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Inferences on the power-law process with applications to repairable systems

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By

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Inferences on the power-law process with applications to repairable systems

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System testing is very time-consuming and costly, especially for complex high-cost and high-reliability systems. For this reason, the number of failures needed for the developmental phase of system testing should be relatively small in general. To assess the reliability growth of a repairable system, the generalized confidence interval and the modified signed log-likelihood ratio test for the scale parameter of the power-law process are studied concerning incomplete failure data. Specifically, some recorded failure times in the early developmental phase of system testing cannot be observed; this circumstance is essential to establish a warranty period or determine a maintenance phase for repairable systems.

For the proposed generalized confidence interval, we have found that this method is not biased estimates which can be seen from the coverage probabilities obtained from this method being close to the nominal level 0.95 for all levels of γ and β. When the performance of the proposed method and the existing method are compared and validated

regarding average widths, the simulation results show that the proposed method is superior to another method due to shorter average widths when the predetermined number of failures is small.

For the proposed modified signed log-likelihood ratio test, we have found that this test performs well in controlling type I errors for complete failure data, and it has desirable powers for all parameters configurations even for the small number of failures. For incomplete failure data, the proposed modified signed log-likelihood ratio test is preferable to the signed log-likelihood ratio test in most situations in terms of controlling type I errors. Moreover, the proposed test also performs well when the missing ratio is up to 30% and $n > 10$. In terms of empirical powers, the proposed modified signed log-likelihood ratio test is superior to another test for most situations.

In conclusion, it is quite clear that the proposed methods, the generalized confidence interval and the modified signed log-likelihood ratio test, are practically useful to save business costs and time during the developmental phase of system testing since only small number of failures is required to test systems, and it yields precise results.

Key words: generalized confidence interval, power-law process, signed log-likelihood ratio test, substitution method, system reliability, third-order approximation

DEDICATION

To my family.

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LIST OF SYMBOLS, ABBREVIATIONS, AND NOMENCLATURE

PLP Power-law process

HPP Homogeneous Poisson process

NHPP Nonhomogeneous Poisson process

CDF Cumulative distribution function

PDF Probability density function

SLRT Signed log-likelihood ratio test

MSLRT Modified signed log-likelihood ratio test

PGCI Proposed generalized confidence interval

CI Confidence interval

CP Coverage probability

CE Coverage error

RB Relative bias

AW Average width

ln Natural logarithm

MLE Maximum likelihood estimate

$U, V, X, ...$ Random variables

T_i Time of the i^{th} failure

t_i Observed time of the i^{th} failure

$N(t)$ Number of failures in the time interval $[0, t]$

$E(N(t))$ Mean of number of failures in the time interval $[0, t]$

$M(t)$ Mean function

$\nu(t)$ Intensity function

γ Scale parameter of the power-law process

β Shape parameter of the power-law process

n Fixed number of failures

F Cumulative distribution function

f Probability density function

L Likelihood function

ln Log-likelihood function

$\hat{\gamma}$ Maximum likelihood estimate of γ

$\hat{\beta}$ Maximum likelihood estimate of β

$\tilde{\beta}$ Unbiased estimate of β

$E(X)$ Expected value of the random variable X

$Var(X)$ Variance of the random variable X

$M_X(t)$ Moment generating function of the random variable X

$X \sim \chi^2_k$ The random variable X has a chi-square distribution with k degrees of freedom.

$N(0,1)$ The standard normal distribution

$X \sim Poisson(\lambda)$ The random variable X has a Poisson distribution with mean λ.

$X_n \xrightarrow{as} X$ X_n almost sure converges to X as $n \to \infty$

$X_n \xrightarrow{d} N(0,1)$ X_n converges in distribution to $N(0,1)$ when $n \to \infty$

α Significance level

$1-\alpha$ Confidence level

$\chi^2_\alpha(k)$ The α^{th} percentile of the chi-square distribution with k degrees of freedom

W_α The α^{th} percentile of the distribution of W

H_0 Null hypothesis

H_1 Alternative hypothesis

I_n Fisher information matrix

W^2_N Cramér Von Mises statistic

C_N^2 Modified version of Cramér Von Mises statistic

r The first failure (r^{th} failure) that can be observed.

$r-1$ Number of missing failure times in the early of development process

Λ Likelihood ratio test statistic

R Signed log-likelihood ratio test statistic

R^* Modified signed log-likelihood ratio test statistic

CHAPTER 1

INTRODUCTION

1.1 Background and Research Motivation

The theory of reliability is mostly focused on non-repairable systems or devices, and the investigation of lifetime models is the main emphasis. By definition, a non-repairable system can fail only once, whereas the distribution of the time when this type of system fails is provided by various lifetime models, for example the Exponential distribution, the Log-normal distribution, and the Weibull distribution. In contrast, it is of course possible for a repairable system to be repaired and returned to normal operation. Therefore, an entire series of failures must be considered by a model for repairable systems, which must also have the ability to take into account the aging of the system which results in the inevitable changes to its reliability. Means of a counting process is frequently used to model a repairable system. Let $N(t)$ represents the number of failures of a repairable system in the specific time interval $[0, t]$, then $N(t)$ is non-negative and integer-valued, and if $t > s$, the difference $N(t) - N(s)$ provides the number of failures occurring within the time interval $(s, t]$. A second aspect can be expressed by the system's successive failure times, $T_1, T_2, ..., T_n$. An approach that is often employed to analyze the data of repairable systems is concerned with parametric assumptions that illustrate significant characteristics of a system that is modeled. An example of this would be a system that, after each failure, is restored to a condition that is "like new". Then, that the times between failures are identically distributed and independent would be a reasonable assumption, which would correspond to assuming that the system is modeled by a *renewal process*. A different type of situation that is commonly encountered with repairable systems involves changes in the reliability of the system as it ages. For example, during the development stage of a repairable system, the initial prototypes usually have flaws in the design, and during the early testing phase, these problems are corrected by changes to the design. If the development program is succeeding, a tendency for there to be longer times between each failure is expected. When this happens, systems are referred to be undergoing reliability growth. On the other hand, in a case where a system is deteriorating and it is given only minimal repairs when it fails, it would be expected that the time between failures would become shorter due to the aging of the system (See Figure 1.1).

For a Poisson process with mean function $M(t)$, the number of occurrences in an interval $(s, t]$ is Poisson distributed, $N(t) - N(s) \sim Poisson[M(t) - M(s)]$. If $M(t)$ is also differentiable, then the derivative, say $\nu(t) = M'(t)$, is called the *intensity function*, or *occurrence rate of failures*.

In the modeling of repairable systems, Poisson processes are frequently used. One characterization of this kind of process is to specify a set of properties or axioms which describe the probabilistic behavior of $N(t)$. For example, the property of independent increments which means that the numbers of failures occurring in disjoint time intervals are essentially stochastically independent. Ross (1983)[37] provides a discussion of a typical set of axioms that define a Poisson process. Cinlar (1975)[7] discussed an interesting

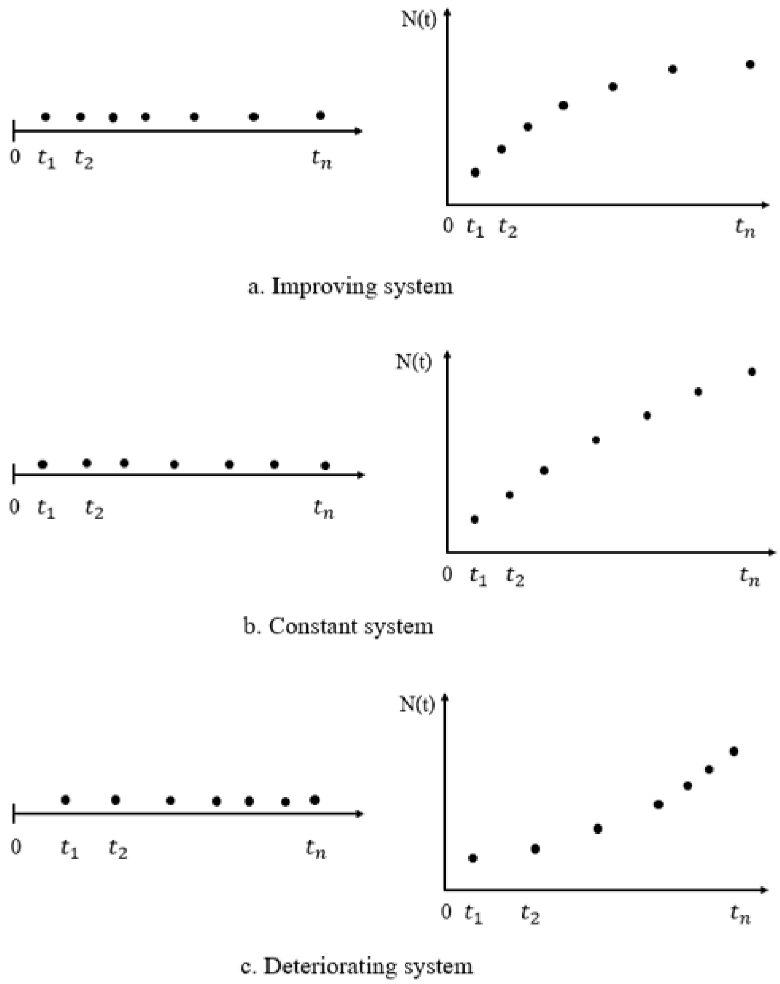


Figure 1.1

Time dot plots (left) and plots of cumulative failure number $N(t)$ against cumulative time (right) of three different types of systems alternate characterization, which is concerned with the mean function $M(t) = E[N(t)]$. If $M(t)$ is continuous, the counting process is known as *regular*. For a regular counting process to be considered a Poisson process, it must have independent increments and not have any simultaneous failures. Meanwhile, for a Poisson process with mean function $M(t)$, the number of occurrences in an interval $(s, t]$ is Poisson distributed, $N(t) - N(s) \sim Poisson[M(t) - M(s)]$. When $M(t)$ is differentiable as well, the derivative, say $\nu(t) = M'(t)$, is called the *intensity function*, or *rate of occurrence of failures*.

The Poisson process that is best known is a homogeneous Poisson process (HPP), which has an intensity function that is constant, such as $\nu(t) = \lambda$. However, Poisson processes with nonconstant intensity functions are the focus of most of the recent work on repairable systems. This type of process is often known as a nonhomogeneous Poisson process (NHPP). As could be expected, an NHPP can be used to model systems that are undergoing either reliability growth or deterioration. In particular, the times between failures tend to be longer if the intensity function $\nu(t)$ is decreasing, and they tend to be shorter if the intensity function $\nu(t)$ is increasing.

The majority of the recent work on modeling and analysis of repairable systems is based on the assumption of a Power-Law process, which is a special type of NHPP. In the literature, this process is also known as a *Weibull process* because it is primarily derived from the similarity of the intensity function to the hazard function of a Weibull distribution. Particularly, the form of the intensity function of this process is $\nu(t) = \gamma\beta t^{\beta-1}$, and the form of the mean value function is $M(t) = \gamma t^{\beta}$.

Over the past decades, there are many papers deal with classical inferences of the power law process for a single system such as point estimation, confidence intervals, and hypothesis testing for unknown parameters of the intensity function, and also goodness-of-fit tests for the power-law process. Finkelstein (1976)[19] obtained confidence intervals for parameters of Weibull process, which is another term for the power law process. Lee and Lee (1978)[26] investigated the results on statistical inference for the Weibull process which focused on current system reliability in the failure truncated case. Engelhardt and Bain (1978)[18] considered statistical analysis of a compound power-law model for repairable systems. Crow (1982)[11] discussed confidence interval procedures for the Weibull process. Kyparisis and Singpurwalla (1985)[25] investigated the current system reliability based on traditional Bayesian approach. Calabria et al. (1988)[5] examined modified maximum likelihood estimators of the expected number of failures in a given time interval and of the failure intensity and compares their mean squared errors with those MLEs. Guida et al. (1989)[22] obtained Bayesian point and interval estimates for a non-homogeneous poisson process with power intensity law. Shaul et al. (1992)[40] reviewed and further developsed Bayesian inference for a power-law process. Park and Pickering (1997)[31] considered statistical analysis of a power law model for repair data. Sen (1998)[39] discussed estimation of current reliability in a Duane-based reliability growth model, and Qiao and Tsokos (1998)[35] obtained the best efficient estimates of intensity function. Gaudoin et al. (2003)[20] discussed goodness-of-fit tests for the power law process, and Ryan (2003)[38] proposed some flexible families of intensities for the NHPP models and discussed their Bayes Inference. Zhao and Wang (2005)[48] discussed goodness-of-fit tests for nonhomogeneous Poisson process models. Gaudoin et al. (2006)[21] proposed asymptotic confidence intervals for the scale parameter of the power law process which were derived from fisher information and theoretical results by Cocozza-Thivent (1997)[8]. Wang et al. (2013)[43] presented a generalized confidence interval for the scale parameter of intensity function and also studied the accuracy of the generalized confidence interval by Monte Carlo simulation.

Although many studies rely on statistical inference for unknown parameters of the power-law process, most works are established on the basis of complete observations in which all failure times can be exactly recorded (Verma and Kapur, 2005[42]). In practical situations, missing data in the power-law process are a common and expected occurrence because of various reasons, and it may have a significant effect on the conclusions about unknown parameters. One of the most frequently encountered circumstances for missing data in the power-law process is that only the cumulative test time and the corresponding cumulative number of failures are observed while the actual failure times are unknown. The literature on missing data in the power-law process was reviewed by Yu et al. (2008)[47]. For example, the early work considered the AMSAA model when all $t_i, i = 1, 2, ..., n$ was missing, and this model was reduced to the famous Duane model. After that, Crow and Basu (1988)[12] investigated another scenario in which irregularities may exist over some interval of the test period, and this might affect in too few or too many failures being reported over that interval. In this case, the maximum likelihood estimators (MLEs) of γ and β were only derived. Then, Yu et al. (2008)[47] considered statistical inference and prediction analyses based on the classical approach to the Weibull process with incomplete observations, especially incomplete observations in the early developmental phase of a testing program which could not be observed, while Tian et al. (2011)[41] considered a Bayesian estimation and prediction for the power-law process in the presence of left-truncated data.

For the power-law process, the exact test and the exact confidence interval for the shape parameter, β, is not troublesome to derive, but the exact test and the exact confidence interval for the scale parameter, γ, is not easy to obtain when β is unknown. Asymptotic distributions, such as the asymptotic normal distribution and the asymptotic chi-square distribution, are therefore used in many previous studies to make a conclusion about parameter γ. Nonetheless, the issue raised by this approach is that we need a sufficiently large sample size (number of failures must be large enough) to produce accurate results. Therefore, this research aims to solve such this problem by developing a statistical method that requires only small number of failures to asses the system's reliability in order to reduce time and cost during the developmental phase of system testing and this contributes to the motivation of this research.

In the next sections, we first provide definitions of key concepts that will be used in this research. Then, we address some fundamental concepts of reliability and some fundamental results on homogeneous Poisson process and nonhomogeneous Poisson process. We also present more details on a particular nonhomogeneous Poisson process, "power-law process", which plays an important role in this research. The results that are discussed in this chapter are basic for the research presented in later chapters.

1.2 Definitions of Key Concepts

The following definitions used in this research are contextual and are therefore defined to clarify and eliminate ambiguities.

1.2.1 Repairable System

A *repairable system* is a system that made up of many components. If one critical component fails, it will bring down the entire system. After that component is repaired or replaced, the system can be restored to its state prior to failure. In the case of a repairable system, after replacement or repairing, it will not place the system back into a like-new condition because there are many other components still working with various ages.

1.2.2 Failure Truncated Data

Data are stated to be *failure truncated* when testing stops after a predetermined number of failures. Suppose that a repairable system is observe till n failures occur (fixed n), so we observe the ordered failure times $t_1 < t_2 < ... < t_n$ where t_i is the time of ith failure. In this case, the number of failures is fixed and the time when the testing stops is random.

1.2.3 Time Truncated Data

Data are stated to be *time truncated* when testing stops at a predetermined time t. We observe a set of failure time $t_1 < t_2 < ... < t_n < t$. In this case, the time when the testing stops is fixed and the number of failures $N(t)$ is random.

1.2.4 Complete Failure Data

In this study, *Complete failure data* is defined as the data in which all failure times during the system testing are recorded. That is, for a predetermined number of failure, n, all failure times $t_1, t_2, ..., t_n$ can be observed (see Figure 1.2).

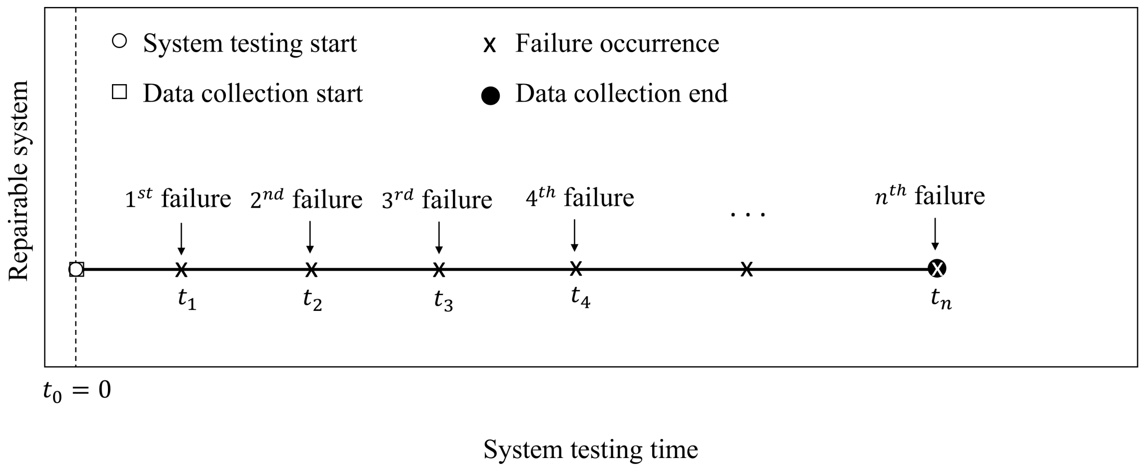


Figure 1.2

Complete failure data with a predetermined number of failures (*n*)

1.2.5 Incomplete Failure Data

In this study, *Incomplete failure data* is defined as the data in which some exact failure times in the early development phase cannot be observed. That is, we assume that $t_1, t_2, ..., t_{r-1}$ are missing, and the observed failure times are defined as $t_r, t_{r+1}, ..., t_n$ [47](see Figure 1.3).

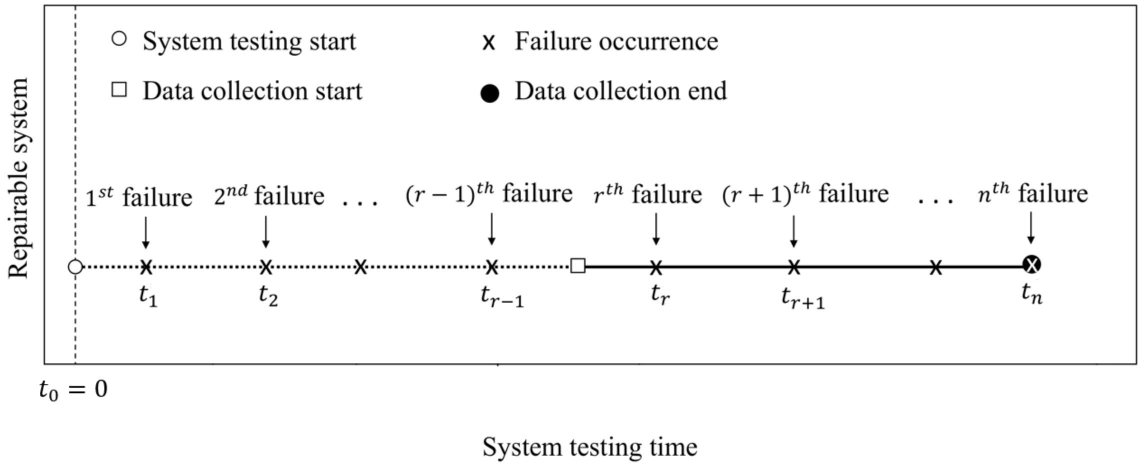


Figure 1.3

Incomplete failure data with a predetermined number of failures (*n*)

1.3 Fundamentals of Reliability

The reliability function (or survival function), denoted by $R(t)$, is the probability that a system will survive beyond time t. Let T denote the failure time since the system was first started ($t = 0$). Let $N(t)$ denote the cumulative number of failures from time 0 to time t. Then the reliability function is defined as

$$\begin{aligned} R(t) &= P[T > t] \\ &= 1 - F(t) \\ &= \int_t^{\infty} f(s)ds, \end{aligned} \tag{1.1}$$

where $F(t)$ and $f(t)$ are the cumulative distribution function (cdf) and the probability density function (pdf) of T, respectively. Notice that the reliability function decreases in t, from 1 at $t = 0$, to 0 at $t = \infty$.

1.4 Counting Process and Poisson Process

In some problems, we observe the occurrences of some types of events over time. For example, counting the number of customers who arrive at a supermarket from 9:00AM to 10:00PM, observing the occurrence of getting myocardial infarction during a year, counting the number of waking up during night, etc. In these scenarios, we are dealing with a *counting process*. The counting process therefore simply the count of the number of events that occurs in any specified time interval, and it is denoted $N(t), t \geq 0$. In the counting process, the most commonly used probabilistic models are homogeneous and nonhomogeneous Poisson processes.

A counting process $N(t)$ is said to be a *Poisson process* if

1. The cumulative number of failures at time 0 is 0, $N(0) = 0$.
2. For $a < b \leq c < d$ the random variables $N(a, b]$ and $N(c, d]$ are independent. This property is called *independent increment*.
3. The intensity function of the Poisson process is defined as:
$$\nu(t) = \lim_{\Delta t \to 0} \frac{P[N(t + \Delta t) - N(t) \geq 1]}{\Delta t}. \tag{1.2}$$
4. The possibility of simultaneous failures is defined as:
$$\lim_{\Delta t \to 0} \frac{P[N(t + \Delta t) - N(t) \geq 2]}{\Delta t} = 0. \tag{1.3}$$

The properties (1) to (4) of the Poisson process imply that

$$P[N(t) = n] = \frac{1}{n!} \left(\int_0^n \nu(x) dx \right)^n \exp \left(- \int_0^t \nu(x) dx \right) \tag{1.4}$$

(Rigdon and Basu, 2000[36]).

1.5 Homogeneous Poisson Process

The counting process $N(t), t \geq 0$ is said to be a homogeneous Poisson process (HPP) if the intensity function $\nu(t)$ is a constant, that is, $\nu(t) = \lambda,\ \gamma > 0$ and

1. The cumulative number of failures at time 0 is 0, $N(0) = 0$.
2. The process has independent increments and stationary increments. A point process has stationary increments if for all k, $P[N(t, t+s] = k]$ is independent of t.

It can be shown that the number of failures in any interval of length $s = t_2 - t_1$ has a Poisson distribution with mean λs, that is

$$P[N(t_2) - N(t_1) = n] = \frac{\exp(-\lambda s)(\lambda s)^n}{n!},\ 0 \leq t_1 \leq t_2,\ n = 0, 1, ... \qquad (1.5)$$

The homogeneous Poisson process has the following properties (Rigdon and Basu, 2000[36]):

Property 1. A process is an HPP with constant intensity function λ, if and only if the times between events are i.i.d. exponential random variables with mean $1/\lambda$.

Property 2. If $0 < T_1 < T_2 < ... < T_n$ are the failure times from an HPP, then the joint pdf of $T_1, T_2, ..., T_n$ is

$$f(t_1, t_2, ..., t_n) = \lambda^n \exp(-\lambda t_n),\ 0 < t_1 < t_2 < ... < t_n. \qquad (1.6)$$

Property 3. The time to the n^{th} failure from a system modeled by an HPP has a gamma distribution with parameter $\alpha = n, \beta = 1/\lambda$.

Property 4. For an HPP, conditional on $N(t) = n$, the failure times $0 < T_1 < T_2 < ... < T_n$ are distributed as order statistics from the uniform distribution on the interval $(0, t)$.

Property 5. The probability of system failure after time t is

$$R(t) = P[T > t] = P[N(t) = 0] = \exp(-\lambda t). \tag{1.7}$$

1.6 Nonhomogeneous Poisson Process

The nonhomogeneous Poisson process is a Poisson process which the intensity function is not a constant. A counting process $N(t), t \geq 0$ has a nonhomogeneous Poisson process if

1. The cumulative number of failures at time 0 is 0, $N(0) = 0$.
2. The process has independent increments.

It can be shown that the number of failures in any interval $(t_1, t_2]$ has a Poisson distribution with mean $\int_{t_1}^{t_2} \nu(t)dt$, that is

$$P[N(t_2) - N(t_1) = k] = \frac{1}{k!}\exp\left(-\int_{t_1}^{t_2} \nu(t)dt\right)\left(\int_{t_1}^{t_2} \nu(t)dt\right)^k. \tag{1.8}$$

For this research, these occurrences in time will be the failure times of repairable systems, and the term *failures* will be used instead of *events* from now on.

The nonhomogeneous Poisson process has the following properties (Rigdon and Basu, 2000[36]):

Property 1. The joint pdf of the failure time $T_1, T_2, ..., T_n$ from an NHPP with intensity function $\nu(t)$ is given by

$$f(t_1, t_2, ..., t_n) = \left(\prod_{i=1}^{n} \nu(t_i)\right) \exp\left(-\int_0^w \nu(x)dx\right), \tag{1.9}$$

where w is the stopping time: $w = t_n$ for the failure truncated case, and $w = t$ for the time truncated case.

