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A Study Of Two Divergent Series With A Convergent Minimum

Alexander James Russo

A STUDY OF TWO DIVERGENT SERIES WITH A CONVERGENT MINIMUM

by

Alexander J Russo

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Abstract

We investigate the phenomenon of divergent series of positive terms having convergent minimum. As entry into this topic, we look at Exercise twenty-three from chapter two of Karl R. Stromberg's "Introduction to Classical Real Analysis", which addresses this very case. The exercise calls for the construction of two infinite divergent series, $\sum a_n$ and $\sum b_n$, having strictly positive, non-increasing terms, such that the series $\sum c_n$, the nth term of which is the minimum of the nth terms of the original two series, converges. We then establish that it is not possible that one of the original two series in such a construction can be the harmonic series. Along the way, we consider Exercise forty-seven, part b from chapter two of the same text, which asks: if we have an infinite, divergent series $\sum d_n$, then what can be said of the infinite series $\frac{d_n}{1+nd_n}$? We also utilize the properties of upper and lower density in formulating the final proof.

Exercise 23 from chapter 2 of Karl R. Stromberg's *An Introduction to Classical Real Analysis* (Stromberg 2015) reads:

"There exists two divergent series $\sum a_n$ and $\sum b_n$ of positive terms with $a_1 \geq a_2 \geq a_3 \geq ...$ and $b_1 \geq b_2 \geq b_3 \geq ...$ such that if $c_n = \min\{a_n, b_n\}$, then $\sum c_n$ converges."

Proof. Put $N_0 = 0$ and $N_k = 2^{k^2}$, $k \geq 1$. That is, put $N_0 = 0$, $N_1 = 2$, $N_2 = 16$, $N_3 = 512$, $N_4 = 65536$,... If $i = 2k$ and $N_i + 1 \leq n \leq N_{i+1}$, set:

$$a_n = \frac{1}{2^{k+1}(N_{2k+1} - N_{2k})}$$

and

$$b_n = \frac{1}{N_{2k+1} - N_{2k}}.$$

Example 1. Consider the case $i = 2$ (that is $k = 1$). If $17 = N_2 + 1 \leq n \leq N_3 = 512$, then $a_n = \frac{1}{2^2(512-16)} = \frac{1}{4(496)} = \frac{1}{1984}$ and $b_n = \frac{1}{512-16} = \frac{1}{496}$. I.e.:

$$a_{17} = a_{18} = \cdots = a_{512} = \frac{1}{1984}$$

and

$$b_{17} = b_{18} = \cdots = b_{512} = \frac{1}{496}.$$

If $i = 2k - 1$, and $N_i + 1 \leq n \leq N_{i+1}$, set:

$$a_n = \frac{1}{N_{2k} - N_{2k-1}}$$

and

$$b_n = \frac{1}{2^{k+1}(N_{2k} - N_{2k-1})}.$$

Example 2. Consider the case $i = 3$ (that is $k = 2$). If $513 = N_3 + 1 \leq n \leq N_4 = 65536$, Then $a_n = \frac{1}{65536-512} = \frac{1}{65024}$ and $b_n = \frac{1}{2^3(65024)} = \frac{1}{520192}$ I.e.:

$$a_{513} = a_{514} = \cdots = a_{65536} = \frac{1}{65024}$$

and

$$b_{513} = b_{514} = \cdots = b_{65536} = \frac{1}{8(65536)} = \frac{1}{524288}.$$

Then $a_{n+1} \leq a_n$ for all n. This is clear when $n \neq N_{2k+1}$.

When $n = N_{2k+1}$,

$$\begin{aligned}
a_{n+1} =& \frac{1}{N_{2k+2} - N_{2k+1}} = \frac{1}{2^{(2k+2)^2} - 2^{(2k+1)^2}} \\
\leq& \frac{1}{2^{(2k+2)^2-1}} = \frac{1}{2^{4k^2+8k+3}} < \frac{1}{2^{k+1}2^{4k^2+4k+1}} \\
=& \frac{1}{2^{k+1}[N_{2k+1}]} \leq \frac{1}{2^{k+1}[N_{2k+1} - N_{2k}]} = a_n.
\end{aligned}$$

Similarly, one may show that $b_{n+1} \leq b_n$ for all n.

