1-1-2020

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OPEN ACCESS
Fractals, Vol. 28, No. 8 (2020) 2040051 (15 pages)

DOI: 10.1142/S0218348X20400514

APPLICATIONS OF BI-FRAMELET SYSTEMS FOR SOLVING FRACTIONAL ORDER DIFFERENTIAL EQUATIONS

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Received February 20, 2020
Accepted April 23, 2020
Published July 6, 2020

Abstract

Framelets and their attractive features in many disciplines have attracted a great interest in the recent years. This paper intends to show the advantages of using bi-framelet systems in the context of numerical fractional differential equations (FDEs). We present a computational method based on the quasi-affine bi-framelets with high vanishing moments constructed using the generalized (mixed) oblique extension principle. We use this system for solving some types of FDEs by solving a series of important examples of FDEs related to many mathematical applications. The quasi-affine bi-framelet-based methods for numerical FDEs show the advantages of using sparse matrices and its accuracy in numerical analysis.

Keywords: Fractional Differential Equations; Quasi-Affine System; Bi-Framelet; Mixed Oblique Extension Principle.

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1. INTRODUCTION

In recent years, fractional differential equations (FDEs) have been widely used in the development of many modern problems in engineering practices and applied sciences. For example, it has applications in the modeling of earthquakes,[1] economics,[2] fluids,[3] dynamic of viscoelastic materials,[4] and many other disciplines, e.g. see Refs. 5–8.

Fractional calculus is a mathematical area that studies and analyzes the properties of the derivative and integration of a non-integer orders. In particular, this area is getting more attention from many researchers to develop new methods for solving differential equations involved by fractional order. Atangana is unique among fractals-theorists in his ability to bring to bear new definitions, theory and ideas on some of the most intractable issues on FDEs. He is the founder of the fractional calculus with nonlocal and non-singular kernels popular in applied mathematics today and has achieved and contributed significantly to the numerical and pure analysis of FDEs. Atangana *et al.* defined the well-known Atangana–Baleanu fractional derivative definition that describes the complicated problems related to the power and exponential laws and free of singularities.[9–17]

Note that, in general, the exact solution of most the FDEs does not exist. Therefore, examining and developing new numerical methods is very important. For example, Laplace and Fourier techniques were proposed,[18,19] Adomian decomposition method in Refs. 20–23, variational iteration method in Ref. 24, and other methods can be found in Refs. 25–28. Interested readers should consult other references therein to have an extra knowledge of the other used methods.

In the context of numerical and computational mathematics, framelet systems have proven as a powerful tool on tackling issues related to the numerical and computational framework. The main aim of this work is to shed some lights on the benefits of using bi-framelets in the area of the FDEs. Some of the FDEs we consider are as follows:

$$\begin{cases} \mathcal{D}_*^\alpha u(t) + c(t)u(t) = f(t), \quad \text{with the conditions,} \\ \left.\dfrac{d^k u}{dt^k}\right|_{t=0} = 0, \quad k = 0, 1, 2, \dots, n-1, \end{cases} \tag{1}$$

for some $n \in \mathbb{N}$ where $n - 1 < \alpha \le n$, $c(t), f(t)$ are known square integrable functions, $\mathcal{D}_*^\alpha u$ is Caputo FDE (see Definition 1.1) operator of u, and $u(t)$ is the unknown function to be approximated.

Definition 1.1. For a real function $u(t)$ where $t, \alpha > 0$, and $n \in \mathbb{N}$, we have the following:

- The Caputo's fractional derivative of order α is defined by

$$\mathcal{D}_*^\alpha u(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t \frac{u^{(n)}(x)}{(t-x)^{\alpha+1-n}} dx \\ \qquad \text{if } n-1 < \alpha \le n, \\ \dfrac{d^n u(t)}{dt^n} \quad \text{if } \alpha = n. \end{cases}$$

- The Riemann–Liouville fractional derivative of order α is defined by

$$\mathcal{D}_t^\alpha u(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \dfrac{d^n}{dt^n} \displaystyle\int_0^t \frac{u(x)}{(t-x)^{\alpha+1-n}} dx \\ \qquad \text{if } n-1 < \alpha \le n, \\ \dfrac{d^n u(t)}{dt^n} \quad \text{if } \alpha = n. \end{cases}$$

- The Riemann–Liouville fractional integral operator (FIO) of order α is defined by

$$\mathcal{I}^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(x)}{(t-x)^{1-\alpha}} dx, \quad n-1 < \alpha \le n. \tag{2}$$

Note that,

$$\mathcal{D}_*^\alpha u(t) = \mathcal{I}^{n-\alpha}\left(\frac{d^n u(t)}{dt^n}\right),$$

$$\mathcal{D}_*^\alpha \mathcal{I}^\alpha u(\cdot) = u(\cdot),$$

and for $t > 0$,

$$\mathcal{I}^\alpha \mathcal{D}_*^\alpha u(\cdot) = u(\cdot) - \sum_{k=0}^{n-1} u^{(k)}(0^+)\frac{t^k}{k!}.$$

2. AN OVERVIEW OF QUASI-AFFINE BI-FRAMELET

Let us recall some definitions and facts to clarify the concept of bi-framelets. It is known that orthonormal bases are non-redundant systems, while their redundant setup are known as *frames*. The redundancy here is very useful in many applications such as the error recovery/correction in transmission of data. Frames are more general than orthogonal systems and provide better representations. They were introduced by Duffin and Schaeffer in Ref. 29. A big development later on has been achieved in Refs. 30–32.

Definition 2.1. Let $L_2(\mathbb{R})$ denote the space of all square integrable functions over $\mathbb{R}$. A family of vectors $\{\psi^v, v \in I\} \subset L_2(\mathbb{R})$ is a *framelet* for $L_2(\mathbb{R})$ if there exist constants $C_1, C_2 > 0$ such that

$$C_1\|g\|^2 \le \sum_{v\in I} |\langle g, \psi^v\rangle|^2 \le C_2\|g\|^2, \quad \forall\, g \in L_2(\mathbb{R}). \tag{3}$$

If $C_1 = C_2 = 1$, it is called a Parseval frame. The importance of such tightness in the framelet system is that they provide a simple and better reconstructions for the elements of the space $L_2(\mathbb{R})$. In framelet analysis, we say that the function ϕ is a refinable function if

$$\phi = \sum_k 2h_0[k]\phi(2\cdot -k), \tag{4}$$

where h_0 is a finitely supported sequence (called the low pass filter of ϕ) such that

$$\hat{h}_0(\xi) = \sum_{k\in\mathbb{Z}} h_0[k]e^{-ik\xi}.$$

Note that, in frequency domain, Eq. (4) can be written as

$$\hat{\phi}(2\,\cdot\,) = \hat{h}_0(\cdot)\hat{\phi}(\cdot), \tag{5}$$

where

$$\hat{\phi}(\xi) = \int_{\mathbb{R}} \phi(x)e^{-ix\xi}dx$$

is the Fourier transform of ϕ.