Property 2. If the failure times of a nonhomogeneous Poisson process are $T_1 < T_2 < ... < T_n$ then conditioned on $T_n = t_n$, the random variables $T_1 < T_2 < ... < T_{n-1}$ are distributed as $n-1$ order statistics from the distribution with cumulative distribution function

$$G(y) = \begin{cases} 0, & y \leq 0, \\ \dfrac{m(y)}{m(t_n)}, & 0 < y \leq t_n, \\ 1, & y > t_n. \end{cases} \tag{1.10}$$

Property 3. If a NHPP with intensity function $\nu(t)$ is observed until time t, and if the failure times are $T_1 < T_2 < ... < T_{N(t)}$ where $N(t)$ is the random number of failures in the interval $(0, t]$, then conditioned on $N(t) = n$, the random variables $T_1 < T_2 < ... < T_n$ are distributed as n order statistics from the distribution with cdf

$$G(y) = \begin{cases} 0, & y \leq 0, \\ \dfrac{m(y)}{m(t)}, & 0 < y \leq t, \\ 1, & y > t. \end{cases} \tag{1.11}$$

Property 4. The probability of system failure occurring after time t is known as the reliability function, $R(t)$. The nonhomogeneous Poisson process assumes that the number of failures in any interval $(t_1, t_2]$ has a Poisson distribution with mean $\int_{t_1}^{t_2} \nu(t)dt$. Hence the reliability function is

$$
\begin{aligned}
R(t) &= P[T > t] \\
&= P[N(t) = 0] \\
&= \frac{\exp\left(-\int_{t_1}^{t_2} \nu(t)dt\right)\left(\int_{t_1}^{t_2} \nu(t)dt\right)^0}{0!} \\
&= \exp\left(-\int_{t_1}^{t_2} \nu(t)dt\right) \\
&= \exp[-(\lambda(t_2) - \lambda(t_1))]. \qquad (1.12)
\end{aligned}
$$

1.7 Power-Law Process

The power-law process, which plays a key role in this research, is widely used to study the reliability or model the reliability growth of repairable systems. Duane (1964)[16] was the first to study and report that the cumulative number of failures of systems up to time t, $N(t)$, often have a "power-law" growth pattern. Then, Crow (1974)[10] formulated the corresponding model as a NHPP having the mean value function is

$$m(t) = E[N(t)] = \gamma t^{\beta},\ t \geq 0,\ \gamma, \beta > 0, \qquad (1.13)$$

and the failure of intensity of the model

$$\nu(t) = \frac{d}{dt} m(t) = \gamma \beta t^{\beta - 1}, \qquad (1.14)$$

where γ is a scale parameter, and β is a shape parameter. This model, known as the PLP, and then it has become the most popular parametric intensity in the repairable systems. In general, parameter β of this intensity function affects how the system improves or deteriorates over time. When $\beta < 1$, there is a decrease in the failure intensity, which indicates reliability growth and that there is improvement in the system; whereas if $\beta > 1$, there is an increase in the failure intensity, which indicates reliability degeneration and that the failures are becoming increasingly more frequent. Moreover, if $\beta = 1$, there is a reduction of the Power-Law process to a homogeneous Poisson process having a constant failure intensity γ (see Figure 1.4).

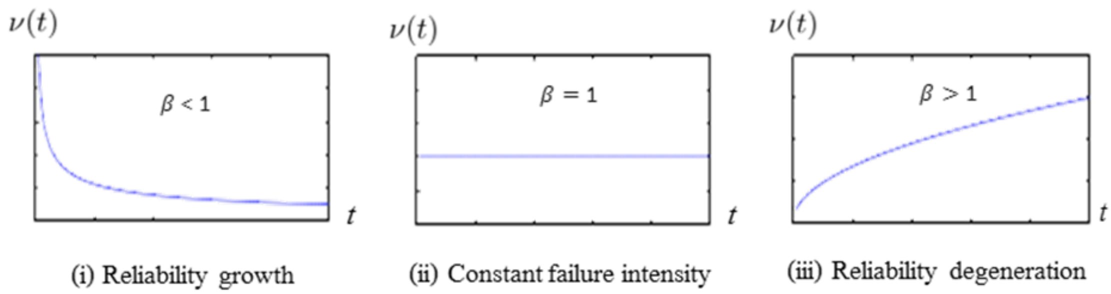


Figure 1.4

Types of system reliability

Therefore, the PLP is a flexible model that can be used to study both reliability growth and reliability deterioration which are two different, but common situations (Gaudoin et al. 2006[21]). The power law model is also known as the Duane model (Duane, 1964[16]), Weibull model, and Army Materials System Analysis Activity (AMSAA) model.

CHAPTER 2

CLASSICAL INFERENCE RESULTS ON THE POWER-LAW PROCESS

In this chapter, we present some traditional inference results on the power-law process for a single system including point estimation, interval estimation, and hypothesis testing for parameters of the PLP. The results that are addressed in this chapter are essential and will be utilized for the research presented afterward. In addition, we also present the intensity function estimation, the mean time between failures estimation, and the goodness of fit tests.

2.1 Point Estimation of the Scale and Shape Parameter

In this section, we present two point estimation methods for parameter γ and β, the maximum likelihood estimation and the unbiased estimation. For the maximum likelihood estimation, it is a standard approach to parameter estimation and inference in statistics because it has many optimal properties in estimation such as sufficiency, consistency, efficiency, and parameterization invariance. However, it is found that the maximum likelihood estimates of parameter γ and β are biased estimates. Therefore, the unbiased estimation which is the adjusted version of this method is also presented.

2.1.1 Likelihood Function for Failure Times

Using Property 1 of the NHPP in the previous chapter, the joint pdf of the failure times, $T_1, T_2, ..., T_n$, from the NHPP with intensity function $\nu(t)$ is given by

$$f(t_1, t_2, ..., t_n) = \left(\prod_{i=1}^{n} \nu(t_i)\right) \exp\left(-\int_0^w \nu(x)dx\right), \tag{2.1}$$

where $w = t_n$ for the failure truncated case, and $w = t$ for the time truncated case.

For failure truncated case with $\nu(t) = \gamma\beta t^{\beta-1}$, the joint pdf of $T_1 < T_2 < ... < T_n$ is satisfying the equation:

$$\begin{aligned} f(t_1, t_2, ..., t_n) &= \left(\prod_{i=1}^{n} \gamma\beta t_i^{\beta-1}\right) \exp\left(-\int_0^{t_n} \gamma\beta x^{\beta-1}dx\right) \\ &= (\gamma\beta)^n \exp(-\gamma t_n^{\beta}) \prod_{i=1}^{n} t_i^{\beta-1}. \end{aligned} \tag{2.2}$$

For time truncated case, the testing stops at a predetermined time t, and we observed the failure times, $t_1 < t_2 < ... < t_{N(t)} < t$. Therefore, the number of failures, $N(t)$, in time truncated interval $(0, t]$ is a random variable. Using Property 3 of the NHPP in the previous chapter that conditional on $N(t) = n$, the failure times $t_1 < t_2 < ... < t_n$ are distributed as n order statistics from the distribution with cdf:

$$F(t_i) = \begin{cases} 0, & t_i \leq 0, \\ \dfrac{m(t_i)}{m(t)}, & 0 < t_i \leq t, \\ 1, & t_i > t. \end{cases} \tag{2.3}$$

Therefore, the pdf of t_i, $f(t_i)$, can be obtained as

$$\begin{aligned} f(t_i) &= F'(t_i) \\ &= \frac{d}{dt_i}\left[\frac{m(t_i)}{m(t)}\right] \\ &= \frac{\nu(t_i)}{\gamma t^\beta} \\ &= \frac{\gamma\beta t_i^{\beta-1}}{\gamma t^\beta} \\ &= \frac{\beta t_i^{\beta-1}}{t^\beta} \end{aligned} \tag{2.4}$$

if $0 \le t_i \le t$, and zero otherwise. Given $N(t) = n$, the conditional density function of the failure times $T_1, T_2, ..., T_n$ is then defined as:

$$\begin{aligned} f(t_1, t_2, ..., t_n|n) &= n! \prod_{i=1}^{n} \frac{\beta t_i^{\beta-1}}{t^\beta} \\ &= n! \left(\frac{\beta}{t^\beta}\right)^n \prod_{i=1}^{n} t_i^{\beta-1} \end{aligned} \tag{2.5}$$

if $0 \le t_1 < t_2 < ... < t_n \le t$, and zero otherwise. Since the random variable $N(t)$ has a Poisson distribution with mean γt^β, then the joint density of $T_1, T_2, ..., T_n$ and $N(t)$ as following:

$$\begin{aligned} f(t_1, t_2, ..., t_n, n) &= f(t_1, t_2, ..., t_n|n) f(n) \\ &= n! \left(\frac{\beta}{t^\beta}\right)^n \prod_{i=1}^{n} t_i^{\beta-1} \left[\frac{\exp(-\gamma t^\beta)(\gamma t^\beta)^n}{n!}\right] \\ &= (\gamma\beta)^n \exp(-\gamma t^\beta) \prod_{i=1}^{n} t_i^{\beta-1}. \end{aligned} \tag{2.6}$$

Notice that for time truncated case, it is possible that there is no failure occurs before time t ($n = 0$). In this case, the joint density of $T_1, T_2, ..., T_n$ and $N(t)$ is reduced to

$$f(t_1, t_2, ..., t_n, n) = \exp(-\gamma t^\beta), \tag{2.7}$$

and we will not consider this case in this study since there is of no inferential interest.

Therefore, the likelihood functions for failure and time truncated case can be written as

$$L(\gamma, \beta) = f(t_1, t_2, ..., t_n, n) = (\gamma\beta)^n \text{exp}(-\gamma w^\beta) \prod_{i=1}^{n} t_i^{\beta-1}, \tag{2.8}$$

where $w = t_n$ for the failure truncated case, and $w = t$ for the time truncated case.

2.1.2 Maximum Likelihood Estimations of the Scale and Shape Parameter

Given the likelihood function as shown in Equation 2.8, the logarithm of the likelihood function can be expressed as

$$\ln L(\gamma, \beta) = n\ln\gamma + n\ln\beta - \gamma w^\beta + (\beta - 1)\sum_{i=1}^{n} \ln t_i, \tag{2.9}$$

where $w = t_n$ for the failure truncated case, and $w = t$ for the time truncated case.

The maximum likelihood estimates (MLEs) for parameter γ and β can be obtained by equating the first partial derivatives (with respect to γ and β) of Equation 2.9 to zero and solving the resulting system. In other words, we can solve for $\gamma = \hat{\gamma}$ and $\beta = \hat{\beta}$ in the system

$$\frac{\partial \ln L(\gamma, \beta)}{\partial \gamma} = \frac{n}{\gamma} - w^\beta = 0, \tag{2.10}$$

$$\frac{\partial \ln L(\gamma, \beta)}{\partial \beta} = \frac{n}{\beta} - \gamma w^\beta \ln w + \sum_{i=1}^{n} \ln t_i = 0. \tag{2.11}$$

Thus, the maximum likelihood estimators of γ and β can be obtained as

$$\hat{\gamma} = \frac{n}{w^{\hat{\beta}}}, \tag{2.12}$$

and

$$\hat{\beta} = \frac{n}{n\ln w - \sum_{i=1}^{n} \ln(t_i)}, \tag{2.13}$$

respectively, where $w = t_n$ for the failure truncated case, and $w = t$ for the time truncated case.

Theorem 1

Suppose that the failure times $T_1, T_2, ..., T_n$ from the PLP with intensity function $\nu(t) = \gamma\beta t^{\beta-1}$ are observed, and $\hat{\gamma}$ and $\hat{\beta}$ are the MLEs of γ and β, respectively. Then:

(i) $U = 2n\beta/\hat{\beta}$ has a chi-square distribution with $2(n-d)$ degrees of freedom, where $d = 1$ for failure truncated case, and $d = 0$ for the time truncated case;

(ii) $V = 2\gamma T_n^{\beta}$ has a chi-square distribution with $2n$ degrees of freedom;

(iii) U and V are independent.

Lemma 1

Let X be a has a chi-square distribution with n degrees of freedom, then

$$E(X^k) = \frac{2^k\Gamma\left(\frac{n}{2}+k\right)}{\Gamma\left(\frac{n}{2}\right)}, \tag{2.14}$$

where k is an integer and $\frac{n}{2} + k > 0$.

Using Theorem 1 and Lemma 1, we can now show that $\hat{\beta}$ is a biased estimator of β, and its expectation and variance are defined as:

$$E(\hat{\beta}) = \frac{n\beta}{(n-d-1)}, \tag{2.15}$$

and

$$Var(\hat{\beta}) = \frac{n^2\beta^2}{(n-d-1)^2(n-d-2)}, \tag{2.16}$$

respectively.

Proof: By Theorem 1 and Lemma 1, we have

$$\begin{aligned}
\hat{\beta} &= \frac{2n\beta}{U} \\
E(\hat{\beta}) &= 2n\beta E\left(\frac{1}{U}\right) \\
&= 2n\beta E(U^{-1}) \\
&= 2n\beta \left[\frac{2^{-1}\Gamma\left(\frac{2n-2d}{2}-1\right)}{\Gamma\left(\frac{2n-2d}{2}\right)}\right] \\
&= \frac{n\beta}{(n-d-1)}.
\end{aligned}$$

We also have

$$\begin{aligned}
\hat{\beta}^2 &= \left(\frac{2n\beta}{U}\right)^2 \\
E(\hat{\beta}^2) &= 4n^2\beta^2 E\left(\frac{1}{U^2}\right) \\
&= 4n^2\beta^2 E(U^{-2}) \\
&= 4n^2\beta^2 \left[\frac{2^{-2}\Gamma\left(\frac{2n-2d}{2}-2\right)}{\Gamma\left(\frac{2n-2d}{2}\right)}\right] \\
&= \frac{n^2\beta^2}{(n-d-1)(n-d-2)}.
\end{aligned}$$

Therefore, the variance of $\hat{\beta}$ can be obtained as

$$\begin{aligned}
Var(\hat{\beta}) &= E(\hat{\beta}^2) - [E(\hat{\beta})]^2 \\
&= \frac{n^2\beta^2}{(n-d-1)(n-d-2)} - \left[\frac{n\beta}{(n-d-1)}\right]^2 \\
&= \frac{n^2\beta^2}{(n-d-1)^2(n-d-2)}.
\end{aligned}$$

■

2.1.3 Unbiased Estimation of the Shape Parameter

Since the MLE of β is biased estimate, we can adjust $\hat{\beta}$ to unbiased estimate, and it is defined as

$$\tilde{\beta} = \left(\frac{n-d-1}{n}\right)\hat{\beta}, \tag{2.17}$$

where $d = 1$ for failure truncated case, and $d = 0$ for the time truncated case.

Thus, the expectation and variance of $\tilde{\beta}$ are

$$E(\tilde{\beta}) = \beta, \tag{2.18}$$

and

$$Var(\tilde{\beta}) = \frac{\beta^2}{(n-d-2)}, \tag{2.19}$$

respectively.

Proof: $E(\tilde{\beta})$ and $Var(\tilde{\beta})$ can easily be shown using the property of expectation and variance as follows:

$$\begin{aligned}
E(\tilde{\beta}) &= E\left[\left(\frac{n-d-1}{n}\right)\hat{\beta}\right] \\
&= \left(\frac{n-d-1}{n}\right)E(\hat{\beta}) \\
&= \left(\frac{n-d-1}{n}\right)\left[\frac{n\beta}{(n-d-1)}\right] \\
&= \beta, \\
Var(\tilde{\beta}) &= Var\left[\left(\frac{n-d-1}{n}\right)\hat{\beta}\right] \\
&= \left(\frac{n-d-1}{n}\right)^2 Var(\hat{\beta}) \\
&= \left(\frac{n-d-1}{n}\right)^2 \left(\frac{n^2\beta^2}{(n-d-1)^2(n-d-2)}\right) \\
&= \frac{\beta^2}{(n-d-2)}.
\end{aligned}$$

■

2.2 Interval Estimation and Hypothesis Test of the Shape Parameter

To construct an exact confidence interval of β, we use the result that $2n\beta/\hat{\beta} \sim \chi^2_{2(n-d)}$, where $d = 1$ for failure truncated case, and $d = 0$ for the time truncated case. Then, the exact $(1-\alpha)100\%$ confidence interval for β is

$$\frac{\hat{\beta}\chi^2_{1-\alpha/2}(2n-2d)}{2n} \leq \beta \leq \frac{\hat{\beta}\chi^2_{\alpha/2}(2n-2d)}{2n}, \tag{2.20}$$

where α is the significance level.

The result that $2n\beta/\hat{\beta} \sim \chi^2_{2(n-d)}$ can also be used to perform a test for

$$H_0 : \beta = \beta_0 \text{ versus } H_1 : \beta \neq \beta_0. \tag{2.21}$$

Then, the test statistic for testing the hypotheses in (2.21) is

$$\chi^2 = \frac{2n\beta_0}{\hat{\beta}}, \tag{2.22}$$

and the null hypothesis is rejected at the level of α if $\chi^2 < \chi^2_{1-\alpha/2}(2n-2d)$ or $\chi^2 > \chi^2_{\alpha/2}(2n-2d)$.

In practical situation, it is useful to test $H_0 : \beta = 1$ versus $H_1 : \beta \neq 1$ to see if the PLP reduces to the HPP. Alternative test can also be stated as $H_1 : \beta < 1$ to see if the system is improving or $H_1 : \beta > 1$ to see if the system is deteriorating.

2.3 Interval Estimation and Hypothesis Test for the Scale Parameter when the Shape Parameter is Known

To construct an exact confidence interval for γ when β is known, we use the result that $2\gamma T_n^{\beta} \sim \chi^2_{2n}$. Then, the exact $(1-\alpha)100\%$ confidence interval for γ when β is known is defined as

$$\frac{\chi^2_{1-\alpha/2}(2n)}{2T_n^{\beta}} \leq \gamma \leq \frac{\chi^2_{\alpha/2}(2n)}{2T_n^{\beta}}, \tag{2.23}$$

where α is the significance level.

The result that $2\gamma T_n^{\beta} \sim \chi^2_{2n}$ can also be used to perform a test for

$$H_0 : \gamma = \gamma_0 \text{ versus } H_1 : \gamma \neq \gamma_0. \tag{2.24}$$

Then, the test statistic for testing the hypotheses in (2.24) is

$$\chi^2 = 2\gamma_0 T_n^{\beta}, \tag{2.25}$$

and the null hypothesis is rejected at the level of α if $\chi^2 < \chi^2_{1-\alpha/2}(2n)$ or $\chi^2 > \chi^2_{\alpha/2}(2n)$.

2.4 Interval Estimation for the Scale Parameter when the Shape Parameter is Unknown

For the PLP, an exact confidence interval for parameter γ is not problematic to derive when β is known, but it cannot be obtained when β is unknown. In 2006, the asymptotic confidence intervals for this scenario with failure truncated case were first proposed by Gaudoin et al. (2006)[21] using theoretical results by Cocozza-Thivent(1997)[8] and Fisher information matrix. Then, Wang et al. (2013)[43] proposed the confidence interval for γ using the generalized pivotal quantity. Details of these methods are shown as following:

2.4.1 Asymptotic Confidence Interval Derived from Finklestein's Formulation

Let $\nu(t) = (t/\eta)^{\beta}$ and $\hat{\eta}_n = \hat{\gamma}_n^{-1/\hat{\beta}_n}$ be the MLE of η where $\eta = \gamma^{-1/\beta}$. Finklestein (1976)[19] showed that $2(\hat{\eta}_n/\eta)^{\beta} n^{\beta/\hat{\beta}_n} = (2/\gamma)T_n^{\beta}$ has the χ^2_{2n} distribution. The following asymptotic properties of $\hat{\eta}_n$ have been studied by Cocozza-Thivent(1997)[8]:

$$\hat{\eta}_n \xrightarrow{as} \eta, \tag{2.26}$$

$$\hat{\beta}_n \frac{\sqrt{n}}{\ln n} \ln \frac{\hat{\eta}_n}{\eta} \xrightarrow{d} N(0,1). \tag{2.27}$$

Then, (2.27) can be written as:

$$\hat{\beta}_n \frac{\sqrt{n}}{\ln n}\left[-\frac{1}{\hat{\beta}_n}\ln\hat{\gamma}_n + \frac{1}{\beta}\ln\gamma\right] = \frac{\sqrt{n}}{\ln n}\ln\frac{\gamma}{\hat{\gamma}_n} - \frac{\sqrt{n}}{\ln n}\left(1 - \frac{\hat{\beta}_n}{\beta}\right)\ln\gamma \xrightarrow{d} N(0,1). \tag{2.28}$$

Based on the asymptotic property of $\hat{\beta}_n$ that $\hat{\beta}_n \xrightarrow{as} \beta$, Gaudoin et al. (2006)[21] obtain the following result

$$\frac{\sqrt{n}}{\ln n}\ln\frac{\gamma}{\hat{\gamma}_n} \xrightarrow{d} N(0,1), \tag{2.29}$$

and the $(1-\alpha)100\%$ confidence interval for γ with positive bounds can be obtained as

$$\hat{\gamma}\,\exp\left(-z_{\alpha/2}\frac{\ln n}{\sqrt{n}}\right) \leq \gamma \leq \hat{\gamma}\,\exp\left(z_{\alpha/2}\frac{\ln n}{\sqrt{n}}\right), \tag{2.30}$$

where $z_{\alpha/2}$ is the $100(\alpha/2)th$ percentile of the standard normal distribution.

2.4.2 Asymptotic Confidence Interval Derived from Fisher Information Matrix

The problem raised by the confidence interval in (2.30) is that this interval is very wide for some values of γ. Therefore, Gaudoin et al. (2006)[21] propose another asymptotic-based confidence interval using the following Fisher information matrix:

$$I_n = \begin{bmatrix} -E\left[\frac{\partial^2 \ln L}{\partial \gamma^2}\right] & -E\left[\frac{\partial^2 \ln L}{\partial \gamma \partial \beta}\right] \\ -E\left[\frac{\partial^2 \ln L}{\partial \gamma \partial \beta}\right] & -E\left[\frac{\partial^2 \ln L}{\partial \beta^2}\right] \end{bmatrix}. \tag{2.31}$$

The likelihood function is given by Equation (2.8), then the information matrix can be obtained as

$$I_n = \begin{bmatrix} \dfrac{n}{\gamma^2} & E\left[(\ln T_n)T_n^{\beta}\right] \\ E\left[(\ln T_n)T_n^{\beta}\right] & \dfrac{n}{\beta^2} + \gamma E\left[(\ln T_n)^2 T_n^{\beta}\right] \end{bmatrix}. \tag{2.32}$$

Gaudoin et al. (2006)[21] show the following asymptotic result:

$$\sqrt{\frac{n}{1+\left(\ln\dfrac{n}{\hat{\gamma}_n}\right)^2}}\ln\frac{\hat{\gamma}_n}{\gamma} \xrightarrow{d} N(0,1), \tag{2.33}$$

and the $(1-\alpha)100\%$ confidence interval for γ is then obtained as

$$\hat{\gamma}\exp\left(\frac{-z_{\alpha/2}}{\sqrt{n}}\sqrt{1+\left(\ln\frac{n}{\hat{\gamma}_n}\right)^2}\right) \leq \gamma \leq \hat{\gamma}\exp\left(\frac{z_{\alpha/2}}{\sqrt{n}}\sqrt{1+\left(\ln\frac{n}{\hat{\gamma}_n}\right)^2}\right). \tag{2.34}$$

2.4.3 Generalized Confidence Interval

The generalized pivotal quantity for γ can be obtained using Theorem 1. Wang et al. (2013)[43] substitute $\hat{\beta}U/2n$ for β in the expression for γ and obtain the following generalized pivotal quantity for γ:

$$W = \frac{V}{2t_n^{U/(2\tau)}} = \frac{\gamma T_n^{\beta}}{t_n^{\beta(\sum_{i=1}^{n-1}\ln(T_n/T_i)/\tau}}, \tag{2.35}$$

where $\tau = \sum_{i=1}^{n-1}\ln(t_n/t_i)$, and $\mathbf{t} = (t_1, t_2, ..., t_n)$ be the observed values of $\mathbf{T} = (T_1, T_2, ..., T_n)$.

As noted in Wang et al. (2013)[43], W is said to be a generalized pivotal quantity since its distribution is free from unknown parameters and $W_{obs} = W(\mathbf{t};\mathbf{t},\gamma,\beta) = \gamma$ does not depend on the nuisance parameter β. Therefore, the $(1-\alpha)100\%$ generalized confidence interval for γ is given by

$$W_{\alpha/2} \leq \gamma \leq W_{1-\alpha/2}, \tag{2.36}$$

where $W_{\alpha/2}$ and $W_{1-\alpha/2}$ denote the $(\alpha/2)^{th}$ and $(1-\alpha/2)^{th}$ percentile of the distribution of W.

2.5 Estimation of Intensity Function

The simplest way to estimate the intensity function of the PLP is using the maximum likelihood estimates of γ and β. Then, the estimate of $\nu(t)$ is defined as

$$\hat{\nu}(t) = \hat{\gamma}\hat{\beta}w^{\hat{\beta}-1} = \left(\frac{n}{w^{\hat{\beta}}}\right)\hat{\beta}w^{\hat{\beta}-1} = \frac{n\hat{\beta}}{w}, \tag{2.37}$$

where $w = t_n$ for the failure truncated case, and $w = t$ for the time truncated case.

2.6 Estimation of Mean Time Between Failure

Mean time between failure (MTBF) refers to the average time that a system or product work without failure, and it is defined as

$$MTBF = \frac{1}{\nu(t)}. \tag{2.38}$$

Similar to the estimation of intensity function, the estimate mean time between failure can be obtained as

$$\widehat{MTBF} = \frac{1}{\hat{\nu}(t)}, \tag{2.39}$$

where $\hat{\nu}(t)$ is defined as (2.37).

2.7 Goodness of Fit Test

In practice, we cannot assume that the failure times of the repairable systems under study always follow the power-law process. We have to test this hypothesis by statistical means. One well known method is to use the Cramér Von Mises goodness statistics.

Suppose that we want to test the following hypotheses:

H_0 : The failure times do not follow a NHPP with intensity function $\nu(t) = \gamma\beta t^{\beta-1}$ and $\beta = \beta_0$,

H_1 : The failure times follow a NHPP with intensity function $\nu(t) = \gamma\beta t^{\beta-1}$ and $\beta = \beta_0$.

To comput the Cramér Von Mises statistic, W_N^2, we use the N transformed failure times which is defined as:

$$t^*_{i_q} = \frac{t_{i_q}}{w_q}, \;\; i = 1, 2, ..., N_q, \;\; q = 1, 2, ..., k, \tag{2.40}$$

where t_{i_q} is the failure times from system, $N = N_1 + N_2 + ... + N_q$, and $w = t_n$ for the failure truncated case and $w = t$ for the time truncated case. Then, we treat the N $t^*_{i_q}$'s as one group and order them in ascending order. These ordered values are called $Z_1, Z_2, ..., Z_N$. That is, Z_1 is the smallest $t^*_{i_q}$, and Z_N is the largest $t^*_{i_q}$. Therefore, the Cramér Von Mises statistic is given by

$$W_N^2 = \frac{1}{12N} + \sum_{j=1}^{N} \left(Z_j^{\beta_0} - \frac{2j-1}{2N} \right)^2. \tag{2.41}$$

The asymptotic significance points for the Cramér Von Mises goodness statistic when H_1 is true can be found in Anderson and Darling (1952)[1]. However, these points are used only when N is moderate large.

