & 8.475103022\times 10^{4} & -2.8950320513\times 10^{4} & -2.939278914\times 10^{6} \end{pmatrix}$$

$$k_{12} = \begin{pmatrix} -2.9387129709\times 10^{6} & -1.9588615354\times 10^{6} & -1.0122882326\times 10^{5} & 9.6920787546\times 10^{3} & -9.7848321354\times 10^{5} \\ -1.9588615354\times 10^{6} & -1.5680686591\times 10^{9} & 1.5384615385\times 10^{7} & -9.6153846154\times 10^{3} & 1.9588615354\times 10^{6} \\ 1.0122882326\times 10^{5} & -1.5384615385\times 10^{7} & -9.2305936966\times 10^{6} & 6.8700185438\times 10^{3} & 8.475103022\times 10^{4} \\ -9.6920787546\times 10^{3} & 9.6153846154\times 10^{3} & 6.8700185438\times 10^{3} & -3.3109884828\times 10^{3} & -2.8950320513\times 10^{4} \\ -9.7997022576\times 10^{5} & 1.9588615354\times 10^{6} & -1.4245135073\times 10^{5} & 2.8950320513\times 10^{4} & -2.939278914\times 10^{6} \end{pmatrix}$$

$$K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$$

APPENDIX B

MATLAB CALCULATION OF THE STIFFNESS MATRIX AND DISPLACEMENTS

Three Degrees of Freedom (DOF) Model

```
clear all
k11=[2.940511E+6      1.9588615E+6  8.1990041E+4  9.5547161E+3  9.8071373E+5
1.9588615E+6  1.5680686E+9  1.5384615E+7  -9.6153846E+3 -1.9588615E+6
8.1990041E+4  1.5384615E+7  9.5330427E+6  -7.2477650E+3 8.475103E+4
9.5547161E+3  -9.6153844E+3 -7.2477650E+3 3.8776083E+3  2.8758012E+4
9.8071373E+5  -1.9588615E+6 8.475103E+4   2.8758012E+4  2.9431429E+6];

k22=[2.940511E+6      1.9588615E+6  -8.1990041E+4 -9.5547161E+3 9.8071373E+5
1.9588615E+6  1.568068E+9   -1.5384615E+7 9.6153846E+3  -1.9588615E+6
-8.1990041E+4 -1.5384615E+7 9.5330427E+6  -7.2477650E+3 -8.475103E+4
-9.5547161E+3 9.6153846E+3  -7.2477650E+3 3.8776083E+3  -2.8758012E+4