Definition 2.2. For a given refinable functions $\phi, \tilde{\phi}$, let $E = \{1, 2, 3, \ldots, r\}$, $\Psi = \{\psi^1, \psi^2, \ldots, \psi^r\} \subset L_2(\mathbb{R})$ and $\tilde{\Psi} = \{\tilde{\psi}^1, \tilde{\psi}^2, \ldots, \tilde{\psi}^r\} \subset L_2(\mathbb{R})$. Then we say that $(X(\Psi), X(\tilde{\Psi}))$, where

$$X(\Psi) = \{\psi^\ell_{j,k}; j, k \in \mathbb{Z}, \ell \in E\},$$
$$X(\tilde{\Psi}) = \{\tilde{\psi}^\ell_{j,k}; j, k \in \mathbb{Z}, \ell \in E\},$$

and

$$\psi^\ell_{j,k} = \begin{cases} 2^{j/2}\psi^\ell(2^jx - k) & \text{if } j \ge 0, \\ 2^j\psi^\ell(2^j(x - k)) & \text{if } j < 0 \end{cases} \tag{6}$$

is a pair of quasi-affine bi-framelet system of $L_2(\mathbb{R})$ if both $X(\Psi)$ and $X(\tilde{\Psi})$ are framelet systems in $L_2(\mathbb{R})$ and

$$\langle g, f\rangle = \sum_{\ell\in E,j,k} \langle \psi^\ell_{j,k}, f\rangle\langle g, \tilde{\psi}^\ell_{j,k}\rangle \quad \text{holds} \quad \forall\, g, f \in L_2(\mathbb{R}). \tag{7}$$

Note that, Eq. (5) is also satisfied for all elements of Ψ and $\tilde{\Psi}$ with finitely supported sequences called high pass filters, $h_\ell[k]$, and $\tilde{h}_\ell[k], \ell \in E$, respectively. For simplicity, we define $\psi^0 = \phi$ and $\tilde{\psi}^0 = \tilde{\phi}$. Thus, for $\xi \in \mathbb{R}$, we have

$$\psi^\ell(2\,\cdot) = \hat{h}_\ell(\cdot)\hat{\phi}(\cdot),$$
$$\hat{\tilde{\psi}}^\ell(2\,\cdot) = \hat{\tilde{h}}_\ell(\cdot)\hat{\tilde{\phi}}(\cdot), \quad \forall\, \ell \in E. \tag{8}$$

Therefore, given a quasi-affine bi-framelet system $X(\Psi)$, we can find a subset of $L_2(\mathbb{R})$, $X(\tilde{\Psi})$, such that (similar to Eq. (3))

$$C_2^{-1}\|g\|^2 \le \sum_{\ell\in E}\sum_{j,k}^{\infty} |\langle g, \tilde{\psi}^\ell_{j,k}\rangle|^2 \le C_2^{-1}\|g\|^2, \quad \forall\, g \in L_2(\mathbb{R}). \tag{9}$$

In particular, if $\phi = \phi$ and $\psi^\ell = \tilde{\psi^\ell}$, $\forall\, \ell \in E$, then it is called a tight framelet system for $L_2(\mathbb{R})$. So, we have the following equation:

$$\|g\|^2 = \sum_{\ell\in E}\sum_{j,k}^{\infty} |\langle g, \psi^\ell_{j,k}\rangle|^2.$$

From Eq. (7), for a bi-framelet system, we have

$$\langle f, g\rangle = \iint f(x)g(t)\mathcal{Z}^\ell_{j,k}(x, t)dxdt, \tag{10}$$

where

$$\mathcal{Z}^\ell_{j,k}(x, t) = \sum_{\ell\in E,j,k} \tilde{\psi}^\ell_{j,k}(x)\psi^\ell_{j,k}(t).$$

Thus, we have the following quasi-affine bi-framelet representation:

$$g = \sum_{\ell\in E,j,k} a^\ell_{j,k}\psi^\ell_{j,k}, \tag{11}$$

where $a^\ell_{j,k} = \langle g, \tilde{\psi}^\ell_{j,k}\rangle$. Note that, the coefficients $a^\ell_{j,k}$ is not unique but it is one of the best choices for a better simulation. Hence, one can consider the following truncated representation from Eq. (11) for f:

$$\mathcal{U}_n g = \sum_{\ell\in E}\sum_{j,k\in\mathbb{Z},j<n} a^\ell_{j,k}\psi^\ell_{j,k}. \tag{12}$$

Sparse representation for a smooth function is of interest in many applications. Therefore, to have such sparsity, it is crucial for h_ℓ to have high vanishing moments, where a function ψ has a vanishing moments of order s if

$$\langle t^k, \psi^\ell\rangle = 0, \quad \text{for all } k = 0, 1, \ldots, s - 1.$$

In literature, there are many principles to construct bi-framelets, such as the the *mixed unitary*

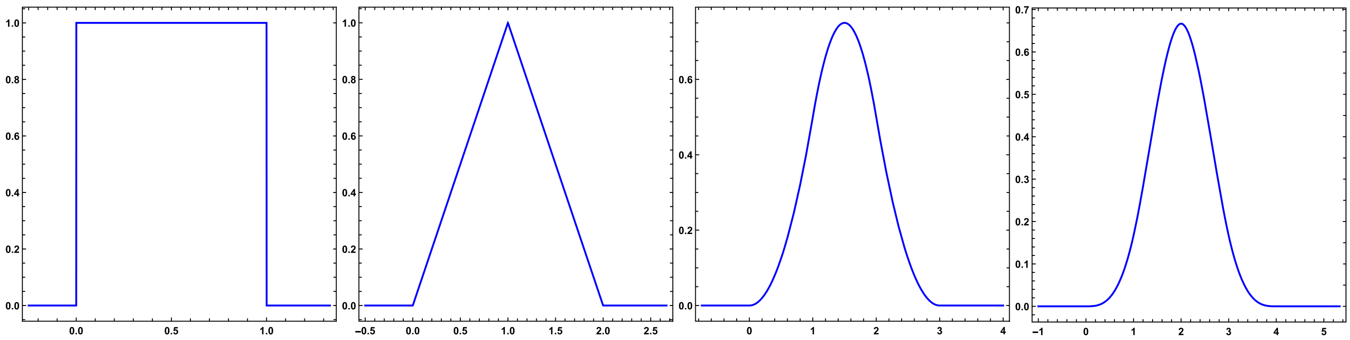

Fig. 1 The first few B-spline functions of order $m = 1$ through $m = 4$.

extension principles (MUEP) (see e.g. Refs. 33–36 and references therein). In this paper, we use the generalization of the MUEP, namely, the *mixed oblique extension principle* (MOEP) presented in Ref. 35. By doing this, we will present some examples of bi-framelet systems that have high vanishing moments. However, to have such property, it is required to have some required constraints on $h_0[k], \tilde{h}_0[k]$. In short, for some trigonometric function $\hat{\Theta}(\cdot)$, if

- $\hat{\Theta}(\cdot)\hat{\phi}(0)\overline{\hat{\tilde{\phi}}(0)} = 1$, and
- $\hat{\Theta}(\cdot)\hat{h}_0(\cdot/2 + \pi\alpha)\overline{\hat{\tilde{h}}_0(\cdot/2)} + \sum_{\ell\in E}\hat{h}_\ell(\cdot/2+\pi\alpha)\overline{\hat{\tilde{h}}_\ell(\cdot/2)} = \Theta(\cdot/2)\delta_\alpha$, where $\alpha \in \{0, 1\}$,

then, the system $(X(\Psi), X(\tilde{\Psi}))$ defined in Definition 2.2 forms a quasi-affine bi-framelet system of $L_2(\mathbb{R})$.

