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Geometric Properties of Cesro Function and Sequence Spaces

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Rozprawę tę dedykuję moim rodzicom, Alinie i Grzegorzowi Kubiakom.

I dedicate this dissertation to my parents, Alina and Grzegorz Kubiak.

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ABSTRACT

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Theory of Banach spaces as a branch of Functional Analysis was founded at the beginning of the 20th century and since then it has been extensively developed and applied. Banach lattices, in particular Banach function and sequence spaces, like classical Lebesgue spaces or Orlicz, Lorentz, Musielak-Orlicz spaces, are of special interest and importance for applications.

In this dissertation we study Cesàro function and sequence spaces which, unlike many other classical spaces, are not rearrangement invariant. This makes them a very interesting object to explore with possible applications to some functional analysis problems.

Cesàro function and sequence spaces appeared for the first time in 1968 in the Dutch Mathematical Society Journal as a problem to find a representation of their dual space. This problem, in case of sequences, was solved in 1974 by Jagers. Since then, Cesàro sequence spaces have gain an attention among Banach space theory specialists. For example, in the nineties several geometric properties of them were studied. During the last decade, more general spaces, Cesàro-Orlicz sequence spaces, have been explored as well. Cesàro function spaces attracted a wider attention only since 2008 when Astashkin and Maligranda published series of interesting papers on their geometrical and topological properties.

In this dissertation we study geometric properties of Cesàro function spaces with general weight.We find an isometric description of the dual of Cesàro function space. This description involves a new concept of essential Ψ-concave majorant of a measurable function which we define and study. We show, among others, that every non empty

relatively open subset (and hence every slice) of the unit ball of Cesàro function space has diameter 2. In particular these spaces do not have the Radon-Nikodym property. Also, they are strictly convex Banach spaces while the unit sphere does not have strongly extreme points. This shows rather unexpected differences between Cesàro function and sequence spaces since the latter are known to have the Radon-Nikodym property and to be locally uniformly rotund.

We also explore Cesàro-Orlicz sequence spaces. We show that they are not B-convex and investigate under what conditions there is an isometric or isomorphic copy of ℓ_∞ in these spaces.

Contents

1 Introduction

In this chapter we give historical background on Cesàro sequence and function spaces. We also set up terminology and notation.

1.1 Introduction

In 1968 the Dutch Mathematical Society posted the following problem labeled "1968. 2."([1], see also [2]):

"Define the Cesàro sequence spaces ces_p as follows:
ces_p is the space of all numerical sequences

$$a = (a_1, a_2, \ldots)$$

with finite norms

$$|a|_p = [\sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} |a_k|^p]^{1/p} \quad \text{for } 1 \leq p < \infty,$$
$$|a|_\infty = \sup_{n \geq 1} \frac{1}{n} \sum_{k=1}^{n} |a_k|.$$

Similarly, the function space Ces_p consists of all (L)-measurable functions f on $[0, \infty]$ with finite norms

$$|f|_p = [\int_0^\infty (\frac{1}{x} \int_0^x |f(y)|\, dy)^p\, dx]^{1/p} \quad \text{for } 1 \leq p < \infty,$$
$$|f|_\infty = \sup_{x>0} \frac{1}{x} \int_0^x |f(y)|\, dy.$$

Investigate the properties of these normed linear spaces and their adjoint spaces (i.e. Banach dual spaces)."

Some basic results regarding both Cesàro sequence and function spaces were obtained in early seventies by Shiue [63, 64], Leibovitz [45] and Hassard and Hussein [30]. In 1974 Jagers gave an explicit isometric description of the dual of ces_p, $1 \leqslant p < \infty$ [34]. Similar result was obtained in 1976 by Ng and Lee for the case

$p = \infty$ [54]. It is worth to mention that in two former papers Cesàro sequence spaces with general positive weight sequence, in place of the weight $(1/n)_{n=1}^{\infty}$, are considered. In the late nineties mathematicians became interested in geometric properties of these spaces. Cui and Płuciennik studied local uniform nonsquareness [23], and, together with Meng, they proved that ces_p has Banach-Saks property and property (β) of Rolewicz [22]. Cui and Hudzik showed that ces_p has fixed point property [18] (see also [23, Part 9]) and obtained the packing constant [19]. Several other geometric properties of ces_p are considered in [17], [23] and [20]. Another look at Cesàro sequence spaces appeared in 2010 [61].

Contrary to the Cesàro sequence spaces their function counterparts did not attract a lot of attention for a long time. In 1987 an interesting description of the dual of Cesàro function spaces appeared. Authors obtained an equivalent norm on the dual of Ces_p, $1 < p < \infty$ using dual of ces_p as an important ingredient [67] (see also [44]). Only recently in a series of papers [4–6], Astashkin and Maligranda started to study thoroughly the structure of Cesàro function spaces. In [4] they proved that Ces_p fails the fixed point property and in [5] they investigated, among others, dual spaces for Ces_p induced by the weight $w(x) = x^{-1}$ for $1 < p < \infty$. Their description can be viewed as being analogous to one given for sequence spaces by Bennett in [9]. They found a Banach space equipped with a norm equivalent to the dual norm, which is an isomorphic representation of the dual space.

Cesàro-Orlicz sequence spaces ces_φ appeared for the first time in 1988 [66] and since then they have been studied by a number of authors. In 2006, Cui et al. studied basic topological and geometric properties of ces_φ [21]. For example, they found conditions on φ under which ces_φ is strictly convex. Maligranda, Petrot and Suantai showed that ces_φ is not B-convex for a wide class of Orlicz functions φ [51]. Local geometric structure of ces_φ has been studied by Foralewski, Hudzik and

Szymaszkiewicz [28]. In 2010 they found, among others, conditions on an Orlicz function φ under which ces_φ is locally uniformly rotund [29].

We consider in this dissertation the problem of existence of order linearly isometric copy of ℓ_∞ in ces_φ under the Luxemburg norm. We also show that all non-trivial ces_φ spaces are not B-convex solving the problem posted in [51].

In Chapter 2 we present an isometric description of the dual of Cesàro function space $C_{p,w}$ where $1 < p < \infty$ and $w > 0$. We introduce a notion of essential Ψ-concave majorant of an arbitrary measurable function and study its properties. This notion is a main ingredient in the description of the dual space. In the last section of this chapter we prove, among other things, that $C_{p,w}$ does not have the Radon-Nikodym property.

In Chapter 3 we show that Cesàro function space is strictly convex, contains almost isometric copy of ℓ_1 and has all weakly relatively open sets of its unit ball of diameter 2. We also present some relations among geometrical properties in general Banach spaces.

Chapter 4 is devoted to Cesàro-Orlicz sequence spaces. We show that these spaces are not B-convex. We present a comparison theorem for them and obtain conditions under which ces_φ contains an order isometric copy of ℓ_∞.

1.2 Preliminaries

As usual, $\mathbb{N}$ and $\mathbb{R}$ denote the set of positive integers and real numbers, respectively. For an interval $I \subset \mathbb{R}$ by $L_0(I)$ we denote the set of all (equivalence classes of extended) real valued Lebesgue (almost everywhere finite) measurable functions on I. The positive cone of $L_0(I)$ is denoted $L_0^+(I) = \{f \in L_0(I) : f \geqslant 0 \text{ a.e.}\}$. By ℓ^0 we denote the set of all real sequences and by c_0 the set of all sequences convergent to 0.

For a Banach space $(X, \|\cdot\|)$ by B_X and S_X, we denote the unit ball and the unit sphere of X, and by X^* the dual space of X. Any Banach space $E = E(I) \subset L_0(I)$ with norm $\|\cdot\|$ satisfying the condition that $f \in E$ and $\|f\| \leqslant \|g\|$ whenever $0 \leqslant f \leqslant g$ a.e., $f \in L_0(I)$ and $g \in E$, is called a *Banach function space* or *Köthe function space*. An element f in a Banach function space E is called *order continuous* if for every $0 \leqslant f_n \leqslant |f|$ a.e. such that $f_n \downarrow 0$ a.e. it holds $\|f_n\| \downarrow 0$. We say that E is *order continuous* if every element in E is order continuous. A Banach function space $(E, \|\cdot\|)$ has the *Fatou property* if for any sequence $(f_n) \subset E$ and any $f \in L_0(I)$ such that $0 \leqslant f_n \leqslant f$ a.e., $f_n \uparrow f$ a.e. and $\sup_n \|f_n\| < \infty$ it holds $f \in E$ and $\|f\| = \lim_n \|f_n\|$.

Similarly for sequence spaces. A Banach space $(X, \|\cdot\|)$ is a *Banach sequence space* (or *Köthe sequence space*) if it is a subspace of ℓ^0, contains an element x such that $x(n) \neq 0$ for all $n \in \mathbb{N}$, and if $x \in \ell^0$ and $y \in X$ with $|x| \leqslant |y|$, i.e. $|x(n)| \leqslant |y(n)|$ for all $n \in \mathbb{N}$, then $x \in X$ and $\|x\| \leqslant \|y\|$. We say that a Banach sequence space X has the *Fatou property* if for any sequence (x_m) of positive elements of X and any $x \in \ell^0$ such that $x_m \uparrow x$ that is for all $n \in \mathbb{N}$, $(x_m(n))_{m=1}^{\infty}$ is increasing and $x_m(n) \to x(n)$, and $\sup_m \|x_m\| < \infty$ we have that $x \in X$ and $\|x_m\| \to \|x\|$ as $m \to \infty$.

Banach sequence and function spaces are examples of Banach lattices.

Throughout this dissertation, terms decreasing or increasing mean non-increasing or non-decreasing, respectively. By m we denote the Lebesgue measure on the real line $\mathbb{R}$.

2 The dual of Cesàro function space

The content of this chapter is published in [38] except Theorem 2.28 which is presented there in a less general case. The main part of this chapter is the isometric description of the dual of Cesàro function space.

2.1 Preliminaries

In 1974, Jagers [34] found an isometric representation of the dual of Cesàro sequence space

$$ces_{p,w} = \left\{ x \in \ell^0 : \|x\|_{ces_{p,w}} := \left[\sum_{n=1}^{\infty} \left(w(n) \sum_{i=1}^{n} |x(n)| \right)^p \right]^{1/p} < \infty \right\},$$

where $1 < p < \infty$ and $(w(n))_{n=1}^{\infty}$ is a (weight) sequence of arbitrary positive numbers. He obtained that the (Köthe) dual of $ces_{p,w}$ is

$$(ces_{p,w})' = \left\{ x \in c_0 : \|x\|_{(ces_{p,w})'} = \left[\sum_{n=1}^{\infty} \left(\frac{\hat{x}(n) - \hat{x}(n-1)}{w(n)} \right)^q \right]^{1/q} \right\},$$

where $1/p + 1/q = 1$ and for a sequence $x \in c_0$, $\hat{x}$ is defined in the following way. Let

$$\Psi(n) = \sum_{k=n}^{\infty} w(n)^p,\ n \in \mathbb{N}.$$

Suppose first that x is nonnegative. Denote $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ and define increasing (finite or infinite) sequence (m_n) of elements of $\overline{\mathbb{N}}$ by

$$m_1 := \max\{k \in \overline{\mathbb{N}} : x(k) = \max_{i \in \mathbb{N}} x(i)\}$$

and, provided that m_n is defined and finite,

$$m_{n+1} := \max\left\{k \in \overline{\mathbb{N}} : k > m_n,\ \frac{x(m_n) - x(k)}{\Psi(m_n) - \Psi(k)} = \min_{m_n < s \leqslant \infty} \frac{x(m_n) - x(s)}{\Psi(m_n) - \Psi(s)}\right\}.$$

Define

$$\hat{x}(j) := x(m_1) \quad \text{for } j \leqslant m_1,$$

and for $m_n < j \leqslant m_{n+1}$, provided m_n is defined and finite, $\hat{x}(j)$ to be such that

$$\frac{x(m_n) - \hat{x}(j)}{\Psi(m_n) - \Psi(j)} = \frac{x(m_n) - x(m_{n+1})}{\Psi(m_n) - \Psi(m_{n+1})}.$$

For arbitrary sequence $x \in c_0$ we define $\hat{x} = \widehat{|x|}$. Given $x \in c_0$, the sequence $\hat{x}$ is always a c_0 decreasing sequence and satisfies the following inequality

$$\frac{\hat{x}(i) - \hat{x}(j)}{\Psi(i) - \Psi(j)} \leqslant \frac{\hat{x}(j) - \hat{x}(k)}{\Psi(j) - \Psi(k)} \quad \text{for all } i < j < k \text{ in } \mathbb{N}. \tag{2.1}$$

This sequence $\hat{x}$ is the smallest sequence satisfying inequality (2.1) and such that $|x| \leqslant \hat{x}$. Any sequence $\hat{x}$ satisfying (2.1) is called a Ψ-concave sequence.

We compute precisely the dual norm of the Cesàro function space $C_{p,w}$ on $(0, l)$, $0 < l \leqslant \infty$, generated by $1 < p < \infty$ and an arbitrary positive weight function w. A description presented in this dissertation resembles the approach of Jagers for sequence spaces; however, the techniques are more involved due to necessity of dealing with functions instead of sequences. One of the difficulties was to find an appropriate definition of a function $\hat{f}$ for an arbitrary measurable function f which satisfies inequality corresponding to (2.1). We define such $\hat{f}$ and explore its properties.

Section 2.1 of this chapter is devoted to Ψ-concave functions and essential Ψ-concave majorants of measurable functions and it can of independent interest. In this section, Ψ is a nonnegative strictly decreasing function on the interval $I = (a, b) \subset \mathbb{R}$. The notion related to Ψ-concavity was defined by Beckenbach in 1937 [7] [58, cf. Section 84, p. 240]. We introduce here also a new notion of essential Ψ-concave majorant $\hat{f}$ of a measurable function f, which is a key to study a representation of dual spaces. We discuss several properties of Ψ-concave functions as well as the existence, continuity or differentiability of Ψ-concave majorant $\hat{f}$ of an arbitrary measurable function f.

In the main section 2.2, we give an isometric description of the dual of Cesàro function space with arbitrary weight function on finite or infinite interval $(0, l)$. We treat finite and infinite case in an unified way, opposite to the isomorphic description given in [5]. It is also worth to mention that in the process of showing our results, we do not use Hardy inequality at all, an essential tool in studying the space Ces_p.

In section 2.3, applying techniques developed in studying duality, we prove that convex combinations of some slices of the unit ball of $C_{p,w}$, $1 < p < \infty$ have diameter 2. From the latter result it follows that every slice of the unit ball $B_{C_{p,w}}$, $1 < p < \infty$, has diameter 2. Consequently, in a final part we state several corollaries as that Cesàro function spaces do not have the Radon-Nikodym property, neither strongly exposed nor denting points, as well as they are not locally uniformly rotund or that they are not dual spaces.

2.2 Ψ-concave functions and essential Ψ-concave majorants

In this section, we fix $I = (a, b) \subset \mathbb{R}$ to be an open (finite or infinite) interval and $\Psi : I \to \mathbb{R}_+$ to be a strictly decreasing function on I. We first collect a number of properties of Ψ-concave functions. Some of them are certainly known [58, cf Section 84, p. 240] but we provide their proofs here for the sake of completeness. Next we introduce a notion of essential Ψ-concave majorant $\hat{f}$ of $f \in L_0(I)$ and discuss a number of its properties like existence, continuity and differentiability. This section can be of independent interest.

Recall a definition of Ψ-concave function [34].

Definition 2.1. A function $f : I \to \mathbb{R}$ is called Ψ-*concave* (respectively *strictly* Ψ-*concave*) on I if for all $x < y < z$ in I,

$$\begin{vmatrix} 1 & 1 & 1 \\ \Psi(x) & \Psi(y) & \Psi(z) \\ f(x) & f(y) & f(z) \end{vmatrix} \geqslant 0 \quad (\text{respectively } > 0). \tag{2.2}$$

It is easy to check that inequality (2.2) is equivalent to

$$\frac{f(x)-f(y)}{\Psi(x)-\Psi(y)} \leqslant \frac{f(y)-f(z)}{\Psi(y)-\Psi(z)} \quad (\text{respectively } <\text{) for all } x<y<z \text{ in } I. \tag{2.3}$$

It is also possible to rewrite (2.3) as

$$\frac{f(x)-f(y)}{\Psi(x)-\Psi(y)} \leqslant \frac{f(x)-f(z)}{\Psi(x)-\Psi(z)} \leqslant \frac{f(y)-f(z)}{\Psi(y)-\Psi(z)} \quad (\text{resp. } <\text{) for all } x<y<z \text{ in } I. \tag{2.4}$$

If the interval $I = (a, b)$ is finite and $\Psi(x) = b - x$, $x \in I$, then Ψ-concavity on I is just usual concavity.

The following definition will be also useful.

Definition 2.2. We say that a function $f : I \mapsto \mathbb{R}$ is Ψ-*affine* on I, if $f(x) = A\Psi(x) + B$, $x \in I$, for some constants A and B.

For arbitrary interval J we say that $f : J \to \mathbb{R}$ is Ψ-*concave on* J if it is Ψ-concave on the interior of J. Similarly in the case of Ψ-affine function.

Now, similarly as done for example in [58] in context of convex functions, we show basic properties of Ψ-concave functions.

Let $f : I \to \mathbb{R}$ be Ψ-concave on I. Define for $x \in I$,

$$D_\Psi^+ f(x) = \lim_{y \to x^+} \frac{f(y) - f(x)}{\Psi(y) - \Psi(x)} \quad \text{and} \quad D_\Psi^- f(x) = \lim_{y \to x^-} \frac{f(y) - f(x)}{\Psi(y) - \Psi(x)}.$$

In order to see the existence and finiteness of the above quantities, it is enough to observe that for any $w < x < y < z < u$ in I, by (2.4) it follows that

$$\frac{f(w) - f(y)}{\Psi(w) - \Psi(y)} \leqslant \frac{f(x) - f(y)}{\Psi(x) - \Psi(y)} \leqslant \frac{f(y) - f(z)}{\Psi(y) - \Psi(z)} \leqslant \frac{f(y) - f(u)}{\Psi(y) - \Psi(u)},$$

and hence, for any fixed $y \in I$, the left side of the inequality

$$\frac{f(x) - f(y)}{\Psi(x) - \Psi(y)} \leqslant \frac{f(y) - f(z)}{\Psi(y) - \Psi(z)}$$

increases as $x \uparrow y$ and the right side decreases as $z \downarrow y$. It follows that $D_\Psi^- f(y)$, $D_\Psi^+ f(y)$ exist and $D_\Psi^- f(y) \leqslant D_\Psi^+ f(y)$ for all $y \in I$. Monotonicity of $D_\Psi^+ f$ and $D_\Psi^- f$ follows again from (2.4). Namely, for any $w < x < y < z$ in I,

$$\begin{aligned} D_\Psi^+ f(w) = \lim_{y \downarrow w} \frac{f(y) - f(w)}{\Psi(y) - \Psi(w)} &\leqslant \frac{f(w) - f(x)}{\Psi(w) - \Psi(x)} \\ \leqslant \frac{f(y) - f(z)}{\Psi(y) - \Psi(z)} &\leqslant \lim_{y \uparrow z} \frac{f(y) - f(z)}{\Psi(y) - \Psi(z)} = D_\Psi^- f(z). \end{aligned}$$

Hence $D_\Psi^- f(w) \leqslant D_\Psi^+ f(w) \leqslant D_\Psi^- f(z) \leqslant D_\Psi^+ f(z)$ for all $w, z \in I$ such that $w < z$.

In fact $D_\Psi^+ f$ is right-continuous if Ψ is right-continuous. Indeed, by monotonicity of $D_\Psi^+ f$ we have that $\lim_{x \to w^+} D_\Psi^+ f(x)$ exists for any $w \in I$. Since for any $y > x$,

$$D_\Psi^+ f(x) \leqslant \frac{f(y) - f(x)}{\Psi(y) - \Psi(x)},$$

and since f and Ψ are right-continuous, $\lim_{x\to w^+} D_\Psi^+ f(x) \leqslant \frac{f(y)-f(w)}{\Psi(y)-\Psi(w)}$, $y > x > w$. It follows that

$$\lim_{x\downarrow w} D_\Psi^+ f(x) \leqslant \lim_{y\downarrow w} \frac{f(y)-f(w)}{\Psi(y)-\Psi(w)} = D_\Psi^+ f(w).$$

On the other hand, we know that $D_\Psi^+ f(w) \leqslant D_\Psi^+ f(x)$ for all $w < x$, and so for all $w \in I$, $\lim_{x\downarrow w} D_\Psi^+ f(x) = D_\Psi^+ f(w)$. Similarly one can show the left-continuity of $D_\Psi^- f$ under assumption of left-continuity on Ψ.

The next proposition summarizes our discussion so far.

Proposition 2.3. *If f is Ψ-concave on I then $D_\Psi^+ f(x)$, $D_\Psi^- f(x)$ exist, are finite and $D_\Psi^- f(x) \leqslant D_\Psi^+ f(x)$ for all $x \in I$. Moreover, $D_\Psi^+ f$, $D_\Psi^- f$ are increasing functions on I. If Ψ is right-continuous on I then so is $D_\Psi^+ f$. Similarly, if Ψ is left-continuous on I then so is $D_\Psi^- f$. For any fixed $y \in I$ the ratio $\frac{f(x)-f(y)}{\Psi(x)-\Psi(y)}$ increases as $x \uparrow y$ and the ratio $\frac{f(y)-f(z)}{\Psi(y)-\Psi(z)}$ decreases as $z \downarrow y$.*

The following basic fact will be also useful.