9.8071373E+5  -1.9588615E+6 -8.475103E+4  -2.8758012E+4 2.9431429E+6];

k21=[-2.9387129E+6    -1.9588615E+6 1.0122886E+5  -9.692078E+3  -9.7997022E+5
-1.9588615E+6 -1.5680686E+9 -1.5384615E+7 9.6153846E+3  1.9588615E+6
-1.0122882E+5 1.5384615E+7  -9.2305936E+6 6.8700185E+3  -1.4245135E+5
9.6920787E+3  -9.6153846E+3 6.8700185E+3  -3.3109884E+3 2.8950320E+4
-9.7848321E+5 1.9588615E+6  8.475103E+4   -2.8950320E+4 -2.939278E+6];

k12=[-2.9387129E+6    -1.9588615E+6 -1.012288E+5  9.6920787E+3  -9.7848321E+5
-1.9588615E+6 -1.5680686E+9 1.538461E+7   -9.6153846E+3 1.9588615E+6
1.0122882E+5  -1.5384615E+7 -9.2305936E+6 6.8700185E+3  8.475103E+4
-9.6920787E+3 9.6153846E+3  6.8700185E+3  -3.3109884E+3 -2.8950320E+4
-9.7997022E+5 1.9588615E+6  -1.4245135E+5 2.8950320E+4  -2.939278E+6];



K=[k11 k12;k21 k22];
k11r=k11(1:3,1:3);
k22r=k22(1:3,1:3);
k12r=k12(1:3,1:3);
k21r=k21(1:3,1:3);

Kr=[k11r k12r;k21r k22r];

nele=80% number of elements
Kg=zeros(3+3*nele,3+3*nele);
Kg(1:6,1:6)=Kr;


for ii=1:nele-1
   j1=4+(ii-1)*3;
   j2=j1+5;
   Kg(j1:j2,j1:j2)=Kg(j1:j2,j1:j2)+Kr;
end
```

```
kgr = Kg(4:3*(nele+1),4:3*(nele+1));

kgri=inv(kgr);

yy=zeros(3*nele,1);

yy(3*nele - 1)=100;

u=kgri*yy
u1(1)=0;
u2(1)=0;
u3(1)=0;
for ii=1:nele
    u1(ii+1)=u(ii*3-2);
    u2(ii+1)=u(ii*3-1);
    u3(ii+1)=u(ii*3-0);
   % u4(ii)=u(ii*5-1);
    %u5(ii)=u(ii*5);
end
nn=[0:nele];
figure(1)
plot(nn,u1)
title('u1: theta, the twist angle')
figure(2)
plot(nn,u2)
title('u2: nu, the vertical displacement')
figure(3)
plot(nn,u3)
title('u3: beta, the shear rotation angle')
%figure(4)
%plot(u4)
%title('u4')
%figure(5)
%plot(u5)
%title('u5')

format long
u(238:240)
format short
```