The MOEP provides a way to construct bi-framelets from refinable functions and it gives us a better approximation orders and reconstruction. Framelets have a great deal of use in many applications due to the features of redundancy (by increasing the number r), and many other properties (see e.g. Ref. 37). In this paper, we use analytic expressions of bi-framelets with high redundancy generated via the MOEB using some analytic refinable functions called B-splines. B-splines are of importance in harmonic theory and have wide range of use for many applications in approximation analysis. It is defined using the convolution product as follows.

Definition 2.3 (Ref. 37). For $m \in \mathbb{N}$, the B-spline of order m, B_m, is defined as

$$B_1 = \chi_{[0,1)}, \quad \text{and}$$

$$B_m = \int_0^1 B_{m-1}(\cdot - x)dx.$$

Figure 1 shows the graphs of the B-splines B_m for $m = 1, \ldots, 4$. Note that, B_m is a piecewise function of polynomials ($B_m \in C^{m-1}(\mathbb{R})$) and is refined as

$$\hat{B}_m(2\xi) = \hat{h}_0(\xi)\hat{B}_m(\xi),$$

where

$$\hat{h}_0(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^m.$$

3. SYSTEMS OF QUASI-AFFINE BI-FRAMELETS

In this section, we use the MOEP to construct quasi-affine bi-framelet systems using B-splines and use it to solve some examples of FDEs.

System A (HAAR bi-framelet). Consider the B-spline of order 1, $B_1 = \chi_{[0,1)}$. Define

$$\hat{\psi}^1(\xi) = \frac{-1208i}{3415\pi\xi} + \frac{3341807ie^{i\xi/2}}{4726360\pi\xi} - \frac{489ie^{i\xi}}{1384\pi\xi},$$

$$\hat{\tilde{\psi}}^1(x) = \frac{529i}{1497\pi\xi} - \frac{57518ie^{i\xi/2}}{81337\pi\xi} + \frac{173ie^{i\xi}}{489\pi\xi}.$$

Hence, the system $(X(\Psi), X(\tilde{\Psi}))$ is a bi-framelet for $L_2(\mathbb{R})$. Figure 2 shows the graphs of the generators, ψ^1 and $\tilde{\psi}^1$ in the time domain.

System B (Linear bi-framelet). For $m = 2$, consider the linear B-spline for ϕ and $\tilde{\phi}$ with the following filters:

$$\hat{h}_0(\xi) = \frac{1}{4}(1+e^{-i\xi})^2,$$

$$\hat{h}_1(\xi) = 1 - \cos\xi, \hat{h}_2(\xi) = \frac{-1}{2}(-1+e^{-i\xi}),$$

$$\hat{\tilde{h}}_1(\xi) = \frac{1}{4}(1+\cos\xi),$$

$$\hat{\tilde{h}}_2(\xi) = \frac{1}{2}e^{-i\xi}.$$

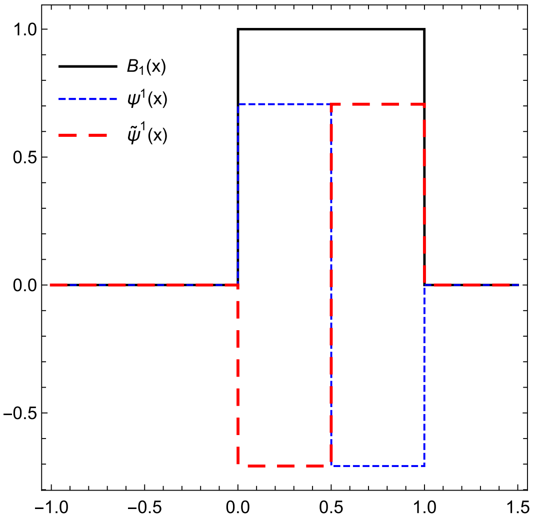

Fig. 2 The graphs of HAAR bi-framelet generators in System A.

Then, this set of filters satisfies the conditions of MOEP and therefore the system $(X(\Psi), X(\tilde{\Psi}))$ forms a quasi-affine bi-framelet system for $L_2(\mathbb{R})$. The generators of this system are depicted in Fig. 3.

System C (Cubic-Linear bi-framelet). Consider the (cubic) B-spline of order 4, $\phi = B_4$, with the following filter:

$$\hat{h}_0(\xi) = \frac{1}{16}(1+e^{-i\xi})^4,$$

and the (linear) B-spline of order 2, $\tilde{\phi} = B_2$, with the following filter:

$$\hat{\tilde{h}}_0(\xi) = \frac{1}{4}(1+e^{-i\xi})^2.$$

Depending on the MOEP, we define explicitly in the time domain the generators $\{\psi^\ell, \tilde{\psi}^\ell, \ell = 1, 2\}$, that generate a bi-framelet system, namely

$$\psi^1(x) = \begin{cases} 1+3x+3x^2+x^3 & \text{if } -1 \le x < -1/2, \\ 1/2(1-6x^2-6x^3) & \text{if } -1/2 \le x < 0, \\ 1/2(1-6x^2+6x^3) & \text{if } 0 \le x < 1/2, \\ 1-3x+3x^2-x^3 & \text{if } 1/2 \le x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi^2(x) = \begin{cases} 1/3(1+3x+3x^2+x^3) & \text{if } -1 \le x < 0, \\ 1/3(1+3x+3x^2-8x^3) & \text{if } 0 \le x < 1/2, \\ 1/3(-1+15x \\ \quad -21x^2+8x^3) & \text{if } 1/2 \le x < 1, \\ 1/3(8-12x+6x^2-x^3) & \text{if } 1 \le x < 2, \\ 0 & \text{otherwise,} \end{cases}$$

$$\tilde{\psi}^1(x) = \begin{cases} 1-2x & \text{if } 1/2 \le x < 1, \\ -7+6x & \text{if } 1 \le x < 3/2, \\ 11-6x & \text{if } 3/2 \le x < 2, \\ -5+2x & \text{if } 2 \le x < 5/2, \\ 0 & \text{otherwise,} \end{cases}$$

$$\tilde{\psi}^2(x) = \begin{cases} 1/4(-1-2x) & \text{if } -1/2 \le x < 1/2, \\ 1/2(-5+8x) & \text{if } 1/2 \le x < 1, \\ 1/2(11-8x) & \text{if } 1 \le x < 3/2, \\ 1/4(-5+2x) & \text{if } 3/2 \le x < 5/2, \\ 0 & \text{otherwise.} \end{cases}$$

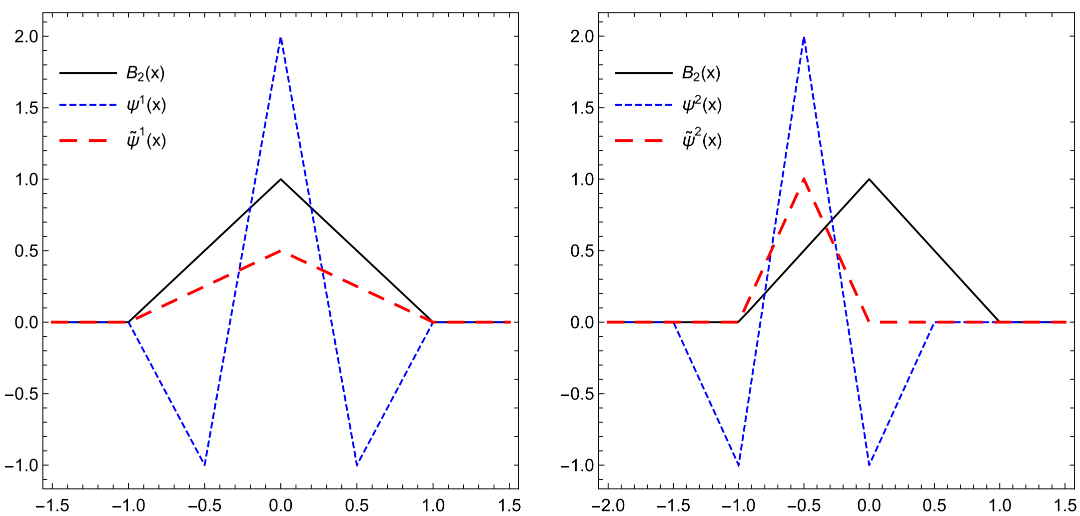

Fig. 3 The graphs of the generators in System B.