Lemma 2.4. *If Ψ is right-, left-, absolutely or Lipschitz continuous then any Ψ-concave function f on I has the same property.*

Proof. Let $[c,d] \subset I$ and $a < c_1 < c$ and $d < d_1 < b$, by Ψ-concavity of f we get

$$\frac{f(c_1)-f(c)}{\Psi(c_1)-\Psi(c)} \leqslant \frac{f(x)-f(y)}{\Psi(x)-\Psi(y)} \leqslant \frac{f(d)-f(d_1)}{\Psi(d)-\Psi(d_1)} \quad \text{for all } c \leqslant x < y \leqslant d,$$

hence

$$\left|\frac{f(x)-f(y)}{\Psi(x)-\Psi(y)}\right| \leqslant \max\left\{\left|\frac{f(c_1)-f(c)}{\Psi(c_1)-\Psi(c)}\right|, \left|\frac{f(d)-f(d_1)}{\Psi(d)-\Psi(d_1)}\right|\right\}.$$

Denoting the right-hand side by K we get that $|f(x)-f(y)| \leqslant K|\Psi(x)-\Psi(y)|$ for all $x, y \in [c,d]$. The claim follows. $\square$

Lemma 2.5. *Let f be Ψ-concave on I. If $D^+_\Psi f(x) \geqslant 0$, $x \in I$, then f is decreasing on I. If $D^+_\Psi f(x) \leqslant 0$, $x \in I$, then f is increasing on I.*

Proof. Let $D^+_\Psi f(x) \geqslant 0$. Since the ratio $\frac{f(z)-f(x)}{\Psi(z)-\Psi(x)}$ decreases as $z \downarrow x$, it follows that $\frac{f(z)-f(x)}{\Psi(z)-\Psi(x)} \geqslant 0$ for $z > x$, and hence $f(z) \leqslant f(x)$ by monotonicity of Ψ. Since $x \in I$ is arbitrary, we get that f is decreasing. The proof of another case is similar. □

Lemma 2.6. *Let function $f \geqslant 0$ be Ψ-concave on I. If $\lim_{x\to a^+} \Psi(x) = \infty$ then $\lim_{x\to a^+} D^+_\Psi f(x) \geqslant 0$.*

Proof. Since function $D^+_\Psi f$ is increasing, $\lim_{x\to a^+} D^+_\Psi f(x)$ exists or is equal to $-\infty$. Suppose that $\lim_{x\to a^+} D^+_\Psi f(x) = C < 0$. It follows that there exists $x_0 > a$ such that $-\infty < D := D^+_\Psi f(x_0) < 0$, so $D^-_\Psi f(x) \leqslant D$ for $x \in (a, x_0)$. It follows that for all $z < x$, $\frac{f(x)-f(z)}{\Psi(x)-\Psi(z)} \leqslant D < 0$, which gives $f(z) \leqslant D\Psi(z) - D\Psi(x) + f(x)$. Now, keeping $x \in (a, x_0)$ fixed and taking $z \to a^+$ we would get that $f(z) < 0$ for z close enough to a, which contradicts the condition $f \geqslant 0$. □

Lemmas 2.5 and 2.6 imply the following corollary.

Corollary 2.7. *If a function $f \geqslant 0$ is Ψ-concave on I and $\lim_{x\to a^+} \Psi(x) = \infty$ then f is decreasing on I.*

Observe that inequality (2.3) can also be equivalently written as

$$f(y) \geqslant \frac{\Psi(y)-\Psi(z)}{\Psi(x)-\Psi(z)} f(x) + \frac{\Psi(x)-\Psi(y)}{\Psi(x)-\Psi(z)} f(z) \text{ for all } x, y, z \in I \text{ with } x < y < z. \tag{2.5}$$

Lemma 2.8. *A function $f : I \to \mathbb{R}$ is Ψ-concave on I if and only if for each $y \in I$ there is at least one function $T(x) = f(y) + A(\Psi(x) - \Psi(y))$ such that $A \in [D^-_\Psi f(y), D^+_\Psi f(y)]$ and $f(x) \leqslant T(x)$ for $x \in I$.*

Proof. If f is Ψ-concave on I and $y \in I$ then for any $A \in [D^-_\Psi f(y), D^+_\Psi f(y)]$,

$$\frac{f(x) - f(y)}{\Psi(x) - \Psi(y)} \geqslant A \quad \text{or} \ \leqslant A,$$

if $x > y$ or $x < y$, respectively. In any case $f(x) \leqslant A\Psi(x) + f(y) - A\Psi(y) = T(x)$ for all $x \in I$.

Conversely, suppose that for each $y \in I$ there is at least one function $T(x) = f(y) + A(\Psi(x) - \Psi(y))$ such that $f(x) \leqslant T(x)$ for $x \in I$. Let $x, y, z \in I$ be such that $x < y < z$. Denoting $\alpha = \frac{\Psi(y)-\Psi(z)}{\Psi(x)-\Psi(z)}$ we get $\Psi(y) = \alpha\Psi(x) + (1-\alpha)\Psi(z)$, $\alpha \in [0,1]$. It follows that $f(y) = T(y) = \alpha T(x) + (1-\alpha)T(z) \geqslant \alpha f(x) + (1-\alpha) f(z)$. Hence, in a view of (2.5), f is Ψ-concave. □

The following lemma will be useful.

Lemma 2.9. *Let f be Ψ-concave on I. The function f is strictly Ψ-concave on I if and only if there is no interval $(c,d) \subset I$ on which f is Ψ-affine. The function f is Ψ-affine on I if and only if $D^+_\Psi f$ is constant on I. The function f is strictly Ψ-concave on I if and only if $D^+_\Psi f$ is strictly increasing on I.*

Proof. We prove only the first part. The proof of the second part is similar.

If there exists an interval $(c,d) \subset I$ on which $f(x) = A\Psi(x) + B$, then $\frac{f(x)-f(y)}{\Psi(x)-\Psi(y)} = A$ for all $x, y \in (c,d)$. Hence f is not strictly Ψ-concave on I. Conversely, if f is not strictly Ψ-concave on I, then from (2.3) we get that there exist $w < x < u$ in I such that

$$A := \frac{f(w) - f(x)}{\Psi(w) - \Psi(x)} = \frac{f(x) - f(u)}{\Psi(x) - \Psi(u)}.$$

It follows from (2.4) that for any y in (x,u),

$$A = \frac{f(w) - f(x)}{\Psi(w) - \Psi(x)} \leqslant \frac{f(w) - f(y)}{\Psi(w) - \Psi(y)} \leqslant \frac{f(w) - f(u)}{\Psi(w) - \Psi(u)} \leqslant \frac{f(x) - f(u)}{\Psi(x) - \Psi(u)} = A.$$

Hence $f(y) = A\Psi(y) + f(w) - A\Psi(w)$ for all $y \in (x, u)$, that is f is Ψ-affine on (x, u), a contradiction. $\square$

In the case when Ψ is a continuous function, we can define Ψ-concavity in one more equivalent way. Namely, denoting $\alpha = \frac{\Psi(y)-\Psi(z)}{\Psi(x)-\Psi(z)}$ for $x < y < z$ in I, we get that $\Psi(y) = \alpha\Psi(x) + (1-\alpha)\Psi(z)$, and so (2.5) can be written as

$$f(\Psi^{-1}(\alpha\Psi(x) + (1-\alpha)\Psi(z))) \geqslant \alpha f(x) + (1-\alpha) f(z). \tag{2.6}$$

If Ψ is continuous then for any $x, z \in I$, say $x \leqslant z$, and any $\alpha \in [0, 1]$ there exists $y \in [x, z]$ such that $\Psi(y) = \alpha\Psi(x) + (1-\alpha)\Psi(z)$. Then inequality (2.6) holds true for all $\alpha \in [0, 1]$ and all $x, z \in I$, that is function $f \circ \Psi^{-1}$ is concave on $\Psi(I)$. Furthermore, in this case, by induction it can be shown that

$$\begin{gathered} f\left(\Psi^{-1}\left(\sum_{i=1}^{n} \alpha_i \Psi(y_i)\right)\right) \geqslant \sum_{i=1}^{n} \alpha_i f(y_i) \\ \text{for all } \alpha_i \geqslant 0,\ \sum_{i=1}^{n} \alpha_i = 1 \text{ and } (y_1, y_2, \ldots, y_n) \in I^n. \end{gathered} \tag{2.7}$$

We have the following lemma.

Lemma 2.10. *If Ψ is a continuous function on I then f is Ψ-concave on I if and only if $f \circ \Psi^{-1}$ is concave on $\Psi(I)$.*

Proof. Only one direction requires proof. Suppose that $f \circ \Psi^{-1}$ is concave on $\Psi(I)$. Let $x, y, z \in I$ be arbitrary, $x < y < z$ and $u = \Psi(x)$, $v = \Psi(z)$. Since Ψ is one-to-one, $x = \Psi^{-1}(u)$ and $z = \Psi^{-1}(v)$. By concavity of $f \circ \Psi^{-1}$ we get that $(f \circ \Psi^{-1})(\alpha u + (1-\alpha)v) \geqslant \alpha(f \circ \Psi^{-1})(u) + (1-\alpha)(f \circ \Psi^{-1})(v)$ for all $\alpha \in [0, 1]$. It follows that $f(\Psi^{-1}(\alpha\Psi(x) + (1-\alpha)\Psi(z))) \geqslant \alpha f(x) + (1-\alpha) f(z)$. Since $y \in (x, z)$

there exists $\alpha \in [0,1]$ such that $\Psi(y) = \alpha\Psi(x) + (1-\alpha)\Psi(z)$. This gives inequality (2.5) and so f is Ψ-concave on I. $\square$

The following notion of essential Ψ-concave majorant is crucial for characterization of the dual space to Cesàro function spaces.

Definition 2.11. For any function $f \in L_0^+(I)$ we define its *essential Ψ-concave majorant* $\hat{f}$ by

$$\hat{f}(y) := \inf\Big\{M > 0 : m^{(n)}\Big\{(y_1, \ldots, y_n) \in I^n : \sum_{i=1}^n \alpha_i f(y_i) > M, \sum_{i=1}^n \alpha_i = 1, \alpha_i \geqslant 0,$$
$$i = 1, \ldots, n, \Psi(y) = \sum_{i=1}^n \alpha_i \Psi(y_i)\Big\} = 0,\ n \in \mathbb{N}\Big\}, \quad y \in I,$$

where $m^{(n)}$ is the Lebesgue product measure on I^n. For arbitrary function $f \in L_0(I)$ we define $\hat{f} = \widehat{|f|}$.

The above definition should be compared to one of concave majorants given in 1970 by Peetre [56].

The remaining results of this section describe several properties of $\hat{f}$. First we give conditions on f under which the essential Ψ-concave majorant $\hat{f}$ is finite on I.

Lemma 2.12. *Let $f \in L_0^+(I)$. If $\operatorname{ess\,sup}_{x\in(y,b)} f(x) < \infty$ and $\operatorname{ess\,sup}_{x\in(a,y)} \frac{f(x)}{\Psi(x)} < \infty$ for all $y \in I$ then $\hat{f} < \infty$ on I.*

Proof. Let $f \in L_0^+(I)$ and $y \in I$. Suppose that

$$A_y := \operatorname{ess\,sup}_{x\in(y,b)} f(x) < \infty \quad \text{and} \quad B_y := \operatorname{ess\,sup}_{x\in(a,y)} \frac{f(x)}{\Psi(x)} < \infty.$$

For any $n \in \mathbb{N}$, if

$$\Psi(y) = \sum_{i=1}^{n} \alpha_i \Psi(y_i),\ \sum_{i=1}^{n} \alpha_i = 1,\ \alpha_i \geqslant 0,\ y_i \in I,\ i = 1, 2, \ldots, n,$$

we have that

$$\begin{aligned}\sum_{i=1}^{n} \alpha_i f(y_i) &= \sum_{y_i<y} \alpha_i f(y_i) + \sum_{y_i \geqslant y} \alpha_i f(y_i) \\ &= \sum_{y_i<y} \alpha_i \Psi(y_i) f(y_i)/\Psi(y_i) + \sum_{y_i \geqslant y} \alpha_i f(y_i) \leqslant B_y \Psi(y) + A_y\end{aligned}$$

except possibly some subset of the set

$$\begin{aligned}C := \bigcup \Big\{ &(y_1, \ldots, y_n) \in I^n : \max_{i \in I_1 : y_i < y} (f(y_i)/\Psi(y_i)) > B_y \Big\} \\ &\cup \Big\{ (y_1, \ldots, y_n) \in I^n : \max_{i \in I_2 : y_i \geqslant y} (f(y_i)) > A_y \Big\},\end{aligned}$$

where the union is taken over all partitions of $\{1, 2, \ldots, n\}$ into two disjoint nonempty sets I_1, I_2. It is not difficult to see that $m^{(n)} C = 0$, whence it follows that $\hat{f}(y) \leqslant B_y \Psi(y) + A_y < \infty$. $\square$

Lemma 2.13. *Let $f \in L_0^+(I)$. If $\hat{f} < \infty$ on I then $\hat{f}$ is Ψ-concave on I.*

Proof. The proof is similar to one for concave majorants [41, p. 47]. Let $x < y < z$ in I. We will show inequality (2.5) for $\hat{f}$. Let $\alpha = \frac{\Psi(y)-\Psi(z)}{\Psi(x)-\Psi(z)}$ and $\epsilon > 0$ be arbitrary. From the definition of $\hat{f}(x)$ it follows that there exist $j \in \mathbb{N}$ and a set

$$B = \left\{ (x_1^\epsilon, x_2^\epsilon, \ldots, x_j^\epsilon) \in I^j : \sum_{i=1}^{j} \alpha_i^\epsilon f(x_i^\epsilon) > \hat{f}(x) - \epsilon/2, \Psi(x) = \sum_{i=1}^{j} \alpha_i^\epsilon \Psi(x_i^\epsilon), \sum_{i=1}^{j} \alpha_i^\epsilon = 1 \right\}$$

with $m^{(j)} B > 0$. Denoting $\alpha_i'^\epsilon = \alpha \alpha_i^\epsilon$, $i = 1, \ldots, j$, we get that for all $(x_1^\epsilon, x_2^\epsilon, \ldots, x_j^\epsilon) \in B$, $\alpha \hat{f}(x) \leqslant \sum_{i=1}^{j} \alpha_i'^\epsilon f(x_i^\epsilon) + \epsilon/2$, where $\sum_{i=1}^{j} \alpha_i'^\epsilon = \alpha$, $\alpha_i'^\epsilon > 0$, $\alpha \Psi(x) = \sum_{i=1}^{j} \alpha_i'^\epsilon \Psi(x_i^\epsilon)$.

Similarly, by definition of $\hat{f}(z)$, there exist $k \in \mathbb{N}$ and a set

$$C = \left\{(z_1^\epsilon, z_2^\epsilon, \dots, z_k^\epsilon) \in I^k : \sum_{i=1}^{k} \beta_i^\epsilon f(z_i^\epsilon) > \hat{f}(z) - \epsilon/2, \Psi(z) = \sum_{i=1}^{k} \beta_i^\epsilon \Psi(z_i^\epsilon), \sum_{i=1}^{k} \beta_i^\epsilon = 1\right\}$$

with $m^{(k)}C > 0$. Denoting $\beta_i'^\epsilon = (1-\alpha)\beta_i^\epsilon$, $i = 1, 2, \dots, k$, we get that for all $(z_1^\epsilon, z_2^\epsilon, \dots, z_k^\epsilon) \in C$, $(1-\alpha)\hat{f}(z) \leqslant \sum_{i=1}^{k} \beta_i'^\epsilon f(z_i^\epsilon) + \epsilon/2$, where $\sum_{i=1}^{k} \beta_i'^\epsilon = 1 - \alpha$, $\beta_i'^\epsilon > 0$, $(1-\alpha)\Psi(z) = \sum_{i=1}^{k} \beta_i'^\epsilon \Psi(z_i^\epsilon)$. Denoting now $\gamma_i^\epsilon = \alpha_i'^\epsilon$, $y_i^\epsilon = x_i^\epsilon$ for $i = 1, 2, \dots, j$, $\gamma_{i+j}^\epsilon = \beta_i'^\epsilon$, $y_{i+j}^\epsilon = z_i^\epsilon$ for $i = 1, 2, \dots, k$, and $n = j + k$ we get that $\sum_{i=1}^{n} \gamma_i^\epsilon = 1$ and $\sum_{i=1}^{n} \gamma_i^\epsilon \Psi(y_i^\epsilon) = \alpha\Psi(x) + (1-\alpha)\Psi(z) = \Psi(y)$. It follows that $\alpha\hat{f}(x) + (1-\alpha)\hat{f}(z) \leqslant \sum_{i=1}^{n} \gamma_i^\epsilon f(y_i^\epsilon) + \epsilon \leqslant \hat{f}(y) + \epsilon$ a.e. Since ϵ was arbitrary the claim follows. $\square$

Recall that if C is a measurable subset of $\mathbb{R}$ and $y \in \mathbb{R}$, then y is called a *point of density* of C if

$$\lim_{\substack{m(x,z)\to 0\\ y\in(x,z)}} \frac{m(C \cap (x,z))}{m(x,z)} = 1.$$

It is known that if C is a measurable subset of $\mathbb{R}$ then almost every $x \in C$ is a point of density of C [65, p. 106] [60, p. 141].

Lemma 2.14. *If Ψ is a continuous function and $\lim_{x\to a^+} \Psi(x) = \infty$ then for any function $f \in L_0^+(I)$ with $\hat{f} < \infty$ on I, it holds that $f \leqslant \hat{f}$ a.e. on I, and $\hat{f}$ is also continuous on I.*

Proof. Suppose there exist $\epsilon > 0$ and a set $C \subset I$ with $mC > 0$ such that $f \geqslant \hat{f} + \epsilon$ on C. Without loss of generality, we assume that all points in C are points of density of C. It follows that for all $y \in C$ and all $x < y < z$ in I, $m(C \cap (x,z)) > 0$.

First we show that for any $x < z$ such that $m(C \cap (x,z)) > 0$ there is $y \in C \cap (x,z)$ for which $m(C \cap (x,y)) > 0$ and $m(C \cap (y,z)) > 0$. Let $c = \sup_{y\in C\cap(x,z)} m(C \cap$

$(x, y)) = 0$ and $d = \inf_{y \in C \cap (x,z)} m(C \cap (y, z)) = 0$. It follows that $c < d$, which gives $m(C \cap (x, y)) > 0$ and $m(C \cap (y, z)) > 0$ for all $c < y < d$.

Since Ψ is a continuous function, by Lemmas 2.13 and 2.4 we get that $\hat{f}$ is continuous. Let $x < z$ in I be such that

$$\hat{f}(x) - \hat{f}(z) < \epsilon/2 \quad \text{and} \quad m(C \cap (x, z)) > 0. \tag{2.8}$$

By the above there is $y \in C \cap (x, z)$ such that $m(C \cap (x, y)) > 0$ and $m(C \cap (y, z)) > 0$. Consider the set $B := \{(y_1, y_2) \in (C \cap (x, z)) \times (C \cap (x, z)) : \Psi(y) = \alpha\Psi(y_1) + (1 - \alpha)\Psi(y_2)$ for some $\alpha \in (0, 1)\}$. It is clear that $B = \{(y_1, y_2) \in (C \cap (x, z)) \times (C \cap (x, z)) : y_1 < y < y_2 \quad \text{or } y_2 < y < y_1\} = ((C \cap (x, y)) \times (C \cap (y, z))) \cup ((C \cap (y, z)) \times (C \cap (x, y)))$ and hence $m^{(2)}B > 0$. Observe that by (2.8) we have $|\hat{f}(y_1) - \hat{f}(y_2)| < \epsilon/2$ for all $y_1, y_2 \in I$ such that $(y_1, y_2) \in B$. Now, for almost all $(y_1, y_2) \in B$, since $\hat{f}$ is decreasing by Corollary 2.7, we have that

$$\hat{f}(y) \geqslant \alpha f(y_1) + (1 - \alpha)f(y_2) \geqslant \alpha\hat{f}(y_1) + (1 - \alpha)\hat{f}(y_2) + \epsilon \geqslant \hat{f}(y) + \epsilon/2.$$

This is impossible, hence $f \leqslant \hat{f}$ a.e. on I. $\square$

Remark 2.15. (1) If $f, g \in L_0^+(I)$ and $f \leqslant g$ a.e. on I then $\hat{f} \leqslant \hat{g}$. In fact $\sum_{i=1}^n \alpha_i f(y_i) \leqslant \sum_{i=1}^n \alpha_i g(y_i)$ for all $(y_1, y_2, \ldots, y_n) \in I^n$ except possibly some set of measure 0.

(2) Let Ψ be a continuous function on I and $\lim_{x \to a^+} \Psi(x) = \infty$. If f is Ψ-concave on I then $f = \hat{f}$. Consequently $\hat{\hat{f}} = \hat{f}$ for any function $f \in L_0(I)$ with $\hat{f} < \infty$ on I. Indeed, from (2.7) it follows that $\sum_{i=1}^n \alpha_i f(y_i) \leqslant f(y)$ whenever $\Psi(y) = \sum_{i=1}^n \alpha_i \Psi(y_i)$ and hence $\hat{f} \leqslant f$ which together with Lemma 2.14 gives that $f = \hat{f}$.

Lemma 2.16. *Let Ψ be a continuous function on I and $\lim_{x\to a^+}\Psi(x)=\infty$. Let $f\in L_0^+(I)$ be such that $\hat{f}<\infty$ on I, $\epsilon>0$ be fixed,*

$$A=\{x\in I: f(x)\geqslant \hat{f}(x)-\epsilon\}$$

and $(u,v)\subset I$ be a finite open interval. If $m(A\cap(u,v))=0$ then $\hat{f}$ is Ψ-affine on (u,v).

Proof. Let $y\in(u,v)$ be fixed. For any $\eta>0$, by definition of $D_\Psi^-\hat{f}(y)$,

$$D_\Psi^-\hat{f}(y)\geqslant \frac{\hat{f}(c)-\hat{f}(y)}{\Psi(c)-\Psi(y)}\geqslant D_\Psi^-\hat{f}(y)-\eta$$

for all $c<y$ close enough to y. Moreover, the ratio $\frac{\hat{f}(c)-\hat{f}(x)}{\Psi(c)-\Psi(x)}$ is a continuous function of x and by Proposition 2.3 it decreases as $x\downarrow y$. Hence for every $\eta>0$ and every $c<y$ there exists $d>y$ arbitrary close to y such that

$$D_\Psi^-\hat{f}(y)+\eta\geqslant \frac{\hat{f}(c)-\hat{f}(d)}{\Psi(c)-\Psi(d)}\geqslant D_\Psi^-\hat{f}(y)-\eta.$$

By the above, we construct an increasing sequence $(a_n)\subset(u,y)$ and a sequence $(b_n)\subset(y,v)$ such that $a_n\to y$, $b_n\to y$ and

$$s_n:=\frac{\hat{f}(a_n)-\hat{f}(b_n)}{\Psi(a_n)-\Psi(b_n)}\to C:=D_\Psi^-\hat{f}(y)\quad \text{as } n\to\infty.$$

Consider the sequence of functions

$$S_n(x)=s_n\Psi(x)+\hat{f}(a_n)-s_n\Psi(a_n),\quad x\in I.$$

It is clear that $S_n(a_n)=\hat{f}(a_n)$, $S_n(b_n)=\hat{f}(b_n)$ and by (2.5), $S_n(x)\leqslant\hat{f}(x)$ for all

$x \in (a_n, b_n)$, $n \in \mathbb{N}$. By Lemma 2.8, $\hat{f}(x) \leqslant C\Psi(x) + (\hat{f}(y) - C\Psi(y))$. Hence

$$\begin{aligned}
\hat{f}(x) - S_n(x) &= \hat{f}(x) - (s_n\Psi(x) + (\hat{f}(a_n) - s_n\Psi(a_n))) \\
&\leqslant C\Psi(x) + \hat{f}(y) - C\Psi(y) - s_n\Psi(x) - \hat{f}(a_n) + s_n\Psi(a_n) \\
&= (C - s_n)\Psi(x) + \hat{f}(y) - \hat{f}(a_n) + s_n\Psi(a_n) - C\Psi(y) \\
&\leqslant |C - s_n|\Psi(a_1) + \hat{f}(y) - \hat{f}(a_n) + s_n\Psi(a_n) - C\Psi(y).
\end{aligned}$$

It follows that for every $\delta > 0$ there exists $N_\delta \in \mathbb{N}$ such that for all $n > N_\delta$ and for all $x \in (a_n, b_n)$,

$$0 \leqslant \hat{f}(x) - S_n(x) \leqslant \delta.$$

By the above for all $y \in (u, v)$ there exist $c_y, d_y \in (u, v)$, $c_y < y < d_y$, such that $\hat{f}(x) \geqslant D\Psi(x) + B$ and $\hat{f}(x) - (D\Psi(x) + B) \leqslant \epsilon$ for all $x \in (c_y, d_y)$ where $D = (\hat{f}(c_y) - \hat{f}(d_y))/(\Psi(c_y) - \Psi(d_y))$ and $B = (\Psi(c_y)\hat{f}(d_y) - \Psi(d_y)\hat{f}(c_y))/(\Psi(c_y) - \Psi(d_y))$. By definition of the set A it follows that $f(x) \leqslant D\Psi(x) + B$ a.e. on (c_y, d_y). Since function $g(t) = \hat{f}\chi_{(c_y,d_y)^c}(t) + (D\Psi(t) + B)\chi_{(c_y,d_y)}(t)$, $t \in I$, is Ψ-concave on I it must be $\hat{f} = g$ by Remark 2.15. But g is Ψ-affine on (c_y, d_y), so $D^+_\Psi \hat{f}$ is constant there by Lemma 2.9.

