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Yerlan Amanbek
Nazarbayev University

Zhibin Du
South China Normal University

Yogi Erlangga
Zayed University

Carlos M. Da Fonseca
Kuwait College of Science & Technology

Bakytzhan Kurmanbek
Nazarbayev University

See next page for additional authors

Rapid Communication

Yerlan Amanbek*, Zhibin Du, Yogi Erlangga, Carlos M. da Fonseca,
Bakytzhan Kurmanbek, and António Pereira

Explicit determinantal formula for a class of banded matrices

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Abstract: In this short note, we provide a brief proof for a recent determinantal formula involving a particular family of banded matrices.

Keywords: determinant, pentadiagonal matrices, Chebyshev polynomials of second kind

MSC 2020: 15A18, 15B05

1 Introduction

It was proved recently in [1] that the determinant of the banded matrix (which is a particular case of a Hessenberg matrix), for any integer $n \geqslant 4$,

$$A_n = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & a & b \\ 1 & 1 & 1 & \ddots & & 0 & a \\ 1 & 1 & \ddots & \ddots & \ddots & & 0 \\ 0 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 1 & 1 \end{pmatrix}_{n\times n} \tag{1.1}$$

is given by

$$\det A_n = \begin{cases} (a-1)^2 & \text{if } n \equiv 0 \pmod 4, \\ a^2 + b + 1 & \text{if } n \equiv 1 \pmod 4, \\ a^2 + 2a - b & \text{if } n \equiv 2 \pmod 4, \\ a^2 & \text{if } n \equiv 3 \pmod 4, \end{cases} \tag{1.2}$$

*** Corresponding author: Yerlan Amanbek,** Nazarbayev University, Department of Mathematics, 53 Kabanbay Batyr Ave., Nur-Sultan, 010000, Kazakhstan, e-mail: yerlan.amanbek@nu.edu.kz
Zhibin Du: School of Software, South China Normal University, Foshan, Guangdong 528225, China, e-mail: zhibindu@126.com
Yogi Erlangga: Zayed University, Department of Mathematics, Abu Dhabi Campus, P.O. Box 144534, Abu Dhabi, United Arab Emirates, e-mail: yogi.erlangga@zu.ac.ae
Carlos M. da Fonseca: Kuwait College of Science and Technology, Doha District, Block 4, P.O. Box 27235, Safat, 13133, Kuwait; University of Primorska, FAMNIT, Glagoljaška 8, 6000, Koper, Slovenia, e-mail: c.dafonseca@kcst.edu.kw, carlos.dafonseca@famnit.upr.si
Bakytzhan Kurmanbek: Nazarbayev University, Department of Mathematics, 53 Kabanbay Batyr Ave., Nur-Sultan, 010000, Kazakhstan, e-mail: bakytzhan.kurmanbek@nu.edu.kz
António Pereira: Departamento de Matemática, Universidade de Aveiro, Aveiro, 3810-193, Portugal, e-mail: antoniop@ua.pt

for any a and b. The proof for this equality is based on several auxiliary results established for particular cases of the matrix (1.1). As a corollary, two conjectures proposed in [2] are proved. For a recent and different approach, the reader is also referred to [3]. In this work, our goal is to provide a proof for (1.2) in a different way than [1]. The explicit formula for the determinant of the non-symmetric matrices can be applied in efficient computations, since several algorithms have been proposed to improve the efficiency of the determinant computation [4,5].