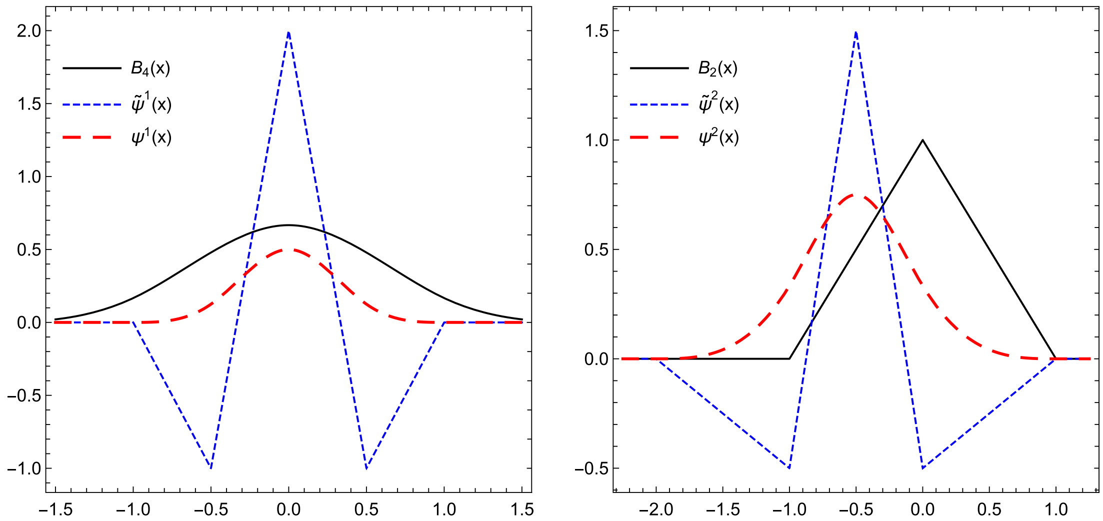

Fig. 4 The graphs of the generators in System C.

The graphs of these generators are shown in Fig. 4.

System D. Now lets consider the B-spline of order four for both ϕ and $\tilde{\phi}$. Again, by applying the setup of the MOEP, we are able to find the generators explicitly as follows:

$$\psi^1(x) = \begin{cases} 8x^3/3 & \text{if } 0 \le x \le 1/2, \\ -8/3(-64 + 48x - 12x^2 + x^3) & \text{if } 7/2 < x < 4, \\ 8/3(-279 + 246x - 72x^2 + 7x^3) & \text{if } 3 \le x \le 7/2, \\ -8(-159 + 170x - 60x^2 + 7x^3) & \text{if } 5/2 < x < 3, \\ 8(-9 + 26x - 24x^2 + 7x^3) & \text{if } 1 \le x \le 3/2, \\ -8/3(-1 + 6x - 12x^2 + 7x^3) & \text{if } 1/2 < x < 1, \\ 8/3(-398 + 540x - 240x^2 + 35x^3) & \text{if } 2 \le x \le 5/2, \\ -8/3(-162 + 300x - 180x^2 + 35x^3) & \text{if } 3/2 < x < 2, \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi^2(x) = \begin{cases} 1/3(2891 - 4050x + 1860x^2 - 280x^3) & \text{if } 2 \le x \le 5/2, \\ 1/3(63 - 186x + 180x^2 - 56x^3) & \text{if } 1 \le x \le 3/2, \\ 2079 - 1882x + 564x^2 - 56x^3 & \text{if } 3 \le x \le 7/2, \\ 1/3(729 - 486x + 108x^2 - 8x^3) & \text{if } 4 \le x \le 9/2, \\ 1/3(-1 + 6x - 12x^2 + 8x^3) & \text{if } 1/2 < x < 1, \\ 1/3(-3367 + 2586x - 660x^2 + 56x^3) & \text{if } 7/2 < x < 4, \\ -231 + 442x - 276x^2 + 56x^3 & \text{if } 3/2 < x < 2, \\ 1/3(-5859 + 6450x - 2340x^2 + 280x^3) & \text{if } 5/2 < x < 3, \\ 0 & \text{otherwise,} \end{cases}$$

$$\tilde{\psi}^1(x) = \begin{cases} (3085325 - 6081780x \\ \quad + 3848631x^2 - 776062x^3)/725760 & \text{if } 3/2 < x < 2, \\ (5581333 - 5351892x \\ \quad + 1694127x^2 - 178370x^3)/725760 & \text{if } 5/2 < x < 3, \\ (2689304 - 2337951x \\ \quad + 654648x^2 - 59555x^3)/725760 & \text{if } 7/2 < x < 4, \\ (1835920 - 1606479x \\ \quad + 445656x^2 - 39651x^3)/725760 & \text{if } 3 \leq x \leq 7/2, \\ (116 - 13531x - 20004x^2 - 7695x^3)/241920 & \text{if } \; -1 \leq x < 0, \\ (1303313 - 644037x \\ \quad + 106251x^2 - 5851x^3)/1451520 & \text{if } 5 \leq x < 6, \\ -311(-343 + 147x - 21x^2 + x^3)/1451520 & \text{if } 6 \leq x < 7, \\ 311(27 + 27x + 9x^2 + x^3)/1451520 & \text{if } \; -3 \leq x < -2, \\ (52717 + 74877x + 36039x^2 + 5851x^3)/1451520 & \text{if } \; -2 \leq x < -1, \\ (-866552 + 542923x - 112344x^2 + 7695x^3)/241920 & \text{if } 4 \leq x < 5, \\ (2836 - 55521x - 30156x^2 + 39651x^3)/725760 & \text{if } 1/2 < x < 1, \\ (348 - 40593x - 60012x^2 + 59555x^3)/725760 & \text{if } 0 \leq x \leq 1/2, \\ (-135883 + 360636x \\ \quad - 446313x^2 + 178370x^3)/725760 & \text{if } 1 \leq x \leq 3/2, \\ (-9331667 + 12543708x \\ \quad - 5464113x^2 + 776062x^3)/725760 & \text{if } 2 \leq x \leq 5/2, \\ 0 & \text{otherwise}, \end{cases}$$

$$\tilde{\psi}^2(x) = \begin{cases} (2968875 - 4307295x + 2038419x^2 - 313856x^3)/181440 & \text{if } 2 \leq x \leq 5/2, \\ (3882467 - 3259287x + 904443x^2 - 83240x^3)/181440 & \text{if } 3 \leq x \leq 7/2, \\ (1709219 - 1396503x + 372219x^2 - 32552x^3)/181440 & \text{if } 7/2 < x < 4, \\ (499 + 253x - 369x^2 - 1922x^3)/60480 & \text{if } 0 \leq x < 1, \\ (1497 + 759x - 1107x^2 - 680x^3)/181440 & \text{if } \; -1 \leq x < 0, \\ -311(-343 + 147x - 21x^2 + x^3)/181440 & \text{if } 6 \leq x < 7, \\ 311(8 + 12x + 6x^2 + x^3)/181440 & \text{if } \; -2 \leq x < -1, \\ (-107383 + 61311x - 11307x^2 + 680x^3)/181440 & \text{if } 5 \leq x < 6, \\ (-247711 + 147587x - 29199x^2 + 1922x^3)/60480 & \text{if } 4 \leq x < 5, \\ (-36821 + 115713x - 116061x^2 + 32552x^3)/181440 & \text{if } 1 \leq x \leq 3/2, \\ (-207893 + 457857x - 344157x^2 + 83240x^3)/181440 & \text{if } 3/2 < x < 2, \\ (-6839125 + 7462305x - 2669421x^2 + 313856x^3)/181440 & \text{if } 5/2 < x < 3, \\ 0 & \text{otherwise}. \end{cases}$$

We depict the graphs of these generators in Fig. 5.

