4-30-2021

Recognizing algebraically constructed graphs which are wreath products.

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A Dissertation
Submitted to the Faculty of
Mississippi State University
in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy
in Mathematics
in the Department of Mathematics and Statistics

Mississippi State, Mississippi

April 2021

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Date of Degree: April 30, 2021

Institution: Mississippi State University

Major Field: Mathematics

Major Professor: Ted Dobson

Title of Study: Recognizing algebraically constructed graphs which are wreath products.

Pages of Study: 53

Candidate for Degree of Doctor of Philosophy

It is known that a Cayley digraph of an abelian group A is isomorphic to a nontrivial wreath product if and only if there is a proper nontrivial subgroup B of A such that the connection set without B is a union of cosets of B in A. We generalize this result to Cayley digraphs of nonabelian groups G by showing that such a digraph is isomorphic to a nontrivial wreath product if and only if there is a proper nontrivial subgroup H of G such that S without H is a union of double cosets of H in G. This result is proven in the more general situation of a double coset digraph (also known as a Sabidussi coset digraph.) We then give applications of this result which include obtaining a graph theoretic definition of double coset digraphs, and determining the relationship between a double coset digraph and its corresponding Cayley digraph. We further expand the result obtained for double coset digraphs to a collection of bipartite graphs called bi-coset graphs and the bipartite equivalent to Cayley graphs called Haar graphs. Instead of considering when this collection of graphs is a wreath product, we consider the more general graph product known as an X-join by showing that a connected bi-coset graph of a group G with respect to some subgroups H_0 and

H_1 of G is isomorphic to an X-join of a collection of empty graphs if and only if the connection set is a union of double cosets of some subgroups K_0 containing H_0 and K_1 containing H_1 in G. The automorphism group of such X-joins is also found. We also prove that disconnected bi-coset graphs are always isomorphic to a wreath product of an empty graph with a bi-coset graph.

Key words: Double coset digraph, bi-coset graph, wreath product, X-join, automorphism group, Cayley digraph

DEDICATION

To the curious students of mathematics, past, present, and future.

ACKNOWLEDGEMENTS

I would like to thank my advisors, Dr. Ted Dobson and Dr. Matthew McBride. I would like to thank Dr. Ted Dobson for all of the encouragement, time, and support he has given me over the years. I have learned much about academic research and being a mathematician under his guidance. I would like to thank Dr. Matthew McBride for picking up the mantle of advisor and mentor after Dr. Dobson moved to another institution. Your advice and encouragement has been invaluable to me.

I would also like to thank the rest of my graduate committee members for their support: Dr. Paul Fabel, Dr. Jonathan Woody, Dr. Vaidyanathan Sivaraman, and Dr. Robert Smith.

I am grateful for all of my teachers and professors throughout the entirety of my academic experiences for the help and advice they have given me. I am especially grateful to Mrs. Amelia Barton for teaching me the joy of learning, to Mrs. Rosa Stokes for teaching me that math is fun, to Dr. Kristen Barnard for being an early role model for me and for your relentless faith, and to Dr. Bangteng Xu for recognizing my talent and reigniting my motivation to pursue a doctoral degree.

I would like to thank my fellow graduate students over the last five years for all of your support and friendship - Justin Smith, Davood Damircheli, Sheida Riahi, Hwamog Kim, and Xiao Wang, among many others. I'm grateful to my academic sister Ágnes Szalai for your friendship and for being my travel buddy during my studies in Europe.

I have immense gratitude for Dennis Wu, for being a safe haven for me during stressful times, and for your love, support, sacrifices, and encouragement during these years. I am grateful to my mother, Delight, my sister, Tiffany, and my godparents, Rudy and Gloria McDonald for their unwavering faith and support.

Last but not least, I want to thank my fur baby Bella for being my furry and warm cuddle buddy and companion as well as for being a cute distraction for me.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I

INTRODUCTION

1.1 Double coset digraphs and wreath products

The purpose of this dissertation is to be able to identify when a double coset digraph is a wreath product. In this section, we collect basic definitions and examples we will need for our main result.

Definition 1

*Let Γ_1 and Γ_2 be digraphs. The **wreath product of** Γ_1 **and** Γ_2, denoted $\Gamma_1 \wr \Gamma_2$, is the digraph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$ and arc set* $\{((u, v), (u, v')) : u \in V(\Gamma_1) \text{ and } (v, v') \in A(\Gamma_2)\} \cup \{((u, v), (u', v')) : (u, u') \in A(\Gamma_1) \text{ and } v, v' \in V(\Gamma_2)\}$.

The wreath product is sometimes referred to as the lexicographic product, graph composition, or the Γ_2-extension of Γ_1. Intuitively, $\Gamma_1 \wr \Gamma_2$ is constructed as follows: First, we have $|V(\Gamma_1)|$ copies of the digraph Γ_2, with these $|V(\Gamma_1)|$ copies indexed by elements of $V(\Gamma_1)$. Next, between corresponding copies of Γ_2 we place every possible arc from one copy to another if in Γ_1 there is an arc between the indexing labels of the copies of Γ_2, and no arcs otherwise. An example of a wreath product is shown in Figure 1.1.

Wreath products of graphs were introduced by Harary [10], mainly to find a graph operation where the automorphism group of the product would be the (group) wreath product of the automorphism groups of the factor graphs. Necessary and sufficient conditions for when the automorphism

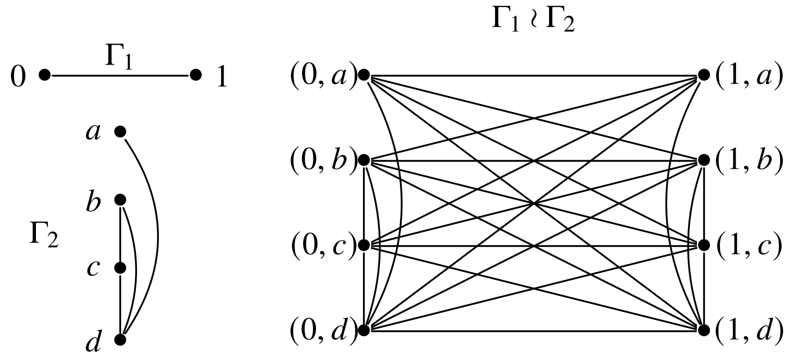

Figure 1.1

Γ_1 and Γ_2 given on the left, $\Gamma_1 \wr \Gamma_2$ on the right.

group of the wreath product is the wreath product of the automorphism groups were given in a sequence of papers first by Sabidussi [21, 22] and then by Hemminger [11], and finally by Ted Dobson and Joy Morris [5] in increasingly general situations.

We give definitions of Cayley and double coset digraphs.

Definition 2

*Let G be a group and $S \subseteq G$. Define a **Cayley digraph** of G, denoted* $\mathrm{Cay}(G, S)$*, to be the digraph with vertex set $V(\mathrm{Cay}(G, S)) = G$ and arc set $A(\mathrm{Cay}(G, S)) = \{(g, gs) : g \in G, s \in S\}$. We call S the **connection set** of* $\mathrm{Cay}(G, S)$*. When $S = S^{-1}$ then it is called a **Cayley graph** of G.*

Figure 1.2 is an example of a Cayley graph of $\mathbb{Z}_{10}$. Note that the connection set $S = \{1, 3, 7, 9\} = S^{-1}$ and so it is a graph and not a digraph.

Definition 3

*Let G be a group and $H, K < G$. For each $g \in G$, $HgK = \{hgk : h \in H, k \in K\}$ is called an (H, K)**-double coset of** g **in** G. The (H, K)-double cosets of G form a partiton of G and need not*

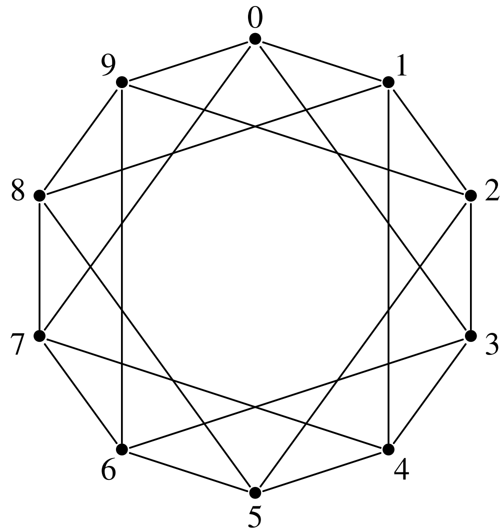

Figure 1.2

The Cayley graph $\mathrm{Cay}(\mathbb{Z}_{10}, \{1, 3, 7, 9\})$.

have the same cardinality. If $H = K$, then the (H, K)-double cosets are referred to as the ***double cosets of*** *H* ***in*** *G or the H* ***double cosets of*** *G.*

Note that the $(\{1\}, K)$-double cosets are the left cosets of K in G and the $(H, \{1\})$-double cosets are just the right cosets of H in G. In general, HgK is a union of right cosets of H as well as the union of left cosets of K.

Definition 4

Let G be a group, $H \leq G$, and $S \subseteq G$ such that $HSH = S$. Define a digraph $\mathrm{Cos}(G, H, S)$ with vertex set $V(\mathrm{Cos}(G, H, S))$ the set of left cosets of H in G, and arc set $(xH, yH) \in A(\mathrm{Cos}(G, H, S))$ if and only if $x^{-1}y \in S$. The digraph $\mathrm{Cos}(G, H, S)$ is called a ***double coset digraph of*** *G with* ***connection set*** *S (or HSH).*

The smallest vertex-transitive non-Cayley graph is the Petersen graph, which can be written as a double coset graph of $\mathrm{AGL}(1,5) \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_4 \cong \langle\rho\rangle \rtimes \langle\tau\rangle$ (see Figure 1.3).

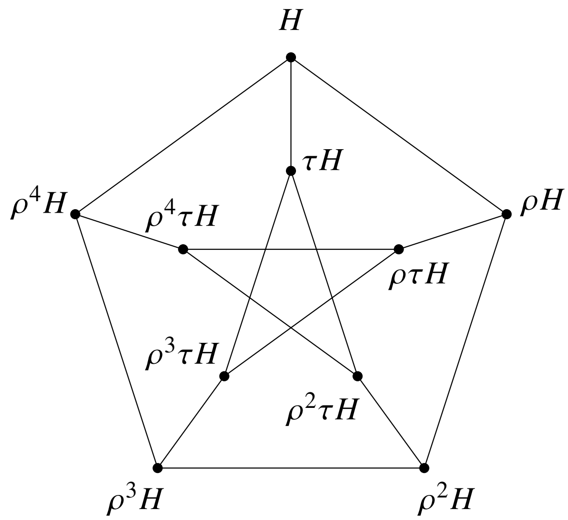

Figure 1.3

The Peterson graph as a double coset graph.

It is customary to impose on H the condition that it is core-free in G. That is, that it contains no nontrivial normal subgroups of G. This ensures that the action of G on the left cosets of H in G is faithful, which is certainly a condition one would want if one were, say, drawing a particular double coset graph. We will not follow this custom - several of the applications of our main results are actually false if this convention is followed. The custom also, inconveniently for us, means that abelian groups, for example, have no double coset digraphs that are not Cayley digraphs as the only core-free subgroup of an abelian group is the trivial group.

We first solve the recognition problem for wreath products of vertex-transitive digraphs. It is known that a Cayley digraph $\mathrm{Cay}(A,S)$ of an abelian group is isomorphic to a wreath product of two smaller digraphs if and only if there exists $1 < B < A$ such that $S \setminus B$ is a union of cosets of B. This was shown explicitly for prime powers in [15, 20] and mentioned without proof in [1]. As every vertex-transitive digraph can be written as a double coset digraph [23, Theorem 2], it suffices to give necessary and sufficient conditions on the connection set of a double coset digraph Γ for Γ to be isomorphic to a non-trivial wreath product. This is done in Theorem 2, where it is shown that a double coset digraph $\mathrm{Cos}(G, H, S)$ can be written as a nontrivial wreath product if and only if there exists $H < K < G$ such that $H(S \setminus K)H$ is a union of double cosets of K in G. As a corollary, a Cayley digraph $\mathrm{Cay}(G, S)$ can be written as a nontrivial wreath product if and only if there exists $1 < K < G$ such that $S \setminus K$ is a union of double cosets of K in G. The full automorphism groups of such wreath products are also found.

We then consider applications of these results to double coset digraphs. Sabidussi showed [23, Theorem 4] that a “multiple" of a double coset digraph $\mathrm{Cos}(G, H, S)$ is isomorphic to a Cayley digraph of G (the multiple is $\mathrm{Cos}(G, H, S) \wr \bar{K}_{|H|}$, where $\bar{K}_n$ is the complement of the complete graph on n vertices). He also showed [23, Lemma 7] that every double coset digraph is a natural quotient of a Cayley digraph. We first unify and strengthen these results by giving a bijective correspondence between irreducible (see Definition 13) double coset digraphs Γ of G (with point stabilizers allowed to vary) and Cayley digraphs of G that can be written in the form $\Gamma \wr \bar{K}_n$ for some irreducible vertex-transitive graph Γ and $n \geq 2$. As every digraph of the form $\Gamma \wr \bar{K}_n$ with Γ irreducible have automorphism group $\mathrm{Aut}(\Gamma) \wr S_n$, double coset digraphs can be interpreted as nothing more than devices for succinctly storing the symmetry information of some Cayley digraphs of G. Finally, we show that the automorphism group of a double coset digraph is determined by the automorphism group of its corresponding Cayley digraph and vice versa, showing that the problem of finding automorphism groups of double coset digraphs is equivalent to finding automorphism groups of Cayley digraphs. We also show a similar, but weaker, relationship between isomorphisms of a double coset digraph of a group G and isomorphisms of its corresponding Cayley digraph of G.

We are able to extend the definition of generalized wreath product digraphs to all double coset digraphs. Generalized wreath product digraphs are a relatively new but very important family of digraphs from the point of view of computing automorphisms groups. They previously were only defined for Cayley digraphs of abelian groups precisely because the recognition problem of wreath products was only solved for Cayley digraphs of abelian groups. This last problem was the original motivation for this work.

1.2 Bi-coset graphs and X-joins

Originally, our next goal was to determine when a Haar graph (formally defined in Definition 6) is isomorphic to a wreath product of a small graph and an empty graph (a graph with no edges) by inspection of its connection set. However, just as quotients of Cayley digraphs need not be Cayley digraphs, quotients of Haar graphs need not be Haar graphs. So the proper context in which to proceed is in the class of graphs which contains all bipartite graphs, and have a group acting transitively on each of the bipartition classes (as Haar graphs have this property - see the discussion of Haar graphs following Definition 6). These graphs are the bi-coset graphs, originally studied by Du and Xu [7], and are the natural analogues to double coset digraphs for bipartite graphs.

Definition 5

Let G be a group, let H_0 and H_1 be subgroups of G, and let S be a union of double cosets of H_0 and

H_1 in G, namely, $S = \bigcup_i H_0 s_i H_1$. Define a bipartite graph $\Gamma = \mathrm{B}(G, H_0, H_1, S)$ with bipartition $V(\Gamma) = G/H_0 \cup G/H_1$ and edge set $E(\Gamma) = \{\{gH_0, gsH_1\} : g \in G, s \in S\}$. This graph is called the ***bi-coset graph*** *with respect to H_0, H_1, and S. We call S the* **connection set** *of Γ.*

In Figure 1.4, we have an example of a bi-coset graph of $\mathbb{Z}_6$ constructed using the subgroups $\langle 3 \rangle$ (the subgroup generated by the element $3 \in \mathbb{Z}_6$) and $\langle 2 \rangle$ with connection set $S = \bigcup \langle 3 \rangle + 1 + \langle 2 \rangle = \mathbb{Z}_6$.

