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## Problems in Extremal Graph Theory

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# PROBLEMS IN EXTREMAL GRAPH THEORY 

by

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A Dissertation<br>Submitted in Partial Fulfillment of the<br>Requirements for the Degree of<br>Doctor of Philosophy

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## Dziękuje


#### Abstract

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This dissertation consists of 6 chapters concerning a variety of topics in extremal graph theory.

Chapter 1 is dedicated to the results in the papers with António Girão, Gábor Mészáros, and Richard Snyder [47, 44]. We say that a graph is path-pairable if for any pairing of its vertices there exist edge disjoint paths joining the vertices in each pair. We study the extremal behavior of maximum degree and diameter in some classes of path-pairable graphs. In particular we show that a path-pairable planar graph must have a vertex of linear degree.

In Chapter 2 we present a joint work with António Girão and Teeradej Kittipassorn [46]. Given graphs $G$ and $H$, we say that a graph $F$ is $H$-saturated in $G$ if $F$ is $H$-free subgraph of $G$, but addition of any edge from $E(G)$ to $F$ creates a copy of $H$. Here we deal with the case when $G$ is a complete $k$-partite graph with $n$ vertices in each class, and $H$ is a complete graph on $r$ vertices. We prove bounds, which are tight for infinitely many values of $k$ and $r$, on the minimal number of edges in a $H$-saturated graph in $G$, for this choice of $G$ and $H$, answering questions of Ferrara, Jacobson, Pfender, and Wenger; and generalizing a result of Roberts.

Chapter 3 is about a joint paper with António Girão and Teeradej Kittipassorn [43]. A coloring of the vertices of a digraph $D$ is called majority coloring if no vertex of $D$ receives the same color as more than half of its outneighbours. This was introduced by van der Zypen in 2016. Recently, Kreutzer, Oum, Seymour, van der Zypen, and Wood posed a number of problems related to this notion: here we solve several of them.

In Chapter 4 we present a joint work with António Girão [45]. We show that given any integer $k$ there exist functions $g_{1}(k), g_{2}(k)$ such that the following holds. For any graph $G$ with maximum degree $\Delta$ one can remove fewer than $g_{1}(k) \sqrt{\Delta}$ vertices from $G$ so that the remaining graph $H$ has $k$ vertices of the same degree at least $\Delta(H)-g_{2}(k)$. It is an


approximate version of conjecture of Caro and Yuster; and Caro, Lauri, and Zarb, who conjectured that $g_{2}(k)=0$.

Chapter 5 concerns results obtained together with Kazuhiro Nomoto, Julian Sahasrabudhe, and Richard Snyder. We study a graph parameter, the graph burning number, which is supposed to measure the speed of the spread of contagion in a graph. We prove tight bounds on the graph burning number of some classes of graphs and make a progress towards a conjecture of Bonato, Janssen, and Roshanbin about the upper bound of graph burning number of connected graphs.

In Chapter 6 we present a joint work with Teeradej Kittipassorn. We study the set of possible numbers of triangles a graph on a given number of vertices can have. Among other results, we determine the asymptotic behavior of the smallest positive integer $m$ such that there is no graph on $n$ vertices with exactly $m$ copies of a triangle. We also prove similar results when we replace triangle by any fixed connected graph.

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## CHAPTER 1

## ON SOME PARAMETERS OF PATH-PAIRABLE GRAPHS

A graph is path-pairable if for any pairing of its vertices there exist edge-disjoint paths joining the vertices in each pair. In this chapter we study the extremal behavior of two graph parameters, maximum degree and diameter, in some classes of path-pairable graphs. We show that any $n$-vertex path-pairable planar graph must contain a vertex of degree linear in $n$. We also obtain sharp bounds on the maximum possible diameter of path-pairable graphs which either have a given number of edges, or are $c$-degenerate. This work is joint with António Girão, Gábor Mészáros, and Richard Snyder.

### 1.1 Introduction

Path-pairability is a graph theoretical notion that emerged from a practical networking problem. This notion was introduced by Csaba, Faudree, Gyárfás, Lehel, and Schelp [26], and further studied by Faudree, Gyárfás, and Lehel [34, 40, 36], and by Kubicka, Kubicki and Lehel [60]. Given a fixed integer $k$ and a simple undirected graph $G$ on at least $2 k$ vertices, we say that $G$ is $k$-path-pairable if, for any pair of disjoint sets of distinct vertices $\left\{x_{1}, \ldots, x_{k}\right\}$ and $\left\{y_{1}, \ldots, y_{k}\right\}$ of $G$, there exist $k$ edge-disjoint paths $P_{1}, P_{2}, \ldots, P_{k}$, such that $P_{i}$ is a path from $x_{i}$ to $y_{i}, 1 \leq i \leq k$. The problems of finding $k$ edge(vertex)-disjoint paths routing some prescribed pairs of vertices in a graph is a well-known problem in algorithmic graph theory and combinatorial optimization (see the surveys [40, 41, 77]). Recently, for a fixed integer $k$, Kawarabayashi, Kobayashi and Reed [55] constructed a $O\left(n^{2}\right)$ time algorithm which for any graph $G$ on $n$ vertices either finds such $k$ vertex-disjoint paths or concludes that no such paths exist. As a corollary they obtained a $O\left(n^{2}\right)$ time algorithm for the edge-disjoint paths problem. This improved upon the seminal work of Robertson and Seymour [73], which initially gave a $O\left(n^{3}\right)$ time algorithm for the vertex-disjoint path problem. Note that the problem of finding
edge(vertex)-disjoint paths between an unbounded number of prescribed pairs of vertices is known to be NP-complete, even when restricted to planar graphs [67].

The concept of $k$-path-pairability is closely related to the well-studied notions of $k$-linkedness and $k$-weak-linkedness. A graph is said to be $k$-(weakly)linked if for any choice $\left\{s_{1}, \ldots, s_{k}, t_{1}, \ldots, t_{k}\right\}$ of $2 k$ vertices (not necessarily distinct) there are vertex(edge) internally disjoint paths $P_{1}, \ldots, P_{k}$ with $P_{i}$ joining $s_{i}$ to $t_{i}, 1 \leq i \leq k$. While any $k$-(weakly)linked graph is $(2 k-1)$-vertex connected ( $k$-edge connected), the same need not hold for $k$-path-pairable graphs. Observe that the $\operatorname{stars} S_{2 k}(k \geq 1)$ are $k$-path-pairable and yet have very low edge density and edge connectivity. On the other hand, a result of Bollobás and Thomason [15] shows that if $G$ is a $2 k$-connected graph with average degree at least $22 k$ then $G$ is $k$-linked. This was later improved by Thomas and Wollan [76] who showed that a $2 k$-connected graph with average degree at least $10 k$ is necessarily $k$-linked. In the context of weakly-linked graphs, a theorem of Hirata, Kubota and Saito [51] states that a $(2 k+1)$-edge connected graph is $(k+2)$-weakly-linked for $k \geq 2$. A few years later, Huck [52] showed that any $(k+2)$-edge-connected graph is $k$-weakly-linked.

A $k$-path-pairable graph on $2 k$ or $2 k+1$ vertices is simply said to be path-pairable.
It is fairly easy to construct path-pairable graphs on $n$ vertices ( $n$ even) with maximum degree linear in $n$ and/or small (constant) diameter. For example, complete graphs $K_{2 n}$ and complete bipartite graphs $K_{m, n}$ are path-pairable for all choices of $m, n \in \mathbb{N}$ with $m+n$ even, $m \neq 2, n \neq 2$.

It is slightly more challenging to construct an infinite family of path-pairable graphs where the maximum degree grows sublinearly or the diameter grows with the number of vertices. We shall now describe such a family. Let $K_{t}$ be the complete graph on $t$ vertices and let $K_{t}^{q}$ be constructed from $K_{t}$ by attaching $q-1$ leaves to each of the original vertices of $K_{t}$. This family was introduced by Csaba, Faudree, Gyárfás, Lehel, and Schelp [26], who also proved that $K_{t}^{q}$ is path-pairable as long as $t \cdot q$ is even and $q \leq\left\lfloor\frac{t}{3+2 \sqrt{2}}\right\rfloor$. The bound on $q$ has been recently improved to $\approx \frac{1}{3} t[49]$. Observe that $n=\left|V\left(K_{t}^{q}\right)\right|=t \cdot q$ and
$\Delta\left(K_{t}^{q}\right)=t+q-2=O(\sqrt{n})$ when $q=\Omega(t)$. Additional path-pairable constructions with maximum degree $c \sqrt{n}$ can be found in [60] and [65]. A construction of path-pairable graphs with unbounded diameter, which is based on a blow-up of a path, is presented in a later section.

The following result due to Faudree [35] shows that the maximum degree of a path-pairable graph has to grow with the order of the graph.

Theorem 1.1. If $G$ is path-pairable on $n$ vertices with maximum degree $\Delta$, then $n \leq 2 \Delta^{\Delta}$.

Letting $\Delta_{\min }(n):=\min \{\Delta(G): G$ is a path-pairable graph on $n$ vertices $\}$, this result is equivalent to

$$
\Delta_{\min }(n) \geq c_{1} \frac{\log n}{\log \log n}
$$

for some constant $c_{1}$. To date, the best known upper bound on $\Delta_{\min }(n)$ is due to Győri, Mezei, and Mészáros, exhibiting a path-pairable graph with maximum degree $\Delta \approx 5.5 \cdot \log n$ [48]. In summary, we have the following general asymptotic bounds on $\Delta_{\text {min }}(n):$

$$
c_{1} \frac{\log n}{\log \log n} \leq \Delta_{\min }(n) \leq c_{2} \log n
$$

The maximum diameter of arbitrary path-pairable graphs was investigated by Mészáros [65] who proved that $d(n) \leq 6 \sqrt{2} \sqrt{n}$.

Recall that the star $K_{1, n-1}$ is path-pairable. This is simply due to the presence of a vertex of large degree. Are there properties we may impose on a general path-pairable graph to force a vertex of large degree, say, linear in $n$ ? Along these lines, Faudree, Gyárfás and Lehel [36] proved that an $n$-vertex path-pairable graph with maximum degree at most $n-2$ must have at least $3 n / 2-\log n-c$ edges, for some absolute constant $c$. Instead of simply imposing a condition on the number of edges, we wished to determine whether or not a structural property like planarity would be enough to force a vertex of linear degree in a path-pairable graph. To formulate this precisely, let us define $\Delta_{\min }^{p}(n)$ to
be

$$
\min \{\Delta(G): G \text { is a path-pairable planar graph on } n \text { vertices }\} .
$$

Our problem, then, is to determine whether or not $\Delta_{\text {min }}^{p}(n)=\Theta(n)$. We first note that a simple application of the Planar Separator Theorem of Lipton and Tarjan [63] shows that every path-pairable planar graph on $n$ vertices must contain a vertex of degree at least $c \sqrt{n}$. Indeed, if $G$ is such a graph, then the Separator Theorem allows us to partition $V(G)$ into three sets $S, A, B$, where $|S|=O(\sqrt{n}),|A| \leq|B| \leq 2 n / 3$, and there are no edges between $A$ and $B$. Now, while path-pairable graphs $G$ need not be highly connected or edge connected, they must satisfy certain connectivity-like conditions. More precisely, they must satisfy the cut-condition: for every subset $X \subset V(G)$ of size at most $n / 2$, there are at least $|X|$ edges between $X$ and $V(G) \backslash X$. Note that the cut-condition is not sufficient to guarantee path-pairability; see [66] for additional details. Accordingly, since $n / 4<|A|<n / 2$ and there are no edges between $A$ and $B$, the cut-condition implies that there are at least $|A|$ edges between $A$ and $S$. We therefore obtain a vertex in $S$ of degree $\Omega(\sqrt{n})$.

Our main theorem, which we state below, shows that we can do much better than this. Namely, every path-pairable planar graph must have a vertex of linear degree.

Theorem 1.2. There exists $c \geq 10^{-10^{10}}$ such that if $G$ is a path-pairable planar graph on $n$ vertices then $\Delta(G) \geq c n$.

We have not made an attempt to optimize the constant $c$ obtained in the proof. The value we give is surely far from the truth.

In the other direction, there are easy examples of path-pairable planar graphs with very large maximum degree; for example, consider the star $K_{1, n-1}$. Our second result finds an infinite family of path-pairable planar graphs with smaller (but of course still linear) degree.

Theorem 1.3. There exist path-pairable planar graphs $G$ on $n$ vertices with $\Delta(G)=\frac{2}{3} n$.

Combining Theorems 1.2 and 1.3 , we have that

$$
10^{-10^{10}} n \leq \Delta_{\min }^{p}(n) \leq \frac{2}{3} n .
$$

For the diameter problem, for a family of graphs $\mathscr{G}$ let us define $d(n, \mathscr{G})$ as follows:

$$
d(n, \mathscr{G})=\max \{d(G): G \in \mathscr{G} \text { and } G \text { is path-pairable on } n \text { vertices }\} .
$$

When $\mathscr{G}$ is the family of path-pairable graphs, we shall simply write $d(n)$ instead of $d(n, \mathscr{G})$.

The maximum diameter of arbitrary path-pairable graphs was investigated by Mészáros [65] who proved that $d(n) \leq 6 \sqrt{2} \sqrt{n}$. Our next aim in this chapter is to investigate the maximum diameter of path-pairable graphs when we impose restrictions on the number of edges and on how the edges are distributed. To state our results, let us denote by $\mathscr{G}_{m}$ the family of graphs with at most $m$ edges. The following result determines $d\left(n, \mathscr{G}_{m}\right)$ for a certain range of $m$.

Theorem 1.4. If $2 n \leq m \leq \frac{1}{4} n^{3 / 2}$ then

$$
\sqrt[3]{\frac{1}{2} m-n} \leq d\left(n, \mathscr{G}_{m}\right) \leq \sqrt[3]{300 m}
$$

We remark that we actually prove a slightly more general bound $d\left(n, \mathscr{G}_{m}\right) \leq \max \left\{\frac{6 m}{n}, \sqrt[3]{300 m}\right\}$ which holds for $m$ in any range but, when $m \geq \sqrt{2} n^{3 / 2}$, the bound obtained by Mészáros [65] is sharper. Determining the behavior of the maximum diameter among path-pairable graphs on $n$ vertices with fewer than $2 n$ edges remains an open problem. In particular, we do not know if the maximum diameter in this range must be bounded (see Section 1.3.4).

Following this line of research, it is very natural to consider the problem of determining the maximum attainable diameter for other classes of graphs. For example,
what is the behavior of the maximum diameter of path-pairable planar graphs? Although we could not give a satisfactory answer to this particular question, we were able to do so for graphs which are $c$-degenerate. As usual, we say that an $n$-vertex graph $G$ is $c$-degenerate if there exists an ordering $v_{1}, \ldots, v_{n}$ of its vertices such that $\left|\left\{v_{j}: j>i, v_{i} v_{j} \in E(G)\right\}\right| \leq c$ holds for all $i=1,2, \ldots, n$. We let $\mathscr{G}_{c \text {-deg }}$ denote the family of $c$-degenerate graphs. Clearly all $c$-degenerate graphs have a linear number of edges, so Theorem 1.4 implies that $d\left(n, \mathscr{G}_{c-\operatorname{deg}}\right)=O(\sqrt[3]{n})$. However, as the next result shows, this bound is far from the truth.

Theorem 1.5. Let $c \geq 5$ be an integer. Then

$$
(4+o(1)) \frac{\log (n)}{\log \left(\frac{c}{c-2}\right)} \leq d\left(n, \mathscr{G}_{c-\operatorname{deg}}\right) \leq(12+o(1)) \frac{\log (n)}{\log \left(\frac{c}{c-2}\right)}
$$

as $n \rightarrow \infty$.

We remark that we have not made an effort to optimize the constants appearing in the upper and lower bounds of Theorems 1.4 and 1.5 .

### 1.2 Maximum degree of path-pairable planar graphs

### 1.2.1 The Construction

Our aim in this section is to prove Theorem 1.3 , which we restate here for convenience.
Theorem 1.3. There exist path-pairable planar graphs $G$ on $n$ vertices with $\Delta(G)=\frac{2}{3} n$.

Proof. Let $G$ be a graph on $n=6 k$ vertices with vertex set
$V(G)=A \cup B \cup C \cup\left\{x_{A B}, x_{B C}, x_{C A}\right\}$ where $|A|=|B|=|C|=2 k-1$, and $x_{A B}, x_{B C}, x_{C A}$ denote three additional vertices forming a triangle such that $x_{A B}, x_{B C}, x_{C A}$ are joined to every vertex in $A \cup B, B \cup C$, and $C \cup A$, respectively, and $A, B, C$ are independent sets. This graph is clearly planar. Let $\mathscr{P}$ be a pairing of the vertices and let $\{u, v\} \in \mathscr{P}$. We describe how to join $u$ and $v$ by a path in all possible cases.

1. If there is an edge between $u$ and $v$, join them by this edge.
2. If $u \in\left\{x_{A B}, x_{B C}, x_{C A}\right\}$ and $v \in A \cup B \cup C$ such that there is no edge between them, join them by the path $u w v$ where the edge $u w$ is consistent with the cyclic ordering $x_{A B}, x_{B C}, x_{C A}$. For example, if $u=x_{A B}$ and $v \in C$, we join $u$ and $v$ by the path $u x_{B C} v$. The remaining cases can be dealt using the same pattern.
3. If $u, v \in A \cup B \cup C$ and they are in the same class, join them by the path $u w v$ where $w$ is an arbitrary common neighbor (out of the two available).
4. If $u, v \in A \cup B \cup C$ and they are in different classes, join them by the path $u w v$ where $w$ is the unique common neighbor.

It is straightforward to check that the above instructions find edge-disjoint paths joining terminals, regardless of the choice of $\mathscr{P}$.

### 1.2.2 The Proof of Theorem 1.2

The aim of this section is to prove our main theorem, Theorem 1.2. Our proof is based on three preparatory lemmas. First, we shall introduce some terminology. Let $G$ be a multigraph. We say that two multiedges $e, f$ of $G$ are at distance $d$ if the shortest path in $G$ joining an endpoint of $e$ and an endpoint of $f$ has length $d$. If two multiedges are at distance 0 , we shall simply say they are incident. Further, we shall refer to a matching of size $k$ as a $k$-matching. We say that a $k$-matching is good if every pair of edges in the matching is at distance exactly 1 . Notice that contracting all the edges of a good $k$-matching results in the complete graph $K_{k}$ (with potential multiple edges and loops).

Our first lemma says that in any multigraph either some multiedges 'cluster' together or many pairs of multiedges are far apart, or one can find a good $k$-matching. We shall need the following inequality.

Fact 1.6. If $k \geq 2$ then $2^{-k}\left(\frac{1+2^{-k-1}}{\left(1-2^{-k}\right)^{2}}\right) \leq 2^{-k+1}$.

The above inequality is easily seen to be equivalent to $\left(2^{-k+2}-1\right)\left(2^{-k-1}-1\right) \geq 0$.

Lemma 1.7. Let $k$ be a natural number and $\varepsilon_{1}, \varepsilon_{2}$ be positive reals such that $\varepsilon_{1}+\varepsilon_{2} \leq 2^{-k}$. Then, for sufficiently large $M=M(k)$, if $G$ is a multigraph on $M$ multiedges, then at least one of the following conditions is satisfied.

1. There is a multiedge in $G$ which is incident with at least $\varepsilon_{1} M$ multiedges;
2. There are at least $\varepsilon_{2}\binom{M}{2}$ pairs of multiedges which are at distance greater than 1 ;
3. $G$ contains a good $k$-matching.

Proof. We shall use induction on $k$. The base case when $k=1$ is trivial - Condition 3 is always satisfied. Assume then that $k \geq 2$ and the lemma is true for $k-1$.

Suppose every multiedge is incident with at most $\varepsilon_{1} M$ multiedges and at most $\varepsilon_{2}\binom{M}{2}$ pairs of multiedges are at distance greater than 1 . We shall show that $G$ contains a good $k$-matching. By an averaging argument there is a multiedge $e$ which is at distance at most 1 from at least $\left(1-\varepsilon_{2}\right) M-1$ multiedges. Let $E^{\prime}$ be the set of those multiedges which are at distance exactly 1 from $e$. It follows from our assumptions that $M^{\prime}=\left|E^{\prime}\right| \geq\left(1-\varepsilon_{1}-\varepsilon_{2}\right) M-1 \geq\left(1-2^{-k}\right) M-1$. Let $G^{\prime}$ be the multigraph spanned by $E^{\prime}$. By assumption, at most $\varepsilon_{2}\binom{M}{2}$ of the multiedges in $G^{\prime}$ are at distance greater than 1. Therefore, since $M \leq \frac{M^{\prime}+1}{1-2^{-k}}$, for large enough $M$ (and hence large enough $M^{\prime}$ ) we have that at most

$$
\begin{aligned}
\varepsilon_{2}\binom{M}{2} \leq \varepsilon_{2}\binom{\frac{M^{\prime}+1}{1-2^{-k}}}{2} & =\frac{\varepsilon_{2}}{\left(1-2^{-k}\right)^{2}}\left(1+\frac{1}{M^{\prime}}\right)\left(1+\frac{1+2^{-k}}{M^{\prime}-1}\right)\binom{M^{\prime}}{2} \\
& \leq \frac{\varepsilon_{2}\left(1+2^{-k-1}\right)}{\left(1-2^{-k}\right)^{2}}\binom{M^{\prime}}{2}
\end{aligned}
$$

pairs of multiedges in $G^{\prime}$ are at distance greater than 1. Also, for $M^{\prime}$ large enough, each multiedge in $G^{\prime}$ is incident with at most $\varepsilon_{1} M \leq \varepsilon_{1} \frac{M^{\prime}+1}{1-2^{-k}}=\frac{\varepsilon_{1}}{1-2^{-k}}\left(1+\frac{1}{M^{\prime}}\right) M^{\prime} \leq \frac{\varepsilon_{1}\left(1+2^{-k-1}\right)}{1-2^{-k}} M^{\prime}$ multiedges. Note that for $k \geq 2$ one
has

$$
\begin{aligned}
\varepsilon_{1} \frac{1+2^{-k-1}}{1-2^{-k}}+\varepsilon_{2} \frac{1+2^{-k-1}}{\left(1-2^{-k}\right)^{2}} & \leq \varepsilon_{1} \frac{1+2^{-k-1}}{\left(1-2^{-k}\right)^{2}}+\varepsilon_{2} \frac{1+2^{-k-1}}{\left(1-2^{-k}\right)^{2}} \\
& \leq 2^{-k} \frac{1+2^{-k-1}}{\left(1-2^{-k}\right)^{2}} \leq 2^{-(k-1)},
\end{aligned}
$$

where the last inequality is precisely Fact 1.6 . Therefore, by the induction hypothesis, $G^{\prime}$ contains a good $(k-1)$-matching. But since $e$ is at distance 1 from any multiedge in $G^{\prime}$, we also have a good $k$-matching in $G$.

Since we shall be operating with planar graphs, we single out the following corollary.
Corollary 1.8. Let $M$ be a sufficiently large integer and let $\varepsilon_{1}, \varepsilon_{2}$ be positive reals such that $\varepsilon_{1}+\varepsilon_{2} \leq \frac{1}{32}$. If $G$ is a planar multigraph with $M$ multiedges then either $G$ has a multiedge which is incident with at least $\varepsilon_{1} M$ multiedges or there are at least $\varepsilon_{2}\binom{M}{2}$ pairs of multiedges at distance greater than 1 .

Proof. If $G$ contained a good 5-matching then it would contain a $K_{5}$ minor.

One strategy in the proof of our main theorem is to consider a suitable bipartition of our path-pairable planar graph, and to exploit the fact that any bipartite planar graph on $n$ vertices has at most $2 n-4$ edges. To exploit this last property we shall need ways of finding pairings of the vertices such that their corresponding edge-disjoint paths contribute 'many' edges to the bipartition. This is formalized in the following lemma.

Lemma 1.9. Let $D$ be a positive integer and $0<\varepsilon \leq 1 / 2$. Then there exists $c>0$ such that the following is true. Suppose $G$ is a path-pairable planar graph on $n>1 / c$ vertices with $\Delta=\Delta(G) \leq c n$. Let $A, U \subset V(G)$ be given with $U \subset A$ such that every vertex in $A$ has degree at most $D,|A| \geq(1-\varepsilon) n$ and $|U| \geq \varepsilon n$. Let $B=V(G) \backslash A$. Then there is a pairing of the vertices in $U$ which contributes to at least $2|U|-16 \varepsilon n$ edges between $A$ and $B$. Proof. We say that a path in $G$ is weak if it begins and ends in $A$, uses no edges inside $B$, and uses at most 2 edges between $A$ and $B$. Now, let $C:=\left\lceil 4 \varepsilon^{-1}\right\rceil$ and note that since
$\varepsilon \leq 1 / 2$ we have that $\frac{3}{C-2} \leq \varepsilon$. For every $x \in U$, let
$U_{x}=\{u \in U: \exists$ weak $x-u$ path in $G$ of length at most $C\}$. We claim that $U_{x}$ is small for every $x \in U$; namely, it is easy to see that

$$
\left|U_{x}\right| \leq D^{C}+D^{C} D \Delta D^{C}=D^{C}\left(1+D^{C+1} \Delta\right)
$$

Choose $c=c(D, \varepsilon)=\frac{\varepsilon}{4 D^{2 C+1}}$ so that $\Delta \leq c n$. Then

$$
\left|U_{x}\right| \leq D^{C}\left(1+D^{C+1} \Delta\right) \leq\left(D^{C}+D^{2 C+1}\right) \Delta \leq 2 D^{2 C+1} \Delta \leq 2 D^{2 C+1} c n \leq \frac{\varepsilon}{2} n
$$

Let us define an auxiliary graph $G_{U}$ with vertex set $U$ where we join two vertices $x, y$ provided $y \notin U_{x}$ (equivalently, $x \notin U_{y}$ ). It is easy to see that $G_{U}$ has a perfect matching (or 'almost' perfect, if $|U|$ is odd; this makes no difference for us). Indeed, the degree of every vertex in $G_{U}$ is at least $|U|-\frac{\varepsilon}{2} n \geq|U| / 2$, and therefore $G_{U}$ has a Hamilton cycle. Fix a perfect matching $\mathscr{M}$ in $G_{U}$ according to this Hamilton cycle and fix a pairing $\mathscr{P}$ of the vertices of $G$ where each edge of $\mathscr{M}$ forms a pair. Finally, since $G$ is path-pairable, choose a collection of edge-disjoint paths $\mathscr{R}$ that realize this pairing. Observe that any path from $\mathscr{R}$ must use an even number of edges between $A$ and $B$. We single out two types of edges $e=x y$ in $\mathscr{M}$ with respect to this realization: either the $x-y$ path in $\mathscr{R}$ is weak but is of length bigger than $C$, or this $x-y$ path uses at least 4 edges between $A$ and $B$. Let $\mathscr{M}=\mathscr{E}_{0} \cup \mathscr{E}_{1} \cup \mathscr{E}_{2}$, where $\mathscr{E}_{0}$ denotes the edges satisfying the former condition, $\mathscr{E}_{1}$ the latter, and $\mathscr{E}_{2}$ denotes the remaining edges. We claim that most of the edges are in $\mathscr{E}_{1}$. Indeed, observe that if $e=x y \in \mathscr{E}_{2}$, then the $x-y$ path must use edges from $B$. By planarity we have $e(B) \leq 3|B|$, and therefore $\left|\mathscr{E}_{2}\right| \leq 3 \varepsilon n$. Using planarity again we have that $e(A) \leq 3|A|$. On the other hand, for each edge in $\mathscr{E}_{0}$ its path in $\mathscr{R}$ uses more than $C$ edges, at most 2 of which are in the cut $\{A, B\}$, and none of which belong to $B$.

Accordingly, since these paths are edge-disjoint, we have that $e(A) \geq(C-2)\left|\mathscr{E}_{0}\right|$ and so

$$
\left|\mathscr{E}_{0}\right| \leq \frac{3}{C-2}|A| \leq \varepsilon|A|
$$

Therefore, $\left|\mathscr{E}_{1}\right| \geq \frac{1}{2}(|U|-1)-\left|\mathscr{E}_{0}\right|-\left|\mathscr{E}_{2}\right| \geq \frac{1}{2}(|U|-1)-\varepsilon n-\left|\mathscr{E}_{2}\right|$. It follows that since every path in $\mathscr{R}$ pairing an edge in $\mathscr{E}_{1}$ contributes at least 4 edges between $A$ and $B$, and these paths must be edge-disjoint, we have

$$
e(A, B) \geq 4\left|\mathscr{E}_{1}\right|+2\left|\mathscr{E}_{2}\right| \geq 2|U|-2-4 \varepsilon n-2\left|\mathscr{E}_{2}\right| \geq 2|U|-2-10 \varepsilon n \geq 2|U|-16 \varepsilon n,
$$

where in the last inequality we used the fact that $n>1 / c \geq 1 / \varepsilon$. This completes the proof of Lemma 1.9 .

Our final lemma allows us to quantify more precisely the degree distribution in any bipartite planar graph.

Lemma 1.10. Let $G$ be a bipartite planar graph on $n$ vertices with parts $A, B$, and let $A^{\prime} \subset A$ be the set of vertices in $A$ with degree at least 3 . Then the following are true.

1. The number of vertices in $A$ with degree 2 is at least $e(A, B)-n-3|B|$;
2. $\left|A^{\prime}\right|<2|B|$;
3. $e\left(A^{\prime}, B\right)<6|B|$.

Proof. For each $i \geq 0$ let $A_{i}, A_{\leq i}$, and $A_{\geq i}$ denote the number of vertices in $A$ that have degree $i$ in $G$, degree at most $i$, and degree at least $i$, respectively. Because of planarity we have that $e\left(A^{\prime}, B\right)<2\left(\left|A^{\prime}\right|+|B|\right)$. Alternatively, $e\left(A^{\prime}, B\right) \geq 3\left|A^{\prime}\right|$ so it follows that $A_{\geq 3}=\left|A^{\prime}\right|<2|B|$, and so $e\left(A^{\prime}, B\right) \leq 2\left(\left|A^{\prime}\right|+|B|\right)<6|B|$, establishing the second and third
items. Further, we can bound the number of edges between $A$ and $B$ as

$$
\begin{aligned}
e(A, B) & \leq A_{\leq 1}+2\left(|A|-A_{\leq 1}-A_{\geq 3}\right)+e\left(A^{\prime}, B\right) \\
& \leq A_{\leq 1}+2\left(|A|-A_{\leq 1}-\left|A^{\prime}\right|\right)+2\left(\left|A^{\prime}\right|+|B|\right) \\
& \leq 2|A|-A_{\leq 1}+2|B| .
\end{aligned}
$$

It follows that $A_{\leq 1} \leq 2|A|+2|B|-e(A, B)$. Finally, we see that $A_{2}=|A|-\left|A^{\prime}\right|-A_{\leq 1}>e(A, B)-|A|-4|B|=e(A, B)-n-3|B|$, as required.

We are now in a position to prove our main theorem, Theorem 1.2. First, let us give a rough sketch of the proof. Let $G$ be a path-pairable planar graph. We first partition the vertex set of $G$ into the set $A$ of vertices of small degree and the set $B$ of vertices of large degree. We can apply Lemma 1.9 to find that there are many edges in this cut. We shall then show that most vertices in $A$ have degree 2 in this bipartite graph. If $Y \subset A$ denotes the vertices of degree 2 , then we define a planar multigraph with vertex set $B$ where we join $x, y \in B$ whenever there is a $v \in Y$ joined to precisely $x$ and $y$. Now, using Corollary 1.8, we are able to either find a vertex of linear degree in $B$, or we can find many pairs of multiedges in our multigraph that are far apart. This, however, allows us to find a pairing which contributes to more than $2 n$ edges between $A$ and $B$, a contradiction to planarity.

We restate Theorem 1.2 for convenience.
Theorem 1.2. There exists $c \geq 10^{-10^{10}}$ such that if $G$ is a path-pairable planar graph on $n$ vertices then $\Delta(G) \geq c n$.

Proof. Suppose $G$ is a path-pairable planar graph and fix some large constant $D$ so that $D^{-1} \leq 8.5 \cdot 10^{-6}$. Partition the vertex set of $G$ into sets $A$ and $B$, where $B=\{v \in V(G): d(v) \geq D\}$ and $A=V(G) \backslash B$. Since $e(G)<3 n$ it easily follows that
$|B| \leq 6 D^{-1} n:=\varepsilon n$. Suppose that $\Delta(G)<c n$, where $c$ is sufficiently small (depending only on $D$ ) given by Lemma 1.9. More precisely, we may take

$$
c=\frac{\varepsilon}{4 D^{2\lceil 4 / \varepsilon\rceil+1}} .
$$

Our aim is to obtain a contradiction to the planarity of $G$, and so there must exist a vertex of degree at least $c n$. Of course, this is trivial if $c n \leq 1$, so we shall assume throughout that $n>1 / c$. By Lemma 1.9 (with $U=A$ ) we have that there are at least $2|A|-16 \varepsilon n \geq 2 n-18 \varepsilon n$ edges between $A$ and $B$.

Next, we shall show that there is a large subset of $A$ which induces a graph with maximum degree at most 2 . To see this, let $A_{0}=A, B_{0}=B$. Suppose $A_{i}, B_{i}$ have been defined already. If there is a vertex $v \in A_{i}$ such that $d_{A_{i}}(v)>d_{B_{i}}(v)$, then let $A_{i+1}=A_{i} \backslash\{v\}$ and $B_{i+1}=B_{i} \cup\{v\}$. Notice that $e\left(A_{i+1}, B_{i+1}\right) \geq e\left(A_{i}, B_{i}\right)+1$, and so $e\left(A_{i+1}, B_{i+1}\right) \geq e(A, B)+i \geq 2 n-18 \varepsilon n+i$. Let $t \geq 0$ be such that there is no $v \in A_{t}$ with more neighbors in $A_{t}$ than in $B_{t}$. Observe that $t \leq 18 \varepsilon n$ (otherwise $e\left(A_{t}, B_{t}\right) \geq 2 n$ ), and accordingly $\left|B_{t}\right|=|B|+t \leq \varepsilon n+18 \varepsilon n=19 \varepsilon n$.

