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Alternate Stable States in Ecological Systems

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By

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In this thesis we study two reaction-diffusion models that have been used to analyze the existence of alternate stable states in ecosystems. The first model describes the steady states of a logistic growth model with grazing in a spatially homogeneous ecosystem. It also describes the dynamics of the fish population with natural predation. The second model describes phosphorus cycling in stratified lakes. The same equation has also been used to describe the colonization of barren soils in drylands by vegetation.

In this study we discuss the existence of multiple positive solutions, leading to the occurrence of S-shaped bifurcation curves. We were able to show that both the models have alternate stable states for certain ranges of parameter values. We also introduce a constant yield harvesting term to the first model and discuss the existence of positive solutions including the occurrence of a Sigma-shaped bifurcation curve in the case of a one-dimensional model. Again we were able to establish that for certain ranges of parameter values the model has alternate stable states. Thus we establish analytically that the above models are capable of describing the phenomena of alternate stable states in ecological systems. We prove our results by the method of sub-super solutions and quadrature method.

Key words: Boundary value problems, Spatial ecology, S-shaped bifurcation, Method of sub-super solutions, Quadrature method

DEDICATION

To my parents and Lakshmi.

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CHAPTER 1

INTRODUCTION

In this thesis we study two reaction-diffusion models that have been used to analyze the existence of alternate stable states in ecosystems. The first of these models is given by

$$\begin{cases} -\Delta u = \lambda[u - \frac{u^2}{K} - c\frac{u^p}{1+u^p}], \quad x \in \Omega \\ \\ u = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta u = div(\nabla u)$ is the Laplacian of u, $\frac{1}{\lambda}$ is the diffusion coefficient, K, c and p are positive constants and $\Omega \subset \mathbb{R}^N$ is a smooth bounded region with $\partial\Omega$ in C^2. We are particularly interested in the case where $p = 2$. Here u is the population density and $u - \dfrac{u^2}{K}$ represents logistics growth.

This model describes grazing of a fixed number of grazers on a logistically growing species (see [32]-[33]). The assumptions are that the ecosystem is spatially homogeneous and the herbivore density is a constant which are valid assumptions for managed grazing systems. The rate of grazing is given by $\dfrac{cu^2}{1+u^2}$. This term saturates to c at high levels of vegetation density as the grazing population is a constant. This model tries to capture the phenomena of bistability and hysteresis and provide qualitative and quantitative information for ecosystem managements. This model has also been applied to describe the dynamics of fish populations. In such cases the term $\dfrac{cu^2}{1+u^2}$ corresponds to natural predation. For more details see [22], [32], [43] and [45]. We also introduce a constant yield harvesting term to this model and discuss the existence of positive solutions including the occurrence of a Σ-shaped bifurcation curve in the case of a one-dimensional model.

The second model under consideration is given by

$$\begin{cases} -\Delta u = \lambda[K - u + c\dfrac{u^4}{1+u^4}], \quad x \in \Omega \\ \\ \quad u = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.2}$$

This model describes phosphorus cycling in stratified lakes (see [10]). It focuses on the loss of phosphorus from the epilimnion (upper layer) and the sudden recycling if the hypolimnion (lower layer) becomes anoxic. Here u is the mass or concentration of phosphorous (P) in the water column and K is the rate of P input from the watershed. The rate of recycling of P is given by $\dfrac{cu^4}{1+u^4}$ where c is the maximum recycling rate. The assumption here is that the recycling is primarily from the sediments. The same equation has also been used to describe the colonization of barren soils in drylands by vegetation (see [37]). In this case, u is amount of barren soil and $\dfrac{cu^4}{1+u^4}$ represents erosion by wind and runoff.

These phenomena have been extensively studied through simple models ignoring the spatial component and the existence of alternative stable states were reported. This study is motivated by the results in [22] and [45] where these models for the case $N = 1$ have been discussed. Here we extend this study for the higher dimension case.

1.1 Brief history and motivation

Population dynamics is the branch of life sciences that studies short and long term changes in the size and age composition of populations, and the biological and environmental processes influencing those changes. The first problem in population dynamics might have been a problem about the rabbit population that appeared as a computational exercise in *Liber Abaci*, a book written by the Italian mathematician Fibonacci in the year 1202. The population was modeled by the recurrence relation $P_{n+1} = P_n + P_{n-1}$, which we now recognize as the renowned Fibonacci sequence. The next significant development in the field must be attributed to Leonhard Euler, the most prolific mathematician of his times. In 1748 he published a treatise titled *Introduction to Analysis of the Infinite*, where he introduces the geometric or exponential model of population dynamics given by $P_{n+1} = (1 + r)P_n$. Here P_n is the population in year n and $r > 0$ is the growth rate.

Euler's idea was adapted by Malthus in his work *An Essay on the Principle of Population*, published in 1798. This work is considered to be the first popular treatment of the subject. In this book Malthus hypothesized that "The power of population is indefinitely greater than the power in the earth to produce subsistence for man" and drew grim conclusions from it. The geometric growth of the human population and the arithmetic growth of subsistence and the resulting check on population were at the core of his thesis. Malthus did not try to translate his theses into mathematical models. Though he was not the one to formulate the geometric growth model, he realized the importance of the study of populations before anyone else and gave publicity to it. This paved the way to the development of field into an important area of research. It is due to these significant contributions that Malthus is considered to be the father of Population Ecology.

In 1835 Belgian mathematician Adolphe Quetelet suggested that populations could not grow geometrically forever because of the constraints noted by Malthus. He thought that by analogy with mechanics this obstacle to growth would be proportional to square of the speed of population growth. This inspired his compatriot Pierre-Francois Verhulst to refine this idea through his logistic model (1838, see [47]):

$$\frac{dP}{dt} = r\Big(1 - \frac{P}{K}\Big)P. \tag{1.3}$$

The general logistic function is characterized by a declining growth rate per capita function. But there are some ecosystems where the growth rate per capita may achieve its peak at a positive density. This is called the Allee effect (see Allee [1], Dennis [14], Lewis Kareiva [30] and Shi-Shivaji [40]). This effect can be caused by shortage of mates (Hopf and Hopf [21], Veit and Lewis [46]), lack of effective pollination (Groom [18]), predator saturation (de Roos et. al. [13]), and cooperative behaviors (Wilson and Nisbet [48]). In this thesis we restrict ourselves to logistic models.

1.1.1 Modeling catastrophic shifts

We expect that small changes in the environment would produce gradual and smooth changes in an ecosystem. However studies have shown that ecosystems may abruptly switch to contrasting alternate stable states ([4], [25], [32], [36] and [39]) i.e. the system might have two or more stable states under the same external conditions. Once a system undergoes a state shift, it remains in the new state until the control variable is changed back to a much lower level. This pattern is known as hysteresis. Examples have been observed in the desertification of Sahara region, shift in Caribbean coral reefs, and the shallow lake eutrophication ([10], [38] and [39]). One of the first models describing such alternative stable states

$$\frac{dV}{dt} = G(V) - cH(V), \tag{1.4}$$

was discussed in [32] and [33]. This describes the effect of grazing pressure on a population that grows logistically. Here $V(t)$ is the vegetation biomass, $G(V)$ is the growth rate of vegetation in absence of grazing, c is the herbivore population density, and $H(V)$ is the per capita consumption rate of vegetation by the herbivore also known as the functional response. There are three different types of functional response curves ([20]). A type I functional response is a linear relationship where as a type II is of the form $H(V) = \frac{V}{1+V}$, a hyperbolic function that saturates due to the time taken to handle the prey. A standard type III functional response is $H(V) = \frac{V^p}{1+V^p}$ with $p > 1$, a sigmoidal curve with low rates at low densities. This function is known as the Hill's function. In [32]-[33] the authors use a functional response term of type III with $p = 2$.

Similarly in [10] and [39] the authors proposed a minimal mathematical model

$$\frac{dP}{dt} = l - sP + r\frac{P^4}{1+P^4} \tag{1.5}$$

to describe the catastrophic regime shifts between alternative stable states. The variable P is the mass/concentration of Phosphorous in the water column, the rate of loss is s and r is the maximum recycling rate.

These models predict the existence of multiple stable states and hysteresis. Though such models are easy to analyze it might fail to capture the complexities of the ecosystem. There is the danger that the alternate stable states predicted by these models might be an artifact of the simplifications. Hence we use reaction-diffusion models to factor in the spatial component and thus provide a more realistic picture.

1.1.2 Reaction-diffusion models

The simple models described in the equations (1.3), (1.4) and (1.5) assume that the environmental effects are negligible. However in order to be realistic the spatial component must be incorporated into these models. This can be done in several ways. Some models treat it implicitly by incorporating it into parameters or by describing the fraction of an environment with a property without specifying the exact arrangement. Examples of such models include the MacArthur-Wilson models for island biogeography and the classical metapopulation model of Levins (see [29] and [31]). The other major class of models treat the space explicitly by describing what is happening at each location at each time. Some of these treat space as a continuum and some as discrete. This thesis is concerned only with reaction-diffusion (continuum) models of the form:

$$\frac{\partial u}{\partial t} = d\Delta u + f(x,t,u), \tag{1.6}$$

where d is the rate of movement and $f(x,t,u)$ is the reaction term. These models could be easily extended into higher dimensions and to several interacting species. The reaction-diffusion models can be derived by proper scaling of models based on random walks. The pioneering work in this topic was done by J.G. Skellam who was the first to combine the diffusive description of dispersal and population dynamics (see [8] and [42]). They can also be derived from Fick's law, from stochastic differential equations or from interacting particle systems.

The reaction-diffusion models can be used to explain the effects of the size, shape and heterogeneity of the spatial environment on the persistence and structure of populations. Kierstead and Slobodkin ([24]) and Skellam ([42]) discovered that reaction-diffusion models can be used to predict the minimal domain size required to sustain a population. Reaction-diffusion models have also been used as a prototype model for pattern formation since the seminal result of Alan Turing ([44]). These models have also been used to study the waves of invasion by exotic species.

In this thesis we study the existence of alternate steady states to reaction-diffusion models 1.1 and 1.2 which are derived from equations (1.4) and (1.5) The 1-D versions of these models have been studied in [22] and [45]. Here we extend it to the the higher dimension case.

In the next few sections we introduce the major results presented in this thesis.

1.2 Existence and multiplicity for the grazing problem

We consider the existence and multiplicity of positive solutions to the steady state reaction diffusion model given in (1.1). Instead of working with the particular reaction term in (1.1) we will prove our results for a class of functions f which satisfy the following hypothesis:

(H1) $f \in C^2([0,\infty))$, $f(0) = 0$, $f'(0) = 1$, $f(u) > 0$ on $(0, r_0)$ and $f(u) < 0$ for $u > r_0$.

Under this hypothesis it is well known that for $\lambda > \lambda_1(\Omega)$ there always exists a positive solution where $\lambda_1(\Omega)$ is the principal eigenvalue of the operator $-\Delta$ with Dirichlet boundary conditions. It is also trivial to prove that there is no solution for $\lambda \leq \lambda_1(\Omega)$. We will prove the existence of at least three positive solutions for a certain range of λ and hence produce a S-shaped bifurcation curve originating from the trivial branch at $(\lambda_1(\Omega), 0)$.The study of

S-shaped bifurcation curves for positone problems ($f(0) > 0$) has a rich history (see [3], [6], [23] and [35]). Even in the case of positone problems proving multiplicity results for nonlinearities with falling zeros (like our reaction terms) is much harder and often remains an open problem (see example (iv) in [6] and [28]). For such problems, the solution space is restricted as $||u||_\infty < r_0$. Here we deal with a situation when $f(0) = 0$ and when f is a rather complicated nonlinearity with a falling zero. In fact, now the solution space is further restricted to $\{(\lambda, \rho)|\lambda < \lambda_1,\ 0 < \rho < r_0\}$ (see Figure 1.1).

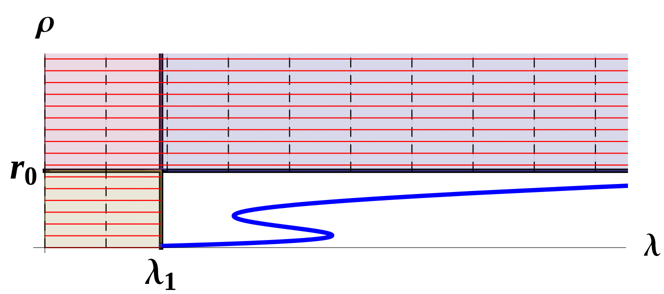

Figure 1.1

Bifurcation diagram for a falling zero problem

To precisely state our multiplicity result, for $0 < a < b$, let

$$Q_1(a, b, \Omega) := \frac{max\{\lambda_1(B_R), \frac{b}{f(b)}(\frac{N+1}{N})^{N+1}\frac{N^2}{R^2}\}}{min\{\frac{a}{||e_\Omega||_\infty f^*(a)}, \frac{2NM}{f(b)R^2}\}}, \tag{1.7}$$

where $B_R = B(0, R)$ is the largest inscribed ball on Ω (see Figure 1.2), e_Ω is the unique positive solution of $-\Delta e = 1$ in Ω, $e = 0$ on $\partial\Omega$ and $f^*(s) = \max_{t\in[0,s]} f(t)$. We establish:

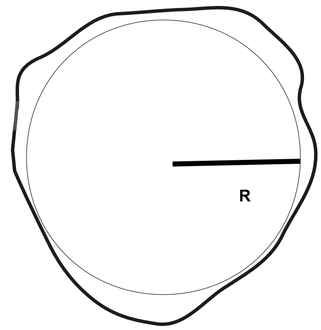

Figure 1.2

B_R - the largest inscribed ball in Ω

Theorem 1

Let $m, M \in (0, r_0)$ *be such that* f *is non-decreasing in* (m, M). *Assume there exists* $b \in [m, M]$ *and* $a \in (0, b)$ *such that* $Q_1(a, b, \Omega) < 1$. *Then* (1.1) *has three positive solutions for* $\lambda \in \left(max\{\lambda_1(B_R), \frac{b}{f(b)}(\frac{N+1}{N})^{N+1}\frac{N^2}{R^2}\}, min\{\frac{a}{||e_\Omega||_\infty f^*(a)}, \frac{2NM}{f(b)R^2}\}\right)$.

We will use the method of sub-supersolutions to prove our results. We prove Theorem 1 and apply our results in the case when $f(u) = u - \dfrac{u^2}{K} - c\dfrac{u^2}{1+u^2}$. This study is motivated by the results in [22] and [45] where such a multiplicity result for the case $N = 1$ was discussed. Here we extend this study for the higher dimension case.

1.3 Existence of alternate stable states in a phosphorous cycling model

We will discuss the existence of alternate stable states to the nonlinear boundary value problem given in (1.2). As with the harvesting model described in the previous section, instead of working with the particular reaction term in (1.2), we will prove our results for a class of functions f which satisfy the following hypothesis:

(H2) $f \in C^2([0,\infty))$, $f(u) > 0$ on $[0, r_0)$ and $f(u) < 0$ for $u > r_0$.

It is known that positone problems ($f(0) > 0$) with a falling zero have a positive solution for all $\lambda > 0$. To state our multiplicity result, for $0 < a < b$, let $Q_2(a, b, \Omega) := \frac{\frac{b}{f(b)}(\frac{N+1}{N})^{N+1}\frac{N^2}{R^2}}{min\{\frac{a}{||e_\Omega||_\infty f^*(a)}, \frac{2NM}{f(b)R^2}\}}$, where $B_R = B(0, R)$ is the largest inscribed ball on Ω, e_Ω is the unique positive solution of $-\Delta e = 1$ in Ω, $e = 0$ on $\partial\Omega$, and $f^*(s) = \max\limits_{t\in[0,s]} f(t)$. We establish:

Theorem 2

Let $m, M \in (0, r_0)$ be such that f is non-decreasing in (m, M). Assume there exists $b \in [m, M]$ and $a \in (0, b)$ such that $Q_2(a, b, \Omega) < 1$. Then (1.2) has three positive solutions for $\lambda \in \left(\frac{b}{f(b)}(\frac{N+1}{N})^{N+1}\frac{N^2}{R^2}, min\{\frac{a}{||e_\Omega||_\infty f^*(a)}, \frac{2NM}{f(b)R^2}\}\right)$.

We will use the method of sub-supersolutions to prove our results. Once we prove the Theorem 1, we apply our results in the case when $f(u) = K - u + c\dfrac{u^4}{1+u^4}$.

This study, when the spatial component is not ignored, is motivated by the results in [22] and [45] where such a multiplicity result for the case $N = 1$ was discussed. Here, we extend this study for the higher dimension case. The arguments used to prove this theorem are similar to those used in the proof of Theorem 1.

1.4 Grazing with constant yield harvesting - An existence result

We also study the existence of a positive solution $u \in C^2(\Omega)\bigcap C(\bar{\Omega})$ to a model with grazing and constant yield harvesting given by

$$\begin{cases} -\Delta u = au - bu^2 - c\dfrac{u^p}{1+u^p} - K, \quad x \in \Omega \\ \\ \quad u = 0, \quad x \in \partial\Omega. \end{cases} \tag{1.8}$$

where $\Delta u = div\big(\nabla u\big)$ is is the Laplacian of u, a, b, c, p, K are positive constants with $p \geq 2$. Ω is a smooth bounded region with $\partial\Omega$ in C^2. Here u is the population density and $au - bu^2$ represents logistics growth. It can be easily shown that (1.8) does not have a positive solution if $a \leq \lambda_1$, where λ_1 is the principal eigenvalue of the operator $-\Delta$ with Dirichlet boundary condition.