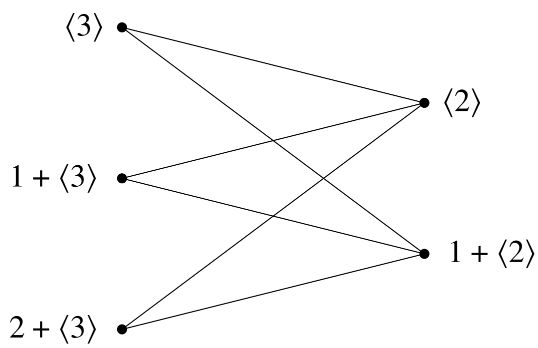

Figure 1.4

Bi-coset graph $\mathrm{B}(\mathbb{Z}_6, \langle 3 \rangle, \langle 2 \rangle, \{0, 1, 2, 3, 4, 5, 6\})$.

It was shown in [7, Lemma 2.3] that the action of G by left multiplication on $V(\Gamma)$ is a semiregular subgroup of $\mathrm{Aut}(\mathrm{B}(G, H_0, H_1, S))$ with two orbits. We denote this subgroup by $\hat{G}$. Notice $\hat{G}^{G/H_0}$ (the induced action of $\hat{G}$ on the left cosets G/H_0) is transitive on G/H_0, and similarly for $\hat{G}^{G/H_1}$. (We use G/H to represent the set of all left cosets of H in G, which is not necessarily a quotient group as we do not assume H is normal in G.)

When $H_0 = H_1$ in a bi-coset graph, we will write the vertex set as $\mathbb{Z}_2 \times G/H_0$. This is only to distinguish the “left partition" of vertices from the “right" partition of vertices.

Definition 6

Let G be a group and $S \subseteq G$. Define the ***Haar graph****, denoted* $\mathrm{Haar}(G, S)$ *with connection set S to be the graph with vertex set $\mathbb{Z}_2 \times G$ and edge set* $\{(\{0, g), (1, gs)\} : g \in G,\ s \in S\}$.

Define $\hat{G}_L = \{\hat{g}_L : g \in G\}$, where $\hat{g}_L : \mathbb{Z}_2 \times G \to \mathbb{Z}_2 \times G$ is given by $\hat{g}_L(i, j) = (i, g_L(j))$. It is straightforward to verify that $\hat{G}_L \leq \mathrm{Aut}(\mathrm{Haar}(G, S))$ for every group G and $S \subseteq G$, and that $\hat{G}_L \cong G_L \cong G$. So $\hat{G}_L$ is a natural subgroup of $\mathrm{Aut}(\mathrm{Haar}(G, S))$ that is semiregular with two orbits and is isomorphic to G.

In Figure 1.5 is an example of a Haar graph. In fact, the graph pictured is the Heawood graph, which is usually constructed as a point-line incidence graph of the Fano plane. The Heawood graph as a Haar graph is $\mathrm{Haar}(\mathbb{Z}_7, \{1, 2, 4\})$.

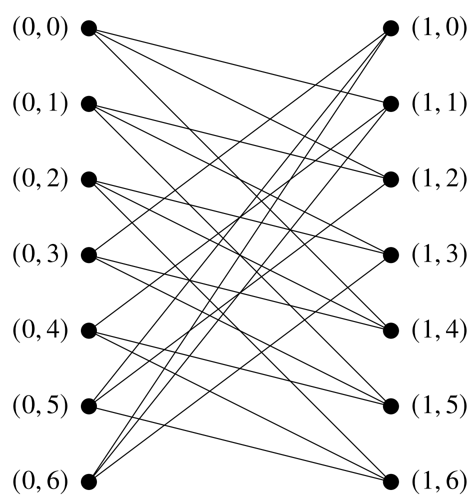

Figure 1.5

$\mathrm{Haar}(\mathbb{Z}_7, \{1, 2, 4\})$, also known as the Heawood graph.

When $H_0 = H_1 = \{1\}$ in the definition of a bi-coset graph, the set S is just a subset of G, and the bi-coset graph with respect to H_0, H_1, and S is simply $\mathrm{Haar}(G, S)$. We remark that some authors refer to Haar graphs as bi-Cayley graphs, and denote them accordingly. As with Cayley graphs, note that $(0, x)(1, y) \in E(\mathrm{Haar}(G, S))$ if and only if $x^{-1}y \in S$. Also note that Haar graphs are the natural bipartite analogues of Cayley digraphs. Haar graphs were introduced in [12].

As the subgroups H_0 and H_1 used to construct a bi-coset graph need not have the same size, the natural product to recover the original bi-coset graph from its quotient will no longer be the wreath product. It is the generalization of the wreath product which allows each vertex to be replaced by arbitrary graphs, called the X-join.

Definition 7

*Let X be a graph, and $Y = \{Y_x : x \in X\}$ a collection of graphs indexed by $V(X)$. The X-**join** of Y is the graph $Z = \bigvee(X, Y)$ with vertex set*

$$V(Z) = \{(x, y) : x \in X, y \in Y_x\} \tag{1.1}$$

and edge set

$$E(Z) = \{\{(x, y), (x', y')\} : \{x, x'\} \in E(X) \text{ or } x = x' \text{ and } \{y, y'\} \in E(Y_x)\}. \tag{1.2}$$

In other words, the graph Z is obtained by replacing each vertex of X by the graph Y_x and inserting either all or none of the possible edges between vertices of Y_u and Y_v depending on whether or not there is an edge between u and v in X. If the Y_x's are all isomorphic, then the X-join of $\{Y_x : x \in X\}$ is the wreath product $X \wr Y$, where $Y \cong Y_x$ for all $x \in X$. Figure 1.6 is an example of an X-join of Y where $X = K_2$ and $Y = \{\bar{K}_2, K_3\}$.

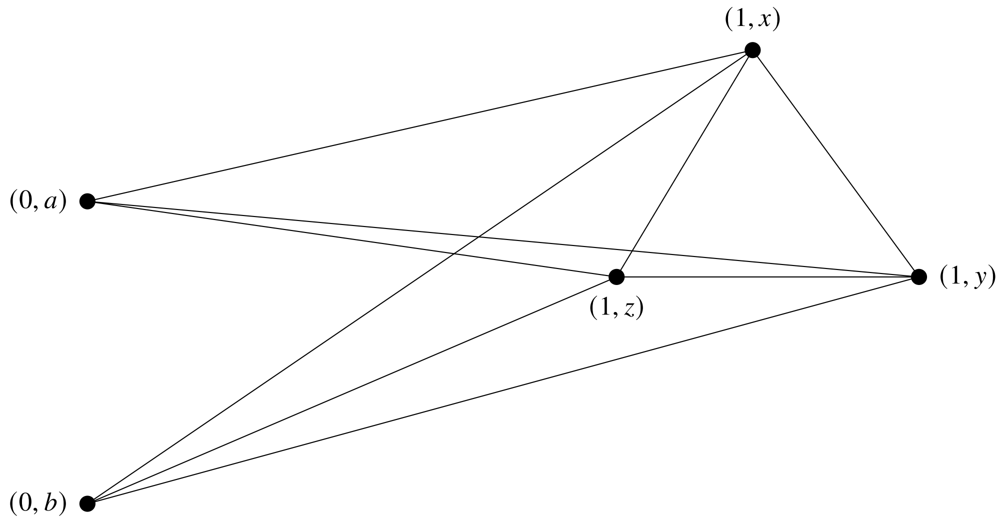

Figure 1.6

$\bigvee(K_2, \{\bar{K}_2, K_3\})$.

We solve the recognition problem for X-joins of connected bi-coset graphs. We give necessary and sufficient conditions on the connection set of a bi-coset graph Z for Z to be isomorphic to an X-join of a collection of empty graphs Y. This is done in Theorem 9, where it is shown that a bi-coset graph $\mathrm{B}(G, H_0, H_1, S)$ can be written as an X-join of empty graphs if and only if there exists $H_0 < K_0 < G$ and $H_1 < K_1 < G$ such that S is a union of (K_0, K_1)-double cosets in G. As a corollary, a Haar graph $\mathrm{Haar}(G, S)$ can be written as an X-join of empty graphs Y if and only if there exists $1 < K_0, K_1 < G$ such that S is a union of (K_0, K_1)-double cosets in G. We also find the full automorphism groups of such X-joins.

We then solve the recognition problem for disconnected bi-coset digraphs. We show that a disconnected bi-coset graph $\mathrm{B}(G, H_0, H_1, S)$ has $[G : K]$ isomorphic components where $\langle SS^{-1}\rangle = K$, and as such is isomorphic to the wreath product of an empty graph and another bi-coset graph. The automorphism group of such graphs is also found.

CHAPTER II

RECOGNIZING VERTEX-TRANSITIVE GRAPHS WHICH ARE WREATH PRODUCTS

2.1 Digraphs as wreath products

In this section, we will introduce some more definitions that will be necessary for our main theorem in this chapter. This section also introduces a lemma and a theorem that were a foundation for our work.

Definition 8

*Let X be a set, $G \leq S_X$ be a transitive group and $B \subseteq X$. B is called a **block** of G if whenever $g \in G$, then $g(B) \cap B = \emptyset$ or B. If $B = \{x\}$ for some $x \in X$ or $B = X$, then B is a **trivial block.***

Note that if B is a block of G, then so is $g(B)$ for every $g \in G$, and is called a **conjugate block of** B. The set of all blocks conjugate to B, denoted $\mathcal{B}$, is a partition of X, and is called a **block system of** G. If $\mathcal{B}$ is the set of orbits of a normal subgroup of G, it is called a **normal** block system of G. We will need a special block system of the wreath product of two transitive permutation groups.

Definition 9

*Let $G \leq S_X$ and $H \leq S_Y$ be transitive groups. The **lexi-partition** of $G \wr H$ with respect to H is the block system $\mathcal{B} = \{\{(x, y) : y \in Y\} : x \in X\}$.*

Definition 10

Suppose that $G \leq S_n$ is a transitive group with a block system $\mathcal{B}$ consisting of m blocks of size k. Then G has an ***induced action on*** *$\mathcal{B}$, which we denote by $G/\mathcal{B}$. Namely, for $g \in G$, we define $g/\mathcal{B}(B) = B'$ if and only if $g(B) = B'$, and set $G/\mathcal{B} = \{g/\mathcal{B} : g \in G\}$. We also define the* ***fixer of*** *$\mathcal{B}$* ***in*** *G, denoted $fix_G(\mathcal{B})$, to be $\{g \in G : g/\mathcal{B} = 1\}$.*

Definition 11

Let Γ be a vertex-transitive digraph whose automorphism group contains a transitive subgroup G with a block system $\mathcal{B}$. Define the ***block quotient digraph of*** *Γ* ***with respect to*** *$\mathcal{B}$, denoted $\Gamma/\mathcal{B}$, to be the digraph with vertex set $\mathcal{B}$ and arc set $A(\Gamma/\mathcal{B}) = \{(B, B') : B \neq B' \in \mathcal{B}$ and $(u, v) \in A(\Gamma), u \in B, v \in B'\}$.*

The following result gives necessary and sufficient conditions to recognize when a vertex-transitive digraph is a wreath product. The first version of this result is by Joseph [13, Lemma 3.11]. The following is a generalization of her result, whose very similar proof is omitted.

Lemma 1

Let Γ be a vertex-transitive digraph whose automorphism group contains a transitive subgroup G that has a block system $\mathcal{B}$. Then $\Gamma \cong \Gamma/\mathcal{B} \wr \Gamma[B_0]$, $B_0 \in \mathcal{B}$, if and only if whenever $B, B' \in \mathcal{B}$ are distinct then there is an arc (x, x') from a vertex $x \in B$ to a vertex $x' \in B'$ if and only if every arc of the form (x, x') with $x \in B$ and $x' \in B'$ is contained in $A(\Gamma)$.

The following theorem was proven in [5, Theorem 5.7], and gives the automorphism group of the wreath product of vertex-transitive digraphs.

Theorem 1

Let Γ_1 and Γ_2 be vertex-transitive digraphs. If $\Gamma = \Gamma_1 \wr \Gamma_2$ *and* $\mathrm{Aut}(\Gamma) \neq \mathrm{Aut}(\Gamma_1) \wr \mathrm{Aut}(\Gamma_2)$, *then there exist positive integers $r > 1$ and $s > 1$ and vertex-transitive digraphs Γ_1' and Γ_2' for which $\Gamma_1 \cong \Gamma_1' \wr K_r$, $\Gamma_2 \cong K_s \wr \Gamma_2'$ or $\Gamma_1 \cong \Gamma_1' \wr \bar{K}_r$, $\Gamma_2 \cong \bar{K}_s \wr \Gamma_2'$. In either case,* $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\Gamma_1') \wr S_{rs} \wr \mathrm{Aut}(\Gamma_2')$.

Suppose $\Gamma_1 \wr \Gamma_2$ is not complete nor the complement of a complete graph, that is, assume that in the statement Γ_1' or Γ_2' have more than one vertex. The above result implies that a wreath product of two vertex-transitive digraphs can always be written as another wreath product of two vertex-transitive digraphs where the automorphism group can also be written as a wreath product of the automorphism groups, that is if $\Gamma \cong \Gamma_1 \wr \Gamma_2$ with $\mathrm{Aut}(\Gamma) \ncong \mathrm{Aut}(\Gamma_1) \wr \mathrm{Aut}(\Gamma_2)$, then there are two nontrivial graphs Γ_1', Γ_2' where $\Gamma \cong \Gamma_1' \wr \Gamma_2'$ with $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma_1') \wr \mathrm{Aut}(\Gamma_2')$.

2.2 Double coset digraphs as wreath products

Before turning to our main result on recognizing wreath products, we have a preliminary result.

Lemma 2

Let G be a group $H \leq G$, and let G act on G/H by left multiplication. Then G acts transitively on G/H and any block system of this action is the set of left cosets of some subgroup $H \leq K \leq G$.

Proof: Note G acts by left multiplication on the left cosets of H in G. Suppose C is a block system of G. By [2, Theorem 1.5A], there is $K \geq H$ such that the block of G that contains H is the orbit of K that contains H. Then the block of G that contains H is the set of all left cosets of H contained in K, which is K. Then any block conjugate to K is simply of the form gK for some $g \in G$, and so $\mathcal{B}$ is the set of left cosets of K in G. ■

Lemma 3

Let G be a group, $H \leq G$, and $S \subseteq G$ such that $S = HSH$. Suppose $\Gamma = \mathrm{Cos}(G, H, S) \cong \Gamma_1 \wr \Gamma_2$ *for* Γ_1 *and* Γ_2 *digraphs with at least two vertices. Let $\mathcal{B}$ be the lexi-partition of* $\mathrm{Aut}(\Gamma_1) \wr \mathrm{Aut}(\Gamma_2)$ *with respect to* $\mathrm{Aut}(\Gamma_2)$. *Then $S \setminus K = K(S \setminus K)K$.*

Proof: By Lemma 2, $\mathcal{B}$ is the set of left cosets of some subgroup $H \leq K \leq G$. Suppose $a \notin K$ and $a \in S$. Let $B = aK \in \mathcal{B}$ such that $H \nsubseteq B$ (here aK is viewed as a union of left cosets of H). Since there is an arc from H to aH if and only if there is an arc from H to every left coset of H contained in B as $\Gamma = \Gamma_1 \wr \Gamma_2$ and $\mathcal{B}$ is the lexi-partition with respect to Γ_2, we conclude that $aK \subseteq HSH$ and $H(S \setminus K)H$ is a union of left cosets of K in G. Then, for every $k, k' \in K$, we have $kH, k'H \subseteq K$, and $(kH, ak'H) \in A(\mathrm{Cos}(G, H, S))$. So $k^{-1}ak' \in HSH$ for every $k, k' \in K$. So $KaK \subseteq HSH$. As $a \notin K$, $k^{-1}ak' \in H(S \setminus K)H$. Thus, $H(S \setminus K)H$ can be written as a union of double cosets of K in G. ∎

With the next result, we are able to recognize when double coset digraphs are wreath products from their connection sets.