Let $X \subset A_{t}$ be the set of vertices in $A_{t}$ with at least 3 neighbors in $A_{t}$. Since every vertex in $A_{t}$ has at least as many neighbors in $B_{t}$ as in $A_{t}$, we have that every vertex in $X$ has at least 3 neighbors in $B_{t}$. Therefore, by Lemma 1.10, $|X| \leq 2\left|B_{t}\right|, e\left(X, B_{t}\right) \leq 6\left|B_{t}\right|$, and there are at least $e\left(A_{t}, B_{t}\right)-n-3\left|B_{t}\right| \geq e(A, B)-n-3\left|B_{t}\right|$ vertices in $A_{t}$ with exactly two neighbors in $B_{t}$. Let $A^{*}=A_{t} \backslash X$ and $B^{*}=B_{t} \cup X$. Now we have that every vertex in $A^{*}$ has at most 2 neighbors in $A^{*}$ and $\left|B^{*}\right| \leq 3\left|B_{t}\right| \leq 57 \varepsilon n$, so $\left|A^{*}\right| \geq n-57 \varepsilon n$. We have to make sure we still have many vertices in $A^{*}$ with exactly two neighbors in $B^{*}$. Notice that if a vertex $v \in A_{t}$ had two neighbors in $B_{t}$ and was not adjacent to any vertex in $X$ then $v \in A^{*}$ and $v$ still has exactly two neighbors in $B^{*}$. Therefore we only have to worry about the vertices in $A_{t}$ which are adjacent to some vertices in $X$. Observe that $e\left(X, A^{*}\right) \leq e\left(X, B_{t}\right) \leq 6\left|B_{t}\right|$, and so there are at least
$e(A, B)-n-9\left|B_{t}\right| \geq(2 n-18 \varepsilon n)-n-9 \cdot 19 \varepsilon n=n-189 \varepsilon n$ vertices in $A^{*}$ with exactly 2 neighbors in $B^{*}$. Hence there are at most $189 \varepsilon n$ vertices in $A^{*}$ which do not have degree 2 in $B^{*}$.

We say that an edge $u v \in G$ is bad if one of the followings holds:

1. (Type I) $u v \in G\left[B^{*}\right]$.
2. (Type II) $u v \in G\left[A^{*}\right]$ and $u$ (or $v$ ) has degree not equal to 2 in $B^{*}$.
3. (Type III) $u v \in G\left[A^{*}\right], d_{B *}(u)=d_{B *}(v)=2$, and $N_{B^{*}}(u) \neq N_{B^{*}}(v)$.
4. (Type IV) $u v \in G$, such that $u \in A^{*}, v \in B^{*}$, and $d_{B^{*}}(u) \geq 3$.

We have the following bound on the number of bad edges.
Claim 1.11. There are at most $1233 \varepsilon n$ bad edges.

Proof. We are going to bound the number of bad edges of each type.
Note that by planarity, there are at most $3\left|B^{*}\right|$ edges in $B^{*}$ so there are at most $3\left|B^{*}\right| \leq 171 \varepsilon n$ edges of Type I.

Now, since every vertex in $A^{*}$ has at most two neighbors in $A^{*}$, each vertex in $A^{*}$ with degree not equal to 2 in $B^{*}$ contributes to at most two bad edges of Type II. As there are at most $189 \varepsilon n$ vertices in $A^{*}$ which do not have degree 2 in $B^{*}$, it follows that there are at most $378 \varepsilon n$ bad edges of Type II.

Let us consider bad edges of Type III. Since $G\left[A^{*}\right]$ has maximum degree 2, we can partition the edges of $G\left[A^{*}\right]$ into at most 3 matchings, $M_{1}, M_{2}, M_{3}$. It is well known (and easy to see) that contracting an edge in a planar graph preserves planarity. It follows that, for $i \in\{1,2,3\}$, we can contract the edges of $M_{i}$ while still preserving planarity. Denote this new graph by $\tilde{G}_{i}$ with vertex set $\tilde{A}_{i} \cup B^{*}$. Since $\tilde{G}_{i}$ is planar, from Lemma 1.10 we have that there are at most $2\left|B^{*}\right|$ vertices in $\tilde{G}_{i}$ with at least 3 neighbors in $B^{*}$. Therefore, at most $2\left|B^{*}\right|$ edges in $M_{i}$ can be bad of Type III. Hence, there are at most $6\left|B^{*}\right| \leq 342 \varepsilon n$ bad edges of Type III.

Finally, by Lemma 1.10 there can be at most $6\left|B^{*}\right| \leq 342 \varepsilon n$ bad edges of Type IV.
So in total there are at most $1233 \varepsilon n$ bad edges of any type.

Let $Y \subseteq A^{*}$ be the set of vertices with degree exactly 2 in $B^{*}$. We now define an auxiliary multigraph $G_{B^{*}}$ in the following way. The vertex set of $G_{B^{*}}$ is $B^{*}$ and for any two vertices $x, y \in B^{*}$, join $x$ to $y$ by an edge for every $v \in Y$ that is joined precisely to $x$ and $y$.

Claim 1.12. $G_{B^{*}}$ is planar.

Proof. This is clear since the bipartite graph $G\left[Y, B^{*}\right]$ between $Y$ and $B^{*}$ is planar, and contracting edges preserves planarity.

Apply Corollary 1.8 to the multigraph $G_{B^{*}}$ with $\varepsilon_{1}=\varepsilon_{2}=1 / 100$. Notice that if an edge in $G_{B^{*}}$ is incident to more than $\frac{1}{100}|Y|$ multiedges then one of its endpoints has degree, in $G$, at least $\frac{1}{200}|Y|$. However, recall that we initially assumed $\Delta(G)<c n$, and certainly $c \leq 1 / 400$ by our choice of $D$. Accordingly, since $|Y| \geq n-189 \varepsilon n \geq n / 2$, we obtain a vertex of degree at least

$$
2 c|Y| \geq c n
$$

a contradiction.
So we may assume that there are at least $\frac{1}{100}\binom{|Y|}{2}$ pairs of edges in $G_{B^{*}}$ which are at distance greater than 1 . Note that if $H$ is any graph on $n$ vertices with edge density at least $\delta$, then it is easy to greedily find a matching of size at least $\frac{\delta}{10} n$. It follows that we may select a collection of pairwise disjoint pairs $\mathscr{P}$ in $Y$, such that $|\mathscr{P}| \geq \frac{1}{1000}|Y| \geq \frac{1}{2000} n$, and such that for every $\{u, v\} \in \mathscr{P}$, their corresponding edges in $G_{B^{*}}$ are at distance greater than 1.

We need the following two claims.
Claim 1.13. Let $P$ be a path contained in $A^{*}$ which has at least two vertices and does not contain any bad edges. Then every vertex $v \in P$ has the same neighborhood (of size 2 ) in $B^{*}$.

Proof. This is immediate from the definition of a bad edge.

Claim 1.14. Let $u, v \in Y$ be two vertices whose corresponding edges in $G_{B^{*}}$ are at distance greater than 1 . Then any path in $G$ joining $u$ and $v$ either contains some bad edges, or uses at least 6 edges between $A^{*}$ and $B^{*}$.

Proof. Suppose $P$ is a path joining $u$ and $v$ which does not use any bad edges and does not use at least 6 edges between $A^{*}$ and $B^{*}$. By definition and using claim 1.13 , all vertices of $V(P) \cap A^{*}$ are in $Y$, it can not have an edge inside $B^{*}$ and it must use 2 or 4 edges between $A^{*}$ and $B^{*}$. We may assume $P$ uses 4 edges as the other case follows from the same argument. Let $P=P_{1} e_{1} e_{2} P_{2} e_{3} e_{4} P_{3}$, where $\left\{e_{1}, e_{2}, e_{3}, e_{4}\right\}$ are edges between $A^{*}$ and $B^{*}$ and $P_{1}, P_{2}, P_{3}$ are paths inside $Y$. From claim 1.13 applied to $P_{1}, P_{2}$ and $P_{3}$ we deduce that the edge of $u$ in $G_{B^{*}}$ is at distance at most 1 to the edge of $v$ in $G_{B^{*}}$.

The proof of Theorem 1.2 is nearly complete. Indeed, since $G$ is path-pairable, there are edge-disjoint paths joining every pair of $\mathscr{P}$, and hence the pairs in $\mathscr{P}$ contribute to at least $6(|\mathscr{P}|-1233 \varepsilon n)$ edges between $A^{*}$ and $B^{*}$.

Let $P$ be the union of the vertices in $\mathscr{P}$ and let $U=A^{*} \backslash P$. Suppose first that $|U|<57 \varepsilon n$. It follows that

$$
2|\mathscr{P}|>(n-57 \varepsilon n)-57 \varepsilon n,
$$

so $|\mathscr{P}|>n / 2-57 \varepsilon n$. Then the above pairing contributes at least $6(n / 2-1290 \varepsilon n)=3 n-7740 \varepsilon n$ edges between $A^{*}$ and $B^{*}$. But this is at least $2 n$ whenever $\varepsilon \leq 7740^{-1}$ which is guaranteed by our choice of $D$, a contradiction. Therefore, we may assume that $|U| \geq 57 \varepsilon n$. By Lemma 1.9 (since $c$ is small enough) there is a pairing of the vertices in $U$ which contributes to at least $2|U|-16 \cdot 57 \varepsilon n=2|U|-912 \varepsilon n$
edges between $A^{*}$ and $B^{*}$. Hence in total the number of edges between $A^{*}$ and $B^{*}$ is

$$
\begin{aligned}
& \geq 6(|\mathscr{P}|-1233 \varepsilon n)+2\left|A^{*}\right|-4|\mathscr{P}|-912 \varepsilon n \\
& \geq 2|\mathscr{P}|+2(n-57 \varepsilon n)-6 \cdot 1233 \varepsilon n-912 \varepsilon n \\
& \geq 2 n+n / 1000-8424 \varepsilon n
\end{aligned}
$$

So by our choice of $D$ we get that $8424 \varepsilon \leq \frac{1}{1000}$, and so there are at least $2 n$ edges between $A^{*}$ and $B^{*}$, a contradiction to the planarity of $G$. It follows that there must exist a vertex of degree at least $c n$.

### 1.2.3 Final Remarks and Open Problems

It is worth observing that our proof relies only on the following three properties of a planar graph $G$ : contracting edges of $G$ preserves planarity, $G$ does not contain a $K_{5}$-minor, and any bipartite subgraph $H$ of $G$ has at most $2|H|$ edges. We remark that it is possible to generalize our result in the following sense. Given integers $t, c$, we say that a graph $G$ is $(t, c)$-good if $G$ is $K_{t}$-minor-free and any bipartite subgraph $H$ of $G$ has at most $2|H|+c$ edges. Moreover, define $\mathscr{G}_{t, c}$ to be the family of $(t, c)$-good graphs $G$ such that contracting edges of $G$ preserves $(t, c)$-goodness.

Theorem 1.15. For any integers $t, c$ there is a positive constant $C=C(t, c)$ such that the following holds. If $G$ is a path-pairable graph on $n$ vertices with $G \in \mathscr{G}_{t, c}$, then $\Delta(G) \geq C n$.

We have the following immediate corollary.
Corollary 1.16. For every non-negative integer $g$ there is a positive constant $C=C(g)$ such that the following holds. If $G$ is a path-pairable graph on $n$ vertices which has a 2-cell embedding on a surface with genus $g$, then $\Delta(G) \geq C n$.

Proof. We claim that $G \in \mathscr{G}_{3 g+5,2 g}$. Indeed, it follows from Euler's formula (see, e.g., [13]) that if $G$ is 2 -cell embedded on a surface of genus $g$ then $n+m-f=2-g$, where $m$
is the number of edges $G$ and $f$ is the number of faces of the embedding. Since $2 m \geq 3 f$ ( $2 m \geq 4 f$ if $G$ is triangle-free) it follows that $e(G) \leq 3 n+3 g-6(e(G) \leq 2 n+2 g-4$ if $G$ is triangle-free). In particular, if $G$ is bipartite then $e(G) \leq 2 n+2 g-4 \leq 2 n+2 g$. Suppose for contradiction that $G$ contains a $K_{3 g+5}$-minor. Then $K_{3 g+5}$ could be 2 -cell embedded on a surface of genus $g$, hence $\binom{3 g+5}{2}=e\left(K_{3 g+5}\right) \leq 12 g+9$, which is easily seen to be a contradiction.

Sketch of a proof of Theorem 1.15 The proof is essentially the same as the proof of Theorem 1.2. Certain changes have to be made in the preparatory lemmas first.

Corollary 1.8 generalizes trivially to multigraphs with no $K_{t}$-minors.
In Lemma 1.9 we only use the fact that any subgraph $H$ of a planar graph has at most $3|H|$ edges. Observe that if $G \in \mathscr{G}_{t, c}$ then any subgraph of $H$ of $G$ has at most $4|H|+2 c$ edges. One can therefore modify the proof, at the expense of a worse constant in front of $\varepsilon n$ in the conclusion of the Lemma.

In the proof of Lemma 1.10 we only use the fact that a bipartite subgraph $H$ of a planar graph does not use more than $2|H|$ edges. The lemma can be therefore modified to work for graphs in $\mathscr{G}_{t, c}$ by introducing some additive constants, depending only on $c$, to the inequalities in every part of the Lemma.

In the proof of Theorem 1.2 all the estimates remain correct by taking $\varepsilon$ small enough.

Note that we also need that $\mathscr{G}_{t, c}$ is closed under edge contractions in order to estimate the number of "bad edges", as in Claim 1.11 (there we used that contracting edges preserves planarity).

We believe that the condition on the number of edges in bipartite subgraphs can be omitted while still ensuring the existence vertex of linear degree. We therefore make the following conjecture.

Conjecture 1.17. For any $t$ there exists a constant $c=c(t)$ such that every path-pairable
graph on $n$ vertices without a $K_{t}$ minor must contain a vertex of degree at least $c n$.

Finally, recall that we defined $\Delta_{\min }^{p}(n)$ to be the minimum of $\Delta(G)$ over all $n$-vertex path-pairable planar graphs $G$. We have shown that $\Delta_{\text {min }}^{p}(n)$ grows linearly in $n$; however, as mentioned in the Introduction, the constants in the upper and lower bounds are quite far apart. We close with the following problem.

Problem 1.18. Determine $\Delta_{\min }^{p}(n)$ for sufficiently large $n$.

We do not know if our construction yielding the upper bound of $2 n / 3$ is optimal, and a significant improvement on our lower bound would be very interesting.

### 1.3 Diameter of path-pairable graphs

### 1.3.1 Path-pairable graphs from blowing up paths

In this section, we shall show how to construct a class of graphs which have large diameter and are path-pairable. Let $G$ be a graph with vertex set $V(G)=\left\{v_{1}, \ldots, v_{k}\right\}$, and let $G_{1}, \ldots, G_{k}$ be graphs. We define the blown-up graph $G\left(G_{1}, \ldots, G_{k}\right)$ as follows: replace every vertex $v_{i}$ in $G$ by the corresponding graph $G_{i}$, and for every edge $v_{i} v_{j} \in E(G)$ insert a complete bipartite graph between the vertex sets of $G_{i}$ and $G_{j}$.

Let $P_{k}$ denote the path on $k$ vertices. The following lemma asserts that if we blow up a path with graphs $G_{1}, \ldots, G_{k}$, such that $G_{i}$ is path-pairable for $i \leq k-1$, and certain properties inherited from the cut-condition hold, then the resulting blow-up is path-pairable.

Lemma 1.19. Let $G_{1}, \ldots, G_{k}$ be graphs on $n_{1}, \ldots, n_{k}$ vertices, respectively, where $G_{i}$ is path-pairable for $i \leq k-1$. Let $n=\sum_{i=1}^{k} n_{i}$ and let $u_{i}=\sum_{j=1}^{i} n_{j}$ for $i=1, \ldots, k-1$. Then $P_{k}\left(G_{1}, \ldots, G_{k}\right)$ is path-pairable if and only if

$$
\begin{equation*}
n_{i} \cdot n_{i+1} \geq \min \left(u_{i}, n-u_{i}\right) \tag{1.1}
\end{equation*}
$$

holds for $i=1, \ldots, k-1$.

Proof. For each $i=1, \ldots, k$, let $U_{i}=\bigcup_{j=1}^{i} V\left(G_{j}\right)$ so that $u_{i}=\left|U_{i}\right|$. Now, if $P_{k}\left(G_{1}, \ldots, G_{k}\right)$ is path-pairable, then we may apply the cut-condition to the cut $\left\{U_{i}, V(G) \backslash U_{i}\right\}$. This implies $n_{i} \cdot n_{i+1} \geq \min \left(u_{i}, n-u_{i}\right)$ must hold for $i=1, \ldots, k-1$. In the remainder, we show that this simple condition is enough to yield the path-pairability of $G:=P_{k}\left(G_{1}, \ldots, G_{k}\right)$. Assume that a pairing $\mathscr{P}$ of the vertices of $G$ is given. If $\{u, v\} \in \mathscr{P}$ we shall say that $u$ is a sibling of $v$ (and vice-versa). We shall define an algorithm that sweeps through the classes $G_{1}, G_{2}, \ldots, G_{k}$ and joins each pair of siblings via edge-disjoint paths.

First we give an overview of the algorithm. We proceed by first joining pairs $\{u, v\} \in \mathscr{P}$ via edge-disjoint paths such that $u$ and $v$ belong to different $G_{i}$ 's, and then afterwards joining pairs that remain inside some $G_{j}$ (using the path-pairability of $G_{j}$ ). Before round 1, we use the path-pairability property of $G_{1}$ to join those siblings which belong to $G_{1}$. During round 1 , we assign to every vertex $u$ of $G_{1}$ a vertex $v$ of $G_{2}$. If $\{u, v\} \in \mathscr{P}$ are siblings, then we simply choose the edge $u v$. Then we join the siblings which are in $G_{2}$ again using the path-pairability property of $G_{2}$. For those paths $u v$ that have not ended (because $\{u, v\} \notin \mathscr{P}$ ) we shall continue by choosing a new vertex $w$ in $G_{3}$ and continue the path with edge $v w$, and so on. We shall call paths which have not finished joining a pair of siblings unfinished; otherwise, we say the path is finished. The last edge which completes a finished path we shall call a path-ending edge. During round $i$, we shall first choose those vertices in $G_{i+1}$ which, together with some vertex of $G_{i}$, form path-ending edges. At the end of round $i$, in $G_{i+1}$ we will have endpoints of unfinished paths and perhaps also some endpoints of finished paths. Note that the vertices of $G_{i+1}$ might be endpoints of several unfinished paths. For $x \in G_{i+1}$ let $w(x)$ denote the number of unfinished paths $P \cup\{x\}$ with $P \subset U_{i}$ at the end of round $i$ which are to be extended by a vertex of $G_{i+2}$ (including the single-vertex path $x$ in the case when $x$ was not joined to its sibling in the latest round). Note that every such path corresponds to a yet not joined vertex in $U_{i+1}$ as well as to another vertex yet to be joined lying in $V(G) \backslash U_{i+1}$. It follows
that

$$
\begin{equation*}
\sum_{x \in G_{i+1}} w(x) \leq \min \left(u_{i+1}, n-u_{i+1}\right) . \tag{1.2}
\end{equation*}
$$

Let us now be more explicit in how we make choices in each round. We shall maintain the following two simple conditions throughout our procedure (the first of which has been mentioned above):
(a) During round $i(1 \leq i \leq k-1)$, if $w \in G_{i}$ is the current endpoint of the path which began at some vertex $u \in U_{i}$ (possibly $u=w$ ), and $\{u, v\} \in \mathscr{P}$ for $v \in G_{i+1}$, then we join $w$ to $v$. Informally, we choose path-ending edges when we can.
(b) $w(x) \leq n_{i+1}$ for all $x \in G_{i}$, for $i=1, \ldots, k-1$.

The second condition above is clearly necessary in order to proceed during round $i$, as $\left|N(x) \cap G_{i+1}\right|=n_{i+1}$ for every $x \in G_{i}$, and hence we cannot continue more than $n_{i+1}$ unfinished paths through $x$.

We claim that as long as both of the above conditions are maintained, the proposed algorithm finds a collection of edge-disjoint paths joining every pair in $\mathscr{P}$. Both conditions are clearly satisfied for $i=1$ as $w(x) \leq 1 \leq n_{2}$ for all $x \in G_{1}$. Let $i \geq 2$ and suppose both conditions hold for rounds $1, \ldots, i-1$. Our aim is to show that an appropriate selection of edges between $G_{i}$ and $G_{i+1}$ exists in round $i$ to maintain the conditions. We start round $i$ by choosing all path-ending edges with endpoints in $G_{i}$ and $G_{i+1}$; this can be done since, by induction, $w(x) \leq n_{i+1}$ for every $x \in G_{i}$. Observe that if $i=k-1$ then the only remaining siblings are in $G_{k}$. Then for every $\{u, v\} \in \mathscr{P}$ such that $u, v \in G_{k}$ we can find a vertex $q$ in $G_{k-1}$ and join $u, v$ with the path $u q v$. When $i<k-1$ then the remaining paths can be continued by assigning arbitrary vertices from $G_{i+1}$ (without using any edge multiple times). We choose an assignment that balances the 'weights' in $G_{i+1}$. More precisely, let us choose an assignment of the vertices that minimizes

$$
\sum_{a \in G_{i+1}}(w(a))^{2}
$$

If for every $x \in G_{i+1}$ we have that $w(x) \leq n_{i+2}$ we are basically done. It remains to find edge-disjoint paths inside $G_{i+1}$ for those pairs $\{x, y\} \in \mathscr{P}$ whose vertices belong to $G_{i+1}$. But this is possible because of the assumption that $G_{i+1}$ is path-pairable.

Suppose then that in the above assignment there exists $x \in G_{i+1}$ with $w(x) \geq n_{i+2}+1$. We first claim that, under this assignment, no other vertex of $G_{i+1}$ has small weight.

Claim 1.20. Every vertex $y \in G_{i+1}$ satisfies $w(y) \geq n_{i+2}-1$.

Proof. Suppose there is $y \in G_{i+1}$ such that $w(y) \leq n_{i+2}-2$. Then, as $w(x)>w(y)+2$, there exist vertices $v_{1}, v_{2} \in G_{i}$ such that a nonempty collection of paths ending at $v_{1}$ and $v_{2}$ have $x$ as their next vertex after round $i$, but no paths using $v_{1}$ or $v_{2}$ are assigned $y$ as their next vertex. Note that at least one of the edges $v_{1} x, v_{2} x$ is not path-ending; we may assume $v_{1} x$ is not path-ending. Replacing our choice of $v_{1} x$ with $v_{1} y$ decreases the square sum $\sum_{a \in G_{i+1}}(w(a))^{2}$-a contradiction.

Therefore, we may assume $w(y) \geq n_{i+2}-1$ for all $y \in G_{i+1}$. In this case, partition the vertices of $G_{i+1}$ into three classes:

$$
\begin{aligned}
& X=\left\{v \in G_{i+1}: w(v) \geq n_{i+2}+1\right\} \\
& Y=\left\{v \in G_{i+1}: w(v)=n_{i+2}-1\right\} \\
& Z=\left\{v \in G_{i+1}: w(v)=n_{i+2}\right\} .
\end{aligned}
$$

Observe first that $1 \leq|X| \leq|Y|$, since otherwise, using 1.2 , we have

$$
n_{i+1} n_{i+2}+1 \leq \sum_{s \in G_{i+1}} w(s) \leq \min \left(u_{i+1}, n-u_{i+1}\right)
$$

contradicting condition (1.1). Notice also that a similar argument as in Claim 1.20 shows
that $w(v) \leq n_{i+2}+1$ for every $v \in G_{i+1}$. Hence, we actually have

$$
X=\left\{v \in G_{i+1}: w(v)=n_{i+2}+1\right\} .
$$

We will need the following claim, which asserts that if there are siblings in $G_{i+1}$, then they must belong to $Z$.

Claim 1.21. If $\{u, v\} \in \mathscr{P}$ and $u, v \in G_{i+1}$, then $u, v \in Z$.

Proof. We first show that every $y \in Y$ is incident to a path-ending edge. Suppose, to the contrary, that there is $y \in Y$ with no path-ending edge ending at $y$. It follows that at most $w(y)$ vertices in $G_{i}$ are joined to $y$ after round $i$. Hence, we can take any $x \in X$ and find $z \in G_{i}$ such that $z$ is not joined to $y$ after round $i$, and such that $z x$ is not path-ending. Replacing $z x$ by $z y$ results in a smaller square sum $\sum_{a \in G_{i+1}}(w(a))^{2}$, which gives a contradiction.

Now, let $\{u, v\} \in \mathscr{P}$ such that $u, v \in G_{i+1}$. Since every $y \in Y$ is incident to a path-ending edge, we have that $u, v \notin Y$. Suppose, for contradiction, that $u \in X$. Then $u$ is joined to $w(u)=n_{i+2}+1$ vertices in $G_{i}$ after round $i$, and so for every $y \in Y$ there is $z \in G_{i}$ which is joined to $u$ but not to $y$. Replacing $z u$ by $z y$ results in a smaller square sum $\sum_{a \in G_{i+1}}(w(a))^{2}$, which again is a contradiction.

Finally, we shall show how to reduce the weights of vertices in $X$ (and pair the siblings inside $G_{i+1}$ ) using the path-pairable property of $G_{i+1}$. For every $x \in X$ pick a different vertex $y_{x} \in Y$ (which we can do, since $|Y| \geq|X|$ ) and let $\mathscr{P}^{\prime}=\left\{\{u, v\} \in \mathscr{P}: u, v \in G_{i+1}\right\} \cup\left\{\left\{x, y_{x}\right\}: x \in X\right\}$. Since $G_{i+1}$ is path-pairable, we can find edge-disjoint paths joining the siblings in $\mathscr{P}^{\prime}$ (note that by Claim 1.21 none of the pairs $\left\{x, y_{x}\right\}$ interfere with any siblings $\{u, v\} \in \mathscr{P}$ with $\left.u, v \in G_{i+1}\right)$. Observe now that for every $x \in X$ one path has been channeled to a vertex $y \in Y$, and so the number of unfinished path endpoints at $x$ has dropped to $n_{i+2}$, as required. This completes the proof of Lemma 1.19

We close this section by pointing out that the path-pairability of certain $G_{i}$ subgraphs in a path-pairable graph $P_{k}\left(G_{1}, \ldots, G_{k}\right)$ cannot be avoided for $k \geq 5$. We do this by giving an example of a blown-up path that satisfies the cut-conditions of Lemma 1.19, but is not path-pairable unless some of the $G_{i}$ 's are path-pairable. For the sake of simplicity we set $k=5$ and prove that $G_{3}$ has to be path-pairable. Let $n=2 t^{2}+t$ for some even $t \in \mathbb{N}$, and let $n_{1}=n_{5}=t^{2}-t, n_{2}=n_{3}=n_{4}=t$, where $n_{i}=\left|G_{i}\right|$ for each $i \in[5]$. Clearly, $P_{5}\left(G_{1}, \ldots, G_{5}\right)$ satisfies the condition 1.1 ) of Lemma 1.19 . Also, observe that any pairing of the vertices in $G_{1} \cup G_{2}$ with the vertices in $G_{4} \cup G_{5}$ has to use all edges between $G_{3}$ and $G_{2} \cup G_{4}$. Therefore, if we additionally pair the vertices inside $G_{3}$, then the paths joining those vertices can only use edges in $G_{3}$. Accordingly, $G_{3}$ must be path-pairable.

### 1.3.2 Proof of Theorem 1.4

Take $x, y \in V(G)$ such that $d(x, y)=d(G)$ and let $V_{i}$ be the set of vertices at distance exactly $i$ from $x$, for every $i$. Observe that $V_{0}=\{x\}$ and $y \in V_{d(G)}$. For $i \in\{1, \ldots, d(G)\}$ define $n_{i}$ to be the size of $V_{i}$ and let $u_{i}=\sum_{j=0}^{i} n_{j}$.

We need the following claim.
Claim 1.22. $u_{2 k+1} \geq\binom{ k+2}{2}$ as long as $u_{2 k+1} \leq \frac{n}{2}$.
Proof. We shall use induction on $k$. For $k=0$ the assertion is clear. Assume that $u_{2 k-1} \geq\binom{ k+1}{2}$. By the cut-condition we have that the number of edges between $V_{2 k}$ and $V_{2 k+1}$ is at least $u_{2 k-1}$, hence $n_{2 k} \cdot n_{2 k+1} \geq u_{2 k-1} \geq\binom{ k+1}{2}$. By the arithmetic-geometric mean inequality, $n_{2 k}+n_{2 k+1} \geq 2 \sqrt{\binom{k+1}{2}} \geq k+1$. As $u_{2 k+1}=u_{2 k-1}+n_{2 k}+n_{2 k+1}$, we have $u_{2 k+1} \geq\binom{ k+2}{2}$.

Now, let $A=\bigcup_{i=0}^{\lfloor d / 3\rfloor} V_{i}, B=\bigcup_{i=\lfloor d / 3\rfloor+1}^{\lfloor 2 d / 3\rfloor} V_{i}, C=\bigcup_{i=\lfloor 2 d / 3\rfloor+1}^{d} V_{i}$. Observe that $|A|,|C| \geq \min \left\{\frac{n}{2}, \frac{d^{2}}{100}\right\}$, so joining vertices in $A$ with vertices in $C$ requires at least
$\min \left\{\frac{n}{2}, \frac{d^{2}}{100}\right\} \cdot \frac{d}{3}$ edges. Hence,

$$
\min \left\{\frac{n}{2}, \frac{d^{2}}{100}\right\} \cdot \frac{d}{3} \leq m
$$

which implies

$$
d \leq \max \left\{\frac{6 m}{n}, \sqrt[3]{300 m}\right\}
$$

Notice that whenever $m \leq n^{3 / 2}$ we have $d \leq \sqrt[3]{300 m}$. Also, if $m \geq \sqrt{2} n^{3 / 2}$, then the upper bound is trivially satisfied by the general upper bound obtained in [65].

For the lower bound, let $n$ and $2 n \leq m \leq \frac{1}{4} n^{3 / 2}$ be given. For any natural number $\ell$ we shall denote by $S_{\ell}$ the star $K_{1, \ell-1}$ on $\ell$ vertices. Consider the graph

$$
G=P_{k+3}\left(G_{1}, \ldots, G_{k+3}\right)
$$

on $n$ vertices, where $k=\left\lfloor\sqrt[3]{\frac{m}{2}-n}\right\rfloor$ and $G_{1}=G_{2}=\cdots=G_{k}=S_{k}, G_{k+1}=S_{k^{2}}$, $G_{k+2}=S_{2}$, and $G_{k+3}$ is an empty graph on $n-2 k^{2}-2$ vertices.

Straightforward calculation shows that

- $u_{i}=i \cdot k$, for $i \leq k, \quad u_{k+1}=2 k^{2}, \quad u_{k+2}=2 k^{2}+2$.
- $n_{1} n_{2}=n_{2} n_{3}=\cdots=n_{k-1} n_{k}=k^{2}, n_{k} n_{k+1}=k^{3}, n_{k+1} n_{k+2}=2 k^{2}$, $n_{k+2} n_{k+3}=2 n-4 k^{2}-4$.

Therefore, for $i \in\{1, \ldots, k+1\}$ we have

$$
n_{i} \cdot n_{i+1} \geq u_{i} \geq \min \left(u_{i}, n-u_{i}\right)
$$

and

$$
n_{k+2} \cdot n_{k+3} \geq n_{k+3} \geq \min \left(u_{k+2}, n-u_{k+2}\right)
$$

Hence, it follows from Lemma 1.19 that $G$ is path-pairable.

It is easy to check that the number of edges in $G$ is at most $2 n+2 k^{3} \leq m$. On the other hand, the diameter of $G$ is $k+2 \geq \sqrt[3]{\frac{m}{2}-n}$. This completes the proof of Theorem 1.4 .

### 1.3.3 Proof of Theorem 1.5

In this section, we investigate the maximum diameter a path-pairable $c$-degenerate graph on $n$ vertices can have. We shall assume that $c$ is an integer and $c \geq 5$.

Let $G$ be a $c$-degenerate graph on $n$ vertices with diameter $d$. We shall show first that $d \leq 4 \log _{\frac{c+1}{c}}(n)+3$. Let $x \in G$ be such that there is $y \in G$ with $d(x, y)=d$. For $i \in\{0, \ldots, d\}$, write $V_{i}$ for the set of vertices at distance $i$ from $x$. Let $n_{i}=\left|V_{i}\right|$ and $u_{i}=\sum_{j=0}^{i} n_{j}$. Observe that $\left|V_{i}\right| \geq 1$ for every $i \in\{0, \ldots, d\}$. Moreover, we can assume that $u_{\left\lfloor\frac{d}{2}\right\rfloor} \leq \frac{n}{2}$ (otherwise, repeat the argument below with $V_{i}^{\prime}=V_{d-i}$ ).

The claimed upper bound on the diameter easily follows from the following claim.
Claim 1.23. $u_{2 k+1} \geq\left(\frac{c+1}{c}\right)^{k}$ as long as $u_{2 k+1} \leq \frac{n}{2}$.
Let us assume the claim and prove the result. Letting $k=\frac{\left\lfloor\frac{d}{2}\right\rfloor-1}{2}$, we have that

$$
\left(\frac{c+1}{c}\right)^{k} \leq u_{2 k+1} \leq n / 2
$$

by the above claim. Hence,

$$
\begin{aligned}
d & \leq 4 \log _{\frac{c+1}{c}}(n)+3=\frac{4 \log (n)}{\log \left(\frac{c+1}{c}\right)}+3 \leq \frac{4 \log (n)}{\log \left(\frac{c+1}{c-1}\right)} \frac{\log \left(\frac{c+1}{c-1}\right)}{\log \left(\frac{c+1}{c}\right)}+3 \\
& \leq 12 \log _{\frac{c+1}{c-1}}(n)+3
\end{aligned}
$$

where the last inequality follows from the easy to check fact that $\frac{\log \left(\frac{c+1}{c-1}\right)}{\log \left(\frac{c+1}{c}\right)} \leq 3$, for all $c \geq 5$. Thus it remains to prove the claim.