We will establish our existence results by the method of sub-super solutions. In the absence of the constant yield harvesting term K, the boundary value problem always has a positive steady state for $a > \lambda_1$.

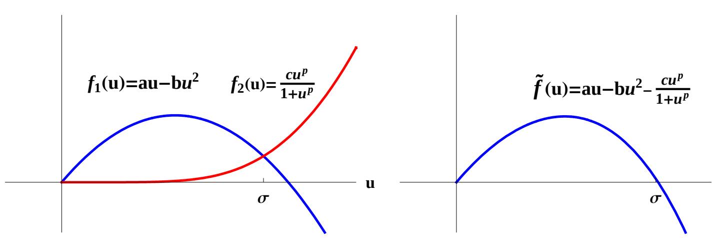

Figure 1.3

Reaction terms with and without grazing.

That is, grazing alone does not eliminate the steady states for all positive 'a' values.

The diffusive logistic equation with constant yield harvesting, in the absence of grazing was studied in [34]. Here if the harvesting rate is too high then there will be no positive steady states. The authors discuss the existence, uniqueness and stability of the maximal steady state solutions to

$$\begin{cases} -\Delta u = au - bu^2 - K, \quad x \in \Omega \\ \\ \quad u = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.9}$$

where $K \geq 0$ represents the harvesting effort. In particular, in [34] it is shown that if $a > \lambda_1$ and $b > 0$ then there exists a $K_1 = K_1(a, b) > 0$ such that for $0 < K < K_1$, (1.9) has a positive solution.

Such problems, i.e., problems of the form

$$\begin{cases} -\Delta u = \lambda f(u), \quad x \in \Omega \\ \\ \quad u = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.10}$$

where $f(0) < 0$, are referred as semipositone problems in the literature. Construction of a subsolution (see Section 2.2.2) is more challenging in semipositone problems (see [5] and [27]). Here our test functions for a positive subsolution must come from positive functions ψ such that $-\Delta\psi < 0$ near $\partial\Omega$ and $-\Delta\psi > 0$ in a large enough interior of Ω so that $\psi > 0$ in Ω.

In our discussions we analyze (1.8), that is the model with grazing and constant yield harvesting and prove the following result:

Theorem 3

Let $a > \lambda_1, b > 0$ and $c > 0$ be fixed. Then there exists a $K_0(a, b, c, p) > 0$ such that for $K < K_0(a, b, c, p)$, (1.8) has a positive solution.

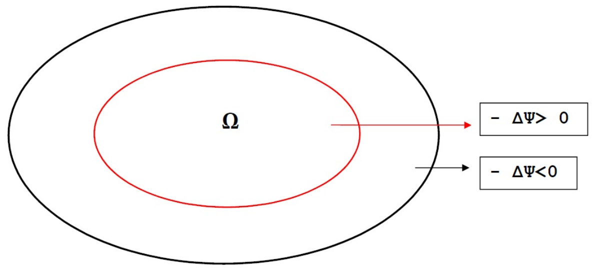

Figure 1.4

Subsolution for semipositone problems.

1.5 An ecological model with a Σ-shaped bifurcation curve

Finally we study the existence of multiple positive solutions in the presence of both grazing and constant yield harvesting. However, our analysis is restricted to the one dimensional model namely

$$\begin{cases} -u'' = \lambda[u - \frac{u^2}{K} - c\frac{u^2}{1+u^2} - \epsilon] =: \tilde{f}(u), \quad x \in (0,1) \\ u(0) = 0 = u(1). \end{cases} \tag{1.11}$$

We first prove:

Theorem 4

Given $c < 2$ there exists $K(c) > 0$ such that for $K > K(c)$ and $\frac{2c}{K^2+1} + \frac{2}{K^3} < \epsilon < \frac{1}{4K}$ the boundary value problem (1.11) has a bifurcation curve that is at least Σ-shaped.

Further, given $c < 2$ our computational results indicate that for $K \gg 1$ there exists a range of λ for which there exists at least four positive solutions. We will provide an analytical proof of this occurence of at least four solutions when $c = \frac{1}{2}$, namely we establish:

Figure 1.5

Σ-shaped bifurcation curve.

Theorem 5

Let $c = \frac{1}{2}$*. Then there exists* $K_0 > 0$ *such that for* $K > K_0$ *and* $\frac{1}{K^2+1} + \frac{2}{K^3} < \epsilon < \frac{1}{4K}$ *the boundary value problem (1.11) has at least four positive solutions for a certain range of* λ*.*

We establish our proof of Theorem 4 and Theorem 5 using the quadrature method discussed in [6] and [27]. We also provide various bifurcation diagrams for positive solutions via Mathematica computations. Here we observe that Σ-shaped bifurcation diagrams occur for any $0 < c < 2$ when ϵ is small and K is large.

We conclude this introduction by providing an outline of this thesis. We introduce some of the required definitions and methods in Chapter 2. In Chapter 3 we prove Theorem 1 and establish that the grazing problem has an at least S-shaped bifurcation curve. Some detailed result for the 1D case is presented as an appendix to this Chapter. In Chapter 4 we discuss existence of alternate stable states in a phosphorous cycling model (1.1) and prove Theorem 2. Again an appendix with detailed one dimensional results is provided. Theorem 3, the existence result for the model with grazing and constant yield harvesting is proved in Chapter 5. In Chapter 6 we prove Theorems 4 and 5 and establish the existence

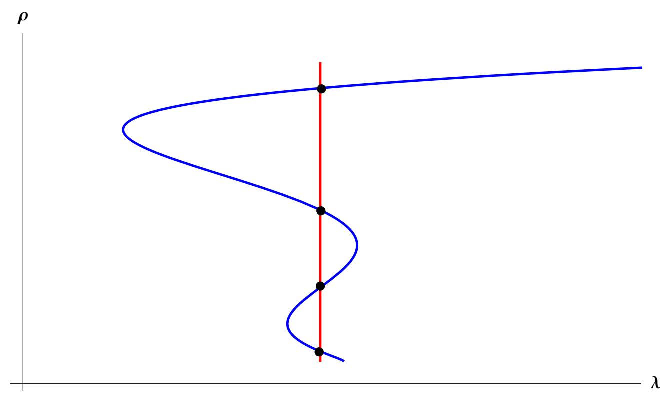

Figure 1.6

At least four solutions for a certain range of λ.

of an at least Σ-shaped bifurcation curve for the one dimensional model with grazing and constant yield harvesting given in (1.11). And finally in Chapter 7 the conclusions and future directions are discussed.

CHAPTER 2

PRELIMINARIES

In this chapter we will discuss some preliminary results on elliptic boundary value problems. In particular, we will discuss maximum principles, anti-maximum principles, quadrature method and the method of sub-super solutions.

2.1 Maximum and anti-maximum principles

In the following discussion Ω is a bounded domain in $\mathbb{R}^n$ and $u \in C^2(\Omega) \cap C(\overline{\Omega})$.

Lemma 1 (Maximum principle)

Let $\Delta u \geq 0$ in Ω. If u attains its maximum at any interior point in Ω, then $u \equiv M$ in Ω.

Lemma 2 (Hopf's maximum principle)

Let $\Delta u \geq 0$ in Ω. Suppose that $u \leq M$ in Ω and $u = M$ at some $p \in \partial\Omega$. Then $\dfrac{\partial u}{\partial \nu} > 0$ at p unless $u \equiv M$ where $\dfrac{\partial}{\partial \nu}$ denotes the outward normal derivative.

Lemma 3 (Anti-maximum principle, Clement and Peletier [12])

There exist a $\delta = \delta(\Omega) > 0$ and a solution z_λ (with $z_\lambda > 0$ in Ω and $\dfrac{\partial z_\lambda}{\partial \nu} < 0$ on $\partial\Omega$, where ν is the outer unit normal to Ω) of

$$\begin{cases} -\Delta z - \lambda z = -1, & x \in \Omega \\ \qquad\quad z = 0, & x \in \partial\Omega, \end{cases} \tag{2.1}$$

for $\lambda \in (\lambda_1, \lambda_1 + \delta)$.

2.2 Nonlinear boundary value problems

Consider the following Dirichlet boundary value problem:

$$\begin{cases} -\Delta u = \lambda f(u), \quad x \in \Omega \\ \\ u = 0, \quad x \in \partial\Omega, \end{cases} \tag{2.2}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded region with $\partial\Omega$ in C^2. In this Chapter we discuss some of the methods and techniques used to study problems of this kind.

2.2.1 Quadrature method

Quadrature method is a simple method developed by Laetsch in [27] to study the two point boundary value problems of the form

$$\begin{cases} -u'' = \lambda f(u), \quad x \in (0,1) \\ \\ u(0) = 0 = u(1). \end{cases} \tag{2.3}$$

Here $\lambda > 0$ and f satisfies the following hypotheses.

(G1) $f \in C^2([0,\infty))$ and $f(u) > 0$ for $0 < u < r_0$ for $r_0 > 0$;

(G2) there exists $k \geq 0$ such that $f(u) - f(v) \geq -k(u-v)$ for all $u, v \in [0, r_0)$ with $u > v$;

(G3) $r_0 < \infty$ and $f(r_0) = 0$.

Since (2.3) is autonomous, u is symmetric with respect to $x = \frac{1}{2}$ and is increasing on $[0, \frac{1}{2})$ and decreasing on $(\frac{1}{2}, 1]$. Thus $u'(\frac{1}{2}) = 0$ and $\sup\{u(x) : x \in [0,1]\} = u(\frac{1}{2})$. Now multiplying (2.3) by $u'(x)$ we obtain

$$-\Big(\frac{u'(x)^2}{2}\Big)' = \lambda\big(F(u(x))\big)', \tag{2.4}$$

where $F(s) := \int_0^s f(t)dt$. Integrating the above equation from x to $\frac{1}{2}$ we get

$$u'(x) = \sqrt{2\lambda(F(\rho) - F(u(x))}; x \in (0, \frac{1}{2}) \tag{2.5}$$

where $\rho = ||u||_\infty$. Here we have used $\rho = u(\frac{1}{2})$ and $u'(\frac{1}{2}) = 0$. Integrating again we have,

$$\int_0^{u(x)} \frac{dz}{\sqrt{[F(\rho) - F(z)]}} = \sqrt{2\lambda}x, \qquad 0 < x < \frac{1}{2}. \tag{2.6}$$

Substituting $x = \frac{1}{2}$ gives,

$$\sqrt{\lambda} = \sqrt{2}\int_0^{\rho} \frac{dz}{\sqrt{[F(\rho) - F(z)]}} := G(\rho). \tag{2.7}$$

Thus if there exists a positive solution u to (2.3) with $||u||_\infty = \rho$, then ρ must be such that $G(\rho)$ exists and satisfy $G(\rho) = \lambda$.

Conversely, let $\rho \in \{u > 0 : f(u) > 0$ and $F(u) > F(s)$ for all $s, 0 \leq s < u\}$ and $\lambda \in (0, \infty)$ be such that $G(\rho) = \sqrt{\lambda}$. Also let $u(x)$ be defined by (2.6). We will show that $u(x)$ is a positive solution of (2.3) with $||u||_\infty = \rho$. The left-hand side of (2.6) is a differentiable function of u which is strictly increasing for $x \in (0, 1)$ as u increases from 0 to ρ. Hence for each $x \in [0, \frac{1}{2})$, there exists a unique $u(x)$ that satisfies (2.6). By the implicit function theorem, $u(x)$ is differentiable as a function of x. Differentiating (2.6), we have

$$u'(x) = \sqrt{2\lambda(F(\rho) - F(u(x))} \tag{2.8}$$

Squaring and differentiating the above we get

$$-u''(x) = \lambda f(u(x)). \tag{2.9}$$

Thus $u(x)$ satisfies the differential equation in (2.3) and $u(0) = 0$. Finally, extending $u(x)$ as a symmetric function to $(0, 1)$, gives a positive solution to (2.3) with $||u||_\infty = \rho$.

2.2.2 Sub-super solutions

By a subsolution of (2.2) we mean a function $\psi \in C^2(\Omega)\bigcap C(\bar{\Omega})$ that satisfies:

$$\begin{cases} -\Delta\psi \leq \lambda f(\psi), \quad x \in \Omega \\ \\ \quad \psi \leq 0, \quad x \in \partial\Omega \end{cases} \tag{2.10}$$

and by a supersolution of (2.2) we mean a function $\phi \in C^2(\Omega)\bigcap C(\bar{\Omega})$ that satisfies:

$$\begin{cases} -\Delta\phi \geq \lambda f(\phi), \quad x \in \Omega \\ \\ \quad \phi \geq 0, \quad x \in \partial\Omega. \end{cases} \tag{2.11}$$

The notion of sub and super solutions can be extended in a weak sense as follows. By a subsolution (supersolution) of (2.2), we mean a function $\psi \in W^{1,2}(\Omega)\bigcap C(\bar{\Omega})$ such that $\psi = 0$ on $\partial\Omega$ and

$$\int_\Omega \nabla\psi.\nabla q \ \ \leq (\geq) \ \ \int_\Omega \lambda f(\psi)q, \tag{2.12}$$

for every $q \in \{\eta \in C_0^\infty(\Omega) : \eta \geq 0 \text{ in } \Omega\}$. Then the following lemma holds (see [2]).

Lemma 4

Assume that $f \in C^1[0,\infty)$. Let ψ be a subsolution of (1.1) and ϕ be a supersolution of (1.1) such that $\psi \leq \phi$. Then (1.1) has a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $\psi \leq u \leq \phi$.

To establish our main multiplicity result we use the following very useful result discussed in [2], [41]. Note here that by $\psi_1 < \psi_2$ we mean that $\psi_1 \leq \psi_2$ and $\psi_1 \neq \psi_2$.

Lemma 5

Let $f \in C^1([0,\infty))$. Suppose there exists a subsolution ψ_1, a strict supersolution Z_1, a strict subsolution ψ_2, and a supersolution Z_2 for (1.1) such that $\psi_1 < Z_1 < Z_2$, $\psi_1 < \psi_2 <$

Z_2 *and* $\psi_2 \not\leq Z_1$. *Then, (1.1) has at least three distinct solutions* u_1, u_2, *and* u_3 *such that* $\psi_1 \leq u_1 < u_2 < u_3 \leq Z_2$.

CHAPTER 3

PROOF OF THEOREM 1

We prove Theorem 1 in Section 3.1. In Section 3.2 we apply our results in the case when $f(u) = u - \frac{u^2}{K} - c\frac{u^2}{1+u^2}$. In Section 3.3 we provide more detailed results for the case $N = 1$ using the quadrature method discussed in [6], [27].

3.1 Proof of Theorem 1

To establish the multiplicity result we have to construct a subsolution ψ_1, a strict supersolution Z_1, a strict subsolution ψ_2 and a supersolution Z_2 for (1.1) such that $\psi_1 < Z_1 < Z_2$, $\psi_1 < \psi_2 < Z_2$ and $\psi_2 \not\leq Z_1$.

Let B_R be the largest inscribed ball in Ω. Define

$$\psi_1(x) = \begin{cases} \tilde{\epsilon}\phi_1(x) & ; \quad x \in B_R \\ 0 & ; \quad x \in \Omega - B_R. \end{cases} \tag{3.1}$$

where $\phi_1 > 0$ is an eigenfunction corresponding to $\lambda_1(B_R)$ and $\tilde{\epsilon} > 0$. Then $\psi_1 \in W^{1,2}(\Omega) \bigcap C(\bar{\Omega})$ and $\psi_1 = 0$ on $\partial\Omega$. Now

$$-\Delta\psi_1 = -\tilde{\epsilon}\Delta\phi_1 = \tilde{\epsilon}\lambda_1(B_R)\phi_1;\ x \in B_R. \tag{3.2}$$

Let $H(s) = \lambda f(s) - \lambda_1(B_R)s$. Then $H'(s) = \lambda f'(s) - \lambda_1(B_R)$, $H(0) = 0$ and $H'(0) = \lambda - \lambda_1(B_R)$. Since we are interested in the λ range $\lambda > \lambda_1(B_R)$, clearly $H'(0) > 0$. So for

$\tilde{\epsilon} \approx 0$ we have $H(\tilde{\epsilon}\phi_1) = \lambda f(\tilde{\epsilon}\phi_1) - \lambda_1(B_R)(\tilde{\epsilon}\phi_1) \geq 0$. Hence from (3.2), for $\tilde{\epsilon} \approx 0$ we have

$$-\Delta\psi_1 = \tilde{\epsilon}\lambda_1(B_R)\phi_1 \leq \lambda f(\tilde{\epsilon}\phi_1) = \lambda f(\psi_1);\ x \in B_R, \tag{3.3}$$

while outside B_R we have $-\Delta\psi_1 = 0 = \lambda f(0) = \lambda f(\psi_1)$. Thus ψ_1 is a positive subsolution.

For the large supersolution choose $Z_2 = r_0$. Then $-\Delta Z_2 = 0 \geq f(r_0) = f(Z_2)$ making Z_2 a positive super solution. For $\tilde{\epsilon} \approx 0$ clearly $\psi_1 \leq Z_2$.