Theorem 2

Let G be a group, $H \leq G$, and $S \subseteq G$ such that $HSH = S$. The double coset digraph $\Gamma = \mathrm{Cos}(G, H, S)$ *is isomorphic to a nontrivial wreath product of two vertex-transitive digraphs of smaller order if and only if there exists $H < K < G$ such that $H(S \setminus K)H$ is a union of double cosets of K in G. If such a $H < K < G$ exists and $\mathcal{B}$ is the set of left cosets of K, then*

$$\mathrm{Cos}(G, H, S) \cong \Gamma/\mathcal{B} \wr \Gamma[K] \cong \mathrm{Cos}(G, K, S \setminus K) \wr \mathrm{Cos}(K, H, (S \cap K)). \tag{2.1}$$

Additionally, if Γ is not complete nor the complement of a complete graph and K is chosen to be maximal in G with the above properties, then

$$\mathrm{Aut}(\mathrm{Cos}(G,H,S)) \cong \mathrm{Aut}(\mathrm{Cos}(G,K,S\setminus K)) \wr \mathrm{Aut}(\mathrm{Cos}(K,H,(S\cap K))). \tag{2.2}$$

Proof: Suppose $\Gamma = \Gamma_1 \wr \Gamma_2$, where both Γ_1 and Γ_2 are nontrivial and not Γ. Then the lexi-partition $\mathcal{B}$ of $\mathrm{Aut}(\Gamma_1) \wr \mathrm{Aut}(\Gamma_2)$ with respect to $\mathrm{Aut}(\Gamma_2)$ is a block system of $\mathrm{Aut}(\Gamma_1) \wr \mathrm{Aut}(\Gamma_2) \le \mathrm{Aut}(\Gamma)$. The result follows by Lemma 3.

Conversely suppose $H < K < G$ such that $H(S\setminus K)H$ is a union of double cosets of K in G. We now show $\Gamma \cong \Gamma/\mathcal{B} \wr \Gamma[K]$. This will complete the “if and only if" part of the proof as well as the first part of the first displayed equation. Suppose $(ak_1H, bk_2H) \in A(\Gamma)$ and ak_1H and bk_2H are not contained in the same left coset of K in G, where $k_1, k_2 \in K$. This gives $k_1^{-1}a^{-1}bk_2 \notin K$. Then $k_1^{-1}(a^{-1}b)k_2 \in K(S\setminus K)K = H(S\setminus K)H$, with the last equality holding as $H(S\setminus K)H$ is a union of double cosets of K in G. Thus $K(a^{-1}b)K \subseteq K(S\setminus K)K$ and $(ak'H, bkH) \in A(\Gamma)$ for all $k, k' \in K$. This means that whenever $B, B' \in \mathcal{B}$ are distinct then there is an arc (x, x') from a vertex $x \in B$ to a vertex $x' \in B'$ if and only if every arc of the form (x, x') with $x \in B$ and $x' \in B'$ is contained in $A(\Gamma)$. By Lemma 1, $\Gamma \cong \Gamma/\mathcal{B} \wr \Gamma[K]$.

We next show $\Gamma/\mathcal{B} = \mathrm{Cos}(G,K,S\setminus K)$. The digraph $\mathrm{Cos}(G,K,S\setminus K)$ is a well-defined double coset digraph as $S\setminus K$ is a union of double cosets of K. Let $a, b \in G$ such that $a^{-1}b \notin K$. Then $(aK, bK) \in A(\Gamma/\mathcal{B})$ if and only if there is $a_1, b_1 \in G$ such that $a_1H \subseteq aK$, $b_1H \subset bK$, and $(a_1H, b_1H) \in A(\Gamma)$. This occurs if and only if $a_1^{-1}b_1H \in S$. As $S\setminus K$ is a union of double cosets of K and $a_1^{-1}b_1H \subseteq a^{-1}bK$, we see $a_1^{-1}b_1H \in S\setminus K$. Thus $(aK, bK) \in A(\Gamma/\mathcal{B})$ if and only if

$a^{-1}bK \in S \setminus K$ (viewing $S \setminus K$ as a union of left cosets of K in G), which occurs if and only if $(aK, bK) \in A(\mathrm{Cos}(G, K, S \setminus K))$. So $\Gamma/\mathcal{B} = \mathrm{Cos}(G, K, S \setminus K)$.

As K is a left coset of itself, $K \in \mathcal{B}$. Then $\mathrm{Cos}(G, H, S)[K] = \mathrm{Cos}(K, H, K \cap S)$, and so $\Gamma[K] = \mathrm{Cos}(K, H, S \cap K)$. This completes the proof of Equation (2.1).

It now only remains to show that if Γ is neither complete nor the complement of a complete graph and K is maximal with the property that $H(S \setminus H)H$ is a union of double cosets of K, then Equation (2.2) holds. Suppose otherwise. Then $\mathrm{Cos}(G, K, S \setminus K)$ can be written as a nontrivial wreath product by Theorem 1. By what we have already shown, $S \setminus K$ is a union of double cosets of some subgroup K'. As the vertices of $\mathrm{Cos}(G, K, S \setminus K)$ are left cosets of K, we have $K' \geq K$. As $\mathrm{Cos}(G, K, S \setminus K)$ can be written as a nontrivial wreath product, $K' > K$. But then S is a union of double cosets of K', contradicting the maximality of K. ■

If one follows the convention that H is core-free in G for double coset digraphs, then with $N = \mathrm{core}_G(H)$, the above equation for the automorphism group of $\mathrm{Cos}(G/N, H/N, \{sN : s \in S\}) \cong \mathrm{Cos}(G, H, S)$ becomes $\mathrm{Cos}(G/N, H/N, \{sN : s \in S\}) \cong \Gamma/C \wr \Gamma[K/N] \cong \mathrm{Cos}(G/N, K/N, T) \wr \mathrm{Cos}(K/N, H/N, (S \cap K)N)$ where $T = \{(sN)(K/N) : s \in H(S \setminus K)H\}$.

When $H = \{1_G\}$, $\mathrm{Cos}(G, H, S) \cong \mathrm{Cay}(G, S)$ and we have a special case of Theorem 2 for Cayley graphs.

Corollary 1

A Cayley digraph $\Gamma = \mathrm{Cay}(G, S)$ *of a group G is isomorphic to a nontrivial wreath product of two vertex-transitive digraphs of smaller order if and only if there exists* $1 < K < G$ *such that* $S \setminus K$ *is* *a union of double cosets of K in G. If such a $1 < K < G$ exists and $\mathcal{B}$ is the block system of G that consists of the left cosets of K, then*

$$\mathrm{Cay}(G, S) \cong \Gamma/\mathcal{B} \wr \Gamma[K] \cong \mathrm{Cos}(G, K, S) \wr \mathrm{Cay}(K, S \cap K). \tag{2.3}$$

Additionally, if Γ is not complete nor the complement of a complete graph and K is chosen to be maximal in G with the above properties, then

$$\mathrm{Aut}(\mathrm{Cay}(G, S)) \cong \mathrm{Aut}(\mathrm{Cos}(G, K, S)) \wr \mathrm{Aut}(\mathrm{Cay}(K, S \cap K)). \tag{2.4}$$

Example 1 *Let $G = D_6$, where $D_6 = \{\tau, \rho : \tau^2 = \rho^6 = 1; \tau\rho = \rho^5\tau\}$ is the dihedral group with 12 elements. Let $K = \langle\tau\rangle$, which is not normal in G as $\rho\tau\rho^5) = \tau\rho^4 \notin \langle\tau\rangle$. Consider the Cayley graph $\Gamma = \mathrm{Cay}(D_6, \{\rho, \rho^5, \tau\rho, \tau\rho^5\})$. The connection set of Γ is exactly the double coset $K\rho K$. So using Corollary 1, we see that $\Gamma \cong \mathrm{Cos}(G, K, S \setminus K) \wr \mathrm{Cay}(K, S \cap K)$. Note that $G/L \cong D_3$, $H/L \cong \langle\tau\rangle$, and $S \cap H = \emptyset$. So $\Gamma \cong \mathrm{Cos}(D_6, \langle\tau\rangle, \{\rho, \rho^5, \tau\rho, \tau\rho^5\}) \wr \mathrm{Cay}(\langle\tau\rangle, \emptyset)$.*

In Figure 2.1 you can see the Cayley graph and the graph re-drawn as a wreath product. The graphs can be identified via the map $(\overline{a}, b) \mapsto (ab)$, where $\overline{a}$ is the left coset of K containing a. Colors have also been added to distinguish each of the blocks in the graphs.

If, in the previous result we have $K \triangleleft G$, then we get a slightly nicer sufficient (but not necessary) condition for a Cayley graph of G to be a wreath product.

Corollary 2

A Cayley digraph $\Gamma = \mathrm{Cay}(G, S)$ of a group G is isomorphic to a nontrivial wreath product of two

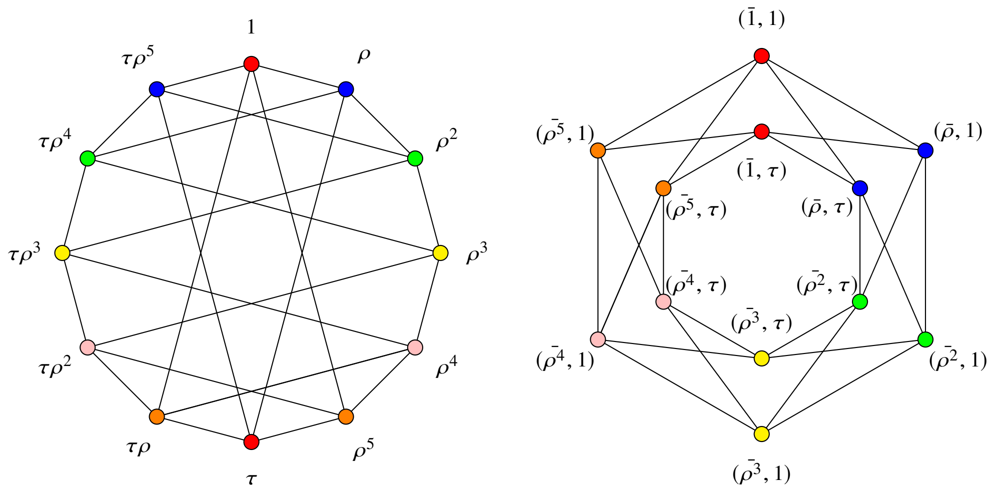

Figure 2.1

$\mathrm{Cay}(D_6, \{\rho, \rho^5, \tau\rho, \tau\rho^5\})$; $\mathrm{Cos}(D_6, \langle\tau\rangle, \{\rho, \rho^5, \tau\rho, \tau\rho^5\}) \wr \mathrm{Cay}(\langle\tau\rangle, \emptyset)$.

vertex-transitive digraphs of smaller order if there exists $1 < H \triangleleft G$ *such that* $S \setminus H$ *is a union of cosets of* H *in* G*. In this case, if* $\mathcal{B}$ *is the block system of* G_L *formed by the orbits of* H*, then*

$$\mathrm{Cay}(G, S) \cong \Gamma/\mathcal{B} \wr \Gamma[H] \cong \mathrm{Cay}(G/H, S_1) \wr \mathrm{Cay}(H, S_2) \tag{2.5}$$

where S_1 *is the set of cosets of* H *contained in* S *and* $S_2 = H \cap S$*. Additionally, if* Γ *is not complete nor the complement of a complete graph and* K *is chosen to be maximal in* G *with the above properties, then*

$$\mathrm{Aut}(\mathrm{Cos}(G, H, S)) \cong \mathrm{Aut}(\mathrm{Cos}(G, K, S)) \wr \mathrm{Aut}(\mathrm{Cos}(K, H, S \cap K)). \tag{2.6}$$

In the case when *all* subgroups of G are normal, the above sufficient condition is also necessary. Groups in which every subgroup is normal are called **Dedekind groups**. Obviously, abelian groups are Dedekind groups, and non-abelian Dedekind groups are called the **Hamilton groups**. Hamilton groups have the form $G = Q_8 \times B \times D$ [9, Theorem 12.5.4], where Q_8 is the quaternion group of order 8, B is an elementary abelian 2-group, and D is a finite abelian group of odd order. This result also generalizes the known result mentioned earlier that Cayley digraphs of abelian groups can be written as nontrivial wreath products of two digraphs of smaller order if and only if its connection set is a union of cosets of some subgroup.

Corollary 3

A Cayley digraph $\mathrm{Cay}(A, S)$ *of a Dedekind group* A *is isomorphic to a nontrivial wreath product of two vertex-transitive digraphs of smaller order if and only if there is* $1 < H < A$ *such that* $S \setminus H$ *is a union of cosets of* H *in* A*. In this case, if* $\mathcal{B}$ *is the block system of* G_L *formed by the orbits of* H*, then*

$$\mathrm{Cay}(A, S) \cong \mathrm{Cay}(A/H, S_1) \wr \mathrm{Cay}(H, S_2) \tag{2.7}$$

where S_1 *is the set of cosets of* H *contained in* S *and* $S_2 = H \cap S$*. If* $\mathrm{Cay}(A, S)$ *is neither complete nor the complement of a complete graph, then choosing* $H \leq A$ *to be the maximal subgroup of* A *such that* $S \setminus H$ *is a union of cosets of* H*, then*

$$\mathrm{Aut}(\mathrm{Cay}(A, S)) = \mathrm{Aut}(\mathrm{Cay}(A/H, S_1)) \wr \mathrm{Aut}(\mathrm{Cay}(H, S_2)). \tag{2.8}$$

2.3 Applications to double coset digraphs

Sabidussi has shown that there is a strong relationship between Cayley digraphs and double coset digraphs. He showed in [23, Theorem 2] that $\mathrm{Cos}(G, H, S) \wr \bar{K}_n$, where $n = |H|$, is isomorphic to a Cayley graph Γ of G (the Cayley graph has connection set a union of left cosets of H in G). He also showed in [23, Theorem 2] that every double coset graph of G is the quotient of a Cayley graph of G with connection set S by the partition of G given by the left cosets of a subgroup of G that is disjoint from S. Examining the proofs of these two results (but not the statements), it turns out the two Cayley graphs Sabidussi constructs in [23, Theorems 2 and 4] are equal. This allows for the use of a stronger quotient than previously defined, which mapped arcs of Γ to arcs of $\Gamma/\mathcal{P}$. The stronger quotient maps arcs of Γ to arcs of $\Gamma/\mathcal{P}$ and non-arcs of Γ to non-arcs of $\Gamma/\mathcal{P}$. This allows more combinatorial and symmetry information to lift from $\Gamma/\mathcal{P}$ to Γ and more to project from Γ to $\Gamma/\mathcal{P}$. This is what we now explore.

While our definition of a double coset digraph is more or less the usual one, it might be better, for the purposes of this section, to think of first fixing the group G, then choosing S, and then letting $H \leq G$ such that $S = HSH$. Such a subgroup always exists, even if it must be $H = 1$, in which case a double coset digraph is isomorphic to a Cayley digraph. In this manner a double coset digraph of G exists for every S, but of course not all of these digraphs will have the same number of vertices.