2 Proof

This new proof is based on the elementary properties of the determinant. First note that when $n = 4, 5, 6, 7$, one can deduce (1.2) by simple computations or by utilizing a Computer Algebra System such as Maple, Mathematica, and SAGE. For the convenience of the reader, we present the matrices for these cases,

$$A_4 = \begin{pmatrix} 1 & 1 & a & b \\ 1 & 1 & 1 & a \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 1 & 1 & 0 & a & b \\ 1 & 1 & 1 & 0 & a \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \quad A_6 = \begin{pmatrix} 1 & 1 & 0 & 0 & a & b \\ 1 & 1 & 1 & 0 & 0 & a \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \quad A_7 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & a & b \\ 1 & 1 & 1 & 0 & 0 & 0 & a \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Let us assume now that $n \geqslant 8$. We used the cofactor expansion of $\det A_n$ along the first column and the subtraction of the first row from second and third rows:

$$\det A_n = \begin{vmatrix} 1 & 1 & 0 & \cdots & \cdots & 0 & a & b \\ 0 & 0 & 1 & \ddots & & 0 & -a & a-b \\ 0 & 0 & 1 & \ddots & \ddots & 0 & -a & -b \\ 0 & 1 & 1 & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 & 0 \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 1 & 1 \\ 0 & \cdots & \cdots & 0 & 0 & 1 & 1 & 1 \end{vmatrix}_{n\times n} \quad \text{(cofactor expansion along the first column)}$$

$$= \begin{vmatrix} 0 & 1 & 0 & \cdots & 0 & -a & a-b \\ 0 & 1 & \ddots & \ddots & 0 & -a & -b \\ 1 & 1 & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 1 & 1 \end{vmatrix}_{(n-1)\times(n-1)} \quad \text{(cofactor expansion along the first column)}$$

$$= \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & -a & a-b \\ 1 & 1 & 0 & \cdots & 0 & -a & -b \\ 1 & 1 & 1 & 1 & \ddots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & 0 \\ \vdots & & \ddots & \ddots & 1 & 1 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 1 & 1 \end{vmatrix}_{(n-2)\times(n-2)} \quad (R_2 - R_1 \text{ and } R_3 - R_1)$$

$$
\begin{aligned}
&= \begin{vmatrix}
1 & 0 & 0 & \cdots & \cdots & 0 & -a & a-b \\
0 & 1 & 0 & \ddots & & 0 & 0 & -a \\
0 & 1 & 1 & 1 & \ddots & 0 & a & b-a \\
0 & 1 & 1 & \ddots & \ddots & \ddots & 0 & 0 \\
0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 & 0 \\
\vdots & & \ddots & \ddots & \ddots & \ddots & 1 & 1 \\
0 & \cdots & \cdots & 0 & 0 & 1 & 1 & 1
\end{vmatrix}_{(n-2)\times(n-2)} \qquad \text{(cofactor expansion along the first column)} \\
&= \begin{vmatrix}
1 & 0 & 0 & \cdots & \cdots & 0 & 0 & -a \\
1 & 1 & 1 & \ddots & & 0 & a & b-a \\
1 & 1 & 1 & 1 & \ddots & 0 & 0 & 0 \\
0 & 1 & 1 & \ddots & \ddots & \ddots & 0 & 0 \\
0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 & 0 \\
\vdots & & \ddots & \ddots & \ddots & \ddots & 1 & 1 \\
0 & \cdots & \cdots & 0 & 0 & 1 & 1 & 1
\end{vmatrix}_{(n-3)\times(n-3)} \qquad (R_2 - R_1 \text{ and } R_3 - R_1) \\
&= \begin{vmatrix}
1 & 0 & 0 & \cdots & \cdots & 0 & 0 & -a \\
0 & 1 & 1 & \ddots & & 0 & a & b \\
0 & 1 & 1 & 1 & \ddots & 0 & 0 & a \\
0 & 1 & 1 & \ddots & \ddots & \ddots & 0 & 0 \\
0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 & 0 \\
\vdots & & \ddots & \ddots & \ddots & \ddots & 1 & 1 \\
0 & \cdots & \cdots & 0 & 0 & 1 & 1 & 1
\end{vmatrix}_{(n-3)\times(n-3)} \qquad \text{(cofactor expansion along the first column)} \\
&= \det A_{n-4}.
\end{aligned}
$$

This means that $\det A_n$ has period 4 and the proof is complete.

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