DISPLACEMENT OUTPUT:

```
ans =

  -0.00006005810641    u1 (twist)
   0.00104041107008    u2 (vertical displacement)
   0.00112584198379    u3 (shear rotation)
```

4 Degrees of Freedom (DOF) Model with distortion

```
clear all
k11=[2.940511E+6      1.9588615E+6  8.1990041E+4  9.5547161E+3  9.8071373E+5
1.9588615E+6  1.5680686E+9  1.5384615E+7  -9.6153846E+3 -1.9588615E+6
8.1990041E+4  1.5384615E+7  9.5330427E+6  -7.2477650E+3 8.475103E+4
9.5547161E+3  -9.6153844E+3 -7.2477650E+3 3.8776083E+3  2.8758012E+4
9.8071373E+5  -1.9588615E+6 8.475103E+4   2.8758012E+4  2.9431429E+6];

k22=[2.940511E+6      1.9588615E+6  -8.1990041E+4 -9.5547161E+3 9.8071373E+5
1.9588615E+6  1.568068E+9    -1.5384615E+7 9.6153846E+3  -1.9588615E+6
-8.1990041E+4 -1.5384615E+7 9.5330427E+6  -7.2477650E+3 -8.475103E+4
-9.5547161E+3 9.6153846E+3  -7.2477650E+3 3.8776083E+3  -2.8758012E+4
9.8071373E+5  -1.9588615E+6 -8.475103E+4  -2.8758012E+4 2.9431429E+6];

k21=[-2.9387129E+6    -1.9588615E+6 1.0122886E+5  -9.692078E+3   -9.7997022E+5
-1.9588615E+6 -1.5680686E+9 -1.5384615E+7 9.6153846E+3  1.9588615E+6
-1.0122882E+5 1.5384615E+7  -9.2305936E+6 6.8700185E+3  -1.4245135E+5
9.6920787E+3  -9.6153846E+3 6.8700185E+3  -3.3109884E+3 2.8950320E+4
-9.7848321E+5 1.9588615E+6  8.475103E+4   -2.8950320E+4 -2.939278E+6];

k12=[-2.9387129E+6    -1.9588615E+6 -1.012288E+5  9.6920787E+3  -9.7848321E+5
-1.9588615E+6 -1.5680686E+9 1.538461E+7   -9.6153846E+3 1.9588615E+6
1.0122882E+5  -1.5384615E+7 -9.2305936E+6 6.8700185E+3  8.475103E+4
-9.6920787E+3 9.6153846E+3  6.8700185E+3  -3.3109884E+3 -2.8950320E+4
-9.7997022E+5 1.9588615E+6  -1.4245135E+5 2.8950320E+4  -2.939278E+6];


k12(3,5)=-k12(5,3);
k21=k12';

k12(5,1)=k12(1,5);
k21=k12';

nn=[1 2 3 5]

K=[k11 k12;k21 k22];
k11r=k11(nn,nn);
k22r=k22(nn,nn);
k12r=k12(nn,nn);
k21r=k21(nn,nn);

Kr=[k11r k12r;k21r k22r];


nele=80% number of elements
Kg=zeros(4+4*nele,4+4*nele);
Kg(1:8,1:8)=Kr;
```

```
for ii=1:nele-1
   j1=5+(ii-1)*4;
   j2=j1+7;
   Kg(j1:j2,j1:j2)=Kg(j1:j2,j1:j2)+Kr;
end

kgr = Kg(5:4*(nele+1),5:4*(nele+1));
kgr1=kgr(1:319,1:319);

kgri=inv(kgr1);

yy=zeros(4*nele-1,1);

yy(4*nele -2)=100;

u=kgri*yy
u1(1)=0;
u2(1)=0;
u3(1)=0;
u4(1)=0;
for ii=1:nele-1
   u1(ii+1)=u(ii*4-3);
   u2(ii+1)=u(ii*4-2);
   u3(ii+1)=u(ii*4-1);
   u4(ii+1)=u(ii*4-0);
  % u4(ii)=u(ii*5-1);
   %u5(ii)=u(ii*5);
end
ii=nele
u1(ii+1)=u(ii*4-3);
u2(ii+1)=u(ii*4-2);
u3(ii+1)=u(ii*4-1);
u4(ii+1)=0;

clear nn
nn=[0:nele];
figure(1)
plot(nn,u1)
title('u1: theta, the twist angle')
figure(2)
plot(nn,u2)
title('u2: nu, the vertical displacement')
figure(3)
plot(nn,u3)
title('u3: beta, the shear rotation angle')
figure(4)
```

```
plot(nn,u4)
title('u4: chi, the distortion variable')
%figure(4)
%plot(u4)
%title('u4')
%figure(5)
%plot(u5)
%title('u5')

format long
u(317:319)
format short
```

Displacement output:

```
ans =

  -0.00021492476561    u1 (twist angle)
   0.00119421235772    u2 (vertical displacement)
   0.00126235449291    u3 (shear rotation)
```