Now for the smaller strict supersolution define $Z_1 = \dfrac{ae_\Omega}{||e_\Omega||_\infty}$, where e_Ω is the unique positive solution of $-\Delta e = 1$ in Ω, $e = 0$ on $\partial\Omega$. Since $\lambda < \frac{a}{||e_\Omega||_\infty f^*(a)}$, $-\Delta Z_1 = \frac{a}{||e_\Omega||_\infty} > \lambda f^*(a) \geq \lambda f^*(\frac{ae_\Omega}{||e_\Omega||_\infty}) \geq \lambda f(\frac{ae_\Omega}{||e_\Omega||_\infty}) = \lambda f(Z_1)$ in Ω. Here $f^*(s) = \max_{t\in[0,s]} f(t)$. Hence Z_1 is a strict supersolution.

We will now construct the strict subsolution ψ_2. Let

$$\tilde{f}(u) = \begin{cases} \hat{f}(u); & u < m \\ f(u) \quad ; & u \geq m \end{cases} \tag{3.4}$$

where $\hat{f}(u)$ is defined so that the function $\tilde{f}(u)$ is strictly increasing on $(0, M)$ and $\tilde{f}(u) \leq f(u)$ (see Figure 3.1).

Let

$$\rho(r) = \begin{cases} 1 & ; \quad r \in [0, \epsilon] \\ 1 - [1 - (\frac{R-r}{R-\epsilon})^\beta]^\alpha; & r \in (\epsilon, R], \alpha, \beta > 1. \end{cases} \tag{3.5}$$

Note that

$$\rho'(r) = \begin{cases} 0 & ; \quad r \in [0, \epsilon] \\ -\alpha\beta[1 - (\frac{R-r}{R-\epsilon})^\beta]^{\alpha-1}(\frac{R-r}{R-\epsilon})^{\beta-1}; & r \in (\epsilon, R], \alpha, \beta > 1 \end{cases} \tag{3.6}$$

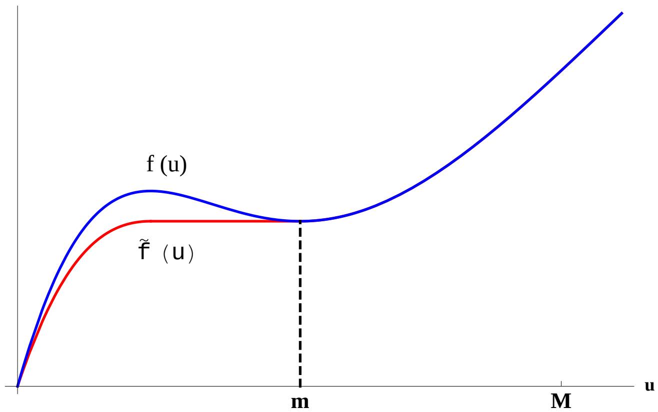

Figure 3.1

Graph of $\tilde{f}(u)$

and $|\rho'(r)| < \frac{\alpha\beta}{R-\epsilon}$. Now define $w(r) := b\rho(r)$ and

$$\psi_2(x) = \begin{cases} \tilde{\psi_2} & ; \quad x \in B_R \\ 0 & ; \quad x \in \Omega - B_R, \end{cases} \tag{3.7}$$

where $\tilde{\psi_2}$ is the solution of

$$\begin{cases} -\tilde{\psi_2}''(r) - \dfrac{N-1}{r}\tilde{\psi_2}'(r) = \lambda\tilde{f}(w(r)), \quad r \in (0,R) \\ \tilde{\psi_2}'(0) = 0 = \tilde{\psi_2}(R). \end{cases} \tag{3.8}$$

Then $\psi_2 \in W^{1,2}(\Omega)\bigcap C(\bar{\Omega})$ and $\psi_2 = 0$ on $\partial\Omega$. We will now establish that $\tilde{\psi_2}(r) \in (w(r), M]$ on $[0,R)$. Then $-\Delta\psi_2 = \lambda\tilde{f}(w(r)) < \lambda\tilde{f}(\tilde{\psi_2}(r)) \leq \lambda f(\psi_2(r))$ on $[0,R)$ while outside B_R we have $-\Delta\psi_2 = 0 = \lambda f(0) = \lambda f(\psi_2)$ and hence ψ_2 will be a strict subsolution.

First we will show that $\tilde{\psi}_2(r) \leq M$. Now

$$(r^{N-1}\tilde{\psi_2}'(r))' = -\lambda r^{N-1}\tilde{f}(w(r)) \tag{3.9}$$

$$\tilde{\psi_2}'(r) = \frac{-\lambda}{r^{N-1}}\int_0^r s^{N-1}\tilde{f}(w(s))ds \tag{3.10}$$

$$\tilde{\psi}_2(t) - \tilde{\psi}_2(0) = -\int_0^t \frac{\lambda}{r^{N-1}}\Big\{\int_0^r s^{N-1}\tilde{f}(w(s))ds\Big\}dr \tag{3.11}$$

But $\tilde{\psi}_2(R) = 0$. Hence we get

$$\begin{aligned}
\tilde{\psi}_2(0) &= \int_0^R \frac{\lambda}{r^{N-1}}\Big\{\int_0^r s^{N-1}\tilde{f}(w(s))ds\Big\}dr && (3.12)\\
&\leq \frac{\lambda\tilde{f}(b)}{N}\int_0^R rds && (3.13)\\
&= \frac{\lambda f(b)}{2N}R^2 \text{ (since } b \geq m \text{ and } \tilde{f}(s) = f(s) \text{ for } s \geq m.\text{)} && (3.14)
\end{aligned}$$

But $\lambda < \frac{2NM}{f(b)R^2}$. Hence $||\tilde{\psi}_2||_\infty = \tilde{\psi}_2(0) < M$.

Next to establish $\tilde{\psi}_2 > w$ on $[0, R]$ we will show that $\tilde{\psi_2}' < w' \leq 0$ on $[0, R]$. This will be sufficient since $\tilde{\psi}_2(R) = w(R) = 0$. Now $w' = 0$ and $\tilde{\psi_2}' < 0$ in the interval $[0, \epsilon)$and hence $\tilde{\psi_2}' < w' \leq 0$ in that interval. For $r > \epsilon$ we have

$$\begin{aligned}
-\tilde{\psi_2}'(r) &= \frac{\lambda}{r^{N-1}}\int_0^r s^{N-1}\tilde{f}(w(s))ds && (3.15)\\
&\geq \frac{\lambda}{r^{N-1}}\int_0^\epsilon s^{N-1}\tilde{f}(w(s))ds && (3.16)\\
&= \frac{\lambda}{r^{N-1}}\int_0^\epsilon s^{N-1}\tilde{f}(b)ds \text{ (since } \rho(s) = 1, s < \epsilon) && (3.17)\\
&\geq \frac{\lambda\tilde{f}(b)}{R^{N-1}}\int_0^\epsilon s^{N-1}ds && (3.18)\\
&= \frac{\lambda f(b)}{R^{N-1}}\frac{\epsilon^N}{N} \text{ (since } b \geq m \text{ and } \tilde{f}(s) = f(s) \text{ for } s \geq m). && (3.19)
\end{aligned}$$

We also know that $|w'(r)| \leq \frac{b\alpha\beta}{R-\epsilon}$. Hence $|\tilde{\psi_2}'(r)| > |w'(r)|$ if $\lambda > \alpha\beta\frac{b}{f(b)}\frac{NR^{N-1}}{(R-\epsilon)\epsilon^N}$. But $\min\limits_{0<\epsilon<R}\frac{1}{(R-\epsilon)\epsilon^N} = \frac{(N+1)^{N+1}}{N^N R^{N+1}}$ and this minimum is achieved at $\epsilon_0 = \frac{NR}{N+1}$. Since $\lambda >$

$\frac{b}{f(b)}\frac{N^2}{R^2}(\frac{N+1}{N})^{N+1} = \frac{b}{f(b)}\frac{NR^{N-1}}{(R-\epsilon_0)\epsilon_0^N}$ we can choose $\epsilon = \epsilon_0$ and $\alpha, \beta > 1$ such that $\lambda > \alpha\beta\frac{b}{f(b)}\frac{NR^{N-1}}{(R-\epsilon_0)\epsilon_0^N}$. Hence $|\tilde{\psi_2}'(r)| > |w'(r)|$ on $(0, R)$. This implies $w < \tilde{\psi_2}$. Thus ψ_2 is a strict subsolution of (1) if $\frac{b}{f(b)}\frac{N^2}{R^2}(\frac{N+1}{N})^{N+1} < \lambda < \frac{2NM}{f(b)R^2}$. Moreover $\tilde{\psi_2}(0) > w(0) = b > a = ||Z_1||_\infty$, i.e. $\psi_2 \not\leq Z_1$. Hence by Lemma 5 Theorem 1 holds.

3.2 Results for the grazing problem

First we will analyze some properties of this nonlinearity. We will show that for large K we can find values of c for which the function $f(u) = u - \frac{u^2}{K} - c\dfrac{u^2}{1+u^2}$ satisfies $(H1)$ and we will also identify m, M such that f is increasing in (m, M). Clearly $f \in C^2([0,\infty))$, $f(0) = 0$ and $f'(0) = 1$. Now we will prove that the other assumptions in Theorem 1 holds in the given example.

Proposition 6 *If* $c > \frac{8}{3\sqrt{3}}$ *then for* K *large there exists three points* m_1, m_2 *and* m_3 *such that* $0 < m_1 < m_2 < m_3$ *and* $f'(m_i) = 0$ *for* $i = 1, 2, 3$.

We have $f'(u) = 1 - \frac{2u}{K} - \frac{2cu}{(1+u^2)^2}$. So $f'(u) = 0$ when $1 - \frac{2u}{K} = \frac{2cu}{(1+u^2)^2}$. Let $g(u) := \frac{2cu}{(1+u^2)^2}$. Here $1 - \frac{2u}{K}$ is a line passing through $(0, 1)$ and with slope $-\frac{2}{K}$. We will prove that for $K >> 1$, this line will cut $g(u)$ at three different points. We have $g(u) \geq 0$, $g(0) = 0$ and $\lim_{u\to\infty} g(u) = 0$. Since $g'(u) = 2c\frac{1-3u^2}{(1+u^2)^3}$, we can see that $g(u)$ achieves a maximum of $\frac{3\sqrt{3}c}{8}$ at $u = \frac{1}{\sqrt{3}}$, if $\max_{x\in(0,\infty)} g(u) = \dfrac{3\sqrt{3}c}{8} > 1$ then for K large the line $1 - \frac{2u}{K}$ will cut $g(u)$ at exactly three different points. Hence if $c > \frac{8}{3\sqrt{3}}$ and K is large, then there are exactly three positive points $m_1 < m_2 < m_3$ such that $f'(m_i) = 0$ for $i = 1, 2, 3$.

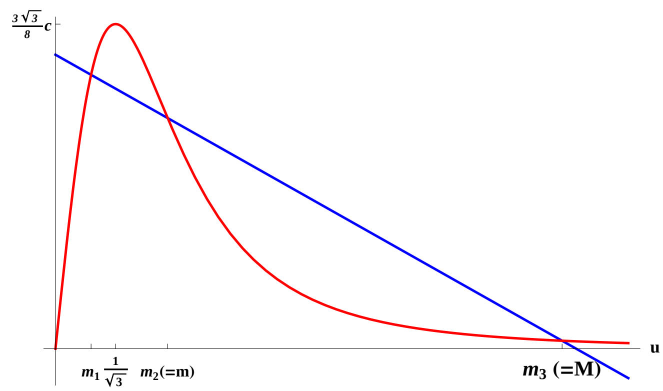

Figure 3.2

Graph of the line $1 - \frac{2u}{K}$ and $g(u)$

Proposition 7 *If $c < 2$ then for $K >> 1$ there exists a unique $r_0 > 0$ such that $f(r_0) = 0$.*

If $c < \frac{8}{3\sqrt{3}}$, then from the geometry discussed above it is clear that for K large $f(u)$ has a unique positive zero. Now consider the case $\frac{8}{3\sqrt{3}} < c < 2$. Since $c > \frac{8}{3\sqrt{3}}$ by Proposition 10 we have for K large, there exist three positive numbers m_1, m_2 and m_3 such that $f'(m_i) = 0$ for $i = 1, 2, 3$. Clearly the function $f(u)$ has a relative minimum at $u = m_2$. We will prove that $f(m_2) > 0$ for $K >> 1$. This implies that $f(u)$ has a unique positive zero. It is clear from Figure 3.2 that there exists a constant M_2 such that $m_2 < M_2$ for all K. In fact $m_2 = m_2(K)$ is a continuous decreasing function of K such that $m_2(K) \in (\frac{1}{\sqrt{3}}, M_2)$. Also $\lim_{K\to\infty} f(m_2) = z - c\dfrac{z^2}{1+z^2}$ for some $z \in (\frac{1}{\sqrt{3}}, M_2)$. But $h(z) = z - c\frac{z^2}{1+z^2} > 0$ for $z > 0$ if $c < 2$. Hence $\lim_{K\to\infty} f(m_2) > 0$. Thus for K large there exists a unique $r_0 > m_3$ such that $f(r_0) = 0$. Thus given $c \in (\frac{8}{3\sqrt{3}}, 2)$ we can find K large so that $f(u)$ is increasing on $(m_2(= m), m_3(= M))$ and there exists a unique $r_0 > 0$ such that $f(r_0) = 0$, i.e. $f(u)$ satisfies $(H1)$.

Next we will select candidates for $b \in [m_2, m_3]$ and $a \in (0, b)$ using which later we will show that $Q_1(a, b, \Omega) < 1$. The point at which the function $\frac{u}{f(u)}$ has a minimum would be an ideal value for b (see Figure 3.4).

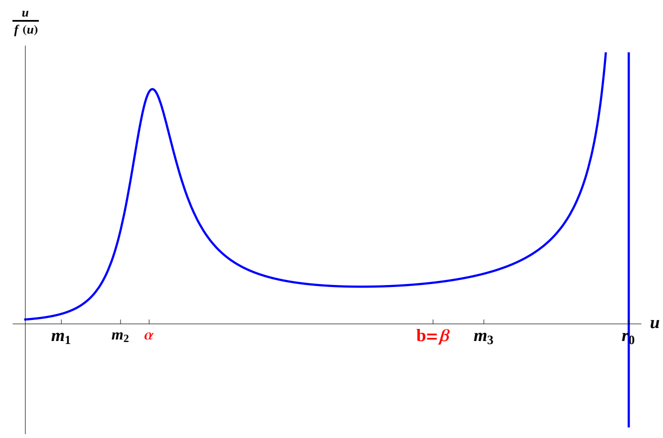

Figure 3.3

Graph of $\frac{u}{f(u)}$

We have $(\frac{u}{f(u)})' = \frac{f(u)-uf'(u)}{(f(u))^2}$. Hence the critical points of $\frac{u}{f(u)}$ are given by $f(u) - uf'(u) = 0$ and in particular the non-zero critical points are given by $\frac{1}{K} - \frac{c}{1+u^2} + \frac{2c}{(1+u^2)^2} = 0$. Solving for u we get the positive critical points as $\alpha = \sqrt{\frac{cK-2-\sqrt{cK(cK-8)}}{2}}$ and $\beta = \sqrt{\frac{cK-2+\sqrt{cK(cK-8)}}{2}}$. Hence $\frac{u}{f(u)}$ has a relative minimum at β. Since $\beta \to \infty$ as $K \to \infty$ and m_2 is bounded we have $\beta \in [m_2, m_3]$ for large K. Choose $b = \beta$. Next we choose $a \in (m_2, b)$ such that $f(a) = f^*(a) = f(m_1)$. This is possible since $f(m_1)$ is bounded while $f(b) \to \infty$ as $K \to \infty$. (See Proposition 13 which follows next where it will be established that $\lim_{K\to\infty} \frac{b}{f(b)} = 1$. But $\lim_{K\to\infty} b = \infty$. Hence $f(b) \to \infty$ as $K \to \infty$).

Proposition 8

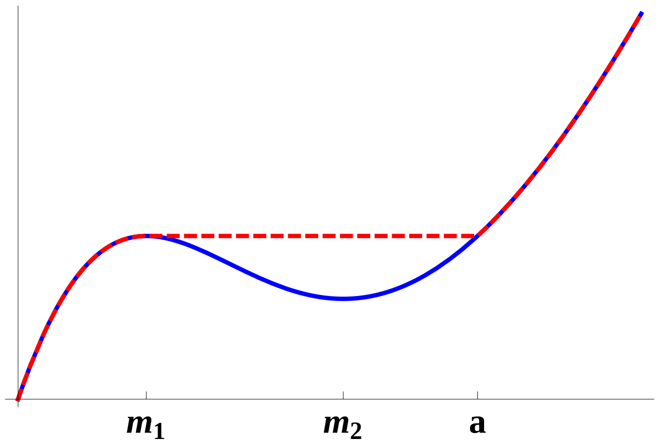

Figure 3.4

Graph of $f^*(u)$ (in red)

(i) $b \le \sqrt{cK}$ *and* $m_3 > \frac{K}{4}$ *for* $K >> 1$.

(ii) $\lim_{K\to\infty} \dfrac{b}{f(b)} = 1$ *and* $\lim_{K\to\infty} \dfrac{m_3}{f(b)} = \infty$.