Definition 12

Let G be a group. Define $\mathrm{Reducible}(G)$ *to be the set of all Cayley digraphs of G that are reducible. That is,* $\mathrm{Reducible}(G) = \{\mathrm{Cay}(G,S) : \mathrm{Cay}(G,S) \cong \Gamma \wr \bar{K}_t, \Gamma \text{ a digraph}, t \geq 1\}$. *Let* $\mathrm{Cos}(G)$ *be the set of all loopless double coset digraphs of G.*

The condition that a double coset digraph $\mathrm{Cos}(G,H,S)$ is loopless means that $H \cap S = \emptyset$. We will implicitly use the fact that if $H \leq K$, $HSH = S$ and $H \cap S = \emptyset$, then $K \cap S = \emptyset$. Note that elements of $\mathrm{Cos}(G)$ need not all have the same order as H need not be core-free in G. If we followed the convention that H is core-free in G, the next result would not be true for abelian groups, for example, as the only core-free subgroup of an abelian group is trivial.

Theorem 3

Let G be a group. Define $\gamma\colon \mathrm{Cos}(G) \to \mathrm{Reducible}(G)$ *by* $\gamma(\mathrm{Cos}(G,H,S)) = \mathrm{Cay}(G,S)$. *Then γ is* *onto, and* $\gamma(\mathrm{Cos}(G, H_1, S_1)) = \gamma(\mathrm{Cos}(G, H_2, S_2))$ *if and only if* $S_1 = S_2$ *and* $\langle H_1, H_2\rangle S\langle H_1, H_2\rangle = S$.

Proof: To see γ is onto, let $\Gamma \in \mathrm{Reducible}(G)$. Then there exists $S \subset G$ such that $\mathrm{Cay}(G, S) \cong \Gamma_1 \wr \bar{K}_t$, where Γ_1 is a digraph and $t \geq 2$. Choose Γ_1 and t so that $\mathrm{Aut}(\mathrm{Cay}(G, S)) \cong \mathrm{Aut}(\Gamma_1) \wr \mathrm{Aut}(\bar{K}_t)$, in which case $\mathrm{Aut}(\mathrm{Cay}(G, S)) \cong \mathrm{Aut}(\Gamma_1) \wr S_t$ by Theorem 1. Then $\mathrm{Aut}(\mathrm{Cay}(G, S))$ has the lexi-partition with respect to S_t, call it $\mathcal{B}$, as a block system with blocks of size t. As $G_L \leq \mathrm{Aut}(\mathrm{Cay}(G, S))$, the blocks of $\mathcal{B}$ are the left cosets of some subgroup K of G. As $\mathrm{Cay}(G, S) = \Gamma_1 \wr \bar{K}_t$ and $\mathcal{B}$ is the lexi-partition, $\mathrm{Cay}(G, S)[gK] \cong \bar{K}_t$ for every $g \in G$ and so $S \cap K = \emptyset$. By Lemma 3, $S = K(S \setminus K)K = KSK$. Then $\mathrm{Cos}(G, K, S) \in \mathrm{Cos}(G)$ and $\gamma(\mathrm{Cos}(G, K, S)) = \mathrm{Cay}(G, S)$. Thus γ is onto.

To see $\gamma(\mathrm{Cos}(G, H_1, S_1)) = \gamma(\mathrm{Cos}(G, H_2, S_2))$ if and only if $S_1 = S_2$ and $\langle H_1, H_2\rangle S\langle H_1, H_2\rangle = S$, let $\mathrm{Cos}(G, H_1, S_1), \mathrm{Cos}(G, H_2, S_2) \in \mathrm{Cos}(G)$. Suppose $\gamma(\mathrm{Cos}(G, H_1, S_1)) = \gamma(\mathrm{Cos}(G, H_2, S_2))$. Then $\mathrm{Cay}(G, S_1) = \mathrm{Cay}(G, S_2)$ and as S_1 and S_2 are both the neighbors of 1_G, $S_1 = S_2$. Set $S = S_1 = S_2$. Then $H_1SH_1 = S$, $H_2SH_2 = S$, and it is then easy to see that $\langle H_1, H_2\rangle S\langle H_1, H_2\rangle = S$. The converse is similarly difficult. ■

Observing that for every group G and connection set S such that there is a subgroup $H \leq G$ with $HSH = S$ there is a maximal subgroup K with $KSK = S$, we may restrict the domain of γ to these unique subgroups K, and obtain a bijection.

Corollary 4

Let G be a group. Let $\mathrm{Cos}_U(G)$ *be the set of all loopless double coset digraphs of G with connection set S such that K is chosen to be maximal such that* $KSK = S$. *That is,* $\mathrm{Cos}_U(G) = \{\mathrm{Cos}(G, K, S) :$

$K \le G, S = KSK$ and $gSg \neq S$ for every $g \in G \setminus K\}$. *Define* $\gamma_U \colon \mathrm{Cos}(G)_U \to \mathrm{Reducible}(G)$ *by* $\gamma_U(\mathrm{Cos}(G, H, S)) = \mathrm{Cay}(G, S)$. *Then* γ_U *is a bijection.*

For our next result, we will need some additional terms.

Definition 13

Let Γ *be a digraph. Define an equivalence relation* R *on* $V(\Gamma)$ *by* $u\ R\ v$ *if and only if the out- and in-neighbors of* u *and* v *are the same. Then* R *is an equivalence relation on* $V(\Gamma)$. *We say* Γ *is* ***irreducible*** *if the equivalence classes of* R *are singletons, and* ***reducible*** *otherwise.*

The equivalence relation above was introduced for graphs by Sabidussi [22, Definition 3], and independently rediscovered by Kotlov and Lovász [14], who call u and v **twins**, and Wilson [24], who calls reducible graphs **unworthy**. It is easy to see that a vertex-transitive digraph Γ is reducible if and only if it can be written as a wreath product $\Gamma_1 \wr \bar{K}_n$ for some positive integer $n \ge 2$. Sabidussi observed in [23] that $\equiv$ is a G-congruence for $G \le \mathrm{Aut}(\Gamma)$.

Theorem 4

Let G *be a group,* $H \le G$, *and* $S \subset G$ *such that* $S \cap H = \emptyset$. *Then* $\mathrm{Cos}(G, H, S)$ *is a well-defined double coset digraph of* G *if and only if the equivalence classes of* R *in* $\mathrm{Cay}(G, S)$ *is refined by* G/H. *Additionally, if* $\mathrm{Cos}(G, H, S)$ *is a well-defined double coset digraph of* G, *then* $\mathrm{Cos}(G, H, S) = \mathrm{Cay}(G, S)/(G/H)$.

Proof: If the equivalence classes of R in $\mathrm{Cay}(G, S)$ is refined by G/H, then $\mathrm{Cay}(G, S)$ can be written as a wreath product $\Gamma_1 \wr \Gamma_2$, and Γ_2 is the empty digraph on H. Then there is a maximum supergraph Γ_2' of $\mathrm{Cay}(G, S)$ that is an empty graph and $\mathrm{Cay}(G, S) \cong \Gamma_1' \wr \Gamma_2'$ for some Γ_1'. As Γ

is a Cayley digraph, we may assume without loss of generality that $\mathrm{Aut}(\Gamma_1' \wr \Gamma_2')$ contains G_L. As Γ_2' is maximal, Γ_1' is irreducible so $\mathrm{Aut}(\Gamma_1' \wr \Gamma_2') \cong \mathrm{Aut}(\Gamma_1') \wr \mathrm{Aut}(\Gamma_2')$. As $G_L \leq \mathrm{Aut}(\Gamma_1' \wr \Gamma_2')$, this gives that the lexi-partition of $\mathrm{Aut}(\Gamma_1' \wr \Gamma_2')$ with respect to $\mathrm{Aut}(\Gamma_2')$ is the set of left cosets of a supergroup K of H, and as Γ_2' is a supergraph of Γ_2, $H \leq K$. As Γ_2' was chosen to be maximal, K is the maximal subgroup of G for which Γ_2' can be chosen to have vertex set a subgroup of G. Applying γ_U^{-1} as defined in Corollary 4, we see $\gamma_U^{-1}(\mathrm{Cay}(G,S)) = \mathrm{Cos}(G,H,S)$ and $S = KSK$. So $S = HSH$ and $\mathrm{Cos}(G,H,S)$ is a well-defined double coset digraph of G.

If $\mathrm{Cos}(G,H,S)$ is a well-defined double coset digraph, then by definition, $S = HSH$ and so $\mathrm{Cay}(G,S)$ is reducible by Corollary 1. Also, $\mathrm{Cay}(G,S) \cong \mathrm{Cay}(G,S)/(G/H) \wr \Gamma[H]$. As $S \cap H = \emptyset$, $\Gamma[H]$ has no arcs. This completes the if and only if statement of the result. To finish, simply observe $\mathrm{Cay}(G,S)/(G/H) \cong \mathrm{Cos}(G,H,S)$. ■

As Theorem 4 is an "if and only if" it gives an alternative definition of a double coset digraph as follows:

Definition 14

Let G be a group and $S \subset G$ such that $\mathrm{Cay}(G,S)$ *is reducible with the equivalence classes of R the left cosets of $K \leq G$. Let $H \leq K$. Define* $\mathrm{Cos}(G,H,S)$ *to be the digraph* $\mathrm{Cay}(G,S)/(G/H)$.

Let G be a group, $1 < H < G$, and $S \subset G$. Theorem 4 also gives an alternative way of computationally checking whether S is a union of double cosets of H. If S is a union of double cosets of H, then Theorem 4 gives that $\mathrm{Cay}(G,S)$ is reducible. This means that the equivalence classes of R are not singleton sets. Thus one only needs to check if, say, 1_G has the same set of in- and out-neighbors as some other vertex in $\mathrm{Cay}(G,S)$.

While this method of determining whether a subset $S \subset G$ defines a double coset digraph, once one has established that $\mathrm{Cos}(G, H, S)$ is well defined (even by checking $S = HSH$), the maximum K for which $S = KSK$ is simply the equivalence class of R which contains 1_G. Computationally, this seems much easier than checking which overgroups K of H in G satisfy $KSK = S$.

The next result shows that all of the symmetry information regarding $\mathrm{Cay}(G, S)$ with S a union of double cosets of a subgroup can be recovered from a double coset digraph with more or less the same connection set via a wreath product. Essentially then, a double coset digraph is simply a way of storing the symmetry information of an appropriate Cayley digraph in a more compact form (as it has fewer vertices and edges than the original Cayley graph). This result is essentially [23, Theorem 4].

Theorem 5

Let G be a group, $H \leq G$, and $S \subset G$ such that $S = HSH$ for some $H \leq G$ and $H \cap S = \emptyset$. Let $H \leq K \leq G$ be maximal such that $S = KSK$, and let $M \leq K$. Let C_M be the left cosets of M in G, $L_M = \mathrm{fix}_G(C_M)$, $n_M = |M|$, and $T_M = \{(sL)(M/L_M) : s \in MSM\}$. Then

$$\mathrm{Cos}(G/L_M, M/L_M, T_M) \wr \bar{K}_{n_M} \cong \mathrm{Cay}(G, S)/C_M \wr \bar{K}_{n_M} \cong \mathrm{Cay}(G, S). \tag{2.9}$$

Proof: First observe that as $M \leq K$ and $S = KSK$, $S = MSM$ as well so $\mathrm{Cos}(G, M, S)$ is well-defined. We apply Theorem 2 with H of that result the subgroup $\{1_G\}$, K of that result M of this result. This gives

$$\begin{aligned} \mathrm{Cos}(G, 1, S) &= \mathrm{Cay}(G, S) \cong \mathrm{Cos}(G/L, M/L, T_M) \wr (\mathrm{Cay}(G, S)[M]) \\ &\cong \mathrm{Cay}(G, S)/C_M \wr (\mathrm{Cay}(G, S)[M]). \end{aligned}$$

As $\mathrm{Cay}(G,S)$ is loopless and $S = MSM$, $1_G \notin S$ and so $M \cap S = \emptyset$. Then $\mathrm{Cay}(G,S)[M] \cong \bar{K}_{n_M}$, and the result follows. ■

The next result says that the automorphism group of a double coset graph can be recovered from its corresponding Cayley digraph. The net effect of this result together with the previous result is that automorphism groups of all double coset digraphs are known if and only if the automorphism groups of their corresponding Cayley digraphs are known. Thus the problems of determining automorphism groups of all Cayley digraphs are equivalent to determining the automorphism groups of all double coset digraphs, and every vertex-transitive digraph is isomorphic to a double coset digraph.

Definition 15

Let $G \leq S_n$ be transitive with block systems $\mathcal{B}$ and C. We write $\mathcal{B} \leq C$ if every block of C is a union of blocks of $\mathcal{B}$, and say $\mathcal{B}$ ***refines*** *C.*

Theorem 6

Let G be a group, $H \leq G$, and $S \subset G$ such that $S = HSH$ and $S \cap H = \emptyset$. If $H \leq K$ is chosen to be maximal such that $S = KSK$, $\mathcal{B}$ is the set of left cosets of K in G, and $n = [K : H]$, then

$$\mathrm{Aut}(\mathrm{Cos}(G,H,S)) \cong (\mathrm{Aut}(\mathrm{Cay}(G,S))/\mathcal{B}) \wr S_n. \tag{2.10}$$

Proof: Let $n_K = |K|$. By Corollary 1, we know that $\mathrm{Aut}(\mathrm{Cay}(G,S)) \cong \mathrm{Aut}(\mathrm{Cos}(G,K,S)) \wr S_{n_K}$, and by Theorem 5, $\mathrm{Cos}(G,H,S) \wr \bar{K}_n \cong \mathrm{Cay}(G,S)$, so $\mathrm{Aut}(\mathrm{Cos}(G,H,S) \wr \bar{K}_n) = \mathrm{Aut}(\mathrm{Cos}(G,K,S)) \wr S_{n_K}$. Notice that the largest subgroup A of $\mathrm{Aut}(\mathrm{Cos}(G,H,S) \wr \bar{K}_n)$ that has the set C of left cosets of H in K as a block system satisfies $A/C = \mathrm{Aut}(\mathrm{Cos}(G,H,S))$, so the largest subgroup B of

$\mathrm{Aut}(\mathrm{Cos}(G,K,S)) \wr S_{n_K}$ that has C as a block system satisfies $B/C = \mathrm{Aut}(\mathrm{Cos}(G,H,S))$. As $H \leq K$, $C_H \preceq C_K$, $B \cong \mathrm{Aut}(\mathrm{Cos}(G,K,S)) \wr S_n \wr S_{n_H}$ and $B/C_H \cong \mathrm{Aut}(\mathrm{Cos}(G,K,S)) \wr S_n$. ■

Our next goal is to show that the isomorphism problem can be solved for all double coset digraphs of G if and only if it can be solved for the corresponding Cayley digraphs of G.

Theorem 7

Let G be a group, $H_1, H_2 \leq G$, and $S_1, S_2 \subseteq G$ such that $H_iS_iH_i$ is a union of double cosets of H_i, $i = 0,1$. Let $K_i \leq G$ be maximal such that $S_i = K_iS_iK_i$, $i = 0,1$. Then $\mathrm{Cos}(G,H_1,S_1) \cong \mathrm{Cos}(G,H_2,S_2)$ *if and only if* $\mathrm{Cay}(G,S_1) \cong \mathrm{Cay}(G,S_2)$.

Proof: If $\mathrm{Cos}(G,H_1,S_1) \cong \mathrm{Cos}(G,H_2,S_2)$ then clearly their vertex-sets have the same cardinality and so $|H_1| = |H_2|$. We first show that $\mathrm{Cay}(G,S_1) \cong \mathrm{Cay}(G,S_2)$.