We now show that $\sum a_n = \infty$:

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n =& \sum_{i=1}^{\infty} \sum_{n=N_i+1}^{N_{i+1}} a_n \\
\geq& \sum_{k=1}^{\infty} \sum_{n=N_{2k-1}+1}^{N_{2k}} a_n \\
=& \sum_{k=1}^{\infty} \sum_{n=N_{2k-1}+1}^{N_{2k}} \frac{1}{N_{2k} - N_{2k-1}} \\
=& \sum_{k=1}^{\infty} (N_{2k} - N_{2k-1}) \frac{1}{N_{2k} - N_{2k-1}} = \infty.
\end{aligned}$$

We next show that $\sum b_n = \infty$:

$$\begin{aligned}
\sum_{n=1}^{\infty} b_n =& \sum_{i=1}^{\infty} \sum_{n=N_i+1}^{N_{i+1}} b_n \\
\geq& \sum_{k=1}^{\infty} \sum_{n=N_{2k}+1}^{N_{2k+1}} b_n \\
=& \sum_{k=1}^{\infty} \sum_{n=N_{2k}+1}^{N_{2k+1}} \frac{1}{N_{2k+1} - N_{2k}} \\
=& \sum_{k=1}^{\infty} (N_{2k+1} - N_{2k}) \frac{1}{N_{2k+1} - N_{2k}} = \infty.
\end{aligned}$$

Finally, we show that $\sum c_n = \sum \min\{a_n, b_n\} < \infty$:

$$\begin{aligned}\sum_{n=1}^{\infty} c_n &= \sum_{i=1}^{\infty} \sum_{n=N_i+1}^{N_{i+1}} c_n \\ &= \sum_{k=1}^{\infty} \sum_{n=N_{2k-1}+1}^{N_{2k}} c_n + \sum_{k=1}^{\infty} \sum_{n=N_{2k}+1}^{N_{2k+1}} c_n \\ &\leq \sum_{k=1}^{\infty} \sum_{n=N_{2k-1}+1}^{N_{2k}} b_n + \sum_{k=1}^{\infty} \sum_{n=N_{2k}+1}^{N_{2k+1}} a_n \\ &= \sum_{k=1}^{\infty} (N_{2k} - N_{2k-1}) \frac{1}{2^{k+1}(N_{2k} - N_{2k-1})} \\ &+ \sum_{k=1}^{\infty} (N_{2k+1} - N_{2k}) \frac{1}{2^{k+1}(N_{2k+1} - N_{2k})} = 1 < \infty.\end{aligned}$$

□

In this thesis, we will consider the question of when, given a single series $\sum a_n$ consistent with the hypotheses of the previous exercise, there exists a second series $\sum b_n$, also consistent with the hypotheses, such that the conclusion holds as well. Exercise 47b. from Karl R. Stromberg's *An Introduction to Classical Real Analysis* appears to be relevant to this question:

> "Suppose that $d_n > 0$ for all $n \in \mathbb{N}$ and $\sum d_n = \infty$. What can be said of the series:
>
> $$\sum_{n=1}^{\infty} \frac{d_n}{1 + nd_n}?"$$

The following claim makes a connection between this exercise and the foregoing one.

Claim 1:
$\sum_{n=1}^{\infty} \frac{d_n}{1+nd_n} < \infty$ if and only if $\sum_{n=1}^{\infty} \min\{d_n, \frac{1}{n}\} < \infty$.

Proof of Claim 1. "$\Leftarrow$" Suppose that $\sum_{n=1}^{\infty} \min\{d_n, \frac{1}{n}\} < \infty$.