The family of sets (c_y, d_y), $y \in (u, v)$ cover each closed subinterval $[u + \epsilon, v - \epsilon]$, $\epsilon > 0$. Using compactness we conclude that $D^+_\Psi \hat{f}$ is constant on (u, v) and by Lemma 2.9, $\hat{f}$ is Ψ-affine on (u, v). $\square$

Recall the following theorem concerning convex functions [55, Corollary 1.3.8].

Theorem 2.17. *If $f_n : I \to \mathbb{R}$ is a pointwise converging sequence of convex functions, then the limit is also convex. Moreover, the convergence is uniform on any compact subinterval included in the interior of I, and (f_n') converges to f' except possibly at countably many points of I.*

Observe that the above theorem works if one replaces words "convex" by "concave". Now, by Lemma 2.10 and Theorem 2.17, we conclude this section by the following result.

Lemma 2.18. *Let Ψ be an absolutely continuous function on each closed subinterval of I with finite and non zero derivative Ψ' a.e. on I. If $f_n : I \to \mathbb{R}$ is a sequence of Ψ-concave functions converging to a function f which is finite on I, then f is Ψ-concave on I and the convergence is uniform on any compact subinterval of I. Moreover, (f_n') converges to f' a.e. on I.*

2.3 Description of the dual space

Let $I = (0, l)$, $0 < l \leqslant \infty$ and $0 < w \in L_0(I)$. The *weighted Cesàro function space* on I is defined to be $(1 \leqslant p < \infty)$

$$C_{p,w} = C_{p,w}(I) := \left\{ f \in L_0(I) : \|f\|_{C_{p,w}} := \left(\int_I \left(w(x) \int_0^x |f(t)|\, dt \right)^p dx \right)^{1/p} < \infty \right\}.$$

Note that for $f \in C_{p,w}(I)$,

$$\|f\|_{C_{p,w}} = \|\mathcal{H}_w f\|_p \quad \text{where} \quad \mathcal{H}_w f(x) = w(x) \int_0^x |f(t)|\, dt, \quad x \in I,$$

and $\|\cdot\|_p$ is the norm in the Lebesgue space $L_p(I)$.

The goal of this section is Theorem 2.27 which gives an isometric description of the Banach dual space $(C_{p,w})^*$. We start with two basic lemmas.

Lemma 2.19. *The space $(C_{p,w}, \|\cdot\|_{C_{p,w}})$ is an order continuous Banach function lattice with the Fatou property.*

Proof. To see that $(C_{p,w}, \|\cdot\|_{C_{p,w}})$ has the Fatou property it is enough to apply Fatou's

Lemma twice. Using the Monotone Convergence Theorem one can show that $C_{p,w}(I)$ is an order continuous space [8, 41]. $\square$

Lemma 2.20. (a) *$C_{p,w}(I) \neq \{0\}$ if and only if $\int_c^l w(x)^p\,dx < \infty$ for some $c \in I$.*

(b) *$C_{p,w}(I)$ is not continuously embedded into $L_1(I)$ whenever it is not trivial.*

Proof. (a) Suppose that $\int_c^l w(x)^p\,dx < \infty$ for some $c \in I$. For all $d \in (c, l)$ we have

$$\|\chi_{(c,d)}\|_{C_{p,w}}^p = \int_c^l (w(x) \int_0^x \chi_{(c,d)}(t)\,dt)^p\,dx \leqslant (d-c)^p \int_c^l w(x)^p\,dx < \infty,$$

whence $\chi_{(c,d)} \in C_{p,w}$. If $C_{p,w}(I) \neq \{0\}$ then $\chi_{(c,d)} \in C_{p,w}$ for some $c, d \in I$, $d > c$. It follows that $\int_d^l w(x)^p\,dx < \infty$.

(b) Let a_n be a strictly increasing sequence in I such that $\int_{a_n}^l w(x)^p\,dx = 1/n^p$, $n \geqslant n_0$, for some large enough $n_0 \in \mathbb{N}$. For $n \geqslant n_0$, let

$$g_n = \begin{cases} \chi_{(a_n, a_n+n)} & \text{if } l = \infty, \\ \frac{n}{a_{n+1}-a_n}\chi_{(a_n, a_{n+1})} & \text{if } l < \infty. \end{cases}$$

Clearly in both cases $\|g_n\|_1 \to \infty$ as $n \to \infty$, $\int_0^x g_n(t)\,dt = 0$ for $x < a_n$ and $\int_0^x g_n(t)\,dt \leqslant n$ for $x \geqslant a_n$. Hence $\|g_n\|_{C_{p,w}}^p \leqslant \int_{a_n}^l n^p w(x)^p\,dx = 1$ for all $n \geqslant n_0$, and the claim follows. $\square$

If $p = 1$ then by Fubini's Theorem

$$\int_0^l w(x) \int_0^x |f(t)|\,dt\,dx = \int_0^l |f(t)| \int_t^l w(x)\,dx\,dt.$$

Hence the space $C_{1,w}(I)$ is just a weighted Lebesgue space with weight $\int_t^l w(x)\,dx$, $t \in I$.

In the sequel we assume that $1 < p < \infty$ is fixed and the weight function w satisfies the following conditions

(i) $w > 0$ a.e. on I,

(ii) $\int_x^l w(t)^p\,dt < \infty$ for all $x \in I$,

(iii) $\int_0^l w(t)^p\,dt = \infty$.

Let further

$$\Psi(x) = \int_x^l w(t)^p\,dt, \quad x \in I.$$

Conditions (i)-(iii) imply that the function Ψ is strictly decreasing on I, $\lim_{x\to l}\Psi(x) = 0$ and $\lim_{x\to 0^+}\Psi(x) = \infty$. Also by absolute continuity of Ψ on each compact subinterval of I, $\Psi' = -w^p < 0$ a.e. on I. Moreover, if $f \in L_0(I)$ is such that $\hat{f} < \infty$ on I then by definition of $D_\Psi^+\hat{f}$, we get that

$$D_\Psi^+\hat{f}(x) = \hat{f}'(x)/\Psi'(x) = -\hat{f}'(x)/w(x)^p \quad \text{for a.a. } x \in I,$$

where $\hat{f}'(x)$ denotes the derivative of $\hat{f}$ at x. Note that this derivative exists a.e. because Ψ is absolutely continuous on every closed subinterval of I, and so is $\hat{f}$ by Lemma 2.4.

It is easy to check that if the weight w is a power function $w(x) = x^s$ then conditions (i)-(iii) are satisfied for $s < -1/p$ if $l = \infty$, and for $s \leqslant -1/p$ if $l < \infty$. If $s = -1$ then the space $C_{p,w}$ is the standard Cesàro function space Ces_p considered by several authors (see [5] and the references given therein).

For $1 < p < \infty$ let q be its conjugate exponent $1/p + 1/q = 1$. Let us denote

$$\hat{\mathcal{H}}_w f(x) = -\hat{f}'(x)/w(x) \quad \text{for a.a. } x \in I.$$

We will show that the Köthe dual space of $C_{p,w}(I)$,

$$(C_{p,w})' = (C_{p,w}(I))' = \left\{ f \in L_0(I) : \int_I f(t)g(t)\,dt < \infty \text{ for all } g \in C_{p,w} \right\},$$

equipped with the usual norm

$$\|f\|_{(C_{p,w})'} = \sup \left\{ \int_I f(t)g(t)\,dt : g \in C_{p,w}, \|g\|_{C_{p,w}} \leqslant 1 \right\},$$

is the space

$$(C_{p,w}(I))' = \left\{ f \in L_0(I) : \hat{f} < \infty \text{ on } I, \quad \lim_{x \to l} \hat{f}(x) = 0 \quad \text{and } \hat{\mathcal{H}}_w f \in L_q(I) \right\},$$

where $\|f\|_{(C_{p,w})'} = \|\hat{\mathcal{H}}_w f\|_q$ and the essential Ψ-concave majorant $\hat{f}$ is obtained with respect to Ψ. Observe that Ψ (and hence $\hat{f}$) depends on both p and w. Since $C_{p,w}$ is an order continuous space with the Fatou property its Köthe dual $(C_{p,w})'$ can be identified with its Banach dual space $(C_{p,w})^*$. In fact each bounded linear functional $F \in (C_{p,w})^*$ is of the integral form $F(g) = \int_I f(t)g(t)\,dt$, $g \in C_{p,w}$, where $f \in (C_{p,w})'$ and $\|F\|_{(C_{p,w})^*} = \|f\|_{(C_{p,w})'}$ [8,41].

We start with several preparatory lemmas.

Lemma 2.21. *If* $0 \leqslant f \in (C_{p,w})'$ *then* $\operatorname{ess\,sup}_{x \in (y,l)} f(x) < \infty$ *and* $\operatorname{ess\,sup}_{x \in (0,y)} \frac{f(x)}{\Psi(x)} < \infty$ *for all* $y \in I$. *Consequently* $\hat{f}(y) < \infty$ *for all* $y \in I$.

Proof. Let $y \in I$ be fixed and $0 \leqslant f \in (C_{p,w})'$. Suppose that $\operatorname{ess\,sup}_{x \in (y,l)} f(x) = \infty$. For all $C > 0$ there exists a set $A \subset (y,l)$ with $0 < mA < \infty$ such that $f > C$ on A. Letting $g = \chi_A/mA$, it follows that $\int_I f(t)g(t)\,dt \geqslant C$. But for all sets $A \subset (y,l)$ of positive and finite measure $\|\chi_A/mA\|_{C_{p,w}} \leqslant \Psi(y)^p$, hence $f \notin (C_{p,w})'$. The latter gives a contradiction. We proved that $\operatorname{ess\,sup}_{x \in (y,l)} f(x) < \infty$.

Now we show that $\int_0^y (w(x)/\Psi(x))^p\,dx < \infty$ for all $y \in I$. Fix $y \in I$. By (ii) and (iii) we can find a sequence (a_n) decreasing to 0, $a_0 = l$ such that

$$n \leqslant \int_{a_{n+1}}^{a_n} w(x)^p\,dx < n+1,\ n = 0, 1, \ldots.$$

Hence, for $x \in [a_{n+1}, a_n)$, $n = 0, 1, \ldots$

$$\Psi(x) \geqslant \int_{a_n}^{l} w(t)^p\,dt = \sum_{i=0}^{n-1} \int_{a_{i+1}}^{a_i} w(t)^p\,dt \geqslant \sum_{i=0}^{n-1} i = \frac{n(n-1)}{2}.$$

Since $y \in [a_{k+1}, a_k)$ for some $k = 0, 1, \ldots$, and $p > 1$ we get that

$$\begin{aligned}\int_0^y \left(\frac{w(x)}{\Psi(x)}\right)^p dx &\leqslant \sum_{n=k}^{\infty} \int_{a_{n+1}}^{a_n} \left(\frac{w(x)}{\Psi(x)}\right)^p dx \\ &\leqslant \sum_{n=k}^{\infty} \left(\frac{2}{n(n-1)}\right)^p \int_{a_{n+1}}^{a_n} w(x)^p\,dx \leqslant \sum_{n=k}^{\infty} \frac{2^p(n+1)}{n^p(n-1)^p} < \infty.\end{aligned}$$

Next, for arbitrary $y \in I$, since $\int_0^y (w(x)/\Psi(x))^p\,dx < \infty$, Ψ is decreasing and $1/\Psi$ is bounded on $(0, y)$, denoting $B = \int_0^y 1/\Psi(x)\,dx$, by (ii) we get that

$$\begin{aligned}\int_I \left(w(x) \int_0^x \frac{\chi_A(t)}{\Psi(t) mA}\,dt\right)^p dx &\leqslant \int_0^y \left(\frac{w(x)}{\Psi(x)} \int_0^x \frac{\chi_A(t)}{mA}\,dt\right)^p dx + \int_y^l (Bw(x))^p\,dx \\ &\leqslant \int_0^y \left(\frac{w(x)}{\Psi(x)}\right)^p dx + B^p \int_y^l w(x)^p\,dx =: E < \infty.\end{aligned}$$

Hence $\|1/((mA)\Psi)\chi_A\|_{C_{p,w}} \leqslant E^{1/p} < \infty$ for all $A \subset (0, y)$ with $0 < mA < \infty$.

Suppose now that $\operatorname{ess\,sup}_{x \in (0,y)} \frac{f(x)}{\Psi(x)} = \infty$. Then for every $C > 0$ there exists a set $A \subset (0, y)$ with $mA > 0$ such that $f(x) \geqslant C\Psi(x)$ for $x \in A$. Let $g = 1/((mA)\Psi)\chi_A$. It follows that $\int_I f(x)g(x)\,dx \geqslant \int_A C\Psi(x)(1/((mA)\Psi(x))\,dx = C$ and hence $f \notin (C_{p,w})'$, which gives a contradiction.

Finally by Lemma 2.12 we have that $\hat{f} < \infty$ on I, and the proof is completed. $\square$

Lemma 2.22. *If $f \in L_0(I)$ is such that $\hat{f} \in (C_{p,w})'$ then* $\lim_{x\to l} \hat{f}(x) = 0$.

Proof. By Corollary 2.7, $\hat{f}$ is decreasing and hence $\lim_{x\to l} \hat{f}(x)$ exists. Suppose that $\lim_{x\to l} \hat{f}(x) = C$ for some $C > 0$. It follows that $\hat{f}(t) \geqslant C$ on I. By Lemma 2.20(b) there is a sequence of functions $g_n \in B_{C_{p,w}}$ such that $\|g_n\|_1 \to \infty$ as $n \to \infty$. Therefore $\int_I \hat{f}(t) g_n(t)\, dt \geqslant C \int_I g_n(t)\, dt \to \infty$, and so $\hat{f} \notin (C_{p,w})'$. □

Lemma 2.23. *If $f \in L_0(I)$ is such that $\hat{\mathcal{H}}_w f \in L_q(I)$ then* $\lim_{x\to 0^+} D^+_\Psi \hat{f}(x) = 0$.

Proof. By Proposition 2.3 function $D^+_\Psi \hat{f}$ is increasing and so $\lim_{x\to 0^+} D^+_\Psi \hat{f}(x)$ exists. Moreover, (iii) and Lemma 2.6 imply that this limit is nonnegative. Suppose that $\lim_{x\to 0^+} D^+_\Psi \hat{f}(x) = C$ for some constant $C > 0$. Since $D^+_\Psi \hat{f}(x) = -\hat{f}'(x)/w(x)^p$ a.e., $-\hat{f}'(x)/w(x)^p \geqslant C > 0$ a.e. It follows $(-\hat{f}'(x)/w(x))^q \geqslant C^q w(x)^p$ a.e., so by (iii) the integral $\int_I (-\hat{f}'(x)/w(x))^q\, dx = \int_I (\hat{\mathcal{H}}_w f)^q(x)\, dx$ diverges, and this is a contradiction. □

Lemma 2.24. *Let $f \in L_0(I)$ with $\hat{f} < \infty$ on I be such that* $\lim_{x\to l} \hat{f}(x) = 0$. *Then $\int_I f(t) g(t)\, dt \leqslant \|\hat{\mathcal{H}}_w f\|_q \|g\|_{C_{p,w}}$ for any $g \in C_{p,w}$. Consequently $\|f\|_{(C_{p,w})'} \leqslant \|\hat{\mathcal{H}}_w f\|_q$.*

Proof. If $\|\hat{\mathcal{H}}_w f\|_q = \infty$ then the claim is clear. Assume that $\|\hat{\mathcal{H}}_w f\|_q < \infty$. In view of $\lim_{x\to l} \hat{f}(x) = 0$, using Fubini's Theorem and the Hölder inequality, for any $g \in C_{p,w}$ we have the following

$$\begin{aligned}
\int_0^l f(t) g(t)\, dt \leqslant \int_0^l |f(t)||g(t)|\, dt \leqslant \int_0^l \hat{f}(t) |g(t)|\, dt \\
&= \int_0^l \int_t^l -\hat{f}'(x)\, dx |g(t)|\, dt = \int_0^l \frac{-\hat{f}'(x)}{w(x)} w(x) \int_0^x |g(t)|\, dt\, dx \\
&\leqslant \|\hat{\mathcal{H}}_w f\|_q \, \|\mathcal{H}_w g\|_p = \|\hat{\mathcal{H}}_w f\|_q \|g\|_{C_{p,w}}.
\end{aligned}$$

From the above it follows that $f \in (C_{p,w})'$ and $\|f\|_{(C_{p,w})'} \leqslant \|\hat{\mathcal{H}}_w f\|_q$. □

Theorem 2.25. *If $f \in L_0(I)$ is such that $\|\hat{\mathcal{H}}_w f\|_q < \infty$ and $\lim_{x\to l} \hat{f}(x) = 0$ then $f \in (C_{p,w})'$ and $\|f\|_{(C_{p,w})'} = \|\hat{\mathcal{H}}_w f\|_q$.*

Proof. Without loss of generality we assume that $\|\hat{\mathcal{H}}_w f\|_q = 1$. By Lemma 2.24 we have that $\int_I f(t)g(t)\,dt \leqslant \|\hat{\mathcal{H}}_w f\|_q \|g\|_{C_{p,w}}$, whence $f \in (C_{p,w})'$ and $\|f\|_{(C_{p,w})'} \leqslant 1$.

The assumption $\|\hat{\mathcal{H}}_w f\|_q = (\int_I (-\hat{f}'(x)/w(x))^q\,dx)^{1/q} < \infty$ gives that $\hat{f} < \infty$ on I. Let $h = (D_\Psi^+ \hat{f})^{q/p}$. Function h is increasing, finite and right-continuous on I and $\lim_{x\to 0^+} h(x) = 0$ as shown in Proposition 2.3 and Lemma 2.23. Note that $h = (-\hat{f}'/w^p)^{q/p}$ a.e. on I.

Fix $\epsilon \in (0,1)$. Our main goal is to define a function g such that $\|g\|_{C_{p,w}} \leqslant 1+\epsilon$ and $\int_I f(t)g(t)\,dt > 1 - 2\epsilon$. The construction of g will involve a special set $A \subset I$ and a carefully chosen subdivision of I. First we find A and then a finite sequence $(a_n) \subset I$ which divides I.

Given $y \in I$ such that $h(y) > 0$ let

$$A_y = \{x \in I : \hat{f}(x) \leqslant |f(x)| + \epsilon/4h(y)\}.$$

Suppose $m(A_y \cap (0,y)) = 0$ for all $y > y_0$ and some $y_0 \in I$. In such case, Lemma 2.16 implies that $\hat{f}$ is Ψ-affine on each interval $(0,y)$, $y \in I$, and hence $\hat{f}$ is Ψ-affine on I. By Lemmas 2.9 and 2.23, $D_\Psi^+ \hat{f}$ is constant on I and $\lim_{x\to 0^-} D_\Psi^+ \hat{f}(x) = 0$. Hence $D_\Psi^+ \hat{f} = 0$ on I and so $\hat{f} = 0$ by $\lim_{x\to l} \hat{f}(x) = 0$, which gives a contradiction with the condition $\|\hat{\mathcal{H}}_w f\|_q = 1$. This shows that for all $y \in I$ there is $b \in (y,l)$ such that $m(A_b \cap (0,b)) > 0$. Since $L_q(I)$ is order continuous we choose $b \in I$ such that

$$\|(\hat{\mathcal{H}}_w f)\chi_{(b,l)}\|_q^q \leqslant \epsilon^p/2, \text{ and } m(A \cap (0,b)) > 0, \text{ where } A = A_b,$$

and b is a point of continuity of h.

Observe now, that if there is $x \in I$ such that $m(A \cap (0,x)) = 0$ then, by Lemma 2.16, $\hat{f}$ is Ψ-affine on $(0,x)$ and so $h = 0$ on $(0,x)$. Suppose now that $m(A \cap (0,x)) > 0$ for all $x \in I$. Then for all $y \in I$, there exists $a < y$ such that $m(A \cap (a,x)) > 0$ whenever $x > a$. Indeed, otherwise there exists $y \in I$ such that for all $a < y$, $m(A \cap (a,x)) = 0$ for $x > a$ close enough to a, say for $x < x_a$. Now the family of sets (a, x_a), $a \in (0,y)$, covers $[\eta, y - \eta]$ for any $\eta > 0$. From this, using compactness, one can infer that $m(A \cap (0,y)) = 0$, which gives a contradiction.