(i) We have $b = \sqrt{\frac{cK-2+\sqrt{cK(cK-8)}}{2}} = \sqrt{\frac{cK-2+cK\sqrt{(1-\frac{8}{cK})}}{2}} \le \sqrt{cK}$. Now $f'(u) = 1 - \frac{2u}{K} - \frac{2cu}{(1+u^2)^2}$. So $f'(\frac{K}{4}) = \frac{1}{2} - \frac{cK}{2(1+\frac{K^2}{16})^2} > 0$ and $f'(\frac{K}{2}) = -\frac{cK}{(1+\frac{K^2}{4})^2} < 0$ for $K >> 1$. Hence $m_3 > \frac{K}{4}$ for $K >> 1$.

(ii) We have $\lim_{K\to\infty} b = \lim_{K\to\infty} \sqrt{\dfrac{cK-2+\sqrt{cK(cK-8)}}{2}} = \infty$. From (i) we have $b \le \sqrt{cK}$. Hence $\lim_{K\to\infty} \dfrac{b}{K} \le \lim_{K\to\infty} \dfrac{\sqrt{cK}}{K} = 0$. Thus $\lim_{K\to\infty} \dfrac{b}{f(b)} = \lim_{K\to\infty} \dfrac{b}{b - \frac{b^2}{K} - c\frac{b^2}{1+b^2}}$ $= \lim_{K\to\infty} \dfrac{1}{1 - \frac{b}{K} - c\frac{b}{1+b^2}} = 1$. Finally $\lim_{K\to\infty} \dfrac{m_3}{f(b)} \ge \lim_{K\to\infty} \dfrac{K}{4(b - \frac{b^2}{K} - c\frac{b^2}{1+b^2})} = \infty$.

Now we will discuss the existence of at least three positive solutions for a certain range of λ (see Theorem 1) by establishing that $Q_1(a, b, \Omega) < 1$. In particular we will analyze the following two cases: (**A**) Ω is a ball in $\mathbb{R}^N$ and (**B**) Ω is a general bounded domain in $\mathbb{R}^N$.

3.2.1 Case A: When Ω is a ball

We will now prove that when Ω is a ball of radius R (i.e. $\Omega = B_R$) in $\mathbb{R}^N$ with $N < 8$ there exists $K >> 1$ and c close to 2 such that $Q_1(a, b, B_R) < 1$..

First for $u \in [0, M_2]$ and $c = 2$ we consider the function $h(u) := \lim_{K\to\infty} f(u) = u - 2\dfrac{u^2}{1+u^2}$. Note that $h'(u) = 1 - \frac{4u}{(1+u^2)^2}$. and solving $h'(u) = 0$ we get $m_1 \approx 0.2956$ and $m_2 = 1$. Solving $f(u) = f(m_1)$, $u \neq m_1$, we get $a \approx 1.5437$ (see Figure 5). Hence $\frac{a}{f^*(a)} = \frac{a}{f(m_1)} \approx 11.4445$.

Our aim is to prove that for $c = 2-\delta$, where $\delta \approx 0$ and $K >> 1$, $Q_1(a, b, B_R) < 1$. We have already seen that $\frac{b}{f(b)} \to 1$ and $\frac{m_3(=M)}{f(b)} \to \infty$ as $K \to \infty$. Hence $Q_1(a, b, B_R) < 1$ if $max\{\lambda_1(B_R), (\frac{N+1}{N})^{N+1}\frac{N^2}{R^2}\} < \frac{a}{||e_{B_R}||_\infty f^*(a)} = \frac{11.4445}{||e_{B_R}||_\infty}$, i.e. if $\lambda_1(B_R)||e_{B_R}||_\infty < 11.4445$ and $(\frac{N+1}{N})^{N+1}\frac{N^2}{R^2}||e_{B_R}||_\infty < 11.4445$.

Next we evaluate $||e_{B_R}||_\infty$. We have $-\Delta e = 1$ in B_R, $e = 0$ on ∂B_R. Then e is radial, radially decreasing and satisfies:

$$\begin{cases} -e''(r) - \dfrac{N-1}{r}e'(r) = 1, \quad r \in (0, R) \\ e'(0) = 0 = e(R). \end{cases} \tag{3.20}$$

Solving this boundary value problem we obtain $e(r) = \frac{1}{2N}(R^2 - r^2)$. From this it follows that $||e_{B_R}||_\infty = e(0) = \frac{R^2}{2N}$. Now the principal eigenvalue λ_1 when Ω is a ball of radius R is given by,

$$\lambda_1(B_R) = \begin{cases} \frac{\pi^2}{4R^2} \approx \frac{2.4674}{R^2}; \quad N = 1 \\ \frac{j_{0,1}^2}{R^2} \approx \frac{5.7832}{R^2}; \quad N = 2 \\ \frac{j_{\frac{N}{2}-1,1}}{R^2} \qquad ; \quad N \geq 3, \end{cases} \tag{3.21}$$

where $j_{n,1}$ is the first zero of the Bessel function of order n (see [19]). From [11] we have $j_{n,1} < (n+1)^{\frac{1}{2}}\big((n+2)^{\frac{1}{2}}+1\big)$ for $n > -1$. Hence for $N \geq 3$, we get $\lambda_1(B_R)||e_{B_R}||_\infty = \frac{j_{\frac{N}{2}-1,1}}{2N} \leq \frac{(\frac{N}{2})^{\frac{1}{2}}\big((\frac{N}{2}+1)^{\frac{1}{2}}+1\big)}{2N} = \frac{1}{2\sqrt{2}}\big(\sqrt{\frac{N+2}{2N}}+\frac{1}{\sqrt{N}}\big) < 1 < \frac{a}{f^*(a)}$. But we also have $\lambda_1(B_R)||e_{B_R}||_\infty = \frac{2.4674}{2}$ and $\lambda_1(B_R)||e_{B_R}||_\infty = \frac{5.7832}{4}$ when $N = 1$ and $N = 2$ respectively. Thus $\lambda_1(B_R)||e_{B_R}||_\infty < \frac{a}{f^*(a)}$ for all N.

Next we have $\frac{N^2}{R^2}(\frac{N+1}{N})^{N+1}||e_{B_R}||_\infty = \frac{N}{2}(\frac{N+1}{N})^{N+1}$. The function $y = \frac{N}{2}(\frac{N+1}{N})^{N+1}$ is increasing for positive N and $\frac{N}{2}(\frac{N+1}{N})^{N+1} < \frac{a}{f^*(a)}(= 11.4445)$ for $N < 8$. Thus for $N < 8$, $Q_1(a,b,B_R) < 1$ when $c \approx 2$ and $K >> 1$.

3.2.2 Case B: When Ω is a general bounded domain

When Ω is a general bounded region we will establish a sufficient condition on the geometry of the region for our multiplicity result to hold. Let $R_1 > 0$, $R_2 > 0$ be such that $B_{R_2} = B(0,R_2) \subseteq \Omega \subseteq B_{R_1} = B(0,R_1)$ (see Figure 3.6).

Let $-\Delta e_{B_{R_1}} = 1$ in B_{R_1}, $e_{B_{R_1}} = 0$ on ∂B_{R_1}. Then $e_{B_{R_1}}$ is a supersolution of the problem $-\Delta e_\Omega = 1$ in Ω, $e_\Omega = 0$ on $\partial\Omega$. Hence $||e_\Omega||_\infty \leq ||e_{B_{R_1}}||_\infty$.

As in the case when the domain is a ball, we will try to prove that for $c = 2-\delta$ (where $\delta > 0$ is very small) and $K >> 1$, $Q_1(a,b,\Omega) < 1$. Now $Q_1(a,b,\Omega) < 1$ if $max\{\lambda_1(B_{R_2}), (\frac{N+1}{N})^{N+1}\frac{N^2}{R_2^2}\} < \frac{a}{||e_\Omega||_\infty f^*(a)} = \frac{11.4445}{||e_\Omega||_\infty}$, i.e. if $\lambda_1(B_{R_2})||e_\Omega||_\infty < 11.4445$ and $(\frac{N+1}{N})^{N+1}\frac{N^2}{R_2^2}||e_\Omega||_\infty < 11.4445$.

By 3.21, for $N \geq 3$, we get $\lambda_1(B_{R_2})||e_\Omega||_\infty \leq \lambda_1(B_{R_2})||e_{B_{R_1}}||_\infty = \frac{j_{\frac{N}{2}-1,1}}{2N}\frac{R_1^2}{R_2^2} \leq \frac{(\frac{N}{2})^{\frac{1}{2}}\big((\frac{N}{2}+1)^{\frac{1}{2}}+1\big)}{2N}\frac{R_1^2}{R_2^2} = \frac{1}{2\sqrt{2}}\big(\sqrt{\frac{N+2}{2N}}+\frac{1}{\sqrt{N}}\big)\frac{R_1^2}{R_2^2} < \frac{R_1^2}{R_2^2}$. If $N = 1$ we have $\lambda_1(B_{R_2})||e_\Omega||_\infty \leq \lambda_1(B_{R_2})||e_{B_{R_1}}||_\infty = \frac{2.4674}{2}\frac{R_1^2}{R_2^2} = 1.2337\frac{R_1^2}{R_2^2}$ and if $N = 2$ we have $\lambda_1(B_{R_2})||e_\Omega||_\infty \leq$

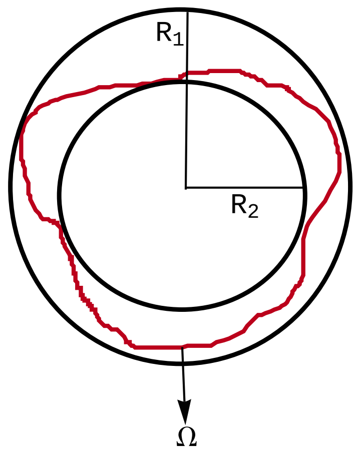

Figure 3.5

General domain Ω

$\lambda_1(B_{R_2})||e_{B_{R_1}}||_\infty = \frac{5.7832}{4}\frac{R_1^2}{R_2^2} = 1.4458\frac{R_1^2}{R_2^2}$. Thus $\lambda_1(B_{R_2})||e_\Omega||_\infty < \frac{a}{f^*(a)} = 11.4445$ for all N whenever $\frac{R_1^2}{R_2^2} < \frac{11.4445}{1.4458} = 7.91569$.

Next we have $(\frac{N+1}{N})^{N+1}\frac{N^2}{R_2^2}||e_\Omega||_\infty \leq (\frac{N+1}{N})^{N+1}\frac{N^2}{R_2^2}||e_{B_{R_1}}||_\infty = \frac{N}{2}(\frac{N+1}{N})^{N+1}\frac{R_1^2}{R_2^2}$. Hence in the general domain case if $\frac{R_1^2}{R_2^2} < min\{7.91569, \frac{2}{N}(\frac{N}{N+1})^{N+1}11.4445\}$ then $Q_1(a,b,\Omega) < 1$ when $c \approx 2$ and $K >> 1$. Note that $\frac{2}{N}(\frac{N}{N+1})^{N+1}11.4445 > 1$ when $N < 8$.

3.3 Analytical and computational results for the case $N = 1$

Consider the two point boundary value problem

$$\begin{cases} -u'' = \lambda f(u), \quad x \in (0,1) \\ u(0) = 0 = u(1), \end{cases} \tag{3.22}$$

where f satisfies the following hypotheses:

(G1) $f \in C^2([0,\infty))$ and $f(u) > 0$ for $0 < u < r_0$ for $r_0 > 0$;

(G2) there exists $k \geq 0$ such that $f(u) - f(v) \geq -k(u-v)$ for all $u, v \in [0, r_0)$ with $u > v$;

(G3) $r_0 < \infty$ and $f(r_0) = 0$.

Using the quadrature method the solution $u = u(x)$ is defined by

$$\int_0^{u(x)} \frac{dz}{\sqrt{[F(\rho) - F(z)]}} = \sqrt{2\lambda}x, \qquad 0 < x < \frac{1}{2}, \tag{3.23}$$

where $F(s) := \int_0^s f(t)dt$, provided

$$\sqrt{\lambda} = \sqrt{2}\int_0^{\rho} \frac{dz}{\sqrt{[F(\rho) - F(z)]}} := G(\rho). \tag{3.24}$$

Here $\rho = u(\frac{1}{2}) = ||u||_\infty$. Since $f(\rho) > 0$ and $F(\rho) > F(z)$ for all $0 \leq z < \rho$, it follows that $G(\rho)$ exists for all $\rho > 0$. Infact $G(\rho)$ is a continuous function. We also have

$$G'(\rho) = \sqrt{2}\int_0^1 \frac{H(\rho) - H(\rho s)}{[F(\rho) - F(\rho s)]^{\frac{3}{2}}} ds \tag{3.25}$$

where $H(u) = F(u) - \frac{u}{2}f(u)$. Then we have the following lemma from [6].

Lemma 6

(a) If the bifurcation curve of (4.20) is S-shaped, then $H'(\rho) < 0$ for some $0 < \rho < r_0$

(b) If there exists positive $\rho_0 < r_0$ such that $H(\rho_0) < 0$, then (4.20) has at least three solutions for a certain range of λ

Consider the case $f(u) = u - \frac{u^2}{K} - c\frac{u^2}{1+u^2}$. Clearly, given $c < 2$ fixed then for $K >> 1$ f satisfies $(G1) - (G3)$ (see Proposition 12). Hence $G(\rho)$ is defined for all $\rho \in S = (0, r_0)$.

Now we will show that there exists $\rho_0 \in (0, r_0)$ such that $H(\rho_0) < 0$. We have $H(u) = F(u) - \frac{u}{2}f(u) = \frac{u^3}{6K} + c\Big(\frac{u^3}{2(1+u^2)} - u + \tan^{-1}(u)\Big)$ and $H'(u) = \frac{1}{2}(f(u) - uf'(u))$. The zeros of $H'(u)$ are the same as the zeros of $(\frac{u}{f(u)})'$ and in Section 3.2 we have already found that the positive roots of $(\frac{u}{f(u)})'$ are $\alpha = \sqrt{\frac{cK-2-\sqrt{cK(cK-8)}}{2}}$ and $\beta = \sqrt{\frac{cK-2+\sqrt{cK(cK-8)}}{2}}$. Hence $H(u)$ has a maximum at $u = \alpha$ and a minimum at $u = \beta$. From Proposition 12 we have $\beta < r_0$ for $K >> 1$. We will now show that $H(\beta) < 0$ for $K >> 1$, giving $H(u)$ the shape shown in Figure 3.6.

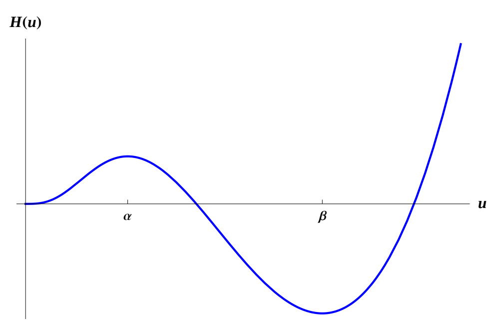

Figure 3.6

Graph of $H(u)$

We have

$$\begin{aligned} H(\beta) &= \frac{\beta^3}{6K} + c\Big(\frac{\beta^3}{2(1+\beta^2)} - \beta + \tan^{-1}(\beta)\Big) && (3.26)\\ &= \frac{\beta^2}{6K}\beta + c\Big(\frac{\beta^2}{2(1+\beta^2)}\beta - \beta + \tan^{-1}(\beta)\Big) && (3.27)\\ &\leq \frac{c}{6}\beta + c(\frac{\beta}{2} - \beta + \frac{\pi}{2}) = c(-\frac{1}{3}\beta + \frac{\pi}{2}) && (3.28) \end{aligned}$$

Clearly $\beta \to \infty$ as $K \to \infty$ and hence $\lim_{K\to\infty} H(\beta) < 0$. Hence by Lemma 6 we have the following result.

Theorem 9

Given $c < 2$ fixed then for $K >> 1$ the boundary value problem (4.20) has at least three solutions for a certain range of λ.

We used Mathematica to compute $\sqrt{\lambda} = G(\rho)$ in the case when $f(u) = u - \dfrac{u^2}{K} - c\dfrac{u^2}{1+u^2}$ and plotted the bifurcation diagrams. In Figure 3.7 and Figure 3.8 bifurcation diagrams for a certain value of c and K are given.

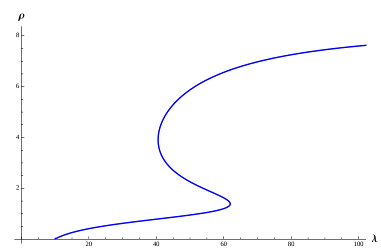

Figure 3.7

Bifurcation Diagram with $c = 1.5$ and $K = 10$. Here $r_0 = 8.19687$

In Figure 3.9 the region of (c, K)-plane that satisfies the hypothesis of Theorem 9 is given. For all values of c and K, that lies in the region enclosed by these curves, the two point boundary value problem given in (4.20) will have at least three different solutions for a certain range of λ values.