Proof of the Claim. We shall prove the claim by induction on $k$. The base case when $k=0$ is trivial as $u_{1} \geq 2$. Suppose the claim holds for every $l \leq k-1$. Since $G$ is $c$-degenerate,
we have that $e\left(V_{2 k}, V_{2 k+1}\right) \leq c\left(n_{2 k}+n_{2 k+1}\right)$. On the other hand, it follows from the cut-condition that $e\left(V_{2 k}, V_{2 k+1}\right) \geq u_{2 k}=u_{2 k-1}+n_{2 k}$. Therefore, by the induction hypothesis, we have

$$
\begin{aligned}
n_{2 k}+n_{2 k+1} \geq \frac{1}{c}\left(u_{2 k-1}+n_{2 k}\right) & \geq \frac{1}{c}\left(\left(\frac{c+1}{c}\right)^{k-1}+n_{2 k}\right) \\
& \geq \frac{1}{c}\left(\frac{c+1}{c}\right)^{k-1}
\end{aligned}
$$

It follows that,

$$
u_{2 k+1}=u_{2 k-1}+n_{2 k}+n_{2 k+1} \geq\left(\frac{c+1}{c}\right)^{k-1}+\frac{1}{c}\left(\frac{c+1}{c}\right)^{k-1} .
$$

But the right-hand side is equal to $\left(1+\frac{1}{c}\right)\left(\frac{c+1}{c}\right)^{k-1}=\left(\frac{c+1}{c}\right)^{k}$, which proves the claim.

We shall prove the lower bound in Theorem 1.5 assuming that $c$ is an odd integer; when $c$ is even we apply the same argument for $c-1$.

To do so, consider the graph $G=P_{2 m^{\prime}-1}\left(G_{1}, \ldots, G_{2 m^{\prime}-1}\right)$ for some $m^{\prime} \in \mathbb{N}$, which we specify later. First, we shall define the sizes of $G_{i}$ for $i \in\left\{1, \ldots, 2 m^{\prime}-1\right\}$. To do so, define a sequence $\left\{n_{i}\right\}_{i \in \mathbb{N}}$ where $n_{1}=1, n_{2 i}=\frac{c-1}{2}$ and $n_{2 i+1}$ is defined recursively in the following way:

$$
\begin{align*}
n_{2 i+1} & =\left\lceil\frac{2}{c-1} \cdot \sum_{j=1}^{2 i} n_{j}\right\rceil \leq\left\lceil\frac{2}{c-1} \sum_{j=1}^{2 i-2}\right\rceil+\left\lceil\frac{2}{c-1}\left(n_{2 i-1}+n_{2 i}\right)\right\rceil  \tag{1.3}\\
& \leq n_{2 i-1}+\left\lceil\frac{2}{c-1} n_{2 i-1}+1\right\rceil \leq \frac{c+1}{c-1} n_{2 i-1}+2 \leq c\left(\frac{c+1}{c-1}\right)^{i}-(c-1), \tag{1.4}
\end{align*}
$$

where the last inequality can be easily proved by induction.
Let $m$ be the largest integer such that $\sum_{j=1}^{m} n_{j} \leq n / 2$. Let $m^{\prime}=m$ when $m$ is odd, and $m^{\prime}=m-1$ when $m$ is even. Moreover, let $\left|G_{m^{\prime}}\right|=n-2 \sum_{j=1}^{m^{\prime}-1} n_{j}$ and let $\left|G_{i}\right|=n_{i}$ for
$1 \leq i<m^{\prime}$ and $\left|G_{m^{\prime}+j}\right|=\left|G_{m^{\prime}-j}\right|$ for $j \in\left\{1, \ldots, m^{\prime}-1\right\}$.
For all $i \in\left\{1, \ldots 2 m^{\prime}-1\right\}$, let $G_{i}=S_{n_{i}}$ be a star on $n_{i}$ vertices. It is easy to check that the graph $P_{2 m^{\prime}-1}\left(G_{1}, \ldots, G_{2 m^{\prime}-1}\right)$ is path-pairable by Lemma 1.19 . It has diameter at least $2 m-4$ and $m \geq 2 \log _{\frac{c+1}{c-1}}(n)-\Theta_{c}(1)$, which follows from 1.3 . Indeed, note that for $i \geq 1$, $\sum_{j=1}^{2 i+1} n_{j} \leq \frac{c-1}{2}\left\lceil\frac{2}{c-1} \sum_{j=1}^{2 i} n_{j}\right\rceil+n_{2 i+1}=\frac{c-1}{2} n_{2 i+1}+n_{2 i+1}=\frac{c+1}{2} n_{2 i+1} \leq \frac{c(c+1)}{2}\left(\frac{c+1}{c-1}\right)^{i}$.

Lastly, it is not too hard to see that $G$ is $c$-degenerate. Indeed, consider some ordering $v_{1}, \ldots, v_{n}$ of the vertices of $G$ such that if $v_{i} \in G_{l}$ and $v_{i^{\prime}} \in G_{l^{\prime}}$ (where $l$ is even and $l^{\prime}$ is odd), then $i<i^{\prime}$; and if $v_{i}, v_{i^{\prime}} \in G_{l}$ and $v_{i}$ is the center of the star $G_{l}$, then $i<i^{\prime}$. In such an ordering, any vertex $v_{i}$ is adjacent to at most $2 \cdot \frac{c-1}{2}+1=c$ vertices $v_{j}$ with $j<i$. This proves that $G$ is $c$-degenerate, and completes the proof of Theorem 1.5 .

### 1.3.4 Final remarks and open problems

We obtained tight bounds on the parameter $d\left(n, \mathscr{G}_{m}\right)$ when $(2+\varepsilon) n \leq m \leq \frac{1}{4} n^{3 / 2}$, for any fixed $\varepsilon>0$. It is an interesting open problem to investigate what happens when the number of edges in a path-pairable graph on $n$ vertices is around $2 n$. We ask the following:

Question 1.24. Is there a function $f$ such that for every $\varepsilon>0$ and for every path-pairable graph $G$ on $n$ vertices with at most $(2-\varepsilon) n$ edges, the diameter of $G$ is bounded by $f(\varepsilon)$ ?

Another line of research concerns determining the behavior of $d(n, \mathscr{P})$, where $\mathscr{P}$ is the family of planar graphs. Since planar graphs are 5-degenerate, it follows from Theorem 1.5 that the diameter of a path-pairable planar graph on $n$ vertices cannot be larger than $c \log n$. This fact makes us wonder whether there are path-pairable planar graphs with unbounded diameter.

Question 1.25. Is there a family of path-pairable planar graphs with arbitrarily large diameter?

The graph constructed in the proof of the lower bound in Theorem 1.5 when $c=5$ is not planar since it contains a copy of $K_{3,3}$. Therefore, it cannot be used to show that the diameter of a path-pairable planar graph can be arbitrarily large (note, however, that this graph does not contain a $K_{7}$-minor nor a $K_{6,6}$-minor). We end by remarking that we were able to construct an infinite family of path-pairable planar graphs with diameter 6, but not larger.

## CHAPTER 2

## PARTITE SATURATION OF COMPLETE GRAPHS

In this chapter we study the problem of determining sat $(n, k, r)$, the minimum number of edges in a $k$-partite graph $G$ with $n$ vertices in each part such that $G$ is $K_{r}$-free but the addition of an edge joining any two non-adjacent vertices from different parts creates a $K_{r}$. Improving recent results of Ferrara, Jacobson, Pfender and Wenger, and generalizing a recent result of Roberts, we define a function $\alpha(k, r)$ such that sat $(n, k, r)=\alpha(k, r) n+o(n)$ as $n \rightarrow \infty$. Moreover, we prove that

$$
k(2 r-4) \leq \alpha(k, r) \leq \begin{cases}(k-1)(4 r-k-6) & \text { for } r \leq k \leq 2 r-3 \\ (k-1)(2 r-3) & \text { for } k \geq 2 r-3\end{cases}
$$

and show that the lower bound is tight for infinitely many values of $r$ and every $k \geq 2 r-1$. This allows us to prove that, for these values, $\operatorname{sat}(n, k, r)=k(2 r-4) n+O(1)$ as $n \rightarrow \infty$. Along the way, we disprove a conjecture and answer a question of the first set of authors mentioned above. This work is joint with António Girão and Teeradej Kittipassorn.

### 2.1 Introduction

Given a graph $H$, the classical Turán-type extremal problem asks for the maximum number of edges in an $H$-free graph on $n$ vertices. While the corresponding minimization problem is trivial, it is interesting to determine the minimum number of edges in a maximal $H$-free graph on $n$ vertices. We say that a graph is $H$-saturated if it is $H$-free but the addition of an edge joining any two non-adjacent vertices creates a copy of $H$. The minimum number sat $(n, H)$ of edges in an $H$-saturated graph on $n$ vertices was first studied in 1949 by Zykov [80] and independently in 1964 by Erdős, Hajnal, and Moon [32] who proved that $\operatorname{sat}\left(n, K_{r}\right)=(r-2)(n-1)-\binom{r-2}{2}$. Soon after this, Bollobás [10] determined exactly $\operatorname{sat}\left(n, K_{r}^{(s)}\right)$ where $K_{r}^{(s)}$ is the complete $s$-uniform
hypergraph on $r$ vertices. Later, in 1986, Kászonyi and Tuza [54] showed that the saturation number $\operatorname{sat}(n, H)$ for a graph $H$ on $r$ vertices is maximized at $H=K_{r}$, and consequently, $\operatorname{sat}(n, H)$ is linear in $n$ for any $H$. For results on the saturation number, we refer the reader to the survey [33].

This concept can be generalized to the notion of $H$-saturated subgraphs which are maximal elements of a family of $H$-free subgraphs of a fixed host graph. A subgraph of a graph $G$ is said to be $H$-saturated in $G$ if it is $H$-free but the addition of an edge in $E(G)$ joining any two non-adjacent vertices creates a copy of $H$. The problem of determining the minimum number $\operatorname{sat}(G, H)$ of edges in an $H$-saturated subgraph of $G$ was first proposed in the above mentioned paper of Erdős, Hajnal, and Moon. They conjectured a value for the saturation number $\operatorname{sat}\left(K_{m, n}, K_{r, r}\right)$ which was verified independently by Bollobás [11, 12] and Wessel [78, 79]. Very recently, Sullivan and Wenger [75] studied the analogous saturation numbers for tripartite graphs within tripartite graphs and determined $\operatorname{sat}\left(K_{n_{1}, n_{2}, n_{3}}, K_{l, l, l}\right)$ for every fixed $l \geq 1$ and every $n_{1}, n_{2}$ and $n_{3}$ sufficiently large. Several other host graphs have been considered, including hypercubes [25, 53, 69] and random graphs [58].

In this chapter, we are interested in the saturation number $\operatorname{sat}(n, k, r)=\operatorname{sat}\left(K_{k \times n}, K_{r}\right)$ for $k \geq r \geq 3$ where $K_{k \times n}$ is the complete $k$-partite graph containing $n$ vertices in each of its $k$ parts. This function was first studied recently by Ferrara, Jacobson, Pfender and Wenger [37] who determined $\operatorname{sat}(n, k, 3)$ for $n \geq 100$. Later, Roberts [72] showed that sat $(n, 4,4)=18 n-21$ for sufficiently large $n$.

For convenience, we say that a $k$-partite graph with a fixed $k$-partition is $K_{r}$-partite-saturated if it is $K_{r}$-free but the addition of an edge joining any two non-adjacent vertices from different parts creates a $K_{r}$. Therefore, $\operatorname{sat}(n, k, r)$ is the minimum number of edges in a $k$-partite graph $G$ with $n$ vertices in each part which is $K_{r}$-partite-saturated.

Our first result states that $\operatorname{sat}(n, k, r)$ is linear in $n$ where the constant $\alpha(k, r)$ in front
of $n$ is defined as follows. Given $k \geq r \geq 3$, consider a $K_{r}$-partite-saturated $k$-partite graph $G$ containing an independent set $X$ of size $k$ consisting of exactly one vertex from each part of $G$. We define $\alpha(k, r)$ to be the minimum number of edges between $X$ and $X^{c}$ taken over all such $G$ and $X$.

Theorem 2.1. For $k \geq r \geq 3$,

$$
\operatorname{sat}(n, k, r)=\alpha(k, r) n+o(n)
$$

as $n \rightarrow \infty$.

Let us shift our focus to the function $\alpha(k, r)$. The next theorem states what we know about it.

Theorem 2.2. For $k \geq r \geq 3$,
(i) $k(2 r-4) \leq \alpha(k, r) \leq \begin{cases}(k-1)(4 r-k-6) & \text { for } r \leq k \leq 2 r-3, \\ (k-1)(2 r-3) & \text { for } k \geq 2 r-3 .\end{cases}$
(ii) $\alpha(k, r)=k(2 r-4)$ if $\left\{\begin{array}{l}k=2 r-3, \text { or } \\ k \geq 2 r-2 \text { and } r \equiv 0 \bmod 2, \text { or } \\ k \geq 2 r-1 \text { and } r \equiv 2 \bmod 3 .\end{array}\right.$
(iii) $\alpha(k, 3)=3(k-1), \alpha(4,4)=18$ and $33 \leq \alpha(5,5) \leq 36$.
(iv) $\alpha(r, r) \geq r(2 r-4)+1$ for $r \geq 4$.

The bounds in $(i)$, together with Theorem 2.1. imply that $\operatorname{sat}(n, k, r)=O(k r n)$, answering a question of Ferrara, Jacobson, Pfender and Wenger [37]. In (ii), we determine exactly $\alpha(k, r)$ for some values of $r$ and every $k$ large enough, allowing us to disprove a conjecture in [37] which states that $\operatorname{sat}(n, k, r)=(k-1)(2 r-3) n-(2 r-3)(r-1)$ for $k \geq 2 r-3$ and sufficiently large $n$. In (iii), we deal with the cases $r=3,4,5$ which have
not been covered by (ii). Finally, (iv) shows that the lower bound in $(i)$, which is attained for certain values of $r$ and $k$ mentioned in (ii), is not tight when $k=r$.

Theorem 2.1 and Theorem 2.2 imply that $\operatorname{sat}(n, k, r)=k(2 r-4) n+o(n)$ for the values of $k$ and $r$ in $(i i)$. We show that, in this case, the $o(n)$ term can be replaced by a constant.

Theorem 2.3. For $k \geq r \geq 3$,

$$
\operatorname{sat}(n, k, r)=k(2 r-4) n+O(1) \text { if }\left\{\begin{array}{l}
k=2 r-3, \text { or } \\
k \geq 2 r-2 \text { and } r \equiv 0 \quad \bmod 2, \text { or } \\
k \geq 2 r-1 \text { and } r \equiv 2 \quad \bmod 3
\end{array}\right.
$$

as $n \rightarrow \infty$.

Now we give a summary of the values of $\operatorname{sat}(n, k, r)$ in the case $r=3,4,5$ which are immediate consequences of the first three results.

Corollary 2.4. (i) $\operatorname{sat}(n, k, 3)=3(k-1) n+o(n)$ for $k \geq 3$ and as $n \rightarrow \infty$.
(ii) $\operatorname{sat}(n, k, 4)= \begin{cases}18 n+o(n) & \text { for } k=4, \text { as } n \rightarrow \infty, \\ 4 k n+O(1) & \text { for } k \geq 5, \text { as } n \rightarrow \infty\end{cases}$
(iii) sat $(n, k, 5) \begin{cases}\in[33 n+o(n), 36 n+o(n)] & \text { for } k=5, \text { as } n \rightarrow \infty, \\ \in[36 n+o(n), 40 n+o(n)] & \text { for } k=6, \text { as } n \rightarrow \infty, \\ \in[48 n+o(n), 49 n+o(n)] \quad \text { for } k=8, \text { as } n \rightarrow \infty, \\ =6 k n+O(1) \quad \text { for } k=7 \text { or } k \geq 9, \text { as } n \rightarrow \infty .\end{cases}$

We note that $(i)$ and the first half of $(i i)$ are not the best known results. In fact, Ferrara, Jacobson, Pfender and Wenger [37] proved that $\operatorname{sat}(n, k, 3)=3(k-1) n-6$ for sufficiently large $n$ and Roberts [72] proved that $\operatorname{sat}(n, 4,4)=18 n-21$ for sufficiently large $n$.

Let us give some more definitions which will be used throughout the chapter. For a $k$-partite $G=V_{1} \cup V_{2} \cup \cdots \cup V_{k}$, we refer to each $V_{i}$ as a part of $G$. We say that an edge (or a non-edge) $u v$ of a $k$-partite graph is admissible if $u, v$ lie in different parts. We say that a non-edge $u v$ of a $K_{r}$-free graph is $K_{r}$-saturated if adding $u v$ to the graph completes a $K_{r}$. In other words, a $k$-partite graph is $K_{r}$-partite-saturated if it is $K_{r}$-free and every admissible non-edge is $K_{r}$-saturated.

The rest of this chapter is organized as follows. Section 2.2 is devoted to the proof of Theorem 2.1. In Section 2.3, we study the function $\alpha(k, r)$ and prove Theorem 2.2(i). In Section 2.4, we prove Theorem 2.2(ii) by describing constructions matching the lower bound $\alpha(k, r) \geq k(2 r-4)$ in Theorem 2.2(i). We prove Theorem 2.2(iii), Theorem 2.2 (iv) and Theorem 2.3 in Section 2.5, Section 2.6 and Section 2.7 respectively. Finally, we conclude the chapter in Section 2.8 with some open problems.

### 2.2 Proof of Theorem 2.1

First we show that the upper bound follows easily from the definition of $\alpha(k, r)$.

Proposition 2.5. For every $k \geq r \geq 3$ and any integer $n \geq \alpha(k, r)+1$, we have $\operatorname{sat}(n, k, r) \leq \alpha(k, r) n+\alpha(k, r)^{2}$.

Proof. Let $G$ be a $K_{r}$-partite-saturated $k$-partite graph containing an independent set $X$ of size $k$ consisting of exactly one vertex from each part of $G$ with $e\left(X, X^{c}\right)=\alpha(k, r)$. We may assume that $\left|X^{c}\right| \leq \alpha(k, r)$. Indeed, since there are $\alpha(k, r)$ edges between $X$ and $X^{c}$, deleting all the vertices in $X^{c}$ with no neighbors in $X$ leaves at most $\alpha(k, r)$ vertices in $X^{c}$. Note that any admissible non-edge with at least one endpoint in $X$ is still $K_{r}$-saturated. We finish by keeping adding admissible edges inside $X^{c}$ until every admissible non-edge inside $X^{c}$ is $K_{r}$-saturated.

Let $V_{1}, V_{2}, \ldots, V_{k}$ be the parts of $G$. It follows that $\left|V_{i}\right|=\left|V_{i} \cap X\right|+\left|V_{i} \cap X^{c}\right| \leq 1+\alpha(k, r) \leq n$, and so we can modify $G$ to have exactly $n$
vertices in each part by blowing up the vertex of $X$ in $V_{i}$ to a class of size $n-\left|V_{i} \cap X^{c}\right|$ for each $i$. The resulting graph is $K_{r}$-partite-saturated and has exactly $n$ vertices in each of its $k$ parts. Moreover, the number of edges is at most
$\alpha(k, r) n+e\left(G\left[X^{c}\right]\right) \leq \alpha(k, r) n+\alpha(k, r)^{2}$.

Now we prove the lower bound $\operatorname{sat}(n, k, r) \geq \alpha(k, r) n+o(n)$.
Let $\varepsilon>0$ and let $G=V_{1} \cup V_{2} \cup \cdots \cup V_{k}$ be a $K_{r}$-partite-saturated $k$-partite graph with $\left|V_{i}\right|=n$ for all $i \in[k]$. We shall show that $e(G) \geq \alpha(k, r) n-\varepsilon n$ for all sufficiently large $n$. Let $d$ be a large natural number to be chosen later. For each $i$, we partition $V_{i}$ into $V_{i}^{+}=\left\{v \in V_{i}: d(x) \geq d\right\}$ and $V_{i}^{-}=\left\{v \in V_{i}: d(x)<d\right\}$. First we show that $V_{i}^{+}$is small. Since $e(G) \geq \frac{d}{2}\left|V_{i}^{+}\right|$, we are done unless $\left|V_{i}^{+}\right| \leq \frac{2 \alpha(k, r)}{d} n$. Now we show that we can delete a constant number of vertices from $\bigcup_{i=1}^{k} V_{i}^{-}$to make it independent.

Lemma 2.6. There exists a subset $U \subset \bigcup_{i=1}^{k} V_{i}^{-}$of size $C_{k, d}$ such that $\left(\bigcup_{i=1}^{k} V_{i}^{-}\right) \backslash U$ forms an independent set in $G$ for some constant $C_{k, d}$.

Let us first show how to finish the proof of Proposition 2.5 using the lemma. For each $1 \leq i \leq k$, let $v_{i}$ be a vertex of smallest degree in $V_{i}^{-} \backslash U$. Since $G$ is a $K_{r}$-partite-saturated $k$-partite graph and $X=\left\{v_{1}, v_{2}, \ldots, v_{k}\right\}$ is an independent set with exactly one vertex in each part of $G$, we have $\sum_{i=1}^{k} d\left(v_{i}\right) \geq \alpha(k, r)$ by the definition of $\alpha(k, r)$. Since $\left(\bigcup_{i=1}^{k} V_{i}^{-}\right) \backslash U$ forms an independent set,

$$
\begin{aligned}
e(G) & \geq \sum_{i=1}^{k} \sum_{v \in V_{i}^{-} \backslash U} d(v) \geq \sum_{i=1}^{k}\left|V_{i}^{-} \backslash U\right| d\left(v_{i}\right) \geq \sum_{i=1}^{k}\left(n-\left|V_{i}^{+}\right|-|U|\right) d\left(v_{i}\right) \\
& \geq\left(n-\frac{2 \alpha(k, r)}{d} n-C_{k, d}\right) \sum_{i=1}^{k} d\left(v_{i}\right) \geq\left(n-\frac{2 \alpha(k, r)}{d} n-C_{k, d}\right) \alpha(k, r) \\
& =\alpha(k, r) n-\left(\frac{2 \alpha(k, r)^{2}}{d}+\frac{\alpha(k, r) C_{k, d}}{n}\right) n \geq \alpha(k, r) n-\varepsilon n
\end{aligned}
$$

by taking $d$ and $n$ sufficiently large. It remains to prove the lemma.

Proof of Lemma 2.6. It is sufficient to show that any matching between $V_{i}^{-}$and $V_{j}^{-}$has
size less than $4^{d^{2}}$ for all $i \neq j$. Indeed, we can take $U$ to be the endpoints of maximal matchings between $V_{i}^{-}$and $V_{j}^{-}$for all $i \neq j$ and $|U|<4^{d^{2}}\binom{k}{2}$.

Suppose for contradiction that $\left\{x_{1} y_{1}, x_{2} y_{2}, \ldots, x_{4^{d^{2}}} y_{4^{d^{2}}}\right\}$ is a matching of size $4^{d^{2}}$ where $X=\left\{x_{1}, x_{2}, \ldots, x_{4^{d^{2}}}\right\} \subset V_{1}^{-}$and $Y=\left\{y_{1}, y_{2}, \ldots, y_{4^{d^{2}}}\right\} \subset V_{2}^{-}$. The strategy of the proof is to iteratively find vertices $x_{t_{1}}, x_{t_{2}}, \ldots, x_{t_{d}}$ of $X$ such that $d\left(x_{t_{i}}\right) \geq i$ for all $1 \leq i \leq d$, which would contradict the fact that $x_{t_{d}} \in V_{1}^{-}$. In fact, we shall find vertices $x_{t_{1}}, x_{t_{2}}, \ldots, x_{t_{d}}$ of $X$ such that
(i) there exists a common neighbor of $x_{t_{i}}$ and $y_{t_{j}}$ which is not a neighbor of

$$
y_{t_{1}}, y_{t_{2}}, \ldots, y_{t_{j-1}} \text { for all } i>j
$$

Clearly, this implies that $d\left(x_{t_{i}}\right) \geq i$ for all $1 \leq i \leq d$. To find such vertices, it is sufficient to find vertices $x_{t_{1}}, x_{t_{2}}, \ldots, x_{t_{d}}$ of $X$ satisfying
(ii) $x_{t_{i}}$ and $y_{t_{j}}$ are not neighbors for all $i>j$, and
(iii) $N\left(x_{t_{i}}\right) \cap N\left(y_{t_{l}}\right)=N\left(x_{t_{j}}\right) \cap N\left(y_{t_{l}}\right)$ for all $i>j>l$.

First we show that (ii) and (iii) imply (i). Let $i>j$. By (ii), $x_{t_{i}} y_{t_{j}}$ is a non-edge. Since $G$ is $K_{r}$-partite-saturated, there exists a clique $W$ of size $r-2$ in the common neighborhood of $x_{t_{i}}$ and $y_{t_{j}}$. Since $r \geq 3$, we are done by picking a required vertex from $W$ unless each vertex in $W$ is joined to some $y_{t_{l}}$ with $l<j$. In this case, $W \cup\left\{x_{t_{j}}, y_{t_{j}}\right\}$ forms a clique of size $r$, contradicting the fact that $G$ is $K_{r}$-free. Indeed, each $w \in W$ belongs to some $N\left(y_{t_{l}}\right)$ with $l<j$, and since $w \in N\left(x_{t_{i}}\right)$, we must have $w \in N\left(x_{t_{j}}\right)$, by (iii).

Now, we find vertices $x_{t_{1}}, x_{t_{2}}, \ldots, x_{t_{d}}$ of $X$ satisfying (ii) and (iii). To help us do so, we shall iteratively construct a nested sequence of sets $X \supset X_{1} \supset X_{2} \supset \cdots \supset X_{d}$ with $x_{t_{i}} \in X_{i}$ for all $2 \leq i \leq d$, satisfying
(iv) $x$ and $y_{t_{i-1}}$ are not neighbors for all $x \in X_{i}$, and
(v) $N(x) \cap N\left(y_{t_{i-1}}\right)=N\left(x^{\prime}\right) \cap N\left(y_{t_{i-1}}\right)$ for all $x, x^{\prime} \in X_{i}$.

Clearly, such vertices $x_{t_{1}}, x_{t_{2}}, \ldots, x_{t_{d}}$ satisfy (ii) and (iii). Start with $x_{t_{1}}=x_{1}$ and $X_{1}=X$. Let $i \leq d$ and suppose that we have found vertices $x_{t_{1}}, x_{t_{2}}, \ldots, x_{t_{i-1}}$ and sets $X_{1} \supset X_{2} \supset \cdots \supset X_{i-1}$ with $x_{t_{j}} \in X_{j}$ for all $j<i$, satisfying (iv) and (v). We delete the neighbors of $y_{t i-1}$ from $X_{i-1}$ and partition the remaining vertices into $2^{d\left(y_{t-1}\right)} \leq 2^{d}$ subsets according to their common neighborhood with $y_{t_{i-1}}$. In other words, $X_{i-1} \backslash N\left(y_{t_{i-1}}\right)$ is partitioned into subsets $\left\{x: N(x) \cap N\left(y_{t_{i-1}}\right)=S\right\}$ for $S \subset N\left(y_{t_{i-1}}\right)$. We choose $X_{i}$ to be such subset of maximum size, i.e. $\left|X_{i}\right| \geq \frac{\left|X_{i-1}\right|-d}{2^{d}}$. Clearly, $X_{i}$ satisfies (iv) and (v). We then choose $x_{t_{i}}$ be any vertex in $X_{i}$. It remains to prove that $\left|X_{i}\right|>0$. Recall that $\left|X_{1}\right|=|X|=4^{d^{2}}$, and we can see, by induction, that $\left|X_{i}\right| \geq 4^{d(d-i)}$ for $i \leq d$. Indeed,

$$
\left|X_{i}\right| \geq \frac{\left|X_{i-1}\right|-d}{2^{d}} \geq \frac{\left|X_{i-1}\right|}{4^{d}} \geq \frac{4^{d(d-i+1)}}{4^{d}} \geq 4^{d(d-i)}
$$

as required.

### 2.3 Bounding $\alpha(k, r)$

In this section, we establish a number of results that will help us prove Theorem 2.2. We shall deduce Theorem $2.2(i)$ at the end of the section.

For $k \geq r \geq 2$ and $1 \leq i \leq k-r+1$, let $\beta_{i}(k, r)$ be the minimum number of vertices in a $K_{r}$-free $k$-partite graph such that the subgraph induced by any $k-i$ parts contains a $K_{r-1}$, i.e. the deletion of any $i$ parts does not destroy all the $K_{r-1}$.

We observe that $\beta_{1}$ and $\beta_{2}$ are useful for bounding $\alpha$.

Proposition 2.7. For $k \geq r \geq 3$,

$$
k \beta_{1}(k-1, r-1) \leq \alpha(k, r) \leq(k-1) \beta_{2}(k, r-1)
$$

Proof. To prove the lower bound, let $G$ be a $K_{r}$-partite-saturated $k$-partite graph containing an independent set $X$ of size $k$ consisting of exactly one vertex from each part of $G$. We
shall show that $e\left(X, X^{c}\right) \geq k \beta_{1}(k-1, r-1)$. It is sufficient to show that each vertex in $X$ has degree at least $\beta_{1}(k-1, r-1)$. Let $x \in X$ and consider the $(k-1)$-partite graph $H=G[N(x)]$. Clearly, it is $K_{r-1}$-free since $G$ is $K_{r}$-free. It remains to show that, for each part $U$ of $H, H \backslash U$ contains a $K_{r-2}$. If $x^{\prime}$ is a vertex of $X$ in the corresponding part of $U$ in $G$ then, since the non-edge $x x^{\prime}$ is $K_{r}$-saturated in $G, H \backslash U$ must contain a $K_{r-2}$. Hence, $|N(x)|=|H| \geq \beta_{1}(k-1, r-1)$.

For the upper bound, let $G_{1}$ be a $K_{r-1}$-free $k$-partite graph on $\beta_{2}(k, r-1)$ vertices such that the subgraph induced by any $k-2$ parts contains a $K_{r-2}$. Let $G_{2}$ be the graph obtained from $G_{1}$ by adding one vertex of $X=\left\{x_{1}, x_{2}, \ldots, x_{k}\right\}$ to each part of $G_{1}$ and joining each $x_{i}$ to every vertex of $G_{1}$ outside its part. By construction, $X$ forms an independent set and $e\left(X, X^{c}\right)=(k-1) \beta_{2}(k, r-1)$ edges. Note that $G_{2}$ is $K_{r}$-free since a clique in $G_{2}$ contains at most one vertex from $X$ and $G_{1}$ is $K_{r-1}$-free. Now, let $G$ be the graph obtained from $G_{2}$ by adding admissible edges inside $X^{c}$, until every admissible non-edge inside $X^{c}$ is $K_{r}$-saturated. To conclude that $G$ is $K_{r}$-partite-saturated, we need to show that every admissible non-edge inside $X$ is $K_{r}$-saturated. Note that, for every pair of distinct vertices $x, x^{\prime} \in X, G_{1}$ contains a $K_{r-2}$ not using vertices from the parts containing $x$ and $x^{\prime}$. Since $x$ and $x^{\prime}$ are joined to every vertex outside their parts, the addition of the edge $x x^{\prime}$ completes a $K_{r}$. Hence, $\alpha(k, r) \leq e\left(X, X^{c}\right)=(k-1) \beta_{2}(k, r-1)$.

In the next sections, the argument above used in the proof of the lower bound will be used several times. Let us state it as a lemma.

Lemma 2.8. Let $G$ be a $k$-partite $K_{r}$-free graph containing an independent set $X$ of size $k$ consisting of exactly one vertex from each part of $G$ such that the non-edges inside $X$ are $K_{r}$-saturated. Then, for each $x \in X, G[N(x)]$ is a $K_{r-1}$-free $(k-1)$-partite graph such that the subgraph induced by any $k-2$ parts contains a $K_{r-2}$. In particular, $d(x) \geq \beta_{1}(k-1, r-1)$ for all $x \in X$.

In the next two subsections, we shall bound $\beta_{1}$ from below and $\beta_{2}$ from above.

### 2.3.1 Upper bounds for $\beta_{i}$

We start with an easy observation which helps us bound $\beta_{i}$ from above.

Lemma 2.9. For $k \geq r \geq 3$ and $1 \leq i \leq k-r+1, \beta_{i}(k, r) \leq \beta_{i}(k-1, r-1)+i+1$.

Proof. Let $H=U_{1} \cup U_{2} \cup \cdots \cup U_{k-1}$ be a $K_{r-1}$-free $(k-1)$-partite graph on $\beta_{i}(k-1, r-1)$ vertices such that the subgraph induced by any $k-i-1$ parts contains a $K_{r-2}$. We shall construct a $K_{r}$-free $k$-partite graph $G=V_{1} \cup V_{2} \cup \cdots \cup V_{k}$ from $H$ with $|G|=|H|+(i+1)$ as follows. First, add new vertices $v_{1}$ to $U_{1}, v_{2}$ to $U_{2}, \ldots, v_{i}$ to $U_{i}$ and $v_{i+1}$ to the new part $V_{k}$. This is possible since $k \geq i+2$. Now, join $v_{i+1}$ to every vertex in $H$ and, for every $1 \leq j \leq i$, join $v_{j}$ to every vertex in $H \backslash U_{j}$. Clearly, $G$ is $K_{r}$-free since $H$ is $K_{r-1}$-free.

Let $\mathscr{C}$ be a collection of $k-i$ parts of $G$. It remains to check that the subgraph of $G$ induced by $\mathscr{C}$ contains a $K_{r-1}$. First, suppose that $V_{k} \in \mathscr{C}$. By the induction hypothesis, the other $(k-1)-i$ parts $\mathscr{C} \backslash\left\{V_{k}\right\}$ induce a subgraph of $H$ containing a $K_{r-2}$. Together with $v_{i+1} \in V_{k}$, they form a $K_{r-1}$ in the subgraph of $G$ induced by $\mathscr{C}$ as required. Now, let us suppose that $V_{k} \notin \mathscr{C}$. Then $\mathscr{C}$ must contain at least one of $V_{1}, V_{2}, \ldots, V_{i}$. Without loss of generality, we may assume that $\mathscr{C}$ contains $V_{1}$. By the induction hypothesis, the other $(k-1)-i$ parts $\mathscr{C} \backslash\left\{V_{1}\right\}$ induce a subgraph of $H$ containing a $K_{r-2}$. Together with $v_{1} \in V_{1}$, they form a $K_{r-1}$ in the subgraph of $G$ induced by $\mathscr{C}$ as required.

Lemma 2.9 immediately implies the following upper bound on $\beta_{i}$.
Corollary 2.10. $\beta_{i}(k, r) \leq(i+1)(r-1)$ for $k \geq r \geq 2$ and $1 \leq i \leq k-r+1$.
Proof. It is clear that $\beta_{i}(k, 2)=i+1$ for $k \geq i+1$ by considering the empty graph on $i+1$ vertices where each vertex is in a different part and the remaining $k-i-1$ parts are empty.