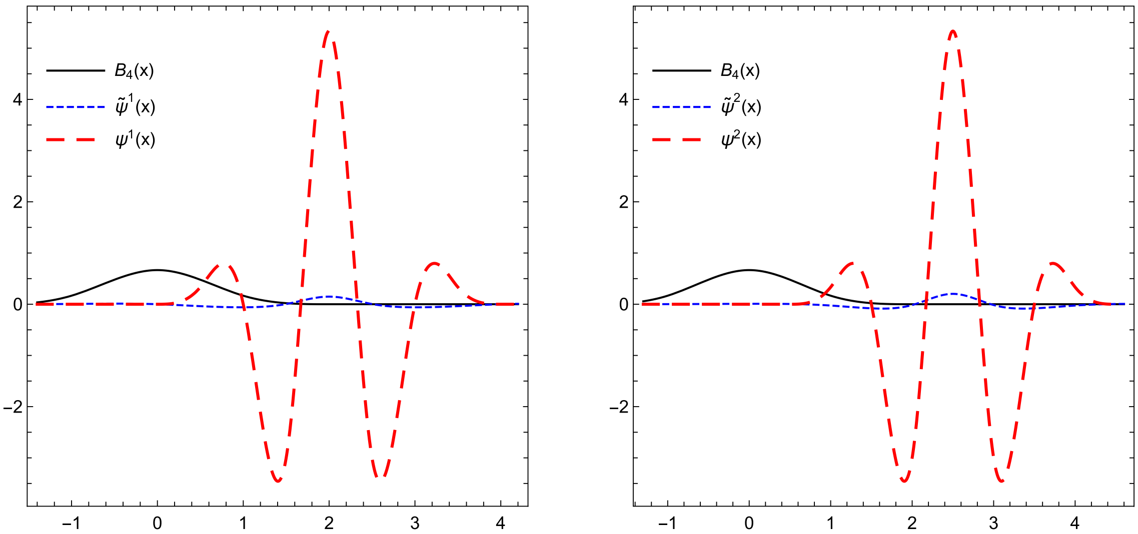

Fig. 5 The graphs of the generators in System D.

4. QUASI-AFFINE BI-FRAMELET OPERATIONAL MATRIX OF FDES

In this section, we introduce the quasi-affine bi-framelet operational matrix of the FDEs defined in Eq. (1) using the function approximation given by Eq. (12).

For simplicity, we present the bi-framelet operational matrix using System A (Haar bi-framelet system $X(\{\psi_1, \psi_2\})$). For other types of bi-framelet systems that generated by higher order of B-splines given in Sec. 3, the procedure will be the same.

Theorem 4.1. *Suppose that $D_*^\alpha \mathfrak{U}_n u(\cdot)$ and the system $(X(\Psi), X(\tilde{\Psi}))$ are obtained using System A based on the MOEP, and the functions $\{u^{(m)}(\cdot), x \in (0,1)\}$, $m \in \mathbb{N}$ are continuous and bounded. Then, for some constant M, we have*

$$\|u - \mathfrak{U}_n u\|_2 \le M 2^{-2n}.$$

Proof. Based on the upper bound of the bi-framelet system using Bessel property, we have

$$\|u - \mathfrak{U}_n u\|_2^2 \le \sum_{j\ge n}\sum_{k\in\mathbb{Z}} (|\langle u, \psi_{j,k}^1\rangle|^2 + |\langle u, \psi_{j,k}^2\rangle|^2).$$

But,

$$\begin{aligned} &|\langle u, \psi_{j,k}^1\rangle| + |\langle u, \psi_{j,k}^2\rangle| \\ &\quad \le \|u\|_\infty (\|\psi_{j,k}^1\|_1 + \|\psi_{j,k}^2\|_1) \\ &\quad = \|u\|_\infty 2^{-j/2}(\|\psi^1\|_1 + \|\psi^2\|_1). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|u - \mathfrak{U}_n u\|_2^2 &\le \|u\|_\infty \left(\max_x \{\|\psi^1\|_1 + \|\psi^2\|_1\}\right) \\ &\quad \times \sum_{j\ge n}\sum_{k\in\mathbb{Z}} 2^{-j}(|\langle u, \psi_{j,k}^1\rangle| \\ &\quad + |\langle u, \psi_{j,k}^2\rangle|). \end{aligned}$$

Using the mean value theorem for integration, one can find ξ_1, ξ_2 such that $\xi_1 < \xi_2$ and that

$$\begin{aligned} |\langle u, \psi_{j,k}^1\rangle| + |\langle u, \psi_{j,k}^2\rangle| &\le 2\left(\int_A |u| dx - \int_B |u| du\right) \\ &\le 2^{-j-1}(|u(\xi_1) - u(\xi_2)|), \end{aligned}$$

where $A = 2^{-j}[k, k+0.5)$ and $B = 2^{-j}[k+0.5, k+1)$. Hence,

$$\begin{aligned} \|u - \mathfrak{U}_n u\|_2^2 &\le \|u\|_\infty \left(\max_x \{\|\psi^1\|_1 + \|\psi^2\|_1\}\right) \\ &\quad \times \sum_{j\ge n} 2^{-2j-1}(|u(\xi_1) - u(\xi_2)|) \\ &\le \frac{2}{3}\|u\|_\infty \left(\max_x \{\|\psi^1\|_1 + \|\psi^2\|_1\}\right) \\ &\quad \times (|u(\xi_1) - u(\xi_2)|) 2^{-2n}. \end{aligned}$$

Then, the result is concluded. □

Notice, Eq. (12) can be written as

$$\mathfrak{U}_n g(t) = A_n^{\mathrm{T}} G_n(t), \tag{13}$$

where the Haar bi-framelet vector A_n (A_n^{T} is the transpose of the bi-framelet vector A_n) and Haar bi-framelet function vector $G_n(t)$ are given as

$$\begin{aligned} A_n =&: [a^1_{-2^n,-n}, a^1_{-2^n+1,-n}, \dots, \\ & a^1_{0,0}, \dots, a^1_{2^n-1,n-1}, a^1_{2^n-1,n}]^{\mathrm{T}}, \\ G_n(t) =&: [\psi^1_{-2^n,-n}(t), \psi^1_{-2^n+1,-n}(t), \dots, \\ & \psi^1_{0,0}(t), \dots, \psi^1_{2^n-1,n-1}(t), \\ & \psi^1_{2^n-1,n}(t)]^{\mathrm{T}}. \end{aligned}$$

Consider the collocation points

$$t_i = \frac{i}{n_p}, \quad i = 1, \dots, n_p,$$

where n_p is the length of interval of $\psi_{j,k}$ that covers the compact support of the bi-framelet system at hand, for example, for the case of Haar bi-framelet system, $n_p = 2^{n+1}(2n+1)$. Define the operational bi-framelet matrix as follows:

$$L_{n_p \times n_p}(t_i) = [G_n(t_i)]_{i=1,\dots,n_p}.$$

Now, considering the system defined by Eq. (1) and in order to solve it using a bi-framelet system via the discretized points, we substitute the truncated expansion given in Eq. (13). Thus

$$\begin{aligned} A_n^{\mathrm{T}}(G_n(t_i) + c(t_i)\ (\mathcal{I}^\alpha G_n)(t_i)) \\ = f(t_i), \quad i = 1, \dots, n_p, \end{aligned} \tag{14}$$

where $\mathcal{I}^\alpha$ is the FIO defined by Eq. (2).