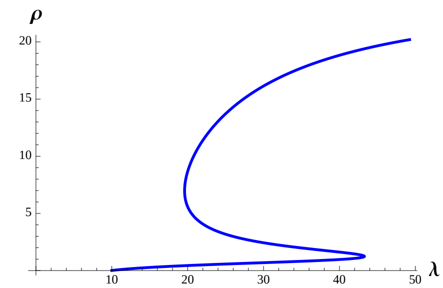

Figure 3.8

Bifurcation Diagram with $c = 1.5$ and $K = 25$. Here $r_0 = 23.4004$

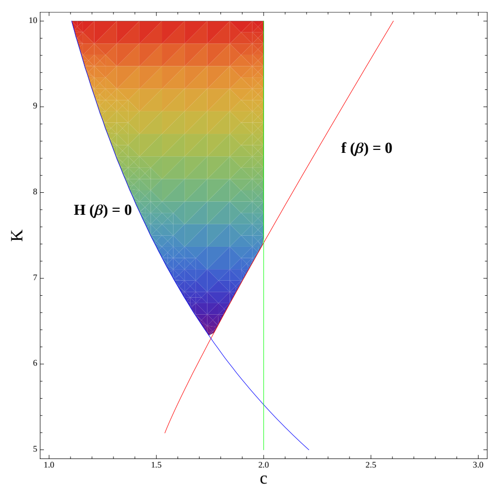

Figure 3.9

Feasible Region

CHAPTER 4

PROOF OF THEOREM 2

We prove Theorem 2 in Section 4.1. The proof is motivated by the arguments in the previous chapter. In Section 4.2, we analyze in detail the phosphorus cycling model when $f(u) = K - u + c\dfrac{u^4}{1+u^4}$ has a unique positive zero r_0. This will be the case when $K > K_0 := \frac{3}{4}\sqrt[4]{\frac{3}{5}} - \frac{1}{4}\sqrt[4]{\frac{3}{5}}^5$ and $c \gg 1$. We will prove that an S-shaped bifurcation curve occurs when $c \gg 1$ and $K_0 < K < \frac{9c}{16}$. This analysis turned out to be quite nontrivial and challenging. We also obtained more detailed analytical and computational results for the case $N = 1$, which are presented in Section 4.3.

4.1 Proof of Theorem 2

To establish the multiplicity result, we have to construct a subsolution ψ_1, a strict supersolution Z_1, a strict subsolution ψ_2, and a supersolution Z_2 for (1.1) such that $\psi_1 < Z_1 < Z_2$, $\psi_1 < \psi_2 < Z_2$ and $\psi_2 \nleq Z_1$. Clearly, $\psi_1 = 0$ is a strict subsolution since $f(0) > 0$. For the large supersolution, choose $Z_2 = M(\lambda)e_\Omega$ where $M(\lambda) > \lambda \max\limits_{t\in[0,r_0]} f(t)$. Then, $-\Delta Z_2 = M(\lambda) \geq \lambda f(Z_2)$ making Z_2 a positive super solution.

Now for the smaller strict supersolution, define $Z_1 = \dfrac{ae_\Omega}{||e_\Omega||_\infty}$. Since $\lambda < \frac{a}{||e_\Omega||_\infty f^*(a)}$, $-\Delta Z_1 = \frac{a}{||e_\Omega||_\infty} > \lambda f^*(a) \geq \lambda f^*(\frac{ae_\Omega}{||e_\Omega||_\infty}) \geq \lambda f(\frac{ae_\Omega}{||e_\Omega||_\infty}) = \lambda f(Z_1)$ in Ω. Here $f^*(s) = \max\limits_{t\in[0,s]} f(t)$. Hence, Z_1 is a strict supersolution.

We will now construct the strict subsolution ψ_2. Let

$$\tilde{f}(u) = \begin{cases} \hat{f}(u); & u < m \\ f(u) & ; \quad u \geq m \end{cases} \tag{4.1}$$

where $\hat{f}(u)$ is defined so that the function $\tilde{f}(u)$ is strictly increasing on $(0, M)$ and $\tilde{f}(u) \leq f(u)$ (see Figure 4.1).

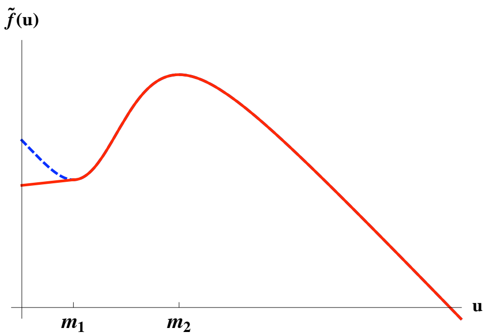

Figure 4.1

Graph of $\tilde{f}(u)$

Let

$$\rho(r) = \begin{cases} 1 & ; \quad r \in [0, \epsilon] \\ 1 - [1 - (\frac{R-r}{R-\epsilon})^{\beta}]^{\alpha}; & r \in (\epsilon, R], \alpha, \beta > 1. \end{cases} \tag{4.2}$$

Note that

$$\rho'(r) = \begin{cases} 0 & ; \quad r \in [0, \epsilon] \\ -\alpha\beta[1 - (\frac{R-r}{R-\epsilon})^{\beta}]^{\alpha-1}(\frac{R-r}{R-\epsilon})^{\beta-1}; & r \in (\epsilon, R], \alpha, \beta > 1 \end{cases} \tag{4.3}$$

and $|\rho'(r)| < \frac{\alpha\beta}{R-\epsilon}$. Now define $w(r) := b\rho(r)$ and

$$\psi_2(x) = \begin{cases} \tilde{\psi_2} & ; \quad x \in B_R \\ 0 & ; \quad x \in \Omega - B_R, \end{cases} \tag{4.4}$$

where $\tilde{\psi_2}$ is the solution of

$$\begin{cases} -\tilde{\psi_2}''(r) - \dfrac{N-1}{r}\tilde{\psi_2}'(r) = \lambda \tilde{f}(w(r)), \quad r \in (0, R) \\ \tilde{\psi_2}'(0) = 0 = \tilde{\psi_2}(R). \end{cases} \tag{4.5}$$

and B_R is the largest inscribed ball in Ω. Then, $\psi_2 \in W^{1,2}(\Omega) \bigcap C(\bar{\Omega})$, and $\psi_2 = 0$ on $\partial\Omega$. We will now establish that $\tilde{\psi_2}(r) \in (w(r), M]$ on $[0, R)$. Then, $-\Delta\psi_2 = \lambda\tilde{f}(w(r)) < \lambda\tilde{f}(\tilde{\psi_2}(r)) \leq \lambda f(\psi_2(r))$ on $[0, R)$ while outside B_R we have $-\Delta\psi_2 = 0 = \lambda f(0) = \lambda f(\psi_2)$, and hence ψ_2 will be a strict subsolution.

First, we will show that $\tilde{\psi_2}(r) \leq M$. Now

$$\begin{aligned} (r^{N-1}\tilde{\psi_2}'(r))' &= -\lambda r^{N-1}\tilde{f}(w(r)) && (4.6)\\ \tilde{\psi_2}'(r) &= \frac{-\lambda}{r^{N-1}}\int_0^r s^{N-1}\tilde{f}(w(s))ds && (4.7)\\ \tilde{\psi_2}(t) - \tilde{\psi_2}(0) &= -\int_0^t \frac{\lambda}{r^{N-1}}\Big\{\int_0^r s^{N-1}\tilde{f}(w(s))ds\Big\}dr && (4.8) \end{aligned}$$

But, $\tilde{\psi_2}(R) = 0$. Hence, we get

$$\begin{aligned} \tilde{\psi_2}(0) &= \int_0^R \frac{\lambda}{r^{N-1}}\Big\{\int_0^r s^{N-1}\tilde{f}(w(s))ds\Big\}dr && (4.9)\\ &\leq \frac{\lambda\tilde{f}(b)}{N}\int_0^R rds && (4.10)\\ &= \frac{\lambda f(b)}{2N}R^2 \text{ (since } b \geq m \text{ and } \tilde{f}(s) = f(s) \text{ for } s \geq m.\text{)} && (4.11) \end{aligned}$$

But $\lambda < \frac{2NM}{f(b)R^2}$. Hence $||\tilde{\psi_2}||_\infty = \tilde{\psi_2}(0) < M$.

Next, to establish $\tilde{\psi_2} > w$ on $[0, R]$, we will show that $\tilde{\psi_2}' < w' \leq 0$ on $[0, R]$. This will be sufficient, since $\tilde{\psi_2}(R) = w(R) = 0$. Now $w' = 0$ and $\tilde{\psi_2}' < 0$ in the interval $[0, \epsilon)$, and hence $\tilde{\psi_2}' < w' \leq 0$ in that interval. For $r > \epsilon$, we have

$$-\tilde{\psi_2}'(r) = \frac{\lambda}{r^{N-1}} \int_0^r s^{N-1} \tilde{f}(w(s)) ds \tag{4.12}$$

$$\geq \frac{\lambda}{r^{N-1}} \int_0^\epsilon s^{N-1} \tilde{f}(w(s)) ds \tag{4.13}$$

$$= \frac{\lambda}{r^{N-1}} \int_0^\epsilon s^{N-1} \tilde{f}(b) ds \text{ (since } \rho(s) = 1, s < \epsilon) \tag{4.14}$$

$$\geq \frac{\lambda \tilde{f}(b)}{R^{N-1}} \int_0^\epsilon s^{N-1} ds \tag{4.15}$$

$$= \frac{\lambda f(b)}{R^{N-1}} \frac{\epsilon^N}{N} \text{ (since } b \geq m \text{ and } \tilde{f}(s) = f(s) \text{ for } s \geq m). \tag{4.16}$$

We also know that $|w'(r)| \leq \frac{b\alpha\beta}{R-\epsilon}$. Hence, $|\tilde{\psi_2}'(r)| > |w'(r)|$ if $\lambda > \alpha\beta \frac{b}{f(b)} \frac{NR^{N-1}}{(R-\epsilon)\epsilon^N}$. But $\min_{0<\epsilon<R} \frac{1}{(R-\epsilon)\epsilon^N} = \frac{(N+1)^{N+1}}{N^N R^{N+1}}$, and this minimum is achieved at $\epsilon_0 = \frac{NR}{N+1}$. Since $\lambda > \frac{b}{f(b)} \frac{N^2}{R^2} (\frac{N+1}{N})^{N+1} = \frac{b}{f(b)} \frac{NR^{N-1}}{(R-\epsilon_0)\epsilon_0^N}$, we can choose $\epsilon = \epsilon_0$ and $\alpha, \beta > 1$ such that $\lambda > \alpha\beta \frac{b}{f(b)} \frac{NR^{N-1}}{(R-\epsilon_0)\epsilon_0^N}$. Hence, $|\tilde{\psi_2}'(r)| > |w'(r)|$ on $(0, R)$. This implies $w < \tilde{\psi_2}$. Thus, ψ_2 is a strict subsolution of (1) if $\frac{b}{f(b)} \frac{N^2}{R^2} (\frac{N+1}{N})^{N+1} < \lambda < \frac{2NM}{f(b)R^2}$. Furthermore, $\tilde{\psi_2}(0) > w(0) = b > a = ||Z_1||_\infty$, i.e. $\psi_2 \nleq Z_1$. Moreover, $M(\lambda)$ can be chosen large enough so that $\psi_2 < Z_2$ and $Z_1 < Z_2$. Hence, by Lemma 5, Theorem 2 holds.

4.2 Results for phosphorous cycling problem

First, we will analyze some properties of this nonlinearity. We will show that for large c we can find values of K for which the function $f(u) = K - u + c\dfrac{u^4}{1+u^4}$ satisfies $(H2)$, and we will also identify m and M such that f is increasing in (m, M). Clearly, $f \in C^2([0,\infty))$, $f(0) = K$ and $f'(0) = -1$.

Proposition 10 *If $c > \frac{16}{5\sqrt[4]{135}}$, then there exists two points m_1 and m_2 such that $0 < m_1 < m_2$ and $f'(m_i) = 0$ for $i = 1, 2$.*

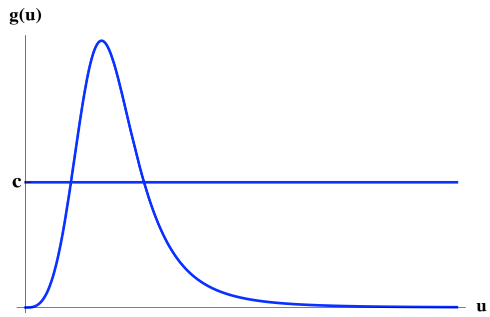

Figure 4.2

$g(u)$ and $h(u)$

We have $f'(u) = -1 + \frac{4cu^3}{(1+u^4)^2}$. So $f'(u) = 0$ when $1 = \frac{4cu^3}{(1+u^4)^2}$. Let $g(u) := \frac{4cu^3}{(1+u^4)^2}$, and let $h(u) := 1$. We have $g(u) \geq 0$, $g(0) = 0$ and $\lim_{u\to\infty} g(u) = 0$. Since $g'(u) = \frac{4cu^2(3-5u^4)}{(1+u^4)^3}$, we can see that $g(u)$ achieves a maximum of $\frac{5c}{16}\sqrt[4]{135}$ at $u = \sqrt[4]{\frac{3}{5}}$. If $\max_{x\in(0,\infty)} g(u) = \dfrac{5c}{16}\sqrt[4]{135} > 1$, then the line $h(u)$ will cut $g(u)$ at exactly two different points. Hence, if $c > \frac{16}{5\sqrt[4]{135}}$, then there are exactly two positive points $m_1 < m_2$ such that $f'(m_i) = 0$ for $i = 1, 2$.

Proposition 11 *If $K > \frac{3}{4}\sqrt[4]{\frac{3}{5}} - \frac{1}{4}\left(\sqrt[4]{\frac{3}{5}}\right)^5 =: K_0$, then there exists a unique $r_0 > 0$ such that $f(r_0) = 0$.*

From Figure 4.3, we can see that if $f(m_1) > 0$ then $f(u)$ has a unique positive zero. Since $f'(m_1) = 0$, we obtain $\frac{cm_1^3}{1+m_1^4} = \frac{1+m_1^4}{4}$. So, $f(m_1) = K - m_1 + \frac{cm_1^4}{1+m_1^4} = K - m_1 + \frac{m_1(1+m_1^4)}{4} = K - \frac{3}{4}m_1 + \frac{m_1^5}{4}$. Hence, $f(m_1) > 0$ if $K > \frac{3}{4}m_1 + \frac{m_1^5}{4}$. On analyzing

$f''(u) = \frac{4cu^2(3-5u^4)}{(1+u^4)^3}$, we see that the positive inflection of $f(u)$ occurs at $u = \sqrt[4]{\frac{3}{5}}$. Thus, $m_1 < \sqrt[4]{\frac{3}{5}}$, and hence $K > K_0$ ensures that there exists a unique r_0 such that $f(r_0) = 0$. Choose $m = m_1$ and $M = m_2$. Thus, given $K > K_0$, we can find c large so that $f(u)$ is increasing on (m, M), and there exists a unique $r_0 > 0$ such that $f(r_0) = 0$, i.e. $f(u)$ satisfies $(H2)$. Now, we will prove that the other assumptions in Theorem 2 hold in the given example.

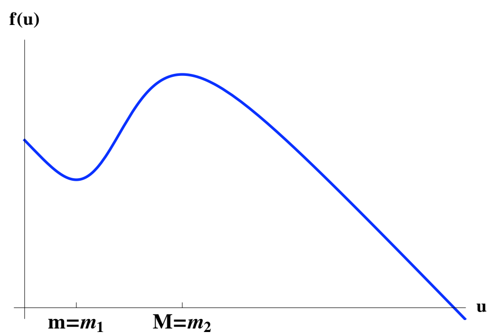

Figure 4.3

Graph of $f(u)$ with m and M marked

We will select $b \in [m, M]$ and $a \in (0, b)$ such that $Q_2(a, b, \Omega) < 1$. The point at which the function $\frac{u}{f(u)}$ has a minimum would be an ideal choice for b.

Proposition 12 *If $K < \frac{9c}{16}$, then $\frac{u}{f(u)}$ has the shape given in Figure 4.4.*

We have $(\frac{u}{f(u)})' = \frac{f(u)-uf'(u)}{(f(u))^2}$. Hence, the critical points of $\frac{u}{f(u)}$ are given by $f(u) - uf'(u) = 0$, and in particular, the non-zero critical points are given by $K + \frac{c(u^8-3u^4)}{(1+u^4)^2} = 0$. Solving for u, we get the positive critical points as $\alpha = \sqrt[4]{\frac{3c-2K-\sqrt{c(9c-16K)}}{2(c+K)}}$ and $\beta = \sqrt[4]{\frac{3c-2K+\sqrt{c(9c-16K)}}{2(c+K)}}$. Note that if $K < \frac{9c}{16}$ then α and β are positive real roots of $(\frac{u}{f(u)})'$

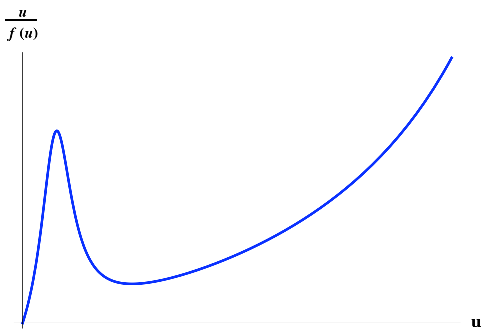

Figure 4.4

Graph of $\frac{u}{f(u)}$

with $\alpha < \beta$. Hence, $\frac{u}{f(u)}$ has a relative maximum at α and a relative minimum at β. Since $\beta \to \sqrt[4]{3}$ as $c \to \infty$ and $m < \sqrt[4]{\frac{3}{5}}$, we have $m < \beta$. Furthermore, it is clear from Figure 4.2 that $M \to \infty$ as $c \to \infty$, so $\beta < M$ for $c \gg 1$. Thus, we have $\beta \in [m, M]$ for large c and we choose $b = \beta$. We also choose $a \in (0, M)$ such that $f(a) = f^*(a) = f(0)$.