Suppose $\delta\colon G/H_1 \to G/H_2$ is an isomorphism between $\mathrm{Cos}(G,H_1,S_1)$ and $\mathrm{Cos}(G,H_2,S_2)$. As δ maps G/H_1 to G/H_2 we may define a map $\tilde{\delta}$ from G to G. First $\tilde{\delta}(gH_1) = \delta(gH_1)$, i.e. $\tilde{\delta}$ maps G/H_1 to G/H_2 in the same fashion as δ to extend δ to $\tilde{\delta}$ map elements of gH_1 to $\delta(gH_2)$ (which will be a bijection from gH_1 to some left coset of H_2) in any fashion. We now show every choice of $\tilde{\delta}$ is an isomorphism from $\mathrm{Cay}(G,S_1)$ to $\mathrm{Cay}(G,S_2)$.

First, $\tilde{\delta}$ is a bijection as it maps G/H_1 to G/H_2 and maps the elements of a left coset of H_1 to the elements of a left coset of H_2. Let $(x,y) \in A(\mathrm{Cay}(G,S_1))$. Then $(xH_1, yH_1) \in A(\mathrm{Cos}(G,H_1,S_1))$ so $(\delta(xH_1), \delta(yH_1)) = (aH_2, bH_2) \in A(\mathrm{Cos}(G,H_2,S_2))$. Then $(ah_2, bh_2') \in A(\mathrm{Cay}(G,S_2))$ for every $h_2, h_2' \in H_2$ so $\tilde{\delta}(x,y) \in A(\mathrm{Cay}(G,S_2))$ and $\tilde{\delta}$ is indeed an isomorphism. So $\mathrm{Cay}(G,S_1) \cong \mathrm{Cay}(G,S_2)$.

Suppose $\phi\colon G \to G$ is an isomorphism between $\mathrm{Cay}(G, H_1, S_1)$ and $\mathrm{Cay}(G, H_2, S_2)$. By Theorem 1, we have by choice K_i that $\mathrm{Aut}(\mathrm{Cay}(G, H_i, S_i)) \cong \mathrm{Aut}(\mathrm{Cos}(G, K_i, S_i) \wr \bar{K}_{k_i}$, where $k_i = |K_i|$, $i = 0, 1$. By Theorem 1 we see that $\mathrm{Cos}(G, K_i, S_i)$ cannot be written as a nontrivial wreath product with the complement of a complete graph, so $\mathrm{Cos}(G, K_i, S_i)$ is irreducible, $i = 0, 1$. This in turn implies $k_0 = k_1$. As G/K_i is the lexi-partition of $\mathrm{Cay}(G, S_i)$ with respect to $\bar{K}_{k_i}$, G/K_i is a block system of $\mathrm{Aut}(\mathrm{Cay}(G, S_i))$ and $\phi(G/K_0)$ is a block system of $\mathrm{Aut}(\mathrm{Cay}(G, S_1))$. Then $\phi(G/K_0) \preceq G/K_1$ or $G/K_1 \preceq \phi(G/K_0)$ by [3, Lemma 5], and so $\phi(G/K_0) = G/K_1$. Clearly then ϕ induces a bijection ϕ' between G/K_0 and G/K_1 and it is straightforward to verify that $\phi'(\mathrm{Cos}(G, H_1, S_1)) = \mathrm{Cos}(G, H_2, S_2)$. ∎

The above result may appear to reduce the isomorphism problem for vertex-transitive digraphs to the isomorphism problem for Cayley digraphs, as every vertex-transitive graph can be written as a double coset digraph. This, however, is not the case, as it is quite possible that a Cayley digraph is isomorphic to a Cayley digraph of more than one group (see [20] for example). The above result will not give isomorphisms between two different representations of a single digraph as Cayley digraphs on different groups.

Corollary 5

Let G_0 and G_1 be groups with $|G_0| = |G_1|$, $H_i \leq G_i$ with $|H_0| = |H_1|$, and $S_i \subseteq G_i$ such that $H_iS_iH_i$ is a union of double cosets of H_i, $i = 0, 1$. Let $\{L_1, \dots, L_t\}$ be the set of all regular subgroups of $\mathrm{Aut}(\mathrm{Cay}(G, S_0))$. *Then* $\mathrm{Cos}(G_0, H_0, S_0) \cong \mathrm{Cos}(G_1, H_1, S_1)$ *if and only if there is some* $1 \leq j \leq t$ *such that* $L_j \cong G_1$ *and* $\mathrm{Cay}(G_0, S_0) \cong \mathrm{Cay}(L_j, T_j)$.

Proof: As $|G_0| = |G_1|$ and $|H_0| = |H_1|$ we have $|V(\mathrm{Cos}(G_0, H_0, S_0))| = |V(G_1, H_1, S_1)|$. As $H_iS_iH_i = S_i$, by Corollary 1, we have $\mathrm{Cos}(G_i, H_i, S_i) \wr \bar{K}_n \cong \mathrm{Cay}(G_i, S_i)$, $i = 0, 1$.

Suppose $\mathrm{Cos}(G_0, H_0, S_0) \cong \mathrm{Cos}(G_1, H_1, S_1)$. Set $n = |H_0| = |H_1|$. Then $\mathrm{Cos}(G_0, H_0, S_0) \wr \bar{K}_n \cong \mathrm{Cos}(G_1, H_1, S_1) \wr \bar{K}_n$. As $\mathrm{Cos}(G_i, H_i, S_i) \wr \bar{K}_n \cong \mathrm{Cay}(G_i, S_i)$, $\mathrm{Aut}(\mathrm{Cay}(G_0, S_0))$ contains a regular subgroup isomorphic G_1, so $G_1 \cong L_j$ for some L_j. Then there exists $T_j \subseteq L_j$ with $\mathrm{Cay}(G_1, S_1) \cong \mathrm{Cay}(T_j, S_j)$, and so $\mathrm{Cay}(G_0, S_0) \cong \mathrm{Cay}(T_j, S_j)$.

Conversely, suppose there is some $1 \leq j \leq t$ such that $L_j \cong G_1$ and $\mathrm{Cay}(G_0, S_0) \cong \mathrm{Cay}(L_j, T_j)$. Then $\mathrm{Cay}(G_1, S_1) \cong \mathrm{Cay}(G_0, S_0)$. As $\mathrm{Cos}(G_i, H_i, S_i) \wr \bar{K}_n \cong \mathrm{Cay}(G_i, S_i)$, $i = 0, 1$, we have $\mathrm{Cos}(G_0, H_0, S_0) \cong \mathrm{Cos}(G_1, H_1, S_1)$. ■

In particular, this shows the importance of the problem of determining when a Cayley digraph is isomorphic to a Cayley digraph of more than one group. We should point out of course that even if this additional problem were solved, the reduction of the isomorphism problem for vertex-transitive digraphs to Cayley digraphs is only of theoretical interest, as the Cayley digraphs corresponding to double coset digraphs have more vertices and edges, and so the automorphism groups of the double coset digraphs would, in practice, be calculated first.

2.4 Generalized wreath products

Generalized wreath products are a fairly new class of digraphs that were introduced to describe automorphism groups of circulant digraphs (Cayley digraphs of cyclic groups). They form one of three broad families of digraphs (the others being deleted wreath product types with a factor a symmetric group, and normal Cayley digraphs of $\mathbb{Z}_n$), and it was stated in [18, Theorem 2.3] that all circulant digraphs fall into at least one of these families. This result is a translation of results proven using results of Schur rings [8, 16, 17] to the language of vertex-transitive digraphs. We will not dwell on a fourth family, namely those with primitive automorphism groups, as for circulant digraphs these digraphs are only the complete graph and its complement. Normal Cayley digraphs were introduced in [25] by M.Y. Xu in 1998 and are those Cayley digraphs of G for which $G_L \triangleleft \mathrm{Aut}(\mathrm{Cay}(G, S))$. Deleted wreath type digraphs were first defined in [1], and are those digraphs whose automorphism group is the same as a deleted wreath product of two smaller digraphs (in general this family should probably be those digraphs whose automorphism group has a factor which is the automorphism group of a digraph of smaller order that is quasiprimitive).

Definition 16

*Let Γ_1 and Γ_2 be digraphs. The **deleted wreath product of** Γ_1 **and** Γ_2, denoted $\Gamma_1 \wr_d \Gamma_2$, is the digraph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$ and arc set*

$$\{((x_1, y_1), (x_2, y_2)) : (x_1, x_2) \in A(\Gamma_1) \text{ and } y_1 \neq y_2 \text{ or } x_1 = x_2 \text{ and } (y_1, y_2) \in A(\Gamma_2)\}. \quad (2.11)$$

The determination of automorphism groups of circulant digraphs is not quite complete. While we do have a classification of circulant digraphs into the three families mentioned above, the automorphism groups of generalized wreath products are not known at this time. The same is true for deleted wreath products, although some partial results are given in [6]. These results do give a general template on how to approach the problem of determining automorphism groups of other classes of vertex-transitive digraphs: prove a classification type result to show that all Cayley digraphs under consideration fall within a certain set of families of digraphs. Then one should determine the automorphism groups of the digraphs in each of the families, and a complete determination of the automorphism groups will be obtained.

Generalized wreath products have thus far only been defined for Cayley digraphs of abelian groups. The reason that they were not defined for Cayley digraphs of nonabelian groups or vertex-transitive digraphs that are not Cayley digraphs is that the recognition problem for when such digraphs are wreath products had not been solved. The idea behind a generalized wreath product Γ is that we do not have any control over which arcs have both endpoints inside a block of a block system C of a transitive subgroup of the automorphism group, but the other arcs form a digraph that is a wreath product, and the lexi-partition of that wreath product refines C. Thus we want to be able to decompose the arc set of the digraph into two sets in such a way that one set of arcs defines a disconnected digraph (which is a wreath product) and the other a wreath product in such a way that the automorphism group of Γ contains the intersection of the automorphism groups of the two wreath products. As up to now we could not determine, by inspection of the connection set, whether the remaining arcs formed a wreath product, we could not extend the definition to other vertex-transitive digraphs. Note that this will imply that $\mathrm{Aut}(\Gamma)$ contains a subgroup isomorphic to the intersection of the automorphism groups of the two wreath product digraphs.

Correcting this defect was the original motivation for this problem. We now define generalized wreath products for all double coset digraphs, and recall that by [23, Theorem 2] that every vertex-transitive digraph is isomorphic to a double coset digraph.

Definition 17

Let G be a group with subgroups $1 \leq H < K \leq L < G$ and $S \subseteq G$ a union of double cosets of H in G such that $S \setminus L$ is a union of double cosets of K. The double coset digraph $\mathrm{Cos}(G, H, S)$ *is called a* (K, L)***-generalized wreath product****.*

Notice that if $K = L$, then by Theorem 2 we have that $\mathrm{Cos}(G, H, S)$ is a wreath product. So generalized wreath products are a generalization of the wreath product construction. It is straightforward to show that if $\mathrm{Aut}(\mathrm{Cos}(G, H, S)$ is a (K, L)-generalized wreath product, then $\mathrm{Aut}(\mathrm{Cos}(G, H, S))$ does contain the intersection of $\mathrm{Aut}(\mathrm{Cos}(G, H, L \cap S))$ and $\mathrm{Aut}(\mathrm{Cos}(G, H, S \setminus L))$, and these digraphs have the properties we were aiming for.

Lemma 4

Let G be a group, and $1 \leq H < K \leq L < G$. Let $S \subset G$, $S_1 = S \cap L$ and $S_2 = S \setminus L$. If $\mathrm{Cos}(G, H, S)$ *is isomorphic to a (K, L)-generalized wreath product then*

$$\mathrm{Aut}(\Gamma) \geq \mathrm{Aut}(\mathrm{Cos}(G, H, S_1)) \cap \mathrm{Aut}(\mathrm{Cos}(G, H, S_2)) \geq (S_r \wr \Gamma[L]) \cap (\Gamma/\mathcal{B} \wr S_t), \qquad (2.12)$$

where $r = [G : L]$, $t = |K|$, and $\mathcal{B}$ is the set of left cosets of H in G.

A good general rule is that symmetry in digraphs is rare; one expects that the automorphism group of a (K, L)-generalized wreath product would be $\mathrm{Aut}(\mathrm{Cos}(G, H, S \cap L)) \cap \mathrm{Aut}(\mathrm{Cos}(G, H, L \setminus K))$. It seems likely that there will be many ways in which the automorphism group will be larger than expected. The following problem is then natural, and its solution is a crucial step in determining automorphism groups of vertex-transitive digraphs.

Problem 1

Determine necessary and sufficient conditions for the automorphism group of a (K, L)-generalized wreath product $\mathrm{Cos}(G, H, S)$ *to have automorphism group* $\mathrm{Aut}(\mathrm{Cos}(G, H, S \cap L)) \cap \mathrm{Aut}(\mathrm{Cos}(G, H, S \setminus L))$ *as expected. Additionally, for each class of generalized wreath products that do not have automorphism group* $\mathrm{Aut}(\mathrm{Cos}(G, H, S \cap L)) \cap \mathrm{Aut}(\mathrm{Cos}(G, H, S \setminus L))$*, determine* *the full automorphism group for each such class. In particular, solve these problems for circulant digraphs.*

It has been shown in [4, Theorem 35] that a (K, L)-generalized wreath product circulant digraph of square-free order n has automorphism group $\mathrm{Aut}(\mathrm{Cay}(\mathbb{Z}_n, S \cap L)) \cap \mathrm{Aut}(\mathrm{Cay}(\mathbb{Z}_n, S \setminus L))$. Also, Theorem 1 solves this problem in the special case when Γ is a wreath product, which is guaranteed when $K = L$.

CHAPTER III

RECOGNIZING BI-COSET GRAPHS WHICH ARE X-JOINS

3.1 The automorphism group of the join of a bipartite graph with empty graphs

Hemminger in his 1968 paper provided the necessary and sufficient conditions for the automorphism group of an X-join of Y to be what he calls the “natural" ones, those maps that permute the components of the X-join and then applying an arbitrary automorphism of each of the components. This will be our tool for determining the automorphism group of a connected bi-coset graph when it is an X-join (specifically of empty graphs). We begin this section by introducing all the necessary definitions and terminology. In this section we do not assume graphs are finite.

Definition 18

*Let Z be an X-join of $\{Y_x\}_{x\in X}$. Then a graph automorphism φ of Z is called **natural** if for each $x_1 \in X$ there is an $x_2 \in X$ such that $\varphi(Y_{x_1}) = Y_{x_2}$. Otherwise φ is called **unnatural**.*

It is important to note that if φ is a natural automorphism of Z, then φ induces an automorphism φ^* of X where $\varphi^*(x_1) = x_2$ if $\varphi(Y_{x_1}) = Y_{x_2}$. Similarly, if ϕ is an automorphism of X such that $Y_x \cong Y_{\phi(x)}$ for all $x \in X$, then ϕ induces a set of natural automorphisms of Z where if ϕ' is one of these natural automorphisms, then $(\phi')^* = \phi$.

Definition 19

Let Γ be a graph. Define D and the equivalence relations R_Γ and S_Γ on $V(\Gamma)$ as follows:

1. $(v, u) \in R_\Gamma$ *if* $N_\Gamma(v) = N_\Gamma(u)$,
2. $(v, u) \in S_\Gamma$ *if* $N_\Gamma(v) \cup \{v\} = N_\Gamma(u) \cup \{u\}$,
3. $D = \{(v, v) : v \in V(\Gamma)\}$.