By induction on $r$ and applying Lemma 2.9 .
$\beta_{i}(k, r) \leq \beta_{i}(k-1, r-1)+i+1 \leq(i+1)(r-2)+i+1=(i+1)(r-1)$ as required.

We remark that there is a straightforward construction proving Corollary 2.10 for the case $k \geq(i+1)(r-1)$, namely, a disjoint union of $i+1$ cliques of size $r-1$ where each
vertex is in a different part and the remaining $k-(i+1)(r-1)$ parts are empty. Clearly, the deletion of any $i$ parts does not destroy all the $K_{r-1}$.

Now we prove a better upper bound for $\beta_{i}(k, r)$ in the case when $i \geq 2$ and $k \geq i(r-1)+1$ by considering the $(r-2)$ th power of the cycle $C_{i(r-1)+1}$.

Proposition 2.11. $\beta_{i}(k, r) \leq i(r-1)+1$ for $k \geq i(r-1)+1$ and $r, i \geq 2$.

Proof. Since $\beta_{i}(k, r)$ is decreasing in $k$ (by adding empty parts), it is enough to show that $\beta_{i}(k, r) \leq i(r-1)+1$ for $k=i(r-1)+1$. Let $G$ be the $(r-2)$ th power of the cycle $C_{i(r-1)+1}$, i.e. $G$ is a graph on $\mathbb{Z}_{i(r-1)+1}$ where $u, v$ are neighbors if $u-v=1,2, \ldots, r-2$. We view $G$ as a $(i(r-1)+1)$-partite graph with one vertex in each part. Clearly, $G$ is $K_{r}$-free if $i \geq 2$. Note that, after deleting any $i$ vertices of $G$, there are at least $r-1$ consecutive vertices remaining in $\mathbb{Z}_{i(r-1)+1}$, which form a $K_{r-1}$ as required.

Proposition 2.11 together with Lemma 2.9 imply a better upper bound than that in Corollary 2.10 for $\beta_{2}(k, r)$ in the remaining cases, i.e when $k<2 r-1$.

Proposition 2.12. $\beta_{2}(k, r) \leq 4 r-k-2$ for $2 \leq r<k \leq 2 r-1$.

Proof. We proceed by induction on $2 r-k$. The base case when $2 r-k=1$ follows from Proposition 2.11. Now, suppose that $2 r-k \geq 2$. Applying Lemma 2.9.

$$
\beta_{2}(k, r) \leq \beta_{2}(k-1, r-1)+3 \leq(4(r-1)-(k-1)-2)+3=4 r-k-2,
$$

by the induction hypothesis, since $2 r-k>2(r-1)-(k-1) \geq 1$,

Let us remark that a similar upper bound for general $\beta_{i}$ can be obtained by the same method. We believe that the bound in Proposition 2.12 is, in fact, an equality.

Conjecture 2.13. $\beta_{2}(k, r)=4 r-k-2$ for $2 \leq r<k<2 r-1$.

For the remaining values of $k$, we shall see in the next subsection that $\beta_{2}(k, r)=2 r-1$ for $k \geq 2 r-1$.

### 2.3.2 Determining $\beta_{1}$

We shall show that the upper bound for $\beta_{1}$ given by Corollary 2.10 is an equality. Recall that the clique number of a graph is the order of a maximum clique.

Proposition 2.14. $\beta_{1}(k, r)=2(r-1)$ for $k \geq r \geq 2$.

The lower bound is a consequence of the following observation which we will prove below.

Proposition 2.15. Let $G$ be a graph on at most $2 s-1$ vertices with clique number $s$. Then there is a vertex which lies in every $K_{S}$ of $G$.

Proof of Proposition 2.14. The upper bound follows from Corollary 2.10. To prove the lower bound, suppose for contradiction that $G$ is a $K_{r}$-free $k$-partite graph on at most $2 r-3$ vertices such that the subgraph induced by any $k-1$ parts contains a $K_{r-1}$. Applying Proposition 2.15 with $s=r-1$, there is a vertex $v$ which lies in every $K_{r-1}$. In particular, the deletion of the part containing $v$ destroys all the $K_{r-1}$. Hence, $\beta_{1}(k, r) \geq 2 r-2$.

Let us remark that Proposition 2.15 is a consequence of the clique collection lemma of Hajnal [50] which states that the sum of the number of vertices in the union and the intersection of a collection of maximum cliques is at least twice the clique number. Our argument below can also be used to give a new proof of Hajnal's clique collection lemma.

Proof of Proposition 2.15. Let $V_{1}, V_{2}, \ldots, V_{m} \subset V(G)$ be the vertex sets of the copies of $K_{s}$ in $G$. For a vertex $v \in V(G)$, let $I_{v}=\left\{i \in[m]: v \in V_{i}\right\}$ be the set of $K_{s}$ containing $v$. For a collection $\mathscr{C} \subset \mathscr{P}([m])$ of subsets of $[m]$, let $V_{\mathscr{C}}=\left\{v \in V(G): I_{v} \in \mathscr{C}\right\}$. Observe that if $\mathscr{C} \subset \mathscr{P}([m])$ is intersecting then $V_{\mathscr{C}}$ induces a clique in $G$. Indeed, $u, v \in V_{\mathscr{C}}$ are neighbors since $I_{u} \cap I_{v} \neq \emptyset$, i.e. there is a clique containing both $u$ and $v$. Therefore, $\left|V_{\mathscr{C}}\right| \leq s$ since $G$ is $K_{s+1}$-free. The following lemma implies the result.

Lemma 2.16. For $m \geq 3$, there exist intersecting families $\mathscr{C}_{1}, \mathscr{C}_{2}, \ldots \mathscr{C}_{m-2} \subset \mathscr{P}([m])$ such that, for $I \subset[m]$, the number of $\mathscr{C}_{j}$ containing $I$ is $\begin{cases}0 & \text { if } I=\emptyset \\ |I|-1 & \text { if } I \neq \emptyset,[m] \\ m-2 & \text { if } I=[m] .\end{cases}$

Proof. The proof is by induction on $m$. For $m=3, \mathscr{C}_{1}=\{\{1,2\},\{2,3\},\{3,1\},\{1,2,3\}\}$ satisfies the required property. For $m \geq 4$, suppose by induction that there exist intersecting families $\mathscr{C}_{1}, \mathscr{C}_{2}, \ldots, \mathscr{C}_{m-3} \subset \mathscr{P}([m-1])$ satisfying the property. We define $\mathscr{D}_{1}, \mathscr{D}_{2}, \ldots, \mathscr{D}_{m-2} \subset \mathscr{P}([m])$ as follows. For $1 \leq j \leq m-3$, let

$$
\mathscr{D}_{j}=\mathscr{C}_{j} \cup\left\{I \cup\{m\}: I \in \mathscr{C}_{j}\right\}
$$

and

$$
\mathscr{D}_{m-2}=\{I \subset[m]: m \in I \text { and }|I| \geq 2\} \cup\{[m-1]\} .
$$

It is easy to check that $\mathscr{D}_{1}, \mathscr{D}_{2}, \ldots, \mathscr{D}_{m-2}$ satisfy the required property.

Let us deduce the result. This is trivial when $m=1,2$ so we may assume that $m \geq 3$. Observe that

$$
\sum_{i=1}^{m}\left|V_{i}\right|=\left|\bigcup_{i=1}^{m} V_{i}\right|+\sum_{j=1}^{m-2}\left|V_{\mathscr{C}_{j}}\right|+\left|\bigcap_{i=1}^{m} V_{i}\right| .
$$

Indeed, a vertex $v$ is counted on both sides $\left|I_{v}\right|$ times by the lemma. Using $\left|V_{i}\right|=s$, $\left|\bigcup_{i=1}^{m} V_{i}\right| \leq 2 s-1$ and $\left|V_{\mathscr{C}_{j}}\right| \leq s$, we have

$$
m s \leq(2 s-1)+(m-2) s+\left|\bigcap_{i=1}^{m} V_{i}\right|
$$

i.e. $\left|\bigcap_{i=1}^{m} V_{i}\right| \geq 1$ as required.

We remark that the fact that $\beta_{1}(k, r)=2(r-1)$ allows us to show that the upper bound for $\beta_{2}(k, r)$ when $k \geq 2 r-1$ in Proposition 2.11 is an equality.

Corollary 2.17. $\beta_{2}(k, r)=2 r-1$ for $k \geq 2 r-1$ and $r \geq 2$.

Proof. Observe that $\beta_{i}(k, r) \geq \beta_{i-1}(k-1, r)+1$. Indeed, if $G$ is a $K_{r}$-free $k$-partite graph on $\beta_{i}(k, r)$ vertices such that the subgraph induced by any $k-i$ parts contains a $K_{r-1}$, then, by deleting a non-empty part of $G$, we obtain a $K_{r}$-free $(k-1)$-partite graph such that the subgraph induced by any $(k-1)-(i-1)$ parts contains a $K_{r-1}$. This graph must contains at least $\beta_{i-1}(k-1, r)$ vertices and therefore, $|G|-1 \geq \beta_{i-1}(k-1, r)$.

Hence, $\beta_{2}(k, r) \geq \beta_{1}(k-1, r)+1=2(r-1)+1=2 r-1$ by Proposition 2.14 .

### 2.3.3 Proof of Theorem 2.2 (i)

The lower bound follows from Proposition 2.7 and Proposition 2.14. The upper bound follows from Proposition 2.7, Proposition 2.12 and Corollary 2.17 .

### 2.4 Proof of Theorem 2.2 (ii)

For $k=2 r-3$, we are done since the lower and upper bounds in Theorem 2.2(i) match, i.e. $\alpha(k, r)=k(2 r-4)=(k-1)(2 r-3)$.

Now we shall describe constructions that match the lower bound $\alpha(k, r) \geq k(2 r-4)$ in Theorem 2.2(i) for the cases when ( $k \geq 2 r-2$ and $r$ is even) and ( $k \geq 2 r-1$ and $r=2$ $\bmod 3)$, i.e. a $K_{r}$-partite-saturated $k$-partite graph $G$ containing an independent set $X$ of size $k$ consisting of exactly one vertex from each part of $G$ with $e\left(X, X^{c}\right)=k(2 r-4)$. Lemma 2.8 tells us that such graph must satisfy $d(x)=2 r-4$, for all $x \in X$.

Note that we do not have to worry about making the admissible non-edges inside $X^{c}$, $K_{r}$-saturated since we can keep adding admissible edges inside $X^{c}$ until every admissible non-edge inside $X^{c}$ is $K_{r}$-saturated.

Let $p \in\{2,3\}$ be a divisor of $r-2$. First we shall construct such $k$-partite graph $G$, for $k=2 r-4+p$. We define $X=\left\{x_{1}, x_{2}, \ldots, x_{k}\right\}$ and $X^{c}=\left\{y_{1}, y_{2}, \ldots, y_{k}\right\}$, where the parts of $G$ are $\left\{x_{i}, y_{i}\right\}$, for $i=1,2, \ldots, k$. There are no edges inside $X$. Let $y_{i} y_{j}$ be an edge iff $i, j$ are not consecutive elements of the circle $\mathbb{Z}_{k}$, and so $G\left[X^{c}\right]$ is the graph $K_{k}$ minus a
cycle $C_{k}$. Let $x_{i} y_{j}$ is an edge iff $i \neq j \bmod \frac{k}{p}$, i.e. $x_{i}$ is joined to all but $p$ equally spaced $y_{j}$. We claim that $G$ satisfies the required properties.

Clearly, we have $d(x)=k-p=2 r-4$ for all $x \in X$ and $e\left(X, X^{c}\right)=k(2 r-4)$. Let us verify that $G$ is $K_{r}$-free. A clique inside $X^{c}$ is a set of non-consecutive elements of $\mathbb{Z}_{k}$, and so a largest clique inside $X^{c}$ has size $\left\lfloor\frac{k}{2}\right\rfloor=r-1$ for $p \in\{2,3\}$. Since a clique which is not inside $X^{c}$ can contain at most one vertex of $X$, it remains to check that the neighborhood of each $x_{i}$ does not contain a clique of size $r-1$. Viewing $X^{c}$ as a circle, $N\left(x_{i}\right)$ consists of $p$ segments of the circle, each of size $\frac{2 r-4}{p}$, separated by gaps of size one. Since $\frac{2 r-4}{p}$ is even, a largest clique in $N\left(x_{i}\right)$ has size $\frac{p(2 r-4)}{2 p}=r-2$.

It remains to show that the admissible non-edges inside $X$, and those between $X$ and $X^{c}$ are $K_{r}$-saturated. Let $x_{i} y_{j}$ be an admissible non-edge, and so $j=i \pm \frac{k}{p}$ in $\mathbb{Z}_{k}$. Clearly, $N\left(x_{i}\right)$ contains $r-2$ vertices which form a non-consecutive set of the circle with $y_{j}$. Therefore, there exists a $K_{r-2}$ in the common neighborhood of $x_{i}$ and $y_{j}$ as required. Now let $x_{i} x_{j}$ be an admissible non-edge. Then the common neighborhood of $x_{i}$ and $x_{j}$ consists of $2 p$ segments of the circle separated by gaps of size one such that they form $p$ pairs where the sum of the sizes of each pair is $\frac{2 r-4}{p}-1$, and so each pair consists of a segment of even size and a segment of odd size. Therefore, a largest non-consecutive set in $N\left(x_{i}\right) \cap N\left(x_{j}\right)$ has size $\frac{p(2 r-4)}{2 p}=r-2$. Hence, there exists a $K_{r-2}$ in $N\left(x_{i}\right) \cap N\left(x_{j}\right)$ as required.

We have constructed such $k$-partite graph $G_{k}$ for $k=2 r-4+p$. Let us obtain $G_{k}$ for $k>2 r-4+p$ from $G_{2 r-4+p}$ by blowing up $x_{1}$ to a class $\left\{x_{1}\right\} \cup\left\{x_{i}: 2 r-3+p \leq i \leq k\right\}$ of size $k-(2 r-4+p)+1$ where each copy of $x_{1}$ (not including itself) forms a part of $G_{k}$ of size one. Clearly, we have $d(x)=2 r-4$ for all $x \in X=\left\{x_{1}, x_{2}, \ldots, x_{k}\right\}$ and $e\left(X, X^{c}\right)=k(2 r-4)$. Since $G_{2 r-4+p}$ is $K_{r}$-free, so is $G_{k}$.

It remains to check that the admissible non-edges inside $X$, and those between $X$ and $X^{c}$ are $K_{r}$-saturated. Any admissible non-edge inside $X$ which is not inside the blow up class of $x_{1}$ is $K_{r}$-saturated by the same property of $G_{2 r-4+p}$. Any admissible non-edge
inside the blow up class of $x_{1}$ is $K_{r}$-saturated since $N\left(x_{1}\right)$ contains a $K_{r-2}$ by the construction of $G_{2 r-4+p}$. Any admissible non-edge $x_{i} y_{j}$ where $j \neq 1$ or $(j=1$ and $i \leq 2 r-4+p)$, is $K_{r}$-saturated by the same property of $G_{2 r-4+p}$. Any admissible non-edge $x_{i} y_{j}$ where $j=1$ and $2 r-3+p \leq i \leq k$, is $K_{r}$-saturated since $N\left(x_{1}\right) \cap N\left(y_{1}\right)$ contains a $K_{r-2}$ by the construction of $G_{2 r-4+p}$.

### 2.5 Proof of Theorem 2.2 (iii)

In this section, we study $\alpha(k, r)$ for $r=3,4,5$. The values of $\alpha(k, 3)$ and $\alpha(k, 4)$ are completely determined while the values of $\alpha(k, 5)$ are unknown for $k=5,6,8$.

### 2.5.1 The function $\alpha(k, 3)$

We shall prove that $\alpha(k, 3)=3(k-1)$ for $k \geq 3$. The upper bound follows from Theorem 2.2(i). Let us prove the lower bound.

Let $G=V_{1} \cup V_{2} \cup \cdots \cup V_{k}$ be a $K_{3}$-partite-saturated $k$-partite graph $G$ containing an independent set $X=\left\{x_{1}, x_{2}, \ldots, x_{k}\right\}$ with $x_{i} \in V_{i}$ for all $i$. By Lemma 2.8, for all $i$, the deletion of any part of $G$ does not destroy all vertices of $N\left(x_{i}\right)$, i.e. $x_{i}$ is joined to at least two parts of $G$. Suppose for contradiction that $e\left(X, X^{c}\right)<3(k-1)$, i.e. $X$ contains at least four vertices of degree 2 , say $x_{1}, x_{2}, x_{3}, x_{4}$. Let $y_{i} \in V_{i}$ and $y_{j} \in V_{j}$ with $1<i<j \leq k$ be the neighbors of $x_{1}$, and so $y_{i}$ and $y_{j}$ are not neighbors otherwise $x_{1} y_{i} y_{j}$ forms a triangle. Since $\{2,3,4\} \backslash\{i, j\} \neq \emptyset$, we may assume that $i, j \neq 2$, i.e. $x_{1}, x_{2}, y_{i}, y_{j}$ are from different parts of $G$. Since any pair in $X$ forms a $K_{3}$-saturated non-edge in $G$, they have a common neighbor. So $x_{1}$ and $x_{2}$ have a common neighbor, say $y_{i}$.

First we suppose that $x_{2} y_{j}$ is a non-edge. Then $x_{2}$ and $y_{j}$ have a common neighbor $y_{l} \in V_{l}$. Since $y_{i}$ and $y_{j}$ are not neighbors, $l \neq i$. We observe that $x_{i} y_{j}$ are neighbors since $x_{1}$ and $x_{i}$ have a common neighbor and $N\left(x_{1}\right)=\left\{y_{i}, y_{j}\right\}$. Similarly, $x_{i} y_{l}$ are neighbors since $x_{2}$ and $x_{i}$ have a common neighbor and $N\left(x_{2}\right)=\left\{y_{i}, y_{l}\right\}$. We obtain a contradiction by observing that $x_{i} y_{j} y_{l}$ forms a triangle.

Now, suppose that $x_{2} y_{j}$ is an edge, and so $N\left(x_{1}\right)=N\left(x_{2}\right)=\left\{y_{i}, y_{j}\right\}$. Then $x_{i} y_{j}$ are neighbors since $x_{1}$ and $x_{i}$ have a common neighbor. Similarly, $x_{j} y_{i}$ are neighbors. We know that $x_{i}$ and $x_{j}$ have a common neighbor $y_{l}$ with $l \neq i, j$. Then either $l \neq 1$ or $l \neq 2$, say $l \neq 1$. Since the non-edge $x_{1} y_{l}$ is $K_{3}$-saturated, $y_{l}$ is joined to either $y_{i}$ or $y_{j}$. This implies a contradiction that either $x_{j} y_{i} y_{l}$ or $x_{i} y_{j} y_{l}$ forms a triangle.

### 2.5.2 The function $\alpha(k, 4)$

As a consequence of Theorem 2.2 (ii), we obtain that $\alpha(k, 4)=4 k$ for $k \geq 5$. For the remaining case $k=4$, we have the bounds $16 \leq \alpha(4,4) \leq 18$ from Theorem 2.2(i). We shall show that $\alpha(4,4)=18$.

Consider the family of graphs appearing in the definition of $\alpha(r, r)$. Let $G=V_{1} \cup V_{2} \cup \cdots \cup V_{r}$ be an $K_{r}$-partite-saturated $r$-partite graph $G$ containing an independent set $X=\left\{x_{1}, x_{2}, \ldots, x_{r}\right\}$ with $x_{i} \in V_{i}$ for all $i$. We shall establish some properties of $G$ which will be useful in this subsection, the next subsection and Section 2.6

We say that a vertex $y \in X^{c}$ is $i$-special if $y$ is the only neighbor of $x_{i}$ in the part of $G$ containing $y$. The special degree of a vertex $y \in X^{c}$ is the number of $i \in[r]$ such that $y$ is $i$-special. We say that a vertex $y \in X^{c}$ is special if the special degree of $y$ is at least one. Let us make some easy observations regarding the special vertices.

Lemma 2.18. Let $G=V_{1} \cup V_{2} \cup \cdots \cup V_{r}$ be an $K_{r}$-partite-saturated $r$-partite graph $G$ containing an independent set $X=\left\{x_{1}, x_{2}, \ldots, x_{r}\right\}$ with $x_{i} \in V_{i}$ for all $i$. The following hold for $r \geq 4$.
(i) A special vertex $y_{i} \in V_{i}$ is joined to every vertex of $X$ except $x_{i}$.
(ii) Each $V_{i}$ contains at most one special vertex.
(iii) If $y_{i} \in V_{i}$ is $i^{\prime}$-special and $y_{j} \in V_{j}$ is $j^{\prime}$-special with $i^{\prime} \neq j$ and $j^{\prime} \neq i$ then $y_{i} y_{j}$ is an edge.
(iv) The number of vertices of special degree at least 2 is at most $r-2$.
(v) If $y_{i} \in V_{i}$ is $i^{\prime}$-special and $y_{j} \in V_{j}$ with $j \neq i, i^{\prime}$ then $y_{j}$ is joined to either $y_{i}$ or $x_{i^{\prime}}$.
(vi) For a special vertex $y_{i} \in V_{i}$, there exist parts $V_{j}$ and $V_{l}$ where $i, j, l$ are distinct such that $N\left(x_{i}\right) \cap V_{j}$ and $N\left(x_{i}\right) \cap V_{l}$ both contain a non-neighbor of $y_{i}$.

Proof. (i) Let $y_{i} \in V_{i}$ be $i^{\prime}$-special and let $j \neq i, i^{\prime}$. Since the non-edge $x_{i^{\prime}} x_{j}$ is $K_{r}$-saturated, the common neighborhood of $x_{i^{\prime}}$ and $x_{j}$ contains a $K_{r-2}$ consisting of one vertex from each part of $G \backslash\left(V_{i^{\prime}} \cup V_{j}\right)$. Then $y_{i}$ is in this $K_{r-2}$ since $y_{i}$ is the only neighbor of $x_{i^{\prime}}$ in $V_{i}$, and so $y_{i}$ is joined to $x_{j}$.
(ii) Suppose for contradiction that $V_{i}$ contains two special vertices $y_{i}$ and $z_{i}$ where $y_{i}$ is $i^{\prime}$-special. Then, by $(i), x_{i^{\prime}}$ is joined to both $y_{i}$ and $z_{i}$ contradicting the fact that $y_{i}$ is the only neighbor of $x_{i^{\prime}}$ in $V_{i}$.
(iii) First, suppose that $i^{\prime} \neq j^{\prime}$. Since the non-edge $x_{i^{\prime}} x_{j^{\prime}}$ is $K_{r}$-saturated, the common neighborhood of $x_{i^{\prime}}$ and $x_{j^{\prime}}$ contains a $K_{r-2}$ consisting of one vertex from each part $G \backslash\left(V_{i^{\prime}} \cup V_{j^{\prime}}\right)$. Since $y_{i}$ is the only neighbor of $x_{i^{\prime}}$ in $V_{i}$ and $y_{j}$ is the only neighbor of $x_{j^{\prime}}$ in $V_{j}$, both $y_{i}$ and $y_{j}$ lie in this $K_{r-2}$. In particular, $y_{i} y_{j}$ is an edge.

Now, suppose that $i^{\prime}=j^{\prime}$. We can pick $l \neq i, j, i^{\prime}$ because $r \geq 4$. Since the non-edge $x_{i^{\prime}} x_{l}$ is $K_{r}$-saturated, the common neighborhood of $x_{i^{\prime}}$ and $x_{l}$ contains a $K_{r-2}$ consisting of one vertex from each part of $G \backslash\left(V_{i^{\prime}} \cup V_{l}\right)$. Since $y_{i}$ is the only neighbor of $x_{i^{\prime}}$ in $V_{i}$ and $y_{j}$ is the only neighbor of $x_{i^{\prime}}$ in $V_{j}$, both $y_{i}$ and $y_{j}$ lie in this $K_{r-2}$. In particular, $y_{i} y_{j}$ is an edge.
(iv) Suppose for contradiction that there exist vertices $y_{1}, y_{2}, \ldots, y_{r-1}$ of special degree at least 2 . By (ii), they lie in different parts of $G$, say $y_{i} \in V_{i}$ for $1 \leq i \leq r-1$. We claim that they form a $K_{r-1}$ which would be a contradiction since, together with $x_{r}$, they form a $K_{r}$ by $(i)$. Now we show that any $y_{i} y_{j}$ is an edge. Since $y_{i}$ and $y_{j}$ have special degree at least 2 , there exist $i^{\prime} \neq j$ and $j^{\prime} \neq i$ such that $y_{i}$ is $i^{\prime}$-special and $y_{j}$ is $j^{\prime}$-special. Therefore, $y_{i} y_{j}$ is an edge by $(i i i)$.
(v) Suppose that $x_{i^{\prime}} y_{j}$ is a non-edge. Then the common neighborhood of $x_{i^{\prime}}$ and $y_{j}$
contains a $K_{r-2}$ consisting of one vertex from each part of $G \backslash\left(V_{i^{\prime}} \cup V_{j}\right)$. Then $y_{i}$ is in this $K_{r-2}$ since $y_{i}$ is the only neighbor of $x_{i^{\prime}}$ in $V_{i}$, and so $y_{i}$ is joined to $y_{j}$.
(vi) Suppose for contradiction that there exists $j \in[r] \backslash\{i\}$ such that $y_{i} \in V_{i}$ is joined to every vertex in $N\left(x_{i}\right) \cap V_{l}$ for all $l \neq i, j$. Since the non-edge $x_{i} x_{j}$ is $K_{r}$-saturated, the common neighborhood of $x_{i}$ and $x_{j}$ contains a $K_{r-2}$ consisting of one vertex from each part of $(G \backslash X) \backslash\left(V_{i} \cup V_{j}\right)$. We obtain a contradiction by observing that this $K_{r-2}$, together with $x_{j}$ and $y_{i}$, form a $K_{r}$. Indeed, by assumption, this $K_{r-2}$ is also in the neighborhood of $y_{i}$ and $x_{j} y_{i}$ is an edge by $(i)$.

Now we are ready to show that $\alpha(4,4) \geq 18$. Suppose for contradiction that $\alpha(4,4) \leq 17$, i.e. there exists a $K_{4}$-partite-saturated 4-partite graph $G=V_{1} \cup V_{2} \cup V_{3} \cup V_{4}$ containing an independent set $X=\left\{x_{1}, x_{2}, x_{3}, x_{4}\right\}$ with $x_{i} \in V_{i}$ for all $i$ such that $\sum_{i=1}^{4} d\left(x_{i}\right) \leq 17$. By Lemma 2.8, $d\left(x_{i}\right) \geq \beta_{1}(3,3)=4$ and each $x_{i}$ has some neighbor in $V_{j}$ for $j \neq i$. Therefore, there are at least three vertices of degree 4 and possibly one of degree 5. Since a vertex of degree 4 in $X$ creates at least two special vertices and a vertex of degree 5 in $X$ creates at least one special vertex, the sum of the special degrees of the vertices in $X^{c}$ is at least $2+2+2+1=7$. By Lemma 2.18(iv), there is a vertex of special degree 3 , say $y_{1} \in V_{1}$.

For $i=2,3,4$, since $y_{1}$ is $i$-special, $x_{i}$ has at least three neighbors in $N\left(y_{1}\right) \cup\left\{y_{1}\right\}$, each in a different part of $G$, by Lemma 2.8. On the other hand, $y_{1}$ has at least two non-neighbors, say $y_{2} \in V_{2}$ and $y_{3} \in V_{3}$, by Lemma 2.18(vi). By Lemma 2.18(v), $x_{i} y_{2}$ is an edge for $i \neq 2$ and $x_{i} y_{3}$ is an edge for $i \neq 3$. So $x_{4}$ has five neighbors, i.e. $y_{2}, y_{3}$ and three vertices in $N\left(y_{1}\right) \cup\left\{y_{1}\right\}$, and $d\left(x_{1}\right)=d\left(x_{2}\right)=d\left(x_{3}\right)=4$. Since $x_{2}$ has four neighbors including $y_{3}$ and it has some neighbor in $\left(N\left(y_{1}\right) \cup\left\{y_{1}\right\}\right) \cap V_{j}$ for each $j=1,3,4$, it has exactly one neighbor in $V_{4}$, say $y_{4}$. Similarly, $x_{3}$ has exactly one neighbor in $V_{4}$ which has to be the same vertex $y_{4}$ by Lemma $2.18(i i)$.

We obtain a contradiction by observing that $x_{1} y_{2} y_{3} y_{4}$ forms a $K_{4}$. First, note that $x_{1} y_{4}$ is an edge by Lemma $2.18(i)$. Now $y_{4}$ is not 1 -special otherwise $y_{4}$ would have
special degree 3 and by repeating the argument above with $y_{1}$ replaced by $y_{4}$, we could deduce that $x_{1}, x_{2}$, or $x_{3}$ had degree 5 . Therefore, the neighbors of $x_{1}$ are $y_{2}, y_{3}, y_{4}$ and a vertex in $V_{4}$. Since $y_{2}, y_{3}$ are both 1 -special and $y_{4}$ is 2,3 -special, $y_{2} y_{3} y_{4}$ forms a triangle by Lemma $2.18($ iii)

### 2.5.3 The function $\alpha(k, 5)$

As a consequence of Theorem $2.2(i)$ and (ii), we obtain that

$$
\begin{gathered}
\alpha(k, 5)=6 k \quad \text { for } k=7 \text { or } k \geq 9, \\
30 \leq \alpha(5,5) \leq 36 \\
36 \leq \alpha(6,5) \leq 40 \\
48 \leq \alpha(8,5) \leq 49
\end{gathered}
$$

We shall improve the lower bound for $\alpha(5,5)$ to 33 .
Suppose for contradiction that $\alpha(5,5) \leq 32$, i.e. there exists a $K_{5}$-partite-saturated 5-partite graph $G=V_{1} \cup V_{2} \cup V_{3} \cup V_{4} \cup V_{5}$ containing an independent set $X=\left\{x_{1}, x_{2}, x_{3}, x_{4}, x_{5}\right\}$ with $x_{i} \in V_{i}$ for all $i$ such that $\sum_{i=1}^{5} d\left(x_{i}\right) \leq 32$. Write $Y_{i}$ for $V_{i} \backslash\left\{x_{i}\right\}$. By Lemma 2.8, $d\left(x_{i}\right) \geq \beta_{1}(4,4)=6$ and each $x_{i}$ has some neighbor in $V_{j}$ for $j \neq i$. Therefore, there are either four vertices in $X$ of degree 6 or there are three vertices of degree 6 and two of degree 7 . Since a vertex of degree 6 in $X$ creates at least two special vertices and a vertex of degree 7 in $X$ creates at least one special vertex, the sum of the special degrees of the vertices in $X^{c}$ is at least 8 , and hence, there exists a vertex of special degree at least two. Let $i$ be such that there is a special vertex $y \in Y_{i}$ with special degree $d_{s}(y)$ at least two where $\left(d\left(x_{i}\right), d_{s}(y)\right)$ is maximum in lexicographical order ${ }^{1}$. Without loss of generality we can assume that $i=1$. Let $N=N(y) \backslash X$. By Lemma 2.18(vi), $x_{1}$ has two neighbors, say $y_{2}, y_{3}$, belonging to two distinct parts of $G$, different from $V_{1}$, which are

[^0]non-neighbors of $y$. Without loss of generality, we can assume that $y_{2} \in Y_{2}$ and $y_{3} \in Y_{3}$.
For a pair of non-adjacent vertices $u, v \in G$ and $S \subset G$, we say that $S$ is $u v$-saturating if adding the edge of $u v$ to $G$ creates a copy $K$ of $K_{5}$ such that $S \subseteq K$. If $S=\{z\}$ then we simply say that $z$ is $u v$-saturating. Notice that if $S$ is $u v$-saturating then $S$ induces a clique.

In the rest of the proof, we shall repeatedly use the following lemma.

Lemma 2.19. Given $i \in\{2,3,4,5\}$ the following hold.
(i) If $j \in\{2,3,4,5\} \backslash\{i\}$ then $x_{i}$ has a neighbor in $V_{j} \cap N$. In particular, $d_{N}\left(x_{i}\right) \geq 3$.
(ii) If $y$ is $i$-special then $x_{i}$ is adjacent to $y_{j}$ for every $j \in\{2,3\} \backslash\{i\}$.
(iii) If $y$ is $x_{i} x_{j}$-saturating, for every $j \in\{2,3,4,5\} \backslash\{i\}$, then $d_{N}\left(x_{i}\right) \geq 4$.
(iv) If $y$ is $i$-special or $d_{s}(y) \geq 3$ then $d_{N}\left(x_{i}\right) \geq 4$.
(v) If $y$ is 2,3 -special and $i \in\{4,5\}$, then $d\left(x_{i}\right) \geq 7$.
(vi) If $i \in\{2,3\}$ and there are $p$ vertices in $X \backslash\left\{x_{1}\right\}$ all of which have neighbors in $Y_{i} \backslash N$ then there is no vertex in $V_{i}$ with special degree bigger than $\max \{1,3-p\}$.
(vii) $Y_{3} \cup Y_{4} \subset N$.

Proof. (i) Observe that we can choose $k \in\{2,3,4,5\} \backslash\{i, j\}$ such that $y$ is either $i$-special or $k$-special. Since there must be a triangle in the common neighborhood of $x_{i}$ and $x_{j}$ which uses $y$, we have that the remaining two vertices belong to $N$. Hence $x_{i}$ has a neighbor in $N \cap x_{j}$.
(ii) This follows directly from Lemma $2.18(v)$,
(iii) We shall show that $d_{N}\left(x_{i}\right) \geq \beta_{1}(3,3)=4$. Take any $j \in\{2,3,4,5\} \backslash\{i\}$. Since $y$ is $x_{i} x_{j}$-saturating then there is an edge in the common neighborhood of $x_{i}$ and $x_{j}$ in $N \backslash\left(V_{i} \cup V_{j}\right)$. Observe that the common neighborhood of $x_{i}$ and $y$ cannot contain a $K_{3}$, hence $d_{N}\left(x_{i}\right) \geq \beta_{1}(3,3)=4$.
(iv) Take any $j \in\{2,3,4,5\} \backslash\{i\}$. Since $y$ is either $i$ - or $j$-special, it follows that $y$ is $x_{i} x_{j}$-saturating. Hence, by $(i i), d_{N}\left(x_{i}\right) \geq 4$.
(v) Without loss of generality we can assume that $i=4$. If $y$ is also 4 -special then it follows from (ii) and (iv) that $d_{N}\left(x_{4}\right) \geq 4$ and $x_{4}$ is adjacent to $y, y_{2}, y_{3}$, therefore $d\left(x_{4}\right) \geq 7$. Hence we can assume that $y$ is not 4 -special. Suppose for contradiction that $d\left(x_{4}\right)=6$. From $(i)$, we have that $d_{N}\left(x_{4}\right) \geq 3$ and since $y$ is not 4 -special we have that $d_{Y_{1}}\left(x_{4}\right) \geq 2$. Moreover, $x_{4}$ has to have at least one neighbor not in $Y_{1} \cup N$ as otherwise there would be a copy of $K_{5}$ in $G$, as seen by considering the non-edge $x_{1} x_{4}$. Therefore,
$d\left(x_{4}\right)=d_{Y_{1}}\left(x_{4}\right)+d_{N}\left(x_{4}\right)+\left|N\left(x_{4}\right) \backslash\left(Y_{1} \cup N\right)\right| \geq 3+2+1=6=d\left(x_{4}\right)$. Hence, $d_{Y_{1}}\left(x_{4}\right)=3, d_{N}\left(x_{4}\right)=4$ and $\left|N\left(x_{4}\right) \backslash\left(Y_{1} \cup N\right)\right|=1$. We shall obtain a contradiction by finding a copy of $K_{5}$ in the graph $G$.