In general, we will not have a fixed value of parameter β, β_0, in mind. Thus, we are usually interested to test the following hypotheses:

H_0 : The failure times do not follow a NHPP with intensity function $\nu(t) = \gamma\beta t^{\beta-1}$, β unspecified,

H_2 : The failure times follow a NHPP with intensity function $\nu(t) = \gamma\beta t^{\beta-1}$, β unspecified.

If hypothesis H_1 is accepted, γ and β are then estimated from the observed failure times.

To test the hypotheses above, the modified version of W_N^2 statistic, C_N^2, is used, and it does not have the same distribution, even asymptotically, as the W_N^2 statistic. Darling (1955)[13] showed that C_N^2 is parameter-free when the proper estimate of β is used for any sample size N. Moreover, the distribution of C_N^2 converges approximately to a distribution with mean 0.09259 and variance 0.00435 as $N \to \infty$ when H_2 is true.

Using $\tilde{\beta}$ as the proper estimate for β, then the modified statistic is given by

$$C_N^2 = \frac{1}{12N} + \sum_{j=1}^{N} \left(Z_j^{\tilde{\beta}} - \frac{2j-1}{2N} \right)^2, \tag{2.42}$$

where

$$\tilde{\beta} = \frac{N-1}{\sum_{q=1}^{k} \sum_{i=1}^{N_q} \ln \left(\frac{w_q}{t_{i_q}} \right)}. \tag{2.43}$$

The critical values of the C_N^2 statistic have been determined at the U.S. Army Material Systems Analysis Activity from Monte Carlo simulation, using 15,000 samples for each N. The various critical values of the C_N^2 are shown in Table 2.1 (Crow, 1974)[10]. If the statistic C_N^2 is greater than the selected critical value, then the hypothesis H_2 is rejected. It means the failure times for k systems follow a NHPP with intensity function $\nu(t) = \gamma\beta t^{\beta-1}$ at the designated significance level.

Table 2.1

Critical Values of C_N^2

N	Level of significance					N	Level of significance				
	0.20	0.15	0.10	0.05	0.01		0.20	0.15	0.10	0.05	0.01
3	0.121	0.135	0.154	0.183	0.231	32	0.127	0.145	0.169	0.214	0.330
4	0.121	0.136	0.156	0.195	0.278	33	0.127	0.144	0.169	0.215	0.337
5	0.123	0.138	0.160	0.202	0.305	34	0.126	0.143	0.171	0.213	0.334
6	0.123	0.139	0.163	0.206	0.315	35	0.127	0.144	0.170	0.215	0.326
7	0.124	0.141	0.166	0.207	0.305	36	0.126	0.144	0.169	0.213	0.331
8	0.124	0.141	0.165	0.209	0.312	37	0.127	0.145	0.170	0.215	0.339
9	0.124	0.141	0.167	0.212	0.324	38	0.127	0.145	0.170	0.217	0.331
10	0.124	0.142	0.169	0.213	0.321	39	0.127	0.145	0.173	0.218	0.334
11	0.124	0.142	0.166	0.216	0.324	40	0.128	0.146	0.172	0.220	0.335
12	0.125	0.143	0.170	0.213	0.323	41	0.128	0.146	0.173	0.218	0.335
13	0.126	0.143	0.168	0.218	0.337	42	0.128	0.146	0.172	0.217	0.333
14	0.126	0.142	0.169	0.213	0.331	43	0.127	0.146	0.172	0.217	0.334
15	0.125	0.144	0.169	0.215	0.335	44	0.128	0.147	0.173	0.218	0.341
16	0.125	0.143	0.169	0.214	0.329	45	0.128	0.146	0.172	0.217	0.342
17	0.126	0.143	0.169	0.216	0.334	46	0.129	0.146	0.172	0.216	0.346
18	0.126	0.143	0.170	0.216	0.339	47	0.128	0.147	0.173	0.216	0.343
19	0.126	0.143	0.169	0.214	0.336	48	0.128	0.145	0.172	0.219	0.343
20	0.127	0.145	0.169	0.217	0.342	49	0.127	0.145	0.171	0.218	0.335
21	0.126	0.145	0.170	0.216	0.332	50	0.127	0.145	0.172	0.219	0.345
22	0.126	0.144	0.171	0.216	0.337	51	0.128	0.146	0.173	0.220	0.344
23	0.127	0.144	0.169	0.217	0.343	52	0.127	0.146	0.172	0.216	0.346
24	0.126	0.143	0.169	0.216	0.339	53	0.127	0.146	0.172	0.218	0.348
25	0.127	0.145	0.170	0.216	0.342	54	0.127	0.146	0.172	0.219	0.351
26	0.127	0.145	0.171	0.215	0.333	55	0.127	0.145	0.173	0.219	0.356
27	0.127	0.144	0.170	0.215	0.335	56	0.127	0.145	0.172	0.221	0.355
28	0.127	0.145	0.170	0.218	0.334	57	0.127	0.145	0.171	0.218	0.352
29	0.127	0.146	0.171	0.217	0.334	58	0.127	0.145	0.171	0.321	0.353
30	0.127	0.145	0.172	0.218	0.328	59	0.128	0.146	0.171	0.222	0.350
31	0.127	0.145	0.170	0.215	0.328	60	0.127	0.146	0.172	0.219	0.352

CHAPTER 3

PROPOSED GENERALIZED CONFIDENCE INTERVAL FOR THE SCALE PARAMETER OF THE POWER-LAW PROCESS WITH INCOMPLETE FAILURE DATA

In this chapter, we present details of the proposed generalized confidence interval (PGCI) for the scale parameter of the PLP with incomplete failure data in case of failure truncated. Then, we compare the proposed method with the existing confidence interval given in Yu et al. (2008)[47] to determine which method is better to assess the system reliability during the developmental phase when number of failures is small and some recorded failure times in the early developmental phase cannot be observed.

3.1 Introduction

In practical situations, incomplete failure data are a common and expected occurrence during the developmental phase of system testing. Various types of incompleteness can and do occur in the aspect of missing positions for the failure data. One of the most frequently encountered scenarios concerns missing position located in the early developmental phase of system testing. This type of missing pattern can be caused by many reasons. For example, a new data-recording engineer may not have the expertise to determine the exact failure times during the early stage of the development process due to lack of experience.

Another example to which this situation can be extended is that an investigation with the objective to forecast building maintenance requirements at a military base. It is discovered that many records of past maintenance activities are available, but in some cases the early data has been lost. In this study, we will consider the situation in which missing data occur in the early developmental phase of system testing. That is, we assume that $t_1, t_2, ..., t_{r-1}$ ($1 \leq r < n$) are missing failure times and the observed failure times are $t_r, t_{r+1}, ..., t_n$.

3.2 Maximum Likelihood Estimates for Parameters of the Power-Law Process with Missing Data

The concept of maximum likelihood estimation (MLE) of parameters with missing data was first proposed by Dempster et al. in 1977 (Dempster et al., 1977)[14]. Complete observations $g(Y_{obs}, Y_{miss}|\gamma, \beta)$ are related to the missing data specification $f(Y_{obs}|\gamma, \beta)$ by

$$f(Y_{obs}|\gamma, \beta) = \int g(Y_{obs}, Y_{miss}|\gamma, \beta) dY_{miss}. \tag{3.1}$$

Here, we assume that the failure process follows the PLP with intensity function

$$\nu(t) = \gamma \beta t^{\beta-1}, \tag{3.2}$$

where $\gamma > 0$ and $\beta > 0$ are a scale and shape parameter, respectively. Therefore, the MLEs of parameters γ and β of the PLP can be obtained by determining values of γ and β which maximize $f(Y_{obs}|\gamma, \beta)$ given an observed observations.

Suppose $t_1, t_2, ..., t_{r-1}$ are missing, then the observed data are defined as $Y_{obs} = t_r, t_{r+1}$, ..., t_n for the failure-truncated case. To obtain the joint probability density function (pdf) of the observed data, we integrate the joint probability density function of complete observations $(t_1, t_2, ..., t_n)$:

$$f(t_1, t_2, ..., t_n) = (\gamma\beta)^n exp(-\gamma t_n^{\beta}) \prod_{i=1}^{n} t_i^{\beta-1}, 0 < t_1 < t_2 < ... < t_n, \quad (3.3)$$

with respect to $t_i, i = 1, 2, ..., r-1$ using two following identities:

$$\int_{D(m;a,b)} dt_1 dt_2 ... dt_m = \frac{(b-a)^m}{m!}, \quad (3.4)$$

and

$$\int_{D(m;a,b)} dF(t_1)dF(t_2)...dF(t_m) = \frac{(F(b)-F(a))^m}{m!}, \quad (3.5)$$

where m is any positive integer, a and b are any real numbers such that $a < b$, $F(t)$ is any increasing and differentiable function, and $D(m; a, b) = (t_1, t_2, ..., t_m)^T$, $a < t_1 < t_2 < ... < t_m < b$ (Yu et al. 2008[47]).

Let $F(t) = t^{\beta}/\beta$, $a = 0$, $b = t_r$, and $m = r-1$, we then obtain the likelihood function of $Y_{obs} = t_r, t_{r+1}, ..., t_n$ using (3.4) and (3.5) as follows:

$$L(\gamma, \beta) = f(t_r, ..., t_n) = \frac{\gamma^n \beta^{n+1-r} \text{exp}(-\gamma t_n^{\beta})}{(r-1)!} t_r^{(r-1)\beta} \prod_{i=r}^{n} t_i^{\beta-1}, 0 < t_r < t_{r+1} < ... < t_n, \quad (3.6)$$

and the log-likelihood function is

$$\text{ln}L(\gamma, \beta) = n\text{ln}\gamma + (n+1-r)\text{ln}\beta - \gamma t_n^{\beta} + (r-1)\beta\text{ln}t_r + (\beta-1)\sum_{i=r}^{n} \text{ln}t_i - \text{ln}(r-1)!. \quad (3.7)$$

By maximizing $\text{ln}L(\gamma, \beta)$ as defined in (3.7), the MLEs of γ and β are found to be

$$\hat{\gamma} = \frac{n}{t_n^{\hat{\beta}}}, \quad (3.8)$$

and

$$\hat{\beta} = \frac{n-r+1}{\sum_{i=r+1}^{n-1} \text{ln}\left(\frac{t_n}{t_i}\right) + r\text{ln}\left(\frac{t_n}{t_r}\right)}, \tag{3.9}$$

respectively.

Theorem 2

Let $U = 2(n-r+1)\beta/\hat{\beta}$ and $V = 2\gamma T_n^{\beta}$ then $U \sim \chi^2_{(2n-2r)}, V \sim \chi^2_{(2n)}$, and they are mutually independent.

Proof: Using the transformation, we first let

$$y_i = \gamma t_i^{\beta}, \tag{3.10}$$

where $i = 1, 2, ..., n$, then y_i become successive failure times from a homogeneous Poisson process (HPP) with a unit intensity function. The property of HPP indicates that the sequence of inter-arrival times, which is denoted by $y_i - y_{i-1}, i = 1, 2, ..., n$, are i.i.d. standard exponential random variables, where $y_0 = 0$. Therefore,

$$Y_n = \sum_{i=1}^{n}(y_i - y_{i-1}) \sim \Gamma(n, 1), \tag{3.11}$$

and the moment generating function (mgf) of Y_n is defined as follows:

$$M_{Y_n}(t) = E[e^{tY_n}] = \left(\frac{1}{1-t}\right)^n, \tag{3.12}$$

where $t < 1$ (Casella and Berger, 2001[6]).

Let $V = 2Y_n$. Then, the moment generating function of V can be obtained as

$$\begin{aligned} M_V(t) &= E[e^{tV}] \\ &= E[e^{2tY_n}] \\ &= \left(\frac{1}{1-2t}\right)^n, \end{aligned} \tag{3.13}$$

where $t < \dfrac{1}{2}$. Thus, $V = 2Y_n$ has a chi-square distribution with $2n$ degrees of freedom.

For the distribution of U, we have

$$U = \frac{2(n-r+1)\beta}{\hat{\beta}} = 2\beta\left[\sum_{i=r+1}^{n-1} \ln\left(\frac{t_n}{t_i}\right) + r\ln\left(\frac{t_n}{t_r}\right)\right]. \tag{3.14}$$

Let $z_i = \ln\left(\dfrac{y_n}{y_{n+r-1-i}}\right),\ i = r, r+1, ..., n-1$, and y_i is defined in (3.7). We have

$$Z = \sum_{i=r}^{n-2} z_i + r z_{n-1} \sim \Gamma(n-r, 1), \tag{3.15}$$

and the MGF of Z is defined as follows:

$$M_Z(t) = E[e^{tZ}] = \left(\frac{1}{1-t}\right)^{n-r}, \tag{3.16}$$

where $t < 1$.

As a result, we get

$$U = 2\left[\sum_{i=r+1}^{n-1} \ln\left(\frac{y_n}{y_i}\right) + r\ln\left(\frac{y_n}{y_r}\right)\right] = 2Z \sim \chi^2_{(2n-2r)}. \tag{3.17}$$

■

Theorem 3

If the MLE of β, $\hat{\beta}$, is defined as (3.9), then $\hat{\beta}$ is a biased estimator of β and its expectation and variance are defined as:

$$E(\hat{\beta}) = \frac{(n-r+1)\beta}{(n-r-1)}, \tag{3.18}$$

and

$$Var(\hat{\beta}) = \frac{(n-r+1)^2\beta^2}{(n-r-1)^2(n-r-2)}, \tag{3.19}$$

respectively.

Proof: By Theorem 2 and Lemma 1 (in Chapter 2), we have

$$\begin{aligned}
\hat{\beta} &= \frac{2(n-r+1)\beta}{U} \\
E(\hat{\beta}) &= 2(n-r+1)\beta E\left(\frac{1}{U}\right) \\
&= 2(n-r+1)\beta E(U^{-1}) \\
&= 2(n-r+1)\beta \left[\frac{2^{-1}\Gamma\left(\frac{2n-2r}{2}-1\right)}{\Gamma\left(\frac{2n-2r}{2}\right)}\right] \\
&= \frac{(n-r+1)\beta}{(n-r-1)}.
\end{aligned}$$

We also have

$$\begin{aligned}
\hat{\beta}^2 &= \left[\frac{2(n-r+1)\beta}{U}\right]^2 \\
E(\hat{\beta}^2) &= 4(n-r+1)^2\beta^2 E\left(\frac{1}{U^2}\right) \\
&= 4(n-r+1)^2\beta^2 E(U^{-2}) \\
&= 4(n-r+1)^2\beta^2 \left[\frac{2^{-2}\Gamma\left(\frac{2n-2r}{2}-2\right)}{\Gamma\left(\frac{2n-2r}{2}\right)}\right] \\
&= \frac{(n-r+1)^2\beta^2}{(n-r-1)(n-r-2)}.
\end{aligned}$$

Therefore, the variance of $\hat{\beta}$ can be obtained as

$$\begin{aligned} Var(\hat{\beta}) &= E(\hat{\beta}^2) - [E(\hat{\beta})]^2 \\ &= \frac{(n-r+1)^2\beta^2}{(n-r-1)(n-r-2)} - \left[\frac{(n-r+1)\beta}{(n-r-1)}\right]^2 \\ &= \frac{(n-r+1)^2\beta^2}{(n-r-1)^2(n-r-2)}. \end{aligned}$$

■

3.3 Unbiased Estimate for the Shape Parameter of the Power-Law Process with Missing Data

Since the MLE of β is biased estimate, we can adjust $\hat{\beta}$ to unbiased estimate, and it is defined as

$$\tilde{\beta} = \left(\frac{n-r-1}{n-r+1}\right)\hat{\beta}. \tag{3.20}$$

Thus, the expectation and variance of $\tilde{\beta}$ are

$$E(\tilde{\beta}) = \beta, \tag{3.21}$$

and

$$Var(\tilde{\beta}) = \frac{\beta^2}{(n-r-2)}, \tag{3.22}$$

respectively.

3.4 Generalized Confidence Intervals

Consider a population represented by an observable random variable X. Let $\mathbf{X}$ = $(X_1, X_2, ..., X_n)$ be a random sample from the population. Suppose the distribution of X is known except for a vector of parameters $\boldsymbol{\zeta} = (\theta, \boldsymbol{\delta})$, where θ is a parameter of interest and $\boldsymbol{\delta}$ is a vector of nuisance parameters. Generally, θ could be a vector of parameters, but we first assume that there is only one parameter of interest and we are interested to find a confidence interval for θ based on observed values of $\mathbf{X}$. The problem is to construct generalized confidence intervals of the form $[A(\mathbf{x}), B(\mathbf{x})] \subset \boldsymbol{\Theta}$, where $A(\mathbf{x})$ and $B(\mathbf{x})$ are functions of the observed data $\mathbf{x}$ (Weerahandi, 2004[45]).

In the classical approach, we find two functions of the observed random vector, $A(\mathbf{X})$ and $B(\mathbf{X})$, such that

$$Pr[A(\mathbf{X}) \leq \theta \leq B(\mathbf{X})] = 1 - \alpha \tag{3.23}$$

is satisfied, where $1 - \alpha$ is the desired confidence level. If it is possible to find $A(\mathbf{X})$ and $B(\mathbf{X})$ that do not depend on unknown parameters, then we compute $a = A(\mathbf{x})$ and $b = B(\mathbf{x})$ using the observed value $\mathbf{x}$ and call $[a, b]$ a $100(1 - \alpha)\%$ confidence interval. The nominal values of $1 - \alpha$ typically used in many situations are 0.9, 0.95, and 0.99. For example, if $1 - \alpha = 0.95$, then the interval $[a, b]$ is called a 95$\%$ confidence interval. The interval obtained in this manner has the property that, in repeated sampling, the interval will contain the true value of parameter θ 100(1-α)$\%$ of the times.

3.4.1 Definition of Generalized Pivotal Quantity

In many situations, it is not easy or impossible to find $A(\mathbf{X})$ and $B(\mathbf{X})$ satisfying (3.23) for all possible values of the nuisance parameters. Weerahandi (1993)[44] showed how this can be achieved by making probability statements relative to the observed sample, but without having to treat unknown parameters as random variables. More specifically, we allow two functions, $A()$ and $B()$, to depend on the observable random vector $\mathbf{X}$ and the observed data **x** both. The construction of regions can be facilitated by generalizing the classical definition of pivotal quantity.

Definition 1 (Pivotal quantity)

A random variable $Q(\mathbf{X}, \theta) = Q(X_1, X_2, ..., X_n, \theta)$ *is a pivotal quantity (or pivot) if the distribution of* $Q(\mathbf{X}, \theta)$ *is independent of all parameters (Casella and Berger, 2001[6]).*

Definition 2 (Generalized pivotal quantity)

A random variable of the form $R = R(\mathbf{X}; \mathbf{x}, \boldsymbol{\zeta})$*, a function of* $\mathbf{X}, \mathbf{x}, \boldsymbol{\zeta}$*, is said to be a generalized pivotal quantity if it has the following two properties:*

(i) The probability distribution of R *does not depend on unknown parameters.*

(ii) The observed pivotal quantity, defined as $r_{obs} = R(\mathbf{x}; \mathbf{x}, \boldsymbol{\zeta})$*, does not depend on the nuisance parameter,* $\boldsymbol{\delta}$*.*

Property (i) is defined to allow us to write probability statements leading to confidence regions that can be assessed regardless of the values of unknown parameters. Property (ii) is defined to guarantee that probability statements based on a generalized pivotal quantity will lead to confidence regions without knowing the values of nuisance parameters.

Suppose we have constructed a generalized pivotal $R = R(\mathbf{X}; \mathbf{x}, \boldsymbol{\zeta})$ for a parameter of interest, and we want to construct a confidence region at the confidence level $1 - \alpha$. Then, a subset $C_{1-\alpha}$ of the sample space of R is defined such that

$$Pr(R \in C_{1-\alpha}) = 1 - \alpha. \tag{3.24}$$

The region defined by 3.24 also specifies a subset $C(\mathbf{x}; \theta)$ of the original sample space satisfying the equation

$$Pr(\mathbf{X} \in C(\mathbf{x}; \theta)) = 1 - \alpha. \tag{3.25}$$

This region depends not only on $1 - \alpha$ and θ, but also on the observed data **x**. If R is a continuous random variable, then $C_{1-\alpha}$ can be found (Weerahandi, 1993[44]).

3.4.2 Substitution Method

This method requires that there is a set of observed statistics with unknown distributions that are equal in number to the number of unknown parameters, $(\delta_1, \delta_2, ..., \delta_k)$. Consider a set of observed statistics $(X_1, X_2, ..., X_k)$ with the observed values $(x_1, x_2, ..., x_k)$. It is assumed that through a set of random variables having distributions free of unknown parameters, the statistics are related to the unknown parameters. In many applications, this would be a set of sufficient statistics with known distributions that can be transformed into distributions free of unknown parameters.

Let $\mathbf{V} = (V_1, V_2, ..., V_k)$ be a set of random variables with distributions free of unknown parameters, and it is assumed that the joint distribution of the random vector **V** is known. To find generalized pivotal quantities, the substitution method is carried out in the following steps:

Step 1. Express the parameter of interest, θ, in terms of the sufficient statistics $(X_1, X_2, ..., X_k)$ and the random variables $(V_1, V_2, ..., V_k)$.

Step 2. Define a potential generalized pivotal quantity, say R, by replacing the statistics $(X_1, X_2, ..., X_k)$ by their observed values $\mathbf{x} = (x_1, x_2, ..., x_k)$ and argue that the distribution of R is free of unknown parameters.

Step 3. Rewrite $(V_1, V_2, ..., V_k)$ terms appearing in R in terms of $\mathbf{X}$ and $\boldsymbol{\delta}$ and show that when $\mathbf{X} = \mathbf{x}$, the observed values of the quantity $R(\mathbf{x}; \mathbf{x}, \boldsymbol{\delta})$ does not depend on the nuisance parameters, where $\mathbf{X} = (X_1, X_2, ..., X_k)$ and $\boldsymbol{\delta} = (\delta_1, \delta_2, ..., \delta_k)$.

3.5 Proposed Generalized Confidence Interval for the Scale Parameter of the Power-Law Process

The generalized confidence interval for the scale parameter γ of the PLP for complete failure data has been proposed by Wang et al. (2013)[43]. In this study, we will derive the generalized confidence interval for γ with incomplete failure data ([**?**]).

From Theorem 3, we use the result that $U = 2(n-r+1)\beta/\hat{\beta}$ and $V = 2\gamma T_n^{\beta}$ has a chi-square distribution with degree of freedom $2(nr)$ and $2n$, respectively. Here, γ is the parameter of interest, and β is the nuisance parameter. Using the substitution method to obtain the generalized pivotal quantity for γ, We first rewrite the result from Theorem 3 as follows:

$$\beta = \frac{\hat{\beta}U}{2(n-r+1)}, \tag{3.26}$$

$$\gamma = \frac{V}{2T_n^{\beta}}. \tag{3.27}$$

Then, we substitute $\hat{\beta}U/2(n-r+1)$ for β in the expression for γ as:

$$\gamma = \frac{V}{2T_n^{U/(2\tau)}}, \tag{3.28}$$

where $\tau = \sum_{i=r+1}^{n-1} \ln\left(\frac{t_n}{t_i}\right) + r\ln\left(\frac{t_n}{t_r}\right)$.

Using Step 2 of the substitution method, we replace the statistics T_n by its observed values t_n and define the following generalized pivotal quantity

$$W = W(\mathbf{T}; \mathbf{t}, \gamma, \beta) = \frac{V}{2t_n^{U/(2\tau)}}. \tag{3.29}$$

We shall now show that the distribution of W is free of unknown parameters.

Theorem 4

Suppose the generalized pivotal quantity W is defined as Equation (3.29), then the cumulative distribution function of W is given by

$$F_W(w) = 1 - \int_0^\infty g(u) e^{-y} \sum_{j=0}^{n-1} \frac{y^j}{j!} du, \tag{3.30}$$

where $y = w t_n^{U/(2\tau)}$, $\tau = \sum_{i=r+1}^{n-1} log\left(\frac{t_n}{t_i}\right) + r log\left(\frac{t_n}{t_r}\right)$, *and* $g(u)$ *is the probability density function (pdf) of* $\chi^2_{(2n-2r)}$ *and* $g(u) = \frac{1}{\Gamma(n-r)2^{n-r}} u^{(n-r)-1} e^{-u/2}$.

Proof: To find the cumulative distribution function of W, we use the definition of the cumulative distribution function which is given by

$$F_W(w) = P(W \leq w). \tag{3.31}$$

Substituting $\frac{V}{2t_n^{U/(2\tau)}}$ for W in Equation (3.31), it yields

$$\begin{aligned} F_W(w) &= P(\frac{V}{2t_n^{U/(2\tau)}} \leq w) \\ &= P(V \leq 2w t_n^{U/(2\tau)}). \end{aligned} \tag{3.32}$$

Notice that the distributions of U and V are known, so $P(V \leq 2w t_n^{U/(2\tau)})$ can be obtained using the conditioning on $U = u$ as follows:

$$F_W(w) = \int_0^\infty P(V \leq 2w t_n^{U/(2\tau)} | U = u) g(u) du. \tag{3.33}$$

Let $y = w t_n^{U/(2\tau)}$, then we get

$$F_W(w) = \int_0^\infty P(V \leq 2y | U = u) g(u) du. \tag{3.34}$$

Recall the cumulative distribution function of $\chi^2_{(2n-2r)}$, and let $F(y)$ be the cumulative distribution function of $\chi^2_{(2n)}$ and $F(y)$ is defined as

$$F(y) = 1 - e^{-y/2} \sum_{j=0}^{n-1} \frac{(y/2)^j}{j!}, \ y > 0 \tag{3.35}$$

(Casella and Berger, 2001). Therefore, Equation (3.34) can be written as

$$\begin{aligned} F_W(w) &= \int_0^\infty F(2y) g(u) du \\ &= \int_0^\infty \left(1 - e^{-(2y)/2} \sum_{j=0}^{n-1} \frac{((2y)/2)^j}{j!} \right) g(u) du, \end{aligned}$$

and the cumulative distribution function of W is then given by

$$F_W(w) = 1 - \int_0^\infty g(u) e^{-y} \sum_{j=0}^{n-1} \frac{y^j}{j!} du. \tag{3.36}$$

■

It is clearly that the distribution of W is free of unknown parameters; it depends only on the degree of freedom of a chi-square distribution.