Hence we fix $a \in I$, without loss of generality a point of continuity of h, such that $\|(\hat{\mathcal{H}}_w f)\chi_{(0,a)}\|_q^q \leqslant \epsilon^p/2$ and either

$$m(A \cap (a,x)) > 0 \text{ for all } x > a, \text{ or} \tag{2.9}$$

$$h(a) = 0 \text{ and } m(A \cap (a,c)) = 0 \text{ for some } c > a. \tag{2.10}$$

It follows that $b > a$, $m(A \cap (a,b)) > 0$ and

$$\|(\hat{\mathcal{H}}_w f)\chi_{(0,a)\cup(b,l)}\|_q^q \leqslant \epsilon^p \tag{2.11}$$

Let $\gamma = \Psi(a)^{1/p}$ and y_i be points of discontinuity of $D_\Psi^+ \hat{f}$ (and hence of h) in (a,b) such that $h(y_i^+) - h(y_i^-) \geqslant \epsilon/4\gamma$. Here $h(y_i^+) = \lim_{x \to y_i^+} h(x)$ and $h(y_i^-) = \lim_{x \to y_i^-} h(x)$. Clearly there is only finite number of them, say $a < y_1 < y_2 < \ldots < y_M < b$. Since $\hat{f}$ is continuous on I for each y_i, $i = 1, 2, \ldots, M$, we can find two points of continuity of h, $\underline{y}_i, \overline{y}_i \in (a,b)$ such that $\underline{y}_i < y_i < \overline{y}_i$, the intervals $[\underline{y}_i, \overline{y}_i]$ are pairwise disjoint,

$$\int_{\underline{y}_i}^{\overline{y}_i} w(x)^p \, dx \leqslant \frac{\epsilon^p}{2^p M (h(y_i^+) - h(y_i^-))^p}, \tag{2.12}$$

$$\hat{f}(\underline{y}_i) - \hat{f}(\overline{y}_i) \leqslant \frac{\epsilon}{4h(b)}, \tag{2.13}$$

$$h(x) - h(y_i^+) \leqslant \frac{\epsilon}{4\gamma} \quad \text{for } x \in (y_i, \overline{y}_i), \quad \text{and} \tag{2.14}$$

$$h(y_i^-) - h(x) \leqslant \frac{\epsilon}{4\gamma} \quad \text{for } x \in (\underline{y}_i, y_i). \tag{2.15}$$

By Lemmas 2.9 and 2.16, $m(A \cap (\underline{y}_i, \overline{y}_i)) > 0$ for all $i = 1, 2, \ldots, M$. Condition (2.22) implies that for all $i = 1, 2, \ldots, M$,

$$\frac{1}{m(A \cap (\underline{y}_i, \overline{y}_i))} \int_{A \cap (\underline{y}_i, \overline{y}_i)} \hat{f}(t)\, dt \geqslant \hat{f}(\underline{y}_i) - \frac{\epsilon}{4h(b)}. \tag{2.16}$$

Now, the set $(a, b) \backslash \cup_{i=1}^{M} [\underline{y}_i, \overline{y}_i]$ is a union of finite number of open disjoint intervals, say $\cup_j (\underline{v}_j, \overline{v}_j)$. Each such interval $(\underline{v}_j, \overline{v}_j)$ can be divided using finite number of points of continuity of h, say u_k, into subintervals (u_k, u_{k+1}) in such a way that the family $(u_k, u_{k+1})_k$ is a partition of $(\underline{v}_j, \overline{v}_j)$, and

$$h(u_{k+1}) - h(u_k) \leqslant \frac{\epsilon}{4\gamma}, \tag{2.17}$$

$$\hat{f}(u_k) - \hat{f}(u_{k+1}) \leqslant \frac{\epsilon}{4h(b)}. \tag{2.18}$$

If $m(A \cap (u_k, u_{k+1})) > 0$ then by (2.27),

$$\frac{1}{m(A \cap (u_k, u_{k+1}))} \int_{A \cap (u_k, u_{k+1})} \hat{f}(t)\, dt \geqslant \hat{f}(u_k) - \frac{\epsilon}{4h(b)}. \tag{2.19}$$

Let $(a_n)_{n=0}^{N+1}$ be a strictly increasing sequence consisting of all points $u_k \in \cup_j (\underline{v}_j, \overline{v}_j)$ obtained above, points a, b and $\underline{y}_i$, $\overline{y}_i$, $i = 1, 2, \ldots, M$. Note that $a_0 = a$, $a_{N+1} = b$ and each interval (a_n, a_{n+1}) contains at most one of the points y_i, $i = 1, 2, \ldots, M$. Note that the union of all sets $(a_n, a_{n+1}]$ is $(a, b]$.

Denote $A_n = A \cap (a_n, a_{n+1})$, $n = 0, 1, \dots, N$. Let $E = \{n \in \{0, 1, \dots, N\} : mA_n > 0\}$. Clearly $E \neq \emptyset$. We can write $E = \{n_1, n_2, \dots, n_k\}$ where $0 \leqslant n_1 < n_2 < \dots < n_k \leqslant N$. Note that, if $n \notin E$ then by Lemmas 2.16 and 2.9, $h(a_{n+1}) - h(a_n) = 0$ since h is constant on (a_n, a_{n+1}) and continuous at each point a_i, $i = 0, 1, \dots, N+1$.

Let $\kappa = 0$ if $n_1 > 0$, i.e. if $mA_0 = 0$ (which is possible only when (2.10) holds true), $\kappa = h(a_0)/mA_0$ if $n_1 = 0$, i.e. if $mA_0 > 0$ (which is always a case when (2.9) holds true). Note that $\kappa mA_0 = h(a_0)$. Define function

$$g = \left(\sum_{i=1}^{k} \frac{h(a_{n_i+1}) - h(a_{n_i})}{mA_{n_i}} \chi_{A_{n_i}} + \kappa \chi_{A_0} \right) \operatorname{sign} f.$$

Now we show that

$$\|g\|_{C_{p,w}} \leqslant 1 + \epsilon.$$

It is clear that $\int_0^x |g(t)|\, dt = 0$ if $x < a_0$. Since h is increasing we get that $\int_0^x |g(t)|\, dt \leqslant h(a_{N+1})$ if $x \geqslant a_{N+1}$ and $\int_0^x |g(t)|\, dt \leqslant h(a_{n+1})$ if $a_n \leqslant x < a_{n+1}$, $n = 0, 1, \dots, N$. If $x \in (a_n, a_{n+1})$ and (a_n, a_{n+1}) does not contain any of points y_i, $i = 1, 2, \dots, M$, then $h(a_{n+1}) - h(x) \leqslant \epsilon/4\gamma$ by (2.26). Similarly, if $x \in (a_n, a_{n+1})$ and (a_n, a_{n+1}) contains point y_i for some $i = 1, 2, \dots, M$, then $h(a_{n+1}) - h(x) \leqslant \epsilon/2\gamma + (h(y_i^+) - h(y_i^-))$ by (2.23) and (2.24). It follows that for $x \in I$,

$$\begin{aligned} \mathcal{H}_w g(x) &= w(x) \int_0^x |g(t)|\, dt \\ &\leqslant w(x)h(x) + \frac{\epsilon}{2\gamma} w(x)\chi_{[a,l)}(x) + w(x) \sum_{i=1}^{M} (h(y_i^+) - h(y_i^-))\chi_{[\underline{y}_i, \overline{y}_i]}(x). \end{aligned}$$

By the triangle inequality, definition of γ and (2.21) we get that

$$\left\| \frac{\epsilon}{2\gamma} w\chi_{(a,l)} + w \sum_{i=1}^{M} (h(y_i^+) - h(y_i^-))\chi_{[\underline{y}_i, \overline{y}_i]} \right\|_p \leqslant \epsilon.$$

Moreover

$$\|wh\|_p = \left(\int_I w(x)^p (-\hat{f}'(x)/w(x)^p)^q \, dx\right)^{1/p} = \left(\|\hat{\mathcal{H}}_w f\|_q^q\right)^{1/p} = 1$$

and hence

$$\|\mathcal{H}_w g\|_p \leqslant \|wh\|_p + \left\|\frac{\epsilon}{2\gamma} w\chi_{(a,l)} + w\sum_{i=1}^{M}(h(y_i^+) - h(y_i^-))\chi_{[\underline{y}_i, \overline{y}_i]}\right\|_p \leqslant 1+\epsilon.$$

Since

$$\int_A |g(t)|\epsilon/4h(b)\, dt \leqslant \epsilon \int_0^b |g(t)|\, dt/4h(b) \leqslant \epsilon/4,$$

by definition of A we have that

$$\begin{aligned}\int_I f(t)g(t)\, dt &= \int_I |f(t)| g(t) \operatorname{sign} f(t)\, dt \\ &\geqslant \int_A (\hat{f}(t) - \epsilon/4h(b)) g(t) \operatorname{sign} f(t)\, dt \\ &\geqslant \int_A \hat{f}(t) g(t) \operatorname{sign} f(t)\, dt - \epsilon/4.\end{aligned}$$

Now by definition of g, (2.25) and (2.28),

$$\begin{aligned}\int_A \hat{f}(t)g(t)\operatorname{sign} f(t)\, dt &= \sum_{i=1}^{k} \frac{h(a_{n_i+1}) - h(a_{n_i})}{mA_{n_i}} \int_{A_{n_i}} \hat{f}(t)\, dt + \kappa \int_{A_0} \hat{f}(t)\, dt \\ &\geqslant \sum_{i=1}^{k} (h(a_{n_i+1}) - h(a_{n_i}))(\hat{f}(a_{n_i}) - \epsilon/4h(b)) \\ &+ h(a_0)(\hat{f}(a_0) - \epsilon/4h(b)).\end{aligned}$$

It is easy to see that

$$\left(\sum_{i=1}^{k}(h(a_{n_i+1}) - (a_{n_i})) + h(a_0)\right)\epsilon/4h(b) \leqslant \epsilon/4.$$

Since $h(a_0) = 0$ if $n_1 > 0$, $h(a_{n+1}) - h(a_n) = 0$ if $n \notin E$ and by (2.11),

$$\sum_{i=1}^{k}(h(a_{n_i+1}) - h(a_{n_i}))\hat{f}(a_{n_i}) + h(a_0)\hat{f}(a_0) = \sum_{n=0}^{N}(h(a_{n+1}) - h(a_n))\hat{f}(a_n) + h(a_0)\hat{f}(a_0)$$

$$= h(a_{N+1})\hat{f}(a_N) + \sum_{n=1}^{N} h(a_n)(\hat{f}(a_{n-1}) - \hat{f}(a_n))$$

$$= \sum_{n=1}^{N} h(a_n)\int_{a_{n-1}}^{a_n}(-\hat{f}'(t))\,dt + h(a_{N+1})\int_{a_N}^{l}(-\hat{f}'(t))\,dt$$

$$\geqslant \sum_{n=1}^{N}\int_{a_{n-1}}^{a_n} h(t)(-\hat{f}'(t))\,dt + \int_{a_N}^{a_{N+1}} h(t)(-\hat{f}'(t))\,dt \geqslant \int_a^b \left(\frac{-\hat{f}'(t)}{w(t)}\right)^q dt \geqslant 1 - \epsilon.$$

Combining all the above together we obtain $\int_I f(t)g(t)\,dt \geqslant 1 - 3\epsilon/2$. Dividing both sides by $1 + \epsilon$ one gets

$$\int_I f(t)g(t)/(1+\epsilon)\,dt \geqslant 1 - 3\epsilon.$$

Finally, by $\|g/(1+\epsilon)\|_{C_{p,w}} \leqslant 1$ it follows that $\|f\|_{(C_{p,w})'} = 1$. □

Lemma 2.26. *If $f \in (C_{p,w})'$ then $\hat{f} \in (C_{p,w})'$ and $\|f\|_{(C_{p,w})'} = \|\hat{f}\|_{(C_{p,w})'} = \|\hat{\mathcal{H}}_w f\|_q$.*

Proof. Consider first the case when $l < \infty$. Let $f \in (C_{p,w})'$ and $f_m = f\chi_{[1/m, l-1/m]}$, $m \in \mathbb{N}$. By Lemma 2.21, $\hat{f} < \infty$ on I. Clearly $\widehat{f_m} \leqslant \hat{f}$. Letting $y \in I$, by definition of $\hat{f}$, for every $\epsilon > 0$ there exist $n \in \mathbb{N}$ and a set $A = \{(y_1, \ldots, y_n) \in I^n : \sum_{i=1}^n \alpha_i f(y_i) > \hat{f}(y) - \epsilon, \sum_{i=1}^n \alpha_i \Psi(y_i) = \Psi(y), \sum_{i=1}^n \alpha_i = 1, \alpha_i \geqslant 0, i = 1, 2, \ldots, n\}$ with $m^{(n)}A > 0$. Let $r > 0$ be such that $1/r < y < l - 1/r$ and $m^{(n)}(A \cap (1/r, l-1/r)^n) > 0$. Since for all $m > r$, $f = f_m$ on $(1/r, l-1/r)$ it follows that $\widehat{f_m}(y) > \hat{f}(y) - \epsilon$. By arbitrariness of ϵ we get that $\widehat{f_m}(y) \to \hat{f}(y)$ as $m \to \infty$. By Lemma 2.18, $\widehat{f_m}' \to \hat{f}'$ a.e. on I.

Note that $D_\Psi^+ \widehat{f_m}(x) = -\widehat{f_m}'(x)/w(x)^p$ is 0 a.e. on $(0, 1/m)$ and constant a.e. on $(l - 1/m, l)$. Hence the function $-\widehat{f_m}'(x)/w(x) = w(x)^{p-1} D_\Psi^+ \widehat{f_m}(x)$ a.e. is in $L_q(I)$, $m \in \mathbb{N}$. By Theorem 2.25 we get that $\|f_m\|_{(C_{p,w})'} = \|\hat{\mathcal{H}}_w f_m\|_q$ for all $m \in \mathbb{N}$. Now by

Lemma 2.24, Lemma 2.18, the Fatou Lemma and the Fatou property of $(C_{p,w})'$ we get that

$$\begin{aligned}\|\hat{f}\|_{(C_{p,w})'} &\leqslant \|\hat{\mathcal{H}}_w f\|_q = \left(\int_I (-\hat{f}'(x)/w(x))^q\,dx\right)^{1/q} \\ &= \left(\int_I (-\lim_n \widehat{f_n}'(x)/w(x))^q\,dx\right)^{1/q} \leqslant \liminf_n \left(\int_I (-\widehat{f_n}'(x)/w(x))^q\,dx\right)^{1/q} \\ &\leqslant \sup_n \|\hat{\mathcal{H}}_w f_n\|_q = \sup_n \|f_n\|_{(C_{p,w})'} = \|f\|_{(C_{p,w})'} < \infty.\end{aligned}$$

So $\|f\|_{(C_{p,w})'} = \|\hat{f}\|_{(C_{p,w})'}$, since $|f| \leqslant \hat{f}$. The above inequality also shows that $\|f\|_{(C_{p,w})'} = \|\hat{\mathcal{H}}_w f\|_q$.

In case when $l = \infty$ we proceed similarly as above taking $f_m = f\chi_{[1/m,m]}$, $m \in \mathbb{N}$. $\square$

Now we are ready to present the main result in this section, isometric description of the dual space $(C_{p,w})^*$. Namely, by Lemma 2.22, Theorem 2.25 and Lemma 2.26 we get the following theorem.

Theorem 2.27. *Let $1 < p < \infty$, $q = \frac{p}{p-1}$, $\Psi(x) = \int_x^l w(t)^p\,dt$, $x \in I = (0,l)$, $0 < l \leqslant \infty$. Then a function $f \in (C_{p,w})'$ if and only if $\hat{f} < \infty$ on I, $\lim_{x\to l} \hat{f}(x) = 0$ and $\|\hat{\mathcal{H}}_w f\|_q < \infty$. Moreover*

$$\|f\|_{(C_{p,w})'} = \|\hat{\mathcal{H}}_w f\|_q \text{ for all } f \in (C_{p,w})'.$$

The Banach dual space $(C_{p,w})^*$ of $C_{p,w}$ is isometrically isomorphic to $(C_{p,w})'$ in the sense that every $F \in (C_{p,w})^*$ is of the form

$$F(g) = \int_I f(t)g(t)\,dt, \quad g \in C_{p,w},$$

for a unique $f \in (C_{p,w})'$ and $\|F\|_{(C_{p,w})^*} = \|f\|_{(C_{p,w})'}$.

2.4 Diameter of slices of the unit ball

Let $(X, \|\cdot\|)$ be a Banach space. Recall that the set $s(x^*;\eta) = \{x \in B_X : x^*x > 1-\eta\}$, where $x^* \in S_{X^*}$, $0 < \eta < 1$, is called a *slice* of B_X. Applying the techniques described in the previous section, we show the following result.

Theorem 2.28. *Let $f_j \in S_{(C_{p,w})'}$, $j = 1, 2, \ldots, r$, $r \in \mathbb{N}$ be such that $\widehat{f_j} = \widehat{f_i}$, $i, j = 1, 2, \ldots, r$. The diameter of a finite convex combination of slices defined by f_j, $j = 1, 2, \ldots, r$, is 2.*

Proof. Let $f = \widehat{f_j}$, $j \in \{1, 2, \ldots, r\}$ and $F_j(g) = \int_I f_j(t)g(t)\,dt$, $g \in C_{p,w}$ be bounded linear functionals on $C_{p,w}$ defined by $f_j \in S_{(C_{p,w})'}$, $j = 1, 2, \ldots, r$, $r \in \mathbb{N}$. Let $S = \sum_{j=1}^r \alpha_j s(F_j;\eta_j)$, $\alpha_j \geqslant 0$, $0 < \eta_j < 1$, $j = 1, 2, \ldots, r$, $\sum_{j=1}^r \alpha_j = 1$ be a convex combination of slices. Let $0 < \epsilon < \min\{\eta_j : j = 1, 2, \ldots, r\}/10$ be arbitrary.

Denote $h = (D_\Psi^+ \hat{f})^{q/p}$. Note that $h = (-\hat{f}'/w^p)^{q/p}$ a.e. on I. Function h is increasing, finite and right-continuous on I and $\lim_{x\to 0^+} h(x) = 0$ as shown in Proposition 2.3 and Lemma 2.23. Moreover h is not constant on I, in particular h is not identically 0 on I. Indeed, if h is constant on I, that is $D_\Psi^+\hat{f}$ is constant on I, then by 2.23, $\lim_{x\to 0^+} D_\Psi^+\hat{f}(x) = 0$. Now, Lemma 2.9 implies that $D_\Psi^+\hat{f} = 0$ on I and so $\hat{f}$ is Ψ-affine on I by Lemma 2.9. We have that $\hat{f} = A\Psi + B$ on I for some constants A and B. Since $D_\Psi^+\hat{f} = 0$ and in view of Lemma 2.22 both $A = B = 0$. Hence $\hat{f} = 0$ on I, which gives a contradiction with the condition $\|f\|_{(C_{p,w})'} = 1$.

Given $y \in I$ such that $h(y) > 0$ let

$$A_y^{(j)} = \{x \in I : |f_j(x)| \geqslant f(x) - \epsilon/4h(y)\},\ j = 1, 2, \ldots, r.$$

First, observe that for each $y \in I$ there is $b \in (y, l)$ such that for all $j \in \{1, 2, \ldots, r\}$ $m(A_b^{(j)} \cap (0, b)) > 0$. Indeed, if there exists $y_0 \in I$ such that for all $b \in (y_0, l)$ there is

$j \in \{1, 2, \ldots, r\}$ for which $m(A_b^{(j)} \cap (0, b)) = 0$ then by Lemma 2.16 $\hat{f}$ is Ψ-affine on $(0, b)$ and hence by Lemma 2.9 $D_\Psi^+ \hat{f}$ is constant on $(0, b)$. The latter and the condition $\lim_{x \to 0^+} D_\Psi^+ \hat{f} = 0$ imply that $h = 0$ on $(0, b)$ for all $b \in (y_0, l)$ and hence $h = 0$ on I, which gives a contradiction.

By the above and by the order continuity of L_q we fix $b \in I$ such that $m(A_b^{(j)} \cap (0, b)) > 0$ for all $j \in \{1, 2, \ldots, r\}$ and $\|(\hat{\mathcal{H}}_w f)\chi_{(b,l)}\|_q^q \leqslant \epsilon^p/2$. Moreover, without loss of generality, we assume that b is a point of continuity of h and such that h is not constant on $(0, b)$. Denote $A^{(j)} = A_b^{(j)}$, $j = 1, 2, \ldots, r$.

Now, we show that either $h = 0$ on some interval near 0 or $h > 0$ on I and for each $y \in I$ there is $a < y$ such that for all $j \in \{1, 2, \ldots, r\}$ and all $x > a$ $m(A^{(j)} \cap (a, x)) > 0$. Indeed, suppose that there exists $y_0 \in I$ such that for all $a \in (0, y_0)$ there are $j \in \{1, 2, \ldots, r\}$ and $x_a \in (a, l)$ for which $m(A^{(j)} \cap (a, x_a)) = 0$. By Lemma 2.16 h is constant on each interval (a, x_a). But the family of intervals (a, x_a), $a \in (0, y_0)$ covers $[\eta, y_0 - \eta]$, $\eta > 0$, whence, using compactness, we infer that h is constant on $(0, y_0)$. Since $\lim_{x \to 0^+} h(x) = 0$ by Lemma 2.23 we get that $h = 0$ on $(0, y_0)$. The claim follows.

By the above, we can find $a \in (0, b)$ such that

$$m(A^{(j)} \cap (a, b)) > 0, \ \|(\hat{\mathcal{H}}_w f)\chi_{(0,a)}\|_q^q \leqslant \epsilon^p/2 \text{ and}$$

$$\text{either } h(a) = 0 \text{ or } m(A^{(j)} \cap (a, x)) > 0 \text{ for all } j = 1, 2, \ldots, r \text{ and all } x > a. \tag{2.20}$$

It follows that $\|(\hat{\mathcal{H}}_w f)\chi_{(0,a)\cup(b,l)}\|_q^q \leqslant \epsilon^p$. Again, without loss of generality, we assume that a is a point of continuity of h and such that h is not constant on (a, b).