Suppose $\left\{z_{1}, z_{2}, z_{3}\right\}$ is $x_{4} x_{5}$ saturating, with $z_{i} \in V_{i}$. We claim that $y \neq z_{1}$ and $\left\{z_{2}, z_{3}\right\} \nsubseteq N$. Suppose for contradiction that it is not the case. If $y$ is $x_{4} x_{5}$-saturating then from (iii) we have that $d_{N}\left(x_{4}\right) \geq 4$ hence we obtain a contradiction. We can therefore assume that $y$ is not $x_{4} x_{5}$-saturating and hence $z_{1} \neq y$. Whence $z_{2}, z_{3} \in N$. Recall that $\left\{z_{1}, z_{2}, z_{3}\right\}$ form a triangle and therefore there is an edge between $z_{2}, z_{3}$. By assumption $z_{2}$ and $z_{3}$ are neighbors of $y$, hence $y, z_{2}, z_{3}$ form a triangle, and therefore $y$ is $x_{4} x_{5}$-saturating since $y, z_{2}, z_{3}$ belong to the common neighborhood of $x_{4}$ and $x_{5}$, which contradicts the assumption that $y$ is not $x_{4} x_{5}$-saturating.

Without loss of generality we can assume that $z_{2} \notin N$. Using $(i)$, we can therefore suppose that $N\left(x_{4}\right) \cap Y_{1}=\left\{y, z_{1}\right\}, N\left(x_{4}\right) \cap Y_{2}=\left\{w, z_{2}\right\}, N\left(x_{4}\right) \cap Y_{3}=\left\{z_{3}\right\}$ and $N\left(x_{4}\right) \cap Y_{5}=\left\{z_{5}\right\}$, for some $w, z_{3}, z_{5} \in N$. We shall obtain a contradiction by observing that $z_{1}, z_{2}, z_{3}, x_{4}, z_{5}$ form a copy of $K_{5}$. First we claim that $\left\{z_{2}, z_{3}, z_{5}\right\}$ is $x_{1} x_{4}$-saturating. Indeed, there must be a triangle in the common neighborhood of $x_{1}$ and $x_{4}$, with one vertex in each $V_{3}, V_{4}, V_{5}$. There are only two candidates for the triangle: $z_{2}, z_{3}, z_{5}$ or $w, z_{3}, z_{5}$. It cannot be $w, z_{3}, z_{5}$ since they are all neighbors of $y$, hence $y, w, z_{3}, x_{4}, z_{5}$ would form a copy of $K_{5}$. Hence we must have that the set $\left\{z_{2}, z_{3}, z_{5}\right\}$ is $x_{1} x_{4}$-saturating. Now,
since $x_{4}$ is not adjacent to $y_{3}$, and $y_{3}$ is not adjacent to $y$ we must have an edge between $z_{1}$ and $z_{5}$. Indeed, there must be a triangle in the common neighborhood of $x_{4}$ and $y_{3}$ with a vertex in each $V_{1}, V_{2}, V_{3}$. Since $x_{4}$ has only one neighbor in $V_{5}$, i.e. $z_{5}$, and $x_{4}$ and $y_{3}$ have only one common neighbor in $V_{1}$, i.e. $z_{1}$, we must have an edge between $z_{1}$ and $z_{5}$.

Therefore we have that $z_{1}, z_{2}, z_{3}$ form a triangle, $z_{2}, z_{3}, z_{5}$ form a triangle, and $z_{1}, z_{5}$ are adjacent. It easy to see now that $z_{1}, z_{2}, z_{3}, x_{4}, z_{5}$ form a copy of $K_{5}$.
( $v i$ ) Let $v$ be a special vertex in $V_{2} \cup V_{3}$, say in $V_{2}$. First observe that if $v$ is 1-special then $x_{3}, x_{4}, x_{5}$ are all adjacent to $y_{2} \in Y_{2} \backslash N$. On the other hand, it follows from (i) that $x_{3}, x_{4}, x_{5}$ all have neighbors in $N \cap Y_{2}$ hence they all have degree at least 2 in $Y_{2}$. It follows that $v$ has special degree 1 . If we assume that $v$ is not 1 -special then $v$ has special degree at most $3-p$, since $p$ of the vertices $x_{3}, x_{4}, x_{5}$ have degree 2 in $Y_{2}$.
(vii) Assume for contradiction that there is $v$, say in $Y_{4} \backslash N$. Observe that if $y$ is $i$-special then it follows from (ii) and (iv) that $d\left(x_{i}\right) \geq 7$, hence if $d_{s}(y) \geq 3$ we obtain contradiction by finding three vertices in $X$ of degree at least 7 . Therefore we can assume that $d_{s}(y)=2$.

If $y$ is $5, i$-special, then from (ii) and (iv) we have that $d\left(x_{5}\right) \geq 8$ and $d\left(x_{i}\right) \geq 7$ hence again we obtain a contradiction. Therefore we can assume that $y$ is not 5 -special. If $y$ is 2,3-special then $d\left(x_{2}\right), d\left(x_{3}\right) \geq 7$ and from (iv) we have that $d\left(x_{4}\right), d\left(x_{5}\right) \geq 7$. Hence we can assume that $y$ is 2,4 -special or 3,4 -special. Suppose that the former is the case. Then $d\left(x_{2}\right), d\left(x_{4}\right) \geq 7$. It follows that $d\left(x_{1}\right)=6$. Therefore by maximality $\left(x_{1}, y\right)$ and from ( $v$ ) we have that every vertex in $Y_{2} \cup Y_{3} \cup Y_{4}$ has special degree at most 1 and no vertex in $Y_{5}$ has special degree bigger than 2 . Which gives a contradiction since the sum of special degree is then at most 7 .

We are now ready to finish showing that $\alpha(5,5) \geq 33$. We consider several cases depending on the special degree of $y$.

Case 1. $d_{s}(y)=4$

Consider the 4-partite graph $H=G[N(y)]$ with an independent set $X^{\prime}=\left\{x_{2}, x_{3}, x_{4}, x_{5}\right\}$. Clearly, $H$ is $K_{4}$-free since $G$ is $K_{5}$-free. We modify $H$ by keeping adding admissible edges inside $H \backslash X^{\prime}$ until every admissible non-edge inside $H \backslash X^{\prime}$ is $K_{4}$-saturated. We claim that $H$ is $K_{4}$-partite-saturated, which would imply that $e\left(X^{\prime}, H \backslash X^{\prime}\right) \geq \alpha(4,4)=18$ by the previous subsection. It remains to show that the admissible non-edges with at least one endpoint in $X^{\prime}$ are $K_{4}$-saturated.

Consider the non-edge $x_{i} y_{j}$ with $y_{j} \in V_{j} \cap H$ (possibly $y_{j}=x_{j}$ ) and distinct $2 \leq i, j \leq 5$. Since the non-edge $x_{i} y_{j}$ is $K_{5}$-saturated in $G$, the common neighborhood in $G$ of $x_{i}$ and $y_{j}$ contains a $K_{3}$ consisting of one vertex from each part of $G \backslash\left(V_{i} \cup V_{j}\right)$. Since $y$ is $i$-special, this $K_{3}$ must contain $y$, and so the common neighborhood in $H$ of $x_{i}$ and $y_{j}$ contains a $K_{2}$, i.e. $x_{i} y_{j}$ is $K_{4}$-saturated in $H$ as required.

Recall that $y$ has two non-neighbors, $y_{2} \in V_{2}$ and $y_{3} \in V_{3}$. By Lemma 2.18(v), $x_{i} y_{2}$ is an edge for $i \neq 2$ and $x_{i} y_{3}$ is an edge for $i \neq 3$. We shall partition the edges between $X$ and $X^{c}$ as follows:

$$
\begin{aligned}
e\left(X, X^{c}\right) & \geq e\left(X^{\prime}, H \backslash X^{\prime}\right)+d\left(x_{1}\right)+e(X, y)+e\left(X^{\prime}, y_{2}\right)+e\left(X^{\prime}, y_{3}\right) \\
& \geq 18+6+4+3+3=34,
\end{aligned}
$$

contradicting the assumption.

Case 2. $d_{s}(y)=3$

If $y$ is 4,5 -special then from Lemma 2.19(ii) and 2.19(iii) we have that $d\left(x_{4}\right), d\left(x_{5}\right) \geq 7$. Otherwise $y$ is 2,3 -special and hence it follows from Lemma 2.19(v) that $d\left(x_{4}\right), d\left(x_{5}\right) \geq 7$. We shall obtain a contradiction by showing that $d\left(x_{1}\right) \geq 7$, hence showing that there are three vertices in $X$ with degrees at least 7 , which is against an assumption made in the beginning of the subsection. It follows from Lemma 2.19(vi) with $p \geq 2$, that the sum of special degrees in $Y_{2} \cup Y_{3}$ is at most 2 . Since the sum of special degrees is at least 8 , it follows that there is a special vertex in $Y_{4} \cup Y_{5}$ with special degree at
least 2 . Therefore from the maximality of $d\left(x_{1}\right)$ we have that $d\left(x_{1}\right) \geq 7$.

Case 3. $d_{s}(y)=2$

We split this case into three subcases.

## Case 3.1. $y$ is 2,3 -special

It follows from Lemma $2.19(v)$ that $d\left(x_{4}\right), d\left(x_{5}\right) \geq 7$. We shall obtain a contradiction by showing that $d\left(x_{1}\right) \geq 7$, hence showing that there are three vertices in $X$ with degrees at least 7, which is against an assumption made in the beginning of the subsection. It follows from Lemma $2.19(v i)$ that the sum of special degrees in $Y_{2} \cup Y_{3}$ is at most 2. Since the sum of special degrees is at least 8 , it follows that there is a special vertex in $Y_{4} \cup Y_{5}$ with special degree at least 2 . Therefore from the maximality of $d\left(x_{1}\right)$ we have that $d\left(x_{1}\right) \geq 7$.

Case 3.2. $y$ is 4,5 -special
It follows from Lemma $2.19(i i)$ and $2.19(i v)$ that $d\left(x_{4}\right), d\left(x_{5}\right) \geq 7$. We shall obtain a contradiction by showing that $d\left(x_{1}\right) \geq 7$, hence showing that there are three vertices in $X$ with degrees at least 7 , which is against an assumption made in the beginning of the subsection. It follows from Lemma $2.19(v i)$ that the sum of special degrees in $Y_{2} \cup Y_{3}$ is at most 3 . Since the sum of special degrees is at least 8 , it follows that there is a special vertex in $Y_{4} \cup Y_{5}$ with special degree at least 2 . Therefore from the maximality of $d\left(x_{1}\right)$ we have that $d\left(x_{1}\right) \geq 7$.

Case 3.3. $y$ is neither 2,3 -special nor 4,5 -special

Without loss of generality we can assume that $y$ is 2,4 -special. It follows from Lemma $2.19(i i)$ and $2.19(i v)$ that $d\left(x_{4}\right) \geq 7$ and from Lemma 2.19(vi) with $p \geq 2$ that there is no special vertex in $Y_{2} \cup Y_{3}$ with special degree bigger than 1. Hence there is either a vertex in $Y_{4}$ with special degree at least 2 or a vertex in $Y_{5}$ with special degree at least 3 . Therefore we can assume that $d\left(x_{1}\right)=7$ as otherwise we obtain a contradiction to the maximality of $\left(d\left(x_{1}\right), d_{s}(y)\right)$.

We shall obtain a contradiction by showing that at least one of $x_{2}, x_{3}$ or $x_{5}$ has degree at least 7, thus finding three vertices with degree at least 7. Suppose $d\left(x_{2}\right)=d\left(x_{3}\right)=d\left(x_{5}\right)=6$. Observe that if there is a vertex in $X_{4}$ of special degree bigger than 2 then we obtain a contradiction to the maximality of $\left(d\left(x_{1}\right), d_{s}(y)\right)$. Therefore there are two vertices in $X$ with at least two neighbors in $X_{4}$. Suppose that $i \in\{3,5\}$ and $x_{i}$ has at least two neighbors in $X_{4}$. Then it follows from Lemma $2.19(i)$ that $d_{N}\left(x_{i}\right) \geq 4$, and hence $x_{i}$ has degree at least 7 as $x_{i}$ has at least three neighbors outside $N$. We can therefore assume that $x_{3}$ and $x_{5}$ have only one neighbor in $X_{4}$. For the same reason we can assume that $x_{3}$ has only one neighbor in $Y_{5}$. If $x_{2}$ has two neighbors in $Y_{5}$ then $d_{N}\left(x_{2}\right) \geq 5$ and therefore $d\left(x_{2}\right) \geq 7$. Hence we can assume that there is $z_{5} \in Y_{5}$ which is 2,3 -special.

Suppose $\left\{z_{1}, z_{2}, z_{4}\right\}$ is $x_{3} x_{5}$-saturating, with $z_{i} \in V_{i}$. We claim that $y \neq z_{1}$ and $z_{2} \notin N$. Suppose for contradiction that it is not the case. If $y$ is $x_{3} x_{5}$-saturating then from (iii) we have that $d_{N}\left(x_{3}\right) \geq 4$ hence we obtain a contradiction. We can therefore assume that $y$ is not $x_{3} x_{5}$-saturating and hence $z_{1} \neq y$. Whence $z_{2} \in N$. Observe that by Lemma 2.19(vii) we have $z_{4} \in N$. Recall that $\left\{z_{1}, z_{2}, z_{4}\right\}$ form a triangle and therefore there is an edge between $z_{2}, z_{4}$. By assumption $z_{2}$ and $z_{4}$ are neighbors of $y$, hence $y, z_{2}, z_{4}$ form a triangle, and therefore $y$ is $x_{3} x_{5}$-saturating since $y, z_{2}, z_{4}$ belong to the common neighborhood of $x_{3}$ and $x_{5}$, which contradicts the assumption that $y$ is not $x_{3} x_{5}$-saturating.

We shall obtain a contradiction by showing that $z_{1}, z_{2}, z_{3}, z_{4}, z_{5}$ form a copy of $K_{5}$. Indeed, by assumption $\left\{z_{1}, z_{2}, z_{4}\right\}$ is $x_{3} x_{5}$-saturating and similar analysis to the one made in the proof of Lemma $2.19(v)$ shows that $\left\{z_{2}, z_{4}, z_{5}\right\}$ is $x_{1} x_{3}$-saturating. Since $y$ is 2 -special it follows that $x_{2}$ is not adjacent to $z_{1}$, and moreover $z_{5}$, as the only neighbor of $x_{2}$ in $Y_{5}$, is $x_{2} z_{1}$-saturating, and therefore there is an edge between $x_{2}$ and $z_{5}$. Hence we have that $z_{2}, z_{4}, z_{5}$ form a triangle, $z_{1}, z_{2}, z_{4}$ form a triangle, and $z_{1}, z_{5}$ are adjacent. It easy to see now that $z_{1}, z_{2}, x_{3}, z_{4}, z_{5}$ form a copy of $K_{5}$.

### 2.6 The diagonal case $\alpha(r, r)$

### 2.6.1 Proof of Theorem 2.2 (iv)

We have seen that the lower bound $\alpha(k, r) \geq k(2 r-4)$ in Theorem $2.2(i)$ is attained for some $k$. In this subsection, we show that this is not the truth for the diagonal case $k=r \geq 4$, i.e. $\alpha(r, r) \geq r(2 r-4)+1$. We shall again use the concept of special vertices introduced in Section 2.5 ,

Suppose for contradiction that for some $r \geq 4, \alpha(r, r)=r(2 r-4)$, i.e. there exists a $K_{r}$-partite-saturated $r$-partite graph $G=V_{1} \cup V_{2} \cup \cdots \cup V_{r}$ containing an independent set $X=\left\{x_{1}, x_{2}, \ldots, x_{r}\right\}$ with $x_{i} \in V_{i}$ for all $i$ such that $\sum_{i=1}^{r} d\left(x_{i}\right)=r(2 r-4)$. Lemma 2.8 tells us that we must have $d\left(x_{i}\right)=2 r-4$ for all $i$ and each $x_{i}$ has some neighbor in $V_{j}$ for $j \neq i$. Therefore, each $x_{i}$ creates at least two special vertices, and so the sum of the special degrees of the vertices in $X^{c}$ is at least $2 r$. By Lemma 2.18(iv), there is a vertex of special degree at least 3 , say $y_{1} \in V_{1}$.

We observe that $y_{1}$ has at least two non-neighbors, say $y_{2} \in V_{2}$ and $y_{3} \in V_{3}$ by Lemma 2.18(vi). Since $y_{1}$ has special degree at least 3 , we can pick $i \geq 4$ such that $y_{1}$ is $i$-special. By Lemma 2.18(v), $y_{2}$ and $y_{3}$ are neighbors of $x_{i}$. Therefore,

$$
\left|N\left(x_{i}\right) \cap N\left(y_{1}\right)\right|=d\left(x_{i}\right)-\left|N\left(x_{i}\right) \backslash N\left(y_{1}\right)\right| \leq(2 r-4)-3=2 r-7 .
$$

On the other hand, we shall obtain a contradiction by showing that the graph $H=G\left[N\left(x_{i}\right) \cap N\left(y_{1}\right)\right]$ contains at least $\beta_{1}(r-2, r-2)=2(r-3)$ vertices. It is sufficient to prove that $H$ is an $(r-2)$-partite $K_{r-2}$-free graph such that the subgraph induced by any $k-3$ parts contains a $K_{r-3}$. Clearly, $H$ is $K_{r-2}$-free since $G$ is $K_{r}$-free. The parts of $H$ are $N\left(x_{i}\right) \cap N\left(y_{1}\right) \cap V_{j}$ for $j \notin[r] \backslash\{1, i\}$. It remains to verify that the deletion of the part $N\left(x_{i}\right) \cap N\left(y_{1}\right) \cap V_{j}$ does not destroy all the $K_{r-3}$. Since the non-edge $x_{i} x_{j}$ is $K_{r}$-saturated in $G$, the common neighborhood in $G$ of $x_{i}$ and $x_{j}$ contains a $K_{r-2}$ consisting of one vertex
from each part of $G \backslash\left(V_{i} \cup V_{j}\right)$. Since $y_{1}$ is $i$-special, this $K_{r-2}$ must contain $y_{1}$, and so the common neighborhood $N\left(x_{i}\right) \cap N\left(y_{1}\right) \cap N\left(x_{j}\right) \subset H$ contains a $K_{r-3}$ not using the vertices of $V_{j}$ as required.

### 2.6.2 Remark on $\beta_{2}(r, r-1)$

Recall from Proposition 2.7 that $\alpha(r, r) \leq(r-1) \beta_{2}(r, r-1)$. Thus, a better estimate on $\beta_{2}$ would translate to a better understanding of the saturation numbers. While we could not find the exact value of $\beta_{2}(r, r-1)$, we suspect that $\beta_{2}(r, r-1)=3 r-6$ as mentioned in Conjecture 2.13. In this subsection, we make an observation about $\beta_{2}(r, r-1)$ which can be viewed as a first step towards determining its exact value. For simplicity of notation, let us write $\beta_{2}(r)=\beta_{2}(r, r-1)$.

Proposition 2.20. Either

- $\beta_{2}(r)=3 r-6$ for all $r \geq 3$, or
- $\beta_{2}(r) \leq(c+o(1)) r$ for some constant $c<3$, as $r \rightarrow \infty$.

Proof. The result is an immediate consequence of the following lemma.
Lemma 2.21. $\beta_{2}\left(r_{1}+r_{2}\right) \leq \beta_{2}\left(r_{1}\right)+\beta_{2}\left(r_{2}\right)+6$ for $r_{1}, r_{2} \geq 3$.

Proof. For $i \in\{1,2\}$, let $G_{i}=V_{i, 1} \cup V_{i, 2} \cup \cdots \cup V_{i, r_{i}}$ be a $K_{r_{i}-1}$-free $r_{i}$-partite graph on $\beta_{2}\left(r_{i}\right)$ vertices such that the subgraph induced by any $r_{i}-2$ parts contains a $K_{r_{i}-2}$. We shall construct a $K_{r_{1}+r_{2}-1}$-free $\left(r_{1}+r_{2}\right)$-partite graph $G$ from $G_{1}$ and $G_{2}$ with $|G|=\left|G_{1}\right|+\left|G_{2}\right|+6$ by starting with the disjoint union of $G_{1}$ and $G_{2}$ and then adding six new vertices $U=\left\{x_{1}, x_{2}, y_{1}, y_{2}, z_{1}, z_{2}\right\}$ as follows: add $x_{i}, y_{i}$ to $V_{i, 1}$ and add $z_{i}$ to $V_{i, 2}$ for $i \in\{1,2\}$. Now, join all admissible pairs between $U$ and $V(G) \backslash U$, and add the edges $x_{1} z_{1}, x_{2} z_{2}, y_{1} y_{2}, z_{1} z_{2}, y_{1} z_{2}, z_{1} y_{2}$ inside $U$.

First, we show that $G$ is $K_{r_{1}+r_{2}-1}$-free. Suppose otherwise. Since $G_{i}$ is $K_{r_{i}-1}$-free for $i \in\{1,2\}$, this $K_{r_{1}+r_{2}-1}$ must contain at least three vertices forming a triangle in $U$,
contradicting the fact that $G[U]$ is triangle-free. It remains to show that the deletion of any two parts does not destroy all the $K_{r_{1}+r_{2}-2}$. Suppose first that both deleted parts are from $G_{1}$. Since $G_{1}$ contains a $K_{r_{1}-2}$ not using these two parts and $G_{2}$ contains a $K_{r_{2}-2}$ not using $V_{2,1}$ and $V_{2,2}$, we obtain a $K_{r_{1}+r_{2}-2}$ not using the deleted parts, formed by these two cliques and $x_{2}, z_{2}$. Now suppose that one of the deleted parts is from $G_{1}$ and the other is from $G_{2}$. For $i \in\{1,2\}$, let $V_{i}$ be a part in $\left\{V_{i, 1}, V_{i, 2}\right\}$ which was not deleted. By construction, $G[U]$ contains an edge between $V_{1, j}$ and $V_{1, l}$ for all $j, l \in\{1,2\}$ and so there exists an edge in $G[U]$ between $V_{1}$ and $V_{2}$, say $e$. Since $G_{1}$ contains a $K_{r_{1}-2}$ not using the deleted part in $G_{1}$ and $V_{1}$, and $G_{2}$ contains a $K_{r_{2}-2}$ not using the deleted part in $G_{2}$ and $V_{2}$, we obtain a $K_{r_{1}+r_{2}-2}$ not using the deleted parts, formed by these two cliques and the endpoints of $e$.

Suppose that $\beta_{2}(s)<3 s-6$ for some $s \geq 3$. We shall show that $\beta_{2}(r) \leq(c+o(1)) r$ with $c=\frac{\beta_{2}(s)+6}{s}<3$. Applying the lemma and induction on $m$, we deduce that $\beta_{2}(m s) \leq c m s-6$ for all positive integer $m$. Hence, writing $r=m s+t$ with $3 \leq t \leq s+2$ and applying the lemma again,

$$
\beta_{2}(r) \leq \beta_{2}(m s)+\beta_{2}(t)+6 \leq c m s+d \leq\left(c+\frac{d}{r}\right) r=(c+o(1)) r
$$

where $d=\max \left\{\beta_{2}(t): 3 \leq t \leq s+2\right\}$.

### 2.7 Proof of Theorem 2.3

Theorems 2.1 and Theorem 2.2(ii) imply that

$$
\operatorname{sat}(n, k, r)=k(2 r-4) n+o(n) \text { if }\left\{\begin{array}{l}
k=2 r-3, \text { or } \\
k \geq 2 r-2 \text { and } r \equiv 0 \quad \bmod 2, \text { or } \\
k \geq 2 r-1 \text { and } r \equiv 2 \quad \bmod 3
\end{array}\right.
$$

In this section, we shall show that the $o(n)$ term can be replaced with $O(1)$. The upper bound follows from Proposition 2.5 and Theorem 2.2 (ii). We prove that the lower bound holds for any $k \geq r \geq 3$ using the fact that $\beta_{1}(k-1, r-1)=2 r-4$.

Proposition 2.22. For $k \geq r \geq 3$, there is an integer $C_{k, r}$ such that sat $(n, k, r) \geq k(2 r-4) n+C_{k, r}$, for every integer $n \geq 0$.

Proof. Suppose, as we may, that $n$ is sufficiently large. Let $G=V_{1} \cup V_{2} \cup \cdots \cup V_{k}$ be a $K_{r}$-partite-saturated $k$-partite graph with $\left|V_{i}\right|=n$ for all $i$. We shall find a subset $U$ of $V(G)$ of constant size such that every vertex in $U^{c}$ has at least $2 r-4$ neighbors in $U$. Then we would be done since $e(G) \geq e\left(U, U^{c}\right) \geq(2 r-4)(k n-|U|)$. Let $v_{1}$ be a vertex of smallest degree in $V_{1}$. Having defined $v_{1}, v_{2}, \ldots, v_{i-1}$, let $v_{i} \in V_{i}$ be a vertex of smallest degree in $V_{i} \backslash\left(N\left(v_{1}\right) \cup N\left(v_{2}\right) \cup \cdots \cup N\left(v_{i-1}\right)\right)$. We shall take $U$ to be $N\left(v_{1}\right) \cup N\left(v_{2}\right) \cup \cdots \cup N\left(v_{k}\right)$. Now we may assume that $d\left(v_{i}\right)<2 k(2 r-4)$ for all $1 \leq i \leq k$. Indeed, if $v_{i}$ is the first vertex in the sequence such that $d\left(v_{i}\right) \geq 2 k(2 r-4)$ then we are done since $e(G) \geq e\left(V_{i}, V_{i}^{c}\right) \geq d\left(v_{i}\right)\left(n-\sum_{j<i} d\left(v_{j}\right)\right) \geq 2 k(2 r-4)(n-2 k(2 r-4)(i-1)) \geq k(2 r-4) n$
for sufficiently large $n$. Therefore, $U$ has size bounded by a function of $k$ and $r$. It remains to show that every vertex $v \in U^{c}$ has at least $2 r-4$ neighbors in $U$. We shall prove that $H=G[N(v) \cap U]$ contains at least $\beta_{1}(k-1, r-1)=2 r-4$ vertices by showing that $H$ is a $K_{r-1}$-free $(k-1)$-partite graph such that the subgraph induced by any $k-2$ parts contains a $K_{r-2}$. Clearly, $H$ is $K_{r-1}$-free since $G$ is $K_{r}$-free. Without loss of generality, $v \in V_{1}$. The parts of $H$ are $N(v) \cap U \cap V_{i}$ for $2 \leq i \leq k$. The deletion of the part $N(v) \cap U \cap V_{i}$ does not destroy all the $K_{r-2}$ since the non-edge $v v_{i}$ is $K_{r-1}$-saturated in $G$, i.e. $N(v) \cap N\left(v_{i}\right) \subset H$ contains a $K_{r-2}$ not using the vertices of $V_{i}$.

### 2.8 Concluding remarks

We have reduced the problem of determining $\operatorname{sat}(n, k, r)$ for large $n$ to that of $\alpha(k, r)$. Although, we have determined $\alpha(k, r)$ for some values of $k$ and $r$, a large number of cases remain unknown. In particular, the seemingly easiest case when $r$ is fixed and $k$ is large, is still open.

Problem 2.23. Determine $\alpha(k, r)$ for $k \geq 2 r-2$ and $r \equiv 1,3 \bmod 6$.

For $k \geq 2 r-2$ and $r \equiv 0,2,4,5 \bmod 6$, we have determined $\alpha(k, r)$ except one missing case when 3 is the smallest divisor of $r-2$ and $k=2 r-2$. Theorem 2.2 (i) implies that $\alpha(2 r-2, r) \in\left\{(2 r-3)^{2},(2 r-3)^{2}-1\right\}$ and we suspect that $\alpha(2 r-2, r)=(2 r-3)^{2}$.

Not only we believe that $\beta_{2}(k, r)=4 r-k-2$ for $r<k \leq 2 r-1$ (see Conjecture 2.13) but we also think that the upper bound $\alpha(k, r) \leq(k-1) \beta_{2}(k, r-1) \leq(k-1)(4 r-k-6)$ in Theorem $2.2(i)$ is the correct value for $\alpha(k, r)$ in this case.

Conjecture 2.24. $\alpha(k, r)=(k-1)(4 r-k-6)$ for $5 \leq r \leq k \leq 2 r-4$.

We have shown that $33 \leq \alpha(5,5) \leq 36$. This is the smallest case for which the value of $\alpha$ is not yet known.

Problem 2.25. Find $\alpha(5,5)$.

To prove the lower and upper bounds on $\alpha(k, r)$, we extensively used the bounds on $\beta_{1}(k, r)$ and $\beta_{2}(k, r)$. We believe that determining the values of $\beta_{i}(k, r)$ is an interesting problem on its own.

Problem 2.26. Determine $\beta_{i}(k, r)$ for $k \geq r \geq 2$ and $2 \leq i \leq k-r+1$.

We end the chapter with a remark on a related problem. Recall that $\operatorname{sat}\left(n, K_{r}\right)$ is the minimum number of edges in a $K_{r}$-free graph on $n$ vertices but the addition of an edge joining any two non-adjacent vertices creates a $K_{r}$. In the pioneer paper of Erdős, Hajnal,
and Moon [32], they determined $\operatorname{sat}\left(n, K_{r}\right)$ by considering a more general problem where the graphs were not required to be $K_{r}$-free. Interestingly, the two problems have the same answer since the extremal graph is $K_{r}$-free. We remark that this phenomenon does not happen for partite saturation. Roberts [72] studied the corresponding more general problem for $\operatorname{sat}\left(K_{r \times n}, K_{r}\right)$ and showed that the minimum number of edges in a $K_{r}$-saturated subgraph of $K_{r \times n}$ where the subgraph is allowed to contain $K_{r}$ is $\binom{r}{2}(2 n-1)$ for $r \geq 4$ and sufficiently large $n$. On the other hand, Theorem 2.1 and Theorem 2.2 imply that $\operatorname{sat}\left(K_{r \times n}, K_{r}\right) \geq r(2 r-4) n+o(n)>\binom{r}{2}(2 n-1)$ for sufficiently large $n$.

## CHAPTER 3

## MAJORITY COLOURINGS OF DIGRAPHS

In this chapter, we solve problems related to a concept called majority coloring recently studied by Kreutzer, Oum, Seymour, van der Zypen and Wood. They raised a problem of determining, for a natural number $k$, the smallest number $m=m(k)$ such that every digraph can be colored with $m$ colors where each vertex has the same color as at most $1 / k$ proportion of its out-neighbors. We show that $m(k) \in\{2 k-1,2 k\}$. We also prove a result supporting the conjecture that $m(2)=3$. Moreover, we prove similar results for a more general concept called majority choosability. This work is joint with António Girão and Teeradej Kittipassorn.

### 3.1 Results

For a natural number $k \geq 2$, a $\frac{1}{k}$-majority coloring of a digraph is a coloring of the vertices such that each vertex receives the same color as at most a $1 / k$ proportion of its out-neighbors. We say that a digraph $D$ is $\frac{1}{k}$-majority $m$-colourable if there exists a $\frac{1}{k}$-majority coloring of $D$ using $m$ colors. The following natural question was recently raised by Kreutzer, Oum, Seymour, van der Zypen and Wood [59].

Question 3.1. Given $k \geq 2$, determine the smallest number $m=m(k)$ such that every digraph is $\frac{1}{k}$-majority $m$-colourable.

In particular, they asked whether $m(k)=O(k)$. Let us first observe that $m(k) \geq 2 k-1$. Consider a tournament on $2 k-1$ vertices where every vertex has out-degree $k-1$. Any $\frac{1}{k}$-majority coloring of this tournament must be a proper vertex-coloring, and hence it needs at least $2 k-1$ colors. Conversely, we prove that $m(k) \leq 2 k$.

Theorem 3.2. Every digraph is $\frac{1}{k}$-majority $2 k$-colourable for all $k \geq 2$.

This is an immediate consequence of a result of Keith Ball (see [18]) about partitions of matrices. We shall use a slightly more general version proved by Alon [3].

Lemma 3.3. Let $A=\left(a_{i j}\right)$ be an $n \times n$ real matrix where $a_{i i}=0$ for all $i, a_{i j} \geq 0$ for all $i \neq j$, and $\sum_{j} a_{i j} \leq 1$ for all $i$. Then, for every $t$ and all positive reals $c_{1}, \ldots, c_{t}$ whose sum is 1 , there is a partition of $\{1,2, \ldots, n\}$ into pairwise disjoint sets $S_{1}, S_{2}, \ldots, S_{t}$, such that for every $r$ and every $i \in S_{r}$, we have $\sum_{j \in S_{r}} a_{i j} \leq 2 c_{r}$.