Γ *is called* ***irreducible*** *if* $R_\Gamma = D$, *otherwise it is called* ***reducible***. *We call the set of equivalence classes of* R_Γ *the* ***unworthy*** *partition of* $V(\Gamma)$. D *is called the* ***diagonal*** *(or identity) relation. Note that* $S_\Gamma = D$ *if and only if* $\bar{\Gamma}$ *is irreducible.*

Definition 20

Let X *be a graph and* Y *a collection of graphs indexed by* $V(X)$. *The partition* $\{\{(x, y) : y \in V(Y_x)\} : x \in V(X)\}$ *is the* ***join partition of*** $\bigvee(X, Y)$.

Theorem 8

Let Γ *be a graph. Suppose* $\Gamma \cong \bigvee(X, Y)$, *where* $Y = \{Y_x : x \in V(X)\}$ *and each* $Y_x \in Y$ *is an empty graph. The following are equivalent:*

1. X *is irreducible,*
2. *the join partition of* $\bigvee(X, Y)$ *is the unworthy partition of* Γ,
3. $\mathrm{Aut}(\Gamma)$ *is the group of the complete set of natural automorphisms induced by* X.

Proof: (1): $\implies$ (2) Suppose X is irreducible. It is clear that if $x \in V(X)$ and Y_x is an empty graph then $u\ R_\Gamma\ v$ for all $u, v \in \{x\} \times V(Y_x)$ (regardless of whether X is irreducible or if the other graphs in Y are empty graphs), so the join partition of Γ is refined by the unworthy partition of Γ. As X is irreducible, for any two distinct vertices $x_1, x_2 \in V(X)$, one of x_1, x_2, say x_1, is adjacent to some vertex x_3 that x_2 is not adjacent to in X. Then no vertex of $\{(x_2, y) : y \in V(Y_{x_2})\}$ is adjacent to any vertex of $\{(x_3, y) : y \in V(Y_{x_3})\}$ while all vertices $\{(x_1, y) : y \in V(Y_{x_1})\}$ are adjacent to every vertex of $\{(x_3, y) : y \in V(Y_{x_3})\}$. We conclude that the cells of the unworthy partition that contains $\{(x_1, y) : y \in V(Y_{x_1})\}$ is $\{(x_1, y) : y \in V(Y_{x_1})\}$. As $x_1 \in V(X)$ was arbitrary, the join partition of Γ is the unworthy partition of Γ.

(2) $\implies$ (3): Suppose the join partition of $\bigvee(X, Y)$ is the unworthy partition of Γ. Let $\gamma \in \mathrm{Aut}(\Gamma)$ and $u, v \in \Gamma$ such that $u\ R_\Gamma\ v$. Then the neighbors in Γ of $\gamma(u)$ and $\gamma(v)$ are the same. Thus $\gamma(u)\ R_\Gamma\ \gamma(v)$. This implies that γ maps the unworthy partition to the unworthy partition, which implies that γ is a natural automorphism of Γ induced by X as the unworthy partition Γ is the join partition of $\bigvee(X, Y)$.

(3) $\implies$ (1): Suppose $\mathrm{Aut}(\Gamma)$ is the group of the complete set of natural automorphisms induced by X. If X is reducible, then there exists vertices $x_1, x_2 \in V(X)$ whose neighbors in X are the same. Then every vertex of $\{(x_1, y) : y \in V(Y_{x_1})\}$ and every vertex of $\{(x_2, y) : y \in V(Y_{x_2})\}$ have the same neighbors. The permutation $((x_1, y_1), (x_2, y_2))$, $y_2 \in V(Y_{x_1})$ and $y_2 \in V(Y_{x_2})$ is an automorphism of Γ and it is an unnatural automorphism, a contradiction. ∎

3.2 Automorphism groups of reducible bi-coset graphs

We will have need of quotients, but as bi-coset graphs need not be vertex-transitive [7], the natural partition by which to quotient has two parts (one for each cell of the natural bipartition), and so is slightly more complicated.

Definition 21

Let Γ be a bi-coset graph $\mathrm{B}(G, H_0, H_1, S)$ where the left partition B_0 consists of the left cosets of H_0 and the right partition B_1 consists of the left cosets of H_1. Let $H_i \leq K_i \leq G$, $i = 0, 1$. Define *the **join-partition of** $V(\Gamma)$ **with respect to** K_0 **and** K_1, denoted $\mathcal{P}(K_0, K_1)$, of the vertices of Γ as follows:*

1. *Let $\mathcal{P}_i$ be the partition of B_i that consists of the left cosets of K_i in G. Note $\mathcal{P}_i$ is a block system of G with its action on B_i by left multiplication, $i = 0, 1$.*
2. *The partition $\mathcal{P}(K_0, K_1)$ of $V(\Gamma)$ is $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1$. This partition of the vertices of Γ does not necessarily form a block system of $\mathrm{Aut}(\Gamma)$ as Γ may not be vertex-transitive.*

With the natural partition in hand, we may now define the appropriate quotient graphs for bi-coset graphs.

Definition 22

*Let Ω be a set, and $\mathcal{P}$ a partition of Ω. Let Γ be a digraph with vertex set Ω. Define the **quotient digraph** of Γ with respect to $\mathcal{P}$, denoted $\Gamma/\mathcal{P}$, by $V(\Gamma/\mathcal{P}) = \mathcal{P}$ and $(P_1, P_2) \in A(\Gamma/\mathcal{P})$ if and only if $(p_1, p_2) \in A(\Gamma)$ for some $p_1 \in P_1$ and $p_2 \in P_2$.*

Lemma 5

Let Γ be a connected bipartite graph with bipartition $\mathcal{B} = \{B_0, B_1\}$. The unworthy partition $\mathcal{P}$ of Γ refines $\mathcal{B}$.

Proof: Suppose otherwise, and so there exists $P_x \in \mathcal{P}$ such that $P_x \cap B_i \neq \emptyset$, $i = 0, 1$. As Γ is connected, $\Gamma/\mathcal{P}$ is connected. So P_x is adjacent in $\Gamma/\mathcal{P}$ to some $P_{x'} \in \mathcal{P}$. Let $i \in \mathbb{Z}_2$ such that $P_{x'} \cap B_i \neq \emptyset$, and $x'' \in P_x$ such that $x'' \in B_i$. Then $x'x'' \in E(\Gamma)$, contradicting the choice of $\mathcal{B}$ as a bipartition of Γ. ■

Lemma 6

Let $\Gamma = B(G, H_0, H_1, S)$ be a connected bi-coset graph. Let $\mathcal{E}$ be the unworthy partition of $V(\Gamma)$, *and $\mathcal{E}_i$ consist of those elements of $\mathcal{E}$ that are contained in B_i, $i = 0, 1$. Then $\mathcal{E}_i$ is the set of orbits of the kernel of the action of F on B_{i+1}, where F is the set-wise stabilizer of $\mathcal{B}$, and are the set of left cosets of some subgroups $H_{i+1} \leq K_{i+1} \leq G$.*

Proof: By Lemma 5, we see that $\mathcal{E}_i$ is a partition of B_i, $i = 0, 1$. Let L_i be the kernel of the action of F on B_{i+1}, $\mathcal{O}$ be an orbit of L_i, $v \in \mathcal{O}$, and $E_i \in \mathcal{E}_i$ with $v \in E_i$. As L_i is the kernel of the action of F on B_{i+1}, L_i fixes every vertex of B_{i+1}. As every neighbor x in Γ of v is contained in B_{i+1} and $L_i^{\mathcal{O}}$ is transitive on $\mathcal{O}$, every element of $\mathcal{O}$ is adjacent in Γ to x. As $v \in \mathcal{O}$ is arbitrary, and $x \in B_{i+1}$ is arbitrary, every element of $\mathcal{O}$ has the same neighbors in Γ. Thus $\mathcal{O} \subseteq E_i$. Conversely, let $E_i \in \mathcal{E}_i$, and for each $\sigma \in S_{E_i}$, define $\bar{\sigma} \in S_{V(\Gamma)}$ by $\bar{\sigma}(v) = \sigma(v)$ if $v \in E_i$ and $\bar{\sigma}(v) = v$ otherwise. Then $\bar{\sigma} \in F$ so $E_i \subseteq \mathcal{O}$, and hence, $E_i = \mathcal{O}$. That the $\mathcal{E}_i$ are the sets of left cosets of subgroups $H_i \leq K_i \leq G$ follows from Lemma 2. ■

We now give the automorphism group of every bi-coset graph which can be written as an X-join of a set of empty graphs Y such that the partition $\mathcal{P} = \{V(Y_x) : x \in X\}$ refines the natural bipartition $\mathcal{B}$.

Corollary 6

Let $\Gamma = B(G, H_0, H_1, S)$ be a connected bi-coset graph that can be written as a nontrivial X-join of empty graphs Y such that $\mathcal{P} = \{V(Y_x) : x \in X\}$ refines the natural bipartition $\mathcal{B}$ of Γ. Then there exists $H_i \leq K_i \leq G$ such that for the (K_0, K_1)-join partition $\mathcal{P}'$ and where Y' is the collection of empty graphs on the cells of $\mathcal{P}'$, we have $\Gamma \cong \bigvee(\Gamma/\mathcal{P}', Y')$ and $\mathrm{Aut}(\bigvee(\Gamma/\mathcal{P}', Y'))$ is the complete set of natural automorphisms induced by $\Gamma/\mathcal{P}'$.

Proof: Let F be the setwise stabilizer of $\mathcal{B}$ in $\mathrm{Aut}(\Gamma)$. Let L_i be the kernel of the action of F on B_{i+1}. By Lemma 6 the orbits of L_i are equivalence classes of R_Γ and are the set of left cosets of $H_i \leq K_i \leq G$, $i = 0, 1$. Consider the action of $\mathrm{Aut}(\Gamma/\mathcal{P}')$ on $\Gamma/\mathcal{P}'$, suppose it is not faithful. Let δ_i be an element of the kernel of the action of $\mathrm{Aut}(\Gamma/\mathcal{P}')$ on B'_{i+1}, $i = 0, 1$, where $\mathcal{B}' = \{B'_0, B'_1\}$ is the natural bipartition of $\Gamma/\mathcal{P}'$. Suppose $\delta_i(K_i) = g_iK_i$ for some $g_i \in G$, $g_i \neq 1_G$, and δ_i fixes all left cosets of K_{i+1}. Thus K_i and g_iK_i must have the same neighbors in the quotient. But this implies that H_i and g_iH_i have the same neighbors in the quotient, and so $H_i \; R_\Gamma \; g_iH_i$. Then K_i must contain g_iK_i, which is a contradiction to our choice of K_i. Thus, the action of $\mathrm{Aut}(\Gamma/\mathcal{P}')$ on $\Gamma/\mathcal{P}'$ must be faithful. So $R_{\Gamma/\mathcal{P}'} = D$, and the result follows by Theorem 8. ∎

Corollary 7

Let $\Gamma = \mathrm{Haar}(G, S)$ be a connected Haar graph that can be written as a nontrivial X-join of empty graphs Y such that $\mathcal{P} = \{V(Y_x) : x \in X\}$. Then there exists $K_i \leq G$ such that for the (K_0, K_1)-join partition $\mathcal{P}'$ and where Y' is the collection of empty graphs on the cells of $\mathcal{P}'$, we have $\Gamma \cong \bigvee(\Gamma/\mathcal{P}', Y')$ and $\mathrm{Aut}(\bigvee(\Gamma/\mathcal{P}', Y'))$ is the complete set of natural automorphisms induced by $\Gamma/\mathcal{P}'$.

Proof: When $H_0 = H_1 = \{1_G\}$, $\Gamma = B(G, H_0, H_1, S) = \mathrm{Haar}(G, S)$. The result follows from Corollary 6. ∎

3.3 Connected bi-coset graphs as X-joins

Lemma 7

Let $\Gamma = B(G, H_0, H_1, S)$ and $\mathcal{P}$ a partition of $V(\Gamma)$ that refines $\mathcal{B}$. Then $\mathcal{P}$ is a G-invariant *partition of $V(\Gamma)$ under the left multiplication action of G if and only if there exists $H_0 \leq K_0 \leq G$ and $H_1 \leq K_1 \leq G$ such that $\mathcal{P}$ is the join-partition of $V(\Gamma)$ with respect to K_0 and K_1.*

Proof: It is clear that the join-partition of $V(\Gamma)$ with respect to K_0 and K_1 is a refinement of $\mathcal{B}$ and invariant under the left multiplication action of G as left multiplication permutes left cosets of any subgroup of G, where $H_0 \leq K_0 \leq G$ and $H_1 \leq K_1 \leq G$. Conversely, suppose that $\mathcal{P}$ refines $\mathcal{B}$ and is invariant under the left multiplication action of G. Let $\mathcal{P}_i$ consists of those subsets of $\mathcal{P}$ that are properly contained in B_0 and B_1. As $\mathcal{P}$ refines $\mathcal{B}$, $\mathcal{P}_i$ is a partition of B_i, $i = 0, 1$. Additionally, G is transitive on B_i, $i = 0, 1$, so $\mathcal{P}_i$ is a block system of G^{B_i}, $i = 0, 1$. Let $P_i \in \mathcal{P}_i$ contain 1_G. By [2, Theorem 1.5A], P_i is an orbit of the left multiplication action of G of some subgroup K_i of G which contains the stabilizer in G of a point, so we assume without loss of generality that this stabilizer is H_i. Then left multiplication of an element of P_i by an element of K_i fixes P_i. Then the vertices in P_i are the sets $\{k_i H_i : k_i \in K_i\}$, $i = 0, 1$, and so P_i consists of all left cosets of H_i that are contained in K_i. So $P_i = K_i$, and as G acts by left multiplication, $\mathcal{P}_i$ is the set of left cosets of K_i in G. So $\mathcal{P}$ is the join-partition of $V(\Gamma)$ with respect to K_0 and K_1. ■

We first prove a variation of Lemma 1 adapted for bi-coset graphs. We remind the reader that bi-coset graphs need not be vertex-transitive, and we do not assume they are here.

Lemma 8

Let G be a group, $H_0 \leq K_0 \leq G$, $H_1 \leq K_1 \leq G$, $m_0 = [K_0 : H_0]$, and $m_1 = [K_1 : H_1]$. Let $S \subseteq G$ such that S is a union of (H_0, H_1)-double cosets in G, and $\Gamma = \mathrm{B}(G, H_0, H_1, S)$. Let $X = \Gamma/\mathcal{P}$ where $\mathcal{P}$ is the join-partition of Γ with respect to K_0 and K_1, $Y_{g,i}$ be the empty graph on the left cosets of H_i contained in gK_i, and $Y = \{Y_{g,i} : g \in G, i \in \mathbb{Z}_2\}$. Then Γ is the X-join of Y if and *only if whenever $P_0 \in \mathcal{P}_0$ and $P_1 \in \mathcal{P}_1$, then there is an edge $\{x_0, x_1\}$ from a vertex $x_0 \in P_0$ to a vertex $x_1 \in P_1$ if and only if every edge of the form $\{x_0, x_1\}$ with $x_0 \in P_0$ and $x_1 \in P_1$ is contained in $E(\Gamma)$.*

Proof: Suppose $\Gamma = \bigvee(X, Y)$. Let $P_0 \in \mathcal{P}_0$, $P_1 \in \mathcal{P}_1$, and $x_0 \in P_0$ and $x_1 \in P_1$. Then

$$\{x_0, x_1\} \in E(\Gamma) \quad \text{if and only if} \quad \{P_0, P_1\} \in E(\Gamma/\mathcal{P}) \tag{3.1}$$

$$\text{if and only if} \quad \{x_0, x_1\} \in E(\Gamma) \forall x_0 \in P_0 \text{ and } x_1 \in P_1. \tag{3.2}$$

by the definition of the X-join of Y.