4 Degrees of Freedom (DOF) model with Warping

```
clear all
k11=[2.940511E+6       1.9588615E+6  8.1990041E+4  9.5547161E+3  9.8071373E+5
1.9588615E+6  1.5680686E+9  1.5384615E+7  -9.6153846E+3 -1.9588615E+6
8.1990041E+4  1.5384615E+7  9.5330427E+6  -7.2477650E+3 8.475103E+4
9.5547161E+3  -9.6153844E+3 -7.2477650E+3 3.8776083E+3  2.8758012E+4
9.8071373E+5  -1.9588615E+6 8.475103E+4    2.8758012E+4  2.9431429E+6];

k22=[2.940511E+6       1.9588615E+6  -8.1990041E+4 -9.5547161E+3 9.8071373E+5
1.9588615E+6  1.568068E+9    -1.5384615E+7 9.6153846E+3  -1.9588615E+6
-8.1990041E+4 -1.5384615E+7 9.5330427E+6  -7.2477650E+3 -8.475103E+4
-9.5547161E+3 9.6153846E+3  -7.2477650E+3 3.8776083E+3  -2.8758012E+4
9.8071373E+5  -1.9588615E+6 -8.475103E+4   -2.8758012E+4 2.9431429E+6];

k21=[-2.9387129E+6    -1.9588615E+6 1.0122886E+5  -9.692078E+3   -9.7997022E+5
-1.9588615E+6 -1.5680686E+9 -1.5384615E+7 9.6153846E+3  1.9588615E+6
-1.0122882E+5 1.5384615E+7  -9.2305936E+6 6.8700185E+3  -1.4245135E+5
9.6920787E+3  -9.6153846E+3 6.8700185E+3  -3.3109884E+3 2.8950320E+4
-9.7848321E+5 1.9588615E+6  8.475103E+4    -2.8950320E+4 -2.939278E+6];

k12=[-2.9387129E+6    -1.9588615E+6 -1.012288E+5   9.6920787E+3  -9.7848321E+5
-1.9588615E+6 -1.5680686E+9 1.538461E+7    -9.6153846E+3 1.9588615E+6
1.0122882E+5  -1.5384615E+7 -9.2305936E+6 6.8700185E+3  8.475103E+4
-9.6920787E+3 9.6153846E+3  6.8700185E+3  -3.3109884E+3 -2.8950320E+4
-9.7997022E+5 1.9588615E+6  -1.4245135E+5 2.8950320E+4  -2.939278E+6];




nn=[1 2 3 4]

K=[k11 k12;k21 k22];
k11r=k11(nn,nn);
k22r=k22(nn,nn);
k12r=k12(nn,nn);
k21r=k21(nn,nn);

Kr=[k11r k12r;k21r k22r];

nele=80% number of elements
Kg=zeros(4+4*nele,4+4*nele);
Kg(1:8,1:8)=Kr;


for ii=1:nele-1
   j1=5+(ii-1)*4;
```

```
    j2=j1+7;
    Kg(j1:j2,j1:j2)=Kg(j1:j2,j1:j2)+Kr;
end

kgr = Kg(5:4*(nele+1),5:4*(nele+1));

kgri=inv(kgr);

yy=zeros(4*nele,1);

yy(4*nele - 2)=100;

u=kgri*yy;
u1(1)=0;
u2(1)=0;
u3(1)=0;
u4(1)=0;
for ii=1:nele
    u1(ii+1)=u(ii*4-3);
    u2(ii+1)=u(ii*4-2);
    u3(ii+1)=u(ii*4-1);
    u4(ii+1)=u(ii*4-0);
   % u4(ii)=u(ii*5-1);
    %u5(ii)=u(ii*5);
end
clear nn
nn=[0:nele];
figure(1)
plot(nn,u1)
title('u1: theta, the twist angle')
figure(2)
plot(nn,u2)
title('u2: nu, the vertical displacement')
figure(3)
plot(nn,u3)
title('u3: beta, the shear rotation angle')
figure(4)
plot(nn,u4)
title('u4: chi, the distortion variable')
%figure(4)
%plot(u4)
%title('u4')
%figure(5)
%plot(u5)
%title('u5')
format long
u(317:320)
format short
```

Displacement Results

ans =

-0.00001493487039	u1 (twist)
0.00110825118590	u2 (vertical displacement)
0.00122368714485	u3 (shear rotation)