Proof of Theorem 3.2. Let $D$ be a digraph on $n$ vertices with vertex set $\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$ and write $d^{+}\left(v_{i}\right)$ for the out-degree of $v_{i}$. Let $A=\left(a_{i j}\right)$ be an $n \times n$ matrix where $a_{i j}=\frac{1}{d^{+}\left(v_{i}\right)}$ if there is a directed edge from $v_{i}$ to $v_{j}$ and $a_{i j}=0$ otherwise. We apply Lemma 3.3 with $t=2 k$ and $c_{i}=\frac{1}{2 k}$ for $1 \leq i \leq 2 k$ obtaining a partition of $\{1,2, \ldots, n\}$ into sets $S_{1}, S_{2}, \ldots, S_{2 k}$, such that for every $r$ and every $i \in S_{r}, \sum_{j \in S_{r}} a_{i j} \leq \frac{1}{k}$. Equivalently, the number of out-neighbors of $v_{i}$ that have the same color as $v_{i}$ is at most $\frac{d^{+}\left(v_{i}\right)}{k}$ where the coloring of $D$ is defined by the partition $S_{1} \cup S_{2} \cup \cdots \cup S_{2 k}$.

Question 3.1 has now been reduced to whether $m(k)$ is $2 k-1$ or $2 k$.

Question 3.4. Is every digraph $\frac{1}{k}$-majority $(2 k-1)$-colourable?

Surprisingly, this is open even for $k=2$. Kreutzer, Oum, Seymour, van der Zypen and Wood [59] gave an elegant argument showing that every digraph is $\frac{1}{2}$-majority 4 -colourable and they conjectured that $m(2)=3$.

Conjecture 3.5. Every digraph is $\frac{1}{2}$-majority 3-colourable.

We provide evidence for this conjecture by proving that tournaments are almost $\frac{1}{2}$-majority 3 -colourable.

Theorem 3.6. Every tournament can be 3-colored in such a way that all but at most 205 vertices receive the same color as at most half of their out-neighbors.

Proof. The proof relies on an observation that in a tournament $T$, the set $S_{i}=\left\{x \in V(T): 2^{i-1} \leq d^{+}(x)<2^{i}\right\}$ has size at most $2^{i+1}$. Indeed, the sum of the out-degrees of the vertices of $S_{i}$ is at least $\binom{\left|S_{i}\right|}{2}$, the number of edges inside $S_{i}$. On the other hand, this sum is at most $\left(2^{i}-1\right)\left|S_{i}\right|$ by the definition of $S_{i}$. Therefore, $\binom{\left|S_{i}\right|}{2} \leq\left(2^{i}-1\right)\left|S_{i}\right|$ and hence, $\left|S_{i}\right| \leq 2^{i+1}-1$.

We proceed by randomly assigning one of three colors to each vertex independently with probability $1 / 3$. Given a vertex $x$, let $B_{x}$ be the number of out-neighbors of $x$ which receive the same color as $x$. We say that $x$ is bad if $B_{x}>d^{+}(x) / 2$. Trivially $\mathbb{E}\left(B_{x}\right)=d^{+}(x) / 3$, and hence by a Chernoff-type bound, it follows that, for $x \in S_{i}$,

$$
\begin{aligned}
\mathbb{P}(x \text { is bad }) & =\mathbb{P}\left(B_{x}>d^{+}(x) / 2\right)=\mathbb{P}\left(B_{x}>(1+1 / 2) \mathbb{E}(B(x))\right) \\
& \leq \exp \left(-\frac{(1 / 2)^{2}}{3} \mathbb{E}\left(B_{x}\right)\right)=\exp \left(-d^{+}(x) / 36\right) \leq \exp \left(-2^{i-1} / 36\right) .
\end{aligned}
$$

Notice that if $i \geq 11$ then $\mathbb{P}(x$ is bad $) \leq 2^{-(2 i-7)}$. Let $X$ denote the total number of bad vertices. Since the vertices of out-degree 0 cannot be bad,

$$
\begin{aligned}
\mathbb{E}(X) & =\sum_{i \geq 1} \sum_{x \in S_{i}} \mathbb{P}(x \text { is bad }) \leq \sum_{i=1}^{10} 2^{i+1} \exp \left(-2^{i-1} / 36\right)+\sum_{i \geq 11} 2^{i+1} 2^{-(2 i-7)} \\
& \leq 205+\sum_{i \geq 11} 2^{-i+8}=205+\frac{1}{4}<206 .
\end{aligned}
$$

Hence, there is a 3-coloring such that all but at most 205 vertices receive the same color as at most half of their out-neighbors.

Observe also that the same argument proves a special case of Conjecture 3.5.

Theorem 3.7. Every tournament with minimum out-degree at least $2^{10}$ is $\frac{1}{2}$-majority 3-colorable.

We remark that Theorem 3.6 can be strengthened (205 can be replaced by 7) by solving a linear programming problem. Recall that the expected number of bad vertices of
out-degree at least 1024 is at most $1 / 4$. We shall use linear programming to show that the expected number of bad vertices of out-degree less than 1024 is less than 7.75. Let $V_{i}$ be the set of vertices of out-degree $i$ for $i \in\{1,2, \ldots, 1023\}$ and note that the expected number of bad vertices of out-degree at most 1023 is $f\left(v_{1}, v_{2}, \ldots, v_{1023}\right)=\sum_{i=1}^{1023} v_{i} p_{i}$ where $v_{i}=\left|V_{i}\right|$ and $p_{i}=\sum_{j=\left\lceil\frac{i+1}{2}\right\rceil}^{i}\binom{i}{j}(1 / 3)^{j}(2 / 3)^{i-j}$. As before, observe that the number of vertices of degree at most $i$ is at most $2 i+1$, and therefore, $\sum_{j=1}^{i} v_{i} \leq 2 i+1$, leading to the following linear program.

> Maximize: $f\left(v_{1}, v_{2}, \ldots, v_{1023}\right)$
> Subject to: $\sum_{j=1}^{i} v_{j} \leq 2 i+1$, for $i \in\{1,2, \ldots, 1023\}$

Subject to: $v_{i} \geq 0$, for $i \in\{1,2, \ldots, 1023\}$.

See Appendix 3.2 for the source code. Similarly, we can replace $2^{10}$ in Theorem 3.7 by 55 , by using the same linear program to show that the expected number of bad vertices of out-degree in $[55,1023]$ is less than $3 / 4$.

Let us now change direction to a more general concept of majority choosability. A digraph is $\frac{1}{k}$-majority $m$-choosable if for any assignment of lists of $m$ colors to the vertices, there exists a $\frac{1}{k}$-majority coloring where each vertex gets a color from its list. In particular, a $\frac{1}{k}$-majority $m$-choosable digraph is $\frac{1}{k}$-majority $m$-colourable. Kreutzer, Oum, Seymour, van der Zypen and Wood [59] asked whether there exists a finite number $m$ such that every digraph is $\frac{1}{2}$-majority $m$-choosable. Anholcer, Bosek and Grytczuk [6] showed that the statement holds with $m=4$. We generalize their result as follows.

Theorem 3.8. Every digraph is $\frac{1}{k}$-majority $2 k$-choosable for all $k \geq 2$.

Theorem 3.8 was independently proved by Fiachra Knox and Robert Šámal [57]. We prove Theorem 3.8 using a slight modification of Lemma 3.3.

Lemma 3.9. Let $A=\left(a_{i j}\right)$ be an $n \times n$ real matrix where $a_{i i}=0$ for all $i, a_{i j} \geq 0$ for all
$i \neq j$, and $\sum_{j} a_{i j} \leq 1$ for all $i$. Then, for every $m$ and subsets $L_{1}, L_{2}, \ldots, L_{n} \subset \mathbb{N}$ of size $m$, there is a function $f:\{1,2, \ldots, n\} \rightarrow \mathbb{N}$ such that, for every $i, f(i) \in L_{i}$ and $\sum_{j \in f^{-1}(r)} a_{i j} \leq \frac{2}{m}$ where $r=f(i)$.

Proof. By increasing some of the numbers $a_{i j}$, if needed, we may assume that $\sum_{j} a_{i j}=1$ for all $i$. We may also assume, by an obvious continuity argument, that $a_{i j}>0$ for all $i \neq j$. Thus, by the Perron-Frobenius Theorem, 1 is the largest eigenvalue of $A$ with right eigenvector $(1,1, \ldots, 1)$ and left eigenvector $\left(u_{1}, u_{2}, \ldots, u_{n}\right)$ in which all entries are positive. It follows that $\sum_{i} u_{i} a_{i j}=u_{j}$. Define $b_{i j}=u_{i} a_{i j}$, then $\sum_{i} b_{i j}=u_{j}$ and $\sum_{j} b_{i j}=u_{i}\left(\sum_{j} a_{i j}\right)=u_{i}$.

Let $f:\{1,2, \ldots, n\} \rightarrow \mathbb{N}$ be a function such that $f(i) \in L_{i}$ and $f$ minimizes the sum $\sum_{r \in \mathbb{N}} \sum_{i, j \in f^{-1}(r)} b_{i j}$. By minimality, the value of the sum will not decrease if we change $f(i)$ from $r$ to $l$ where $l \in L_{i}$. Therefore, for any $i \in f^{-1}(r)$ and $l \in L_{i}$, we have

$$
\sum_{j \in f^{-1}(r)}\left(b_{i j}+b_{j i}\right) \leq \sum_{j \in f^{-1}(l)}\left(b_{i j}+b_{j i}\right) .
$$

Summing over all $l \in L_{i}$, we conclude that

$$
m \sum_{j \in f^{-1}(r)}\left(b_{i j}+b_{j i}\right) \leq \sum_{j \in f^{-1}\left(L_{i}\right)}\left(b_{i j}+b_{j i}\right) \leq \sum_{j=1}^{n}\left(b_{i j}+b_{j i}\right)=2 u_{i} .
$$

Hence, $\sum_{j \in f^{-1}(r)} u_{i} a_{i j}=\sum_{j \in f^{-1}(r)} b_{i j} \leq \sum_{j \in f^{-1}(r)}\left(b_{i j}+b_{j i}\right) \leq \frac{2 u_{i}}{m}$. Dividing by $u_{i}$, the desired result follows.

Proof of Theorem 3.8. The proof is the same as that of Theorem 3.2, using Lemma 3.9 instead of Lemma 3.3.

In fact, the same statement also holds when the size of the lists is odd.

Corollary 3.10. Every digraph is $\frac{2}{m}$-majority $m$-choosable for all $m \geq 2$.

This statement generalizes a result of Anholcer, Bosek and Grytczuk [6] where they prove the case $m=3$ which says that, given a digraph with color lists of size three assigned to the vertices, there is a coloring from these lists such that each vertex has the same color as at most two thirds of its out-neighbors.

We have established that the $\frac{1}{k}$-majority choosability number is either $2 k-1$ or $2 k$. Let us end this chapter with an analogue of Question 3.4.

Question 3.11. Is every digraph $\frac{1}{k}$-majority $(2 k-1)$-choosable?

### 3.2 Linear program

We use the toolkit [1] to solve the linear program with the following source code:

```
param N := 1024;
param comb ' n choose k' {n in 0..N, k in 0..n} :=
    if k = 0 or k = n then 1 else comb[n-1,k-1] + comb[n-1,k];
param prob 'probability' {n in 0..N} :=
    sum{k in (floor (n/2)+1)\ldotsn} comb[n, k] * ((1/3)^k)* ((2/3)^(n-k));
var }x{1..N}, integer, >= 0
subject to constraint{i in 1..N}: sum{j in 1..i} x[j] <= 2*i+1;
maximize expectation: sum{i in 1..N} x[i]*prob[i];
solve;
end ;
```


## CHAPTER 4

## LARGE INDUCED SUBGRAPHS WITH $K$ VERTICES OF ALMOST MAXIMUM DEGREE

In this chapter, we prove that for every integer $k$, there exist constants $g_{1}(k)$ and $g_{2}(k)$ such that the following holds. If $G$ is a graph on $n$ vertices with maximum degree $\Delta$ then it contains an induced subgraph $H$ on at least $n-g_{1}(k) \sqrt{\Delta}$ vertices, such that $H$ contains $k$ vertices of the same degree of order at least $\Delta(H)-g_{2}(k)$. This solves an approximate version of a conjecture of Caro and Yuster which states that $g_{2}(k)$ can be taken to be 0 for every $k$. This work is joint with António Girão.

### 4.1 Introduction

Given a graph $G$, let the repetition number, denoted by $\operatorname{rep}(G)$, be the maximum multiplicity of a vertex degree. Trivially, any graph $G$ of order at least two contains at least two vertices of the same degree, i.e. $\operatorname{rep}(G) \geq 2$. This parameter has been widely studied by several researchers (e.g., [7, 14, 21, 24, 23]), in particular, by Bollobás and Scott, who showed that for every $k \geq 2$ there exist triangle-free graphs on $n$ vertices with $\operatorname{rep}(G) \leq k$ for which $\alpha(G)=(1+o(1)) n / k([14])$. As there are infinitely many graphs having repetition number two, it is natural to ask what is the smallest number of vertices one needs to delete from a graph in order to increase the repetition number of the remaining induced subgraph. This question was partially answered by Caro, Shapira and Yuster in [20], indeed, they proved that for every $k$ there exists a constant $C(k)$ such that given any graph on $n$ vertices one needs to remove at most $C(k)$ vertices and thus obtain an induced subgraph with at least $\min \{k, n-C(k)\}$ vertices of the same degree. Related to this question, Caro and Yuster ([22]) considered the problem of finding the largest induced subgraph $H$ of a graph $G$ which contains at least $k$ vertices of degree $\Delta(H)$. To do so they defined $f_{k}(G)$ to be the smallest number of vertices one needs to remove from a graph $G$
such that the remaining induced subgraph has its maximum degree attained by at least $k$ vertices. They found examples of graphs on $n$ vertices for which $f_{2}(G) \geq(1-o(1)) \sqrt{n}$ and conjectured $f_{k}(G) \leq O(\sqrt{n})$ for every graph $G$ on $n$ vertices. In the same paper they established the conjecture for $k \leq 3$.

The following more general conjecture was posed recently by Caro, Lauri and Zarb in [19].

Conjecture 4.1. For every $k \geq 2$ there is a constant $g(k)$ such that given a graph $G$ with maximum degree $\Delta$, one can remove at most $g(k) \sqrt{\Delta}$ vertices such that the remaining subgraph $H \subseteq G$ has at least $k$ vertices of degree $\Delta(H)$.

Let us define $g(k, \Delta)=\max \left\{f_{k}(G): \Delta(G) \leq \Delta\right\}$. In the same paper, they proved that $g(2, \Delta)=\left\lceil\frac{3+\sqrt{8 \Delta+1}}{2}\right\rceil$ and stated that $g(3, \Delta) \leq 42 \sqrt{\Delta}$. We should point out that, if true, the conjecture is best possible, as there are graphs on $n$ vertices found in [19] for which any induced subgraph on more than $n-\frac{k}{2} \sqrt{\Delta}$ does not contain $k$ vertices of the same maximum degree. We shall present such constructions in Section 4.3 ,

In this chapter we prove the following approximate version of Conjecture 4.1

Theorem 4.2. For every positive integer $k$, there exist constants $g_{1}(k)$ and $g_{2}(k)$ such that the following holds. If $G$ is a graph on $n$ vertices with maximum degree $\Delta$ then it contains an induced subgraph $H$ on at least $n-g_{1}(k) \sqrt{\Delta}$ vertices, such that $H$ has $k$ vertices of the same degree at least $\Delta(H)-g_{2}(k)$.

### 4.2 Proofs

First, we shall introduce the following definitions. Let $n$ be an integer and $A_{1} \cup A_{2} \cup \ldots \cup A_{t}$ be a partition of the set $\{1,2, \ldots, n\}$ into $t$ sets. Moreover, let $r_{1}>r_{2}>r_{3}>\ldots>r_{t}$ be a strictly decreasing sequence of non-negative integers. We shall say that a multiset $\mathscr{A}$ consisting of subsets of $[n]$ is an $\left(r_{1}, r_{2}, \ldots, r_{t}\right)$-uniform cover of
$\{1,2 \ldots, n\}$ if for every $i \in\{1, \ldots, t\}$ and $j \in A_{i}$, we have $|\{A \in \mathscr{A}: j \in A\}|=r_{i}$. Note that in a multiset we allow repetitions.

We call an $\left(r_{1}, r_{2}, \ldots, r_{t}\right)$-uniform cover $\mathscr{A}$ of $\{1,2, \ldots, n\}=A_{1} \cup A_{2} \cup \ldots \cup A_{t}$ irreducible if there is no proper $\left(r_{1}^{\prime}, \ldots, r_{t}^{\prime}\right)$-uniform cover $\mathscr{B} \subset \mathscr{A}$, for some strictly decreasing sequence of non-negative integers $r_{1}^{\prime}>r_{2}^{\prime}>\ldots>r_{t}^{\prime}$.

Given a uniform cover $\mathscr{A}$ of $\{1,2, \ldots, n\}$ and a subset $B \subseteq\{1,2, \ldots, n\}$ we define $w_{\mathscr{A}}(B)$ to be the number of times $B$ appears in $\mathscr{A}$.

Lemma 4.3. For all $n \in \mathbb{N}$, there exists $f(n)$ such that for any $1 \leq t \leq n$ and any partition of $\{1,2, \ldots, n\}$ into $t$ sets $A_{1}, A_{2}, \ldots, A_{t}$, every $\left(r_{1}, r_{2}, \ldots, r_{t}\right)$-uniform cover $\mathscr{A}$ of $\{1,2, \ldots, n\}$ contains a $\left(r_{1}^{\prime}, r_{2}^{\prime}, \ldots, r_{t}^{\prime}\right)$-uniform sub-cover $\mathscr{B} \subset \mathscr{A}$ with $r_{1}^{\prime} \leq f(n)$. Proof. We shall prove there are only finitely many irreducible covers. For otherwise, let us assume there exists an infinite sequence $\left\{B_{i}\right\}_{i \in \mathbb{N}}$ of irreducible uniform covers. Since there are only finitely many partitions of $\{1,2, \ldots, n\}$, we may pass to an infinite subsequence $\left\{B_{l_{i}}\right\}_{i \in \mathbb{N}}$ of uniform covers of the same partition of $\{1,2, \ldots, n\}$. Now, choose $A \subseteq\{1,2, \ldots, n\}$ and consider the sequence of non-negative integers $\left\{w_{B_{l_{i}}}(A)\right\}_{i \in \mathbb{N}}$, clearly it must contain an infinite non-decreasing subsequence $w_{B_{l_{i_{1}}}}(A) \leq w_{B_{l_{l_{2}}}}(A) \leq \ldots$. We restrict our attention to this subsequence of the uniform covers $B_{l_{i_{1}}}, B_{l_{i_{2}}}, \ldots$ and iteratively apply the same argument for the remaining subsets of $\{1,2, \ldots, n\}$, always passing to a subsequence of the previous sequence of uniform covers. After we have done it for every subset of $\{1,2, \ldots, n\}$, we must end up with two distinct irreducible uniform covers (actually an infinite sequence) $\mathscr{A}, \mathscr{B}$ for which $w_{\mathscr{A}}(F) \leq w_{\mathscr{B}}(F)$ for every $F \subseteq\{1,2, \ldots, n\}$. This implies $\mathscr{A} \subseteq \mathscr{B}$, which is a contradiction. Take $f(n)$ to be the maximum $r_{1}$ over all irreducible uniform covers of $\{1,2, \ldots, n\}$.

Lemma 4.4. For every $n \in \mathbb{N}$, there exists $f(n)$ such that the following holds. Let $G=(A, B)$ be a bipartite graph with $A=\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$. Then there exists a subset $W \subseteq V(B)$ of size at most $n \cdot f(n)=f^{\prime}(n)$, such that the induced bipartite graph
$G^{\prime}=G[A,(B \backslash W)]$ has the property that

$$
\text { if } d_{G}\left(x_{i}\right)>d_{G}\left(x_{j}\right) \text { then } d_{G}\left(x_{i}\right)-d_{G^{\prime}}\left(x_{i}\right)>d_{G}\left(x_{j}\right)-d_{G^{\prime}}\left(x_{j}\right) \text {. }
$$

Proof. Partition $A$ into $A_{1}, \ldots, A_{t}$, so that two vertices belong to the same part if they have the same degree. Let $r_{i}$ be the degree of the vertices in $A_{i}$. We may assume that $r_{1}>r_{2}>\cdots>r_{t}$. The lemma follows as a corollary of Lemma 4.3. Indeed, for every vertex $w \in B$, let $A_{w} \subseteq\{1,2, \ldots, n\}$ such that $i \in A_{w}$ if $x_{i}$ is a neighbor of $w$ in $G$. Note that $\mathscr{A}=\left\{A_{w}: w \in B\right\}$ is an $\left(r_{1}, r_{2}, \ldots, r_{t}\right)$-uniform cover of $\{1,2, \ldots, n\}$. Applying now Lemma 4.3, we can find a $\left(r_{1}^{\prime}, r_{2}^{\prime}, \ldots, r_{t}^{\prime}\right)$-uniform sub-cover $\mathscr{B} \subseteq \mathscr{A}$ with $r_{1}^{\prime} \leq f(n)$. Let $W=\left\{w \in B: A_{w} \in \mathscr{B}\right\}$ and $G^{\prime}=G[A,(B \backslash W)]$. It is easy to see that $|W| \leq n \cdot f(n)$ and that the property is satisfied by the definition of uniform cover.

Given a positive integer $k$ and a graph $G$ with the vertex set $\left\{x_{1}, \ldots, x_{n}\right\}$ such that $d\left(x_{1}\right) \geq \cdots \geq d\left(x_{n}\right)$, let $r_{k}(G):=\Delta(G)-d_{G}\left(x_{k}\right)$ be the difference between the maximum degree and the degree of vertex $x_{k}$.

Theorem 4.5. For every positive integer $k$ there exists $h(k)$ such that the following holds. If $G$ is a graph on $n$ vertices with maximum degree $\Delta$ then it contains an induced subgraph $H$ on at least $n-(h(k)+k) \sqrt{\Delta}$ vertices, such that $r_{k}(H) \leq h(k) \cdot k$.

Proof. The proof consists of two parts. Firstly, we shall show that we can remove at most $k \sqrt{\Delta}$ vertices from $G$ so that in the remaining graph $H^{\prime}$ we have $r_{k}\left(H^{\prime}\right) \leq \sqrt{\Delta}$. Then we iteratively apply Lemma 4.4 (at most $\sqrt{\Delta}$ times) in order to obtain an induced subgraph $H$ of $H^{\prime}$ on at least $n-(h(k)+k) \sqrt{\Delta}$ vertices such that $r_{k}(H) \leq h(k) \cdot k$. We may take $h(k)$ to be $f^{\prime}(k)$ from Lemma 4.4. We start with the first part of the proof.

Claim 4.6. There is an induced subgraph $H^{\prime}$ of $G$ on at least $n-k \sqrt{\Delta}$ vertices such that $r_{k}\left(H^{\prime}\right) \leq \sqrt{\Delta}$.

Proof of Claim 4.6. The idea of the proof is to keep removing some $k$ vertices of highest
possible degrees and observe that the maximum degree on the induced remaining graph must have decreased considerably. Indeed, consider the following procedure. Let $G_{0}=G$ and suppose that $G_{0} \supset \cdots \supset G_{i}$ have been defined. If $G_{i}$ does not have the required property then, let $G_{i+1}$ be obtained from $G_{i}$ by removing some $k$ vertices with largest degrees in $G_{i}$. Notice that $\Delta\left(G_{i+1}\right) \leq \Delta\left(G_{i}\right)-\sqrt{\Delta}$ since, by assumption, there were at most $k$ vertices in $G_{i}$ having degrees in the range $\left[\Delta\left(G_{i}\right), \Delta\left(G_{i}\right)-\sqrt{\Delta}\right]$. Also $\left|G_{i+1}\right|=\left|G_{i}\right|-k$. Observe that the procedure will stop after at most $\sqrt{\Delta}$ steps, as otherwise the obtained graph would have maximum degree 0 . Since $\left|G_{i}\right| \geq n-i \cdot k$ we have that $\left|H^{\prime}\right| \geq n-k \sqrt{\Delta}$.

We now proceed to the second part of the proof and iteratively apply Lemma 4.4. In each step we remove at most $h(k)$ vertices from $H^{\prime}$ while decreasing the value of $r_{k}$ and we stop when $r_{k}$ is at most $k \cdot h(k)$.

Let $H_{0}=H^{\prime}$ and suppose that $H_{0}, \ldots, H_{i}$ have already been defined. If $r_{k}\left(H_{i}\right) \leq k \cdot h(k)$ then we are done, so we may assume that $r_{k}\left(H_{i}\right)>k \cdot h(k)$. Let $A=\left\{x_{1}, \ldots, x_{k}\right\}$ be a set of $k$ vertices with the largest degrees in $H_{i}$ and write $B$ for $H_{i} \backslash A$. Without loss of generality we may assume that $d_{H_{i}}\left(x_{1}\right) \geq \cdots \geq d_{H_{i}}\left(x_{k}\right)$. Since $r_{k}\left(H_{i}\right)>k \cdot h(k)$ there must exist $l \in\{2, \ldots, k\}$ such that $d_{H_{i}}\left(x_{l}\right)>d_{H_{i}}\left(x_{l-1}\right)+h(k)$. Now consider the bipartite subgraph $K=H_{i}[A, B]$. By Lemma 4.4, with $G=K$ and $n=k$, we can remove a set $W \subset B$ of at most $f^{\prime}(k)=h(k)$ vertices from $B$, and obtain $K^{\prime}=H_{i}[A,(B \backslash W)]$ such that

$$
\begin{equation*}
\text { for any } x, y \in A \text {, if } d_{K}(x)<d_{K}(y) \text { then } d_{K}(x)-d_{K^{\prime}}(x)<d_{K}(y)-d_{K^{\prime}}(y) \tag{4.1}
\end{equation*}
$$

Let $H_{i+1}=H_{i} \backslash W$ (hence $\left|H_{i+1}\right| \geq\left|H_{i}\right|-|W| \geq\left|H_{i}\right|-h(k)$ ). The following claim asserts that the above procedure will stop after at most $\sqrt{\Delta}$ steps.

Claim 4.7. $r_{k}\left(H_{i+1}\right)<r_{k}\left(H_{i}\right)$.

Proof of Claim 4.7 Let $z$ be a vertex with the maximum degree and $w$ a vertex with the
$k^{\prime}$ 'th largest degree in $H_{i+1}$. Observe that $z=x_{t}$ for some $t \geq l$ and $d_{H_{i+1}}(w) \geq d_{H_{i+1}}\left(x_{s}\right)$ for some $s<l$. First, notice that $d_{H_{i}}\left(x_{t}\right)-d_{H_{i}}\left(x_{s}\right) \leq d_{H_{i}}\left(x_{1}\right)-d_{H_{i}}\left(x_{k}\right)=r_{k}\left(H_{i}\right)$. Hence, $r_{k}\left(H_{i+1}\right)=d_{H_{i+1}}(z)-d_{H_{i+1}}(w) \leq d_{H_{i+1}}\left(x_{t}\right)-d_{H_{i+1}}\left(x_{s}\right)<d_{H_{i}}\left(x_{t}\right)-d_{H_{i}}\left(x_{s}\right) \leq r_{k}\left(H_{i}\right)$, where the strict inequality follows from (4.1) since $d_{K}\left(x_{t}\right)>d_{K}\left(x_{s}\right)$.

As in each iteration the value of $r_{k}$ decreases, we must stop after at most $r_{k}\left(H^{\prime}\right)=\sqrt{\Delta}$ steps thus getting an induced subgraph $H \subset H^{\prime}$ with $r_{k}(H) \leq k \cdot h(k)$ and $|H| \geq\left|H^{\prime}\right|-h(k) \sqrt{\Delta} \geq n-(h(k)+k) \sqrt{\Delta}$.

In order to prove Theorem 4.2 we need the following theorem of Caro, Shapira and Yuster, appearing in [20], whose proof is inspired by the one used by Alon and Berman in [4].

Theorem 4.8. For positive integers $r, d, q$, the following holds. Any sequence of $n \geq(\lceil q / r\rceil+2)(2 r d+1)^{d}$ elements of $[-r, r]^{d}$ whose sum, denoted by $z$, is in $[-q, q]^{d}$ contains a subsequence of length at most $(\lceil q / r\rceil+2)(2 r d+1)^{d}$ whose sum is $z$.

As usual, we write $R(k)$ (see e.g. [13]) for the two colored Ramsey number, the least integer $n$ such that in any two coloring of the edges of the complete graph on $n$ vertices, there is a monochromatic $K_{k}$.

Proof of Theorem 4.2. Firstly, we apply Theorem 4.5 with $k=R(k)$ to find a large induced subgraph $G^{\prime} \subset G$ of order at least $n^{\prime} \geq n-(h(R(k))+R(k)) \sqrt{\Delta}$ and with vertex set $\left\{x_{1}, \ldots, x_{n^{\prime}}\right\}$ where $d\left(x_{1}\right) \geq d\left(x_{2}\right) \geq \cdots \geq d\left(x_{n^{\prime}}\right)$ and $d\left(x_{1}\right)-d\left(x_{R(k)}\right) \leq h(R(k)) \cdot R(k)=M$. Now we follow the proof of Theorem 1.1 in [20]. By the definition of $R(k)$ we can find a set $S$ of $k$ vertices in $\left\{x_{1}, \ldots, x_{R(k)}\right\}$ that induces either a complete graph or an independent set.

Without loss of generality, assume that $S=\left\{v_{n^{\prime}-k+1}, \ldots, v_{n^{\prime}}\right\}$ and $V(G) \backslash S=\left\{v_{1}, \ldots, v_{n^{\prime}-k}\right\}$. Let $e\left(v_{i}, v_{j}\right)$ be equal to 1 if there is an edge between $v_{i}$ and $v_{j}$, and 0 otherwise. We construct a sequence $X$ of $n^{\prime}-k$ vectors $w_{1}, \ldots, w_{n^{\prime}-k}$ in $[-1,1]^{k-1}$
as follows. The coordinate $j$ of $w_{i}$ is $e\left(v_{n^{\prime}-k+j}, v_{i}\right)-e\left(v_{n^{\prime}}, v_{i}\right)$ for $i=1, \ldots, n^{\prime}-k$ and $j=1, \ldots, k-1$. It is clear that $e\left(v_{n^{\prime}-k+j}, v_{i}\right)-e\left(v_{n^{\prime}}, v_{i}\right) \in[-1,1]$ as required. Consider the sum of all the $j$ 'th coordinates,

$$
\begin{aligned}
\sum_{i=1}^{n^{\prime}-k}\left(e\left(v_{n^{\prime}-k+j}, v_{i}\right)-e\left(v_{n^{\prime}}, v_{i}\right)\right) & =\sum_{i=1}^{n^{\prime}-k} e\left(v_{n^{\prime}-k+j}, v_{i}\right)-\sum_{i=1}^{n^{\prime}-k} e\left(v_{n^{\prime}}, v_{i}\right) \\
& =\left(d\left(v_{n^{\prime}-k+j}\right)-a\right)-\left(d\left(v_{n^{\prime}}\right)-a\right) \\
& =d\left(v_{n^{\prime}-k+j}\right)-d\left(v_{n^{\prime}}\right) \leq M,
\end{aligned}
$$

where $a=k-1$ if $G^{\prime}[S]$ is complete, and $a=0$ otherwise. Hence,

$$
z=\sum_{i=1}^{n^{\prime}-k} w_{i} \in[-M, M]^{k-1}
$$

By Theorem 4.8, with $d=k-1$ and $q=M$, there is a subsequence of $X$ of size at most $(M+2)(2 k-1)^{k-1}$ whose sum is $z$. Deleting the vertices of $G^{\prime}$ corresponding to the elements of this subsequence results in an induced subgraph $H \subset G^{\prime}$ in which all the $k$ vertices of $S$ have the same degree of order at least $\Delta(H)-\left(M+(M+2)(2 k-1)^{k-1}\right)$. Choosing $g_{1}(k)=g_{2}(k)=h(R(k))(4 k)^{k}$ we conclude the theorem.

### 4.3 Remarks

In the previous section, we proved that every graph contains a large induced subgraph with at least $k$ vertices having the same degree of order almost the maximum degree. Note that Theorem4.2 is sharp up to the size of the functions $g_{1}(k)$ and $g_{2}(k)$. Indeed, there are graphs for which one needs to remove "roughly" $\frac{k}{2} \sqrt{\Delta}$ vertices to force the remaining subgraph to have $k$ vertices with the same degree "near" the maximum degree. For any $k$ and $\Delta$, let $G^{\Delta}$ be the disjoint union of the stars $K_{1, n_{1}}, \ldots, K_{1, n_{t}}$, where $n_{i}=i \cdot \sqrt{\Delta}$, for $i \in\{1, \ldots, t=\sqrt{\Delta}\}$ and let $G_{k}^{\Delta}$ to be the disjoint union of $k / 2$ copies of $G^{\Delta}$. It is easy to
see that, for any constant $D$, one needs to remove at least $\frac{k}{2} \sqrt{\Delta}-\frac{k}{2} D$ vertices from $G_{k}^{\Delta}$ in order to obtain an induced graph $H$ with $k$ vertices of the same degree of order at least $\Delta(H)-D$.

Whether removing $C(k) \sqrt{\Delta}$ vertices is enough to force the remaining induced subgraph to have at least $k$ vertices of exactly maximum degree remains an interesting open question.

## CHAPTER 5

## BOUNDS ON THE GRAPH BURNING NUMBER

In this chapter we prove a few results concerning the burning number, $b(G)$, of a graph $G$ which is a graph parameter defined by Bonato, Janssen, and Roshanbin [16] which, supposedly, measures the speed of the spread of contagion in a graph. We show that for any connected graph $G$ on $n$ vertices, its burning number is bounded above by $\left\lceil\sqrt{\frac{4}{3} n}\right\rceil$. This makes a progress towards a conjecture of Bonato, Janssen, and Roshanbin who conjectured that any connected graph burns in at most $\sqrt{n}$ rounds. Moreover, we prove that if $G$ is a disjoint union of $k$ paths then $b(G) \leq\left\lceil\sqrt{|G|+(k-1)^{2}}\right\rceil$, which we later use to show that $b(S) \leq\lceil\sqrt{n}\rceil$, for any spider graph $S$ on $n$ vertices. This work is joint with Kazuhiro Nomoto, Julian Sahasrabudhe, and Richard Snyder.

### 5.1 Introduction

Graph burning is a deterministic process defined on a graph, which was introduced by Bonato, Janssen, and Roshanbin [16], and which is supposed to model the expansion of a fire in a graph. In each step, first the fire spreads from burning vertices to their neighbors that are not already burning, then a new fire starts at some, not yet burning, vertex. The burning number of a graph $G$, denoted by $b(G)$, is the smallest possible number of steps needed to burn the whole graph $G$. The process was inspired by other graph processes like firefighting, graph cleaning, bootstrap percolation (see for example [38, 5, 8]).