Let $\gamma = \Psi(a)^{1/p}$ and y_i be points of discontinuity of $D_\Psi^+ \hat{f}$ (and hence of h) in (a, b) such that $h(y_i^+) - h(y_i^-) \geqslant \epsilon/4\gamma$. Here $h(y_i^+) = \lim_{x \to y_i^+} h(x)$ and $h(y_i^-) =$

$\lim_{x\to y_i^-} h(x)$. Clearly there is only finite number of them, say $a < y_1 < y_2 < \ldots < y_M < b$. Since $\hat{f}$ is continuous on I for each y_i, $i = 1, 2, \ldots, M$, we can find two points of continuity of h, $\underline{y}_i$, $\overline{y}_i \in (a, b)$ such that $\underline{y}_i < y_i < \overline{y}_i$, the intervals $[\underline{y}_i, \overline{y}_i]$ are pairwise disjoint,

$$\int_{\underline{y}_i}^{\overline{y}_i} w(x)^p \, dx \leqslant \frac{\epsilon^p}{2^p M (h(y_i^+) - h(y_i^-))^p}, \tag{2.21}$$

$$\hat{f}(\underline{y}_i) - \hat{f}(\overline{y}_i) \leqslant \frac{\epsilon}{4h(b)}, \tag{2.22}$$

$$h(x) - h(y_i^+) \leqslant \frac{\epsilon}{4\gamma} \quad \text{for } x \in (y_i, \overline{y}_i), \quad \text{and} \tag{2.23}$$

$$h(y_i^-) - h(x) \leqslant \frac{\epsilon}{4\gamma} \quad \text{for } x \in (\underline{y}_i, y_i). \tag{2.24}$$

By Lemmas 2.9 and 2.16, $m(A^{(j)} \cap (\underline{y}_i, \overline{y}_i)) > 0$ for all $i = 1, 2, \ldots, M$, $j = 1, 2, \ldots, r$. Condition (2.22) implies that for all $i = 1, 2, \ldots, M$ and $j = 1, 2, \ldots, r$

$$\frac{1}{m(A^{(j)} \cap (\underline{y}_i, \overline{y}_i))} \int_{A^{(j)} \cap (\underline{y}_i, \overline{y}_i)} \hat{f}(t) \, dt \geqslant \hat{f}(\underline{y}_i) - \frac{\epsilon}{4h(b)}. \tag{2.25}$$

Now, the set $(a, b) \backslash \cup_{i=1}^{M} [\underline{y}_i, \overline{y}_i]$ is a union of finite number of open disjoint intervals, say $\cup_j (\underline{v}_j, \overline{v}_j)$. Each such interval $(\underline{v}_j, \overline{v}_j)$ can be divided using finite number of points of continuity of h, say u_k, into subintervals (u_k, u_{k+1}) in such a way that the family $(u_k, u_{k+1})_k$ is a partition of $(\underline{v}_j, \overline{v}_j)$, and

$$h(u_{k+1}) - h(u_k) \leqslant \frac{\epsilon}{4\gamma}, \tag{2.26}$$

$$\hat{f}(u_k) - \hat{f}(u_{k+1}) \leqslant \frac{\epsilon}{4h(b)}. \tag{2.27}$$

If $m(A^{(j)} \cap (u_k, u_{k+1})) > 0$, $j \in \{1, 2, \ldots, r\}$ then by (2.27),

$$\frac{1}{m(A^{(j)} \cap (u_k, u_{k+1}))} \int_{A^{(j)} \cap (u_k, u_{k+1})} \hat{f}(t) \, dt \geqslant \hat{f}(u_k) - \frac{\epsilon}{4h(b)}. \tag{2.28}$$

Let $(a_n)_{n=0}^{N+1}$ be a strictly increasing sequence consisting of all points $u_k \in \cup_j(\underline{v}_j, \overline{v}_j)$ obtained above, points a, b and $\underline{y}_i$, $\overline{y}_i$, $i = 1, 2, \ldots, M$. Note that $a_0 = a$, $a_{N+1} = b$ and each interval (a_n, a_{n+1}) contains at most one of the points y_i, $i = 1, 2, \ldots, M$. Moreover $\cup_{n=0}^{N}(a_n, a_{n+1}] = (a, b]$.

Denote

$$A_n^{(j)} = A^{(j)} \cap (a_n, a_{n+1}),\ n = 0, 1, \ldots, N,\ j = 1, 2, \ldots, r.$$

Let

$$E^{(j)} = \{n \in \{0, 1, \ldots, N\} : mA_n^{(j)} > 0\}, j = 1, 2, \ldots, r.$$

Clearly $E^{(j)} \neq \emptyset$, $j = 1, 2, \ldots, r$. Let $E = \cap_{j=1}^{r} E^{(j)}$. We have that $E \neq \emptyset$. Indeed, if $E = \emptyset$ then for each $n \in \{0, 1, \ldots, N\}$ there is $j \in \{1, 2, \ldots, r\}$ such that $m(A^{(j)} \cap (a_n, a_{n+1}) = 0$ and hence by Lemma 2.16 $\hat{f} = \hat{f}_j$ is Ψ-affine on each interval (a_n, a_{n+1}), $n \in \{0, 1, \ldots, N\}$. By the latter and Lemma 2.9 we would have that h is constant on (a, b) which is not the case.

We can write $E = \{n_1, n_2, \ldots, n_k\}$ where $0 \leqslant n_1 < n_2 < \ldots < n_k \leqslant N$. Note that, if $n \notin E$ then $n \notin E^{(j)}$ for some j and hence by Lemmas 2.16 and 2.9, $h(a_{n+1}) - h(a_n) = 0$ since h is constant on (a_n, a_{n+1}) and continuous at each point a_i, $i = 0, 1, \ldots, N+1$.

Now, for each $n \in E$ let sets $B_n^{(j)}, C_n^{(j)} \subset A_n^{(j)}$ be such that $m(B_n^{(j)}) > 0$, $m(C_n^{(j)}) > 0$, for $j = 1, 2, \ldots, r$, $\cup_{j=1}^{r} A_n^{(j)} = \cup_{j=1}^{r}(B_n^{(j)} \cup C_n^{(j)})$ and all sets $B_n^{(i)}, C_n^{(j)}$ are pairwise disjoint $i, j = 1, 2, \ldots, r$.

Observe that if $0 \notin E$ then by (2.20), $h(a_0) = 0$ and in this case we interpret expressions $\frac{h(a_0)}{mB_0^{(j)}}\chi_{B_0^{(j)}}$ and $\frac{h(a_0)}{mC_0^{(j)}}\chi_{C_0^{(j)}}$ as 0 (also when sets $B_0^{(j)}$, $C_0^{(j)}$ are not formally defined).

For $j = 1, 2, \dots, r$ define functions

$$g_{1,j} = \left(\sum_{i=1}^{k} \frac{h(a_{n_i+1}) - h(a_{n_i})}{mB_{n_i}^{(j)}} \chi_{B_{n_i}^{(j)}} + \frac{h(a_0)}{mB_0^{(j)}} \chi_{B_0^{(j)}} \right) \operatorname{sign} f_j$$

and

$$g_{2,j} = \left(\sum_{i=1}^{k} \frac{h(a_{n_i+1}) - h(a_{n_i})}{mC_{n_i}^{(j)}} \chi_{C_{n_i}^{(j)}} + \frac{h(a_0)}{mC_0^{(j)}} \chi_{C_0^{(j)}} \right) \operatorname{sign} f_j.$$

Now, we show that

$$\|g_{s,j}\|_{C_{p,w}} \leqslant 1 + \epsilon \text{ for } s = 1, 2 \text{ and } j = 1, 2, \dots, r.$$

It is clear that $\int_0^x |g_{s,j}(t)|\, dt = 0$ if $x < a_0$. Since h is increasing we get that $\int_0^x |g_{s,j}(t)|\, dt \leqslant h(a_{N+1})$ if $x \geqslant a_{N+1}$ and $\int_0^x |g_{s,j}(t)|\, dt \leqslant h(a_{n+1})$ if $a_n \leqslant x < a_{n+1}$, $n = 0, 1, \dots, N$. For a fixed $n \in \{0, 1, \dots, N\}$, if $x \in (a_n, a_{n+1})$ and (a_n, a_{n+1}) does not contain any of points y_i, $i = 1, 2, \dots, M$, then $h(a_{n+1}) - h(x) \leqslant \epsilon/4\gamma$ by (2.26). Similarly, if $x \in (a_n, a_{n+1})$ and (a_n, a_{n+1}) contains point y_i for some $i = 1, 2, \dots, M$, then $h(a_{n+1}) - h(x) \leqslant \epsilon/2\gamma + (h(y_i^+) - h(y_i^-))$ by (2.23) and (2.24). It follows that for $x \in I$,

$$\begin{aligned} \mathcal{H}_w g_{s,j}(x) &= w(x) \int_0^x |g_{s,j}(t)|\, dt \\ &\leqslant w(x)h(x) + \frac{\epsilon}{2\gamma} w(x) \chi_{[a,l)}(x) + w(x) \sum_{i=1}^{M} (h(y_i^+) - h(y_i^-)) \chi_{[\underline{y}_i, \overline{y}_i]}(x). \end{aligned}$$

By the triangle inequality, definition of γ and (2.21) we get that

$$\left\| \frac{\epsilon}{2\gamma} w \chi_{(a,l)} + w \sum_{i=1}^{M} (h(y_i^+) - h(y_i^-)) \chi_{[\underline{y}_i, \overline{y}_i]} \right\|_p \leqslant \epsilon. \tag{2.29}$$

Moreover

$$\|wh\|_p = \left(\int_I w(x)^p(-\hat{f}'(x)/w(x)^p)^q\,dx\right)^{1/p} = \left(\|\hat{\mathcal{H}}_w f\|_q^q\right)^{1/p} = 1$$

and hence

$$\|\mathcal{H}_w g_{s,j}\|_p \leqslant \|wh\|_p + \left\|\frac{\epsilon}{2\gamma} w\chi_{(a,l)} + w\sum_{i=1}^{M}(h(y_i^+) - h(y_i^-))\chi_{[\underline{y}_i,\overline{y}_i]}\right\|_p \leqslant 1+\epsilon.$$

Next we show that for $s = 1,2$, $g_{s,j} \in s(F_j;\eta_j)$, $j \in 1,2,\ldots,r$.

Denote $B^{(j)} = \cup_{i=1}^k B_{n_i}^{(j)}$ the support of $g_{1,j}$, $j = 1,2,\ldots,r$. Let $j \in \{1,2,\ldots,r\}$ be arbitrary. Since

$$\int_{B^{(j)}} |g_{1,j}(t)|\epsilon/4h(b)\,dt \leqslant \epsilon\int_0^b |g_{1,j}(t)|\,dt/4h(b) \leqslant \epsilon/4,$$

by definition of $B^{(j)}$ we have that

$$\begin{aligned}\int_I f_j(t)g_{1,j}(t)\,dt &= \int_I |f_j(t)|g_{1,j}(t)\operatorname{sign} f_j(t)\,dt\\ &\geqslant \int_{B^{(j)}}(f(t)-\epsilon/4h(b))g_{1,j}(t)\operatorname{sign} f_j(t)\,dt\\ &\geqslant \int_{B^{(j)}} f(t)g_{1,j}(t)\operatorname{sign} f_j(t)\,dt - \epsilon/4.\end{aligned}$$

Now by definition of $g_{1,j}$, (2.25) and (2.28),

$$\begin{aligned}\int_{B^{(j)}} f(t)g_{1,j}(t)\operatorname{sign} f_j(t)\,dt &= \sum_{i=1}^k \frac{h(a_{n_i+1})-h(a_{n_i})}{mB_{n_i}^{(j)}}\int_{B_{n_i}^{(j)}} f(t)\,dt + \int_I \frac{h(a_0)}{mB_0^{(j)}}\chi_{B_0^{(j)}} f(t)\,dt\\ &\geqslant \sum_{i=1}^k (h(a_{n_i+1})-h(a_{n_i}))(f(a_{n_i})-\epsilon/4h(b))\\ &\quad + h(a_0)(f(a_0)-\epsilon/4h(b)).\end{aligned}$$

It is easy to see that

$$\left(\sum_{i=1}^{k}(h(a_{n_i+1})-h(a_{n_i}))+h(a_0)\right)\epsilon/4h(b)\leqslant \epsilon/4.$$

Since $h(a_0)=0$ if $0\notin E$, $h(a_{n+1})-h(a_n)=0$ if $n\notin E$ and by (2.11),

$$\sum_{i=1}^{k}(h(a_{n_i+1})-h(a_{n_i}))f(a_{n_i})+h(a_0)f(a_0)=\sum_{n=0}^{N}(h(a_{n+1})-h(a_n))f(a_n)+h(a_0)f(a_0)$$
$$=h(a_{N+1})f(a_N)+\sum_{n=1}^{N}h(a_n)(f(a_{n-1})-f(a_n))$$
$$=\sum_{n=1}^{N}h(a_n)\int_{a_{n-1}}^{a_n}(-f'(t))\,dt+h(a_{N+1})\int_{a_N}^{l}(-f'(t))\,dt$$
$$\geqslant \sum_{n=1}^{N}\int_{a_{n-1}}^{a_n}h(t)(-f'(t))\,dt+\int_{a_N}^{a_{N+1}}h(t)(-f'(t))\,dt\geqslant\int_a^b\left(\frac{-f'(t)}{w(t)}\right)^q dt\geqslant 1-\epsilon.$$

Combining all the above together we obtain $\int_I f_j(t)g_{1,j}(t)\,dt\geqslant 1-3\epsilon/2$. Dividing both sides by $1+\epsilon$ one gets

$$\int_I f_j(t)g_{1,j}(t)/(1+\epsilon)\,dt\geqslant 1-3\epsilon\geqslant 1-\eta_j,\ j=1,2,\ldots,r.$$

Similarly one can show that $g_{2,j}\in s(F_j;\eta_j)$, $j=1,2,\ldots,r$.

Let $g_1=\sum_{j=1}^r\alpha_j g_{1,j}$ and $g_2=\sum_{j=1}^r\alpha_j g_{2,j}$ be convex combinations of functions $g_{1,j}$ and $g_{2,j}$, $j\in\{1,2,\ldots,r\}$, respectively, where $\alpha_j\geqslant 0$, $j=1,2,\ldots,r$ and $\sum_{j=1}^r\alpha_j=1$. Clearly $g_1,g_2\in S$. Observe that $\int_0^x|g_{s,j}(t)|\,dt=0$ if $x<a_0$, $s=1,2$, $j=1,2,\ldots,r$. Let $n\in\{0,1,\ldots,N\}$ be arbitrary. If $x\in(a_n,a_{n+1})$ and $y_i\notin(a_n,a_{n+1})$ for all $i=1,2,\ldots,M$, then $\int_0^x|g_{s,j}(t)|\,dt\geqslant h(a_n)\geqslant h(a_{n+1})-\epsilon/2\gamma$ by (2.26), $s=1,2$, $j=1,2,\ldots,r$. Similarly, if $x\in(a_n,a_{n+1})$ and $y_i\in(a_n,a_{n+1})$ for some $i=1,2,\ldots,M$, then $\int_0^x|g_{s,j}(t)|\,dt\geqslant h(a_n)\geqslant h(a_{n+1})-\epsilon/2\gamma-(h(y_i^+)-h(y_i^-))$ by (2.23) and (2.24),

$s=1,2,\ j=1,2,\dots,r$. It follows that for all $x\in(a,b)$, $s=1,2,\ j=1,2,\dots,r$,

$$\int_0^x |g_{s,j}(t)|\,dt \geqslant h(x)\chi_{(a,b)}(x)-\frac{\epsilon}{2\gamma}\chi_{(a,b)}(x)-\sum_{i=1}^{M}(h(y_i^+)-h(y_i^-))\chi_{(\underline{y}_i,\overline{y}_i)}(x). \quad (2.30)$$

Observe that

$$h(x)-\sum_{i=1}^{M}(h(y_i^+)-h(y_i^-))\chi_{(\underline{y}_i,\overline{y}_i)}(x)\geqslant 0 \text{ for all } x\in I.$$

Let $c=\sup\{x\in I: h(x)<\epsilon/2\gamma\}$. If $c>a$ then by definition of γ,

$$\|wh\chi_{(a,c)}\|_p^p \leqslant (\epsilon/2\gamma)^p\|w\chi_{(a,l)}\|_p^p\leqslant(\epsilon/2)^p. \quad (2.31)$$

Let $d=\max\{a,c\}$. Now, for $x\in(d,b)$ the right-hand side of 2.30 is non-negative.

Since function $g_{s,j}$, $s=1,2,\ j=1,2,\dots,r$ have disjoint supports, we get that for all $x\in(a,b)$,

$$\begin{aligned} w(x)\int_0^x |g_1(t)-g_2(t)|\,dt &= w(x)\sum_{j=1}^{r}\alpha_j\left(\int_0^x|g_{1,j}(t)|\,dt+\int_0^x|g_{2,j}(t)|\,dt\right)\\ &\geqslant 2w(x)h(x)\chi_{(a,b)}(x)-\frac{\epsilon}{\gamma}w(x)\chi_{(a,b)}(x)-2w(x)\sum_{i=1}^{M}(h(y_i^+)-h(y_i^-))\chi_{(\underline{y}_i,\overline{y}_i)}(x).\end{aligned}$$

It follows that

$$\begin{aligned}\|g_1-g_2\|_{C_{p,w}} &\geqslant \left\|2wh\chi_{(d,b)}-\left(\frac{\epsilon}{\gamma}w\chi_{(d,b)}+2w\sum_{i=1}^{M}(h(y_i^+)-h(y_I^-))\chi_{(\underline{y}_i,\overline{y}_i)}\right)\right\|_p\\ &\geqslant \left|2\|wh\chi_{(d,b)}\|_p-\left\|\frac{\epsilon}{\gamma}w\chi_{(d,b)}+2w\sum_{i=1}^{M}(h(y_i^+)-h(y_i^-))\chi_{(\underline{y}_i,\overline{y}_i)}\right\|_p\right|.\end{aligned}$$

Since $\|wh\|_p=1$, $\|wh\chi_{(0,a)\cup(b,l)}\|_p^p=\|(\hat{\mathcal{H}}_w f)\chi_{(0,a)\cup(b,l)}\|_q^q\leqslant\epsilon^p$ and by (2.31) we get that

$$
\begin{aligned}
\|wh\chi_{(d,b)}\|_p &= \|wh - (wh\chi_{(0,a)\cup(b,l)} + wh\chi_{(a,c)})\|_p \\
&\geqslant \big| \|wh\|_p - \|wh\chi_{(0,a)\cup(b,l)}\|_p - \|wh\chi_{(a,c)}\|_p \big| \\
&= 1 - \|(\hat{\mathcal{H}}_w f)\chi_{(0,a)\cup(b,l)}\|_q^{q/p} - \epsilon/2 \geqslant 1 - 3\epsilon/2.
\end{aligned}
$$

By the above and (2.29) we get that $\|g_1 - g_2\|_{C_{p,w}} \geqslant 2 - 4\epsilon$. Dividing now both sides by $1+\epsilon$ we obtain that $\|(g_1 - g_2)/(1+\epsilon)\|_{C_{p,w}} \geqslant 2 - 6\epsilon$. Since ϵ can be taken arbitrarily small we obtain that the diameter of S is 2. □

Corollary 2.29. *Every slice of $B_{C_{p,w}}$ has diameter 2.*

2.5 The Radon-Nikodym property

Recall that a Banach space $(X, \|\cdot\|)$ is called *locally uniformly convex* if for any $x \in S_X$ and any sequence $(x_n) \subset B_X$, $\lim_{n\to\infty} \|x + x_n\| = 2$ implies that $\lim_{n\to\infty} \|x - x_n\| = 0$. A point $x \in S_X$ is said to be *strongly exposed* if there is $x^* \in S_{X^*}$ such that $x^*x = 1$, $x^*y < 1$ for all $y \in B_X \setminus \{x\}$, and $x^*x_n \to 1$ implies that $\|x - x_n\| \to 0$ as $n \to \infty$ for any sequence $(x_n) \subset B_X$.

A point $x \in S_X$ is called a *denting point* of B_X if $x \notin \overline{co}\{B_X \setminus (x + \epsilon B_X)\}$ for each $\epsilon > 0$. It is easy to see that if the unit ball B_X has denting points then it has slices of arbitrary small diameter [12, Proposition 2.3.2, p. 28]. Also any strongly exposed point is a denting point [57, p. 227] and in locally uniformly convex space all points of its unit sphere are denting [27]. The Radon-Nikodym property can be characterized in terms of denting points. Namely, a Banach space X has the Radon-Nikodym property if and only if for every equivalent norm in X the respective unit ball B_X has a denting point [12, p. 30]. For definition and more details on Radon-Nikodym property, we refer to [12]. Consequently by Corollary 2.29 we get the following corollaries.

Corollary 2.30. *The space* $(C_{p,w}, \|\cdot\|_{C_{p,w}})$ *does not have the Radon-Nikodym property.*

Corollary 2.31. *The unit sphere of* $(C_{p,w}, \|\cdot\|_{C_{p,w}})$ *does not have strongly exposed points.*

Corollary 2.32. *The unit sphere of* $(C_{p,w}, \|\cdot\|_{C_{p,w}})$ *does not have denting points.*

Corollary 2.33. *The space* $(C_{p,w}, \|\cdot\|_{C_{p,w}})$ *is not locally uniformly convex.*

Corollary 2.34. *The space* $(C_{p,w}, \|\cdot\|_{C_{p,w}})$ *is not a dual space.*

Proof. It is known that every separable dual space has the Kreĭn-Milman Property [10] and that the latter is equivalent to the Radon-Nikodym Property in Banach lattices [11, 15]. Since $C_{p,w}$ is a separable Banach lattice without the Radon-Nikodym Property it cannot be a dual space. $\square$

3 Other geometric properties of Cesàro function spaces

In this chapter we show that Cesàro function space is strictly convex, contains an asymptotically isometric copy of ℓ_1 and has all relatively weakly open sets of its unit ball of diameter 2. We also show that no point of this space is uniformly non-square, and what follows, there are no strongly extreme points nor H-points and the space is not uniformly convex in every direction. As in Chapter 2, $I = (0, l)$, $0 < l \leqslant \infty$, is a finite or infinite interval on which we consider the space $C_{p,w}$, where $1 < p < \infty$ and the weight function w satisfies conditions (i)-(iii) from page 22.

3.1 Strict convexity

A point $x \in S_X$ is called *extreme* if for every $y \in X$ the condition $\|x \pm y\| = 1$ implies that $y = 0$. Equivalently, if $y, z \in B_X$ and $\|(y + z)/2\| = 1$ then $y = z = x$. If all points of the unit sphere S_X are extreme then the space $(X, \|\cdot\|)$ is called *strictly convex.*

In this section, we show that space $C_{p,w}$ is strictly convex.

Theorem 3.1. *The space $(C_{p,w}, \|\cdot\|_{C_{p,w}})$ is strictly convex.*

Proof. Let $f \in S_{C_{p,w}}$ and suppose that $\|f \pm g\|_{C_{p,w}} = \|f\|_{C_{p,w}}$. Since

$$|f| = \frac{|f + g + f - g|}{2} \leqslant \frac{1}{2}|f + g| + \frac{1}{2}|f - g|,$$

we get that

$$\|f\| = \left\| \frac{1}{2}(f + g + f - g) \right\| \leqslant \frac{1}{2}\|f + g\| + \frac{1}{2}\|f - g\| = \|f\|.$$

It follows that

$$\int_I \left(w(x) \int_0^x \frac{|f(t)+g(t)| + |f(t)-g(t)|}{2}\, dt \right)^p dx - \int_I \left(w(x) \int_0^x |f(t)|\, dt \right)^p dx = 0.$$

Since $(|f+g|+|f-g|)/2 \geqslant |f|$, we get that

$$w(x) \int_0^x \frac{|f(t)+g(t)| + |f(t)-g(t)|}{2}\, dt = w(x) \int_0^x |f(t)|\, dt \text{ for a.a } x \in I.$$

Since $L_p(I)$ space is strictly convex for $1 < p < \infty$ and, by assumption,

$$w(x) \int_0^x |f(t)|\, dt,\ w(x) \int_0^x |f(t)+g(t)|\, dt,\ w(x) \int_0^x |f(t)-g(t)|\, dt \in S_{L_p}$$

we get that $\int_0^x |f(t) \pm g(t)| - |f(t)|\, dt = 0$ for all $x \in I$ and by [59, Lemma 8, p. 105], $|f(t) \pm g(t)| = |f(t)|$ a.e. on I. This gives $g = 0$. □

The above result can be also obtained by applying [33, Corollary 1].