Next, we rewrite the random variable U and V appearing in W in terms of $\mathbf{T} = (T_r, T_{r+1}, ..., T_n)$ and β as follows:

$$\begin{aligned} W &= \frac{V}{2t_n^{U/(2\tau)}} \\ &= \frac{2\gamma T_n^\beta}{2t_n^{[2(n-r+1)\beta/\hat{\beta}]/(2\tau)}} \\ &= \frac{\gamma T_n^\beta}{t_n^{\beta[\sum_{i=r+1}^{n-1} \ln(T_n/T_i) + r \ln(T_n/T_r)]/\tau}}, \end{aligned} \tag{3.37}$$

where $\tau = \sum_{i=r+1}^{n-1} \ln\left(\frac{t_n}{t_i}\right) + r\ln\left(\frac{t_n}{t_r}\right)$. When $\mathbf{T} = (T_r, T_{r+1}, ..., T_n)$ is replaced by its observed value $\mathbf{t} = (t_r, t_{r+1}, ..., t_n)$, we obtain the following result

$$\begin{aligned} W_{obs} &= W(\mathbf{t}; \mathbf{t}, \gamma, \beta) \\ &= \frac{\gamma t_n^{\beta}}{t_n^{\beta[\sum_{i=r+1}^{n-1} \ln(t_n/t_i)+r\ln(t_n/t_r)]/\tau}} \\ &= \gamma. \end{aligned} \quad (3.38)$$

The result (3.38) shows that $W_{obs} = W(\mathbf{t}; \mathbf{t}, \gamma, \beta) = \gamma$ does not depend on the nuisance parameter β. Therefore, W is a generalized pivotal quantity.

Based on the distribution of W, a $100(1-\alpha)\%$ generalized confidence interval for parameter γ can be obtained as follows:

$$[W_{\alpha/2}, W_{1-\alpha/2}], \quad (3.39)$$

where $W_{\alpha/2}$ and $W_{1-\alpha/2}$ denote the $(\alpha/2)^{th}$ and $(1-\alpha/2)^{th}$ percentile of the distribution of W, respectively. Consequently, a test for the hypotheses $H_0 : \gamma = \gamma_0$ versus $H_1 : \gamma > \gamma_0$ is to reject H_0 if $\hat{\gamma} > W_{\alpha}$.

Notice that the $(\alpha/2)^{th}$ and $(1-\alpha/2)^{th}$ percentile of W ($W_{\alpha/2}$ and $W_{1-\alpha/2}$) can be estimated from the quantity $W = V/2t_n^{U/(2\tau)}$ using the following simulation algorithm.

Algorithm 1

Step 1: For given failure times $t_r, t_{r+1}, ..., t_n$, compute $\tau = \sum_{i=r+1}^{n-1} log\left(\frac{t_n}{t_i}\right) + rlog\left(\frac{t_n}{t_r}\right)$.

Step 2: Generate $U \sim \chi^2_{(2n-2r)}$ and $V \sim \chi^2_{(2n)}$, independently. Then, compute $W = \frac{V}{2t_n^{U/(2\tau)}}$.

Step 3: Repeat Step 2 m times.

Notice that a numerical error results by computing the improper integral for $F_W(w)$ in (3.30), when we truncate the integral at some finite value. For example, if the integration range is from 0 to m, then the error is given by

$$\epsilon_m = \int_m^\infty g(u)e^{-y}\sum_{j=0}^{n-1}\frac{y^j}{j!}du. \tag{3.40}$$

For any specified $\epsilon > 0$, we can choose a sufficiently large value for m to make $\epsilon_m < \epsilon$. Then, we can evaluate the improper integral for $F_W(w)$ in (3.30) to the desired accuracy.

Step 4: Arrange all W values in ascending order: $W_{(1)} < W_{(2)} < ... < W_{(m)}$. Then, the α^{th} percentile of W is estimated by $W_{(\alpha m)}$.

3.6 Existing Confidence interval for the Scale Parameter of the Power-Law Process

Yu et al. (2008) consider the intensity function $(\beta/\theta)(t/\theta)^{\beta-1}$ which reduces to (1.17) when $\theta = \gamma^{-1/\beta}$, and the MLE of θ is given by

$$\hat{\theta} = \frac{t_n}{n^{1/\hat{\beta}}}. \tag{3.41}$$

Let $Z = (\hat{\theta}/\theta)^{\hat{\beta}} = (1/n)(V/2)^{2(n-r+1)/U}$. Yu et al. (2008) show that the distribution of Z is free from unknown parameters. Then, a two-sided $(\alpha^*)100\%$ confidence interval for θ is given by

$$(\hat{\theta}(z_{(1+\alpha^*)/2})^{-1/\hat{\beta}}, \hat{\theta}(z_{(1-\alpha^*)/2})^{-1/\hat{\beta}}), \tag{3.42}$$

where $\hat{\beta}$ is the MLE of β and Z_{α^*} denotes the α^* quantile of the distribution of Z (see Appendix B), and the confidence interval for γ can also be obtained using the invariance property of the MLE.

3.7 Simulation Study

In this section, we design a simulation study to investigate the influence of the predetermined number of failures (n) and the number of missing failures in the early testing stage ($r-1$) toward the proposed generalized confidence interval (PGCI) and the existing confidence interval (CCI) for the scale parameter γ.

On the basis of the most common parameters in practical cases, we select some of the parameter configurations of Wang et al. (2013)[**?**], $\beta = 1, \gamma = 0.1$, and propose other levels of parameters as $\beta = 0.5, 1.5, \gamma = 0.05, 0.5$, and calculate the 95$\%$ confidence intervals (α = 0.05) as two-sided with equal tail probabilities. Numbers of failures considered are 10, 20, 30, and 40, and numbers of missing failures considered are 0, 1, 2, ..., (0.2*n* - 1) or $r = 1, 2, ..., 0.2n$, respectively. For each parameter configuration, we generate 10,000 random samples from *W* and *Z* distributions and use 1,000 simulation replications.

The coverage probability is accepted as the most important attribute of a confidence interval. Thus, we prefer to use this criterion to evaluate the performance of the confidence interval with probability close to the nominal level $1-\alpha$. Moreover, we also determine the bias by examining how the confidence interval fails when it does not cover the true parameter γ. When the confidence interval is more likely to be lower than the true parameter, we consider the method to be *negatively biased*. When the confidence interval is more likely to be higher than the true parameter, we consider the method to be *positively biased*. Another criterion is the width of the confidence interval. If two or more intervals have similar coverage probabilities, the shortest one is superior to the others because it indicates the preciseness. Therefore, to compare performances of the PGCI and CCI methods, the following criteria are considered: coverage probability, coverage error, relative bias, and average width of the resulting confidence intervals.

3.7.1 Simulation Procedure

We use the following procedure to investigate the accuracy of proposed methods.

1. Generate the NHPP power law data with $r-1$ missing failure times (see Appendix A).
2. Calculate confidence intervals using PGCI and CCI methods.
3. Consider if the true value of parameter γ falls in the interval in 2).
4. Repeat 1) to 3) 1,000 times for each situation.
5. Calculate the coverage probability, coverage error, relative bias, and average width.

(i) Coverage probability (CP) is defined as

$$CP = \frac{1}{1000}\sum_{i=1}^{1000} C_i, \tag{3.43}$$

where C_i is 0 when the interval from the i^{th} replication does not cover the true value of the parameter, and C_i is 1 when the interval from the i^{th} replication covers the true value of the parameter. To compare the coverage probability with the nominal level, we conclude that the coverage probability is close to the nominal level $1-\alpha$ if it falls in the interval

$$[P_0 - Z_{1-\alpha/2}\text{se}(\hat{P}), P_0 + Z_{1-\alpha/2}\text{se}(\hat{P})],$$

where P_0 is a confidence level, $\hat{P}$ is an approximate coverage probability from the simulation method, and $\text{se}(\hat{P})$ is a standard error of estimation, and given by

$$\text{se}(\hat{P}) = \sqrt{\frac{P_0(1-P_0)}{1000}}.$$

Therefore, for 95$\%$ confidence level, we conclude that the coverage probability is close to the nominal level if the coverage probability falls in the interval [0.9365, 0.963].

(ii) Coverage error (CE) is defined as

$$CE = |CP - 0.95|, \tag{3.44}$$

where 0.95 is the nominal level used in this study.

(iii) Relative bias (RB) is defined as

$$RB = \frac{\%CI < \gamma - \%CI > \gamma}{\%CI < \gamma + \%CI > \gamma}, \tag{3.45}$$

where $\%CI < \gamma$ and $\%CI > \gamma$ represent the percentage of the intervals falling below and above the true parameter γ, respectively.

(iv) Average width (AW) is defined as

$$AW = \sum_{i=1}^{s} L_i/s, \tag{3.46}$$

where L_i is the interval width from the i^{th} replication, and s is the total number of intervals that cover the true value of the parameter.

3.7.2 Simulation Results

The results of performance evaluations for the proposed generalized confidence interval (PGCI) and the classical confidence interval (CCI) using simulations are shown in Tables 3.1-3.4 and Figure 4-7.

It is quite clear that the PGCI is not biased estimates or overly estimates since its coverage probabilities are close to the nominal level 0.95 for all levels of γ and β, even for

Table 3.1

Coverage probability (CP), coverage error (CE), relative bias (RB), and average width (AW) of 95% CI for γ obtained from PGCI and CCI when $n = 10$

β	γ	r	PGCI				CCI			
			CP	CE	RB	AW	CP	CE	RB	AW
0.5	0.05	1	0.949	0.001	-0.0169	0.9678	0.958	0.008	-0.0492	3.3542
		2	0.943	0.007	0.0000	1.0881	0.945	0.005	-0.0313	6.2703
	0.10	1	0.955	0.005	+0.0667	1.3158	0.953	0.003	+0.0213	6.8065
		2	0.957	0.007	+0.0698	1.4526	0.957	0.007	+0.1163	14.4752
	0.50	1	0.945	0.005	+0.0545	2.6296	0.947	0.003	+0.0189	42.136
		2	0.951	0.001	+0.1429	2.8152	0.952	0.002	+0.2083	47.212
1.0	0.05	1	0.947	0.003	+0.0943	0.9809	0.940	0.010	+0.1667	3.3586
		2	0.939	0.011	+0.0820	1.0986	0.941	0.009	+0.0169	7.0585
	0.10	1	0.956	0.006	+0.1364	1.3140	0.951	0.001	+0.1020	6.0921
		2	0.952	0.002	+0.0833	1.4541	0.952	0.002	+0.1250	12.6410
	0.50	1	0.951	0.001	+0.1020	2.6625	0.957	0.007	+0.1163	29.853
		2	0.958	0.008	+0.1429	2.8346	0.957	0.007	+0.1163	51.935
1.5	0.05	1	0.947	0.003	+0.1321	0.9794	0.953	0.003	+0.1064	2.9781
		2	0.950	0.000	+0.0400	1.1214	0.950	0.000	+0.0400	6.1279
	0.10	1	0.948	0.002	+0.0385	1.2991	0.954	0.004	+0.0435	5.9064
		2	0.953	0.003	+0.0638	1.4582	0.951	0.001	+0.0612	11.4015
	0.50	1	0.958	0.008	+0.2381	2.6891	0.957	0.007	+0.2093	29.593
		2	0.955	0.005	+0.2000	2.8696	0.955	0.005	+0.2889	61.673

rather small failure numbers. It can be seen in Tables 1-3 that the coverage probabilities obtained from PGCI fall in the interval (0.9365, 0.9635) for a 95$\%$ confidence level, and the coverage errors are close to 0.

Likewise, the CCI is not biased estimate for all levels of γ and β, except when *n* = 20, γ = 0.05, β = 0.5, and *r* = 3. In this case, the confidence intervals obtained from CCI are more likely to be less than the true value of parameter γ, which can be seen from the positive sign of the relative bias (+0.0303) in Table 3.2. In this scenario, the CCI is considered as negatively biased.

Regarding average widths, Tables 3.1, 3.2 and 3.4 demonstrate that the average widths of confidence intervals obtained from PGCI increase slightly as the value of parameter γ and *r* increases for all levels of β (see Figure 3.1 and Figure 3.2), while confidence intervals obtained from CCI increase significantly (see Figure 3.3 and Figure 3.4). Moreover, the average widths of confidence intervals obtained from both methods also decrease as the predetermined number of failures (*n*) increases. When the predetermined numbers of failures are small ($n < 30$), the PGCI yields confidence intervals that have shorter average widths than CCI for all levels of γ and β. On the other hand, the average widths of confidence intervals obtained from CCI are shorter than the average widths obtained from PGCI for all levels of β when γ = 0.05 and 0.1, and the predetermined numbers of failures are large ($n \geq 30$).

When the performance of the two confidence intervals are compared and validated regarding average widths, the PGCI is superior to CCI due to the shorter average widths when the predetermined numbers of failures are small ($n < 30$). For large numbers of

Table 3.2

Coverage probability (CP), coverage error (CE), relative bias (RB), and average width (AW) of 95% CI for γ obtained from PGCI and CCI when $n = 20$

β	γ	r	PGCI				CCI			
			CP	CE	RB	AW	CP	CE	RB	AW
0.5	0.05	1	0.948	0.002	+0.1538	0.6361	0.943	0.007	+0.1930	0.6755
		2	0.950	0.000	+0.2000	0.6747	0.946	0.004	+0.1852	0.7674
		3	0.939	0.011	-0.0164	0.6973	0.934	0.016	+0.0303	0.8431
		4	0.943	0.007	+0.0175	0.7501	0.937	0.013	-0.0476	0.9839
	0.1	1	0.944	0.006	-0.1429	0.8818	0.948	0.002	0.0000	1.2408
		2	0.944	0.006	+0.0357	0.9396	0.945	0.005	+0.0545	1.3964
		3	0.946	0.004	+0.0370	0.9962	0.945	0.005	+0.0909	1.6768
		4	0.950	0.000	+0.2400	1.0603	0.950	0.000	+0.2000	1.9854
	0.5	1	0.955	0.005	-0.2444	2.1989	0.956	0.006	-0.2273	6.1866
		2	0.951	0.001	-0.0612	2.3109	0.947	0.003	-0.0943	7.1504
		3	0.950	0.000	0.0000	2.4105	0.949	0.001	-0.0196	8.3321
		4	0.950	0.000	0.0000	2.4931	0.951	0.001	0.0204	9.8216
1.0	0.05	1	0.957	0.007	0.0233	0.6125	0.952	0.002	0.0000	0.6280
		2	0.956	0.006	-0.1818	0.6363	0.954	0.004	-0.1304	0.6965
		3	0.958	0.008	-0.0952	0.6832	0.954	0.004	-0.0870	0.8098
		4	0.954	0.004	-0.0870	0.7293	0.954	0.004	-0.0435	0.9792
	0.1	1	0.949	0.001	-0.0588	0.8883	0.953	0.003	+0.1064	1.2736
		2	0.948	0.002	-0.0385	0.9309	0.947	0.003	+0.0566	1.4021
		3	0.949	0.001	+0.0588	0.9737	0.950	0.000	+0.0400	1.5874
		4	0.948	0.002	+0.1385	1.0747	0.937	0.013	+0.1429	1.8918
	0.5	1	0.947	0.003	-0.0303	2.2668	0.946	0.004	+0.0462	4.6466
		2	0.953	0.003	+0.0333	2.3387	0.949	0.001	+0.0154	5.1306
		3	0.953	0.003	-0.0169	2.4195	0.952	0.002	-0.0357	5.6422
		4	0.953	0.003	-0.1148	2.5250	0.953	0.003	-0.1525	6.3882
1.5	0.05	1	0.951	0.001	+0.0204	0.6110	0.956	0.006	+0.1364	0.6358
		2	0.949	0.001	+0.0588	0.6438	0.956	0.006	+0.0909	0.7180
		3	0.954	0.004	+0.0435	0.6884	0.955	0.005	-0.0222	0.8233
		4	0.949	0.001	-0.0196	0.7280	0.956	0.006	-0.0455	0.9514
	0.1	1	0.950	0.000	-0.2000	0.9437	0.951	0.001	-0.2245	1.4089
		2	0.952	0.002	-0.1667	0.9946	0.955	0.005	-0.1556	1.6393
		3	0.947	0.003	-0.2453	1.0375	0.957	0.007	-0.1163	1.9245
		4	0.958	0.008	-0.1905	1.1007	0.952	0.002	-0.0833	2.1620
	0.5	1	0.954	0.004	+0.0435	2.2528	0.953	0.003	+0.0638	6.3625
		2	0.951	0.001	-0.0612	2.3318	0.951	0.001	-0.0612	7.3670
		3	0.949	0.001	+0.0196	2.4374	0.946	0.004	+0.0370	8.7654
		4	0.951	0.001	+0.0204	2.5362	0.949	0.001	-0.0196	10.2482