For the converse, as each vertex of $\Gamma/\mathcal{P}$ is of the form gK_i where $g \in G$ and $i \in \mathbb{Z}_2$, and each $Y_{g,i}$ is the empty graph on the left cosets of H_i contained in gK_i, $V(\bigvee(X, Y)) = V(\Gamma)$. We now need only show that the edges of Γ are the same as the edges of $\bigvee(X, Y)$. As Γ is bipartite with bipartition G/H_0 and G/H_1, every edge of Γ is from some vertex $xH_0 \in xK_0 \in \mathcal{P}_0$ to some vertex $yH_1 \in yK_1 \in \mathcal{P}_1$. By hypothesis, this occurs if and only if every edge of the form $\{xH_0, yH_1\}$ with $xH_0 \in xK_0$ and $yH_1 \in yK_1$ is contained in $E(\Gamma)$. Then

$$E(\Gamma) = \{\{xH_0, yH_1\} : xH_0 \in \mathcal{P}_0, yH_1 \in \mathcal{P}_1; \{xK_0, yK_1\} \in E(\Gamma/\mathcal{P})\} \tag{3.3}$$

$$= \{\phi\{x_0, x_1\} : \{P_0, P_1\} \in E(\Gamma/\mathcal{P}); x_0 \in P_0 \in \mathcal{P}_0, x_1 \in P_1 \in \mathcal{P}_1\} \tag{3.4}$$

$$= \{\{(P_0, i), (P_1, j)\} : \{P_0, P_1\} \in E(\Gamma/\mathcal{P}); i \in \mathbb{Z}_{m_0}, j \in \mathbb{Z}_{m_1}\}. \tag{3.5}$$

Then $\phi(E(\Gamma)) = E(\bigvee(X, Y))$, and $\phi(\Gamma) = \bigvee(X, Y)$. ∎

We are now ready to prove our main theorem regarding recognition of bi-coset graphs that are X-joins using their connection set. This is the analogue of Theorem 2 for bi-coset graphs. We only concern ourselves here with recognizing X-joins from their connection sets as we determined in the previous sections the automorphism group of such graphs. We need an additional term.

Definition 23

Let X be a set, and $G \leq S_X$ (we note that G need not be transitive). A partition $\mathcal{P}$ of X will be called a G-invariant partition of X if $g(P) \in \mathcal{P}$ for every $P \in \mathcal{P}$.

We observe that if G is transitive, a G-invariant partition is simply a block system of G. We will only use this terminology though when the group G is intransitive.

Similar to double coset digraphs, the action of G on $(G/H_0) \cup (G/H_1)$ is faithful if and only if $\mathrm{core}_G(H_0) \cap \mathrm{core}_G(H_1) = \{1_G\}$. The next result shows that we may implicitly assume $\mathrm{core}_G(H_0) \cap \mathrm{core}_G(H_1) = \{1_G\}$.

Lemma 9

Let G be a group, $H_0, H_1 \leq G$, $S \subset G$ such that S is a union of (H_0, H_1)-double cosets, and $\Gamma = B(G, H_0, H_1, S)$. Let $N = \mathrm{core}_G(H_0) \cap \mathrm{core}_G(H_1)$. Then

$$B(G, H_0, H_1, S) \cong B(G/N, H_0/N, H_1/N, \{sN : s \in S\}). \tag{3.6}$$

Proof: Define $\gamma : (G/H_0) \cup (G/H_1) \to ((G/N)/(H_0/N)) \cup ((G/N)/(H_0/N))$ by $\gamma(gH_i) = gN(H_i/N)$. Suppose $\gamma(gH_i) = \gamma(g'H_i)$. Then $gN(H_i/N) = g'N(H_i/N)$ or $(g^{-1}g'N)H_i/N = H_i/N$. Hence $g^{-1}g'N \in H_i/N$ and $g^{-1}g' \in H_i$. Thus $gH_i = g'H_i$ and γ is well-defined. As $|G/H_i| = |(G/N)/(H_i/N)|$ and G is finite, we see γ is a bijection. Let $T = \{sN : s \in S\}$. As $S = H_0SH_1$ and $N \triangleleft H_i$, $i \in \mathbb{Z}_2$, we see that $(H_0/N)T(H_1/N) = T$ as $h_0NsNh_1N = h_0sh_1N \in T$ for every $h_i \in H_i$, $i = 0, 1$. Thus $B(G/N, H_0/N, H_1/N, T)$ is a well-defined graph. Let $\{gH_0, gsH_1\} \in E(B(G, H_0, H_1, S))$, so $s \in S$. Then

$$\gamma(\{gH_0, gsH_1\}) = \{gN(H_0/N), gsN(H_1/N)\} \in B(G/N, H_0/N, H_1/N, T) \tag{3.7}$$

as $sN \in T$ as $s \in S$. The result follows. ■

Theorem 9

Let $\Gamma = B(G, H_0, H_1, S)$ be a connected bi-coset graph, $H_i \leq K_i \leq G$, $i = 0, 1$, and $\mathcal{P} = \mathcal{P}(K_0, K_1)$ be the join-partition of $V(\Gamma)$ with respect to K_0 and K_1. Let $X = \Gamma/\mathcal{P}$. For $gK_i \in \mathcal{P}$, let $Y_{g,i}$ the empty graph with vertex set gK_i, and let $Y = \{Y_{g,i} : g \in G, i \in \mathbb{Z}_2\}$. Then Γ is the X-join of Y if and only if S is a union of (K_0, K_1)-double cosets in G. If such a $K_0, K_1 \leq G$ exists, then

$$B(G, H_0, H_1, S) = \bigvee(\Gamma/\mathcal{P}, Y) \cong \bigvee(B(G, K_0, K_1, S), Y) \tag{3.8}$$

Proof: We use the notation of the statement of the theorem. Suppose Γ is isomorphic to the X-join of X and Y. Let $s \in S$. Then H_0 is adjacent to some vertex of sK_1. Then every vertex of K_0 is adjacent to every vertex of sK_1. Let $k_0^{-1} \in K_0$ and $k_1 \in K_1$. Then $\{k_0^{-1}H_0, sk_1H_1\} \in E(\Gamma)$ as k_1H_1 is a vertex in K_1, and so $k_0sk_1 \in S$. Hence as k_0 and k_1 were arbitrary, $K_0sK_1 \subseteq S$ and as s is arbitrary, S is a union of (K_0, K_1)-double cosets.

Conversely, suppose that $H_0 \leq K_0 \leq G$ and $H_1 \leq K_1 \leq G$ such that S is a union of (K_0, K_1)-double cosets. Suppose that $\{xH_0, yH_1\} \in E(\Gamma)$, where $x, y \in G$. Then $x^{-1}y \in S$, and as S is a union of (K_0, K_1)-double cosets, we see $k_0^{-1}x^{-1}yk_1 \in S$ for every $k_0 \in K_0$ and $k_1 \in K_1$ and so $K_0x^{-1}yK_1 \subseteq S$. Then $\{xk_0H_0, yk_1H_1\} \in E(\Gamma)$. for every $k_0 \in K_0$ and $k_1 \in K_1$. So every vertex contained in xK_0 is adjacent in Γ to every vertex in yK_1 (here K_0 and K_1 are viewed as unions of cosets of H_0 and H_1, respectively) for every $x, y \in G$. As $\Gamma[gK_i]$ has no edges, we see that $\Gamma = \bigvee(\Gamma/\mathcal{P}, Y)$.

We next show $\Gamma/\mathcal{P} = \mathrm{B}(G, K_0, K_1, S)$. The graph $\mathrm{B}(G, K_0, K_1, S)$ is a well-defined bi-coset digraph as S is a union of (K_0, K_1)-double cosets in G. Let $a, b \in G$ such that $a^{-1}b \notin K_1$. Then

$(aK_0, bK_1) \in E(\Gamma/\mathcal{P})$ if and only if there is $a', b' \in G$ such that $a'H_0 \subseteq aK_0$, $b'H_1 \subseteq bK_1$, and $(a'H_0, b'H_1) \in E(\Gamma)$. This occurs if and only if $(a')^{-1}b'H_1 \in S$. As S is a union of (K_0, K_1)-double cosets in G and $(a')^{-1}(b')H_1 \subseteq (a')^{-1}bK_1$, we see $(a')^{-1}b'H_1 \in S$. Thus $(aK_0, bK_1) \in E(\Gamma/\mathcal{P})$ if and only if $a^{-1}bK_1 \in S$ (viewing S as a union of left cosets of K_1 in G), which occurs if and only if $(aK_0, bK_1) \in E(\mathrm{B}(G, K_0, K_1, S))$. So $\Gamma/\mathcal{P} = \mathrm{B}(G, K_0, K_1, S)$. ■

Some observations are in order. First, it is possible that a given bi-coset graph $\Gamma = B(G, H_0, H_1, S)$ is isomorphic to an X-join of Y, where $X = B(G', H_0', H_1', S)$ is a bi-coset graph and Y is a set of empty graphs, but that there is no relationship at all between G and G', as the next example shows. So the result above does not capture *all* the ways a bi-coset graph can be isomorphic to an X-join of a set of empty graphs, but rather only those where X is a quotient of Γ using a partition of $V(\Gamma)$ which is a join-partition of $V(\Gamma)$ with respect to K_0 and K_1.

Example 2 *Let $G = \mathbb{Z}_5$, $H_0 = 1$, $H_1 = G$, and $S = \{0\}$. Then $B(G, H_0, H_1, S)$ is a star on 6 vertices (that is, it has six vertices with one vertex of degree 5 and 5 vertices of degree 1). Let $X = (\mathbb{Z}_2, 1, \mathbb{Z}_2, \{0\})$ (so X is the star on 3 vertices), and label the vertices of X with element of $\{x, y, z\}$ such that the unique vertex of X of degree 2 is x. Set $Y_x = K_1$, $Y_y = \bar{K}_2$, $Y_z = \bar{K}_3$, and $Y = \{Y_x, Y_y, Y_z\}$. Then $B(G, H_0, H_1, S)$ is isomorphic to the X-join of Y.*

Corollary 8

Let $\Gamma = B(G, H_0, H_1, S)$ be a connected bi-coset graph, $H_i \le K_i \le G$, $i = 0, 1$, and $\mathcal{P} = \mathcal{P}(K_0, K_1)$ be the join-partition of $V(\Gamma)$ with respect to K_0 and K_1. Let $X = \Gamma/\mathcal{P}$. For $gK_i \in \mathcal{P}$, let $Y_{g,i}$ the empty graph with vertex set gK_i, and let $Y = \{Y_{g,i} : g \in G, i \in \mathbb{Z}_2\}$. Assume S is a union of (K_0, K_1)-double cosets. Then $\Gamma \cong \Gamma/\mathcal{P} \wr \bar{K}_m$ if and only if $m = [K_0 : H_0] = [K_1 : H_1]$.

Proof: The graph $\bigvee(X,Y) \cong \Gamma/\mathcal{P} \wr \bar{K}_m$ if and only if each $Y_{g,i}$ is isomorphic which occurs if and only if $m = [K_0 : H_0] = [K_1 : H_1]$. ■

Notice that it is possible for a bi-coset graph Γ to be simultaneously isomorphic to a wreath product and an X-join of empty graphs that is not written as a wreath product.

Example 3 *Let $n \geq 3$ and G a group of order n. Then $K_{n,n} \cong B(G,1,1,G) \cong K_2 \wr \bar{K}_n$. Let $\{S_1, S_2\}$ be a partition of G into two subsets of different sizes, $P_1 = \{(0,p_1) : p_1 \in S_1\}$, $P_2 = \{(0,p_2) : p_2 \in S_2\}$, and $P_3 = \{(1,g) : g \in G\}$. Set $\mathcal{P}_0 = \{P_1, P_2\}$ and $\mathcal{P}_1 = \{P_3\}$. Then $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_2$ is a partition of $V(B(G,1,1,G))$. Let X be the graph with $V(X) = \mathcal{P}$ and edge set $\{\{P_1,P_3\},\{P_2,P_3\}$, and let Y_{P_i} be the graph with vertex set P_i and no edges. Then $B(G,1,1,G) = \bigvee(X,Y)$ is not a wreath product as $|S_1| \neq |S_2|$ (it is though, of course, isomorphic to a wreath product).*

When $H_0 = H_1 = \{1_G\}$, $B(G,H_0,H_1,S) \cong \mathrm{Haar}(G,S)$, and we have a special case of Theorem 9 for Haar graphs.

Corollary 9

Let $\Gamma = \mathrm{Haar}(G,S)$ be a connected Haar graph, $K_i \leq G$, $i = 0,1$, and $\mathcal{P} = \mathcal{P}(K_0,K_1)$ be the join-partition of $V(\Gamma)$ with respect to K_0 and K_1. Let $X = \Gamma/\mathcal{P}$. For $gK_i \in \mathcal{P}$, let $Y_{g,i}$ be the empty graph with vertex set gK_i, and let $Y = \{Y_{g,i} : g \in G, i \in \mathbb{Z}_2\}$. Then Γ is the X-join of Y if and only if S is a union of (K_0,K_1)-double cosets in G. If such a $K_0, K_1 \leq G$ exists, then

$$Haar(G,S) = \bigvee(\Gamma/\mathcal{P}, Y) \cong \bigvee(B(G,K_0,K_1,S),Y). \tag{3.9}$$

For Haar graphs, $H_0 = H_1 = 1$ so $[K_i : H_i] = |K_i|$. So we have a slightly simpler form of Corollary 8.

Corollary 10

Let $\Gamma = \mathrm{Haar}(G, S)$ *be a connected Haar graph,* $K_i \leq G$*,* $i = 0, 1$*, and* $\mathcal{P} = \mathcal{P}(K_0, K_1)$ *be the join-partition of* $V(\Gamma)$ *with respect to* K_0 *and* K_1*. Let* $X = \Gamma/\mathcal{P}$*. For* $gK_i \in \mathcal{P}$*, let* $Y_{g,i}$ *the empty graph with vertex set* gK_i*, and let* $Y = \{Y_{g,i} : g \in G, i \in \mathbb{Z}_2\}$*. Assume* S *is a union of* (K_0, K_1)*-double cosets. Then* $\Gamma \cong \Gamma/\mathcal{P} \wr \bar{K}_m \cong \mathrm{B}(G, K_0, K_1, S) \wr \bar{K}_m$ *if and only if* $m = |K_0| = |K_1|$*.*

Example 4 *Let* $\Gamma = \mathrm{Haar}(D_3, \{1, \tau, \rho, \tau\rho\}))$*, where* D_3 *is the dihedral group with six elements. Looking closely at the connection set S, we see that it is exactly the double coset* $\langle\tau\rangle\tau\langle\tau\rho\rangle$*. Also* $|\langle\tau\rangle| = |\langle\tau\rho\rangle| = 2$*. Then by Corollary 10 we know that* Γ *is isomorphic to the wreath product* $\mathrm{B}(D_3, \langle\tau\rangle, \langle\tau\rho\rangle, \{1, \tau, \rho, \tau\rho\}) \wr \bar{K}_2$*. Figure 3.1 shows the Haar graph and its corresponding quotient digraph using the join-partition with respect to* $\langle\tau\rangle$ *and* $\langle\tau\rho\rangle$*. Colors have been added to the vertices so we can easily recognize the partition sets.*

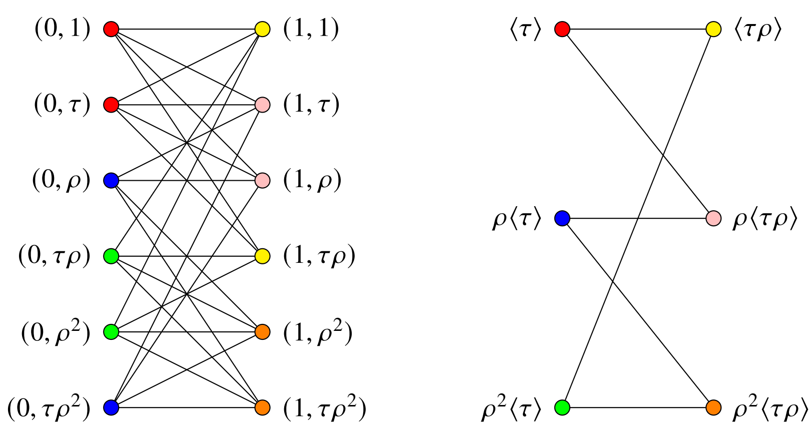

Figure 3.1

$\mathrm{Haar}(D_3, \{1, \tau, \rho, \tau\rho\})$ (left) and its quotient digraph (right).