3.2 Copy of ℓ_1

Recall [26] that a Banach space $(X, \|\cdot\|)$ contains an *asymptotically isometric copy of* ℓ_1 if there exists a sequence $(\epsilon_n) \subset (0,1)$, $\epsilon_n \to 0$ as $n \to \infty$ and a sequence $(x_n) \subset X$ such that for arbitrary $(\alpha_n) \in \ell_1$ we have

$$\sum_{n=1}^{\infty} (1-\epsilon_n)|\alpha_n| \leqslant \|\sum_{n=1}^{\infty} \alpha_n x_n\| \leqslant \sum_{n=1}^{\infty} |\alpha_n|. \tag{3.1}$$

In 2008 Astashkin and Maligranda proved that the Cesàro function space with standard weight $w(x) = 1/x$ contains an asymptotically isometric copy of ℓ_1 [4]. In fact their proof works well also for arbitrary weights.

Theorem 3.2. *The space $(C_{p,w}, \|\cdot\|_{C_{p,w}})$ contains an asymptotically isometric copy of ℓ_1.*

Proof. Let $(b_n)_{n=1}^{\infty}$ be any strictly increasing sequence in I such that $\lim_{n\to\infty} b_n = b < l$. Denote $a = b_1$. Let $g_n = \chi_{(b_n,b_{n+1})}$ and $f_n = g_n/\|g_n\|_{C_{p,w}}$, $n \in \mathbb{N}$. It is easy to check for any function $f \in L_1(I)$ with $\operatorname{supp} f \subset [c,d]$ for some bounded interval $[c,d] \subset I$,

$$\Psi(d)^{1/p}\|f\|_{L_1} \leqslant \|f\|_{C_{p,w}} \leqslant \Psi(c)^{1/p}\|f\|_{L_1}. \tag{3.2}$$

Let $(\alpha_n)_{n=1}^{\infty} \in \ell_1$. Since $\operatorname{supp} \sum_{n=1}^{\infty} \alpha_n f_n \subset [a,b]$ and $\operatorname{supp} g_n \subset [b_n, b_{n+1}]$, by (3.2) we get that

$$\begin{aligned}
\|\sum_{n=1}^{\infty}\alpha_n f_n\|_{Cp,w} &\geqslant \Psi(b)^{1/p}\|\sum_{n=1}^{\infty}\alpha_n f_n\|_{L_1} \\
&= \Psi(b)^{1/p}\sum_{n=1}^{\infty}|\alpha_n|\|f_n\|_{L_1} = \Psi(b)^{1/p}\sum_{n=1}^{\infty}|\alpha_n|\|g_n\|_{L_1}/\|g_n\|_{C_{p,w}} \\
&\geqslant \sum_{n=1}^{\infty}|\alpha_n|\Psi(b)^{1/p}/\Psi(b_n)^{1/p} = \sum_{n=1}^{\infty}(1-\epsilon_n)|\alpha_n|,
\end{aligned}$$

where $\epsilon_n = 1 - \Psi(b)^{1/p}/\Psi(b_n)^{1/p}$. Clearly $\epsilon_n \to 0$ as $n \to \infty$. The other inequality is obvious. □

Let X be a Banach space. Recall that a mapping $T : K \to K$ is called *nonexpansive* if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in K \subset X$. A Banach space X has the *(weak) fixed point property* if every nonexpansive mapping of every (nonempty weakly compact convex) closed bounded convex subset K into itself has a fixed point.

Similarly as in [4] by results in [25], we conclude this section with the following corollary.

Corollary 3.3. *Cesàro function space $(C_{p,w}, \|\cdot\|_{C_{p,w}})$ and its dual $((C_{p,w})', \|\cdot\|_{(C_{p,w})'})$*

fail the fixed point property. Moreover $((C_{p,w})', \|\cdot\|_{(C_{p,w})'})$ *contains an isometric copy of* $L_1[0,1]$, *hence it even fails the weak fixed point property.*

3.3 Some results in general Banach spaces

Let $(X, \|\cdot\|)$ be a (real) Banach space. For any $x^* \in S_{X^*}$ and $\epsilon > 0$ the set $s(x^*;\epsilon) = \{x \in B_X : x^*x > 1-\epsilon\}$ is called *a slice* determined by x^* and ϵ.

Following Schaffer [62, p. 131], we say that point $x \in X$ is *uniformly non-square* if there exists $\rho > 1$ such that

$$\rho \min\{\|x\|, \|y\|\} \leqslant \max\{\|x+y\|, \|x-y\|\} \text{ for all } y \in X.$$

If all points of the unit sphere S_X are uniformly non-square then the space $(X, \|\cdot\|)$ is called *locally uniformly non-square* (a LUNS space for short).

The following simple observation is known, however we provide its proof for completeness.

Lemma 3.4. *Let* $(X, \|\cdot\|)$ *be a Banach space.*

(i) *If* $x \in X$ *is a uniformly non-square point then so is* λx *for any* $\lambda > 0$.

(ii) *All points* $x \in X$ *are uniformly non-square if and only if all points* $x \in S_X$ *are uniformly non-square.*

Proof. (i) Let $x \in X$ be uniformly non-square and $\lambda > 0$. Let $y \in X$ be arbitrary and $y' = y/\lambda$. By definition we get

$$\begin{aligned}\max(\|\lambda x + y\|, \|\lambda x - y\|) &= \max(\|\lambda x + \lambda y'\|, \|\lambda x - \lambda y'\|)\\ &= \lambda \max(\|x + y'\|, \|x - y'\|) \geqslant \rho\lambda \min(\|x\|, \|y'\|)\\ &= \rho \min(\|\lambda x\|, \|\lambda y'\|) = \rho \min(\|\lambda x\|, \|y\|).\end{aligned}$$

(ii) Suppose that every $x \in S_X$ is uniformly non-square. Clearly $x = 0$ is uniformly non-square. For $0 \neq x \in X$, since $x/\|x\|$ is uniformly non-square so is x by (i).

$\square$

Proposition 3.5. *Let $(X, \|\cdot\|)$ be a Banach space. The following are equivalent.*

(i) *A point $x \in S_X$ is not uniformly non-square.*

(ii) *There exists a sequence $(y_n) \subset X$ such that $\|y_n\| \to 1$ and $\|x \pm y_n\| \to 1$ as $n \to \infty$.*

(iii) *There exists a sequence $(y_n) \subset B_X$ such that $\|y_n\| \to 1$ and $\|x \pm y_n\| \to 1$ as $n \to \infty$.*

(iv) *There exists a sequence $(y_n) \subset S_X$ such that $\|x \pm y_n\| \to 1$ as $n \to \infty$.*

(v) *There exists a sequence $(y_n) \subset B_X$ such that $\|y_n\| \to 1$ and $\|\lambda x \pm y_n\| \to 1$ as $n \to \infty$ for all $\lambda \in [0, 1]$.*

Proof. (i) $\Longrightarrow$ (ii) First observe that, for any $x, y \in X$,

$$\|x\| = \left\| \frac{x-y}{2} + \frac{x+y}{2} \right\| \leqslant \frac{1}{2}\|x+y\| + \frac{1}{2}\|x-y\|.$$

Hence

$$\|x+y\| + \|x-y\| \geqslant 2\|x\|. \tag{3.3}$$

Changing the roles of x and y we can infer that $\|x+y\| + \|x-y\| \geqslant 2\max(\|x\|, \|y\|)$. Hence

$$\max(\|x+y\|, \|x-y\|) \geqslant \max(\|x\|, \|y\|), \text{ for all } x, y \in X. \tag{3.4}$$

It follows that

$$\frac{\max(\|x+y\|, \|x-y\|)}{\min(\|x\|, \|y\|)} \geqslant 1, \text{ for all } x, y \in X. \tag{3.5}$$

Let $x \in S_X$ be a point which is not uniformly non-square. By the definition of x and inequality (3.5), we get that there exists sequence $(y_n) \subset X$ such that

$$\frac{\max(\|x+y_n\|, \|x-y_n\|)}{\min(1, \|y_n\|)} \to 1 \text{ as } n \to \infty. \tag{3.6}$$

By (3.4) we have that $\max(\|x+y_n\|, \|x-y_n\|) \geqslant 1$ for all $n \in \mathbb{N}$ and hence it must be $\|y_n\| \to 1$ as $n \to \infty$. Indeed, if there is a subsequence y_{n_k} such that $\|y_{n_k}\| \leqslant \eta < 1$ for all $k \in \mathbb{N}$ then $\max(\|x+y_{n_k}\|, \|x-y_{n_k}\|)/\min(1, \|y_{n_k}\|) \geqslant 1/\eta > 1$. Similarly, if $\|y_{n_k}\| \geqslant \eta > 1$ for all $k \in \mathbb{N}$ then by (3.4) $\max(\|x+y_{n_k}\|, \|x-y_{n_k}\|)/\min(1, \|y_{n_k}\|) \geqslant \eta > 1$. Which is a contradiction with (3.6). Now it follows from (3.6) that $\max(\|x+y_n\|, \|x-y_n\|) \to 1$ as $n \to \infty$. Since $\|x\| = 1$, by (3.3) we get that $\|x \pm y_n\| \to 1$ as $n \to \infty$.

(ii) $\implies$ (iv) By (ii) we can find a subsequence of (y_n), again called (y_n), such that $\|y_n\| \to 1$, $\|x \pm y_n\| \to 1$ as $n \to \infty$ and either $(y_n) \subset B_X$ or $(y_n) \subset X \setminus B_X$.

Suppose first that $(y_n) \subset B_X$. Let $\gamma, \gamma_n \geqslant 0$ be such that $\gamma_n \to \gamma$ as $n \to \infty$. We have

$$\|(1+\gamma_n)y_n \pm x\| = \|(2+\gamma_n)y_n - (y_n \mp x)\| \geqslant |(2+\gamma_n)\|y_n\| - \|y_n \mp x\|| \to 2+\gamma-1 = 1+\gamma.$$

Hence $\underline{\lim}_{n\to\infty}\|(1+\gamma_n)y_n \pm x\| \geqslant 1+\gamma$. But $\|(1+\gamma_n)y_n \pm x\| \leqslant \gamma_n\|y_n\| + \|y_n \pm x\| \to 1+\gamma$ whence $\overline{\lim}_{n\to\infty}\|(1+\gamma_n)y_n \pm x\| \leqslant 1+\gamma$ and so $\lim_{n\to\infty}\|(1+\gamma_n)y_n \pm x\| = 1+\gamma$. Equivalently

$$\lim_{n\to\infty} \frac{1+\gamma_n}{1+\gamma}\|(1+\gamma_n)^{-1}x \pm y_n\| = 1.$$

Since $(1+\gamma_n)/(1+\gamma) \to 1$ as $n \to \infty$ we get that $\lim_{n\to\infty}\|(1+\gamma_n)^{-1}x \pm y_n\| = 1$.

Now taking $\gamma_n = 1/\|y_n\| - 1$ we obtain that $\gamma_n \to 0$ and

$$\left\| x \pm \frac{y_n}{\|y_n\|} \right\| = \frac{1}{\|y_n\|} \| \|y_n\| x \pm y_n\| = (1+\gamma_n)\|(1+\gamma_n)^{-1}x \pm y_n\| \to 1.$$

In case when $(y_n) \subset X \setminus B_X$ we proceed similarly. Let $0 \leqslant \gamma_n \leqslant 1$, $n \in \mathbb{N}$ be such that $\gamma_n \to 0$ as $n \to \infty$. We have

$$\|(1-\gamma_n)y_n \pm x\| = \|(2-\gamma_n)y_n - (y_n \mp x)\| \geqslant |(2-\gamma_n)\|y_n\| - \|y_n \mp x\|| \to 2-1 = 1.$$

Hence $\underline{\lim}_{n\to\infty}\|(1-\gamma_n)y_n \pm x\| \geqslant 1$. But $\|(1-\gamma_n)y_n \pm x\| \leqslant \|\gamma_n y_n\| + \|y_n \pm x\| \to 1$ whence $\overline{\lim}_{n\to\infty}\|(1-\gamma_n)y_n \pm x\| \leqslant 1$ and so $\lim_{n\to\infty}\|(1-\gamma_n)y_n \pm x\| = 1$. Equivalently

$$\lim_{n\to\infty}(1-\gamma_n)\|(1-\gamma_n)^{-1}x \pm y_n\| = 1.$$

Now taking $\gamma_n = 1 - 1/\|y_n\|$ we obtain that $\gamma_n \to 0$ and

$$\left\| x \pm \frac{y_n}{\|y_n\|} \right\| = \frac{1}{\|y_n\|} \| \|y_n\| x \pm y_n\| = (1-\gamma_n)\|(1-\gamma_n)^{-1}x \pm y_n\| \to 1.$$

Implications (iv) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (i) are obvious. Hence conditions (i)-(iv) are equivalent.

The implication (v) $\Longrightarrow$ (iii) is clear. While (iii) $\Longrightarrow$ (v) can be proved in the same way as the first part of (ii) $\Longrightarrow$ (iv) by taking $\gamma_n = 1/\lambda - 1$ to be a constant sequence and $\gamma = 1/\lambda - 1$ in case when $\lambda \in (0,1]$. For $\lambda = 0$ the claim is clear. $\square$

Observe that $x \in S_X$ is uniformly non-square if and only if there exists $\delta > 0$ such that

$$\max\{\|x+y\|, \|x-y\|\} \geqslant 1+\delta \text{ for all } y \in S_X. \tag{3.7}$$

Clearly, if $x \in S_X$ is uniformly non-square then (3.7) is satisfied. If $x \in S_X$ is not uniformly non-square then by Proposition 3.5 (iv) there exists a sequence $(y_n) \subset S_X$ such that $\|x \pm y_n\| \to 1$. Hence (3.7) is not satisfied.

A point $x \in S_X$ is called *strongly extreme* or *midpoint locally uniformly rotund* (a MLUR point) if for every sequence $(x_n) \subset B_X$ the condition $\|x \pm x_n\| \to 1$ as $n \to \infty$ implies that $x_n \to 0$ as $n \to \infty$. If all points of the unit sphere S_X are strongly extreme points then the space $(X, \|\cdot\|)$ is called *midpoint locally uniformly rotund* (a MLUR space). A Banach space $(X, \|\cdot\|)$ is called *uniformly rotund in every direction* (URED space) if $x_n, z \in X$, $\|x_n\| \to 1$, $\|x_n + z\| \to 1$ and $\|2x_n + z\| \to 2$ implies that $z = 0$.

Lemma 3.6. *If $x \in S_X$ is a strongly extreme point then it is a uniformly non-square point.*

Proof. If $x \in S_X$ is not a uniformly non-square point then by Proposition 3.5 (iv) there exists a sequence $(y_n) \subset S_X$ such that $\|x \pm y_n\| \to 1$. Since $\|y_n\| = 1$ for all $n \in \mathbb{N}$ we see that x is not a strongly extreme point. □

Proposition 3.7. *If a Banach space $(X, \|\cdot\|)$ is uniformly rotund in every direction then it is locally uniformly non-square.*

Proof. Let $(X, \|\cdot\|)$ be an URED space. Suppose that $x \in S_X$ is not an uniformly non-square point. By Proposition 3.5 (iv) there exists a sequence $(y_n) \subset S_X$ such that $\|x \pm y_n\| \to 1$ as $n \to \infty$. We get that

$$\|2y_n + x\| = \|3y_n - (y_n - x)\| \geqslant |3\|y_n\| - \|y_n - x\|| \to 2 \text{ as } n \to \infty.$$

Hence $\underline{\lim}_n \|2y_n + x\| \geqslant 2$. But $\|2y_n + x\| \leqslant \|y_n\| + \|y_n + x\| \to 2$ as $n \to \infty$, which gives $\overline{\lim}_n \|2y_n + x\| \leqslant 2$. It follows that $\lim_n \|2y_n + x\| = 2$. Since X is an URED space we get that $x = 0$ which is a contradiction with $x \in S_X$. $\square$

Proposition 3.8. *If there are no uniformly non-square points in the unit sphere S_X then all slices of the unit ball B_X have diameter 2.*

Proof. Let $x^* \in S_{X^*}$ and $\eta > 0$ be arbitrary. Let $\delta < \eta/4$ be positive and choose $x \in S_X$ such that $x^*x > 1 - \delta$. In particular $x \in s(x^*;\eta)$. By assumption x is not a uniformly non-square point, so by Proposition 3.5 (iii) there exists a sequence $(y_n) \subset B_X$ such that $\|x \pm y_n\| \to 1$ and $\|y_n\| \to 1$ as $n \to \infty$. Since

$$x^*(x \pm y_n) \leqslant |x^*(x \pm y_n)| \leqslant \|x^*\|_{X^*}\|x \pm y_n\| \to 1,$$

there exists $N \in \mathbb{N}$ such that $x^*(x \pm y_n) < 1 + \delta$ for all $n > N$. It follows that $|x^*y_n| < 2\delta$ for all $n > N$. Now, since $\|x \pm y_n\| \to 1$ as $n \to \infty$, for all n large enough

$$\left\|\frac{x \pm y_n}{1+\delta}\right\| \leqslant 1.$$

Moreover

$$x^*\left(\frac{x \pm y_n}{1+\delta}\right) = \frac{1}{1+\delta}(x^*x \pm x^*y_n) > \frac{1}{1+\delta}(1 - \delta - 2\delta) > 1 - 4\delta > 1 - \eta,$$

hence $\frac{x \pm y_n}{1+\delta} \in s(x^*;\eta)$ for all n large enough. We also get that

$$\left\|\frac{x + y_n}{1+\delta} - \frac{x - y_n}{1+\delta}\right\| = \frac{1}{1+\delta}\|2y_n\| \to \frac{2}{1+\delta} \text{ as } n \to \infty.$$

Since δ can be taken arbitrarily small we get that diameter of $s(x^*;\eta)$ is equal 2. $\square$

The following result is a part of the proof of Theorem 2.5 in [3]. We provide it here for reader's convenience.

Lemma 3.9. *Let $(X, \|\cdot\|)$ be an infinite dimensional Banach space. If for every $x \in S_X$ there is a sequence $(y_n) \subset B_X$ such that $\|x \pm y_n\| \to 1$, $\|y_n\| \to 1$ and $y_n \to 0$ weakly in X as $n \to \infty$ then all nonempty relatively weakly open subsets of the unit ball B_X have diameter 2.*

Proof. Let $W \neq \emptyset$ be a relatively weakly open subset of B_X. Since X is infinite dimensional there is $x \in W$ such that $\|x\| = 1$. By assumption there exists sequence $(y_n) \subset B_X$ such that $\|x \pm y_n\| \to 1$, $\|y_n\| \to 1$ and $y_n \to 0$ weakly in X as $n \to \infty$. Let $x_n^{'} = x + y_n$ and $x_n^{''} = x - y_n$. Clearly $x_n^{'}, x_n^{''} \to x$ weakly in X, $\|x_n^{'}\|$, $\|x_n^{''}\| \to 1$ and $\|x_n^{'} - x_n^{''}\| = 2\|y_n\| \to 2$ as $n \to \infty$. Taking $z_n = x_n^{'}/\|x_n^{'}\|$ and $w_n = x_n^{''}/\|x_n^{''}\|$, we clearly have that $z_n, w_n \to x$ weakly in X, $z_n, w_n \in B_X$ and $\|z_n - w_n\| \to 2$ as $n \to \infty$. $\square$

Recall that a point $x \in S_X$ is called an *H-point* of B_X if for every sequence $(x_n) \subset X$ such that $\|x_n\| \to \|x\|$ and $x_n \to x$ weakly it follows that $\|x - x_n\| \to 0$ as $n \to \infty$. If all points of the unit sphere are H-points then $(X, \|\cdot\|)$ is said to have *Kadec-Klee property* or *H property* or *Radon-Riesz property.*

Similarly as above we can prove the following.

Lemma 3.10. *If for $x \in S_X$ there is a sequence $(y_n) \subset B_X$ such that $y_n \nrightarrow 0$, $y_n \to 0$ weakly in X and either $\|x + y_n\| \to 1$ or $\|x - y_n\| \to 1$ as $n \to \infty$ then x is not an H-point of B_X.*

Proof. It is enough to take $x_n = x + y_n$ or $x_n = x - y_n$. By assumption $\|x_n\| \to 1 = \|x\|$, $x_n \to x$ weakly and $\|x_n - x\| = \|y_n\| \nrightarrow 0$. $\square$

3.4 Several geometric properties of Cesàro function spaces

First, we define a sequence of Rademacher-like functions on an arbitrary measurable set of finite and positive Lebesgue measure.

Definition 3.11. Let $A \subset \mathbb{R}$ be a Lebesgue measurable set with $0 < mA < \infty$. A sequence of Rademacher functions r_n, $n \in \mathbb{N}$, on A is defined as

$$r_n(t) = \begin{cases} 1 \text{ if } t \in \bigcup_{i=0}^{2^n-1} A_{2i}^{(n)}, \\ -1 \text{ if } t \in \bigcup_{i=0}^{2^n-1} A_{2i+1}^{(n)}, \end{cases}$$

where $A_0^{(0)} = A$ and given $A_i^{(n-1)}$, $i = 0, 1, \ldots, 2^{n-1} - 1$, sets $A_i^{(n)}$ are any sets which satisfy

$$A_i^{(n-1)} = A_{2i}^{(n)} \cup A_{2i+1}^{(n)}, \quad A_{2i}^{(n)} \cap A_{2i+1}^{(n)} = \emptyset, \quad mA_{2i}^{(n)} = mA_{2i+1}^{(n)},$$

for $i = 0, 1, \ldots, 2^{n-1} - 1$. Note that $A = \cup_{i=0}^{2^n-1} A_i^{(n)}$ for all $n \in \mathbb{N}$, and $mA_i^{(n)} = mA/2^n$ for all $n \in \mathbb{N}$, and $i \in \{0, 1, \ldots, 2^n - 1\}$.

The following fact is known but we provide its proof for completeness.