The following estimates hold for a and M for $c \gg 1$.

Proposition 13 *For $c \gg 1$:*

(i) $\sqrt[5]{c} < M < \sqrt[4]{c}$.

(ii) $\frac{1}{\sqrt[3]{c}} < a < \frac{1}{\sqrt[4]{c}}$

(i) By the shape of the graph of $f(u)$ established in Proposition 10 and 12 (see Figure 4.3), it is enough if we prove that $f'(\sqrt[5]{c}) > 0$ and $f'(\sqrt[4]{c}) < 0$. We have $f'(\sqrt[5]{c}) = -1 + \frac{4c^{\frac{8}{5}}}{(1+c^{\frac{4}{5}})^2} = -1 + \frac{4c^{\frac{8}{5}}}{1+2c^{\frac{4}{5}}+c^{\frac{8}{5}}} = -1 + \frac{4}{\frac{1}{c^{\frac{8}{5}}}+\frac{2}{c^{\frac{4}{5}}}+1} > 0$, and $f'(\sqrt[4]{c}) = -1 + \frac{4c^{\frac{7}{4}}}{(1+c)^2} = -1 + \frac{4}{\frac{1}{c^{\frac{7}{4}}}+\frac{2}{c^{\frac{3}{4}}}+c^{\frac{1}{4}}} < 0$ for $c \gg 1$. Thus, $\sqrt[5]{c} < M < \sqrt[4]{c}$ for large c.

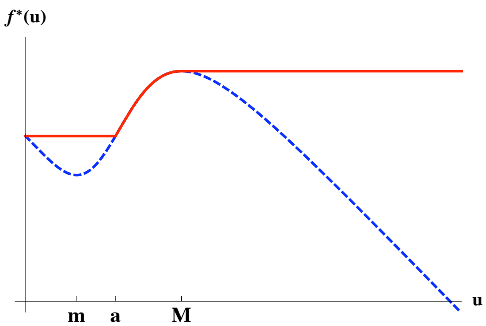

Figure 4.5

Graph of $f^*(u)$

(ii) We have $f(a) = K$, simplifying which we get $a^4 - ca^3 + 1 = 0$. Define $j(u) := u^4 - cu^3 + 1$. If $u < a$, then $j(u) > 0$, and if $u \in (a, M)$, then $j(u) < 0$. We have $j(\frac{1}{\sqrt[3]{c}}) = (\frac{1}{\sqrt[3]{c}})^4 - c(\frac{1}{\sqrt[3]{c}})^3 + 1 = \frac{1}{c^{\frac{4}{3}}} > 0$, and $j(\frac{1}{\sqrt[4]{c}}) = (\frac{1}{\sqrt[4]{c}})^4 - c(\frac{1}{\sqrt[4]{c}})^3 + 1 = \frac{1}{c} - \sqrt[4]{c} + 1 < 0$ for $c \gg 1$. Hence, $\frac{1}{\sqrt[3]{c}} < a < \frac{1}{\sqrt[4]{c}}$ for large c. Note that since $b = \beta \to \sqrt[4]{3}$ and $a < \frac{1}{\sqrt[4]{c}} \to 0$ as $c \to \infty$, $a \in (0, b)$ for $c \gg 1$.

Now we will discuss the existence of at least three positive solutions for a certain range of λ (see Theorem 2). Our aim is to prove that for $c \gg 1$ and $K_0 < K < \frac{9c}{16}$, $Q_2(a, b, \Omega) < 1$. It is enough if we prove that

$$\frac{b}{f(b)} \left(\frac{N+1}{N}\right)^{N+1} \frac{N^2}{R^2} < min\left\{\frac{a}{||e_\Omega||_\infty f^*(a)}, \frac{2NM}{f(b)R^2}\right\}. \tag{4.17}$$

Note that for $c \gg 1$ $\frac{b}{f(b)} = \frac{\sqrt[4]{3}}{K - \sqrt[4]{3} + \frac{3c}{4}}$ and $\frac{M}{f(b)} = \frac{M}{K - \sqrt[4]{3} + \frac{3c}{4}} > \frac{\sqrt[5]{c}}{K - \sqrt[4]{3} + \frac{3c}{4}}$. Also, $\frac{a}{f^*(a)} = \frac{a}{K} > \frac{1}{K\sqrt[3]{c}}$. Applying the estimates we obtained for $\frac{a}{f^*(a)}$, $\frac{b}{f(b)}$, and $\frac{M}{f(b)}$ to the above inequality, we get the following:

$$\left(\frac{\sqrt[4]{3}}{k - \sqrt[4]{3} + \frac{3c}{4}}\right) \left(\frac{N+1}{N}\right)^{N+1} \frac{N^2}{R^2} < min\left\{\frac{1}{||e_\Omega||_\infty k\sqrt[3]{c}}, \frac{2N}{R^2}\left(\frac{\sqrt[5]{c}}{k - \sqrt[4]{3} + \frac{3c}{4}}\right)\right\} \tag{4.18}$$

Simplifying the above, we can see that $Q_2(a, b, \Omega) < 1$ if

$$\left(\frac{\sqrt[4]{3}}{\frac{k}{c} - \frac{\sqrt[4]{3}}{c} + \frac{3}{4}}\right)\left(\frac{N+1}{N}\right)^{N+1}\frac{N^2}{R^2} < min\left\{\frac{c^{\frac{2}{3}}}{||e_\Omega||_\infty k}, \frac{2N}{R^2}\left(\frac{c^{\frac{1}{5}}}{\frac{k}{c} - \frac{\sqrt[4]{3}}{c} + \frac{3}{4}}\right)\right\}. \quad (4.19)$$

Clearly, this inequality is true for $c \gg 1$; hence, Theorem 2 holds.

4.3 Analytical and computational results for $N = 1$

Consider the two point boundary value problem

$$\begin{cases} -u'' = \lambda f(u), \quad x \in (0, 1) \\ u(0) = 0 = u(1), \end{cases} \quad (4.20)$$

where f satisfies the following hypotheses:

(G1) $f \in C^2([0, \infty)),\ f(u) > 0$ for $0 < u < r_0$ and $f(u) < 0$ for $u > r_0$ for some $r_0 > 0$.

(G2) there exists $k \geq 0$ such that $f(u) - f(v) \geq -k(u - v)$ for all $u, v \in [0, r_0)$ with $u > v$;

Using the quadrature method (see [27]), it follows that (4.20) has a positive solution iff

$$\sqrt{\lambda} = \sqrt{2}\int_0^\rho \frac{dz}{\sqrt{[F(\rho) - F(z)]}} := G(\rho). \quad (4.21)$$

where $F(s) := \int_0^s f(t)dt$ and $\rho = u(\frac{1}{2}) = ||u||_\infty$. Further, $u(x)$ is symmetric about $x = \frac{1}{2}$ and is given by

$$\int_0^{u(x)} \frac{dz}{\sqrt{[F(\rho) - F(z)]}} = \sqrt{2\lambda}x, \qquad 0 < x < \frac{1}{2}. \quad (4.22)$$

(4.21) describes the bifurcation curve of positive solutions of (4.20) and it follows by results in [27] that $\lim_{\rho \to 0^+} G(\rho) = 0$ and $\lim_{\rho \to r_0^-} G(\rho) = \infty$. Furthermore, from [6] we have

$$G'(\rho) = \sqrt{2}\int_0^1 \frac{H(\rho) - H(\rho s)}{[F(\rho) - F(\rho s)]^{\frac{3}{2}}}ds \quad (4.23)$$

where $H(u) = F(u) - \frac{u}{2}f(u)$. Note that $H(0) = 0$ and $H'(0) = \frac{1}{2}f(0) > 0$. Hence, if there exists a point $\rho_0 \in (0, r_0)$ such that $H(\rho_0) < 0$, then $G'(\rho) < 0$ for certain range of ρ; thus the bifurcation diagram must be at least S-shaped. We will now prove that such a ρ_0 exists when $K > K_0$ and $c > 5.20626K$.

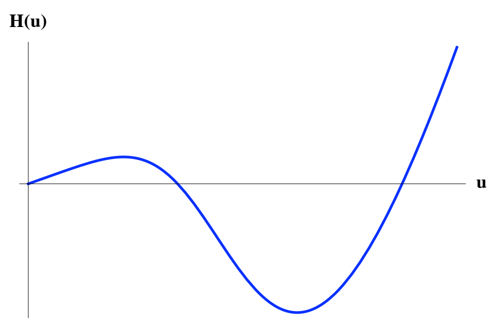

Figure 4.6

Graph of $H(u)$

Consider the case $f(u) = K - u + c\frac{u^4}{1+u^4}$. Clearly, given $K > K_0$, then for $c \gg 1$ f satisfies $(G1) - (G2)$ (see Proposition 12). Hence, $G(\rho)$ is defined for all $\rho \in S = (0, r_0)$.

We have $H(u) = F(u) - \frac{u}{2}f(u) = \frac{Ku}{2} + cu - c\frac{u^5}{2(1+u^4)} - \frac{c}{4\sqrt{2}}\{-2\tan^{-1}(1-\sqrt{2}u) + 2\tan^{-1}(1+\sqrt{2}u) - \ln\frac{1+\sqrt{2}u+u^2}{1-\sqrt{2}u+u^2}\}$. Clearly, $\sqrt[4]{\frac{3}{5}} < r_0$ (see Proposition 12); choose $\rho_0 = \sqrt[4]{\frac{3}{5}}$. Thus, $H(\rho_0) = .440056K - .0845244c$ and hence $H(\rho_0) < 0$ if $c > 5.20626K$.

We finally used Mathematica to compute $\sqrt{\lambda} = G(\rho)$ in the case when $f(u) = K - u + c\frac{u^4}{1+u^4}$ and plotted the bifurcation diagrams. We found that the bifurcation diagrams are, in fact, exactly S-shaped when multiplicity occurred. Figures Figure 4.7 and Figure 4.8 describe the bifurcation diagrams for a certain value of c and K.

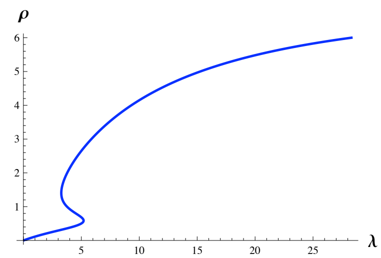

Figure 4.7

Bifurcation Diagram with $K = 1$ and $c = 6$.

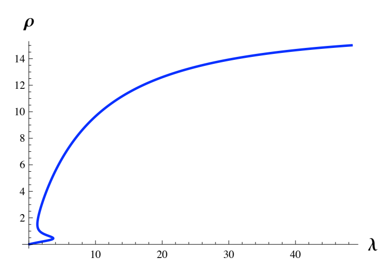

Figure 4.8

Bifurcation Diagram with $K = 1$ and $c = 15$.

CHAPTER 5

PROOF OF THEOREM 3

We provide the proof for Theorem 3 in this chapter.

5.1 Proof of Theorem 3

We start with the construction of a positive subsolution for (1.8). To get a positive subsolution, we can apply an anti-maximum principle by Clement and Peletier [12], from which we know that there exist a $\delta = \delta(\Omega) > 0$ and a solution z_λ (with $z_\lambda > 0$ in Ω and $\frac{\partial z_\lambda}{\partial \nu} < 0$ on $\partial\Omega$, where ν is the outer unit normal to Ω) of

$$\begin{cases} -\Delta z - \lambda z = -1, & x \in \Omega \\ z = 0, & x \in \partial\Omega, \end{cases} \tag{5.1}$$

for $\lambda \in (\lambda_1, \lambda_1 + \delta)$. Fix $\lambda^* \in (\lambda_1, min\{a, \lambda_1 + \delta\})$. Let z^*_λ be the solution of (5.1) when $\lambda = \lambda^*$ and $\alpha = ||z^*_\lambda||_\infty$.

Define $\psi = \mu K z^*_\lambda$ where $\mu \geq 1$ is to be determined later. We will choose μ and $K > 0$ properly so that ψ is a subsolution. We know $-\Delta\psi = -\Delta(\mu K z^*_\lambda) = \lambda^*\psi - \mu K$. Hence ψ is a subsolution if $\lambda^*\psi - \mu K \leq a\psi - b\psi^2 - c\frac{\psi^p}{1+\psi^p} - K$. That is if

$$(a - \lambda^*)\psi - b\psi^2 - c\frac{\psi^p}{1+\psi^p} + (\mu - 1)K \geq 0. \tag{5.2}$$

Consider

$$r(t) = (a - \lambda^*)t - bt^2 - c\frac{t^p}{1+t^p} + (\mu - 1)K. \tag{5.3}$$

It can be written as $r(t) = r_1(t) + r_2(t)$ where

$$
\begin{aligned}
r_1(t) &= (a - \lambda^*)t - bt^2 - ct^p + (\mu - 1)K \text{ and} \\
r_2(t) &= \frac{t^{2p}}{1 + t^p}.
\end{aligned}
\tag{5.4}
$$

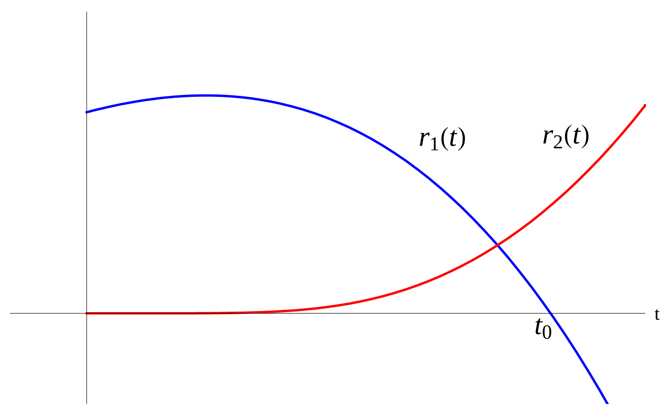

Figure 5.1

Graphs of $r_1(t)$ and $r_2(t)$

Clearly $r_2(t) \geq 0$ for all $t \geq 0$. So if we can find K and μ such that $r_1(t) \geq 0$ for $0 \leq t \leq \mu K \alpha$ then ψ will be a subsolution. Now $r_1(0) = (\mu - 1)K$, $r_1''(t) = -2b - cp(p-1)t^{p-2} < 0$ and there exists a unique t_0 such that $r_1(t_0) = 0$. This means that ψ is a subsolution if $r_1(\mu K \alpha) \geq 0$, i.e. if

$$
(a - \lambda^*)\mu K \alpha - b(\mu K \alpha)^2 - c(\mu K \alpha)^p + (\mu - 1)K \geq 0. \tag{5.5}
$$

Let

$$
G(K) = (a - \lambda^*)\mu\alpha - b(\mu\alpha)^2 K - c(\mu\alpha)^p K^{p-1} + (\mu - 1). \tag{5.6}
$$

Then $G(0) = (a-\lambda^*)\mu\alpha + (\mu - 1) > 0$ since $\mu \geq 1$ and $a > \lambda^*$. Also, we have $G'(K) = -b(\mu\alpha)^2 - c(p-1)(\mu\alpha)^p K^{p-2} < 0$. Hence given μ and p there exists a unique $K^* = K^*(a, b, c, \mu, p) > 0$ with $G(K^*) = 0$.

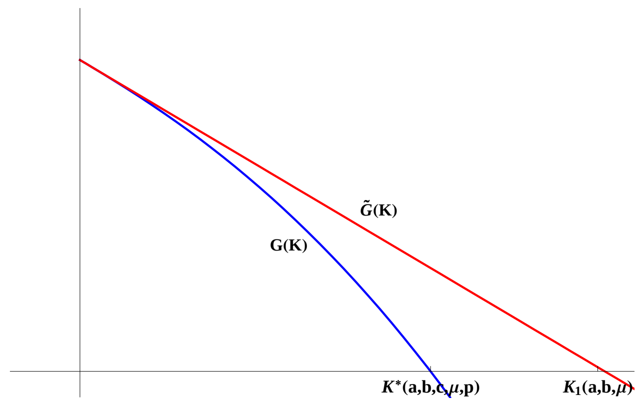

Figure 5.2

Graph of $G(K)$

Since $G(K) \leq (a-\lambda^*)\mu\alpha - b(\mu\alpha)^2 K + (\mu - 1) = \tilde{G}(K)$ we see that

$$K^* \leq \frac{(a-\lambda^*)\mu\alpha + (\mu-1)}{b\mu^2\alpha^2} := K_1(a, b, \mu). \tag{5.7}$$

Note that $K_1(a, b, \mu)$ is bounded for $\mu \in [1, \infty)$. Hence K^* is bounded for $\mu \in [1, \infty)$. Let $K_0(a, b, c, p) = \sup_{\mu \geq 1} K^*(a, b, c, \mu, p)$. Now let $\tilde{K} < K_0(a, b, c, p)$. By definition there will exist a $\tilde{\mu} \geq 1$ such that $\tilde{K} < K^*(a, b, c, \tilde{\mu}, p) < K_0(a, b, c, p)$. Choose $\psi = \tilde{\mu}\tilde{K}z$. With $\mu = \tilde{\mu}$ we have $G(\tilde{K}) \geq 0$ and hence

$$(a-\lambda^*)\tilde{\mu}\tilde{K}\alpha - b(\tilde{\mu}\tilde{K}\alpha)^2 - c(\tilde{\mu}\tilde{K}\alpha)^p + (\tilde{\mu} - 1)\tilde{K} \geq 0. \tag{5.8}$$

Thus ψ is a subsolution to (1.8).