Then we have

$$\frac{d_n}{1 + nd_n} \leq d_n$$

(since $1 + nd_n \geq 1$) and

$$\frac{d_n}{1 + nd_n} \leq \frac{d_n}{nd_n} = \frac{1}{n}.$$

Hence
$$\frac{d_n}{1+nd_n} \le \min\Big\{d_n, \frac{1}{n}\Big\}.$$

So $\sum_{n=1}^{\infty} \frac{d_n}{1+nd_n} < \infty$ by the comparison test.

"$\Rightarrow$" Suppose that $\sum_{n=1}^{\infty} \frac{d_n}{1+nd_n} < \infty$.

Fact: For all n, either $\frac{1}{n} \le 10\frac{d_n}{1+nd_n}$ or $d_n \le 10\frac{d_n}{1+nd_n}$.

Suppose that Fact is false. Then there exists an n such that:
$$\frac{1}{n} > 10\frac{d_n}{1+nd_n}$$
and
$$d_n > 10\frac{d_n}{1+nd_n}.$$
Therefore:
$$1+nd_n > 10nd_n$$
and
$$d_n + nd_n^2 > 10d_n.$$

So
$$1 > 9nd_n$$
and
$$nd_n^2 > 9d_n.$$
Thus,
$$\frac{1}{9} > nd_n$$
and
$$nd_n > 9.$$

$\Rightarrow\Leftarrow$

This contradiction establishes Fact.

Therefore (by Fact), $10\frac{d_n}{1+nd_n} \ge \min\{d_n, \frac{1}{n}\}$, so $\sum \min\Big\{d_n, \frac{1}{n}\Big\} < \infty$ by the comparison test. $\square$

One may wonder whether it is possible to satisfy $\sum \min\{\frac{1}{n}, d_n\} < \infty$ jointly with $\sum d_n = \infty$ for strictly positive d_n; it is.

Consider the following example: Let

$$d_n = \begin{cases} \frac{1}{2^n} & \text{if n is not a perfect square,} \\ 1 & \text{if n is a perfect square.} \end{cases}$$

Then $\sum d_n = \infty$, since $d_n = 1$ for infinitely many n. Moreover,

$$\begin{aligned} &\sum \min\{\frac{1}{n}, d_n\} \\ \leq &\sum_{k=1}^{\infty} \frac{1}{k^2} + \sum_{n=1}^{\infty} \frac{1}{2^n} \\ = &\frac{\pi^2}{6} + 1. \end{aligned}$$

One may notice, however, that the previous example is not monotone, whereas $\{a_n\}$ and $\{b_n\}$ from Exercise 23 are monotone. Therefore, it is natural to ask the following question:

Main Question:
Is there a sequence $\{d_n\}$, with $\sum d_n = \infty$, $d_n > 0$, and $d_1 \geq d_2 \geq d_3 \geq ...$ such that $\sum \min\{\frac{1}{n}, d_n\} < \infty$?

In order to address the Main Question, it will be helpful to introduce some notions of "largeness" for sets of natural numbers.

Definition:
Let $A \subseteq \mathbb{N} = \{1, 2, 3, 4, \cdots\}$. The upper density of A is defined by:

$$\overline{d}(A) = \limsup_{N\to\infty} \frac{|A \cap \{1, \cdots, N\}|}{N}.$$

The lower density of A is defined by:

$$\underline{d}(A) = \liminf_{N\to\infty} \frac{|A \cap \{1, \cdots, N\}|}{N}.$$

If $\overline{d}(A) = \underline{d}(A)$, then we can denote this value as $d(A)$ and call it the density of A.

The following proposition (see for example McCutcheon 1999) establishes basic properties of densities that will be used later.

Proposition. Let $E, F \subset \mathbb{Z}$. Then:

1. $\underline{d}(E) \leq \overline{d}(E)$.
2. $\overline{d}(E \cup F) \leq \overline{d}(E) + \overline{d}(F)$.

3. $\overline{d}(E) = 1 - \underline{d}(E^c)$.
4. If $d(E)$, $d(F)$ exist, $E \cap F = \emptyset$. Then $d(E \cup F) = d(E) + d(F)$.
5. $\exists E \subset \mathbb{N}$ with $\overline{d}(E) = 1$, $\underline{d}(E) = 0$.