Lemma 3.12. *Let $A \subset I$ be a Lebesgue measurable set with $0 < mA < \infty$. A sequence of Rademacher functions r_n on A converges to 0 weakly in $L_1(I)$.*

Proof. Let $0 \leqslant g \in L_\infty(I)$. Let $\epsilon > 0$ be arbitrary and sets $B_i \subset A$, $i = 1, 2, \ldots, N$, be disjoint and such that $A = \cup_{i=1}^N B_i$ and $\operatorname{ess\,sup}_{t,s \in B_i} |g(t) - g(s)| < \epsilon/mA$, $i = 1, 2, \ldots, N$. For large enough n's and for all $i \in \{1, 2, \ldots, N\}$ there exist disjoint sets $C_i^{(n)}$, $D_i^{(n)}$ and $E_i^{(n)}$ such that $B_i^{(n)} = C_i^{(n)} \cup D_i^{(n)} \cup E_i^{(n)}$, $mC_i^{(n)} = mD_i^{(n)}$, $r_n(C_i^{(n)}) = \{1\}$, $r_n(D_i^{(n)}) = \{-1\}$ and $m(E_i^{(n)}) \to 0$ as $n \to \infty$. Observe that

$$\int_{\cup_{i=1}^N E_i^{(n)}} |g(t) r_n(t)| \, dt \leqslant \|g\|_\infty m(\cup_{i=1}^N E_i^{(n)}) \to 0 \text{ as } n \to \infty,$$

and for all $i \in \{1, 2, \ldots, N\}$

$$\left|\int_{B_i \setminus E_i^{(n)}} g(t) r_n(t)\, dt\right| = \left|\int_{C_i^{(n)}} g(t)\, dt - \int_{D_i^{(n)}} g(t)\, dt\right|$$
$$\leqslant mC_i^{(n)} \operatorname*{ess\,sup}_{t \in B_i} g(t) - mD_i^{(n)} \operatorname*{ess\,inf}_{t \in B_i} g(t) < \epsilon m B_i / mA.$$

Now, for n's large enough,

$$\begin{aligned}\left|\int_I g(t) r_n(t)\, dt\right| &= \left|\int_A g(t) r_n(t)\, dt\right| \\ &= \sum_{i=1}^{N} \left|\int_{B_i} g(t) r_n(t)\, dt\right| \\ &\leqslant \sum_{i=1}^{N} \left|\int_{B_i \setminus E_i^{(n)}} g(t) r_n(t)\, dt\right| + \sum_{i=1}^{N} \int_{E_i^{(n)}} |g(t)|\ dt \leqslant 2\epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we get that $\int_I g(t) r_n(t)\, dt \to 0$ as $n \to \infty$. If $g \in L_\infty(I)$ is arbitrary then one writes $g = g^+ - g^-$, where $g^+, g^- \geqslant 0$ and applies the above result to both g^+, g^- and obtains the hypothesis. □

Lemma 3.13. *For any function $f \in L_0(I)$ there exists sequence $(p_n) \subset L_1(I)$, $p_n(I) = \{-1, 1\}$ such that $\int_I f \chi_B p_n \to 0$ as $n \to \infty$ for any measurable set B for which $f\chi_B \in L_1(I)$.*

Proof. Suppose first that $m(\operatorname{supp} f) < \infty$. Let $p_n = r_n$ be a sequence of Rademacher functions defined on $\operatorname{supp} f$ and (A_m) be an increasing sequence of measurable sets such that $\cup_m A_m = \operatorname{supp} f$ and f is essentially bounded on each A_m. Since $L_1(I)$ is order continuous for every $\epsilon > 0$ there exists $M \in \mathbb{N}$ such that $\int_I |f(t)| \chi_{B \cap A_m^c}(t)\, dt < \epsilon$ for all $m > M$. It follows that $\left|\int_I |f(t)| \chi_{B \cap A_m^c}(t) p_n(t)\, dt\right| < \epsilon$ for all $m > M$ and all

$n \in \mathbb{N}$. Since $p_n \to 0$ weakly in $L_1(I)$, we get that

$$\begin{aligned}\left|\int_I f(t)\chi_B(t)p_n(t)\,dt\right| &= \left|\int_I f(t)\chi_{B\cap A_m}(t)p_n(t)\,dt + \int_I f(t)\chi_{B\cap A_m^c}(t)p_n(t)\,dt\right| \\ &\leqslant \left|\int_I f(t)\chi_{B\cap A_m}(t)p_n(t)\,dt\right| + \int_I |f(t)|\chi_{B\cap A_m^c}(t)\,dt \\ &\leqslant \left|\int_I f(t)\chi_{B\cap A_m}(t)p_n(t)\,dt\right| + \epsilon \to \epsilon \quad \text{as } n \to \infty.\end{aligned}$$

It follows that $\left|\int_I f(t)\chi_B(t)p_n(t)\,dt\right| \leqslant 2\epsilon$ for all n's large enough. Hence

$$\int_I f(t)\chi_B(t)p_n(t)\,dt \to 0 \text{ as } n \to \infty.$$

Suppose now that $m(\operatorname{supp} f) = \infty$. Similarly as above let (A_m) be an increasing sequence of measurable sets such that $\cup_m A_m = \operatorname{supp} f$ and f is essentially bounded on each A_m. Since $L_1(I)$ is order continuous for every $\epsilon > 0$ there exists $M \in \mathbb{N}$ such that $\int_I |f(t)|\chi_{B\cap A_m^c}(t)\,dt < \epsilon$ for all $m > M$. For each $m \in \mathbb{N}$ let $p_n^{(m)}$ be sequence of Rademacher functions defined on A_m. By the first part $\int_I f(t)\chi_{A_m}(t)\chi_B(t)p_n^{(m)}(t)\,dt \to 0$ as $n \to \infty$ for each $m \in \mathbb{N}$. We have that for each $m \in \mathbb{N}$ there exists $N_m \in \mathbb{N}$ such that for all $n \geqslant N_m$,

$$\left|\int_I f(t)\chi_{A_m}(t)\chi_B(t)p_n^{(m)}(t)\,dt\right| = \left|\int_I f(t)\chi_B(t)p_n^{(m)}(t)\,dt\right| < 1/m.$$

Taking $p_n = p_{N_n}^{(n)}$ finishes the proof. $\square$

Lemma 3.14. *For every function $f \in S_{Cp,w}$ there exists a sequence $(g_n) \subset B_{C_{p,w}}$ such that $\|f \pm g_n\|_{C_{p,w}} \to 1$, $\|g_n\|_{C_{p,w}} \to 1$ and $g_n \to 0$ weakly in $C_{p,w}$ as $n \to \infty$.*

Proof. Since $f \in S_{Cp,w}$ by Lemma 3.13 there exists sequence $(p_n) \subset L_1(I)$, $p_n(I) = \{-1, 1\}$ such that $\int_I f\chi_B p_n \to 0$ as $n \to \infty$ for any measurable set B for which $f\chi_B \in L_1(I)$. Let $g_n = fp_n$. Since $C_{p,w}$ is order continuous and $\operatorname{supp} p_n \subset \operatorname{supp} f$,

$m(\operatorname{supp} f \setminus \operatorname{supp} p_n) \to 0$ it is clear that $\|g_n\|_{C_{p,w}} \to 1$ as $n \to \infty$ and $\|g_n\|_{C_{p,w}} \leqslant \|f\|_{C_{p,w}} = 1$.

Now, by Lemma 3.13, for all $x \in I$ we get that

$$\begin{aligned}\int_0^x |f(t) \pm g_n(t)|\, dt &= \int_0^x |f(t) \pm f(t)p_n(t)|\, dt = \int_0^x |f(t)|(1 \pm p_n(t))\, dt \\ &= \int_0^x |f(t)|\, dt \pm \int_0^x |f(t)|p_n(t)\, dt \to \int_0^x |f(t)|\, dt \text{ as } n \to \infty.\end{aligned}$$

By the Lebesgue Dominated Convergence Theorem we get that $\|f \pm g_n\|_{C_{p,w}} \to 1$ as $n \to \infty$.

Observe that for any $f \in C_{p,w}$, $g_n = fp_n \to 0$ weakly in $C_{p,w}$. Indeed, for every function $g \in (C_{p,w})'$, $\int_I g(t)f(t)p_n(t)\, dt \to 0$ as $n \to \infty$ since $fg \in L_1(I)$, by Lemma 3.14. $\square$

Lemmas 3.14, 3.9, 3.10 imply the following.

Corollary 3.15. *All nonempty relatively open subsets of the unit ball of the space $C_{p,w}$ have diameter 2. The unit sphere $S_{C_{p,w}}$ does not have uniformly non-square points nor strongly extreme points nor H-points. The space $(C_{p,w}, \|\cdot\|_{C_{p,w}})$ is neither locally uniformly non-square nor midpoint locally uniformly rotund nor uniformly rotund in every direction.*

4 Cesàro-Orlicz sequence spaces

In this chapter we study Cesàro-Orlicz sequence spaces. We explore the influence of the growth condition δ_2 of φ on the geometric structure of these spaces. Moreover, we present the comparison theorem for these spaces and show that they are not B-convex.

All results presented in this chapter are published in [42] and [37].

4.1 Preliminaries

Cesàro sequence spaces ces_p, $1 \leqslant p < \infty$, appeared in 1968, as mentioned in Chapter 1. It seems that their generalization, Cesàro-Orlicz sequence spaces ces_φ, were defined for the first time in 1988, when Lim and Yee found their dual spaces [66]. Recently Cui, Hudzik, Petrot, Suantai and Szymaszkiewicz obtained important properties of spaces ces_φ [21]. In 2007 Maligranda, Petrot and Suantai showed that ces_φ is not B-convex, if φ satisfies δ_2 condition and $ces_\varphi \neq \{0\}$ [51]. The extreme points and strong U-points of ces_φ have been characterized by Foralewski, Hudzik and Szymaszkiewicz in [28]. They also considered local uniform convexity and Kadec-Klee property of ces_φ [29]. Although the spaces ces_φ have been studied by several mathematicians, some essential and basic properties remain still unknown.

In this dissertation, we present characterizations of some of them. In section 4.2, under the assumption that the lower index $\alpha_\varphi > 1$, we shall present that δ_2 condition is necessary and sufficient for the space of all order continuous elements $(ces_\varphi)_a$ to coincide with ces_φ. In section 4.3, given functions φ_1 and φ_2, under the assumption that $\alpha_{\varphi_1} > 1$, we show that $ces_{\varphi_1} \subset ces_{\varphi_2}$ if and only if $\ell_{\varphi_1} \subset \ell_{\varphi_2}$, that is, there exist $b > 0$ and $t_0 > 0$ such that $0 < \varphi_1(t_0) < \infty$ and $\varphi_2(t) \leqslant \varphi_1(bt)$ for all t with $0 < t \leqslant t_0$.

In section 4.4 we consider the problem of existence of order linearly isometric copy of ℓ_∞ in ces_φ under the Luxemburg norm. Recall that ℓ_∞ is an order isometric copy in Orlicz space ℓ_φ equipped with the Luxemburg norm if and only if φ does not satisfy condition δ_2 [47]. It is expected that a similar result remains true in ces_φ. However such factors as lack of symmetry or the presence of averaging operator in the definition of these spaces cause that this problem in the context of Cesàro-Orlicz spaces is more involved than in Orlicz spaces. Here we present a solution of this problem for comparatively large class of Orlicz functions φ. In section 4.4 we prove that such a copy exists in ces_φ whenever φ does not satisfy condition δ_2 and the Orlicz class $\{x : I_\varphi(x) < \infty\}$ is closed under the averaging operator G, that is $\{x : I_\varphi(x) < \infty\} \subset \{x : I_\varphi(Gx) < \infty\}$. We also present several conditions under which the latter inclusion is satisfied and discuss their relations to Matuszewska-Orlicz indices of φ. We show among others that whenever $\varphi^{1/p}$ is strongly equivalent to a convex function for some $p > 1$, then the Orlicz class is closed under the averaging operation. The latter condition is also fulfilled whenever the Hardy inequality for the Orlicz function φ holds true. We finish this section by presenting an example of Orlicz function φ for which the Hardy inequality is not satisfied but the space ces_φ contains an order isometric copy of ℓ_∞.

In 2007, Maligranda, Petrot and Suantai showed that ces_φ is not B-convex if $\varphi \in \delta_2$ and $ces_\varphi \neq \{0\}$ [51]. In the last section we show that the n-th (strong) James constant of non-trivial space ces_φ equipped with either the Luxemburg or Orlicz norm equals n (which, in particular implies that the space is not B-convex), extending the family of functions φ for which it is satisfied and solving the problem posed in [51].

We shall use the following notation in the sequel. For any $a \in \mathbb{R}$, $\lceil a \rceil$ is the smallest integer greater than a. By H_n we denote the n-th *harmonic number*, that is $H_n = \sum_{i=1}^n i^{-1}$. A function $\varphi : [0, \infty) \to [0, \infty]$ is called an *Orlicz function* if it is convex, right-continuous at 0, left-continuous on $(0, \infty)$, $\varphi(0) = 0$, and $\varphi(u) > 0$ for some $u > 0$. If, in addition, φ satisfies the conditions $\lim_{u\to 0} \varphi(u)/u = 0$ and $\lim_{u\to\infty} \varphi(u)/u = \infty$ then it is called an N-function.

By ℓ^0 we denote the linear space of all real sequences $x = (x(n))_{n=1}^{\infty}$. By e_n we denote unit vectors in ℓ^0. The convex modular $I_\varphi(x) = \sum_{n=1}^{\infty} \varphi(|x(n)|)$ defined on the whole ℓ^0 gives rise of the *Orlicz sequence space* ℓ_φ with the Luxemburg norm $\|x\|_\varphi = \inf\{\varepsilon > 0 : I_\varphi(\varepsilon^{-1}x) \leqslant 1\}$. We say that the Orlicz function φ satisfies the δ_2 condition (we will write $\varphi \in \delta_2$) if there are $K > 0$ and $u_0 > 0$ such that $\varphi(u_0) > 0$, and $\varphi(2u) \leqslant K\varphi(u)$ for all $u \in [0, u_0]$. It follows that $\varphi(u_0) < \infty$. This condition plays crucial role in the theory of Orlicz sequence spaces. The function $\varphi^*(v) = \sup\{uv - \varphi(u) : u > 0\}$, $v \geqslant 0$, is called a *complementary function* to φ. Two Orlicz functions φ_1 and φ_2 are said to be *equivalent* if there exist a, b, $u_0 > 0$ such that $\varphi_2(u_0) > 0$ and $\varphi_1(au) \leqslant \varphi_2(u) \leqslant \varphi_1(bu)$ for all $u \in [0, u_0]$. Two Orlicz functions φ_1 and φ_2 are said to be *strongly equivalent* if there exist A, B, $u_0 > 0$ such that $\varphi_1(u_0) > 0$, and $A\varphi_1(u) \leqslant \varphi_2(u) \leqslant B\varphi_1(u)$ for all $u \in [0, u_0]$.

Orlicz sequence spaces are thoroughly discussed in [47] (see also [16]), and the most comprehensive exposition of Orlicz functions is presented in [40] and [16]. The information on modular spaces can be found in [53].

For any $x \in \ell^0$ we denote by Gx the sequence of averages of x, that is

$$Gx(n) = \frac{1}{n} \sum_{i=1}^{n} |x(i)|, \quad n \in \mathbb{N}.$$

Given an Orlicz function φ, the modular

$$I_{ces_\varphi}(x) = I_\varphi(Gx) = \sum_{n=1}^{\infty} \varphi(Gx(n))$$

is convex and defines Cesàro-Orlicz sequence space

$$ces_{\varphi} = \left\{x \in \ell^0 : I_{ces_{\varphi}}(\lambda x) < \infty \quad \text{for some } \lambda > 0\right\}$$

with the Luxemburg norm given by

$$\|x\|_{ces_{\varphi}} = \inf\left\{\varepsilon > 0 : I_{ces_{\varphi}}\left(\varepsilon^{-1}x\right) \leqslant 1\right\} = \|Gx\|_{\varphi}.$$

In ces_{φ} we also define the *Orlicz norm* in the Amemyia form

$$\|x\|^0_{ces_{\varphi}} = \inf_{k>0} \frac{1}{k}\left(1 + I_{ces_{\varphi}}(kx)\right) = \|Gx\|^0_{\varphi} = \inf_{k>0} \frac{1}{k}\left(1 + I_{\varphi}(kGx)\right).$$

It is well known that for any $x \in ces_{\varphi}$, $\|x\|_{ces_{\varphi}} \leqslant \|x\|^0_{ces_{\varphi}} \leqslant 2\|x\|_{ces_{\varphi}}$ [32].

Unless we state explicitly otherwise, we shall consider further the space ces_{φ} equipped with the Luxemburg norm.

In the case when $\varphi(u) = u^p$, $1 \leqslant p < \infty$, the space ces_{φ} is just a Cesàro sequence space ces_p, with the norm given by

$$\|x\|_{ces_p} = \left[\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{i=1}^{n}|x(i)|\Big)^p\right]^{1/p}.$$

If $\varphi(u) = 0$ for $u \in [0,1]$ and $\varphi(u) = \infty$ for $u \in (1,\infty)$, then ces_{φ} is denoted by ces_{∞} and

$$ces_{\infty} = \{x \in \ell^0 : \sup_{n\in\mathbb{N}} \frac{1}{n}\sum_{i=1}^{n}|x(i)| < \infty\},$$

where $\|x\|_{ces_{\varphi}} = \|x\|^0_{ces_{\varphi}} = \sup_{n\in\mathbb{N}} \frac{1}{n}\sum_{i=1}^{n}|x(i)|$.

Note that, if we define in a similar way the space ces_{φ} for a function $\varphi : [0,\infty) \to [0,\infty)$ which is concave and such that $\varphi(0) = 0$, we get a trivial space if φ vanishes only at zero. Indeed, if there is $N \in \mathbb{N}$ such that $x(N) \neq 0$ then for any $\lambda > 0$, by concavity of φ we have

$$\begin{aligned} I_{ces_\varphi}(\lambda x) &= \sum_{n=1}^{\infty} \varphi\left(\frac{\lambda}{n}\sum_{i=1}^{n}|x(i)|\right) \geqslant \sum_{n=N}^{\infty} \varphi\left(\frac{\lambda}{n}\sum_{i=1}^{n}|x(i)|\right) \\ &\geqslant \sum_{n=N}^{\infty} \varphi\left(\frac{\lambda|x(N)|}{n}\right) \geqslant \sum_{n=N}^{\infty} \frac{1}{n}\varphi(\lambda|x(N)|) = \infty, \end{aligned}$$

hence $ces_\varphi = \{0\}$. It is well known that $ces_1 = \{0\}$ [45], which also follows from the fact that $\varphi(u) = u$ is concave and our remark mentioned above. Note that, if $\lim_{u\to 0^+} \varphi(u)/u > 0$ then $ces_\varphi = \{0\}$ since φ is then equivalent to a linear function and so $ces_\varphi = ces_1$. Hence as long as we deal with a non trivial space ces_φ we may assume that φ is an N-function (in the case when $\lim_{u\to\infty} \varphi(u)/u < \infty$ we can always find an N-function which is equivalent to φ).

Let φ be an Orlicz function. Let us mention some well known facts about the ces_φ spaces. The space ces_φ is not trivial if and only if for every $k > 0$ there exist $n_k \in \mathbb{N}$ such that $\sum_{n=n_k}^{\infty} \varphi\left(\frac{k}{n}\right) < \infty$ (this is also equivalent to the condition $\sum_{n=n_1}^{\infty} \varphi\left(\frac{1}{n}\right) < \infty$ for some $n_1 \in \mathbb{N}$) [21].

The space ces_φ (as well as ℓ_φ) is a Köthe sequence space with the Fatou property. For a proof of the Fatou property in ces_φ we refer to [21], and for details on Köthe spaces see [39].

Recall that the Matuszewska-Orlicz lower index α_φ and upper index β_φ of an Orlicz function φ are defined as follows

$$\alpha_\varphi = \sup\left\{p > 0 : \exists_{K>0,\, v:\varphi(v)>0} \forall_{0<t\leqslant 1,\ 0<\lambda\leqslant v}\, \varphi(\lambda t) \leqslant K t^p \varphi(\lambda)\right\},$$

$$\beta_\varphi = \inf\left\{p > 0 : \exists_{K>0,\, v:\varphi(v)>0} \forall_{0<t\leqslant 1,\ 0<\lambda\leqslant v}\, \varphi(\lambda t) \geqslant K t^p \varphi(\lambda)\right\}.$$

Note, that in the case when the function φ vanishes only at zero, we can write

$$\begin{aligned} \alpha_\varphi &= \sup\{p > 0 : \exists_{K>0}\,\forall_{0<\lambda,t\leqslant 1}\varphi(\lambda t) \leqslant Kt^p\varphi(\lambda)\} \\ \beta_\varphi &= \inf\{p > 0 : \exists_{K>0}\,\forall_{0<\lambda,t\leqslant 1}\varphi(\lambda t) \geqslant Kt^p\varphi(\lambda)\}. \end{aligned} \tag{4.1}$$

These indices were introduced by Matuszewska and Orlicz in [52] in a different, but equivalent way. By convexity of φ, $1 \leqslant \alpha_\varphi \leqslant \beta_\varphi \leqslant \infty$. It is well known that the condition $\beta_\varphi < \infty$ is equivalent to $\varphi \in \delta_2$, for an N-function φ we have $\alpha_\varphi^{-1} + \beta_{\varphi^*}^{-1} = \alpha_{\varphi^*}^{-1} + \beta_\varphi^{-1} = 1$ [49,50], and that $\alpha_\varphi > 1$ is equivalent to $\varphi^* \in \delta_2$ [49].

It turns out that $\alpha_\varphi = p(\ell_\varphi)$ and $\beta_\varphi = q(\ell_\varphi)$ where $p(\ell_\varphi)$ and $q(\ell_\varphi)$ are the lower and upper Boyd indices of the Orlicz sequence space ℓ_φ (see [48, Proposition 2.b.5 and Remark 2 on page 140]). We also have that the appropriate indices of two equivalent functions coincide. For necessary definitions and more information about Boyd indices, Matuszewska-Orlicz indices (as well as for relations among them) and Boyd indices of Orlicz spaces we refer to [14,35] and chapter 4 of [49]; see also two classical books [47,48].

For a bounded sequence x its decreasing rearrangement x^* is defined by

$$x^*(n) = \inf\{\,\lambda : \#\{i \in \mathbb{N} : |x(i)| > \lambda\} < n\}\,,\ n \in \mathbb{N},$$

where for a set A, $\#A$ denotes the number of elements in A if A is a finite set, or ∞ otherwise. Let us mention one important, direct consequence of Theorem 1 and its proof in [13].

Theorem 4.1. *The Hardy operator $H : \ell_\varphi \to \ell_\varphi$ defined by*

$$Hx(n) = \frac{1}{n}\sum_{i=1}^{n} x^*(i) \quad \textit{for all } n \in \mathbb{N},$$

where x^* is a decreasing rearrangement of x, is bounded if and only if $p(\ell_\varphi) > 1$.

As a further consequence of Theorem 4.1, we have the following.