Now for a supersolution choose $\phi = Me$, where

$$\begin{cases} -\Delta e = 1, & x \in \Omega \\ \\ e = 0, & x \in \partial\Omega, \end{cases} \tag{5.9}$$

and $M > 0$ is such that $f(u) = au - bu^2 - c\dfrac{u^p}{1+u^p} - K \leq M$ for all $u \geq 0$. Clearly $-\Delta\phi = M \geq f(\phi)$ and ϕ is a supersolution.

Since by the Hopf maximum principle $\dfrac{\partial e}{\partial \nu} < 0$ on $\partial\Omega$ (where ν is the outer unit normal to Ω), we can choose $M >> 1$ so that $\phi = Me \geq \psi$. Hence by Lemma 4 the problem has a positive solution for all $K < K_0(a, b, c, p)$.

5.2 Corollary

Corollary 1

Let $p = 2$ then the boundary value problem (1.8) has a solution for all $K < \tilde{K}_0(a, b, c)$, where $\tilde{K}_0(a, b, c) = \sup_{\mu \geq 1} \dfrac{(a - \lambda^*)\mu\alpha + \mu - 1}{(b+c)(\mu\alpha)^2}$

In this case

$$G(K) = (a - \lambda^*)\mu\alpha - b(\mu\alpha)^2 K - c(\mu\alpha)^2 K + (\mu - 1). \tag{5.10}$$

and $K^* = K^*(a, b, c, \mu) = \dfrac{(a - \lambda^*)\mu\alpha + \mu - 1}{(b+c)(\mu\alpha)^2}$. Hence (1.8) has a solution for all $K < \tilde{K}_0(a, b, c) = \sup_{\mu \geq 1} \dfrac{(a - \lambda^*)\mu\alpha + \mu - 1}{(b+c)(\mu\alpha)^2}$.

CHAPTER 6

PROOF OF THEOREMS 4 AND 5

We establish our proof of Theorem 4 in Section 6.1 using the quadrature method discussed in [6] and [27]. In Section 6.2 we prove Theorem 5 and in Section 6.3 we provide various bifurcation diagrams for positive solutions via mathematica computations.

6.1 Proof of Theorem 4

Let $c < 2$, $\frac{2c}{K^2+1} + \frac{2}{K^3} < \epsilon < \frac{1}{4K}$ and $K \gg 1$. We will first show that $\tilde{f}(u)$ and $\tilde{F}(u) = \int_0^s \tilde{f}(t)dt$ have the shapes as given in the figure below.

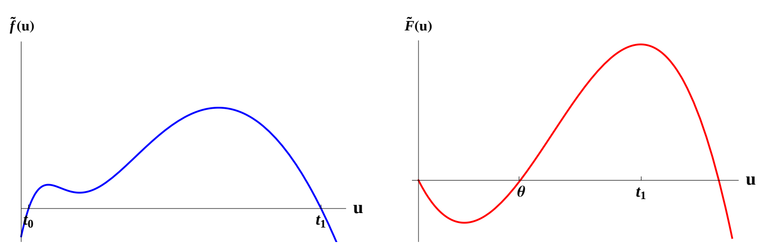

Figure 6.1

Graph of $\tilde{f}(u)$ and $\tilde{F}(u)$

Next multiplying (1.11) by $u'(x)$ and integrating we obtain

$$-\frac{[u'(x)]^2}{2} = \lambda \tilde{F}(u(x)) + C \tag{6.1}$$

Since we are dealing with positive solutions, $u(x)$ has to be symmetric with respect to $x = \frac{1}{2}$ and $u'(x) > 0$ for $x \in (0, \frac{1}{2})$. In fact, if $\rho = ||u||_\infty$ then $u(\frac{1}{2}) = \rho,\ \theta \leq \rho \leq t_1$, and substituting $x = \frac{1}{2}$ in (6.1) we obtain

$$u'(x) = \sqrt{2\lambda[\tilde{F}(\rho) - \tilde{F}(u)]};\ x \in [0, \frac{1}{2}]. \tag{6.2}$$

Integrating (6.2) from 0 to x we get

$$\int_0^{u(x)} \frac{dz}{\sqrt{[\tilde{F}(\rho) - \tilde{F}(z)]}} = \sqrt{2\lambda}x; \qquad x \in [0, \frac{1}{2}], \tag{6.3}$$

and substituting $x = \frac{1}{2}$ in the above equation we have

$$\sqrt{\lambda} = \sqrt{2}\int_0^{\rho} \frac{dz}{\sqrt{[\tilde{F}(\rho) - \tilde{F}(z)]}} := G(\rho). \tag{6.4}$$

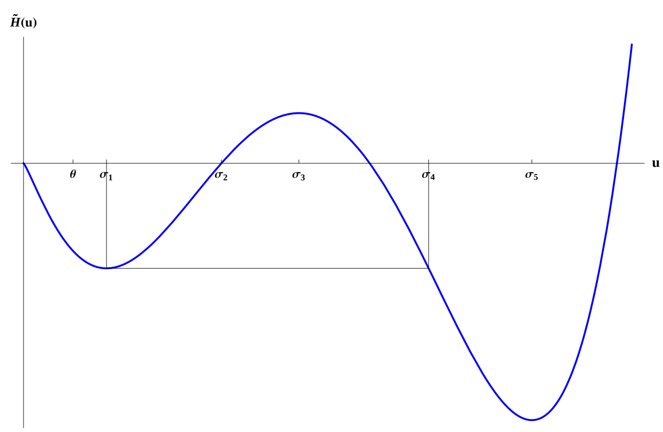

Figure 6.2

Graph of $\tilde{H}(u)$

For given λ such that $G(\rho) = \sqrt{\lambda}$ for some $\rho \in (\theta, t_1)$, it follows that (1.11) has a positive solution $u(x)$ given by (6.3) with $||u||_\infty = u(\frac{1}{2}) = \rho$. Further it follows that $G(\rho)$ is a continuous function and differentiable for $\rho \in (\theta, t_1)$ with

$$G'(\rho) = \sqrt{2}\int_0^1 \frac{\tilde{H}(\rho) - \tilde{H}(\rho s)}{[\tilde{F}(\rho) - \tilde{F}(\rho s)]^{\frac{3}{2}}} ds \tag{6.5}$$

where $\tilde{H}(u) = \tilde{F}(u) - \frac{u}{2}\tilde{f}(u)$.

Next we will show that $\tilde{H}(u)$ has the shape given in Figure 6.2. In particular we will show that $\theta < \sigma_1$. From the figure we see that $\tilde{H}(\theta) - \tilde{H}(\theta s) < 0$ for $s \in (0,1)$ and hence $G'(\theta) < 0$. In fact we can see that $\tilde{H}(\rho) - \tilde{H}(\rho s) < 0$ for $s \in (0,1)$ when $\rho \in [\theta, \sigma_1)$, $\tilde{H}(\rho) - \tilde{H}(\rho s) > 0$ for $s \in (0,1)$ when $\rho \in (\sigma_2, \sigma_3]$ and $\tilde{H}(\rho) - \tilde{H}(\rho s) < 0$ for $s \in (0,1)$ when $\rho \in (\sigma_4, \sigma_5]$. We can also see that $\tilde{H}(\rho) - \tilde{H}(\rho s) > 0$ when ρ large for $s \in (0,1)$. Consequently $G'(\rho) < 0$ for $\rho \in [\theta, \sigma_1)$, $G'(\rho) > 0$ for $\rho \in (\sigma_2, \sigma_3]$, $G'(\rho) < 0$ for $\rho \in (\sigma_4, \sigma_5]$ and $G'(\rho) > 0$ for ρ large which results in a Σ-shaped bifurcation diagram.

To obtain the required shape of $\tilde{f}$ we let $f(u) = u - \dfrac{u^2}{K} - c\dfrac{u^2}{1+u^2}$. Then $f'(u) = 1 - \frac{2u}{K} - \frac{2cu}{(1+u^2)^2} = 0$ when $1 - \frac{2u}{K} = \frac{2cu}{(1+u^2)^2}$. Let $g(u) := \frac{2cu}{(1+u^2)^2}$. Here $1 - \frac{2u}{K}$ is a line passing through $(0,1)$ and with slope $-\frac{2}{K}$. Note that $g(u) \geq 0$, $g(0) = 0$ and $\lim_{u\to\infty} g(u) = 0$.

Since $g'(u) = 2c\frac{1-3u^2}{(1+u^2)^3}$, we can see that $g(u)$ achieves a maximum of $\frac{3\sqrt{3}c}{8}$ at $u = \frac{1}{\sqrt{3}}$.

Now if $\frac{8}{3\sqrt{3}} \leq c < 2$ and K is large, then there are exactly three positive points $m_1 < m_2 < m_3$ such that $f'(m_i) = 0$ for $i = 1, 2, 3$. Clearly the function $f(u)$ has a relative minimum at $u = m_2$. We will prove that $f(m_2) > 0$ for $K >> 1$. This implies that $f(u)$ has a unique positive zero. It is clear from Figure 6.3 that there exists a constant

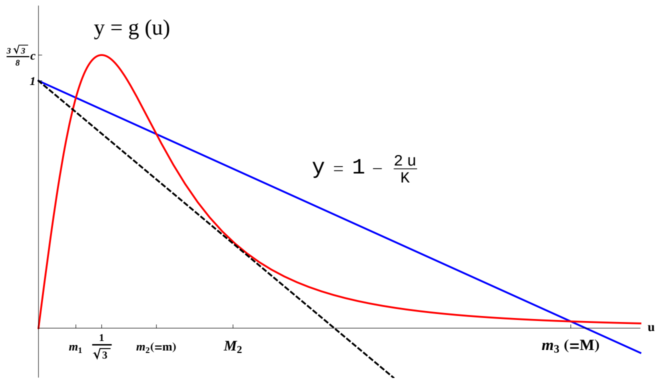

Figure 6.3

$\frac{8}{3\sqrt{3}} \leq c < 2$

M_2 such that $m_2 < M_2$ for all K. Moreover $m_2 = m_2(K)$ is a continuous decreasing function of K such that $m_2(K) \in (\frac{1}{\sqrt{3}}, M_2]$. Also $\lim\limits_{K\to\infty} f(m_2) = z - c\dfrac{z^2}{1+z^2} := h(z,c)$ for some $z \in (\frac{1}{\sqrt{3}}, M_2)$.

But $h(z,c) > 0$ for $z \geq \frac{1}{\sqrt{3}}$ and $c < 2$. Hence $\lim\limits_{K\to\infty} f(m_2) \geq \min\limits_{z \geq \frac{1}{\sqrt{3}}} f(z) > 0$. Thus for $K \gg 1$ we have $\epsilon < \frac{1}{4K} < \min\limits_{z\geq\frac{1}{\sqrt{3}}} f(z)$ and hence the function $\tilde{f}(u) = f(u) - \epsilon$ will have the shape given in Figure 6.4.

Next if $c < \frac{8}{3\sqrt{3}}$, then $g(\frac{1}{\sqrt{3}}) < 1$ and for $K \gg 1$ clearly there exists a unique $M > 0$ such that $f'(M) = 0$ (see Figure 6.5). Consequently, for K large $f(u)$ has a unique positive zero. Also for $K \gg 1$ we have $\tilde{f}(\frac{K}{2}) = \frac{K}{4} - c\frac{K^2}{4+K^2} - \epsilon > 0$ and hence the function $\tilde{f}(u)$ will have the shape given in Figure 6.6.

To obtain the required shape for $\tilde{F}$ we only need to show that $\tilde{F}(u) = \frac{u^2}{2} - \frac{u^3}{3K} - c(u - \tan^{-1} u) - \epsilon u > 0$ for some $u > 0$. But $\tilde{F}(K) = \frac{K^2}{6} - (c+\epsilon)K + c\tan^{-1} K > 0$ for $K \gg 1$. Hence from the above discussion we see that there exists constants θ, t_1 with $\tilde{F}(s) > 0$ in (θ, t_1) and $\tilde{F}(\theta) = 0$.

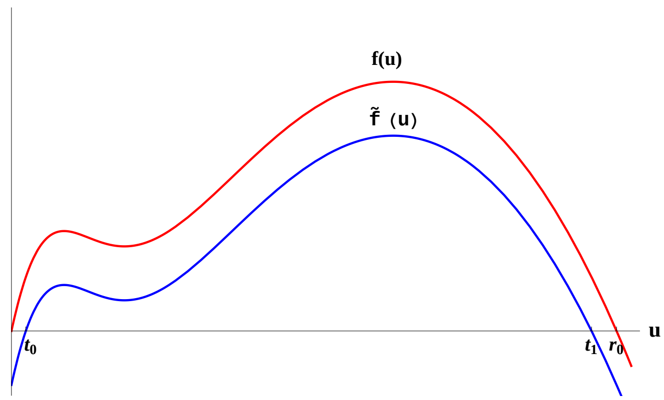

Figure 6.4

Graph of $\tilde{f}(u)$ for $\frac{8}{3\sqrt{3}} \leq c < 2$ and $K \gg 1$

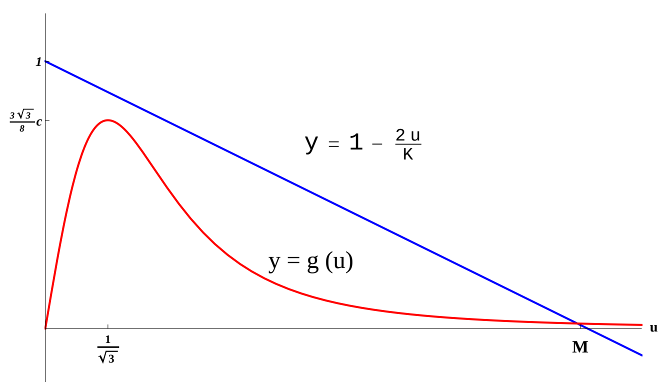

Figure 6.5

The case when $c < \frac{8}{3\sqrt{3}}$

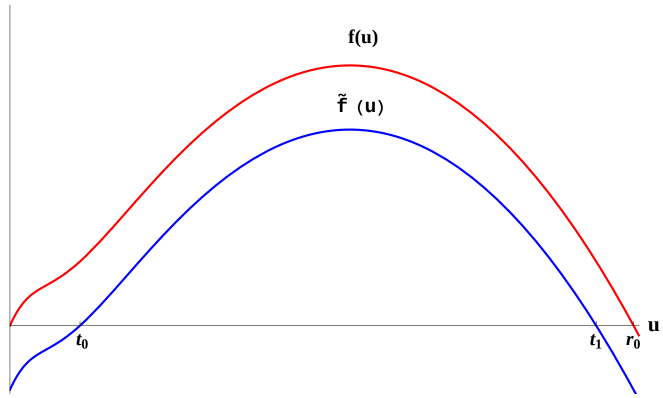

Figure 6.6

Graph of $\tilde{f}(u)$ for $c < \frac{8}{3\sqrt{3}}$ and $K \gg 1$

Now we will establish that $\tilde{H}(u)$ has the shape given in Figure 6.2. In [28] (see also [22] for related results) we prove that given $\epsilon = 0$ and $c < 2$ fixed then for $K \gg 1$ the boundary value problem (1.11) has at least three solutions for a certain range of λ. In fact we prove that if $c < 2$ then there exists $K_1(c)$ such that whenever $K > K_1(c)$, the function $H(u) = F(u) - \frac{u}{2}f(u) = \frac{u^3}{6K} + c\left(\frac{u^3}{2(1+u^2)} - u + \tan^{-1} u\right)$ has the shape given in Figure 6.7, where $F(s) = \int_0^s f(t)dt$. But $\tilde{H}(u) = H(u) - \frac{\epsilon u}{2}$ and $\tilde{H}'(u) = H'(u) - \frac{\epsilon}{2} = \frac{1}{2}(f(u) - uf'(u)) - \frac{\epsilon}{2}$. In order to prove that $\tilde{H}(u)$ has the shape given in Figure 6.2 we first observe that $\tilde{H}'(0) = 0 - \frac{\epsilon}{2} < 0$. Now we show that $\tilde{H}'(u) = 0$ has three positive roots.

The zeros of $H'(u)$ are given by $f(u) - uf'(u) = 0$ and in particular the non-zero critical points are given by $\frac{1}{K} - \frac{c}{1+u^2} + \frac{2c}{(1+u^2)^2} = 0$. Solving for u we get the positive critical points as $\alpha_1 = \sqrt{\frac{cK-2-\sqrt{cK(cK-8)}}{2}}$ and $\alpha_2 = \sqrt{\frac{cK-2+\sqrt{cK(cK-8)}}{2}}$ (see Figure 6.7). $\tilde{H}'(u) = 0$ will have three positive roots if $\max\limits_{x\in(0,\alpha_1)} H'(u) > \dfrac{\epsilon}{2}$.