Proof of Proposition.

1. $\underline{d}(E) \leq \overline{d}(E)$.

   This is trivial given the definitions of supremum and infimum.

2. $\overline{d}(E \cup F) \leq \overline{d}(E) + \overline{d}(F)$.

$$\begin{aligned}
\overline{d}(E \cup F) =& \limsup_{N\to\infty} \frac{|(E \cup F) \cap \{1, ..., N\}|}{N} \\
\leq& \limsup_{N\to\infty} \frac{|E \cap \{1, ..., N\}| + |F \cap \{1, ..., N\}|}{N} \\
\leq& \limsup_{N\to\infty} \frac{|E \cap \{1, ..., N\}|}{N} + \limsup_{N\to\infty} \frac{|F \cap \{1, ..., N\}|}{N} \\
=& \overline{d}(E) + \overline{d}(F),
\end{aligned}$$

   since $(E \cup F) \cap \{1, ..., N\} = (E \cap \{1, ..., N\}) \cup (F \cap \{1, ..., N\})$.

3. $\overline{d}(E) = 1 - \underline{d}(E^c)$.

$$\begin{aligned}
\overline{d}(E) =& \limsup_{N\to\infty} \frac{|E \cap \{1, ..., N\}|}{N} \\
=& \limsup_{N\to\infty} \frac{N - |E^c \cap \{1, ..., N\}|}{N} \\
=& 1 + \limsup_{N\to\infty} \frac{-|E^c \cap \{1, ..., N\}|}{N} \\
=& 1 - \liminf_{N\to\infty} \frac{|E^c \cap \{1, ..., N\}|}{N} \\
=& 1 - \underline{d}(E^c),
\end{aligned}$$

   since $\limsup_N (-x_n) = -\liminf x_n$ for any real-valued sequence (x_n).

4. If $d(E), d(F)$ exist and $E \cap F = \emptyset$, then $d(E \cup F) = d(E) + d(F)$.

$$
\begin{aligned}
d(E \cup F) &= \lim_{N\to\infty} \frac{|(E\cup F)\cap\{1,\ldots,N\}|}{N} \\
&= \lim_{N\to\infty} \Big[\frac{|E\cap\{1,\ldots,N\}|}{N} + \frac{|F\cap\{1,\ldots,N\}|}{N}\Big] \\
&= \lim_{N\to\infty} \frac{|E\cap\{1,\ldots,N\}|}{N} + \lim_{N\to\infty} \frac{|F\cap\{1,\ldots,N\}|}{N} \\
&= d(E) + d(F).
\end{aligned}
$$

5. $\exists E \subset \mathbb{N}$ with $\overline{d}(E) = 1$, $\underline{d}(E) = 0$.

   Define E as follows:

$$
1 \in E^c \text{ and } \begin{cases} k \in E \text{ if } 2^{2^n} \le k < 2^{2^{n+1}} \text{ for some n even,} \\ k \in E^c \text{ if } 2^{2^n} \le k < 2^{2^{n+1}} \text{ for some n odd.} \end{cases}
$$

   Then

$$
\begin{aligned}
\overline{d}(E) &= \limsup_{N\to\infty} \frac{|E\cap\{1,\ldots,N\}|}{N} \\
&\ge \limsup_{n\to\infty,\ \text{n is even}} \frac{|E\cap\{1,2,\ldots,2^{2^{n+1}}\}|}{2^{2^{n+1}}} \\
&\ge \limsup_{n\to\infty,\ \text{n is even}} \frac{2^{2^{n+1}} - 2^{2^n}}{2^{2^{n+1}}} = 1
\end{aligned}
$$

   and

$$
\begin{aligned}
\underline{d}(E) &= \liminf_{N\to\infty} \frac{|E\cap\{1,\ldots,N\}|}{N} \\
&\le \liminf_{n\to\infty,\ \text{n is odd}} \frac{|E\cap\{1,2,\ldots,2^{2^{n+1}}\}|}{2^{2^{n+1}}} \\
&\le \liminf_{n\to\infty,\ \text{n is odd}} \frac{2^{2^n}}{2^{2^{n+1}}} = 0.
\end{aligned}
$$