For a vertex $v \in G$ and a non-negative integer $r$, define, $B_{G}(v, r)$ to be the set of vertices in $G$ which are at distance at most $r$ from $v$. For brevity we shall drop the subscript and simply write $B(v, r)$, when there is no risk of confusion.

It is easy to see that the problem of determining the burning number of a graph $G$ is equivalent to finding the smallest integer $k$ such that one can cover the vertices of $G$ with some graphs $H_{0}, \ldots, H_{k-1}$ such that $H_{i}$ has radius $i$. Alternatively, we can define $b(G)$ to be
the smallest integer $k$ such that there is a sequence of vertices $x_{0}, \cdots, x_{k-1}$ in $G$ such that

$$
V(G)=B\left(x_{0}, 0\right) \cup B\left(x_{1}, 1\right) \cup \cdots \cup B\left(x_{k-1}, k-1\right),
$$

or, equivalently, for every $y \in G$ there is $i \in\{0, \cdots, k-1\}$ such that $d\left(x_{i}, y\right) \leq i$.
In their original paper, Bonato, Janssen, and Roshanbin asked the following conjecture, which attracted considerable attention.

Conjecture 5.1 (Bonato, Janssen, Roshanbin [16]). Let $G$ be a connected graph on $n$ vertices. Then

$$
b(G) \leq\lceil\sqrt{n}\rceil .
$$

If the conjecture is true then the result is best possible, as seen by considering a path on $n$ vertices. The conjecture remains open but, nevertheless, certain progress towards it has been made. In the original paper [16] the authors proved that for any connected graph $G$ on $n$ vertices we have $b(G) \leq 2 \sqrt{n}-1$. This was later improved by Bessy, Bonato, Janssen, Rautenbach, and Roshanbin [9] who showed that $b(G) \leq \sqrt{\frac{32}{19} \frac{n}{1-\varepsilon}}+\sqrt{\frac{27}{19 \varepsilon}}$ and $b(G) \leq \sqrt{\frac{12 n}{7}}+3 \sim 1.309 \sqrt{n}$, for every $\varepsilon \in(0,1)$. Finally, Land and Lu [61] showed the bound $b(G) \leq\left\lceil\frac{-3+\sqrt{24 n+33}}{4}\right\rceil \sim 1.22 \sqrt{n}$. We make a further improvement and show the following.

Theorem 5.2. Let $G$ be a connected graph on $n$ vertices. Then

$$
b(G) \leq\left\lceil\sqrt{\frac{4}{3} n}\right\rceil \sim 1.15 \sqrt{n}
$$

The burning number has been also studied for other classes of graphs. Mitsche, Prałat, and Roshanbin [68] considered random binomial graphs, random geometric graphs, and the Cartesian products of paths. Fitzpatrick and Wilm [39] studied the graph burning numbers of circulant graphs. Sim, Tan, Wong [74] gave asymptotically tight bounds for the class of generalized Petersen graphs. Bonato, and Lidbetter [17] proved the
following two bounds on the burning number of (disjoint) union of paths.

Theorem 5.3 (Bonato, Lindbetter [17]). Let $G$ be a union of $k$ paths on $n$ vertices in total. Then $b(G) \leq\left\lfloor\frac{n}{2 k}\right\rfloor+k$. Moreover, when $k \leq\lceil\sqrt{n}\rceil$ then $b(G) \leq\left\lceil\sqrt{n}+\frac{k-1}{2}\right\rceil$.

They also showed that spider graphs (trees with exactly one vertex of degree at least 3) burn in $\lceil\sqrt{n}\rceil$ rounds. Here we obtain better than in Theorem 5.3 bounds on the burning numbers of unions of paths.

Theorem 5.4. Let $G$ be a union of $k$ paths on $n$ vertices in total. Then

$$
b(G) \leq\left\lceil\sqrt{n+(k-1)^{2}}\right\rceil
$$

Observe that when $k \geq \frac{n+1}{2}$, then Theorem 5.4 implies that $b(G) \leq k$, which together with the trivial lower bound $b(G) \geq k$, gives $b(G)=k$. Note also that the above theorem is tight for every $n$ and $k \leq \frac{n}{2}$ as well, since it takes $\left\lceil\sqrt{n+(k-1)^{2}}\right\rceil$ steps to burn a union of $k-1$ paths of order 2 and one path of order $n-2 k+2$. Consequently, we give an alternative proof of the fact that $b(G) \leq\lceil\sqrt{n}$ when $G$ is a spider graph.

Theorem 5.5. Let $G$ be a spider on $n$ vertices. Then

$$
b(G) \leq\lceil\sqrt{n}\rceil .
$$

### 5.2 Tight bounds on burning numbers of spiders

A spider graph is a tree in which exactly one vertex has degree at least three and all the other vertices have degrees at most two. We shall call the unique vertex of degree at least 3 the head of the spider. The paths obtained by removing the head from the spider graph shall called the legs of the spider graph. The aim of this section is to prove Theorem 5.5 and Theorem 5.4 . Theorem 5.4 is a corollary of the following lemma.

Lemma 5.6. Suppose $n=n_{1}+\cdots+n_{k}$ where $n_{i}$ is a positive integer, for every $i \in[k]$. Then for any positive integer $t$ such that $t^{2} \geq n+(k-1)^{2}$ there is a partition of integers $\{0, \cdots, t-1\}$ into $A_{1}, \cdots, A_{k}$ such that, for every $i \in[k]$

$$
\sum_{a \in A_{i}}(2 a+1) \geq n_{i} .
$$

Let us first deduce Theorem 5.4 from Lemma 5.6 .

Proof of Theorem 5.4. Let $G$ be a union of $k$ paths $P_{1}, P_{2}, \ldots, P_{k}$. Write $n_{i}$ for $\left|P_{i}\right|$ and $|G|=: n=n_{1}+n_{2}+\ldots n_{k}$. By Lemma 5.6. for $t=\left\lceil\sqrt{n+(k-1)^{2}}\right\rceil$, there is a partition of the integers $\{0, \ldots, t-1\}$ into sets $A_{1}, \ldots, A_{k}$ such that for every $i \in\{1, \ldots, k\}$, we have $\sum_{a \in A_{i}}(2 a+1) \geq n_{i}$. It is easy to see that it is possible to cover the vertices of $P_{i}$ using balls of radii in $A_{i}$.

Proof of Lemma 5.6. Given a set $S$ let $f(S)=\sum_{s \in S}(2 s+1)$. Let $\mathscr{A}=\left\{A_{1}, \cdots, A_{k}\right\}$ be a partition of $\{0, \cdots, t-1\}$ minimizing the function

$$
\begin{equation*}
F(\mathscr{A})=\sum_{i=1}^{k}\left(f\left(A_{i}\right)-n_{i}\right)^{2}, \tag{5.1}
\end{equation*}
$$

subject to every element of the partition being non-empty, i.e., $A_{i} \neq \emptyset$, for every $i$. We shall show that this partition satisfies the conclusion of the lemma. Assume for the sake of contradiction that the partition $\mathscr{A}$ does not satisfy the conclusion of the lemma, which means that for some $i$ we have $f\left(A_{i}\right)-n_{i}<0$.

For $p \in\{0, \cdots, t-1\}$ let $A^{(p)}=A_{i}$ and $n^{(p)}=n_{i}$, for $i$ such that $p \in A_{i}$. Now, define $g(p)=f\left(A^{(p)}\right)-n^{(p)}$. We claim that $g(p)$ does not grow too fast as a function of $p$.

Claim 5.7. For any $p \in\{0, \cdots, t-2\}, g(p+1) \leq g(p)+2$.

Proof. Suppose for contradiction that there is $p \in\{0, \cdots, t-2\}$ with $g(p+1) \geq g(p)+3$. We shall construct another partition with a smaller square sum 5.1, contradicting the
minimality of $\mathscr{A}$. For brevity, let us write $B=A^{(p)}$ and $C=A^{(p+1)}$. Let $\mathscr{A}^{\prime}=\left\{A_{1}^{\prime}, \cdots, A_{k}^{\prime}\right\}$ where

$$
A_{i}^{\prime}= \begin{cases}\left(A_{i} \backslash\{p\}\right) \cup\{p+1\}, & \text { if } A_{i}=B \\ \left(A_{i} \backslash\{p+1\}\right) \cup\{p\}, & \text { if } A_{i}=C \\ A_{i}, & \text { otherwise. }\end{cases}
$$

It is easy to check that $F(\mathscr{A})-F\left(\mathscr{A}^{\prime}\right)=4(g(p+1)-g(p)-2) \geq 4(3-2)=4>0$.

Our second claim says that 0 belongs to a set $A_{i}$ such that $f\left(A_{i}\right)-n_{i} \leq 0$, otherwise we could remove 0 from $A_{i}$ and put in a set $A_{j}$, for some $j$ such that $f\left(A_{j}\right)-n_{j}<0$, decreasing the square sum and obtaining a contradiction.

Claim 5.8. $g(0) \leq 0$.

Proof. Assume for contradiction that $g(0)>0$. Suppose $0 \in A_{i}$ and let $j$ be any integer such that $A_{j}-n_{j}<0$. Observe that since $n_{i}$ is a positive integer and $f\left(A_{i}\right)>n_{i}$, and $0 \in A_{i}$, it follows that $\left|A_{i}\right| \geq 2$. We can therefore move 0 from $A_{i}$ to $A_{j}$ which will result in a smaller square sum 5.1. This contradicts the minimality of $\mathscr{A}$.

Without loss of generality we can assume that $f\left(A_{i}\right)-n_{i} \leq f\left(A_{j}\right)-n_{j}$ for $i<j$. Note that $f\left(A_{1}\right)-n_{1}<0$. It follows from the above two claims that $f\left(A_{i}\right)-n_{i} \leq 2 i-3$. Therefore

$$
t^{2}=\sum_{i=1}^{k} f\left(A_{i}\right)=n+\sum_{i=1}^{k}\left(f\left(A_{i}\right)-n_{i}\right) \leq n+\sum_{i=1}^{k}(2 i-3)=n+(k-1)^{2}-1,
$$

contradicting the assumption that $t^{2} \geq n+(k-1)^{2}$.

Now we are ready to deduce Theorem 5.5 from Theorem 5.4 .
Proof of Theorem 5.5. Let $G$ be a spider with the head $v$, on $n=k^{2}+\ell$ vertices, where $1 \leq \ell \leq 2 k+1$. We shall show, using induction on $n$, that $G$ can be burned in at most $k+1$
steps. The base case, when $n \leq 4$, is trivial as the star $K_{1,3}$ is the only spider on fewer than 5 vertices, in which case $G$ burns in two steps. Suppose that the theorem holds for every integer $n^{\prime}$ such that $4 \leq n^{\prime}<n$.

Observe that when $G$ has a leg of length (where length of a leg is the number of vertices on the leg, not counting the head of the spider) at least $\ell$, then we can cover $\ell$ vertices of the leg with a ball of radius $k$, obtaining a spider or a path on $k^{2}$ vertices, which by induction burns in at most $k$ steps. Hence, in total, $G$ burns in at most $k+1$ steps. We can therefore assume that every leg of $G$ has length at most $\ell-1 \leq 2 k$.

Let $t$ be the number of legs of length at least $k+1$ in $G$. It follows from an easy calculation that $t \leq k$. We shall consider two cases, first when $t<k$ and second when $t=k$. Suppose first that $t<k$.

Consider the ball $B=B(v, k)$ and let $H=G \backslash B$. Let $w$ be the total number of vertices on legs of length at most $k$ in $G$. Observe that any leg in $G$ of length at most $k$, is completely covered by the ball $B$. Therefore $H$ is a union of $t$ paths, with total number of vertices $n-t k-1-w \leq k^{2}+2 k-t k-w$. On the other hand, by the assumption that every leg has fewer than $l \leq 2 k+1$ vertices, we have that $H$ has at most $t(l-1-k) \leq t k$ vertices. Therefore, combining these two bounds we obtain that

$$
|H| \leq \min \left\{t k, k^{2}+2 k-t k-w\right\} .
$$

Applying Theorem 5.4. we see that

$$
b(H) \leq\left\lceil\sqrt{\min \left\{t k, k^{2}+2 k-t k-w\right\}+(t-1)^{2}}\right\rceil
$$

Therefore as long as $f_{k}(t)=\min \left\{t k, k^{2}+2 k-t k-w\right\}+(t-1)^{2}$ does not exceed $k^{2}$ we are done. Easy calculation shows that $f_{k}(1) \leq k \leq k^{2}$, for $k \geq 2$, and $f_{k}(2) \leq 2 k+1 \leq k^{2}$,
for $k \geq 3$. For $3 \leq t \leq k-1$ and $k \geq 4$ we have

$$
\begin{aligned}
f_{k}(t) & \leq g_{k}(t)=k^{2}+2 k-t k+(t-1)^{2} \leq g_{k}(3)=g_{k}(k-1) \\
& =k^{2}-k+4 \leq k^{2}
\end{aligned}
$$

Therefore we can assume that $t=k$. When $w \geq 1$ then

$$
f_{k}(k) \leq 2 k-w+(k-1)^{2}=k^{2}+1-w \leq k^{2} .
$$

We can hence assume that no leg has length smaller than $k+1$. We shall consider two cases depending on the distribution of the lengths of the legs of $G$.

Case 1 - there in a leg of length exactly $k+1$. Place a ball of radius $k$ at a vertex on a leg of length exactly $k+1$, at distance 1 from the head of $G$. The remaining, uncovered, graph $H$ consists of $k-1$ paths with total number vertices at most $n-(k+1+1+(k-1)(k-1))=n-k^{2}+k-3 \leq(k+1)^{2}-k^{2}+k-3 \leq 3 k-2$. By Theorem 5.4 we have $b(H) \leq\left\lceil\sqrt{3 k-2+(k-2)^{2}}\right\rceil=\left\lceil\sqrt{k^{2}-k+2}\right\rceil \leq k$ (note that here we need the assumption that $k \geq 2$ ), hence we are done.

Case 2 - all legs have length $k+2$. Place a ball of radius $k$ at any vertex at distance exactly 2 from the head of the spider $G$. It is easy to see that the remaining graph consists of $k-1$ paths of length exactly 4 . We can cover one of the paths using balls of radius 0 and 1 , and the remaining $k-2$ paths by the balls of radii $2, \ldots, k-1$.

It is easy to check that there are no other cases.

### 5.3 General Bound on the Burning Numbers of Connected Graphs

In this section we prove Theorem 5.2, i.e., that any connected graph on $n$ vertices can be burned in $\left\lceil\sqrt{\frac{4}{3} n}\right\rceil$ rounds. It is clear that it is enough to verify the bound for trees.

Let $T$ be a tree of diameter $d$ and let $P=\left\{x_{1}, \cdots, x_{d+1}\right\}$ be a longest path in $T$. For
each $x_{i} \in P$ let $T_{i}$ be the tree rooted at $x_{i}$ consisting of $x_{i}$ and all the vertices which can be reached from $x_{i}$ not using any other vertex of $P$. It is easy to see that for any $i$ the height of $T_{i}$ is strictly less than $i$, as otherwise there would be a path in $T$ strictly longer than $P$, contradicting the maximality of $P$. We say that a ball $B \subset V(T)$ covers $A$ nicely if $A \subseteq B$ and $T \backslash A$ remains connected.

The following, rather technical lemma, is the heart of the proof.
Lemma 5.9. Fix a non-negative integer $a$. Let $X=\left\{r_{0}, \cdots, r_{s}\right\}$ with $a \leq r_{i}<r_{i+1}<\frac{d-1}{2}$ be a set of non-negative integers. Assume that for every $r_{i} \in X$ there is no set of at least $2 r_{i}+1-a$ vertices which can be covered nicely with a ball of radius $r_{i}$. Let $B_{i}=B\left(x_{r_{i}+1}, r_{i}\right)$ be the (closed) ball of radius $r_{i}$ centered at $x_{1+r_{i}}$ and let $j_{i}$ be the smallest non-negative integer such that $T_{j_{i}} \nsubseteq B_{i}$. Then the following is true for any $t \in\{0, \ldots, s\}$.

1. $r_{t}+1<j_{t} \leq 2 r_{t}+2-2 t-a$ (and hence $r_{t} \geq 2 t+a$ ),
2. $\sum_{i=1}^{j_{t}}\left|T_{i} \cap B_{t}\right| \geq 2 t+a+j_{t}$.

Proof. We will use induction on $t$. The first inequality in the first part holds for any $t$ trivially because $T_{i}$ has height strictly less than $i$, for any $i$, hence $T_{i} \subseteq B_{t}$, for every $i \leq r_{t}+1$. Let us consider the base case when $t=0$. It is easy to show that $j_{0} \leq 2 r_{0}+1-a$, as otherwise we obtain a contradiction by observing that $B_{0}$ covers nicely at least $2 r_{0}+1-a$ vertices. To prove the second part, suppose $j_{0}=r_{0}+1+b$, for some non-negative integer $b$. We have that $T_{j_{0}}$ has height bigger than $r_{0}-b$ as otherwise $T_{j_{0}}$ would be covered by $B_{0}$. Hence

$$
\sum_{i=1}^{j_{0}}\left|T_{i} \cap B_{0}\right| \geq r_{0}+1+b+r_{0}-b=2 r_{0}+1 \geq j_{0}+a
$$

Now suppose the statement of the lemma is true for some non-negative integer $t<s$. We shall show it is true for $t+1$. Assume first that the first part does not hold, hence

$$
j_{t+1}>2 r_{t+1}+2-2(t+1)-a=2 r_{t+1}-2 t-a \geq 2 r_{t}+2-2 t-a \geq j_{t}
$$

where the last inequality follows by the induction hypothesis. It means that $B_{t+1}$ covers $T_{i}$ nicely, for every $i \leq 2 r_{t+1}-2 t-a$, hence the number of nicely covered vertices is at least

$$
\begin{aligned}
\sum_{i=1}^{2 r_{t+1}-2 t-a}\left|T_{i}\right| & \geq \sum_{i=1}^{j_{t}}\left|T_{i}\right|+2 r_{t+1}-2 t-a-j_{t} \\
& \geq \sum_{i=1}^{j_{t}}\left|T_{i} \cap B_{t}\right|+1+2 r_{t+1}-2 t-a-j_{t} \\
& \geq 2 t+a+j_{t}+1+2 r_{t+1}-2 t-a-j_{t} \\
& =2 r_{t+1}+1 \geq 2 r_{t+1}+1-a
\end{aligned}
$$

where the third inequality holds by induction and in the second inequality we used the fact that $B_{t+1}$ covers $T_{j_{t}}$ completely, whereas $B_{t}$ did not, hence there is at least one vertex in $T_{j_{t}}$ uncovered by $B_{t}$ but covered by $B_{t+1}$. We therefore obtain a contradiction to the assumption that no $B_{t}$ covers nicely more than $2 r_{t}-a$ vertices. Hence $j_{t+1} \leq 2 r_{t+1}+2-2(t+1)-a$ and the first part holds.

To prove the second part we shall consider two cases depending on $j_{t+1}$.
It is easy to check then when $j_{t+1}=j_{t}$ then $B_{t+1}$ covers at least $2\left(r_{t+1}-r_{t}\right)$ more vertices of $T_{j_{t}}$ than $B_{t}$. Therefore

$$
\left|T_{j_{t+1}} \cap B_{t+1}\right| \geq\left|T_{j_{t+1}} \cap B_{t}\right|+2\left(r_{t+1}-r_{t}\right) \geq\left|T_{j_{t}} \cap B_{t}\right|+2
$$

Hence, again by the induction hypothesis,

$$
\begin{aligned}
\sum_{i=1}^{j_{t+1}}\left|T_{i} \cap B_{t+1}\right| & \geq \sum_{i=1}^{j_{t}}\left|T_{i} \cap B_{t}\right|+2 \geq 2 t+a+j_{t}+2 \\
& =2(t+1)+a+j_{t+1}
\end{aligned}
$$

On the other hand, when $j_{t}<j_{t+1}$ then $T_{j_{t}} \cap B_{t+1}=T_{j_{t}}$ and hence $\left|T_{j_{t}} \cap B_{t+1}\right| \geq\left|T_{j_{t}} \cap B_{t}\right|+1$. Observe that
$\left|T_{j_{t+1}} \cap B_{t+1}\right| \geq 2 r_{t+1}+2-j_{t+1} \geq 2 r_{t+1}+2-\left(2 r_{t+1}+2-2(t+1)-a\right)=2(t+1)+a \geq 2$,
by the first part (which we already proved for $t+1$ ), and hence

$$
\begin{aligned}
\sum_{i=1}^{j_{t+1}}\left|T_{i} \cap B_{t+1}\right| & =\sum_{i=1}^{j_{t}}\left|T_{i} \cap B_{t+1}\right|+\sum_{i=j_{t}+1}^{j_{t+1}-1}\left|T_{i} \cap B_{t+1}\right|+\left|T_{j_{t+1}} \cap B_{t+1}\right| \\
& \geq\left(\sum_{i=1}^{j_{t}}\left|T_{i} \cap B_{t}\right|+1\right)+\left(j_{t+1}-j_{t}-1\right)+2 \\
& \geq 2 t+a+j_{t}+2+j_{t+1}-j_{t}=2(t+1)+a+j_{t+1}
\end{aligned}
$$

Which completes the proof.

We have an immediate corollary to that lemma which, roughly speaking, says that given a tree and a "big" set of distinct non-negative integers there is a ball of radius in that set which covers nicely "many" vertices of the tree.

Corollary 5.10. Let $T$ be a tree and $X=\left\{r_{0}, \cdots, r_{s}\right\}$ be a set of non-negative integers with $r_{i}<r_{i+1}$. Then either

1. $T$ can be covered by a ball of radius $r_{s}$, or
2. For $a=\max \left\{r_{s}+1-2 s, 0\right\}$ there is a ball $B$ of radius $r \in X$ which covers nicely at least $2 r+1-a$ vertices.

Proof. Suppose for contradiction that $T$ cannot be covered by a ball of radius $r_{s}$ and for $a=\max \left\{r_{s}+1-2 s, 0\right\}$ there is no set of $2 r+1-a$ vertices which can be covered nicely by a ball of radius $r \in X$. It follows that $r_{s}<\frac{\operatorname{diam}(T)-1}{2}$, hence we can apply Lemma 5.9 and conclude that $r_{t} \geq 2 t+a$, for every $t \in\{0, \cdots, s\}$. In particular $r_{s} \geq 2 s+a \geq 2 s+r_{s}+1-2 s=r_{s}+1$, which is a contradiction.

Lemma 5.11. Fix $M$. Let $T$ be any tree and $X$ be a set of distinct non-negative integers with $\max X \leq M$. Suppose $T$ cannot be covered by balls with radii from $X$. Then, using balls of radii in $X$, we can cover at least

$$
\sum_{r \in X}(2 r+1)-\frac{M^{2}}{4}
$$

vertices of $T$.

Proof. We shall first show that in the case when the cardinality of $X$ is not too large, i.e., when $|X| \leq \frac{M+1}{2}$, we can cover nicely at least

$$
\sum_{r \in X}(2 r+1)+|X|^{2}-|X| M
$$

vertices of $T$. The proof will use induction on the cardinality of $|X|$. The base case $|X|=0$ is trivial. Suppose the result holds for any set of non-negative integers $X^{\prime}$ with $\max X^{\prime} \leq M$ and $\left|X^{\prime}\right|<|X|$. It follows from the assumption that $|X| \leq \frac{M+1}{2}$ and Corollary 5.10, that there is a ball $B$ of radius $r \in X$ which, for $a \leq M+1-2|X|$, covers nicely at least $2 r+1-a \geq 2 r+2|X|-M$ vertices of $T$. Let $A$ be a set of $2 r+2|X|-M$ vertices which are covered nicely by $B$ and let $T^{\prime}=T \backslash A$ and $X^{\prime}=X \backslash\{r\}$. By the induction hypothesis we can cover nicely $\sum_{r^{\prime} \in X^{\prime}}\left(2 r^{\prime}+1\right)+\left|X^{\prime}\right|^{2}-\left|X^{\prime}\right| M$ vertices in $T^{\prime}$ using balls of radii from $X^{\prime}$. Hence in total we can cover at least

$$
\begin{aligned}
& \sum_{r^{\prime} \in X^{\prime}}\left(2 r^{\prime}+1\right)+\left|X^{\prime}\right|^{2}-\left|X^{\prime}\right| M+2 r+2|X|-M \\
= & \sum_{r^{\prime} \in X^{\prime}}\left(2 r^{\prime}+1\right)+|X|^{2}+2 r+1-|X| M \\
= & \sum_{r \in X}(2 r+1)+|X|^{2}-|X| M
\end{aligned}
$$

vertices of $T$ with balls of radii in $X$. This finishes the claim. An instant corollary of that claim is that when $|X| \leq \frac{M+1}{2}$ then the conclusion of the lemma holds, as

$$
|X|^{2}-|X| M=|X|(|X|-M) \geq-\frac{M^{2}}{4}
$$

Suppose now that $|X| \geq \frac{M+1}{2}$. Observe that in this case it follows from Corollary 5.10 that there is a ball of radius $r \in X$ which covers nicely at least $2 r+1$ vertices of $T$. Applying this observation iteratively we can conclude that there is a subset
$Y$ of $X$ such that $|X \backslash Y| \leq \frac{M+1}{2}$ and we can cover nicely $L^{\prime} \geq \sum_{r \in Y}(2 r+1)$ vertices of $T$. Let $T^{\prime}$ be the uncovered subtree of $T$ (hence $\left|T^{\prime}\right| \leq T-L^{\prime}$ ) and $X^{\prime}=X \backslash Y$. Since $\left|X^{\prime}\right| \leq \frac{M+1}{2}$ we can use the claim made in the paragraph above to conclude that we can cover nicely $L^{\prime \prime} \geq \sum_{r \in X^{\prime}}(2 r+1)+\left|X^{\prime}\right|^{2}+\left|X^{\prime}\right| M$ vertices of $T^{\prime}$. In total we have covered at least

$$
\begin{aligned}
L^{\prime}+L^{\prime \prime} & \geq \sum_{r \in Y}(2 r+1)+\sum_{r \in X \backslash Y}(2 r+1)+\left|X^{\prime}\right|^{2}-\left|X^{\prime}\right| M \\
& =\sum_{r \in X}(2 r+1)+\left|X^{\prime}\right|^{2}-\left|X^{\prime}\right| M \\
& \geq \sum_{r \in X}(2 r+1)-\frac{M^{2}}{4}
\end{aligned}
$$

We can now easily deduce Theorem 5.2 from Lemma 5.11 .
Proof of Theorem 5.2. Let $X=\{0, \ldots, m-1\}$, where $m=\left\lceil\sqrt{\frac{4}{3} n}\right\rceil$. Suppose for contradiction that $T$ cannot be burned in $m$ rounds. Then, by Lemma 5.11, we can cover $\sum_{r=0}^{m-1}(2 r+1)-\frac{(m-1)^{2}}{4} \geq m^{2}-\frac{m^{2}}{4}=\frac{3}{4} m^{2} \geq n$ vertices using the balls of radii at most $m$. This gives a contradiction.

## CHAPTER 6

## ON POSSIBLE NUMBERS OF COPIES OF A FIXED GRAPH

In this chapter we investigate the set $T_{n}$ of possible number of triangles in a graph on $n$ vertices. The first main result says that every natural number less than $\binom{n}{3}-(\sqrt{2}+o(1)) n^{3 / 2}$ belongs to $T_{n}$. On the other hand, we show that there is a number $m=\binom{n}{3}-(\sqrt{2}+o(1)) n^{3 / 2}$ which is not a member of $T_{n}$. In addition, we prove that there are two interlacing sequences $\binom{n}{3}-(\sqrt{2}+o(1)) n^{3 / 2}=c_{1} \leq d_{1} \leq c_{2} \leq d_{2} \leq \cdots \leq c_{s} \leq d_{s}=\binom{n}{3}$ with $\left|c_{t}-d_{t}\right|=n-2-\binom{s-t+1}{2}$ such that $\left(c_{t}, d_{t}\right) \cap T_{n}=\emptyset$ for all $t$. Moreover, we obtain a generalization of these results for the set of possible number of copies of a connected graph $H$ in a graph on $n$ vertices. This work is joint with Teeradej Kittipassorn.

### 6.1 Introduction

We ask the following natural question: given a graph $H$ and a natural number $n$, what are the possible values of $m$ such that there exists a graph on $n$ vertices with exactly $m$ copies of $H$ ? Surprisingly, very little is known about this problem.

A question of this flavor was first considered by Kittipassorn and Mészáros [56] who studied the set $F_{n}$ of possible number of frustrated triangles, i.e. triples of vertices inducing an odd number of edges. They proved that about two thirds of the numbers in $\left[0, n^{3 / 2}\right]$ do not appear in $F_{n}$ and every even number between $(1+o(1)) n^{3 / 2}$ and $\binom{n}{3}-(1+o(1)) n^{3 / 2}$ is a member of $F_{n}$ for sufficiently large and even $n$.

Much more attention has been given to the problem of maximizing or minimizing the number of subgraphs of certain type in graphs of given number of vertices and edges. For example, Rademacher proved that every graph with $\left\lfloor n^{2} / 4\right\rfloor+1$ edges contains at least $\lfloor n / 2\rfloor$ triangles. Erdős [29] posed a conjecture, which was later proved by Lovasz and Simonovits [64], that a graph of size $\left\lfloor n^{2} / 4\right\rfloor+k$ contains at least $k\lfloor n / 2\rfloor$ triangles if
$k<n / 2$. On the other hand, Alon [2] investigated the maximum number of subgraphs isomorphic to some given graph where the maximum is taken over all graphs of certain size. We refer interested readers to [42, 27, 62, 31, 28, 30, 70, 71] for similar results.

We consider a fixed connected graph $H$. For a graph $G$, we define $k_{H}(G)$ to be the number of copies of $H$ in $G$. The main object of interest of this chapter is

$$
S_{H}^{(n)}=\left\{k_{H}(G):|G|=n\right\},
$$

the set of possible number of copies of $H$ in a graph on $n$ vertices. Our first result says that almost every number (in the appropriate range) is realizable as a number of copies of $H$ in some graph of order $n$.

Theorem 6.1. As $n \rightarrow \infty$, the following holds

$$
\left[0,(1-o(1)) k_{H}\left(K_{n}\right)\right] \subset S_{H}^{(n)}
$$

It is not unreasonable to expect that $o(1)$ above could be replaced by 0 . As it happens, this is not the case. Before we state the second result, let us introduce some notations which play a major role in the rest of the chapter. For a graph $G=(V, E)$ of order $n$, let $f_{H}(G)$ be the number of subgraphs $H$ of $K(V)$ isomorphic to $F$ such that $E(H) \cap E \neq \emptyset$, where $K(V)$ denotes the complete graph on the vertex set $V$. For instance, $f_{K_{3}}(G)$ is the number of triples of vertices in $G$ inducing at least one edge. Observe that $k_{H}(G)=k_{H}\left(K_{n}\right)-f_{H}(\bar{G})$, and therefore we shall work instead with the complement of $G$ which is easier to draw when $G$ is dense. We shall write $a_{H}^{(n)}(t)=f_{H}\left(S_{t}^{(n)}\right)$ and $b_{H}^{(n)}(t)=f_{H}\left(M_{t}^{(n)}\right)$ where $S_{t}^{(n)}$ is a graph on $n$ vertices with $t$ edges forming a star with $t \leq n-1$ and $M_{t}^{(n)}$ is a graph on $n$ vertices with $t$ edges forming a matching with $t \leq\lfloor n / 2\rfloor$. We prove the existence of some forbidden intervals in $S_{H}^{(n)}$.

Theorem 6.2. For all sufficiently large $n$ and all $t \geq 0$, we have

$$
\left|\left(k_{H}\left(K_{n}\right)-a_{H}^{(n)}(t+1), k_{H}\left(K_{n}\right)-b_{H}^{(n)}(t)\right) \cap S_{H}^{(n)}\right|=o\left(n^{h-2}\right) .
$$

We shall see in Section 6.4 that when $t<c \sqrt{n}$, where $c$ is some nonnegative constant depending only on $H$, then $a_{H}^{(n)}(t+1)-b_{H}^{(n)}(t)$ is of order $n^{h-2}$.

In the case when $H$ is a triangle, we prove a sharp analog of Theorems 6.1 and 6.2 .

Theorem 6.3. The following hold.
(i) $\left[0,\binom{n}{3}-(\sqrt{2}+o(1)) n^{3 / 2}\right] \subset S_{K_{3}}^{(n)}$ as $n \rightarrow \infty$.
(ii) $\left.\binom{n}{3}-a_{K_{3}}^{(n)}(t+1),\binom{n}{3}-b_{K_{3}}^{(n)}(t)\right) \cap S_{K_{3}}^{(n)}=\emptyset$ for all $n, t \geq 0$.

We shall remark that $a_{K_{3}}^{(n)}(t)=t(n-2)-\binom{t}{2}$ and $b_{K_{3}}^{(n)}(t)=t(n-2)$, and so it is easy to check that the interval in the second part of the theorem is not empty as long as $t \lesssim \sqrt{2 n}$, whence there exists a number $m=\binom{n}{3}-(\sqrt{2}+o(1)) n^{3 / 2}$ which is not a member of $T_{n}$. Therefore the first part of Theorem 6.3 is sharp.

The rest of this chapter is organized as follows. In Section6.2, we prove some preliminary lemmas. In Section 6.3, we prove the sharp results, Theorem 6.3 for triangles. The proofs of Theorems 6.1 and 6.2 are presented in Section 6.4 . We conclude the chapter in Section 6.5 with some open problems.

### 6.2 Complete graphs

In this section we shall consider the case when $H=K_{r}$ is a complete graph and prove some basic lemmas which we shall later use to prove some of our main theorems.

Let $P_{n, r}$ be the set of possible number of copies of $K_{r}$ in an $r$-partite graph on $n$ vertices. Clearly $P_{n, r} \subseteq S_{K_{r}}^{(n)}$.

We shall start by showing that the first $\left(\left\lfloor\frac{n-r^{2}}{r}\right\rfloor\right)^{r}$ natural numbers are realizable, i.e. for every $k \leq\left(\left\lfloor\frac{n-r^{2}}{r}\right\rfloor\right)^{r}$ there exists an $r$-partite graph on $n$ vertices with exactly $k$ copies
of $K_{r}$.
Lemma 6.4. For natural numbers $n \geq r \geq 2$ and any non-negative integer $k \leq\left(\left\lfloor\frac{n-r^{2}}{r}\right\rfloor\right)^{r}$ there is an $r$-partite graph with $k$ copies of $K_{r}$.