The nonzero roots of $H''(u) = -\frac{1}{2}uf''(u) = 0$ are given by $f''(u) = -\frac{2}{K} - 2c\frac{1-3u^2}{(1+u^2)^3} = 0$. Clearly for large K the first positive root of $H''(u)$ is approximately equal to $\frac{1}{\sqrt{3}}$. Hence $\max_{x\in(0,\alpha_1)} H'(u) \geq H'(\frac{1}{\sqrt{3}}) = \frac{1}{6K} + \frac{3c}{8} > \frac{\epsilon}{2}$ and

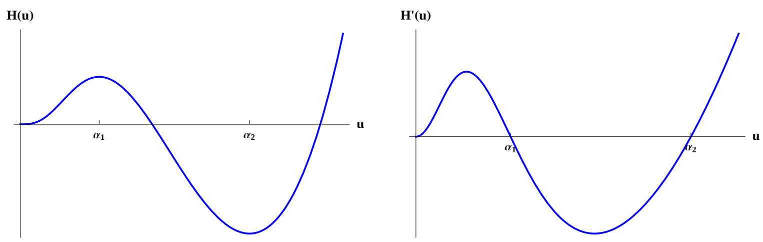

Figure 6.7

Graphs of $H(u)$ and $H'(u)$

there exist three positive numbers σ_1, σ_3 and σ_5 such that $0 < \sigma_1 < \sigma_3 < \alpha_1 < \alpha_2 < \sigma_5$ and $\tilde{H}'(\sigma_i) = 0$ for $i = 1, 3, 5$. Note that $\sigma_5 > \alpha_2 > 1$. Now if we show that $\tilde{H}(u) > 0$ for some $u < \sigma_5$ then $\tilde{H}(u)$ will have the shape given in Figure 6.2. We have $\tilde{H}(1) = \frac{1}{6K} + \frac{c(\pi-3)}{4} - \frac{\epsilon}{2} > 0$ since $\epsilon < \frac{1}{4K}$. Thus we have shown that $\tilde{H}(u)$ has the shape given in Figure 6.2.

In order to prove that $\theta < \sigma_1$, it is enough if we show that $\theta < \sigma_3$ and $\tilde{H}'(\theta) < 0$. We have seen that for large K the first positive root of $H''(u)$ is approximately equal to $\frac{1}{\sqrt{3}}$ and thus it is clear that σ_3 is bounded below (see Figure 6.8). Hence for $K \gg 1$, $\dfrac{1}{K} < \sigma_3$. We will prove that $\theta < \dfrac{1}{K}$ and $\tilde{H}'(\frac{1}{K}) < 0$.

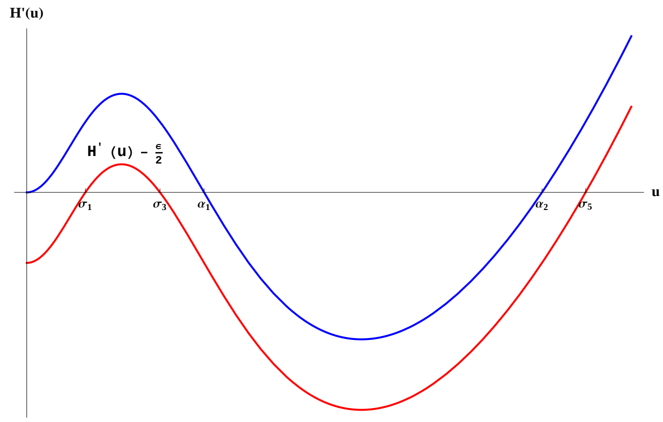

Figure 6.8

Graph of $H'(u)$

$$\begin{aligned}
\tilde{F}(\frac{1}{K}) &= \frac{1}{2K^2} - \frac{1}{3K^4} - c(\frac{1}{K} - \tan^{-1}\frac{1}{K}) - \frac{\epsilon}{K} && (6.6)\\
&> \frac{1}{4K^2} - \frac{1}{3K^4} - c(\frac{1}{K} - \tan^{-1}\frac{1}{K}) \text{ (since } \epsilon < \tfrac{1}{4K}). && (6.7)
\end{aligned}$$

Consider the function $\dfrac{c(v - \tan^{-1} v) + \frac{v^4}{3}}{\frac{v^2}{4}}$. We have

$$\begin{aligned}
\lim_{v\to 0}\frac{c(v - \tan^{-1} v) + \frac{v^4}{3}}{\frac{v^2}{4}} &= \lim_{v\to 0}\frac{c(1 - \frac{1}{1+v^2}) + \frac{4}{3}v^3}{\frac{v}{2}} && (6.8)\\
&= \lim_{v\to 0}\frac{2(3cv^2 + 4(1+v^2)v^3)}{3v(1+v^2)} && (6.9)\\
&= 0 && (6.10)
\end{aligned}$$

Hence $\tilde{F}(\frac{1}{K}) > 0$ and $\theta < \dfrac{1}{K}$ for $K \gg 1$. Next we have

$$\begin{aligned}
\tilde{H}'(\frac{1}{K}) &= \frac{1}{K^3} + c\frac{\frac{1}{K^2}}{1+\frac{1}{K^2}}(\frac{2}{1+\frac{1}{K^2}} - 1) - \frac{\epsilon}{2} && (6.11)\\
&< \frac{1}{K^3} + c\frac{1}{K^2+1} - \frac{\epsilon}{2} && (6.12)\\
&< 0 \quad \text{since} \quad \epsilon > \frac{2c}{K^2+1} + \frac{2}{K^3}. && (6.13)
\end{aligned}$$

Hence we have $\theta < \frac{1}{K} < \sigma_1$ and the graph of $\tilde{H}(u)$ is as given in Figure 6.2. Thus it follows that for $c < 2$ there exists a constant $K(c)$ such that for $K > K(c)$ and $\frac{2c}{K^2+1} + \frac{2}{K^3} < \epsilon < \frac{1}{4K}$ the boundary value problem (1.11) has at least a Σ-shaped bifurcation curve.

6.2 Proof of Theorem 5

Let $c = \frac{1}{2}$. By Theorem 4 there exists a constant $K(\frac{1}{2})$ such that for $K > K(\frac{1}{2})$ and $\frac{1}{K^2+1} + \frac{2}{K^3} < \epsilon < \frac{1}{4K}$, $\tilde{H}(u)$ is as in Figure 6.2 and the boundary value problem (1.11) has a Σ-shaped bifurcation curve. If in addition $G(\theta) > G(\sigma_5)$ we have the desired four solution result in a certain range of λ (see Figure 6.9).

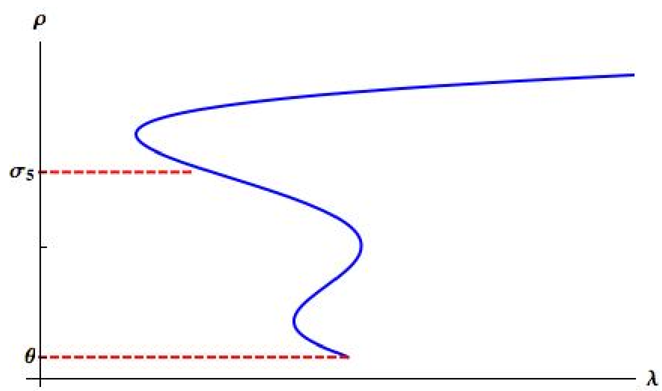

Figure 6.9

$G(\theta) > G(\sigma_5)$

Note that

$$G(\sigma_5) = \frac{\sqrt{2}\sigma_5}{\sqrt{\tilde{F}(\sigma_5)}} \int_0^1 \frac{dv}{\sqrt{1 - \frac{\tilde{F}(\sigma_5 v)}{\tilde{F}(\sigma_5)}}}. \tag{6.14}$$

Let $L(v) := \frac{\tilde{F}(\sigma_5 v)}{\tilde{F}(\sigma_5)}$. Then $L(0) = 0$, $L(1) = 1$ and $L'(v) = \frac{\tilde{f}(\sigma_5 v)\sigma_5}{\tilde{F}(\sigma_5)}$. We have $L'(0) < 0$ and $L'(v)$ has only one zero in $[0, 1]$. Hence we get $L(v) \leq v$ for $v \in [0, 1]$. Consequently from (6.14)

$$G(\sigma_5) \leq \frac{\sqrt{2}\sigma_5}{\sqrt{\tilde{F}(\sigma_5)}} \int_0^1 \frac{dv}{\sqrt{1-v}} = 2\sqrt{2}\frac{\sigma_5}{\sqrt{\tilde{F}(\sigma_5)}} \tag{6.15}$$

But $\left(\frac{u}{\sqrt{\tilde{F}(u)}}\right)' = \frac{\sqrt{\tilde{F}(u)} - u\frac{\tilde{f}(u)}{2\sqrt{\tilde{F}(u)}}}{\tilde{F}(u)} = \frac{\tilde{H}(u)}{\tilde{F}(u)^{\frac{3}{2}}} < 0$ in $[\alpha_2, \sigma_5]$ since $\tilde{H}(\alpha_2) < 0$ and $\sigma_3 < \alpha_2 < \sigma_5$. So from (6.15) we have $G(\sigma_5) < 2\sqrt{2}\frac{\alpha_2}{\sqrt{\tilde{F}(\alpha_2)}} \to 4$ as $K \to \infty$. Hence $G(\sigma_5) \leq 4$ for K large. Now consider $G(\theta)$. We have

$$G(\theta) = \sqrt{2}\int_0^\theta \frac{dv}{\sqrt{\tilde{F}(\theta) - \tilde{F}(v)}} \tag{6.16}$$

$$= \sqrt{2}\int_0^\theta \frac{dv}{\sqrt{-\tilde{F}(v)}dv} \tag{6.17}$$

$$\geq \sqrt{2}\int_0^\theta \frac{dv}{\sqrt{-\frac{v^2}{2} + \frac{1}{2}(v - \tan^{-1}(v)) + \epsilon v}} \tag{6.18}$$

Let $h(v) = v^2 - (v - \tan^{-1} v)$. Then $h(0) = 0$ and $h'(v) = 2v - \frac{v^2}{1+v^2} > 0$ for $v > 0$. Hence $v^2 > v - \tan^{-1} v$ for $v > 0$. So we have

$$G(\theta) > \sqrt{2}\int_0^\theta \frac{dv}{\sqrt{\epsilon v}} \tag{6.19}$$

$$= 2\sqrt{2}\sqrt{\frac{\theta}{\epsilon}}. \tag{6.20}$$

(6.21)

It is easy to see that $\tilde{F}(2\epsilon) = \frac{4\epsilon^2}{2} - \frac{8\epsilon^3}{3K} - \frac{1}{2}(2\epsilon - \tan^{-1}(2\epsilon)) - 2\epsilon^2 < 0$. Consequently $\theta > 2\epsilon$ and $G(\theta) > 4$. Thus we have proved that $G(\sigma_5) < G(\theta)$ and the boundary value problem (1.11) has at least four positive solutions for a certain range of λ.

6.3 Bifurcation diagrams via Mathematica computations

Here we provide some exact bifurcation diagrams in the case when $f(u) = u - \frac{u^2}{K} - c\frac{u^2}{1+u^2} - \epsilon$ using Mathematica computations of $\sqrt{\lambda} = G(\rho)$. In Section 2 we had proved that under certain conditions the bifurcation diagrams will be at least Σ-shaped. But these calculations show that the bifurcation diagrams are exactly Σ-shaped. In Figure 6.10 we provide the bifurcation diagram in the case when $c = \frac{1}{2}$, $\epsilon = \frac{1}{5K}$ and $K = 30$. Our bifurcation diagrams indicate not only that there is a range of λ where the problem has four solutions but also that it is exactly Σ-shaped.

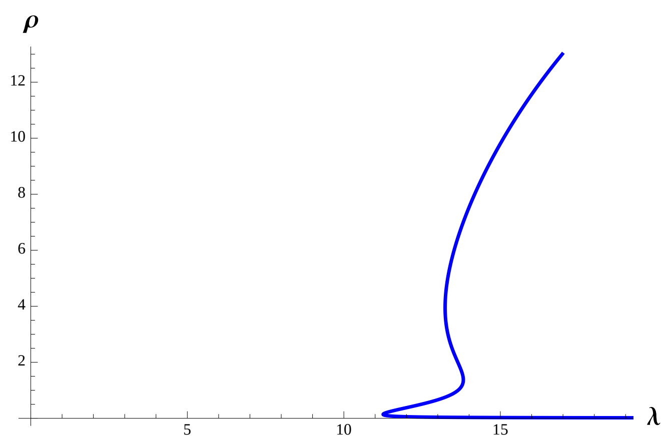

Figure 6.10

Bifurcation Diagram with $c = \frac{1}{2}$, $\epsilon = \frac{1}{5K}$ and $K = 30$

In Figure 6.11 and Figure 6.12 we provide the bifurcation diagrams for $c = 0.1$ and $c = 1.9$ respectively. Based on these we conjecture that the problem has at least four solutions for any $c \in (0, 2)$ for some ϵ and K and they are also exactly Σ-shaped.

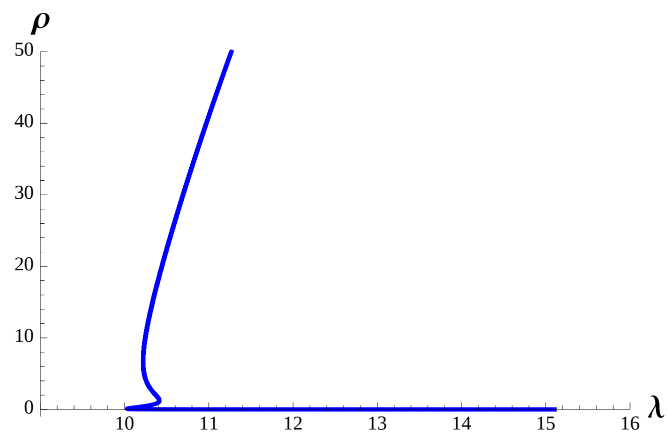

Figure 6.11

Bifurcation Diagram with $c = .1$, $\epsilon = \dfrac{1}{5K}$ and $K = 350$

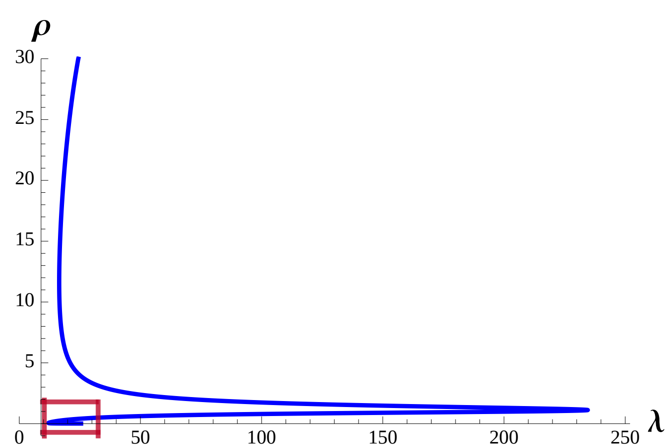

Figure 6.12

Bifurcation Diagram with $c = 1.9$, $\epsilon = \dfrac{1}{5K}$ and $K = 50$

CHAPTER 7

CONCLUSIONS AND FUTURE DIRECTIONS

7.1 Conclusions

We studied two classes of reaction-diffusion models and our results on alternate stable states are applicable to two models considered in the recent literature on ecosystems. It was shown that for certain parameter values the systems can have at least three solutions. We also introduced a constant yield harvesting into the first model and established the existence of a positive solution. In the case $N = 1$ we obtained the interesting result that the bifurcation diagram is at least Σ-shaped.

7.2 Future Directions

The general logistic function is characterized by a declining growth rate per capita function. But there are some ecosystems where the growth rate per capita may achieve its peak at a positive density. This is called the Allee effect (see Allee [1], Dennis [14], Lewis Kareiva [30] and Shi-Shivaji[40]). This effect can be caused by shortage of mates (Hopf and Hopf [21], Veit and Lewis [46]), lack of effective pollination (Groom [18]), predator saturation (de Roos et. al. [13]), and cooperative behaviors (Wilson and Nisbet [48]). We would like to investigate the existence of alternate stable states if the logistic term is replaced with Allee effect in the grazing problem.

We would like to look at the grazing problems from a control theory perspective to discover optimal strategies for managed ecosystems. The control could be either on the growth rate of the population (see [15]) or the harvesting term (see [26, 16]). This opens up various management problems, which require different control sets and objective functionals.

We would also like to look at the above models with nonlinear boundary conditions of the form:

$$\alpha(x,u)\frac{\partial u}{\partial \eta} + [1-\alpha(x,u)]u = 0 \quad \text{on } \partial\Omega, \tag{7.1}$$

where $\frac{\partial u}{\partial \eta}$ is the outward normal derivative, and $\alpha(x,u) : \Omega \times \mathbb{R} \to [0,1]$ is a nondecreasing C^1 function. The nonlinear boundary condition (7.1) has only been recently studied by such authors as [7, 8, 9, 17], among others. Here

$$\alpha(x,u) = \alpha(u) = \frac{u}{u - \frac{1}{\lambda}\frac{\partial u}{\partial \eta}}$$

represents the fraction of the population that remains on the boundary when reached. For the case when $\alpha(x,u) \equiv 0$, (7.1) becomes the well known Dirichlet boundary condition. If $\alpha(x,u) \equiv 1$ then (7.1) becomes the Neumann boundary condition. We are interested in the study of positive steady state solutions when

$$\alpha(x,u) = \frac{u}{u + \frac{g(u)}{\lambda}} \quad \text{on } \partial\Omega.$$

Here $g(u)$ is a smooth function with some growth conditions.

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