□

Theorem 1. Let $\{d_n\}$ be a non-increasing sequence of positive reals satisfying $\sum d_n = \infty$. Then the following are equivalent:

1. There exists a non-increasing sequence of positive reals $\{c_n\}$ with $\sum c_n = \infty$ and $\sum \min\{d_n, c_n\} < \infty$.
2. There is a set $E \subset \mathbb{N}$ with $\overline{d}(E) = 1$ such that $\sum_{n\in E} d_n < \infty$.

Proof. *(1)* $\implies$ *(2).* Assume that there is a non-increasing sequence $\{c_n\}$ such that

$$\sum c_n = \infty$$

and

$$\sum_{n=1}^{\infty} \min\{d_n, c_n\} < \infty.$$

Let

$$E = \{n : \min\{d_n, c_n\} = d_n\}.$$

Then clearly

$$\sum_{n \in E} d_n = \sum_{n \in E} \min\{d_n, c_n\} < \infty.$$

We need to show that $\overline{d}(E) = 1$. Equivalently, we need to show that $\underline{d}(E^c) = 0$. We know that

$$\sum_{n \in E^c} c_n = \sum_{n \in E^c} \min\{d_n, c_n\} < \infty.$$

Suppose that

$$\underline{d}(E^c) > 0.$$

We will obtain a contradiction: Choose $k \in \mathbb{N}$ such that

$$\underline{d}(E^c) > \frac{1}{k}.$$

Write

$$E = \{a_1, a_2, \ldots\} \text{ with } a_1 < a_2 < a_3 < \cdots$$

and

$$E^c = \{b_1, b_2, \ldots\} \text{ with } b_1 < b_2 < b_3 < \cdots .$$

Claim 2. The set $S = \{n : b_n > a_{kn}\}$ is finite.

Proof of Claim 2: Suppose S is infinite. Then for any $n \in S$,

$$\frac{|E^c \cap \{1, 2, \ldots, b_n\}|}{b_n} \leq \frac{1}{k+1}.$$

This is because

$$\{b_1, \ldots, b_n\} \,\dot\cup\, \{1, \ldots, a_{kn}\} \subset \{1, 2, \ldots, b_n\},$$

which implies that $b_n \geq kn + n = (k+1)n$, whereas

$$|E^c \cap \{1, 2, \ldots, b_n\}| = n.$$

So, since S is infinite,

$$\underline{d}(E^c) \leq \frac{1}{k+1},$$

which is a contradiction. This proves Claim 2.

□

By Claim 2, then, there is some $M \in \mathbb{N}$ such that for all $n \geq M$, $b_n < a_{kn}$. This implies that

$$c_{b_n} \geq c_{a_{kn}} \geq c_{a_{kn+1}} \geq \cdots \geq c_{a_{kn+k-1}},$$

which in turn implies that

$$c_{b_n} \geq \frac{1}{k}(c_{a_{kn}} + c_{a_{kn+1}} + \cdots + c_{a_{kn+k-1}}).$$

Therefore,

$$\sum_{n=M}^{\infty} c_{b_n} \geq \frac{1}{k} \sum_{n=M}^{\infty} [c_{a_{kn}} + c_{a_{kn+1}} + \cdots + c_{a_{kn+k-1}}] \geq \frac{1}{k} \sum_{i=kM}^{\infty} c_{a_i} = \infty.$$

$\Rightarrow\Leftarrow$

(2) $\implies$ *(1)*. Assume that there is a set $E \subset \mathbb{N}$ with $\overline{d}(E) = 1$ such that

$$\sum_{n \in E} d_n < \infty.$$

We will construct a sequence

$$c_1 \geq c_2 \geq c_3 \geq \cdots$$

such that

$$\sum_{n \in E} c_n = \infty$$

and

$$\sum_{n \in E^c} c_n < \infty,$$

thus satisfying *(1)*.