For $i = 1, \dots, n_p$, Eq. (14) can be written as

$$L^{\mathrm{T}}_{n_p \times n_p}(t_i) A_n = F_{n_p}(t_i), \tag{15}$$

where $F_{n_p}(t_i) = [f(t_i)]_{i=1,\dots,n_p}$ is a matrix function vector of order $n_p \times 1$. The coefficient bi-framelet vector A_n in Eq. (15), and so the approximated solution, can be calculated using, for example, Mathematica software.

5. NUMERICAL APPLICATIONS

In this section, we use the bi-framelet systems presented in Sec. 3 to illustrate and show the efficiency of the proposed method by using some examples of FDEs.

Example 1. Consider the following FDEs:

$$\begin{cases} \mathcal{D}^\alpha_* u(t) + c(t)u(t) = f(t), \quad \text{where} \\ \alpha = \dfrac{1}{3}, \quad c(t) = \sqrt[3]{t}, \\ f(t) = 1.10773\sqrt[3]{t^2} + \sqrt[3]{t^4}, \\ u(0) = 0. \end{cases}$$

The exact solution for this FDE is $u(t) = t$.

By applying the procedure in Sec. 4 and solving the resulting system, we obtain the approximated solution of the FDE using System A, for example, when $n = 12, 13$, respectively, we have

$$\mathcal{U}u_{12}(x) = \begin{cases} 4.44399 \times 10^{-16} \\ \quad + 8.51319 \times 10^{-13}x, & \dfrac{1}{2} < x \le 1, \\ -3.19679 \times 10^{-17} \\ \quad + 8.52429 \times 10^{-13}x, & \dfrac{1}{4} < x < \dfrac{1}{2}, \\ 8.52429 \times 10^{-13}x, & 0 < x \le \dfrac{1}{4}, \end{cases}$$

$$\mathcal{U}u_{13}(x) = \begin{cases} 7.58912 \times 10^{-14} + x, & \dfrac{3}{4} < x \le \dfrac{7}{8}, \\ 3.21889 \times 10^{-14} + x, & \dfrac{3}{8} < x \le \dfrac{1}{2}, \\ 6.08208 \times 10^{-15} + x, & \dfrac{1}{8} < x \le \dfrac{1}{4}, \\ -1.56248 \times 10^{-14} + x, & \dfrac{5}{8} < x \le \dfrac{3}{4}, \\ -1.74251 \times 10^{-14} + x, & \dfrac{1}{2} < x \le \dfrac{5}{8}, \\ -3.84196 \times 10^{-14} + x, & \dfrac{7}{8} < x \le 1, \\ -2.04705 \times 10^{-14} + x, & \dfrac{1}{4} < x \le \dfrac{3}{8}, \\ x, & 0 < x \le \dfrac{1}{8}. \end{cases}$$

We also present the matrix $L_{n_p \times n_p}(n = 1)$ in Eq. (15) using System A where

$$L_{n_p \times n_p} = \begin{bmatrix} 0.123526 & -0.000125493 & -0.000177473 & -0.000250985 & 0. \\ 0.213044 & -0.000216492 & -0.000306166 & -0.000432984 & 0. \\ 0.297806 & -0.000302682 & -0.000428058 & -0.000605365 & 0. \\ 0.38092 & -0.000387215 & -0.000547605 & -0.000851786 & 0. \\ 0.463514 & -0.000471231 & -0.000666422 & -0.00107302 & 0. \\ 0.546115 & -0.000555266 & -0.000785265 & -0.00129082 & 0. \\ 0.629008 & -0.000639607 & -0.000962684 & -0.00114703 & 0.280299 \\ 0.712356 & -0.000724419 & -0.00112202 & -0.00110508 & 0.479997 \\ 0.79626 & -0.000809802 & -0.0012794 & -0.00108281 & 0.66531 \\ 0.880779 & -0.000895818 & -0.0014365 & -0.00106985 & 0.267067 \\ 0.965953 & -0.000982504 & -0.00159398 & -0.00106223 & 0.0286871 \\ 1.051800 & -0.00106988 & -0.00175219 & -0.00105799 & -0.180348 \end{bmatrix}.$$

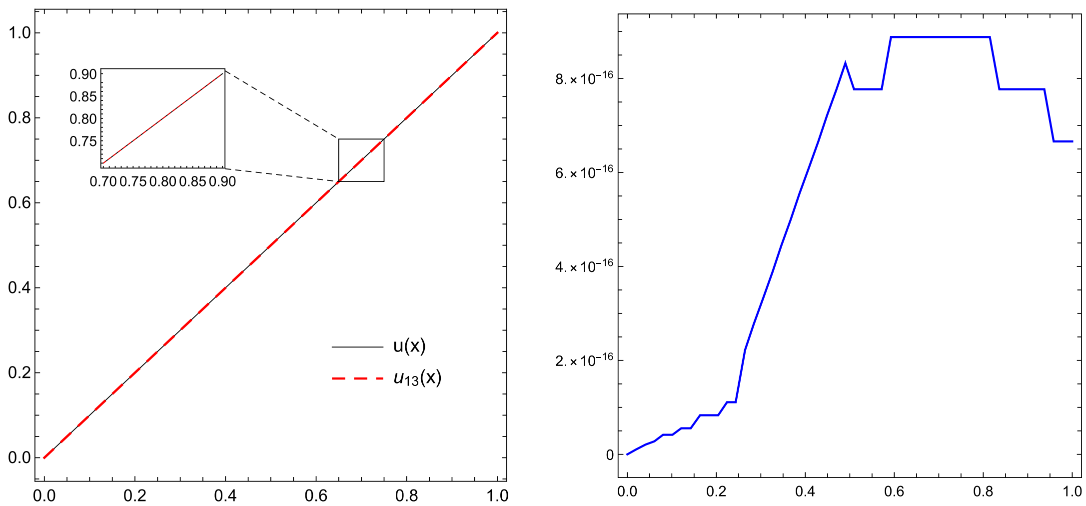

Fig. 6 Numerical and exact solution with the error graph using System A for $n = 13$.