3.4 Disconnected bi-coset graphs as X-joins

It has been shown [19, Lemma 2.1] that if G is a group and $S \subseteq G$, then for every $\alpha \in G$ and $a, b \in G$, $\mathrm{Haar}(G, S) \cong \mathrm{Haar}(G, a\alpha(G)b)$. We will have need of a version of this for bi-coset graphs. For Haar graphs, one can show that there are three types of isomorphisms can be used to prove the statement. The first type is induced by automorphisms of G, and for $\alpha \in \mathrm{Aut}(G)$ is given by $(i, g) \mapsto (i, \alpha(g))$. The other two can be though of as multiplication on the right and left by elements of G: for $a, b \in G$, we have the maps

$$(0, g) \mapsto (0, g) \text{ and } (1, g) \mapsto (1, ag) \tag{3.10}$$

and

$$(1, g) \mapsto (0, g) \text{ and } ((1, g) \mapsto (1, gb). \tag{3.11}$$

Unfortunately, multiplication on the right cannot be generalized to bi-coset graphs as the vertex set consists of left cosets of subgroups of G, not right cosets. However, multiplication on the right can be thought of as composition of multiplication on the left, and an appropriate inner automorphism of G. We will use this trick to generalize the above fact about isomorphisms of Haar graphs to bi-coset graphs. Note that the isomorphisms we will obtain will usually *not* have the same vertex-sets as in the result for Haar graphs, but will still be an isomorphism of bi-coset graphs of G (but H_0 and H_1 may change).

Definition 24

Let G be a group and $H_0, H_1 \leq G$. Set $V_{H_0,H_1} = \{(0, gH_0 : g \in G\} \cup \{(1, gH_1) : g \in G\}$. Thus V_{H_0,H_1} is the vertex set of a bi-coset graph $B(G, H_0, H_1, S)$.

Lemma 10

Let G be a group, $a, b \in G$ and $\alpha \in \mathrm{Aut}(G)$. Then

$$B(G, H_0, H_1, S) \cong B(G, a\alpha(H_0)a^{-1}, b^{-1}\alpha(H_1)b, a\alpha(S)b) \tag{3.12}$$

by the map $\phi : V_{H_0,H_1} \mapsto V_{H_0,H_1}$ given by

$$\phi(0, gH_0) = (0, g\alpha(H_0)) \text{ and } \phi(1, g) = (1, gab^{-1}\alpha(H_1)b). \tag{3.13}$$

Proof: We will first show that there are two kinds of bijections from V_{H_0,H_1} which map bi-coset graphs of G to bi-coset graphs of G. Both of these will change the vertices of V_{H_0,H_1} and connection sets as it maps a bi-coset graph of G to its image which is also a bi-coset graph of G. We will then compose maps of each kind to obtain the isomorphism ϕ. Let $\Gamma = B(G, H_0, H_1, S)$ be a bi-coset of G with $H_0, H_1 \leq G$ and S a union of (H_0, H_1)-double cosets.

For the first kind of isomorphism, let $\alpha \in \mathrm{Aut}(G)$. Define $\tilde{\alpha} : V_{H_0,H_1} \mapsto V_{\alpha(H_0),\alpha(H_1)}$ by $\tilde{\alpha}(i, gH_i) = (i, \alpha(g)\alpha(H_i))$. Clearly $\tilde{\alpha}$ is a well-defined bijection. As $\alpha \in \mathrm{Aut}(G)$, $\alpha(H_0)$ and $\alpha(H_1)$ are subgroups of G, and as S is a union of (H_0, H_1)-double cosets, $\alpha(S)$ is a union of $(\alpha(H_0), \alpha(H_1))$-double cosets. Let $\Gamma' = B(G, \alpha(H_0), \alpha(H_1), \alpha(S))$, so Γ' is a bi-coset graph of G. Additionally, $\tilde{\alpha}(V(\Gamma)) = V(\Gamma')$. Note that for every $g \in G$, and $s \in G$

$$\tilde{\alpha}((0, gH_0)(1, gsH_1)) = (0, \alpha(g)\alpha(H_0), \alpha(g)\alpha(s)\alpha(H_1)). \tag{3.14}$$

Hence $(0, gH_0)(1, gsH_1) \in E(\Gamma)$ if and only if $\tilde{\alpha}((0, gH_0)(1, gsH_1)) \in E(\Gamma')$ so $\tilde{\alpha}(\Gamma) = \Gamma'$.

For the second kind of isomorphism, let $b \in G$. Define $\bar{b}_L : V_{H_0,H_1} \mapsto V_{H_0,b^{-1}H_1 b}$ be given by

$$\bar{b}_L(0, gH_0) = (0, gH_0) \text{ and } \bar{b}_L(1, gH_1) = (1, gb \cdot b^{-1}H_1 b). \tag{3.15}$$

Note that $\bar{b}_L$ maps V_{H_0,H_1} to $V_{H_0,b^{-1}H_1b}$. Clearly $\bar{b}_L$ is a well-defined bijection on $\{(0, gH_0) : g \in G\}$. Suppose $\bar{b}_L(g_1H_1) = \bar{b}_L(g_2H_1)$. Then $g_1b \cdot b^{-1}H_1b = g_2b \cdot b^{-1}H_1b$ or equivalently, $g_1H_1 = g_2H_1$, and $\bar{b}_L$ is well-defined, and is clearly a bijection from $\{(1, gH_1) : g \in G\}$ to $\{(1, gb^{-1}Hb) : g \in G\}$. Also, let $H_0sH_1 \in S$. Then $H_0sb \cdot b^{-1}H_1b = H_0sH_1b$, so Sb is a union of $(H_0, b^{-1}H_1b)$-double cosets. Let $\Gamma' = \mathcal{B}(G, H_0, b^{-1}H_1b, Sb)$. Note that for every $g \in G$, and $s \in G$,

$$\bar{b}_L((0, gH_0)(1, gsH_1)) = (0, gH_0)(1, gsb \cdot b^{-1}H_1)b). \tag{3.16}$$

Hence $(0, gH_0)(1, gsH_1) \in E(\Gamma)$ if and only if $\bar{b}_L((0, gH_0)(1, gsH_1)) \in E(\Gamma')$ so $\bar{b}_L(\Gamma) = \Gamma'$.

To obtain ϕ, let $\alpha \in \mathrm{Aut}(G)$ and $a, b \in G$. Let $\delta \in \mathrm{Aut}(G)$ be defined by $\delta(g) = a\alpha(g)a^{-1}$. Put another way, δ is the automorphism of G obtained by composing the inner automorphism of G induced by conjugation by a^{-1} with α. Let $\phi = \overline{ab}_L\tilde{\delta}$. Then $\phi(H_0) = a\alpha(H_0)a^{-1} \leq G$ and $\phi(H_1) = (ab)^{-1}a\alpha(H_1)a^{-1}(ab) = b^{-1}\alpha(H_1)b \leq G$. Also, $\delta(S) = a\alpha(S)a^{-1}$, and $\overline{ab}_L(a\alpha(S)a^{-1}) = a\alpha(S)b$. The result follows. ■

Lemma 11

Let G be a group, $H_0, H_1 \leq G$ and $S \subseteq G$ such that S is a union of (H_0, H_1)-double cosets in G. Then $\Gamma = \mathrm{B}(G, H_0, H_1, S)$ is disconnected if and only if $S \subseteq Kg$ for some subgroup $H_0, H_1 \leq K < G$ and $g \in G$.

Proof: Suppose Γ is disconnected, then $\langle SS^{-1}\rangle < G$. Let $t \in S$. Then for every $s \in S$, $st^{-1} = a \in \langle SS^{-1}\rangle$ or $s = at$. So $S \subseteq \langle SS^{-1}\rangle t$. Let $h \in H_0$. Then there exists $s \in S$, $g \in G$, and $h' \in H_1$ such that $s = hgh'$. Then $s(gh')^{-1} = h \in \langle SS^{-1}\rangle$ as $gh' = 1_{H_0}gh' \in S$. Thus, $H_0 \leq \langle SS^{-1}\rangle$. Similarly, $H_1 \leq \langle SS^{-1}\rangle$.

Conversely, suppose there is $g \in G$ and $H_0, H_1 \leq K < G$ such that $S \subseteq Kg$. Let $s_1, s_2 \in S$ with $k_1, k_2 \in K$ such that $s_1 = k_1 g$ and $s_2 = k_2 g \in S$. Then the product $(k_1 g)(k_2 g)^{-1} = k_1 k_2^{-1} \in K$. Thus, $\langle SS^{-1} \rangle \subseteq K < G$, and Γ is disconnected. ∎

It was shown in [7, Lemma 2.3 (iii)] that a bi-coset graph, $\Gamma = \mathrm{B}(G, H_0, H_1, S)$ is disconnected if and only if $\langle SS^{-1} \rangle \neq G$. First observe that any element of $\langle SS^{-1} \rangle$ is simply a walk starting at $(1, 1_G)$. Hence $\langle SS^{-1} \rangle$ is a component of Γ. The next result then says that the "right hand" bipartition sets of the components of Γ are the right cosets of K in G.

Lemma 12

Let $\Gamma = \mathrm{B}(G, H_0, H_1, S)$ *be a disconnected bi-coset graph. Then the number of components of* Γ *is* $[G : K]$ *where* $\langle SS^{-1} \rangle = K < G$.

Proof: Since Γ is disconnected, $\langle SS^{-1} \rangle = K < G$. Let $k \in K$ and consider the left coset kH_0. Since $\langle SS^{-1} \rangle = K$, then there are $s_0, \ldots, s_r \in S$ such that $s_0 s_1^{-1} s_2 s_3^{-1} \cdots s_{r-1} s_r^{-1} = k$. Then, $H_0, s_0 H_1, s_0 s_1^{-1} H_0 \ldots, s_0 s_1^{-1} \cdots s_{r-1} s_r^{-1} H_0$ is a path from H_0 to kH_0. Since k was arbitrary, there exists a path from H_0 to every left coset of H_0 in K. Similarly, there is a path from gH_0 to gkH_0 for $g \in G \backslash K$ and $k \in K$, therefore the size of the vertex set of the left partition of each component is $[K : H_0]$. Since the size of the vertex set of the left partition of Γ is $[G : H_0]$, the number of components of Γ must be $[G : K]$. ∎

As a consequence from the proof of the above lemma, we know that the component that contains H_0 as a vertex also contains all left cosets of H_0 in K. This fact will be important in order to prove the next lemma.

Lemma 13

Let $\Gamma = \mathrm{B}(G, H_0, H_1, S)$ *be a disconnected bi-coset graph. Then each component of* Γ *is isomorphic.*

Proof: Let Γ_g denote the component of Γ that contains the left coset gH_0, $g \in G$ as a vertex. Define the mapping $\phi_g : \Gamma_1 \to \Gamma_g$ by $\phi_g(xH_i) = gxH_i$ for $i = 0, 1$ and $x \in G$. Clearly ϕ_g is a well-defined bijection.

We next show that ϕ_g is a graph isomorphism. By Lemma 12 we know that Γ_1 contains all left cosets of H_0 in K, where $K = \langle SS^{-1} \rangle$. So, every edge in Γ_1 is of the form $\{kH_0, ksH_1\}$, where $s \in S, k \in K$. Now, $\phi_g(\{kH_0, ksH_1\}) = \{gkH_0, gksH_1\} \in E(\Gamma_g)$ if and only if $(gks)^{-1}(gk) = s \in S$. Thus ϕ_g is a graph isomorphism from Γ_1 to Γ_g. Since $g \in G$ was arbitrary, we have shown that each component of Γ is isomorphic to Γ_1, thus all components of Γ are isomorphic. ∎

Theorem 10

Let $\Gamma = \mathrm{B}(G, H_0, H_1, S)$ *be a disconnected bi-coset graph where* $S \subseteq Kt$, $t \in G$ *and* $K = \langle SS^{-1} \rangle$. *Then*

$$\Gamma \cong \bar{K}_n \wr \mathrm{B}(K, H_0, tH_1t^{-1}, St^{-1}) \tag{3.17}$$

and

$$\mathrm{Aut}(\Gamma) \cong S_n \wr \mathrm{Aut}(\mathrm{B}(K, H_0, tH_1t^{-1}, St^{-1})) \tag{3.18}$$

where $n = [G : K]$.

Proof: In order for $\mathrm{B}(K, H_0, tH_1t^{-1}, St^{-1})$ to be defined, it must be the case that $tH_1t^{-1} \le K$. As Γ is defined, we have $H_0SH_1 = S$. This is true if and only if $H_0SH_1t^{-1} = St^{-1}$, which is true if and only if $H_0S(t^{-1}t)H_1t^{-1} = St^{-1}$. Rewriting this equality, we have $H_0 \cdot St^{-1} \cdot tH_1t^{-1} = St^{-1}$. As Γ

is defined, $H_0, H_1 \leq K$ and by hypothesis, $S \subseteq Kt$. Hence $St^{-1} \subseteq K$. So $H_0 \cdot St^{-1} \subseteq K$, and so $tH_1t^{-1} \leq K$.

We now with to apply ϕ of Lemma 10 with $\alpha = 1$, $a = 1$, and $b = t^{-1}$ to Γ to obtain $\phi(\Gamma) = B(H_0, tH_1t^{-1}, St^{-1})$. As $\langle SS^{-1} \rangle = K$, we have $\langle St^{-1}(St^{-1})^{-1} \rangle = K$. Also, as $St^{-1} \leq K$, every walk in $\phi(\Gamma)$ starting at a vertex in $\{(0, k) : k \in K\}$ or $\{(1, k) : k \in K\}$ will end at a vertex in $\{(i, k) : i \in \mathbb{Z}_2, k \in K\}$. As $\langle St^{-1}(St^{-1})^{-1} \rangle$, for every $k \in K$ there is a walk in $\phi(\Gamma)$ starting at $(1, 1_G)$ which ends at $(1, k)$. Similarly, for every $k \in K$ there is a walk in $\phi(\Gamma)$ starting at $(0, 1_G)$ which ends at $(0, k)$. We conclude that the component of $\phi(\Gamma)$ that contains any element of $\{(i, k) : i \in \mathbb{Z}_2, k \in K\}$ is $\{(i, k) : i \in \mathbb{Z}_2, k \in K\}$. Thus the partition of $V(\phi(\Gamma))$ determined by its components is $\mathcal{P}(K, K)$. Thus there are n components of $\phi(\Gamma)$, each isomorphic (by an element of $\hat{G}_L$, we see $n = [G : K]$. Then $\mathrm{Aut}(\phi(\Gamma)) \cong S_n \wr \mathrm{Aut}(\mathrm{B}(G, H_0, tH_1t^{-1}, St^{-1}))$, and so $\mathrm{Aut}(\Gamma) \cong S_n \wr \mathrm{Aut}(\mathrm{B}(G, H_0, tH_1t^{-1}, St^{-1}))$. ■

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