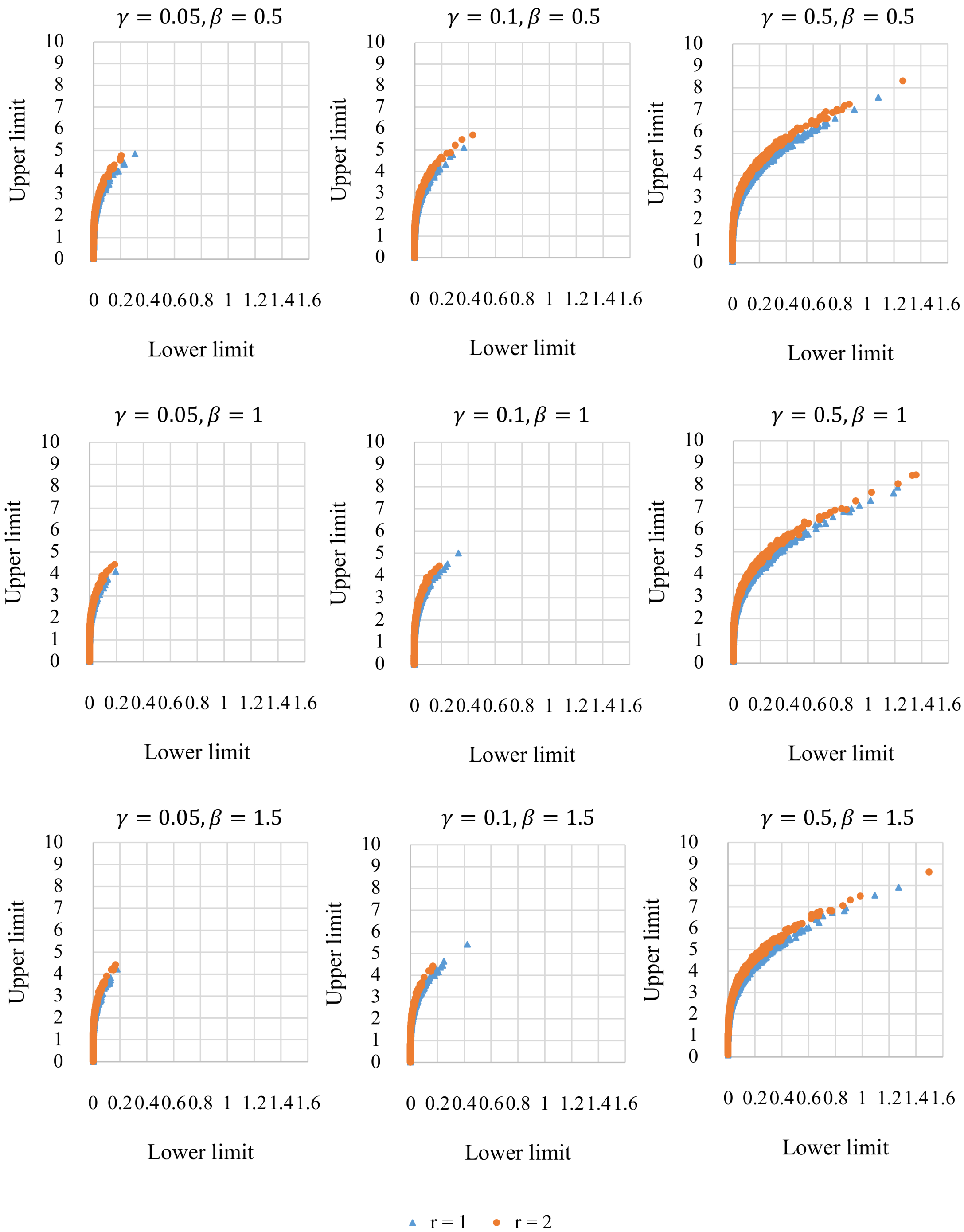


Figure 3.1

Lower limits and upper limits of 95% CI for γ obtained from PGCI when $n = 10$

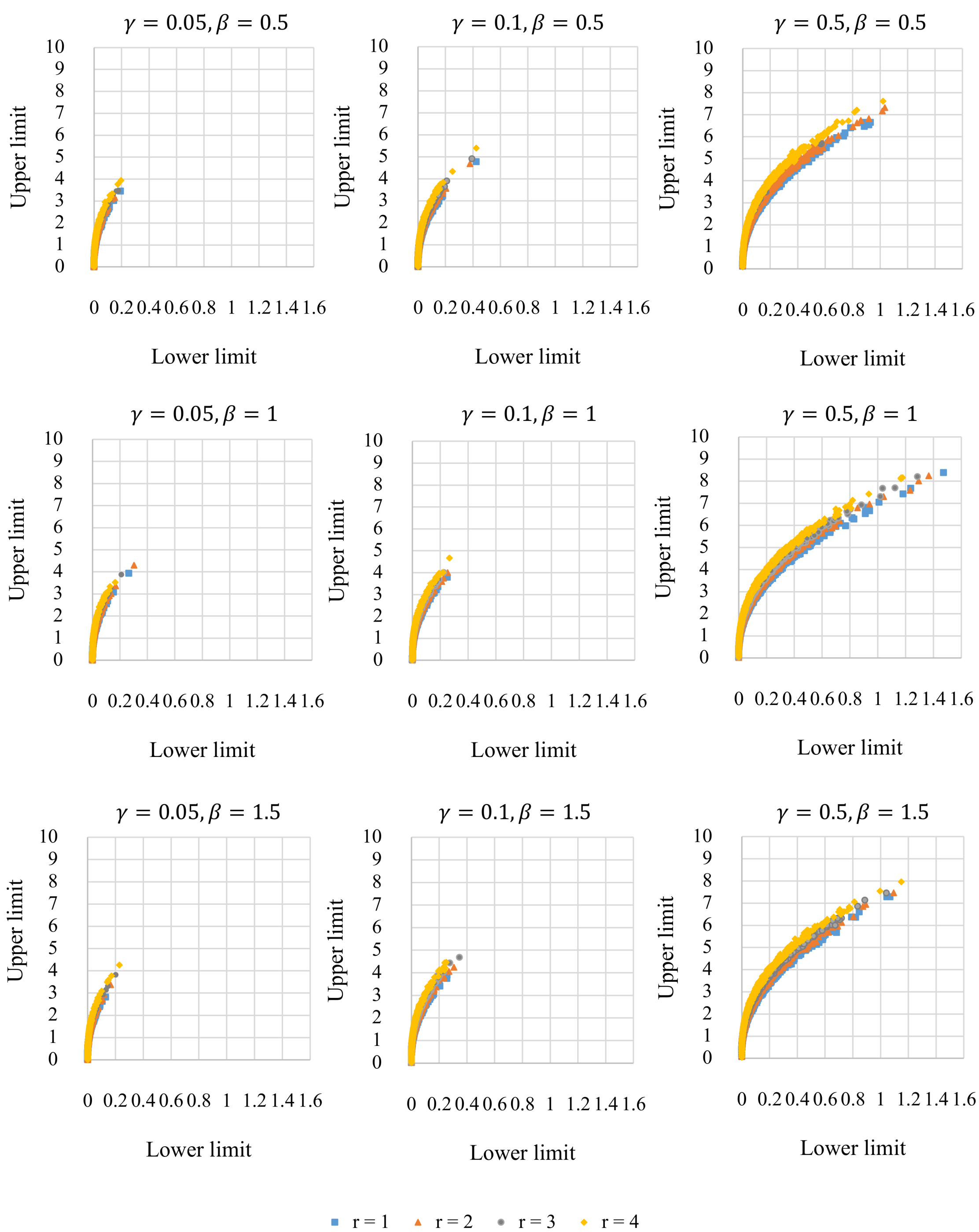


Figure 3.2

Lower limits and upper limits of 95% CI for γ obtained from PGCI when $n = 20$

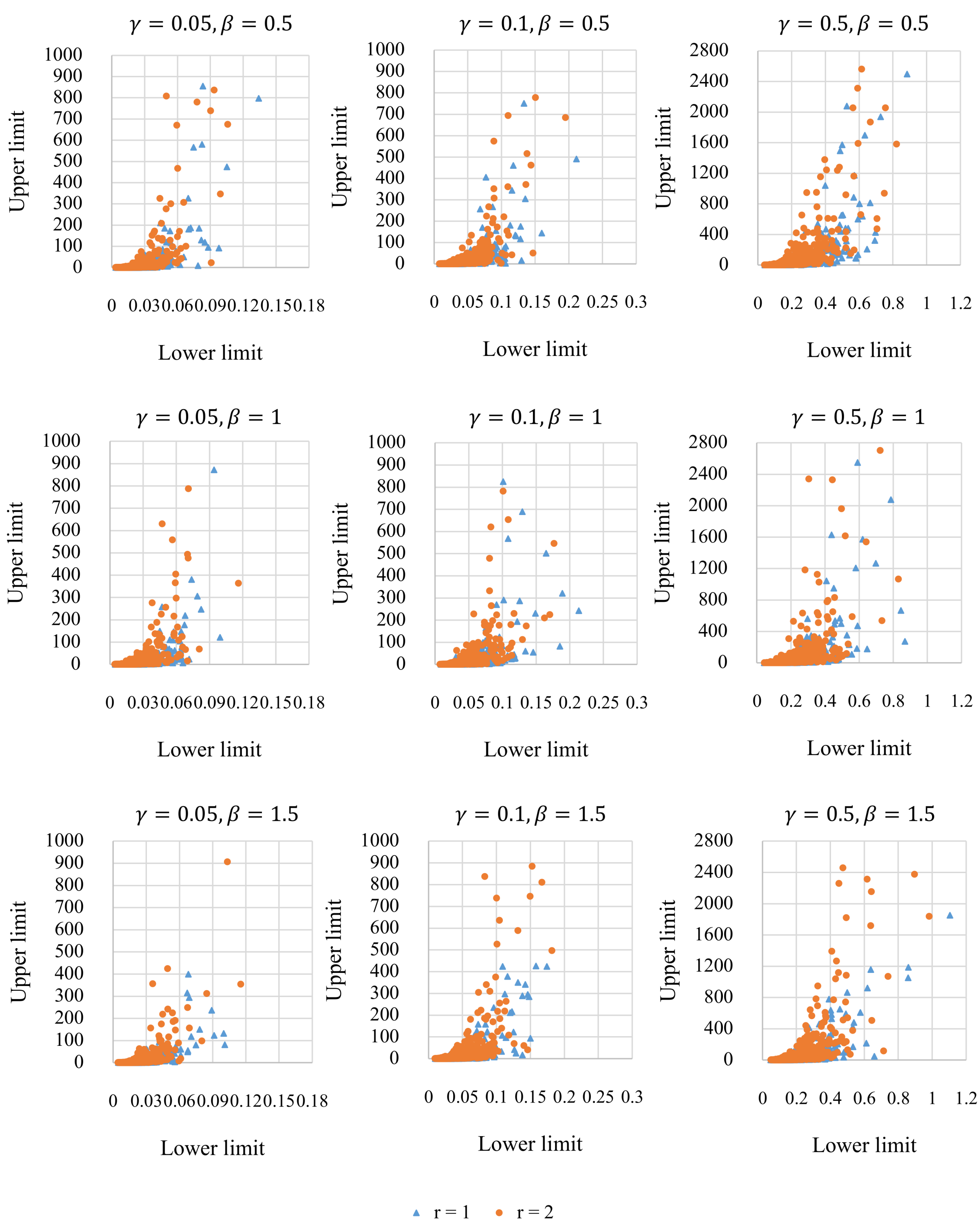


Figure 3.3

Lower limits and upper limits of 95% CI for γ obtained from CCI when $n = 10$

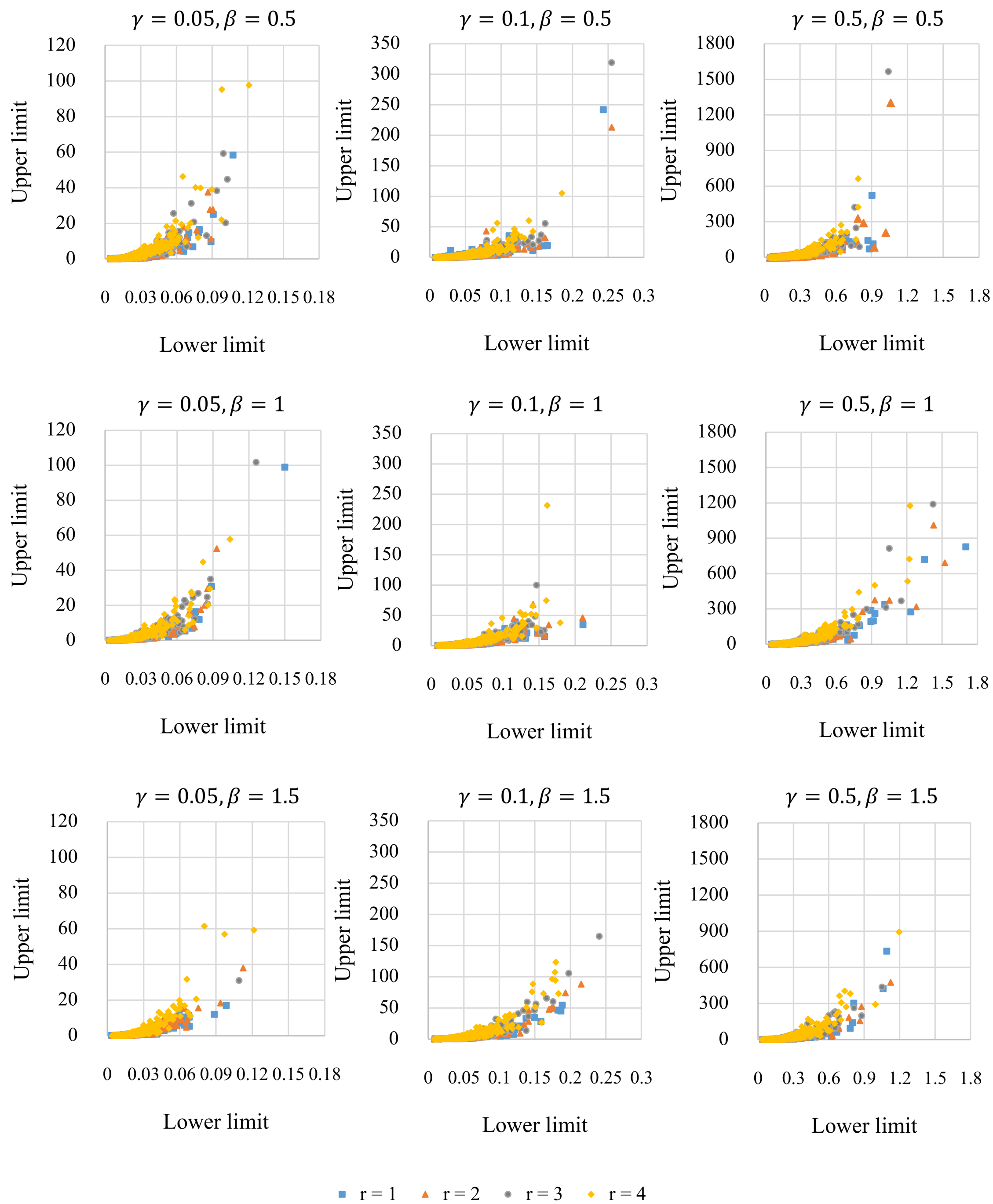


Figure 3.4

Lower limits and upper limits of 95% CI for γ obtained from CCI when $n = 20$

Table 3.3

Coverage probability of 95% CI for γ obtained from PGCI and CCI when n is large

n	r	Method	$\beta = 0.5$			$\beta = 1$			$\beta = 1.5$		
			$\gamma = 0.05$	$\gamma = 0.1$	$\gamma = 0.5$	$\gamma = 0.05$	$\gamma = 0.1$	$\gamma = 0.5$	$\gamma = 0.05$	$\gamma = 0.1$	$\gamma = 0.5$
30	1	PGCI	0.953	0.946	0.953	0.950	0.959	0.956	0.949	0.943	0.951
		CCI	0.956	0.945	0.952	0.953	0.959	0.955	0.949	0.942	0.949
	2	PGCI	0.955	0.948	0.959	0.953	0.959	0.962	0.950	0.944	0.952
		CCI	0.956	0.951	0.957	0.951	0.958	0.961	0.955	0.940	0.953
	3	PGCI	0.959	0.948	0.956	0.942	0.953	0.962	0.950	0.941	0.949
		CCI	0.958	0.949	0.958	0.937	0.958	0.962	0.952	0.937	0.947
	4	PGCI	0.955	0.946	0.956	0.942	0.962	0.957	0.956	0.943	0.945
		CCI	0.954	0.948	0.954	0.942	0.960	0.956	0.957	0.943	0.945
	5	PGCI	0.954	0.953	0.951	0.951	0.957	0.956	0.957	0.945	0.947
		CCI	0.955	0.954	0.953	0.946	0.960	0.957	0.956	0.945	0.948
	6	PGCI	0.956	0.952	0.950	0.948	0.958	0.957	0.956	0.938	0.953
		CCI	0.959	0.953	0.951	0.951	0.961	0.956	0.954	0.940	0.950
40	1	PGCI	0.952	0.951	0.942	0.949	0.950	0.947	0.944	0.950	0.947
		CCI	0.950	0.950	0.944	0.947	0.948	0.946	0.943	0.948	0.947
	2	PGCI	0.945	0.947	0.941	0.948	0.945	0.950	0.942	0.952	0.949
		CCI	0.946	0.951	0.944	0.950	0.944	0.949	0.944	0.958	0.946
	3	PGCI	0.948	0.950	0.944	0.951	0.950	0.950	0.946	0.952	0.954
		CCI	0.945	0.950	0.946	0.953	0.945	0.950	0.938	0.948	0.950
	4	PGCI	0.954	0.952	0.946	0.948	0.946	0.954	0.942	0.950	0.952
		CCI	0.951	0.955	0.944	0.948	0.947	0.952	0.942	0.952	0.949
	5	PGCI	0.953	0.951	0.949	0.951	0.947	0.954	0.942	0.950	0.957
		CCI	0.951	0.951	0.948	0.950	0.942	0.956	0.939	0.950	0.958
	6	PGCI	0.950	0.952	0.945	0.949	0.945	0.952	0.946	0.956	0.950
		CCI	0.954	0.948	0.944	0.947	0.943	0.953	0.944	0.950	0.950
	7	PGCI	0.956	0.954	0.943	0.943	0.940	0.953	0.946	0.946	0.953
		CCI	0.956	0.955	0.941	0.941	0.939	0.950	0.942	0.952	0.951
	8	PGCI	0.955	0.952	0.941	0.941	0.941	0.955	0.948	0.948	0.954
		CCI	0.951	0.948	0.937	0.941	0.942	0.954	0.948	0.950	0.953

failures ($n \geq 30$), the CCI is better than PGCI when $\gamma < 0.5$. Therefore, the PGCI is practically useful to save business costs and time during the developmental phase of system testing since only small numbers of failures are required to test systems, and it yields precise results.

3.8 Numerical Examples

In this section, we use two real examples from an engine system development program provided by Zhou and Weng in 1992 and the failure times in hours for an aircraft generator provided by Rigdon and Basu in 1989 to illustrate the proposed methods.

3.8.1 Engine Failure Data

The total number of failure times (hours) in the engine system development testing is fixed at 40 and were reported as follows: *, *, *, 171, 234, 274, 377, 530, 533, 941, 1074, 1188, 1248, 2298, 2347, 2347, 2381, 2456, 2456, 2500, 2913, 3022, 3038, 3728, 3873, 4724, 5147, 5179, 5587, 5626, 6824, 6983, 7106, 7106, 7568, 7568, 7593, 7642, 7928, 8063, where * represents the unknown exact failure times which occur in the early phase of the test.

Here, we have $n = 40$ and $r = 4$. We thus obtain the maximum likelihood estimates of γ and β using (3.8) and (3.9) and get $\hat{\gamma} = 0.0914$ and $\hat{\beta} = 0.6761$.

To determine whether the engine system is improving, we can also perform a test with null (H_0) and alternative (H_1) hypotheses as follows:

H_0: The engine system is not improving ($\beta = 1$),

H_1: The engine system is improving ($\beta < 1$).

Table 3.4

Average width of 95% CI for γ obtained from PGCI and CCI when n is large

n	r	Method	$\beta = 0.5$			$\beta = 1$			$\beta = 1.5$		
			$\gamma = 0.05$	$\gamma = 0.1$	$\gamma = 0.5$	$\gamma = 0.05$	$\gamma = 0.1$	$\gamma = 0.5$	$\gamma = 0.05$	$\gamma = 0.1$	$\gamma = 0.5$
30	1	PGCI	0.4644	0.7030	2.0008	0.4701	0.6948	1.9274	0.4723	0.6928	1.9478
		CCI	0.3822	0.6510	3.1827	0.3886	0.6433	2.9854	0.3849	0.6262	3.0449
	2	PGCI	0.4812	0.7229	2.0557	0.4907	0.7161	1.9765	0.4845	0.7222	1.9947
		CCI	0.4043	0.6769	3.3752	0.4199	0.6752	3.1689	0.4096	0.6750	3.2257
	3	PGCI	0.4922	0.7502	2.1066	0.5070	0.7295	2.0371	0.5022	0.7384	2.0275
		CCI	0.4320	0.7141	3.5536	0.4451	0.6977	3.3674	0.4278	0.7002	3.3285
	4	PGCI	0.5145	0.7841	2.1603	0.5237	0.7656	2.0748	0.5136	0.7593	2.0971
		CCI	0.4629	0.7824	3.7504	0.4669	0.7359	3.5172	0.4615	0.7494	3.5784
	5	PGCI	0.5315	0.8145	2.2175	0.5526	0.7887	2.1296	0.5394	0.7837	2.1520
		CCI	0.4920	0.8316	4.0031	0.5086	0.7805	3.7462	0.4963	0.7896	3.8009
	6	PGCI	0.5554	0.8458	2.2706	0.5716	0.8217	2.2045	0.5580	0.8128	2.2121
		CCI	0.5235	0.8877	4.2316	0.5676	0.8657	4.0095	0.5222	0.8381	4.0542
40	1	PGCI	0.3847	0.6060	1.7320	0.3872	0.6005	1.7686	0.3790	0.6392	1.7417
		CCI	0.2834	0.5234	2.3748	0.2841	0.5141	2.4299	0.2764	0.6154	2.3721
	2	PGCI	0.3949	0.6167	1.7696	0.3907	0.6165	1.7947	0.3853	0.6490	1.7684
		CCI	0.2924	0.5387	2.4741	0.2967	0.5319	2.5136	0.2833	0.6384	2.4336
	3	PGCI	0.4055	0.6274	1.7860	0.4044	0.6326	1.8171	0.3959	0.6639	1.7958
		CCI	0.2987	0.5491	2.5220	0.3081	0.5506	2.5770	0.2914	0.6497	2.5268
	4	PGCI	0.4184	0.6418	1.8170	0.4107	0.6475	1.8397	0.4089	0.6687	1.8198
		CCI	0.3179	0.5717	2.6027	0.3127	0.5731	2.6461	0.3070	0.6709	2.5877
	5	PGCI	0.4260	0.6576	1.8505	0.4262	0.6640	1.8623	0.4189	0.6904	1.8614
		CCI	0.3302	0.5880	2.6951	0.3342	0.5884	2.7251	0.3175	0.7047	2.7053
	6	PGCI	0.4403	0.6783	1.8723	0.4360	0.6802	1.9093	0.4269	0.7053	1.8789
		CCI	0.3468	0.6101	2.7629	0.3450	0.6160	2.8393	0.3295	0.7158	2.7599
	7	PGCI	0.4570	0.7032	1.9172	0.4483	0.7002	1.9430	0.4376	0.7222	1.9210
		CCI	0.3672	0.6489	2.8902	0.3601	0.6413	2.9477	0.3454	0.7456	2.8905
	8	PGCI	0.4708	0.7278	1.9441	0.4611	0.7178	1.9683	0.4530	0.7448	1.9424
		CCI	0.3825	0.6845	2.9692	0.3774	0.6749	3.0402	0.3739	0.7857	2.9668

Let $L_0 = L(\hat{\gamma}, \beta = 1)$ and $L_1 = L(\hat{\gamma}, \hat{\beta})$ be the maximum likelihoods of the engine failure data under H_0 and H_1 respectively. The likelihood ratio test in this case is then defined as the statistic $\chi^2 = -2\ln(L_0/L_1)$ which, under H_0, follows χ^2 distribution with 1 degree of freedom (L_0 assumes 1 parameter less than L_1), and the null hypothesis is rejected if χ^2 is greater than the critical value $\chi^2_{1,\alpha}$. Using the joint probability density function of $Y_{obs} = t_r, t_{r+1}, ..., t_n$ in (3.6), we obtain lnL_0 = -819.0254 and lnL_1 = -235.3695. Therefore, the likelihood ratio test statistic is $\chi^2 = -2\ln(L_0/L_1)$ = 1167.3117, which is greater than $\chi^2_{1,0.05}$ = 3.841. Thus, there is sufficiently strong evidence to support the hypothesis that the engine system is improving at a significance level of 0.05 (see Figure 3.5).

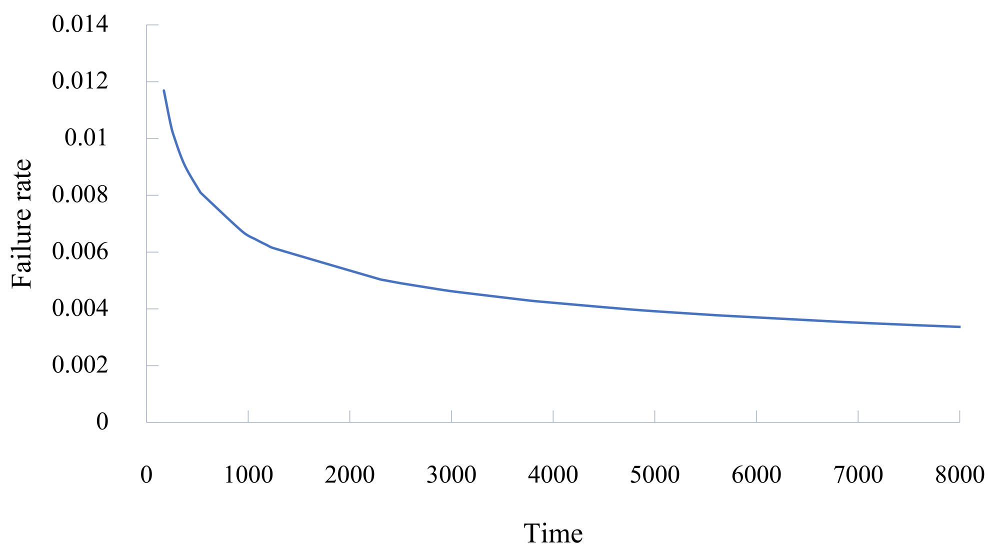


Figure 3.5

Failure rate estimate for the engine system

To illustrate the PGCI and CCI methods, we use the engine failure data to construct 95% confidence intervals for γ, and the results appear in Table 3.5.

Table 3.5

95% CIs and interval widths for γ using engine failure data

Method	95% CI for γ	Interval width
PGCI	[0.0129, 0.6309]	0.6179
CCI	[0.0362, 0.5077]	0.4715

Table 3.5 shows that the interval width obtained from CCI is a bit shorter than the interval width obtained from PGCI, and this result is similar to the result based on the simulated data when n = 40, r = 4, β = 0.6761, and γ = 0.0914 (0.6146 and 0.5796 for PGCI and CCI, respectively). The distributions of W and Z obtained from the simulation algorithm using $m = 10,000$ are demonstrated in Figure 3.6.

3.8.2 Failure Times of an Aircraft Generator

In this example, we consider the failure times of the aircraft generator when the testing was stopped after the 13th failure. The observed failure times are as follows: 55*, 166*, 205*, 341, 488, 567, 731, 1308, 2050, 2453, 3115, 4017, and 4596.

For illustrative purposes, we assume that the exact failure times for the first three failures cannot be observed. The maximum likelihood estimates, 95% confidence intervals, and interval widths obtained from the PGCI and CCI methods are tabulated in Table 3.6.

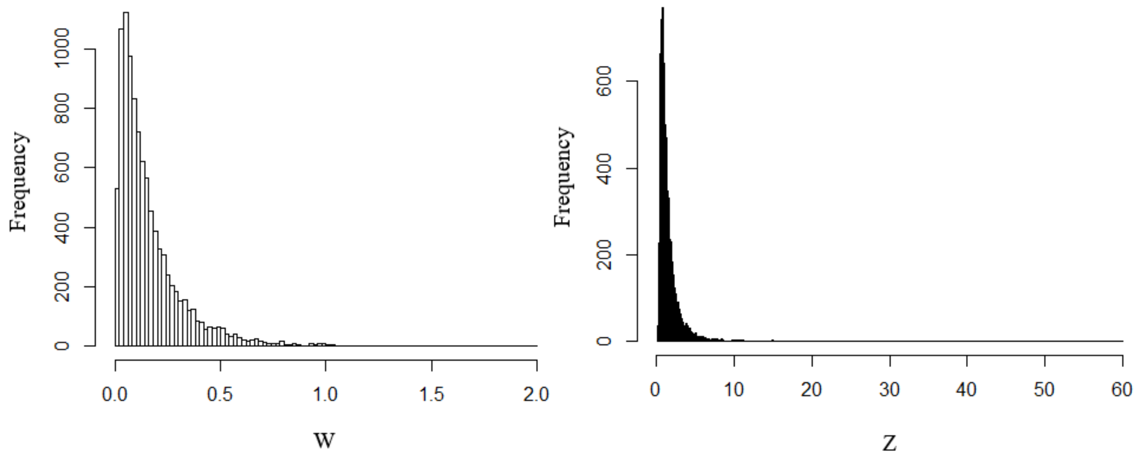


Figure 3.6

Distributions of W and Z

Table 3.6

Estimates of parameters and interval widths for complete ($r = 1$) and incomplete failure data ($r \geq 2$)

r	$\hat{\gamma}$	$\hat{\beta}$	PGCI		CCI	
			95% CI for γ	Width	95% CI for γ	Width
1	0.1072	0.5690	[0.0081, 1.4244]	1.4164	[0.0394, 2.1045]	2.0651
2	0.1238	0.5519	[0.0103, 1.6140]	1.6037	[0.0463, 2.6811]	2.6348
3	0.1676	0.5159	[0.0137, 2.0741]	2.0605	[0.0590, 5.2652]	5.2062
4	0.1835	0.5052	[0.0151, 2.3437]	2.3286	[0.0656, 7.2116]	7.1460

Similar to the engine failure data, we can perform a statistical hypothesis testing to determine whether the aircraft generator is improving with the null (H_0) and alternative (H_1) hypotheses as follows:

H_0: The aircraft generator is not improving ($\beta = 1$),

H_1: The aircraft generator is improving ($\beta < 1$).

Based on the MLEs of γ and β in Table 3.6, we obtained the likelihood ratio test statistics for r = 1, 2, 3, and 4 as 869.9073, 1018.9382, 1414.5336, and 1558.1749, respectively. Therefore, we can conclude that there is sufficiently strong evidence to support the hypothesis that the aircraft generator is improving for all situations (r = 1, 2, 3, and 4) at a significance level of 0.05 ($\chi^2_{1,0.05}$ = 3.841). The failure rate estimate during the testing time is shown in Figure 3.7.

Moreover, we also observe that the interval widths obtained from both methods increase as the number of missing failures increases, and that the PGCI is superior to the CCI regarding the shorter interval widths (see Figure 3.8). This result is similar to the result based on the simulated data when the predetermined number of failures (n) is small.

3.9 Conclusions and Discussions

In this research, we proposed the generalized confidence interval (PGCI) for the scale parameter of the PLP with the specific type of incomplete failure data when missing failure times occur only in the early developmental phase of system testing. This type of incompleteness becomes essential to establish a warranty period or determine a maintenance phase for repairable systems. The performance of the proposed generalized confidence

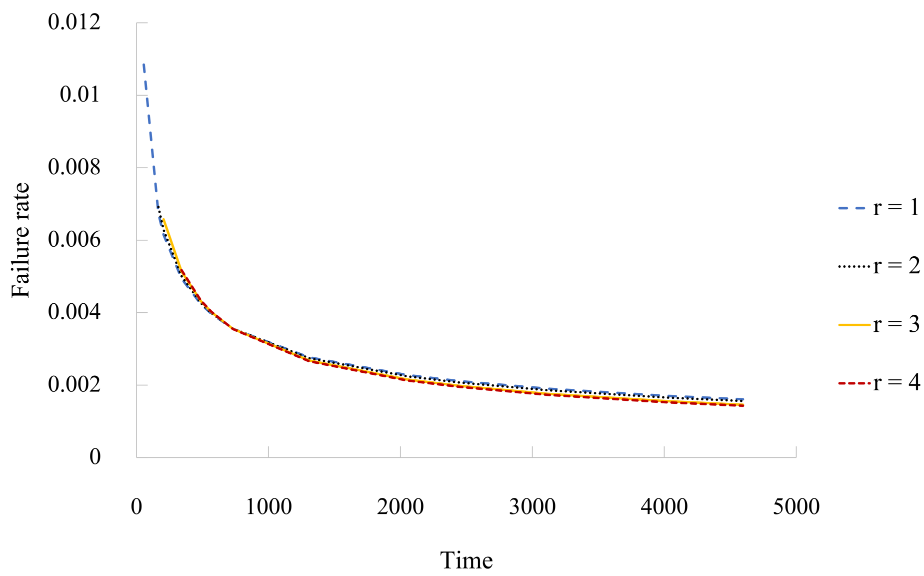


Figure 3.7

Failure rate estimate for the aircraft generator

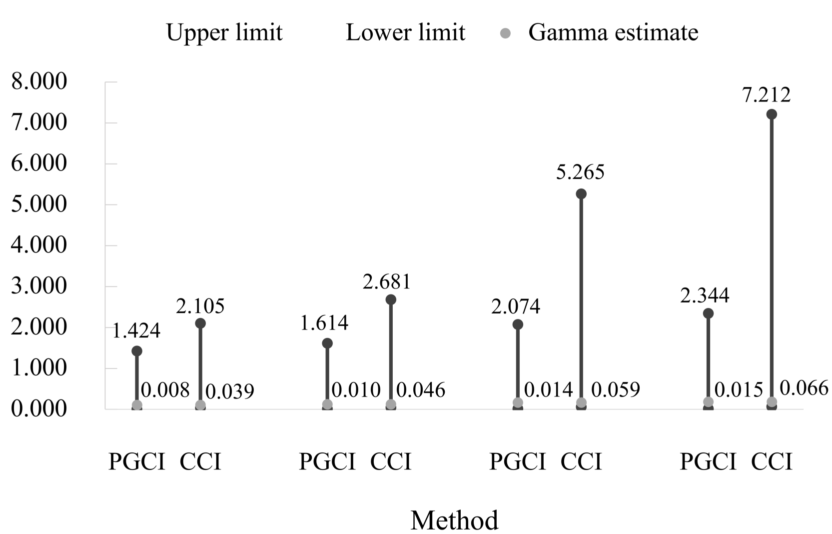


Figure 3.8

95% CI for γ obtained from PGCI and CCI using the failure times of an aircraft generator interval is validated and compared to the classical confidence interval (CCI) given by Yu et al. using the following criteria: coverage probability, coverage error, relative bias, and average width.