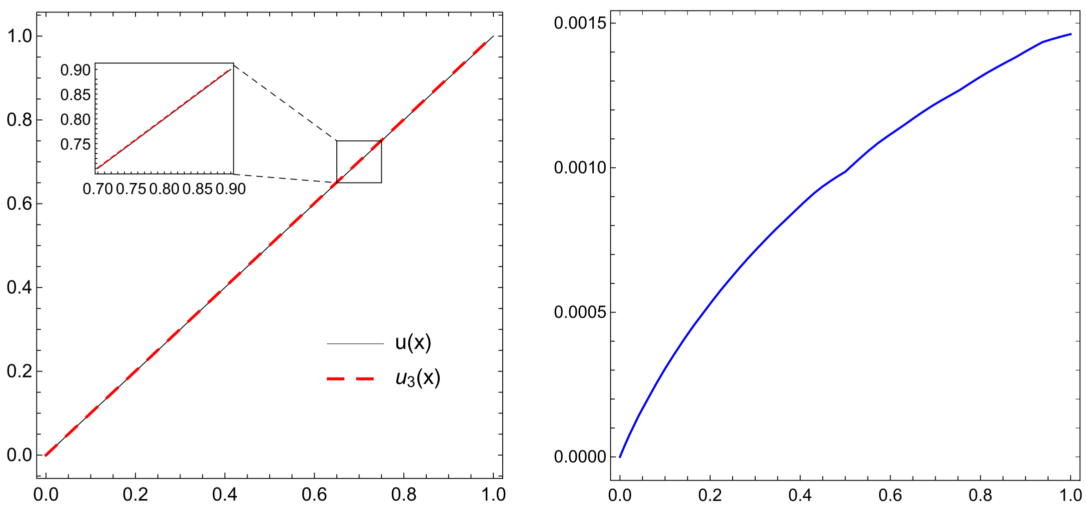

Fig. 7 Numerical and exact solution with the error graph using System B for $n = 3$.

Table 1 Numerical Results for $\mathcal{U}u_3$ Using Systems A through D.

$\mathcal{U}u_3(x)$ of Example 1					
x	Exact	System A	System B	System C	System D
0.1	0.1	0.1005346898001	0.100304753855	0.10006672850	0.1000042543
0.2	0.2	0.2009938636138	0.200529195474	0.20005975472	0.2000024200
0.3	0.3	0.3013962039710	0.300713551069	0.30045594550	0.3000034660
0.4	0.4	0.4017562049210	0.400868129463	0.40002791531	0.4000077620
0.5	0.5	0.5020926227360	0.500985983964	0.50001001520	0.5000940020
0.6	0.6	0.6024023892389	0.600115377357	0.60102998071	0.6000102330
0.7	0.7	0.7026903928032	0.700220677682	0.70102124144	0.7000174420
0.8	0.8	0.8029661945575	0.800314535807	0.80107057352	0.8000861630
0.9	0.9	0.9032226752090	0.900401982061	0.90203838642	0.9000234220
1.0	1.0	1.0034649540220	1.000461888702	1.00347076090	1.0008222021

Table 2 Error Results of Example 1 for Different Values on n.

n	System A	System B	System C	System D
1	5.12×10^{-2}	4.84×10^{-4}	2.25×10^{-5}	6.45×10^{-6}
2	3.85×10^{-3}	1.02×10^{-5}	5.65×10^{-6}	2.67×10^{-7}
3	1.93×10^{-5}	2.01×10^{-6}	2.67×10^{-7}	1.97×10^{-9}
4	3.75×10^{-6}	1.97×10^{-7}	5.01×10^{-9}	2.20×10^{-10}
5	8.24×10^{-7}	1.83×10^{-9}	3.56×10^{-10}	3.73×10^{-12}

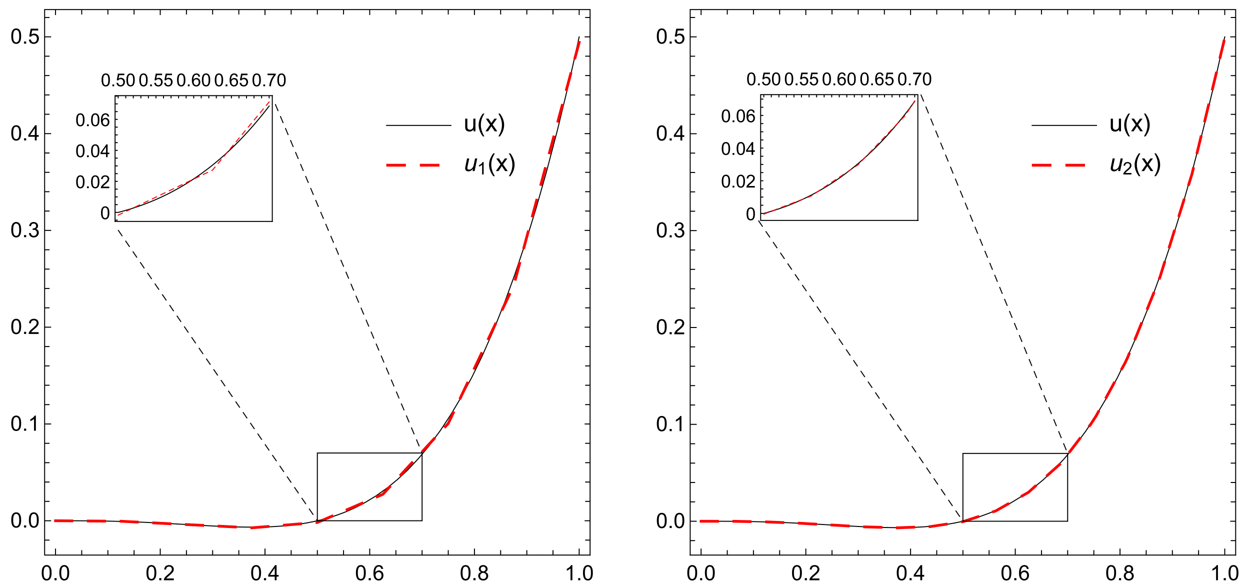

Fig. 8 Numerical and exact solution using System A for $n = 1, 2$, respectively.

The graphical and numerical results of the Systems A and B, for different values of n, are presented in Figs. 6 and 7. The obtained results are in an excellent agreement with the exact solution. Also, the error bounds are controlled as detailed in Tables 1 and 2.

Example 2. Consider the following FDE that generated from a Newtonian fluid application[38]:

$$\begin{cases} \mathcal{D}_*^{\alpha} u(t) + c(t)u(t) = f(t), \quad \text{where}, \\ \alpha = \dfrac{1}{4}, \quad c(t) = 1, \\ f(t) = -0.678274t^{\frac{11}{4}} - 0.5t^3 + 1.44699t^{\frac{15}{4}} + t^4, \\ u(0) = 0. \end{cases}$$

The exact solution for the given FDE is $u(t) = \frac{1}{2}(2t^4 - t^3)$. Again, based on on the numerical scheme presented in Sec. 4, we conclude the following numerical and graphical results based on System A through D in Tables 3 and 4, Figs. 8 and 9.

Figure 10 shows the convergence behavior of the numerical results obtained in Examples 1 and 2 based on the Systems A through D.