5 Degrees of Freedom (DOF) Model with both warping and distortion

```
clear all
k11=[2.940511E+6      1.9588615E+6  8.1990041E+4  9.5547161E+3  9.8071373E+5
1.9588615E+6  1.5680686E+9  1.5384615E+7  -9.6153846E+3 -1.9588615E+6
8.1990041E+4  1.5384615E+7  9.5330427E+6  -7.2477650E+3 8.475103E+4
9.5547161E+3  -9.6153844E+3 -7.2477650E+3 3.8776083E+3  2.8758012E+4
9.8071373E+5  -1.9588615E+6 8.475103E+4   2.8758012E+4  2.9431429E+6];

k22=[2.940511E+6      1.9588615E+6  -8.1990041E+4 -9.5547161E+3 9.8071373E+5
1.9588615E+6  1.568068E+9   -1.5384615E+7 9.6153846E+3  -1.9588615E+6
-8.1990041E+4 -1.5384615E+7 9.5330427E+6  -7.2477650E+3 -8.475103E+4
-9.5547161E+3 9.6153846E+3  -7.2477650E+3 3.8776083E+3  -2.8758012E+4
9.8071373E+5  -1.9588615E+6 -8.475103E+4  -2.8758012E+4 2.9431429E+6];

k21=[-2.9387129E+6    -1.9588615E+6 1.0122886E+5  -9.692078E+3  -9.7997022E+5
-1.9588615E+6 -1.5680686E+9 -1.5384615E+7 9.6153846E+3  1.9588615E+6
-1.0122882E+5 1.5384615E+7  -9.2305936E+6 6.8700185E+3  -1.4245135E+5
9.6920787E+3  -9.6153846E+3 6.8700185E+3  -3.3109884E+3 2.8950320E+4
-9.7848321E+5 1.9588615E+6  8.475103E+4   -2.8950320E+4 -2.939278E+6];

k12=[-2.9387129E+6    -1.9588615E+6 -1.012288E+5  9.6920787E+3  -9.7848321E+5
-1.9588615E+6 -1.5680686E+9 1.538461E+7   -9.6153846E+3 1.9588615E+6
1.0122882E+5  -1.5384615E+7 -9.2305936E+6 6.8700185E+3  8.475103E+4
-9.6920787E+3 9.6153846E+3  6.8700185E+3  -3.3109884E+3 -2.8950320E+4
-9.7997022E+5 1.9588615E+6  -1.4245135E+5 2.8950320E+4  -2.939278E+6];


k12(3,5)=-k12(5,3);
k21=k12';

k12(5,1)=k12(1,5);
k21=k12';

nn=[1 2 3 4 5]

K=[k11 k12;k21 k22];
k11r=k11(nn,nn);
k22r=k22(nn,nn);
k12r=k12(nn,nn);
k21r=k21(nn,nn);

Kr=[k11r k12r;k21r k22r];


nele=80% number of elements
Kg=zeros(5+5*nele,5+5*nele);
Kg(1:10,1:10)=Kr;
```

```
for ii=1:nele-1
   j1=6+(ii-1)*5;
   j2=j1+9;
   Kg(j1:j2,j1:j2)=Kg(j1:j2,j1:j2)+Kr;
end

kgr = Kg(6:5*(nele+1),6:5*(nele+1));
kgr1=kgr(1:398,1:398);

kgri=inv(kgr1);

yy=zeros(5*nele-2,1);

yy(5*nele -3)=100;

u=kgri*yy
u1(1)=0;
u2(1)=0;
u3(1)=0;
u4(1)=0;
u5(1)=0;
for ii=1:nele-1
   u1(ii+1)=u(ii*5-4);
   u2(ii+1)=u(ii*5-3);
   u3(ii+1)=u(ii*5-2);
   u4(ii+1)=u(ii*5-1);
   u5(ii+1)=u(ii*5-0);
% u4(ii)=u(ii*5-1);
   %u5(ii)=u(ii*5);
end
ii=nele


u1(ii+1)=u(ii*5-4);
u2(ii+1)=u(ii*5-3);
u3(ii+1)=u(ii*5-2);
u4(ii+1)=0;
u5(ii+1)=0;

clear nn
nn=[0:nele];
figure(1)
plot(nn,u1)
title('u1: theta, the twist angle')
figure(2)
plot(nn,u2)
```

```
title('u2: nu, the vertical displacement')
figure(3)
plot(nn,u3)
title('u3: beta, the shear rotation angle')
figure(4)
plot(nn,u4)
title('u4: U, the warping variable')
figure(5)
plot(nn,u5)
title('u5: chi, the distortion variable')

format long
u(396:398)
format short
```

Displacement output

```
ans =

  -0.00074939073837    u1 (twist)
   0.00182963917027    u2 (vertical displacement)
   0.00180828955409    u3 (shear rotation
```