The results of the simulations indicate that the PGCI and CCI methods are not biased estimates, which can be seen from the coverage probabilities obtained from both methods being close to the nominal level 0.95 for all levels of γ and β. The simulation results also demonstrate that the average widths of confidence intervals obtained from PGCI increase slightly as the value of parameter γ and r increases for all levels of β, while confidence intervals obtained from CCI increase greatly. The reason is that the proposed generalized confidence interval does not depend on the shape parameter β, and as a result, the average widths will depend only on n, r, and γ. On the other hand, the classical confidence interval depends on the shape parameter β, and the MLE of β is a biased estimator. Therefore, $\hat{\beta}$ significantly affects the width of CCI for small failure numbers, but it will improve with increasing failure numbers. Moreover, the average widths of confidence intervals obtained from both methods also decrease as the predetermined number of failures (n) increases. These results correspond to the results presented by Wang et al. (2013)[43] for complete failure data ($r = 1$).

When the performance of the two confidence intervals are compared and validated regarding average widths, the PGCI is superior to CCI due to shorter average widths when the predetermined numbers of failures are small ($n < 30$). Based on this result, it is quite clear that the proposed method is practically useful to save business costs and time during the developmental phase of system testing since only small numbers of failures are required to test systems, and it yields precise results.

Finally, this study only focuses on failure truncated case when the number of failures is predetermined and considers only the situation when missing failure times occur in the early developmental phase of system testing. Thus, the potential future research can be extended to time truncated case, consider other scenarios of missing failure times, or apply this proposed method to more than one system similar to the study presented by Il and Woojin (2017)[29].

CHAPTER 4

PROPOSED MODIFIED SIGNED LOG-LIKELIHOOD RATIO TEST FOR THE SCALE PARAMETER OF THE POWER-LAW PROCESS

In this chapter, we present details of the proposed modified signed log-likelihood ratio test (MSLRT) for testing the scale parameter of the PLP applicable on both complete and incomplete failure data for failure truncated cases. To compare the proposed with the signed log-likelihood ratio test (SLRT), the empirical type I errors and the empirical powers for testing two-sided hypotheses are investigated.

4.1 Hypothesis Testing for the Scale Parameter of the Power-Law Process with Complete Failure Data

In this section, we first review the method of likelihood ratio test (LRT) and signed log-likelihood ratio test (SLRT), and then we propose the modified version of signed log-likelihood ratio test (MSLRT).

4.1.1 Likelihood Ratio Test for the Scale Parameter of the Power-Law Process with Complete Failure Data

For the problem considered in this study, we are interested to make a conclusion about the scale parameter (γ) of the PLP. To perform a two-sided test about γ, the null (H_0) and alternative (H_1) hypotheses are stated as follows:

$$H_0 : \gamma = \gamma_0 \text{ versus } H_1 : \gamma \neq \gamma_0. \tag{4.1}$$

Let $\Omega = \{(\gamma, \beta) : \gamma > 0, \beta > 0\}$ denote the entire parameter space, and $\Omega_0 = \{(\gamma, \beta) : \gamma = \gamma_0, \beta = \beta_{\gamma_0} > 0\}$ denote the null hypothesis parameter space. Then, the likelihood functions of the observed failure times $t_1, t_2, ..., t_n$ under the entire parameter space and the null hypothesis parameter space are given by

$$L(\Omega) = L(\gamma, \beta) = (\gamma\beta)^n \text{exp}(-\gamma t_n^{\beta}) \prod_{i=1}^{n} t_i^{\beta-1} \tag{4.2}$$

and

$$L(\Omega_0) = L(\gamma = \gamma_0, \beta_{\gamma_0}) = (\gamma_0\beta_{\gamma_0})^n \text{exp}(-\gamma_0 t_n^{\beta_{\gamma_0}}) \prod_{i=1}^{n} t_i^{\beta_{\gamma_0}-1}, \tag{4.3}$$

respectively.

Let the maximum of $L(\Omega)$ in Ω be denoted by $L(\hat{\Omega})$ and let the maximum of $L(\Omega_0)$ in Ω_0 be denoted by $L(\hat{\Omega}_0)$. Then the criterion for the test of H_0 against H_1 is the likelihood ratio

$$\Lambda = \frac{L(\hat{\Omega}_0)}{L(\hat{\Omega})}. \tag{4.4}$$

Here, $L(\Omega)$ obtains its maximum value at $\hat{\gamma}$ and $\hat{\beta}$ which are given in (2.10) and (2.11), respectively. Thus

$$\max L(\Omega) = L(\hat{\Omega}) = (\hat{\gamma}\hat{\beta})^n \text{exp}(-n) \prod_{i=1}^{n} t_i^{\hat{\beta}-1}. \tag{4.5}$$

Similarly, $L(\Omega_0)$ obtains its maximum value at γ_0 and β_{γ_0}. Unfortunately, an analytic solution for β_{γ_0} is not available. Therefore, the following algorithm is used to find a numerical solution for β_{γ_0}.

Algorithm 2

Step 1: Find the maximum likelihood estimator for parameter β (as shown in 3.9) and use it as an initial value for β_{γ_0}. That is, we set $\beta_{\gamma_0}^{(0)} = \hat{\beta}$.

Step 2: Evaluate the first partial derivatives $l'(\Omega_0)$ at $\beta_{\gamma_0} = \beta_{\gamma_0}^{(0)}$,

$$l'(\gamma_0, \beta_{\gamma_0}^{(0)}) = \frac{n}{\beta_{\gamma_0}^{(0)}} - \gamma_0 t_n^{\beta_{\gamma_0}^{(0)}} \ln(t_n) + \sum_{i=1}^{n} \ln t_i.$$

Step 3: Find the second partial derivatives $l''(\Omega_0)$ at $\beta_{\gamma_0} = \beta_{\gamma_0}^{(0)}$,

$$l''(\gamma_0, \beta_{\gamma_0}^{(0)}) = -\frac{n}{(\beta_{\gamma_0}^{(0)})^2} - \gamma_0 t_n^{\beta_{\gamma_0}^{(0)}} (\ln(t_n))^2.$$

Step 4: Compute the current estimate of the $(k+1)^{th}$ iteration ($k = 0, 1, 2, ...$) for parameter β_{γ_0},

$$\begin{aligned}\beta_{\gamma_0}^{(k+1)} &= \beta_{\gamma_0}^{(k)} - [l''(\gamma_0, \beta_{\gamma_0}^{(k)})]^{-1} l'(\gamma_0, \beta_{\gamma_0}^{(k)}) \\ &= \beta_{\gamma_0}^{(k)} - \frac{\beta_{\gamma_0}^{(k)} \left(n - \gamma_0 \beta_{\gamma_0}^{(k)} t_n^{\beta_{\gamma_0}^{(k)}} \ln(t_n) + \beta_{\gamma_0}^{(k)} \sum_{i=1}^{n} \ln t_i \right)}{-n - \gamma_0 (\beta_{\gamma_0}^{(k)})^2 t_n^{\beta_{\gamma_0}^{(k)}} (\ln(t_n))^2}.\end{aligned}$$

Step 5: Repeat step (2) through (4) until the estimates meet a convergence criterion. That is, we stop and obtain the final estimator, $\hat{\beta}_{\gamma_0} = \beta_{\gamma_0}^{(k)}$, when $|l(\gamma_0, \beta_{\gamma_0}^{(k+1)}) - l(\gamma_0, \beta_{\gamma_0}^{(k)})| \leq \epsilon$ where ϵ is the desired level of error ($\epsilon = 0.000000000000001$).

Thus, the maximum of $L(\Omega_0)$ is

$$L(\hat{\Omega}_0) = (\gamma_0 \hat{\beta}_{\gamma_0})^n \exp(-\gamma_0 t_n^{\hat{\beta}_{\gamma_0}}) \prod_{i=1}^{n} t_i^{\hat{\beta}_{\gamma_0} - 1}. \quad (4.6)$$

Therefore, the likelihood ratio statistic can be expressed as

$$\Lambda = \frac{L(\hat{\Omega}_0)}{L(\hat{\Omega})} = \frac{(\gamma_0\hat{\beta}_{\gamma_0})^n \exp(-\gamma_0 t_n^{\hat{\beta}_{\gamma_0}}) \prod_{i=1}^{n} t_i^{\hat{\beta}_{\gamma_0}-1}}{(\hat{\gamma}\hat{\beta})^n \exp(-n) \prod_{i=1}^{n} t_i^{\hat{\beta}-1}}. \tag{4.7}$$

To test hypotheses $H_0 : \gamma = \gamma_0$ versus $H_1 : \gamma \neq \gamma_0$, we base the test on the following statistic:

$$Q = -2\ln\Lambda, \tag{4.8}$$

which under H_0, we have approximately that $Q \sim \chi^2_{(1)}$. Therefore, an approximate size α test is to reject H_0 if $Q \geq \chi^2_{(1)}$ (Max and Lee, 1992[17]).

4.1.2 Signed Log-Likelihood Ratio Test for the Scale Parameter of the Power-Law Process

The idea of signed log-likelihood ratio (SLRT) has been proposed and discussed by McCullagh (1982, 1984)[27][28], Petersen (1981)[32], Pierce and Schafer (1986)[34], and Barndorff-Nielsen (1980, 1984, 1986)[2][3][4].

4.1.2.1 Signed Log-Likelihood Ratio Test

Suppose the log-likelihood function based on sample data is $l(\theta) = l(\psi, \lambda)$ where $\theta = (\psi, \lambda)$, ψ is a parameter of interest and λ is a nuisance parameter.

For testing the null hypothesis $H_0 : \psi = \psi_0$, a conclusion can be made based on the statistic

$$\begin{aligned} \Lambda &= -2[l(\psi, \hat{\lambda}_\psi) - l(\hat{\psi}, \hat{\lambda})] \\ &= 2[l(\hat{\psi}, \hat{\lambda}) - l(\psi, \hat{\lambda}_\psi)], \end{aligned} \tag{4.9}$$

where $\hat{\theta} = (\hat{\psi}, \hat{\lambda})$ denotes the maximum likelihood estimator of $\theta = (\psi, \lambda)$ and $\hat{\theta}_\psi = (\psi, \hat{\lambda}_\psi)$ denotes the constrained maximum likelihood estimator of θ for a fixed ψ. Under H_0, it is well known that Λ follows χ^2 distribution with 1 degree of freedom ($l(\psi, \hat{\lambda}_\psi)$ assumes 1 parameter less than $l(\hat{\psi}, \hat{\lambda})$), so that an approximate size α test is to reject H_0 if $LR \geq \chi^2_{1,\alpha}$.

Based on the statistic in (4.9), it is easily verified that the signed log-likelihood ratio statistic, say $R(\psi)$, to test the null hypothesis $H_0 : \psi = \psi_0$ has the following form

$$R(\psi) = sign(\hat{\psi} - \psi)\sqrt{2[l(\hat{\psi}, \hat{\lambda}) - l(\psi, \hat{\lambda}_\psi)]}, \tag{4.10}$$

where $sign(\hat{\psi} - \psi) = 1$, if $(\hat{\psi} - \psi) > 0$ and $sign(\hat{\psi} - \psi) = -1$, if $(\hat{\psi} - \psi) < 0$.

$R(\psi)$ is in general known to be approximately distributed as a standard normal distribution up to an order of $O(n^{-1/2})$ (Cox and Hinkley, 1974[9]), and a two-sided p-value for testing the null hypothesis $H_0 : \psi = \psi_0$ can be obtained from $R(\psi)$ by

$$p - \text{value} = 2P(R(\psi) > |R(\psi)_0|) \approx 2(1 - \phi(|R(\psi)_0|)), \tag{4.11}$$

where $R(\psi)_0$ is the observed value of the statistic $R(\psi)$ and $\phi(.)$ is the standard normal distribution function. Additionally, the approximate 100(1-α)% confidence interval for ψ can be obtained from

$$(\psi; |R(\psi)| \leq z_{\alpha/2}), \tag{4.12}$$

where $z_{\alpha/2}$ is the 100(1-$\alpha/2$)th percentile of the standard normal distribution (Wu et al., (2002)[46]).

4.1.2.2 Signed Log-Likelihood Ratio Test for the Scale Parameter of the Power-Law Process with Complete Failure Data

Suppose the likelihood ratio test for $H_0 : \gamma = \gamma_0$ versus $H_1 : \gamma \neq \gamma_0$ is defined as (4.8), then the SLRT is given by

$$R(\gamma) = sign(\hat{\gamma} - \gamma)\sqrt{Q}, \tag{4.13}$$

where $sign(\hat{\gamma} - \gamma) = 1$ if $(\hat{\gamma} - \gamma) > 0$ and $sign(\hat{\gamma} - \gamma) = -1$ if $(\hat{\gamma} - \gamma) < 0$, and $R(\gamma)$ is asymptotically distributed as a standard normal to the first order. Therefore, an approximate size α test is to reject H_0 if $|R(\gamma)| \geq z_{1-\alpha/2}$, where $z_{1-\alpha/2}$ is the $(1 - \alpha/2)^{th}$ percentile of the standard normal distribution.

4.1.3 Modified Signed Log-Likelihood Ratio Test for the Scale Parameter of the Power-Law Process with Complete Failure Data

It has been found that the SLRT is not very accurate (see Figure 4.1), especially when the sample size is small[33]. Therefore, some improvements are required in order to increase the accuracy of the SLRT. In this study, we propose the modified signed log-likelihood ratio test (MSLRT) for the problem of testing the scale parameter of the PLP and it has following form

$$R^*(\gamma) = \frac{R(\gamma) - m[\tilde{R}(\gamma)]}{\sqrt{v[\tilde{R}(\gamma)]}}, \tag{4.14}$$

where $\tilde{R}(\gamma)$ is the signed log-likelihood ratio statistic assessed at the constrained MLEs, $m[\tilde{R}(\gamma)]$ and $v[\tilde{R}(\gamma)]$ are the mean and variance of the statistic $\tilde{R}(\gamma)$, respectively. Then, $R^*(\gamma)$ is asymptotically distributed as a standard normal up to an error of $O(n^{-3/2})$[15].

As noted in DiCiccio et al. (2001)[15], $R^*(\gamma)$ is approximately distributed as a standard normal up to an error of $O(n^{-3/2})$, while $R(\gamma)$ follows a standard normal distribution to the first order. Furthermore, the asymptotic formula for $m[R(\gamma)]$ and $v[R(\gamma)]$ are complex expression involving expectation of high-order derivatives of the log-likelihood. Therefore, a simpler alternative method of calculation can be based on a parametric bootstrap (PB). In this study, we will use the PB approach to approximate the mean and variance of the $R(\gamma)$ test statistic. Details of finding the value of MSLRT statistic are given in the following simulation algorithm.

Algorithm 3

Step 1: For given NHPP power-law data $(t_1, t_2, ..., t_n)$, compute the unconstrained MLEs, $\hat{\gamma}$ and $\hat{\beta}$, using (3.8) and (3.9), and the constrained MLE, $\hat{\beta}_{\gamma_0}$, using Algorithm 2. Then, compute the SLRT statistic, $R(\gamma)$, using (4.13).

Step 2: Generate the Bootstrap NHPP power-law data $(t_1^*, t_2^*, ..., t_n^*)$ with parameter $\gamma = \gamma_0$ and $\beta = \hat{\beta}_{\gamma_0}$.

Step 3: Compute the unconstrained MLEs, $\hat{\beta}^*$ and $\hat{\gamma}^*$, and the constrained MLE, $\hat{\beta}^*_{\gamma_0}$. Then, compute the SLRT statistic, $\tilde{R}(\gamma)$, based on $(t_1^*, t_2^*, ..., t_n^*)$.

Step 4: Repeat steps 2-3 b times ($b = 10,000$).

Step 5: Compute the sample mean and sample variance of $\tilde{R}(\gamma)$

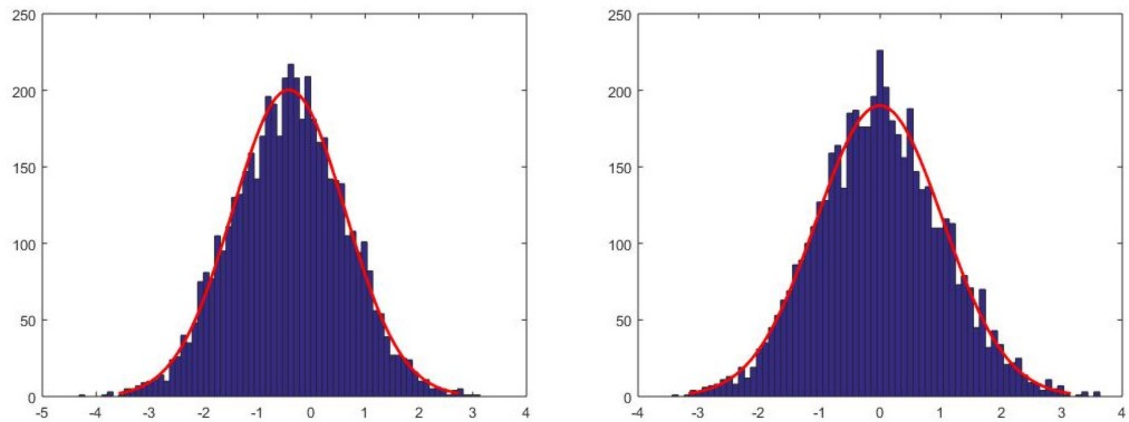
$$m[\tilde{R}(\gamma)] = \frac{1}{b}\sum_{i=1}^{b}\tilde{R}(\gamma)_i, \tag{4.15}$$

and

$$v[\tilde{R}(\gamma)] = \frac{1}{b-1}\sum_{i=1}^{b}(\tilde{R}(\gamma) - m[\tilde{R}(\gamma)])^2, \tag{4.16}$$

and then compute the MSLRT statistic, $R^*(\gamma)$, using (4.14).

For testing $H_0 : \gamma = \gamma_0$ versus $H_1 : \gamma \neq \gamma_0$, the null hypothesis is rejected at the level of α if $|R^*(\gamma)| \geq z_{1-\alpha/2}$, where $z_{1-\alpha/2}$ is the $(1-\alpha/2)^{th}$ percentile of the standard normal distribution.



SLRT MSLRT

Figure 4.1

Distributions of SLRT and MSLRT statistics when n = 8

4.2 Signed Log-Likelihood Ratio Test and Modified Signed Log-Likelihood Ratio Test for the Scale Parameter of the Power-Law Process with Incomplete Failure Data

Let define the incomplete failure data as the data in which some exact failure times in the early of system development process cannot be observed. That is, we assume that $t_1, t_2, ..., t_{r-1}$ are missing, and the observed failure times are defined as $Y_{obs} = t_r, t_{r+1}, ..., t_n$.

Recall the likelihood function for $Y_{obs} = t_r, t_{r+1}, ..., t_n$ (see Chapter 3 for details):

$$L(\gamma, \beta) = \frac{\gamma^n \beta^{n+1-r} \exp(-\gamma t_n^\beta)}{(r-1)!} t_r^{(r-1)\beta} \prod_{i=r}^{n} t_i^{\beta-1}, 0 < t_r < t_{r+1} < ... < t_n. \quad (4.17)$$

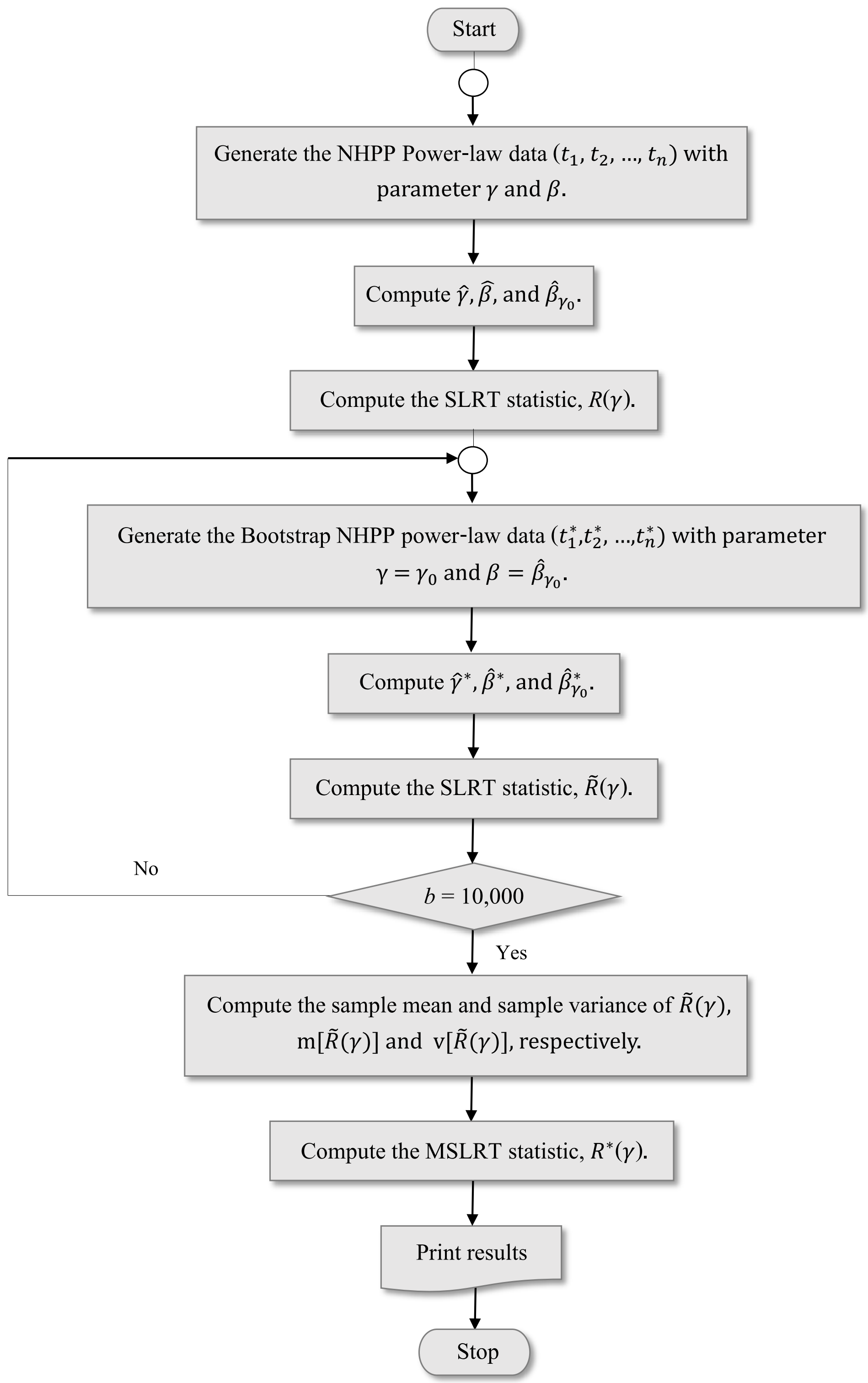


Figure 4.2

MSLRT Algorithm diagram

Similarly to complete failure case, to test hypothesis $H_0 : \gamma = \gamma_0$ against $H_1 : \gamma \neq \gamma_0$, we let $\Omega = \{(\gamma, \beta) : \gamma > 0, \beta > 0\}$ denote the entire parameter space, and $\Omega_0 = \{(\gamma, \beta) : \gamma = \gamma_0, \beta = \beta_{\gamma_0} > 0\}$ denote the null hypothesis parameter space. Then, the likelihood functions of the observed failure times $Y_{obs} = t_r, t_{r+1}, ..., t_n$ under the entire parameter space and the null hypothesis parameter space are given by

$$L(\Omega) = L(\gamma, \beta) = \frac{\gamma^n \beta^{n+1-r} \text{exp}(-\gamma t_n^{\beta})}{(r-1)!} t_r^{(r-1)\beta} \prod_{i=r}^{n} t_i^{\beta - 1} \tag{4.18}$$

and

$$L(\Omega_0) = L(\gamma = \gamma_0, \beta_{\gamma_0}) = \frac{\gamma^n \beta_{\gamma_0}^{n+1-r} \text{exp}(-\gamma t_n^{\beta_{\gamma_0}})}{(r-1)!} t_r^{(r-1)\beta_{\gamma_0}} \prod_{i=r}^{n} t_i^{\beta_{\gamma_0} - 1}, \tag{4.19}$$

respectively.

Let the maximum of $L(\Omega)$ in Ω be denoted by $L(\hat{\Omega})$ and let the maximum of $L(\Omega_0)$ in Ω_0 be denoted by $L(\hat{\Omega}_0)$. Then, the likelihood ratio statistic for testing $H_0 : \gamma = \gamma_0$ versus $H_1 : \gamma \neq \gamma_0$, can be expressed as

$$\begin{aligned} \Lambda &= \frac{L(\hat{\Omega}_0)}{L(\hat{\Omega})} \\ &= \frac{\gamma^n \hat{\beta}_{\gamma_0}^{n+1-r} \text{exp}(-\gamma t_n^{\hat{\beta}_{\gamma_0}}) t_r^{(r-1)\hat{\beta}_{\gamma_0}} \prod_{i=r}^{n} t_i^{\hat{\beta}_{\gamma_0} - 1}}{\hat{\gamma}^n \hat{\beta}^{n+1-r} \text{exp}(-n) t_r^{(r-1)\hat{\beta}} \prod_{i=r}^{n} t_i^{\hat{\beta} - 1}}, \end{aligned} \tag{4.20}$$

where $\hat{\gamma}$ and $\hat{\beta}$ are MLEs of γ and β and can be obtained using (3.8) and (3.9), respectively, and $\hat{\beta}_{\gamma_0}$ can be obtained using the following algorithm.

Algorithm 4

Step 1: Find the maximum likelihood estimator for parameter β (as shown in Equation (3.9)) and use it as an initial value for β_{γ_0}. That is, we set $\beta_{\gamma_0}^{(0)} = \hat{\beta}$.