Put $N_0 = 0$ Since $\overline{d}(E) = 1$, one can find

$$N_1 < N_2 < N_3 < \cdots$$

such that $N_{k+1} > 2N_k, k = 1, 2, \ldots,$

$$|E \cap \{1, 2, \ldots, N_1\}| > (1 - \frac{1}{2})N_1,$$

$$|E \cap \{N_1 + 1, N_1 + 2, \ldots, N_2\}| > (1 - \frac{1}{3})(N_2 - N_1),$$

and, more generally,

$$|E \cap \{N_k + 1, N_k + 2, \ldots, N_{k+1}\}| > (1 - \frac{1}{k+2})(N_{k+1} - N_k)$$

for $k > 1$. For $k = 0, 1, \ldots$, let

$$c_n = \frac{1}{(k+1)(N_{k+1} - N_k)} \text{ for } N_k < n \leq N_{k+1}.$$

Then

$$\sum_{n=1}^{\infty} c_n = \sum_{k=0}^{\infty} \sum_{n=N_k+1}^{N_{k+1}} c_n$$
$$\geq \sum_{k=0}^{\infty} \frac{1}{(k+1)(N_{k+1}-N_k)}(N_{k+1}-N_k) = \sum_{k=0}^{\infty} \frac{1}{k+1} = \infty.$$

Meanwhile,

$$\sum_{n\in E^c} c_n = \sum_{k=0}^{\infty} \sum_{N_k<n\leq N_{k+1}, n\in E^c} c_n$$
$$= \sum_{k=0}^{\infty} \frac{1}{(k+1)(N_{k+1}-N_k)} |E^c \cap \{N_k+1, \ldots, N_{k+1}\}|$$
$$\leq \sum_{k=0}^{\infty} \frac{1}{(k+1)(N_{k+1}-N_k)} \cdot \frac{(N_{k+1}-N_k)}{(k+2)} = \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} = 1 < \infty.$$

□

Theorem 2. If $E \subset \mathbb{N}$ and $\sum_{n\in E} \frac{1}{n} < \infty$, then $\overline{d}(E) = 0$.

Proof. Suppose that $\overline{d}(E) > 0$. We will obtain a contradiction.

Choose $k \in \mathbb{N}$ such that $\overline{d}(E) > \frac{1}{k}$. Choose next $N > 0$ such that

$$\sum_{n>N, n\in E} \frac{1}{n} < -\log(1 - \frac{1}{k}).$$

Now choose M to be a multiple of k large enough that

$$\frac{|E \cap \{N+1, \ldots, M\}|}{M} > \frac{1}{k}.$$

Then

$$\sum_{n>N, n\in E} \frac{1}{n} \geq \sum_{n=M(1-\frac{1}{k})}^{M} \frac{1}{n} > \int_{M(1-\frac{1}{k})}^{M} \frac{1}{x} dx$$
$$= \log M - \log M(1 - \frac{1}{k}) = -\log(1 - \frac{1}{k})$$

$\Rightarrow\Leftarrow$

□

We can now answer our Main Question, in the negative. That is, we can show that there is no sequence $\{d_n\}$, with

$$\sum d_n = \infty,$$

$$d_n > 0,$$

and

$$d_1 \geq d_2 \geq d_3 \geq \cdots$$

such that $\sum \min\{\frac{1}{n}, d_n\} < \infty$. For if there were, then Theorem 1 would imply the existence of $E \subset \mathbb{N}$ with $\overline{d}(E) = 1$ such that

$$\sum_{n \in E} \frac{1}{n} < \infty.$$

But this violates Theorem 2. Hence, there exists no such sequence.

We also remark, in conclusion, that we can now say something substantive about Exercise 47b as well. (See above.) Namely, our results establish that if $d_n > 0$ for all n, $\sum d_n = \infty$ *and* (d_n) is non-increasing, then $\sum_{n=1}^{\infty} \frac{d_n}{1+nd_n} = \infty$. (This follows immediately from Claim 1 and the negative result we obtained for our Main Question.)

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