Proof. We shall use induction on $r$. The base case $r=2$ is trivial. Suppose the assertion holds for some $r \geq 2$. Let $G=\left(V_{1} \sqcup V_{2} \sqcup V_{3}, E\right)$ where $\left|V_{1}\right|=\left\lceil\frac{r}{r+1}(n-r)\right\rceil$, $\left|V_{2}\right|=\left\lfloor\frac{1}{r+1}(n-r)\right\rfloor$, every vertex in $V_{1}$ is joint to every vertex in $V_{2}$, and $V_{3}$ induces $K_{r}$. By the induction hypothesis we can replace $V_{1}$ by an $r$-partite graph having $k$ copies of $K_{r}$, therefore obtaining an $(r+1)$-partite graph having $\left\lfloor\frac{1}{r+1}(n-r)\right\rfloor k$ copies of $K_{r+1}$, for any $k \leq\left(\frac{\frac{r(n-r)}{r+1}-r^{2}}{r}\right)^{r}=\left(\frac{n-r}{r+1}-r\right)^{r}$. Therefore, we get an increasing sequence in $P_{n, r+1}$ starting with 0 and ending with
$\left\lfloor\frac{n-r}{r+1}\right\rfloor\left(\left\lfloor\frac{n-r}{r+1}-r\right\rfloor\right)^{r} \geq\left(\left\lfloor\frac{n-r}{r+1}-r\right\rfloor\right)^{r+1}=\left(\left\lfloor\frac{n-r(r+2)}{r+1}\right\rfloor\right)^{r+1} \geq\left(\left\lfloor\frac{n-(r+1)^{2}}{r+1}\right\rfloor\right)^{r+1}$ such that the difference between consecutive terms is equal to $\left|V_{2}\right|$. To obtain the missing numbers between consecutive terms, notice that it is enough to join needed number of vertices in $V_{2}$ to every vertex in $V_{3}$. Clearly all graphs in the sequence are $(r+1)$-partite.

We shall fix $n, r \geq 2$ and for brevity write $f(G)=f_{K_{r}}(G), a_{t}=a_{K_{r}}^{(n)}(t), b_{t}=b_{K_{r}}^{(n)}$. Recall that the number of copies of $K_{r}$ in $G$ is equal to $\binom{n}{r}-f(\bar{G})$ where $f(G)$ is the number of $r$-sets of vertices in $G$ which induce at least one edge. Therefore, we shall later work instead with the complement of $G$ which is easier to deal with when $G$ is dense.

Observe that $a_{t}=\sum_{i=1}^{t}\binom{n-1-i}{r-2}$ and hence $a_{t+1}-a_{t}=\binom{n-1-(t+1)}{r-2}$.
Lemma 6.5. For a graph $G$ on $n$ vertices and $e \leq \frac{n-1}{2}$ edges, $f(G) \in\left[a_{e}, b_{e}\right]$.

Proof. We shall show this by induction on the number of edges $e$. In the base case when $e \leq 1$, there is nothing to show. For $e>1$, assume that $f\left(G^{\prime}\right) \in\left[a_{e-1}, b_{e-1}\right]$ for any graph $G^{\prime}$ on $n$ vertices and $e-1$ edges.

First we shall show that $f(G) \leq b_{e}$. Take an edge $x y \in G$ such that $d(y)>1$ and an isolated vertex $w \in G$. Let $G^{\prime}$ be a graph obtained by removing $x y$ from $G$ and replacing it
by $x w$. Notice that $f\left(G^{\prime}\right) \geq f(G)$, hence repeating this process for any nonindependent edge, we eventually obtain a matching, without decreasing the value of $f$.

To show that $f(G) \geq a_{e}$, pick an edge $x y \in G$ and let $G^{\prime}$ be a graph obtained by removing $x y$ from $G$. We shall show that $f(G)-f\left(G^{\prime}\right) \geq\binom{ n-1-e}{r-2}=a_{e}-a_{e-1}$ which will complete the proof, as then $f(G) \geq f\left(G^{\prime}\right)+a_{e}-a_{e-1} \geq a_{e}$, by the induction hypothesis applied to $G^{\prime}$. Let $A=V(G) \backslash\left(N_{G}(x) \cup N_{G}(y)\right)$ (observe that $x, y \neq A$ ). Write $M$ for a largest independent set contained in $A$ and $e_{A}$ for the number of edges induced by $A$. It is easy to show that $|M| \geq|A|-e_{A}$. Therefore $f(G)-f\left(G^{\prime}\right) \geq\binom{|M|}{r-2} \geq\binom{ n-|N(x) \cup N(y)|-e_{A}}{r-2} \geq\binom{ n-e-1}{r-2}$.

We remark that Lemma 6.5 does not imply that $f(G) \notin\left(b_{t}, a_{t+1}\right)$. However, the result follow immediately from the monotonicity of $f$.

Lemma 6.6. For any graph $G$ on $n$ vertices and any $t \geq 0, f(G) \notin\left(b_{t}, a_{t+1}\right)$.

Proof. We shall write $t_{\max }=\max \left\{t: b_{t-1}+1<a_{t}\right\}$ for the last $t$ where there is a gap between the intervals $\left[a_{t-1}, b_{t-1}\right]$ and $\left[a_{t}, b_{t}\right]$. It is enough to show that $f(G) \in\left[a_{t}, b_{t}\right]$ for some $t \leq t_{\max }$ or $f(G) \geq a_{t_{\max }}$. By Lemma 6.5, we are done if $e(G) \leq t_{\max }$. So we can assume that $e(G)>t_{\max }$. Let $G^{\prime}$ be a graph obtained from $G$ by deleting some edges until there are exactly $t_{\max }$ edges left. By monotonicity of $f$, we have $f(G)>f\left(G^{\prime}\right) \geq a_{t_{\max }}$ by Lemma 6.5,

We remark that $t_{\max }=\Theta(\sqrt{n})$.

### 6.3 Triangle

We shall now consider the case when $H=K_{3}$, i.e. when $H$ is a triangle. For brevity, let us write $T_{n}=S_{K_{3}}^{(n)}, f(G)=f_{K_{3}}(G), a_{t}=a_{K_{3}}^{(n)}(t)$ and $b_{t}=b_{K_{3}}^{(n)}(t)$.

In order to improve Lemma 6.4 to Theorem $6.3(i)$, let us change the direction to Theorem 6.3 (ii) and look for nonmembers of $T_{n}$. Recall that the number of triangles in $G$ is equal to $\binom{n}{3}-f(\bar{G})$ where $f(G)$ is the number of triples of vertices in $G$ which induce
at least one edge. Therefore, we shall work instead with the complement of $G$ which is easier to deal with when $G$ is dense. Notice that we have a simple formula $f(G)=e(G)(n-2)-n_{c}+n_{t}$ where $n_{c}$ is the number of cherries (i.e. paths with two edges, $P_{2}$ ) and $n_{t}$ is the number of triangles in $G$. This comes from the fact that each edge is contained in exactly $n-2$ triples, but we double count the triples which contain more than one edge. Using this formula it is easy to see that $a_{t}=t(n-2)-\binom{t}{2}$ and $b_{t}=t(n-2)$.

We have shown in Lemma 6.6 that $f(G) \notin\left(b_{t}, a_{t+1}\right)$ for all $t \geq 0$. On the other hand, we shall prove that any number bigger than $(\sqrt{2}+o(1)) n^{3 / 2}$ is realizable.

Lemma 6.7. For any sufficiently large $n$, if $m \in\left[(\sqrt{2}+o(1)) n^{3 / 2},(1-o(1))\binom{n}{3}\right]$ then there is a graph $G$ on $n$ vertices such that $f(G)=m$.

Proof. Given natural number $n$ and an integer $m \in\left[(\sqrt{2}+o(1)) n^{3 / 2},(1-o(1))\binom{n}{3}\right]$ we shall construct graph $G$ on $n$ vertices such that $f(G)=m$. Let us partition $G$ into four parts, so $V(G)=V_{1} \sqcup V_{2} \sqcup V_{3} \sqcup V_{4}$ where

- $G\left[V_{1}\right]$ is empty of order $n^{\prime}=n-2 \sqrt{2 n}-4(8 n)^{1 / 4}$,
- $G\left[V_{2}\right], G\left[V_{3}\right], G\left[V_{4}\right]$ are matchings of sizes $\sqrt{2 n},(8 n)^{1 / 4},(8 n)^{1 / 4}$ respectively,
- there are no edges between the classes, i.e. $E\left(V_{i}, V_{j}\right)=\emptyset$, for $i \neq j$.

We shall consider a sequence of graphs obtained by adding edges one by one to $G\left[V_{1}\right]$, i.e. $G_{0}=G$ and $G_{i}\left[V_{1}\right]=G_{i-1}\left[V_{1}\right] \cup e$ for some edge $e \notin G_{i-1}\left[V_{1}\right]$, for any $0<i \leq\binom{ n^{\prime}}{2}$. Observe that for sufficiently large $n$ we have $f\left(G_{0}\right)=(n-2)\left(\sqrt{2 n}-2(8 n)^{1 / 4}\right)=(\sqrt{2}+o(1)) n^{3 / 2}$ and $f\left(G_{\binom{n^{\prime}}{2}}\right)=(1+o(1))\binom{n}{3}$. Therefore, in order to prove the lemma, it suffices to show that any number in the interval $\left(f\left(G_{i-1}\right), f\left(G_{i}\right)\right)$ is realizable. We shall achieve that by moving edges within each of $G_{i}\left[V_{2}\right], G_{i}\left[V_{3}\right], G_{i}\left[V_{4}\right]$, but not across them. As an edge is contained in $n-2$ triples, it follows easily that $f\left(G_{i}\right)-f\left(G_{i-1}\right) \leq n-2$.

Let us fix $i \geq 1$. By construction, $V\left(G_{i}\right)=V_{1} \sqcup V_{2} \sqcup V_{3} \sqcup V_{4}$. Let $\left\{x_{1} y_{1}, \cdots x_{\sqrt{2 n}} y_{\sqrt{2 n}}\right\}$ be the matching inside $V_{2}$. We shall construct another sequence $G_{i, 1}, G_{i, 2}, \cdots, G_{i, \sqrt{2 n}}$ where $G_{i, k+1}$ is obtained from $G_{i, k}$ by deleting edge $x_{k+1} y_{k+1}$ and adding edge $x_{1} y_{k+1}$ (see Figure 1). Notice that


Figure 1: Star accumulation of $V_{2}$ and $V_{3}$.
$f\left(G_{i, \sqrt{2 n}}\right)=f\left(G_{i}\right)-\binom{\sqrt{2 n}}{2}=f\left(G_{i}\right)-\frac{\sqrt{2 n}(\sqrt{2 n}-1)}{2}=f\left(G_{i}\right)-n+\sqrt{2 n} / 2$ and $f\left(G_{i, k}\right)-f\left(G_{i, k+1}\right)=k$. Hence we obtain a refinement of the sequence with the gaps between consecutive terms bounded by $\sqrt{2 n}$.

Let us fix $i \geq 1$ and $j \geq 2$. Let $\left\{x_{1} y_{1}, \cdots x_{(8 n)^{1 / 4}} y_{(8 n)^{1 / 4}}\right\}$ be the matching inside $V_{3}$. We shall construct another sequence $G_{i, j, 1}, G_{i, j, 2}, \cdots, G_{i, j,(8 n)^{1 / 4}}$ where $G_{i, j, k+1}$ is obtained from $G_{i, j, k}$ by deleting edge $x_{k+1} y_{k+1}$ and adding edge $x_{1} y_{k+1}$. Notice that $f\left(G_{i, j,(8 n)^{1 / 4}}\right)=f\left(G_{i, j}\right)-\binom{(8 n)^{1 / 4}}{2}=f\left(G_{i}\right)-\frac{(8 n)^{1 / 4}\left((8 n)^{1 / 4}-1\right)}{2}=\sqrt{2 n}-(8 n)^{1 / 4} / 2$ and $f\left(G_{i, j, k}\right)-f\left(G_{i, j, k+1}\right)=k$. Hence we obtain a refinement of the sequence with the gaps between consecutive terms bounded by $(8 n)^{1 / 4}$.

Finally, let us fix $i, j, k$. Let $\left\{x_{1} y_{1}, \cdots x_{(8 n)^{1 / 4}} y_{(8 n)^{1 / 4}}\right\}$ be the matching inside $V_{4}$. We shall construct another sequence $G_{i, j, k, 1}, G_{i, j, k, 2}, \cdots, G_{i, j, k,(8 n)^{1 / 4}}$ where $G_{i, j, k, l+1}$ is obtained from $G_{i, j, k, l}$ by deleting edge $x_{k+1} y_{k+1}$ and adding edge $y_{k} y_{k+1}$ (see Figure 2).


Figure 2: Path accumulation of $V_{4}$.

Notice that $f\left(G_{i, j, k,(8 n)^{1 / 4}}\right)=f\left(G_{i, j, k}\right)-(8 n)^{1 / 4}$ and $f\left(G_{i, j, k, l}\right)-f\left(G_{i, j, k, l+1}\right)=1$. Hence we obtain a refinement of the sequence with no gaps.

We are now ready to deduce Theorem 6.3 from Lemmas 6.4, 6.6 and 6.7 .

Proof of Theorem 6.3. (i) Recall that the number of triangles in $G$ is equal to $\binom{n}{3}-f(\bar{G})$. Therefore, Lemma 6.7 implies that $\left[o\left(\binom{n}{3}\right),\binom{n}{3}-(\sqrt{2}+o(1)) n^{3 / 2}\right] \subset T_{n}$. Together with Lemma 6.4 which, for $r=3$, says that $\left[0,\left(\frac{n-9}{3}\right)^{3}\right] \subset T_{n}$, we conclude that $\left[0,\binom{n}{3}-(\sqrt{2}-o(1)) n^{3 / 2}\right] \subseteq T_{n}$ for sufficiently large $n$.
(ii) Since the number of triangles in $G$ is equal to $\binom{n}{3}-f(\bar{G})$, we obtain, using Lemma 6.6, that $\left.\binom{n}{3}-a_{t+1},\binom{n}{3}-b_{t}\right) \cap T_{n}=\emptyset$ for all $n, t \geq 0$.

### 6.4 General H

Now, let us consider the case when $H$ is an arbitrary connected graph on $h \geq 3$ vertices. We shall start by showing that when $n$ goes to infinity then the first $(1-o(1)) k_{H}\left(K_{n}\right)$ numbers are realizable.

Our strategy will be to recursively partition the vertex set into two subsets and modify the edges between vertices in each of the classes, but without adding diagonal edges. Let $g_{H}=g_{H}^{(n)}$ be the maximum number of new copies of $H$ obtained by adding an edge to a graph, over all graphs on $n$ vertices. We claim that there is a constant $c_{H}>0$ such that $g_{H} \leq c_{H} n^{h-2}$. Indeed, a new copy must contain both the endvertices of the newly added edge, and there are $\binom{n-2}{h-2} h$-sets of vertices in $G$ containing two fixed vertices, and each $h$-set may contain at most $c_{H}^{\prime}$ copies of $H$, for some $c_{H}^{\prime}$ independent of $n$, therefore $g_{H} \leq c_{H}^{\prime}\binom{n-2}{h-2} \leq c_{H} n^{h-2}$.

The next two lemmas are needed in our construction.
Lemma 6.8. If $\left[0, c n^{\alpha}\right] \subset S_{H}^{(n)}$ for all sufficiently large $n$, where $\alpha \leq h-2$, then for all
sufficiently large $n$ and some new constant $c_{1}>0$,

$$
\left[0, c_{1} n^{\alpha h /(h-2)}\right] \subset S_{H}^{(n)}
$$

Proof. Consider an empty graph $G$ with vertex set $V=V_{1} \sqcup V_{2}$, where $n_{1}=\left|V_{1}\right|=c^{\prime} n^{\alpha /(h-2)}$ and $n_{2}=\left|V_{2}\right|=n-\left|V_{1}\right|$, where $c^{\prime}>0$ will be chosen later. Let $G_{0}=G$ and let $G_{i+1}$ be a graph obtained by adding an edge between vertices of $V_{1}$ in $G_{i}$, then

$$
k_{H}\left(G_{i+1}\right)-k_{H}\left(G_{i}\right) \leq g_{H}\left(n_{1}\right) \leq c_{H} n_{1}^{h-2}=c_{H} c^{\prime h-2} n^{\alpha} .
$$

Therefore we obtain an increasing sequence in $S_{H}^{(n)}$ starting with 0 and ending with $k_{H}\left(G_{\binom{n_{1}}{2}}\right)$ such that the differences between consecutive terms are at most $c_{H} c^{h-2} n^{\alpha}$. We shall modify $G_{i}\left[V_{2}\right]$ to obtain the missing numbers between consecutive terms. By the hypothesis, we can modify $G_{i}\left[V_{2}\right]$, to obtain $G_{i}^{\prime}\left[V_{2}\right]$ containing any number $k$ of copies of $H$, where $k \in\left[0, c n_{2}^{\alpha}\right]$. Hence it suffices to find $c^{\prime}>0$ such that $c_{H} c^{\prime h-2} n^{\alpha}<c n_{2}^{\alpha}$. Let us consider two cases depending on $\alpha$.

1. if $\alpha=h-2$ then $c n_{2}^{\alpha}=c\left(n-c^{\prime} n\right)^{\alpha}=c\left(1-c^{\prime}\right)^{\alpha} n^{\alpha}$, therefore it suffices to choose $c^{\prime}>0$ such that $c_{H} c^{\prime h-2}<c\left(1-c^{\prime}\right)$. Hence we have $g_{H}^{\left(n_{1}\right)}<c n_{2}^{\alpha}$.
2. if $\alpha<h-2$ then $c n_{2}^{\alpha}=c(n-o(n))^{\alpha} \sim c n^{\alpha}$, hence if we choose $c^{\prime}>0$ such that $c^{\prime}<\left(c / c_{H}\right)^{1 /(h-2)}$, then for sufficiently large $n$ we will have $g_{H}^{\left(n_{1}\right)}<c n_{2}^{\alpha}$.

Therefore, any number less than $k_{H}\left(G_{\binom{n_{1}}{2}}\right)=k_{H}\left(K_{n_{1}}\right)>c^{\prime \prime} n_{1}^{h}>c_{1} n^{h}$ is realizable.
From the next lemma we learn that for sufficiently large $n$ we can construct a graph on $n$ vertices with $k$ of copies of $H$ for any $k \leq(1-o(1)) k_{H}\left(K_{n}\right)$

Lemma 6.9. If $\left[0, c n^{\alpha}\right] \subset S_{H}^{(n)}$ for sufficiently large $n$ where $\alpha>h-2$ then for sufficiently large $n$,

$$
\left[0,(1-o(1)) k_{H}\left(K_{n}\right)\right] \subset S_{H}^{(n)}
$$

Proof. We shall proceed similarly as in Lemma 6.8. Choose $\beta \in((h-2) / \alpha, 1))$ and let $n_{2}=\left|V_{2}\right|=n^{\beta}$ and $n_{1}=\left|V_{1}\right|=n-\left|V_{2}\right|$. Notice that $g_{H}^{\left(n_{1}\right)}=O\left(n^{h-2}\right)$ and by the hypothesis we can modify $G\left[V_{2}\right]$ to obtain any number of copies of $H$ up to $n_{2}^{\alpha}$, where $n_{2}^{\alpha}=\omega\left(n^{h-2}\right)$. Therefore any number in the interval $\left[0, k_{H}\left(n_{1}\right)\right]$ is realizable. But $n_{1}=(1-o(1)) n$, whence $k_{H}\left(n_{1}\right)=(1-o(1)) k_{H}$.

We shall use Lemmas 6.8 and 6.9 to prove one of the two main theorems of this section.

Proof of Theorem 6.1. We start by showing that trivially $[0,\lfloor n / h\rfloor] \subset S_{H}^{(n)}$. To achieve that notice that for any $k \leq\lfloor n / h\rfloor$ we can simply construct a graph on $n$ vertices consisting of $k$ disjoint copies of $H$.

Let $k_{\text {max }}$ be the largest integer $k$ such that $\left(\frac{h}{h-2}\right)^{k} \leq h$ (note that $\left.\left(\frac{h}{h-2}\right)^{k_{\text {max }}} \in(h-2, h]\right)$. We claim that $\left[0, c_{k} n^{(h /(h-2))^{k}}\right] \subset S_{H}^{(n)}$ for every positive integer $k \leq k_{\text {max }}$. We shall show the claim by induction on $k$. For $k=0$, we already showed that $\left[0, c_{0} n\right] \subset S_{H}^{(n)}$. Suppose, that $\left[0, c_{k} n^{(h /(h-2))^{k}}\right] \in S_{H}^{(n)}$ and $k<k_{\max }$. Observe first that $(h /(h-2))^{k} \leq h-2$, as otherwise $(h /(h-2))^{k_{\max }}$ would be greater than $h$, hence we can apply Lemma 6.8 and conclude that $\left[0, c_{k+1} n^{(h /(h-2))^{k+1}}\right] \in S_{H}^{(n)}$ for large enough $n$. Note that we apply Lemma 6.8 only finitely many times hence $n$ remains finite.

Therefore for $n$ large enough we have $\left[0, c n^{\alpha}\right] \subset S_{H}^{(n)}$ with $\alpha=\left(\frac{h}{h-2}\right)^{k_{\max }} \in(h, h-2]$, hence we can apply Lemma 6.9 and conclude that for $n$ sufficiently large $\left[0,(1-o(1)) k_{H}\right] \subset S_{H}^{(n)}$.

Let us recall few major notations. For a graph $G=(V, E)$ of order $n$, let $f_{H}(G)$ be the number of subgraphs $H$ of $K(V)$ isomorphic to $F$ such that $E(H) \cap E \neq \emptyset$, where $K(V)$ denotes the complete graph on the vertex set $V$. Then the number of copies of $H$ in $G$ is equal to $k_{H}\left(K_{n}\right)-f_{H}(\bar{G})$.

In the next lemma we shall describe the formula for $f_{H}$.

Lemma 6.10. For any graph $H$ on $h$ vertices and $G$ on $n$ vertices, we have

$$
f_{H}(G)=c_{H} e(G)\binom{n-2}{h-2}-\sum_{k=2}^{e(H)} \sum_{\substack{e(F)=k \\ \delta(F)>0}}(-1)^{k+1} c_{H}(F) k_{F}(G)\binom{n-|V(F)|}{h-|F|}
$$

Proof. Let $E(G)=\left\{e_{1}, \cdots, e_{m}\right\}$, where $m=e(G)$. We define $A_{i}=\left\{F: e_{i} \in E(F), F \cong H\right\}$ to be the set of subgraphs of the complete graph isomorphic to $H$ containing the edge $e_{i}$. Notice that $f_{H}(G)=\left|\bigcup_{i=1}^{m} A_{i}\right|$, therefore by the inclusion-exclusion principle we can write

$$
f_{H}(G)=\sum_{k=1}^{e(G)} \sum_{i_{1}<\ldots<i_{k}}(-1)^{k+1}\left|A_{i_{1}} \cap \ldots \cap A_{i_{k}}\right| .
$$

For a graph $F$ on at most $h$ vertices, let $c_{H}(F)$ be the number of copies of $H$ in the complete graph $K_{h}$ containing all the edges of a fixed subgraph of the complete graph $K_{h}$, isomorphic to $F$. Let $F=G\left[e_{i_{1}}, \ldots, e_{i_{k}}\right]$ be the graph induced by the edges $e_{i_{1}}, \cdots, e_{i_{k}}$. Observe that $\left|A_{i_{1}} \cap \cdots \cap A_{i_{k}}\right|=c_{H}(F)\binom{n-|F|}{h-|F|}$, therefore

$$
\begin{aligned}
f_{H}(G) & =\sum_{k=1}^{e(G)} \sum_{i_{1}<\ldots<i_{k}}(-1)^{k+1}\left|A_{i_{1}} \cap \ldots \cap A_{i_{k}}\right| \\
& =\sum_{k=1}^{e(G)} \sum_{\substack{F \subseteq H \\
e(F)=k \\
\delta(F)>0}}(-1)^{k+1} k_{F}(G) c_{H}(F)\binom{n-|F|}{h-|F|} \\
& =\sum_{k=1}^{e(H)} \sum_{\substack{F \subset H \\
e(F)=k \\
\delta(F)>0}}(-1)^{k+1} k_{F}(G) c_{H}(F)\binom{n-|F|}{h-|F|} \\
& =c_{H} e(G)\binom{n-2}{h-2}-\sum_{k=2}^{e(H)} \sum_{e(F)=k}(-1)^{k+1} c_{H}(F) k_{F}(G)\left(\begin{array}{l}
n-|F| \\
\\
h-|F|
\end{array}\right)
\end{aligned}
$$

The following easy lemma gives us an upper bound for $k_{F}(G)$.

Lemma 6.11. If $H$ is a graph on $h$ vertices with no isolated vertices then for every graph $G$ on $e$ edges the number of copies of $H$ in $G$ is at most $e^{h-1}$.

Proof. We shall proceed by induction. The base case $h=2$ is trivial. Assume then that $h>2$ and let us consider two cases. If $H$ is a matching on $h=2 l$ vertices then the result follows easily - the number of copies of $H$ in $G$ is at most $\binom{e}{l} \leq e^{l} \leq e^{2 l-1}=e^{h-1}$. In the other case, when $H$ is not a matching, there exists a vertex $v \in H$ such that $H^{\prime}=H-v$ has no isolated vertices. By the induction hypothesis there are at most $e^{h^{\prime}-1}$ copies of $H^{\prime}$ in $G$, where $h^{\prime}=\left|H^{\prime}\right|=h-1$. Each copy of $H^{\prime}$ in $G$ can be extended to at most $e$ copies of $H$ in $G$, since by assumption $v$ must be adjacent to some vertex of $H^{\prime}$. Therefore there are at most $e^{h^{\prime}}=e^{h-1}$ copies of $H$ in $G$.

Lemma 6.12. We have

$$
f_{H}(G)=c_{H} e(G)\binom{n-2}{h-2}-c_{H}\left(P_{2}\right) k_{P_{2}}(G)\binom{n-3}{h-3}+c_{H}\left(K_{3}\right) k_{K_{3}}(G)\binom{n-3}{h-3}+o\left(n^{h-2}\right) .
$$

Proof. Let us consider the term $c_{H}(F) k_{F}(G)\binom{n-|F|}{h-|F|}$ where $F$ is a graph on at least four vertices. It follows from Lemma 6.11 that $k_{F}(G) \leq e^{|F|-1}$ and therefore $k_{F}(G)\binom{n-|F|}{h-|F|} \leq e^{|F|-1} n^{h-|F|}$. Hence, under the assumption that $e=O\left(n^{1 / 2}\right)$, we have $k_{F}(G)\binom{n-|F|}{h-|F|}=O\left(n^{1 / 2|F|-1 / 2} n^{h-|F|}\right)=O\left(n^{h-1 / 2-1 / 2|F|}\right)=O\left(n^{h-5 / 2}\right)=o\left(n^{h-2}\right)$. As there are only three graphs on fewer than four vertices with no isolated vertices, namely $K_{2}, K_{3}$ and $P_{2}$, and the number of terms in the summation in $f(G)$ depends only on $H$, we can write

$$
f_{H}(G)=c_{H} e(G)\binom{n-2}{h-2}-c_{H}\left(P_{2}\right) k_{P_{2}}(G)\binom{n-3}{h-3}+c_{H}\left(K_{3}\right) k_{K_{3}}(G)\binom{n-3}{h-3}+o\left(n^{h-2}\right) .
$$

The next lemma, which we shall later use to prove that there are gaps in $S_{H}^{(n)}$, tells us that for sufficiently large $n$, stars and matchings are asymptotically extremal examples of
graphs for $f_{H}(G)$, i.e. for a graph $G$ on $t$ edges we have

$$
a_{H}^{(n)}(t)-o\left(n^{h-2}\right)=f_{H}\left(S_{t}^{(n)}\right)-o\left(n^{h-2}\right) \leq f_{H}(G) \leq f_{H}\left(M_{t}^{(n)}\right)+o\left(n^{h-2}\right)=b_{H}^{(n)}(t)+o\left(n^{h-2}\right) .
$$

Lemma 6.13. Let $G$ be a graph on $n$ vertices with $e=O\left(n^{1 / 2}\right)$ edges. Then the following hold.

1. $f_{H}(G) \geq a_{H}^{(n)}(e)-o\left(n^{h-2}\right)=c_{H} e\binom{n-2}{h-2}-c_{H}\left(P_{2}\right)\binom{e}{2}\binom{n-3}{h-3}+o\left(n^{h-2}\right)$; and
2. $f_{H}(G) \leq b_{H}^{(n)}(e)+o\left(n^{h-2}\right)=c_{H} e\binom{n-2}{h-2}+o\left(n^{h-2}\right)$,
as $n$ goes to infinity.

Proof. This an immediate corollary of Lemma 6.12. Observe that that $k_{K_{3}}\left(S_{e}^{(n)}\right)=0$, as stars contain no triangles, and $k_{P_{2}}\left(S_{e}^{(n)}\right)=\binom{e}{2}$. Therefore, by Lemma 6.12

$$
a_{H}^{(n)}(e)=c_{H} e \cdot\binom{n-2}{h-2}-c_{H}\left(P_{2}\right)\binom{e}{2}\binom{n-3}{h-3}+o\left(n^{h-2}\right) .
$$

On the other hand, matchings contain no copies of $K_{3}$ nor $P_{2}$, hence, again, by Lemma 6.12

$$
b_{H}^{(n)}(e)=c_{H} e \cdot\binom{n-2}{h-2}+o\left(n^{h-2}\right) .
$$

Now, since for any graph $G$ on $e$ edges we have $k_{P_{2}}(G) \leq\binom{ e}{2}$ it follows from the above estimate on $a_{H}^{(n)}(e)$ and from Lemma 6.12 that

$$
f_{H}(G) \geq c_{H} e \cdot\binom{n-2}{h-2}-c_{H}\left(P_{2}\right)\binom{e}{2}\binom{n-3}{h-3}+o\left(n^{h-2}\right) \geq a_{H}^{(n)}(e)-o\left(n^{h-2}\right) .
$$

On the other hand, it follows from easy to see fact that for any graph $G$ we have $c_{H}\left(P_{2}\right) k_{P_{2}}(G) \geq c_{H}\left(K_{3}\right) k_{K_{3}}(G)$ and Lemma 6.12 that

$$
f_{H}(G) \leq c_{H} e \cdot\binom{n-2}{h-2}+o\left(n^{h-2}\right) \leq b_{H}^{(n)}(e)+o\left(n^{h-2}\right)
$$

We can now prove the second main theorem of the section.
Proof of Theorem 6.2. Let $t_{\max }=\max \left\{t: b_{H}^{(n)}(t)<a_{H}^{(n)}(t+1)\right\}$. We shall first show that $t_{\max }=\Theta(\sqrt{n})$. Indeed, we have

$$
\begin{aligned}
a_{H}^{(n)}(t+1)-b_{H}^{(n)}(t) & =c_{H}\binom{n-2}{h-2}-c_{H}\left(P_{2}\right)\binom{t+1}{2}\binom{n-3}{h-3}+o\left(n^{h-2}\right) \\
& =c_{1} n^{h-2}-c_{2} t^{2} n^{h-3}+o\left(n^{h-2}\right),
\end{aligned}
$$

for some constants $c_{1}, c_{2}>0$ depending depending only on $H$. It follows that there is a constant $C$ depending only on $H$, such that for all sufficiently large $n$ we have $a_{H}^{(n)}(t+1)-b_{H}^{(n)}(t) \geq 0$ if $t \geq C \sqrt{n}$ and $a_{H}^{(n)}(t+1)-b_{H}^{(n)}(t) \leq 0$ otherwise. From Lemma 6.13 we know that for all sufficiently large $n$ we have

$$
f_{H}(G) \in\left[a_{H}^{(n)}(t)-o\left(n^{h-2}\right), b_{H}^{(n)}(t)+o\left(n^{h-2}\right)\right] .
$$

Therefore, for any $t<t_{\max }$, the number of integers $m$ in the interval $\left(b_{H}^{(n)}(t), a_{H}^{(n)}(t+1)\right)$ such that there is a graph $G$ on $n$ vertices with $f_{H}(G)=m$ is at most $o\left(n^{h-2}\right)$. Whence $\left|\left(k_{H}\left(K_{n}\right)-a_{t+1}, k_{H}\left(K_{n}\right)-b_{t}\right) \cap S_{H}^{(n)}\right|=o\left(n^{h-2}\right)$, for every $t$. Notice that when $t<\frac{t_{\text {max }}}{2}$, the gap between $a_{t+1}$ and $b_{t}$ is of the order $n^{h-2}$, hence

$$
\frac{\left|\left(k_{H}\left(K_{n}\right)-a_{t+1}, k_{H}\left(K_{n}\right)-b_{t}\right) \cap S_{H}^{(n)}\right|}{a_{t+1}-b_{t}}=o(1),
$$

as $n$ goes to infinity.

### 6.5 Open problems

We conclude this chapter with some open problems that we feel would merit further study. Let $\phi_{H}^{(n)}=\min \left\{m \geq 0: m \notin S_{H}^{(n)}\right\}$ be the smallest nonmember of $S_{H}^{(n)}$. In this chapter we
have proved that $\phi_{K_{3}}^{(n)}=\binom{n}{3}-(\sqrt{2}+o(1)) n^{3 / 2}$. We would like to remark that we were also able to prove the following two results, whose proofs we omit, as they are very similar to the proof of Theorem 6.3 .

- $\phi_{P_{2}}^{(n)}=3\binom{n}{3}-(4+o(1)) n^{3 / 2}$,
- $\binom{n}{4}-(c+o(1)) n^{8 / 3} \leq \phi_{K_{4}}^{(n)} \leq\binom{ n}{4}-\left(\frac{1}{2}+o(1)\right) n^{5 / 2}$.

The ultimate goal is to determine $S_{H}^{(n)}$, in particular we ask the following question.
Problem 6.14. What is the asymptotic behavior of $\binom{n}{r}-\phi_{K_{r}}^{(n)}$ ?

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[^0]:    ${ }^{1}$ We say that $(a, b) \preccurlyeq(c, d)$ if $a<c$ or $a=c$ and $b \leq d$, where $\preccurlyeq$ denotes the lexicographical order relation.