Step 2: Evaluate the first partial derivatives $l'(\Omega_0)$ at $\beta_{\gamma_0} = \beta_{\gamma_0}^{(0)}$,

$$l'(\gamma_0, \beta_{\gamma_0}^{(0)}) = \frac{n+1-r}{\beta_{\gamma_0}^{(0)}} - \gamma_0 t_n^{\beta_{\gamma_0}^{(0)}} \ln t_n + (r-1)\ln t_r + \sum_{i=r}^{n} \ln t_i.$$

Step 3: Find the second partial derivatives $l''(\Omega_0)$ at $\beta_{\gamma_0} = \beta_{\gamma_0}^{(0)}$,

$$l''(\gamma_0, \beta_{\gamma_0}^{(0)}) = -\frac{n+1-r}{[\beta_{\gamma_0}^{(0)}]^2} - \gamma_0 t_n^{\beta_{\gamma_0}^{(0)}} (\ln t_n)^2.$$

Step 4: Compute the current estimate of the $(k+1)^{th}$ iteration ($k = 0, 1, 2, ...$) for parameter β_{γ_0},

$$\beta_{\gamma_0}^{(k+1)} = \beta_{\gamma_0}^{(k)} - [l''(\gamma_0, \beta_{\gamma_0}^{(k)})]^{-1} l'(\gamma_0, \beta_{\gamma_0}^{(k)})$$

$$= \beta_{\gamma_0}^{(k)} - \frac{\beta_{\gamma_0}^{(k)} \left(n+1-r - \gamma_0 \beta_{\gamma_0}^{(k)} t_n^{\beta_{\gamma_0}^{(k)}} \ln t_n + (r-1)\beta_{\gamma_0}^{(k)} \ln t_r + \beta_{\gamma_0}^{(k)} \sum_{i=r}^{n} \ln t_i \right)}{-(n+1-r) - \gamma_0 [\beta_{\gamma_0}^{(k)}]^2 t_n^{\beta_{\gamma_0}^{(k)}} (\ln t_n)^2}.$$

Step 5: Repeat step (2) through (4) until the estimates meet a convergence criterion. That is, we stop and obtain the final estimator, $\hat{\beta}_{\gamma_0} = \beta_{\gamma_0}^{(k)}$, when $|l(\gamma_0, \beta_{\gamma_0}^{(k+1)}) - l(\gamma_0, \beta_{\gamma_0}^{(k)})| \leq \epsilon$ where ϵ is the desired level of error ($\epsilon = 0.000000000000001$).

Therefore, the SLRT and the MSLRT can be obtained using Equation (4.13) and (4.14) as described in the previous section.

4.3 Simulation Study

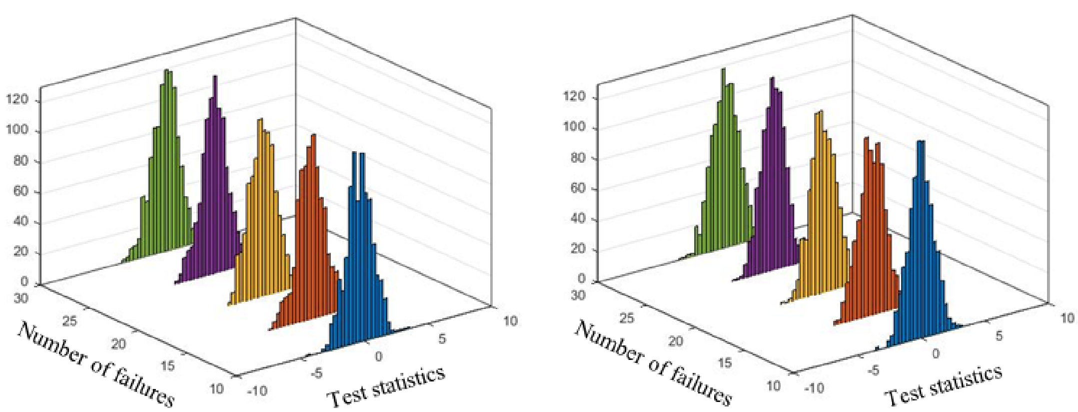
In this section, we carry out a simulation study with 5,000 replications to assess the accuracy of the proposed MSLRT methods. We investigate the empirical type I errors and empirical powers for the two-sided hypotheses testing to compare the performance of the proposed method with the existing SLRT.

To cover all three types of system reliability (reliability growth, constant failure rate, and reliability degeneration), we select some values of the most common parameters in practical cases as $\beta = 0.8, 1.0$, and 1.2 and $\gamma = 0.05, 0.1$, and 0.5. Numbers of failures (n) considered in this study are 10, 20, 30, 40, 50, 60, 70, 80, 90, and 100 for complete failures case, and 10, 20, 30, 40, and 50 for incomplete failures case with missing ratio 10%, 20%, and 30%.

4.3.1 Simulation Results for Complete Failure Data

Table shows the results of empirical type I errors for testing hypothesis $H_0 : \gamma = \gamma_0$ versus $H_1 : \gamma \neq \gamma_0$ at nominal level α = 0.05 when complete failure data are reported.

The results in Table 4.1 show that for all parameter configurations, the empirical type I errors of the modified signed log-likelihood ratio test (MSLRT) are satisfactory because they are close to the nominal level 0.05. However, the signed log-likelihood ratio test (SLRT) also performs satisfactorily when $n \geq 30$.



(i) SLRT

(ii) MSLRT

Figure 4.3

Distributions of test statistics when $\gamma = 0.5$ and $\beta = 0.8$

Table 4.1

Empirical type I errors for testing $H_0 : \gamma = \gamma_0$ versus $H_1 : \gamma \neq \gamma_0$

n	Methods	$\beta = 0.8$			$\beta = 1.0$			$\beta = 1.2$		
		$\gamma_0 = 0.05$	$\gamma_0 = 0.1$	$\gamma_0 = 0.5$	$\gamma_0 = 0.05$	$\gamma_0 = 0.1$	$\gamma_0 = 0.5$	$\gamma_0 = 0.05$	$\gamma_0 = 0.1$	$\gamma_0 = 0.5$
10	SLRT	0.0680	0.0710	0.0742	0.0752	0.0718	0.0782	0.0752	0.0764	0.0756
	MSLRT	0.0534	0.0580	0.0574	0.0540	0.0520	0.0598	0.0530	0.0562	0.0580
20	SLRT	0.0638	0.0660	0.0548	0.0680	0.0588	0.0690	0.0628	0.0646	0.0604
	MSLRT	0.0526	0.0530	0.0548	0.0574	0.0512	0.0522	0.0554	0.0562	0.0546
30	SLRT	0.0572	0.0730	0.0632	0.0574	0.0596	0.0560	0.0620	0.0584	0.0620
	MSLRT	0.0500	0.0550	0.0582	0.0500	0.0528	0.0506	0.0510	0.0514	0.0496
40	SLRT	0.0614	0.0580	0.0566	0.0602	0.0590	0.0602	0.0602	0.0580	0.0606
	MSLRT	0.0528	0.0534	0.0494	0.0586	0.0522	0.0536	0.0544	0.0530	0.0564
50	SLRT	0.0576	0.0480	0.0574	0.0568	0.0614	0.0534	0.0590	0.0542	0.0534
	MSLRT	0.0532	0.0560	0.0512	0.0562	0.0566	0.0502	0.0542	0.0520	0.0510
60	SLRT	0.0530	0.0582	0.0506	0.0530	0.0544	0.0526	0.0578	0.0558	0.0564
	MSLRT	0.0506	0.0556	0.0510	0.0506	0.0536	0.0522	0.0546	0.0516	0.0544
70	SLRT	0.0510	0.0534	0.0524	0.0522	0.0558	0.0550	0.0514	0.0552	0.0522
	MSLRT	0.0518	0.0500	0.0512	0.0514	0.0540	0.0514	0.0498	0.0520	0.0516
80	SLRT	0.0570	0.0552	0.0524	0.0556	0.0558	0.0594	0.0538	0.0540	0.0552
	MSLRT	0.0534	0.0516	0.0498	0.0516	0.0548	0.0568	0.0520	0.0508	0.0514
90	SLRT	0.0578	0.0528	0.0520	0.0576	0.0550	0.0530	0.0510	0.0586	0.0492
	MSLRT	0.0534	0.0514	0.0494	0.0530	0.0514	0.0526	0.0506	0.0570	0.0498
100	SLRT	0.0528	0.0616	0.0508	0.0530	0.0560	0.0482	0.0538	0.0566	0.0508
	MSLRT	0.0498	0.0558	0.0494	0.0512	0.0548	0.0500	0.0522	0.0562	0.0494

The empirical powers clearly indicate that the MSLRT is very satisfactory for all number of failures (n) and parameters configurations. Moreover, the empirical powers of both methods increase as n and k increases. Note that k is defined as function of the ratio γ_1/γ_0. The results are reported in Table 4.2-4.4 and Figure 4.3.

4.3.2 Simulation Results for Incomplete Failure Data

The results of empirical type I errors for testing hypothesis $H_0 : \gamma = \gamma_0$ versus $H_1 : \gamma \neq \gamma_0$ at nominal level 0.05 are reported in Table 4.5-4.7.

In terms of controlling type I errors, the MSLRT is preferable to the SLRT in most situations which can be seen that most empirical type I errors are close to the nominal level

Table 4.2

Empirical powers for testing $H_0 : \gamma = 0.05$ versus $H_1 : \gamma = \gamma_1$ when $\beta = 0.8$

n	Methods	$\gamma_1 = 0.05$ (k = 1)	$\gamma_1 = 0.1$ (k = 2)	$\gamma_1 = 0.2$ (k = 4)	$\gamma_1 = 0.3$ (k = 6)	$\gamma_1 = 0.5$ (k = 10)	$\gamma_1 = 1$ (k = 20)
10	SLRT	0.0740	0.0760	0.1220	0.1940	0.3570	0.6500
	MSLRT	0.0610	0.0850	0.1780	0.2920	0.4750	0.7400
20	SLRT	0.0610	0.0840	0.1960	0.3170	0.5260	0.8160
	MSLRT	0.0480	0.1050	0.2530	0.4060	0.6220	0.8670
30	SLRT	0.0560	0.0770	0.2310	0.3820	0.6260	0.8950
	MSLRT	0.0550	0.1060	0.2880	0.4690	0.6860	0.9230
40	SLRT	0.0430	0.0650	0.2780	0.4690	0.7250	0.9440
	MSLRT	0.0380	0.0930	0.3430	0.5470	0.7750	0.9620
50	SLRT	0.0570	0.0900	0.2920	0.5270	0.7700	0.9650
	MSLRT	0.0630	0.1150	0.3610	0.5920	0.8180	0.9750
60	SLRT	0.0620	0.1200	0.3580	0.5710	0.8190	0.9750
	MSLRT	0.0570	0.1380	0.4280	0.6200	0.8480	0.9800
70	SLRT	0.0490	0.1270	0.4010	0.6360	0.8550	0.9890
	MSLRT	0.0460	0.1650	0.4400	0.6840	0.8840	0.9920
80	SLRT	0.0520	0.1350	0.4170	0.6740	0.8970	0.9940
	MSLRT	0.0560	0.1720	0.4760	0.7310	0.9130	0.9950
90	SLRT	0.0500	0.1380	0.4680	0.7280	0.9010	0.9950
	MSLRT	0.0490	0.1700	0.5180	0.7620	0.9240	0.9960
100	SLRT	0.0510	0.1510	0.4770	0.7650	0.9300	0.9960
	MSLRT	0.0520	0.1780	0.5300	0.7950	0.9410	0.9970

Table 4.3

Empirical powers for testing $H_0 : \gamma = 0.05$ versus $H_1 : \gamma = \gamma_1$ when $\beta = 1$

n	Methods	$\gamma_1 = 0.05$ (k = 1)	$\gamma_1 = 0.1$ (k = 2)	$\gamma_1 = 0.2$ (k = 4)	$\gamma_1 = 0.3$ (k = 6)	$\gamma_1 = 0.5$ (k = 10)	$\gamma_1 = 1$ (k = 20)
10	SLRT	0.0740	0.0760	0.1240	0.1950	0.3650	0.6650
	MSLRT	0.0610	0.0850	0.1940	0.2960	0.4790	0.7430
20	SLRT	0.0610	0.0840	0.1750	0.3090	0.5190	0.8230
	MSLRT	0.0480	0.1050	0.2410	0.3950	0.6190	0.8730
30	SLRT	0.0560	0.0770	0.2280	0.3620	0.6180	0.8880
	MSLRT	0.0550	0.1060	0.2920	0.4490	0.6850	0.9200
40	SLRT	0.0430	0.0920	0.2430	0.4670	0.7080	0.9380
	MSLRT	0.0380	0.1270	0.3380	0.5370	0.7700	0.9610
50	SLRT	0.0570	0.1130	0.3350	0.5240	0.7690	0.9550
	MSLRT	0.0630	0.1360	0.3980	0.5770	0.8170	0.9680
60	SLRT	0.0620	0.1200	0.3600	0.5780	0.8380	0.9840
	MSLRT	0.0570	0.1380	0.4130	0.6300	0.8640	0.9890
70	SLRT	0.0490	0.1270	0.3890	0.6430	0.8710	0.9860
	MSLRT	0.0460	0.1650	0.4480	0.6930	0.8940	0.9880
80	SLRT	0.0520	0.1350	0.4130	0.6820	0.8850	0.9960
	MSLRT	0.0560	0.1720	0.4590	0.7240	0.9190	0.9970
90	SLRT	0.0500	0.1380	0.4690	0.7040	0.9140	0.9890
	MSLRT	0.0490	0.1700	0.5160	0.7530	0.9290	0.9910
100	SLRT	0.0510	0.1650	0.4930	0.7250	0.9280	0.9960
	MSLRT	0.0520	0.1840	0.5410	0.7610	0.9420	0.9970

Table 4.4

Empirical powers for testing $H_0 : \gamma = 0.05$ versus $H_1 : \gamma = \gamma_1$ when $\beta = 1.2$

n	Methods	$\gamma_1 = 0.05$ (k = 1)	$\gamma_1 = 0.1$ (k = 2)	$\gamma_1 = 0.2$ (k = 4)	$\gamma_1 = 0.3$ (k = 6)	$\gamma_1 = 0.5$ (k = 10)	$\gamma_1 = 1$ (k = 20)
10	SLRT	0.0740	0.0760	0.1220	0.1940	0.3650	0.6430
	MSLRT	0.0610	0.0850	0.1790	0.2920	0.4790	0.7490
20	SLRT	0.0610	0.0840	0.1840	0.3170	0.5190	0.7870
	MSLRT	0.0480	0.1050	0.2430	0.4060	0.6190	0.8500
30	SLRT	0.0560	0.0770	0.2230	0.3820	0.6180	0.8850
	MSLRT	0.0550	0.1060	0.2760	0.4690	0.6850	0.9190
40	SLRT	0.0430	0.0900	0.2700	0.4690	0.7080	0.9260
	MSLRT	0.0380	0.1240	0.3240	0.5470	0.7700	0.9430
50	SLRT	0.0570	0.0920	0.3310	0.5270	0.7690	0.9730
	MSLRT	0.0630	0.1240	0.3830	0.5920	0.8170	0.9770
60	SLRT	0.0620	0.1200	0.3430	0.5710	0.8380	0.9760
	MSLRT	0.0570	0.1380	0.4130	0.6200	0.8640	0.9820
70	SLRT	0.0490	0.1270	0.3930	0.6160	0.8710	0.9860
	MSLRT	0.0460	0.1650	0.4460	0.6610	0.8940	0.9900
80	SLRT	0.0520	0.1350	0.4210	0.6740	0.8850	0.9930
	MSLRT	0.0560	0.1720	0.4710	0.7310	0.9190	0.9950
90	SLRT	0.0500	0.1380	0.4670	0.6950	0.9140	0.9980
	MSLRT	0.0490	0.1700	0.5080	0.7580	0.9290	0.9990
100	SLRT	0.0510	0.1370	0.4970	0.7530	0.9280	0.9970
	MSLRT	0.0520	0.1680	0.5470	0.7810	0.9420	0.9990

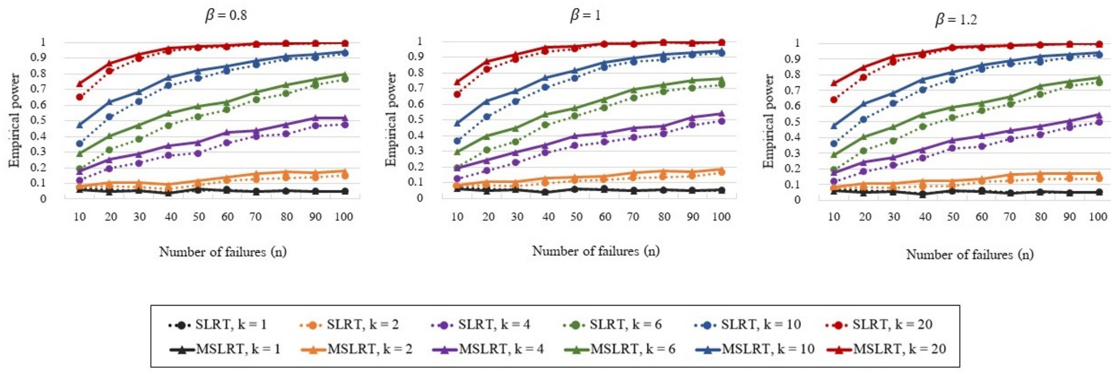


Figure 4.4

Empirical powers as function of the ratio γ_1/γ_0 of proposed methods with complete failure data

Table 4.5

Empirical type I errors for testing $H_0 : \gamma = \gamma_0$ versus $H_1 : \gamma \neq \gamma_0$ when $\beta = 0.8$

Missing ratio (%)	n	$\gamma_0 = 0.05$		$\gamma_0 = 0.1$		$\gamma_0 = 0.3$	
		SLRT	MSLRT	SLRT	MSLRT	SLRT	MSLRT
10	10	0.0800	0.0520	0.0800	0.0590	0.0820	0.0560
	20	0.0570	0.0550	0.0600	0.0590	0.0610	0.0590
	30	0.0650	0.0520	0.0610	0.0540	0.0560	0.0510
	40	0.0540	0.0510	0.0640	0.0590	0.0640	0.0540
	50	0.0590	0.0580	0.0630	0.0540	0.0610	0.0550
20	10	0.0830	0.0720	0.0800	0.0530	0.0820	0.0540
	20	0.0660	0.0510	0.0690	0.0540	0.0580	0.0560
	30	0.0520	0.0490	0.0550	0.0530	0.0600	0.0560
	40	0.0550	0.0550	0.0640	0.0500	0.0600	0.0500
	50	0.0640	0.0530	0.0670	0.0500	0.0580	0.0530
30	10	0.0840	0.0830	0.0870	0.0630	0.0790	0.0630
	20	0.0680	0.0520	0.0650	0.0540	0.0690	0.0540
	30	0.0690	0.0540	0.0640	0.0580	0.0600	0.0570
	40	0.0630	0.0520	0.0620	0.0510	0.0530	0.0500
	50	0.0540	0.0570	0.0560	0.0500	0.0670	0.0550

Table 4.6

Empirical type I errors for testing $H_0 : \gamma = \gamma_0$ versus $H_1 : \gamma \neq \gamma_0$ when $\beta = 1$

Missing ratio (%)	n	$\gamma_0 = 0.05$		$\gamma_0 = 0.1$		$\gamma_0 = 0.3$	
		SLRT	MSLRT	SLRT	MSLRT	SLRT	MSLRT
10	10	0.0740	0.0516	0.0792	0.0576	0.0754	0.0590
	20	0.0674	0.0560	0.0590	0.0510	0.0674	0.0548
	30	0.0592	0.0524	0.0606	0.0548	0.0616	0.0508
	40	0.0560	0.0500	0.0538	0.0512	0.0510	0.0500
	50	0.0600	0.0520	0.0610	0.0560	0.0540	0.0510
20	10	0.0860	0.0490	0.0840	0.0550	0.0850	0.0570
	20	0.0660	0.0550	0.0650	0.0550	0.0670	0.0550
	30	0.0440	0.0500	0.0530	0.0520	0.0560	0.0560
	40	0.0620	0.0540	0.0580	0.0540	0.0580	0.0520
	50	0.0670	0.0560	0.0680	0.0570	0.0670	0.0590
30	10	0.0600	0.0830	0.0590	0.0640	0.0830	0.0510
	20	0.0610	0.0520	0.0610	0.0510	0.0560	0.0520
	30	0.0570	0.0510	0.0570	0.0530	0.0620	0.0510
	40	0.0480	0.0530	0.0480	0.0530	0.0510	0.0510
	50	0.0590	0.0500	0.0600	0.0510	0.0600	0.0540

Table 4.7

Empirical type I errors for testing $H_0 : \gamma = \gamma_0$ versus $H_1 : \gamma \neq \gamma_0$ when $\beta = 1.2$

Missing ratio (%)	n	$\gamma_0 = 0.05$		$\gamma_0 = 0.1$		$\gamma_0 = 0.3$	
		SLRT	MSLRT	SLRT	MSLRT	SLRT	MSLRT
10	10	0.0830	0.0670	0.0850	0.0680	0.0840	0.0730
	20	0.0620	0.0510	0.0610	0.0510	0.0600	0.0520
	30	0.0550	0.0510	0.0620	0.0550	0.0710	0.0520
	40	0.0650	0.0580	0.0670	0.0570	0.0640	0.0550
	50	0.0560	0.0540	0.0540	0.0510	0.0690	0.0590
20	10	0.0740	0.0630	0.0740	0.0610	0.0790	0.0550
	20	0.0720	0.0500	0.0710	0.0500	0.0700	0.0500
	30	0.0640	0.0560	0.0650	0.0550	0.0640	0.0570
	40	0.0620	0.0550	0.0630	0.0540	0.0600	0.0580
	50	0.0580	0.0600	0.0570	0.0600	0.0540	0.0580
30	10	0.0680	0.0600	0.0870	0.0630	0.0690	0.0530
	20	0.0660	0.0560	0.0640	0.0540	0.0780	0.0550
	30	0.0630	0.0510	0.0610	0.0520	0.0610	0.0500
	40	0.0610	0.0560	0.0660	0.0580	0.0630	0.0520
	50	0.0580	0.0520	0.0650	0.0510	0.0630	0.0590

0.05, except when number of failures is very small ($n = 10$). Moreover, the MSLRT also performs well when the missing ratio is up to 30%.

The empirical powers of the SLRT, and MSLRT are reported in Table 8-10. We observe that the MSLRT is superior to the SLRT for most situations, except when number of failures is very small ($n = 10$) and missing ratio is large (30$\%$). Moreover, we also observe that the empirical powers of both methods tend to decrease as the missing ratio increase when $k > 2$, and increase as n and k increase.

4.4 Case Study

We use the failure-time data of a copy machine to illustrate some of the results of the proposed methods. For this machine, time is measured by the number of copies made, and the time at installation is defined to be 0 (Ni et al. (2007)[30]). A test is stopped after the 8^{th} failure, and the observed failure times are shown in Table 4.11.

Here, we have $n = 8$, $t_n = 19694$, lnt_n = 9.8881, and $\sum_{i=1}^{n}$ lnt_i = 63.4407. We thus obtain the maximum likelihood estimates of γ and β using Equation (2.10) and (2.11), and obtain the unbiased estimate of β using Equation (2.17). Then, the maximum likelihood estimates of γ and β are $\hat{\gamma} = 0.0513$ and $\hat{\beta} = 0.5107$, respectively, and the unbiased estimate of β is $\tilde{\beta} = 0.3830$.

Using the maximum likelihood estimates ($\hat{\beta}$) and the unbiased estimate ($\tilde{\beta}$) of β as initial values in Algorithm 1 to find the constrained MLE of β_{γ_0}, we obtain the iteration estimate, $\hat{\beta}_{\gamma_0}$, as shown in Table 4.12.