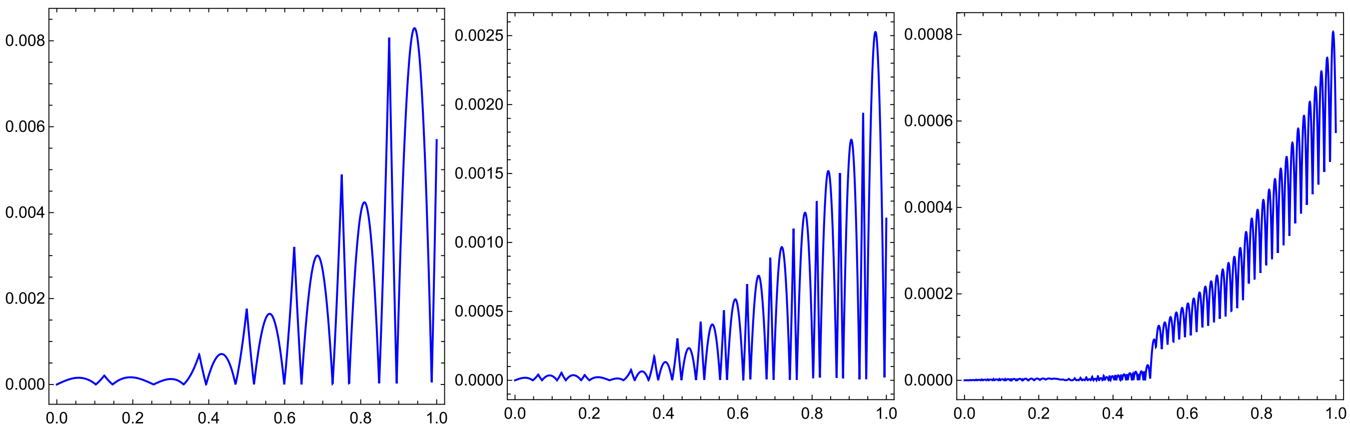

Fig. 9 The error graph using System A for $n = 1, 2, 3$, respectively.

Table 3 Numerical Results for $\mathcal{U}u_3$ Using Systems A through D.

	$\mathcal{U}u_3(x)$ of Example 2				
x	Exact	System A	System B	System C	System D
0.1	−0.0004	−0.00041860817	−0.000403556327	−0.000402299	−0.00040197654
0.2	−0.0024	−0.00256530153	−0.002401047694	−0.002387746	−0.00240284943
0.3	−0.0054	−0.00527270464	−0.005383112986	−0.005400365	−0.00540105671
0.4	−0.0064	−0.00618757695	−0.006312109453	−0.006457773	−0.00639123865
0.5	0.0	0.000177081303	0.0001475890703	0.000001993	0.000002750685
0.6	0.0216	0.021484411843	0.0260873667288	0.0214862543	0.021647366728
0.7	0.0686	0.007125184395	0.0637654888274	0.0217741345	0.068818651729
0.8	0.1536	0.153581303000	0.1533247096767	0.1537444675	0.153953548984
0.9	0.2916	0.291359969675	0.2917539974832	0.2916787765	0.292175856955
1.0	0.5	0.498312385646	0.4993664893789	0.4995883653	0.500574441403

Table 4 Error Results of Example 2 for Different Values on n.

n	System A	System B	System C	System D
1	6.44×10^{-2}	1.35×10^{-3}	2.35×10^{-4}	7.67×10^{-6}
2	3.02×10^{-3}	6.45×10^{-5}	4.45×10^{-6}	0.88×10^{-6}
3	4.56×10^{-4}	6.73×10^{-6}	3.35×10^{-7}	2.55×10^{-9}
4	6.45×10^{-6}	7.33×10^{-7}	3.01×10^{-8}	5.68×10^{-10}
5	1.24×10^{-8}	0.46×10^{-8}	2.55×10^{-10}	7.75×10^{-12}

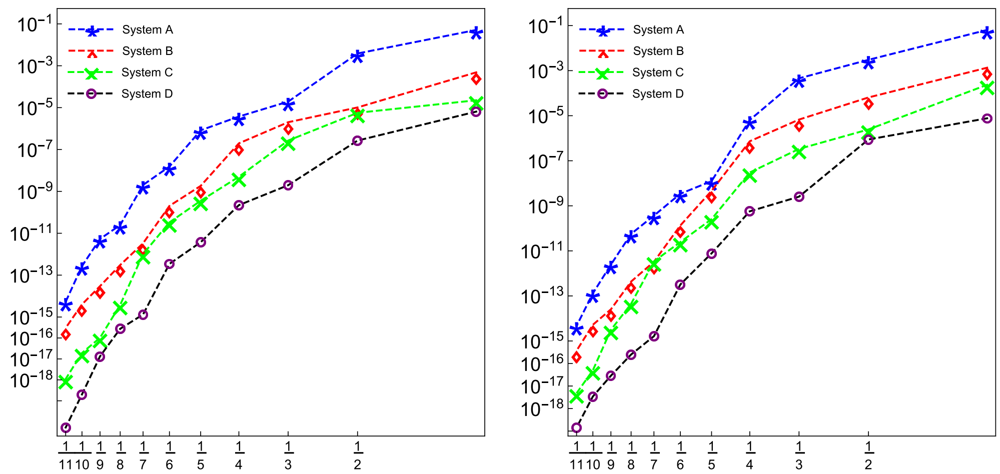

Fig. 10 Convergence rate of Examples 1 and 2, respectively based on Systems A through D.

We provide the approximate solution of this FDE based on system A for $n = 2$, namely

$$\mathcal{U}u_2(x) = \begin{cases} 0.00365168 - 0.0302517x, & \frac{3}{16} \le x \le \frac{1}{4}, \\ 0.00363541 - 0.0301866x, & \frac{1}{4} < x \le \frac{5}{16}, \\ 0.00201247 - 0.0215092x, & \frac{1}{8} \le x < \frac{3}{16}, \\ -0.000951834 - 0.0155074x, & \frac{5}{16} < x < \frac{3}{8}, \\ 0.000547547 - 0.00978986x, & \frac{1}{16} < x < \frac{1}{8}, \\ -0.0141255 + 0.0196224x, & \frac{3}{8} \le x \le \frac{7}{16}, \\ -0.0413182 + 0.0817772x, & \frac{7}{16} < x \le \frac{1}{2}, \\ -0.0887531 + 0.176647x, & \frac{1}{2} < x < \frac{9}{16}, \\ -0.162199 + 0.307218x, & \frac{9}{16} \le x < \frac{5}{8}, \\ -0.272311 + 0.483397x, & \frac{5}{8} \le x < \frac{11}{16}, \\ -0.427621 + 0.709302x, & \frac{11}{16} \le x < \frac{3}{4}, \\ -0.638948 + 0.991072x, & \frac{3}{4} \le x < \frac{13}{16}, \\ -0.0010291x, & 0 < x \le \frac{1}{16}, \\ -0.917987 + 1.3345x, & \frac{13}{16} \le x < \frac{7}{8}, \\ -1.27376 + 1.7411x, & \frac{7}{8} \le x < \frac{15}{16}, \\ -1.74605 + 2.24487x, & \frac{15}{16} \le x \le 1. \end{cases}$$

6. CONCLUSION

The purpose of this work is to develop an efficient method for solving FDEs. We derive a new numerical scheme for solving important types of FDEs based on the quasi-affine bi-framelet operational matrices of fractional integration formula.

The Systems A through D were generated using B-spline functions of different orders and based upon the popular OEP that increase the accuracy orders of the approximated function by increasing the vanishing moments property. The utilized numerical scheme can be used to solve various types of FDEs including the nonlinear case that will be considered in the future work. Two examples of FDEs have been considered to test the validity of the proposed scheme and demonstrate the powerfulness of the method. The problems have been reduced to solving a system of algebraic equations. The obtained numerical and graphical results show an excellent agreement with the exact values and the theoretical estimation. Increasing the partial sums and the order of the B-splines being used to generate the systems result an increase in the accuracy and efficiency of the method.

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