Table 4.8

Empirical powers for testing $H_0 : \gamma = 0.05$ versus $H_1 : \gamma = \gamma_1$ when $\beta = 0.8$

Missing ratio (%)	n	$\gamma_1 = 0.1$ (k = 2)		$\gamma_1 = 0.3$ (k = 6)		$\gamma_1 = 0.5$ (k = 10)		$\gamma_1 = 1$ (k = 20)	
		SLRT	MSLRT	SLRT	MSLRT	SLRT	MSLRT	SLRT	MSLRT
10	10	0.0700	0.0880	0.1980	0.2760	0.3730	0.4780	0.6480	0.7090
	20	0.0840	0.1060	0.2890	0.3870	0.4860	0.5760	0.7930	0.8530
	30	0.0770	0.0980	0.3590	0.4410	0.6170	0.6860	0.8640	0.8950
	40	0.0740	0.0910	0.4170	0.4980	0.6980	0.7470	0.9090	0.9330
	50	0.0970	0.1250	0.4850	0.5420	0.7470	0.7970	0.9380	0.9560
	60	0.0820	0.1130	0.5400	0.6060	0.7780	0.8180	0.9760	0.9830
	70	0.0840	0.1260	0.5580	0.6190	0.8210	0.8570	0.9830	0.9870
	80	0.1280	0.1520	0.6290	0.6690	0.8740	0.8930	0.9870	0.9940
	90	0.1290	0.1460	0.6400	0.6860	0.8990	0.9160	0.9930	0.9950
	100	0.1250	0.1600	0.6930	0.7270	0.9050	0.9270	0.9960	0.9980
20	10	0.0740	0.0750	0.1840	0.2510	0.3230	0.3810	0.5880	0.6210
	20	0.0850	0.0970	0.2690	0.3450	0.4500	0.5310	0.7520	0.8090
	30	0.0870	0.1140	0.3330	0.4200	0.5520	0.6220	0.8290	0.8630
	40	0.0850	0.1020	0.3860	0.4620	0.6130	0.6810	0.8790	0.9090
	50	0.0960	0.1170	0.4520	0.5250	0.7090	0.7510	0.9290	0.9460
	60	0.0920	0.1160	0.4960	0.5600	0.7530	0.8000	0.9420	0.9500
	70	0.1090	0.1340	0.5410	0.5990	0.7710	0.8130	0.9640	0.9750
	80	0.0960	0.1180	0.5710	0.6320	0.8150	0.8420	0.9790	0.9820
	90	0.1040	0.1260	0.6170	0.6800	0.8560	0.8810	0.9820	0.9900
	100	0.1150	0.1390	0.6370	0.6800	0.8630	0.8830	0.9880	0.9900
30	10	0.0630	0.0780	0.1660	0.2180	0.2840	0.3160	0.5340	0.4680
	20	0.0660	0.0790	0.2130	0.2990	0.3830	0.4930	0.6700	0.7460
	30	0.0750	0.0980	0.3050	0.3860	0.5190	0.6010	0.7760	0.8350
	40	0.0800	0.1000	0.3470	0.4360	0.5230	0.5990	0.8230	0.8700
	50	0.0810	0.1020	0.3840	0.4570	0.6270	0.6880	0.9000	0.9210
	60	0.0940	0.1230	0.4400	0.5040	0.6860	0.7540	0.9180	0.9330
	70	0.0970	0.1250	0.4530	0.5290	0.7360	0.7750	0.9400	0.9600
	80	0.0960	0.1240	0.5170	0.5850	0.7830	0.8220	0.9630	0.9700
	90	0.1090	0.1260	0.5440	0.5920	0.7800	0.8220	0.9740	0.9780
	100	0.1150	0.1430	0.5810	0.6320	0.8270	0.8590	0.9820	0.9860

Table 4.9

Empirical powers for testing $H_0 : \gamma = 0.05$ versus $H_1 : \gamma = \gamma_1$ when $\beta = 1$

Missing ratio (%)	n	$\gamma_1 = 0.1$ (*k* = 2)		$\gamma_1 = 0.3$ (*k* = 6)		$\gamma_1 = 0.5$ (*k* = 10)		$\gamma_1 = 1$ (*k* = 20)	
		SLRT	MSLRT	SLRT	MSLRT	SLRT	MSLRT	SLRT	MSLRT
10	10	0.0780	0.0940	0.1980	0.2760	0.3600	0.4460	0.6540	0.7240
	20	0.0850	0.1110	0.2890	0.3870	0.5010	0.5860	0.7900	0.8450
	30	0.0890	0.1220	0.3590	0.4410	0.6130	0.6850	0.8610	0.8920
	40	0.0810	0.1150	0.4170	0.4980	0.6740	0.7340	0.9110	0.9310
	50	0.0910	0.1160	0.4930	0.5570	0.7400	0.7860	0.9420	0.9540
	60	0.1020	0.1260	0.5580	0.6020	0.7860	0.8300	0.9740	0.9810
	70	0.0940	0.1230	0.5800	0.6360	0.8330	0.8630	0.9860	0.9900
	80	0.1140	0.1440	0.6310	0.6780	0.8570	0.8870	0.9810	0.9890
	90	0.1120	0.1430	0.6740	0.7110	0.9090	0.9260	0.9900	0.9920
	100	0.1450	0.1660	0.6750	0.7270	0.8980	0.9150	0.9940	0.9970
20	10	0.0750	0.0820	0.1860	0.2490	0.3140	0.3880	0.6090	0.6480
	20	0.0600	0.0840	0.2550	0.3330	0.4560	0.5670	0.7280	0.8090
	30	0.0710	0.1010	0.3120	0.3970	0.5590	0.6400	0.8500	0.8840
	40	0.0990	0.1220	0.4060	0.4750	0.6240	0.6940	0.9000	0.9220
	50	0.0960	0.1180	0.4620	0.5190	0.6990	0.7470	0.9360	0.9540
	60	0.0820	0.1050	0.4810	0.5400	0.7530	0.8020	0.9500	0.9620
	70	0.0910	0.1150	0.5310	0.6020	0.7750	0.8220	0.9550	0.9670
	80	0.0970	0.1190	0.5490	0.6170	0.8220	0.8610	0.9740	0.9840
	90	0.1070	0.1300	0.6030	0.6560	0.8620	0.8860	0.9850	0.9880
	100	0.1100	0.1400	0.6510	0.6900	0.8840	0.9080	0.9890	0.9910
30	10	0.0740	0.0840	0.1720	0.2220	0.2870	0.3160	0.5640	0.4940
	20	0.0630	0.0830	0.2300	0.3190	0.4080	0.5160	0.6950	0.7640
	30	0.0660	0.0870	0.2640	0.3470	0.5270	0.6170	0.7980	0.8500
	40	0.0890	0.1160	0.3170	0.3900	0.5740	0.6470	0.8290	0.8740
	50	0.0810	0.1050	0.3540	0.4310	0.6440	0.7050	0.8890	0.9200
	60	0.1080	0.1280	0.4370	0.4990	0.6650	0.7280	0.9160	0.9310
	70	0.1090	0.1250	0.4510	0.5140	0.7500	0.7980	0.9540	0.9700
	80	0.0920	0.1140	0.4880	0.5550	0.7730	0.8220	0.9680	0.9710
	90	0.0980	0.1300	0.5670	0.6160	0.7990	0.8440	0.9720	0.9830
	100	0.1240	0.1610	0.5660	0.6240	0.8480	0.8700	0.9800	0.9880

Table 4.10

Empirical powers for testing $H_0 : \gamma = 0.05$ versus $H_1 : \gamma = \gamma_1$ when $\beta = 1.2$

Missing ratio (%)	n	$\gamma_1 = 0.1$ (*k* = 2)		$\gamma_1 = 0.3$ (*k* = 6)		$\gamma_1 = 0.5$ (*k* = 10)		$\gamma_1 = 1$ (*k* = 20)	
		SLRT	MSLRT	SLRT	MSLRT	SLRT	MSLRT	SLRT	MSLRT
10	10	0.0630	0.0770	0.1870	0.2630	0.3510	0.4510	0.6540	0.7190
	20	0.0770	0.1080	0.3070	0.3810	0.4940	0.5870	0.7830	0.8390
	30	0.0770	0.1010	0.3570	0.4420	0.6010	0.6710	0.8850	0.9150
	40	0.0800	0.1130	0.4290	0.4930	0.6610	0.7230	0.9160	0.9420
	50	0.1140	0.1390	0.4830	0.5410	0.7550	0.8000	0.9610	0.9770
	60	0.1090	0.1340	0.5340	0.5870	0.7980	0.8380	0.9690	0.9760
	70	0.1110	0.1280	0.5740	0.6320	0.8240	0.8570	0.9860	0.9900
	80	0.1070	0.1390	0.6220	0.6700	0.8460	0.8820	0.9830	0.9900
	90	0.1270	0.1570	0.6890	0.7270	0.8900	0.9150	0.9930	0.9950
	100	0.1410	0.1670	0.6900	0.7330	0.9060	0.9230	0.9900	0.9920
20	10	0.0590	0.0790	0.1720	0.2520	0.3230	0.3810	0.5730	0.6250
	20	0.0650	0.0820	0.2630	0.3500	0.4500	0.5310	0.7330	0.8020
	30	0.0760	0.0990	0.3110	0.4110	0.5520	0.6220	0.8340	0.8850
	40	0.0780	0.1100	0.3990	0.4760	0.6130	0.6810	0.8870	0.9210
	50	0.0760	0.0990	0.4440	0.5130	0.7090	0.7510	0.9150	0.9420
	60	0.0910	0.1120	0.4780	0.5420	0.7530	0.8000	0.9460	0.9640
	70	0.1230	0.1460	0.5120	0.5800	0.7710	0.8130	0.9740	0.9830
	80	0.1190	0.1490	0.5820	0.6360	0.8150	0.8420	0.9860	0.9880
	90	0.1200	0.1470	0.6100	0.6470	0.8560	0.8810	0.9780	0.9830
	100	0.1160	0.1400	0.6440	0.6850	0.8630	0.8830	0.9910	0.9930
30	10	0.0730	0.0930	0.1720	0.2160	0.2840	0.3160	0.4940	0.4770
	20	0.0740	0.0850	0.2130	0.3050	0.3830	0.4930	0.6690	0.7540
	30	0.0740	0.0940	0.2670	0.3530	0.4980	0.5830	0.7680	0.8270
	40	0.0940	0.1220	0.3310	0.4270	0.5460	0.6260	0.8340	0.8740
	50	0.0830	0.1080	0.3680	0.4490	0.6140	0.6750	0.8860	0.9130
	60	0.0730	0.0950	0.4200	0.4940	0.6940	0.7540	0.9380	0.9480
	70	0.0950	0.1270	0.4670	0.5260	0.7380	0.7880	0.9430	0.9630
	80	0.0920	0.1200	0.5170	0.5700	0.7520	0.7940	0.9550	0.9670
	90	0.1040	0.1320	0.5760	0.6220	0.7970	0.8290	0.9670	0.9740
	100	0.1240	0.1560	0.5820	0.6250	0.8230	0.8550	0.9770	0.9820

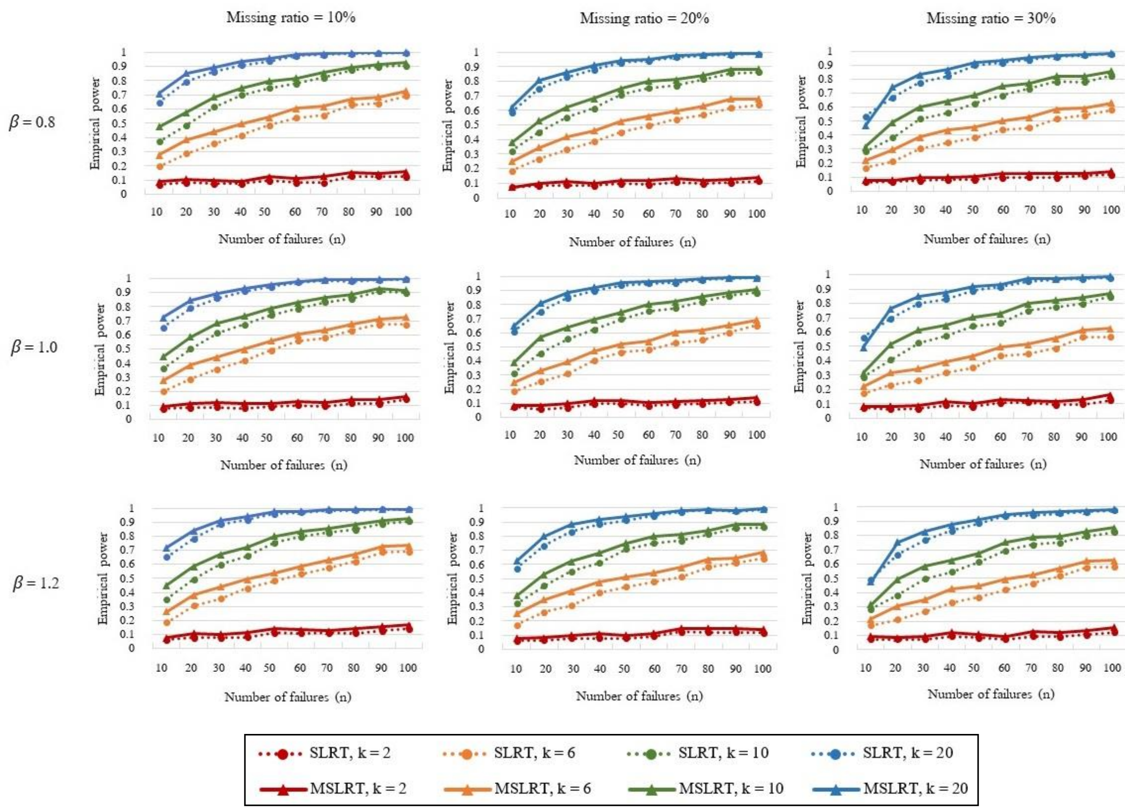


Figure 4.5

Empirical powers as function of the ratio γ_1/γ_0 of proposed methods with incomplete failure data

Table 4.11

Copy machine failure data

Failure number (i)	Failure time (t_i)
1	452
2	472
3	2467
4	2517
5	3727
6	4537
7	8079
8	19694

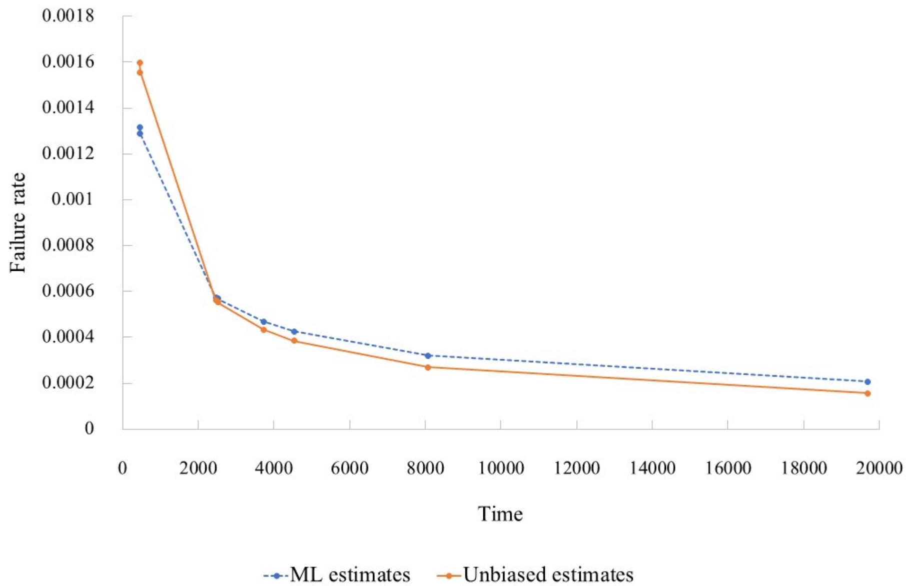


Figure 4.6

Failure rate estimates for the copy machine

Table 4.12

Iteration estimates for parameter β_{γ_0}

Number of iterations (k)	Starting value	
	$\hat{\beta} = 0.5107$	$\tilde{\beta} = 0.3830$
1	0.513195230366468	0.614211030400670
2	0.513167231233629	0.549751554428545
3	0.513167227559253	0.518857933071618
4	0.513167227559253	0.513316731976915
5	0.513167227559253	0.513167332260168
6	0.513167227559253	0.513167227559304
7	0.513167227559253	0.513167227559253
8	0.513167227559253	0.513167227559253
9	0.513167227559253	0.513167227559253
10	0.513167227559253	0.513167227559253

In Table 4.12, the iteration estimation result for $\beta_{\gamma 0}$ using initial value $\beta_{\gamma 0}^{(0)} = \hat{\beta} = 0.5107$ and $\epsilon = 0.000000000000001$ is $\hat{\beta}_{\gamma 0} = 0.513167227559253$ with number of iterations of 3, and the iteration estimation result for $\beta_{\gamma 0}$ using $\beta_{\gamma 0}^{(0)} = \tilde{\beta} = 0.3830$ is $\hat{\beta}_{\gamma 0} = 0.513167227559253$ with number of iterations of 7. Therefore, it can be concluded that both initial values yield the same estimation results, but using the unbiased estimate ($\tilde{\beta}$) has more iteration number than using maximum likelihood estimates ($\hat{\beta}$).

Other statistics that are required for finding the MSLRT are obtained as follows. The SLRT statistic, $R(\gamma)$, is 0.0137, the parametric bootstrap estimate for $m[R(\gamma)]$ is -0.3829, and the parametric bootstrap estimate for $v[R(\gamma)]$ is 1.0629. Using Equation (4.14), the MSLRT statistic, $R^*(\gamma)$, can be then obtained as

$$R^*(\gamma) = \frac{0.0137 - (-0.3829)}{\sqrt{1.0629}} = 0.3847. \quad (4.21)$$

To test hypothesis $H_0 : \gamma = 0.05$ versus $H_1 : \gamma \neq 0.05$, the test statistics, p-value, and conclusion of the proposed methods are reported are reported in Table 4.13.

Table 4.13

Test results for $H_0 : \gamma = 0.05$ versus $H_1 : \gamma \neq 0.05$ at nominal level 0.05

Methods	Test statistics	p-value	Conclusion
SLRT	0.0137	0.9890	Do not reject H_0.
MSLRT	0.3847	0.7005	Do not reject H_0.

The results in Table 11 show that both methods do not reject the null hypothesis H_0 : $\gamma = 0.05$. Therefore, the data do not provide sufficient evidence to conclude that the scale parameter of the PLP is not equal to 0.05.

4.5 Conclusions and Discussions

In this research, we propose a modified version of the signed log-likelihood ratio test (MSLRT) for testing the scale parameter of the PLP with complete and incomplete failure data. The accuracy of the proposed method is evaluated and compared with the signed log-likelihood ratio test (SLRT). To compare them, the simulation technique is used to investigate their empirical type I errors and empirical powers for two-sided hypotheses testing.

As shown in simulation studies in Section 4.4, the modified signed log-likelihood ratio test (MSLRT) performs well in controlling type I errors, and it is superior to the SLRT for complete failure data. Moreover, the MSLRT works satisfactorily because it has desirable powers and does better than the SLRT for all parameters configurations even for small number of failures. Our results correspond to the results presented by Krishnamoorthy and Lee, 2014[24] and Kazemi and Jafari (2015)[23] that the SLRT statistic is not very accurate when the sample size is small. For incomplete failure data, we observe that the MSLRT is preferable to the SLRT in most situations in terms of controlling type I errors, except when number of failures is very small ($n = 10$). However, the MSLRT also performs well when the missing ratio is up to 30% and $n > 10$. In terms of empirical powers, the MSLRT is superior to the SLRT for most situations, except when number of failures is very small ($n = 10$) and missing ratio is large (30%).

For the SLRT and MSLRT, however, the maximum of likelihood function under the null hypothesis parameter space cannot be solved analytically. Therefore, the iteration algorithm such as the Newton Raphson Algorithm, the Fisher Scoring Algorithm, and the Expectation Maximization Algorithm are necessary to obtain the constrained MLE.

Finally, this study only focuses on failure-truncated cases when predetermining the number of failures and considering only one system. As Engelhardt and Bain (1992)[17] noted, the chi-square approximation for the likelihood ratio test (LRT) does not work well for time-truncated cases with left-censored data, so potential future research may be extended to this scenario or this proposed method may be applied by more than one system.

CHAPTER 5

CONCLUSIONS

5.1 Conclusion

In this dissertation, two statistical methods, the generalized confidence interval and the modified signed log-likelihood ratio test for the scale parameter of the power-law process are developed to assess reliability growth of repairable systems concerning the situation that some recorded failure times in the early developmental phase of system testing cannot be observed. As mentioned in Chapter 1, for the power-law process, the exact test and the exact confidence interval for the shape parameter, β, is not troublesome to derive, but the exact test and the exact confidence interval for the scale parameter, γ, is not easy to obtain when β is unknown. Asymptotic distributions, such as the asymptotic normal distribution and the asymptotic chi-square distribution, are therefore used in many previous studies to make a conclusion about parameter γ. However, the issue raised by this approach is that we need a sufficiently large number of failures to produce accurate results. This research aims to solve such this problem in order to reduce time and cost during the developmental phase of system testing and this contributes to the motivation of this research.

Before going to more details of research presented in this dissertation, in Chapter 1, we provide definitions of key concepts that are therefore defined to clarify and eliminate ambiguities, and then we address some fundamental concepts of reliability and some fundamental results on homogeneous Poisson process and nonhomogeneous Poisson process. We also present some concepts on a particular nonhomogeneous Poisson process, "power-law process", which plays an important role in this research.

In Chapter 2, we present some traditional inference results on the power-law process for a single system including maximum likelihood estimates of γ and β, the unbiased estimate of β, the interval estimation and hypothesis testing of β, the interval estimation and hypothesis testing of γ, the estimation of intensity function and mean time between failure, and the goodness of fit test. The results that are covered and addressed in Chapter 2 are essential and will be utilized for the research presented in Chapter 3 and Chapter 4.

In Chapter 3, we present details of the first proposed method, the generalized confidence interval for the scale parameter of the power-law process with incomplete failure data, including the background of this study, the maximum likelihood estimates for parameter γ and β with missing data, and the definition and some details of generalized confidence interval. A simulation study and numerical examples are also conducted and presented to evaluate the performance of the proposed method. In this study, we have found that the proposed generalized confidence interval are not biased estimates, which can be seen from the coverage probabilities obtained from this method being close to the nominal level 0.95 for all levels of γ and β. Moreover, the average widths of the proposed method increase slightly as the value of parameter γ and r increases for all levels of β, and decrease as the predetermined number of failures (n) increases. When the performance of the proposed method and the existing method are compared and validated regarding average widths, the simulation results show that the proposed method is superior to the another one due to shorter average widths when the predetermined numbers of failures are small ($n < 30$). Based on this result, it is quite clear that the proposed method is practically useful to save business costs and time during the developmental phase of system testing since only small numbers of failures are required to test systems, and it yields precise results.

In Chapter 4, details of the second proposed method, the modified signed log-likelihood ratio test for the scale parameter of the power-law process, are presented. In this study, we have found that for complete failure data, the modified signed log-likelihood ratio test performs well in controlling type I errors, and it has desirable powers for all parameters configurations even for the small number of failures. For incomplete failure data, the modified signed log-likelihood ratio test is preferable to the signed log-likelihood ratio test in most situations in terms of controlling type I errors, except when the number of failures is very small ($n = 10$). However, the modified signed log-likelihood ratio test also performs well when the missing ratio is up to 30$\%$ and $n > 10$. In terms of empirical powers, the modified signed log-likelihood ratio test is superior to the signed log-likelihood ratio test for most situations, except when the number of failures is very small ($n = 10$) and the missing ratio is large (30$\%$).

5.2 Discussion and Future Works

In this research, we developed two statistical methods, the generalized confidence interval and the modified signed log-likelihood ratio test for the scale parameter of the power-law process concerning the situation that some recorded failure times in the early developmental phase of system testing cannot be observed. However, our research focuses on only one system and considers only the failure truncated case. Nowadays, there are many works on the power-law process with applications to multiple repairable systems. In this research, we carry out some useful detailed research and our findings can be extended to multiple repairable systems case.

For multiple repairable systems, most works that deal with classical inferences of the power law process such as the interval estimation always base on an asymptotic distribution, especially the asymptotic normal distribution. The most popular method to construct an approximate confidence interval is using the local Fisher information matrix as follows.

$$\hat{I}_n = -\begin{bmatrix} \dfrac{\partial^2 \ln \hat{L}}{\partial \gamma^2} & \dfrac{\partial^2 \ln \hat{L}}{\partial \gamma \partial \beta} \\ \dfrac{\partial^2 \ln \hat{L}}{\partial \gamma \partial \beta} & \dfrac{\partial^2 \ln \hat{L}}{\partial \beta^2} \end{bmatrix}, \tag{5.1}$$

where L is the likelihood function for multiple repairable systems, $\dfrac{\partial^2 \ln \hat{L}}{\partial \gamma^2}$, $\dfrac{\partial^2 \ln \hat{L}}{\partial \beta^2}$, and $\dfrac{\partial^2 \ln \hat{L}}{\partial \gamma \partial \beta}$ are the partial derivatives evaluated at $\gamma = \hat{\gamma}$ and $\beta = \hat{\beta}$. Equation (5.1) implies that $-\dfrac{\partial^2 \ln \hat{L}}{\partial \gamma^2}$, $-\dfrac{\partial^2 \ln \hat{L}}{\partial \beta^2}$, and $-\dfrac{\partial^2 \ln \hat{L}}{\partial \gamma \partial \beta}$ can be used as the estimators of Var($\hat{\gamma}$), Var($\hat{\beta}$), and Cov($\hat{\gamma}, \hat{\beta}$), respectively. These estimators respectively denoted by $\widehat{Var}(\hat{\gamma})$, $\widehat{Var}(\hat{\beta})$, and $\widehat{Var}(\hat{\gamma}, \hat{\beta})$. Therefore, an approximate $100(1-\alpha)\%$ confidence interval for γ and β are given by

$$[max(0, \hat{\gamma} - z_{\alpha/2}\sqrt{\widehat{Var}(\hat{\gamma})}, \hat{\gamma} + z_{\alpha/2}\sqrt{\widehat{Var}(\hat{\gamma})}], \tag{5.2}$$

and

$$[max(0, \hat{\beta} - z_{\alpha/2}\sqrt{\widehat{Var}(\hat{\beta})}, \hat{\beta} + z_{\alpha/2}\sqrt{\widehat{Var}(\hat{\beta})}], \tag{5.3}$$

respectively, where $z_{\alpha/2}$ is the $(\alpha/2)^{th}$ percentile of the standard normal distribution (Il and Woojin, 2017[29]). However, this approach works well only when number of failures from each system is large enough. Therefore, the generalized confidence interval and the modified signed log-likelihood ratio test may be applied to multiple repairable systems to reduce the number of failures used for systems testing.

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APPENDIX A

GENERATING NHPP POWER-LAW DATA

A.1 Generating NHPP Power-Law Complete Failure Data

The NHPP power-law complete failure data can be simulated using the following algorithm:

Step 1: Input the positive constants of the scale parameter (γ), the shape parameter (β), and the number of failure (n).

Step 2: Simulate n uniform (0,1) random numbers, $u_1, u_2, ..., u_n$.

Step 3: Calculate $t_1 = \left(-\frac{1}{\gamma} ln(u_1)\right)^{1/\beta}$.

Step i: Calculate $t_i = \left(t_{i-1}^{\beta} - \frac{1}{\gamma} ln(u_i)\right)^{1/\beta}$ for $i = 2, 3, ..., n$.

Then, we obtain $t_1, t_2, ..., t_n$ as complete failure times simulated from the NHPP Power-Law process with parameters γ and β.

A.2 Generating NHPP Power-Law Incomplete Failure Data

The NHPP power-law incomplete failure data can be simulated using the following algorithm:

Step 1: Input the positive constants of the scale parameter (γ), the shape parameter (β), the number of failure (n), and the first observed failure (r).

Step 2: Simulate n uniform (0,1) random numbers, $u_1, u_2, ..., u_n$.

Step 3: Calculate $t_1 = \left(-\frac{1}{\gamma} ln(u_1)\right)^{1/\beta}$.

Step i: Calculate $t_i = \left(t_{i-1}^{\beta} - \frac{1}{\gamma} ln(u_i)\right)^{1/\beta}$ for $i = 2, 3, ..., n$.

We obtain $t_1, t_2, ..., t_n$ as complete failure times simulated from an NHPP Power-Law process with parameters γ and β. In this case, we assume that the first $r - 1$ failure times

$(t_1, t_2, ..., t_{r-1})$ are missing. Therefore, the observed failure times are $t_r, t_{r+1}, ..., t_n$.

APPENDIX B

FINDING PERCENTILE OF Z

B.1 Finding the α^{th} Percentile of Z

The $((1-\alpha^{*})/2)^{th}$ and $((1+\alpha^{*})/2)^{th}$ percentile of Z ($Z_{(1-\alpha^{*})/2}$ and $Z_{(1+\alpha^{*})/2}$) can be estimated from the quantity $Z = (1/n)(V/2)^{2(n-r+1)/U}$ using the following algorithm:

Step 1: Generate $U \sim \chi^2_{(2n-2r)}$ and $V \sim \chi^2_{(2n)}$, independently. Then, compute

$$Z = \left(\frac{1}{n}\right)\left(\frac{V}{2}\right)^{2(n-r+1)/U}$$

Step 2: Repeat Step 1 m times.

Step 3: Arrange all Z values in ascending order: $Z_{(1)} < Z_{(2)} < ... < Z_{(m)}$ Then, the α^{th} percentile of Z is estimated by $Z_{(\alpha m)}$.