It is clear that for any sequence $x \in ces_\varphi$, $\|x\|_{ces_\varphi} = \|Gx\|_\varphi$. Suppose that $p(\ell_\varphi) > 1$, then

$$\|Gx\|_\varphi \leqslant \|Gx^*\|_\varphi = \|Hx\|_\varphi \leqslant C\|x^*\|_\varphi = C\|x\|_\varphi,$$

hence $x \in \ell_\varphi$ implies $x \in ces_\varphi$. Conversely, suppose that there is $C > 0$ such that $\|Gx\|_\varphi \leqslant C\|x\|_\varphi$ for every $x \in \ell_\varphi$. Then it also holds for every x^*, and hence $\|Hx\|_\varphi \leqslant C\|x\|_\varphi$, which implies $p(\ell_\varphi) > 1$. Since $p(\ell_\varphi) = \alpha_\varphi$, we get

Corollary 4.2. *For any Orlicz function φ we have $\alpha_\varphi > 1$ if and only if $\ell_\varphi \subset ces_\varphi$. In particular, if $\alpha_\varphi > 1$ then $ces_\varphi \neq \{0\}$.*

The following example shows that it is possible that $\alpha_\varphi = 1$ and $ces_\varphi \neq \{0\}$, as well as that $ces_\varphi = \{0\}$ although φ is not equivalent to a linear function.

Example 4.3. Let $a \geqslant 1$ and

$$\varphi_a(t) = \begin{cases} 0 & \text{if } t = 0, \\ \frac{t}{(-\ln(t))^a} & \text{if } 0 < t \leqslant \frac{1}{e}, \\ \frac{1}{2}e(a^2+2a)t^2 + (1-a-a^2)t + \frac{a^2}{2e} & \text{if } t > \frac{1}{e}. \end{cases}$$

It is easy to see that each φ_a is a strictly convex N-function. Moreover, for $p > 1$ (applying de L'Hospital rule $\lceil a \rceil$ times), we get

$$\lim_{t\to 0^+} \frac{\varphi_a(\lambda t)}{\varphi_a(\lambda)t^p} = \lim_{t\to 0^+} \frac{(-\ln\lambda)^a t^{1-p}}{(-\ln \lambda t)^a} = \left(\frac{p-1}{a}\right)^{\lceil a \rceil} \lim_{t\to 0^+} \frac{(-\ln\lambda)^a t^{1-p}}{(\ln \lambda t)^{a-\lceil a \rceil}} = \infty$$

for all $\lambda > 0$. Hence $\sup_{0<\lambda,t\leqslant 1} \frac{\varphi_a(\lambda t)}{\varphi_a(\lambda)t^p} = \infty$ for all $p > 1$, which together with convexity of φ_a gives $\alpha_{\varphi_a} = 1$ (hence $\beta_{\varphi_a^*} = \infty$ which is equivalent to $\varphi_a^* \notin \delta_2$). We also have $\varphi_a \in \delta_2$, since

$$\limsup_{t\to 0^+} \frac{\varphi_a(2t)}{\varphi_a(t)} = \limsup_{t\to 0^+} \frac{2(-\ln t)^a}{(-\ln(2t))^a} \leqslant 2 < \infty.$$

Hence $\beta_{\varphi_a} < \infty$ (in fact $\beta_{\varphi_a} = 1$). For $a > 1$ the space $ces_{\varphi_a} \neq \{0\}$ since $\sum_{n=3}^{\infty} \varphi_a(\frac{1}{n}) = \sum_{n=3}^{\infty} \frac{1}{n(\ln(n))^a} < \infty$ by the integral test. Notice that $ces_{\varphi_1} = \{0\}$ since $\sum_{n=3}^{\infty} \frac{1}{n(\ln(n))} = \infty$, but the function φ_1 is not equivalent to a linear function.

The above example also shows that Matuszewska-Orlicz index α_φ is not fine enough to determine the validity of $ces_\varphi \neq \{0\}$.

4.2 The condition δ_2 in the Cesàro-Orlicz sequence space

Let φ be an Orlicz function and

$$(ces_\varphi)_a = \left\{ x \in ces_\varphi : \forall_{k>0} \exists_{n_k \in \mathbb{N}} \sum_{n=n_k}^{\infty} \varphi\left(\frac{k}{n} \sum_{i=1}^{n} |x(i)| \right) < \infty \right\}.$$

It turns out that $(ces_\varphi)_a$ is the subspace of all order continuous elements of ces_φ [21].

Remark 4.4. Let ψ be an Orlicz function, such that $\psi(u) > 0$ for all $u > 0$. For an Orlicz function φ defined by

$$\varphi(u) = \begin{cases} 0 & \text{for } u \in [0, a], \\ \psi(u-a) & \text{for } u > a, \end{cases}$$

for some $a > 0$, we have $ces_\varphi \neq (ces_\varphi)_a$. Indeed, taking $x = (a, a, \ldots)$ we have $I_{ces_\varphi}(x) = 0$ and so $x \in ces_\varphi$. Moreover $\sum_{n=n_2}^{\infty} \varphi\left(\frac{2}{n}\sum_{i=1}^{n} |x(i)|\right) = \sum_{n=n_2}^{\infty} \varphi(2a) = \infty$ for any $n_2 \in \mathbb{N}$, which implies that $x \notin (ces_\varphi)_a$.

Theorem 2.4 in [21] states that if $\varphi \in \delta_2$ then $ces_\varphi = (ces_\varphi)_a$. We will show the partial converse of this theorem.

Theorem 4.5. *If $(ces_\varphi)_a = ces_\varphi$ and $\alpha_\varphi > 1$ then $\varphi \in \delta_2$. In particular, if $\alpha_\varphi > 1$ then $(ces_\varphi)_a = ces_\varphi$ if and only if $\varphi \in \delta_2$.*

Proof. Let $(ces_\varphi)_a = ces_\varphi$ and $\alpha_\varphi > 1$. Note that we have $ces_\varphi \neq \{0\}$ by Corollary 4.2, and that $\varphi(u) > 0$ for all $u > 0$ by Remark 4.4. Let u_0 be such that $\varphi(2u_0) < \infty$. Suppose that $\varphi \notin \delta_2$. Then for all $K > 0$ and $u > 0$ there exists $v \in [0, u]$, such that $\varphi(2v) > K\varphi(v)$. So there exists $u_1 \in (0, u_0]$ such that $\varphi(2u_1) > 2\varphi(u_1)$. Let $c_1 = \lceil \frac{1}{2\varphi(u_1)} \rceil$. We can find a decreasing sequence $(u_n) \subset (0, u_0]$ such that

$$\varphi(2u_n) > 2^n \varphi(u_n). \tag{4.2}$$

Let

$$c_n = \left\lceil \frac{1}{2^n \varphi(u_n)} \right\rceil \text{ for } n > 1.$$

It follows that

$$\varphi(u_n) < \frac{\varphi(2u_n)}{2^n} \leqslant \frac{\varphi(2u_0)}{2^n} \text{ for all } n \in \mathbb{N}. \tag{4.3}$$

Let $c_0 = 0$. Define sets for $n \in \mathbb{N}$,

$$E_n = \{c_0 + c_1 + c_2 + \ldots + c_{n-1} + 1, c_0 + c_1 + \ldots + c_{n-1} + 2, \ldots, c_0 + c_1 + \ldots + c_n\}.$$

The sets E_n are pairwise disjoint, $\mu E_n = c_n$ for all $n \in \mathbb{N}$ and $\bigcup_{n=1}^{\infty} E_n = \mathbb{N}$.

Define the sequence $(x(i))_{i=1}^{\infty}$ by

$$x(i) = u_n \text{ if } i \in E_n \text{ for some } n \in \mathbb{N}.$$

We have that

$$
\begin{aligned}
I_\varphi(x) &= \sum_{i=1}^{\infty}\varphi(|x(i)|) = \sum_{n=1}^{\infty}\varphi(u_n)\mu E_n = \sum_{n=1}^{\infty}\varphi(u_n)\left\lceil\frac{1}{2^n\varphi(u_n)}\right\rceil \\
&\leqslant \sum_{n=1}^{\infty}\varphi(u_n)\left(\frac{1}{2^n\varphi(u_n)}+1\right) \leqslant \sum_{n=1}^{\infty}\left(\frac{1}{2^n}+\frac{\varphi(2u_0)}{2^n}\right) \\
&= 1+\varphi(2u_0) < \infty
\end{aligned}
$$

by inequality (4.3). Hence $x \in l_\varphi$, so $x \in ces_\varphi$ since $\ell_\varphi \subset ces_\varphi$ by the condition $\alpha_\varphi > 1$.

But, by inequality (4.2) and the fact that x is a decreasing sequence we have

$$
\begin{aligned}
\sum_{n=n_2}^{\infty}\varphi\left(\frac{2}{n}\sum_{i=1}^{n}|x(i)|\right) &\geqslant \sum_{n=n_2+1}^{\infty}\sum_{i\in E_n}\varphi(2u_n) \\
&= \sum_{n=n_2+1}^{\infty}\varphi(2u_n)\mu E_n \\
&> \sum_{n=n_2+1}^{\infty}2^n\varphi(u_n)\left\lceil\frac{1}{2^n\varphi(u_n)}\right\rceil \geqslant \sum_{n=n_2+1}^{\infty}1 = \infty
\end{aligned}
$$

for all $n_2 \in \mathbb{N}$, so $x \notin (ces_\varphi)_a$. Hence we get a contradiction.

The equivalence stated in the second part of this theorem follows clearly from the first part and from Theorem 2.4 in [21]. □

We would like to be able to replace condition $\alpha_\varphi > 1$ in Theorem 4.5 by $ces_\varphi \neq \{0\}$. It would be possible if we manage to show that the conditions $\alpha_\varphi = 1$ and $\beta_\varphi = \infty$ imply, either $ces_\varphi = \{0\}$ or $(ces_\varphi)_a \neq ces_\varphi$. Unfortunately these conditions do not imply $ces_\varphi = \{0\}$, which shows the following example.

Example 4.6. Let

$$p(t) = \begin{cases} 0 & \text{if } t = 0, \\ \frac{1}{n!} & \text{if } t \in \left[\frac{1}{(n+1)!}, \frac{1}{n!}\right) \text{ for } n \in \mathbb{N}, \\ t & \text{if } t \geqslant 1. \end{cases}$$

The function $\varphi(u) = \int_0^u p(t)dt$ is an N-function such that $\varphi \notin \delta_2$ and $\varphi^* \notin \delta_2$, that is $\beta_\varphi = \infty$ and $\alpha_\varphi = 1$. Indeed, let $u_n = \frac{1}{n!}$ for all $n \in \mathbb{N}$. It is easy to see that $u_n^2(1 - (n-1)^{-1}) \leqslant \varphi(u_n) \leqslant u_n^2$. We have that

$$\frac{\varphi(2u_n)}{\varphi(u_n)} > \frac{\int_{1/n!}^{2/n!} p(t)dt}{\varphi(u_n)} \geqslant \frac{(n!(n-1)!)^{-1}}{(n!n!)^{-1}} = n$$

and

$$\begin{aligned} \frac{\varphi\left(2^{-1}u_n\right)}{\varphi(u_n)} &\geqslant \frac{\varphi(u_n) - 2^{-1}\varphi(u_n) - (n!(n+1)!)^{-1}}{\varphi(u_n)} \\ &= \frac{1}{2} - \frac{(n!(n+1)!)^{-1}}{(n!)^{-2}(1-(n-1)^{-1})} \geqslant \frac{1}{2} - \frac{1}{n-2} \quad \text{for } n > 2. \end{aligned}$$

So $\varphi(2u_n) > n\varphi(u_n)$ for all $n > 2$, which gives $\varphi \notin \delta_2$. Similarly $\varphi^* \notin \delta_2$, since $\varphi(2^{-1}u_n)/\varphi(u_n) \to 2^{-1}$ as $n \to \infty$. We also have, for all $n \in \mathbb{N}$,

$$\begin{aligned} \varphi\left(\frac{1}{n!}\right) &= \sum_{k=n+1}^{\infty} \left(\frac{1}{(k-1)!} - \frac{1}{k!}\right) \frac{1}{(k-1)!} \\ &= \sum_{k=n}^{\infty} \left(\frac{1}{k!}\right)^2 \left(1 - \frac{1}{k+1}\right) < \sum_{k=n}^{\infty} \left(\frac{1}{k!}\right)^2, \end{aligned}$$

and

$$\sum_{m=n!}^{(n+1)!-1} \varphi\left(\frac{1}{m}\right) \leqslant ((n+1)! - n!)\, \varphi\left(\frac{1}{n!}\right) = nn!\varphi\left(\frac{1}{n!}\right).$$

Hence

$$\sum_{n=1}^{\infty}\varphi\left(\frac{1}{n}\right)=\sum_{n=1}^{\infty}\sum_{m=n!}^{(n+1)!-1}\varphi\left(\frac{1}{m}\right)\leqslant\sum_{n=1}^{\infty}\sum_{k=n}^{\infty}\frac{nn!}{(k!)^2}$$
$$\leqslant e+\sum_{n=2}^{\infty}\frac{1}{(n-1)!}\left(1+\frac{1}{(n+1)^2}+\frac{1}{(n+1)^2(n+2)^2}+\dots\right)$$
$$\leqslant e+\sum_{n=2}^{\infty}\frac{1}{(n-1)!}\sum_{k=1}^{\infty}\frac{1}{n^k}=e+\sum_{n=1}^{\infty}\frac{1}{n!n}<2e<\infty,$$

which means that $ces_\varphi\neq\{0\}$ (note that also $ces_{\varphi^*}\neq\{0\}$).

4.3 The comparison theorem for the Cesàro-Orlicz sequence spaces

We start this section with a basic observation.

Proposition 4.7. *Let φ_1 and φ_2 be Orlicz functions. If there exist b, $t_0>0$ such that $\varphi_2(t_0)>0$ and $\varphi_2(t)\leqslant\varphi_1(bt)$ for all $t\in[0,t_0]$ then $ces_{\varphi_1}\subset ces_{\varphi_2}$.*

Proof. We may assume that $b\geqslant 1$, and substituting $u=bt$ we get that

$$\varphi_2(b^{-1}u)\leqslant\varphi_1(u)\quad\text{for all } u\in[0,bt_0]. \tag{4.4}$$

Let $x\in ces_{\varphi_1}$, i.e. there exists $\lambda>0$, such that $I_{ces_{\varphi_1}}(\lambda x)<\infty$. The set $A_x=\left\{n\in\mathbb{N}:\frac{\lambda}{n}\sum_{i=1}^{n}|x(i)|>bt_0\right\}$ is finite, because otherwise we would get

$$I_{ces_{\varphi_1}}(\lambda x)\geqslant\sum_{n\in A_x}\varphi_1\left(\frac{\lambda}{n}\sum_{i=1}^{n}|x(i)|\right)>\sum_{n\in A_x}\varphi_1\left(bt_0\right)>\sum_{n\in A_x}\varphi_2(t_0)=\infty$$

by the inequality (4.4). Taking $\tilde{\lambda}=\frac{c}{b}$ for c small enough, we get that $I_{ces_{\varphi_2}}(\tilde{\lambda}x)\leqslant I_{ces_{\varphi_1}}(cx)\leqslant I_{ces_{\varphi_1}}(\lambda x)<\infty$, and so $x\in ces_{\varphi_2}$. $\square$

Corollary 4.8. *If functions φ_1 and φ_2 are equivalent then $ces_{\varphi_1} = ces_{\varphi_2}$ as sets and the norms of these spaces are equivalent.*

Proof. It follows from Proposition 4.12 and the fact that ces_φ is a Köthe sequence (function) space with the Fatou property for any Orlicz function φ (see [8, Theorem 1.8] and [39]). $\square$

We use a similar approach as in the proof of Theorem 4.5 to show the following comparison theorem.

Theorem 4.9. *Let φ_1 and φ_2 be Orlicz functions such that $\varphi_1(u) > 0$, $\varphi_2(u) > 0$ for all $u > 0$ and $\alpha_{\varphi_1} > 1$. If $ces_{\varphi_1} \subset ces_{\varphi_2}$ then there exist $b > 0$ and $t_0 > 0$ such that $\varphi_2(t) \leqslant \varphi_1(bt)$ for all t with $0 < t \leqslant t_0$.*

Proof. By convexity of φ_2, the condition stated in the hypothesis is equivalent to the following one: there exist a, b, $u_0 > 0$ such that

$$\varphi_2(u) \leqslant a\varphi_1(bu) \quad \text{for all } u \in (0,\, u_0]. \tag{4.5}$$

Assume that $ces_{\varphi_1} \subset ces_{\varphi_2}$. Suppose that condition (4.5) is not satisfied. Let $u_0 > 0$ be such that $\varphi_2(u_0) < \infty$. We can find decreasing sequence $(u_n)_{n=1}^{\infty} \subset (0, u_0]$ such that $\varphi_2(u_1) > 2\varphi_1(2u_1)$ and

$$\varphi_2(u_n) > (c_1 + c_2 + \ldots + c_{n-1})2^n \varphi_1((c_1 + c_2 + \ldots + c_{n-1})2^n u_n), \tag{4.6}$$

where $c_1 = \left\lceil \frac{1}{2\varphi_1(2u_1)} \right\rceil$, and

$$c_n = \left\lceil \frac{1}{(c_1 + \ldots + c_{n-1})2^n \varphi_1((c_1 + \ldots + c_{n-1})2^n u_n)} \right\rceil \quad \text{for} \quad n > 1. \tag{4.7}$$

We have $1 \leqslant c_n < \infty$ for all $n \in \mathbb{N}$ by (4.6) and since $\varphi_1(u) > 0$ for all $u > 0$. Denote $d_1 = 1$ and $d_n = \sum_{i=1}^{n-1} c_i$ for $n > 1$. Note that the sequence $(d_j)_{j=1}^{\infty}$ is strictly increasing. Define the sets $E_1 = \{1, 2, \ldots, c_1\}$ and $E_n = \{d_n + 1, d_n + 2, \ldots, d_{n+1}\}$ for $n > 1$. The sets E_n are pairwise disjoint, $\mu E_n = c_n$ for all $n \in \mathbb{N}$ and $\bigcup_{n=1}^{\infty} E_n = \mathbb{N}$. Define the sequence $(x(i))_{i=1}^{\infty}$ by

$$x(i) = d_n 2^n u_n \quad \text{if} \quad i \in E_n \quad \text{for some} \quad n \in \mathbb{N}.$$

We have that

$$\begin{aligned} I_{\varphi_1}(x) &= \sum_{n=1}^{\infty} \varphi_1(d_n 2^n u_n) \mu E_n \leqslant \sum_{n=1}^{\infty} \varphi_1(d_n 2^n u_n) \left(\frac{1}{d_n 2^n \varphi_1(d_n 2^n u_n)} + 1 \right) \\ &\leqslant \sum_{n=1}^{\infty} \left(2^{-n} + 2^{-n} \varphi_2(u_0) \right) = 1 + \varphi_2(u_0) < \infty, \end{aligned}$$

by inequality (4.6). Hence $x \in l_{\varphi_1}$, so $x \in ces_{\varphi_1}$ since $\ell_{\varphi_1} \subset ces_{\varphi_1}$ by the assumption $\alpha_{\varphi_1} > 1$.

Let now $\lambda > 0$. Since for $n \in E_j$ we have

$$\frac{n - d_j}{n} d_j = \left(1 - \frac{d_j}{n}\right) d_j \geqslant \left(1 - \frac{d_j}{d_j + 1}\right) d_j = \frac{d_j}{d_j + 1} \to 1 \quad \text{as } j \to \infty.$$

The latter and inequality (4.6) imply that

$$\begin{aligned} I_{ces_{\varphi_2}}(\lambda x) &= \sum_{j=1}^{\infty} \sum_{n \in E_j} \varphi_2 \left(\frac{\lambda}{n} \sum_{i=1}^{n} |x(i)| \right) \geqslant \sum_{j=1}^{\infty} \sum_{n \in E_j} \varphi_2 \left(\frac{\lambda}{n} \sum_{i=d_j+1}^{n} d_j 2^j u_j \right) \\ &= \sum_{j=1}^{\infty} \sum_{n \in E_j} \varphi_2 \left(\lambda \frac{n - d_j}{n} d_j 2^j u_j \right) \geqslant \sum_{j=j_0}^{\infty} \varphi_2 \left(\lambda \frac{1}{2} 2^j u_j \right) \mu E_j \\ &\geqslant \sum_{j=j_1}^{\infty} \varphi_2 (u_j) \mu E_j > \sum_{j=j_1}^{\infty} d_j 2^j \varphi_1 \left(d_j 2^j u_j \right) \mu E_j = \sum_{j=j_1}^{\infty} 1 = \infty \end{aligned}$$

for sufficiently large indices $j_1 \geqslant j_0 \geqslant 1$. Since λ was arbitrary, we get $x \notin ces_{\varphi_2}$, and this gives a contradiction. □

Corollary 4.10. *Let φ_1 and φ_2 be Orlicz functions such that $\varphi_1(u) > 0$, $\varphi_2(u) > 0$ for all $u > 0$, and $\alpha_{\varphi_1} > 1$. The following conditions are equivalent.*

(i) $ces_{\varphi_1} \subset ces_{\varphi_2}$.

(ii) *There exist b, $t_0 > 0$ such that $\varphi_2(t) \leqslant \varphi_1(bt)$ for all $t \in [0, t_0]$.*

(iii) *There exists $C > 0$ such that $\|x\|_{ces_{\varphi_2}} \leqslant C\|x\|_{ces_{\varphi_1}}$ for all $x \in ces_{\varphi_1}$.*

Proof. It follows from Lemma 4.12, Corollary 4.8 and Theorem 4.9. □

4.4 Order isometric copy of ℓ_∞ in Cesàro-Orlicz sequence spaces

In this section we provide some sufficient conditions under which the space ces_φ contains an order isometric copy of ℓ_∞. Recall that if $\varphi \in \delta_2$ then ces_φ is order continuous [21, Theorem 2.4], and thus in view of [39, Theorem 4, p. 295], ces_φ does not contain any isomorphic copy of ℓ_∞. On the other hand, if $\varphi \notin \delta_2$ and $\alpha(\varphi) > 1$ then ces_φ is not order continuous (Theorem 4.5) and again by [39, Theorem 4, p. 295], ces_φ contains an order isomorphic copy of ℓ_∞. It is also well known that the Orlicz sequence space ℓ_φ under the Luxemburg norm has an order isometric copy of ℓ_∞ if and only if $\varphi \notin \delta_2$ [36]. It is expected that the similar result remains true in the case of ces_φ spaces. We prove here the desired result for quite large family of Orlicz functions, namely for $\varphi \notin \delta_2$ and such that the Orlicz class is closed under the operation G, in particular for such φ that $\varphi^{1/p}$ is strongly equivalent to a convex function for some $p > 1$.

We start with the main result in this section.

Theorem 4.11. *If $\varphi \notin \delta_2$ and the Orlicz class $\{x : I_\varphi(x) < \infty\}$ is closed under the averaging operator G, then ces_φ contains an order isometric copy of ℓ_∞.*

Proof. We will consider two cases. First we assume that $\varphi(u) > 0$ for all $u > 0$. Let $b = \sup\{u \geqslant 0 : \varphi(u) < \infty\}$. It is well known that $\varphi \in \delta_2$ if and only if there exist $L > 1$ and $K, u_0 > 0$ such that $\varphi(u_0) > 0$, and $\varphi(Lu) \leqslant K\varphi(u)$ for all $u \in [0, u_0]$ (see [16] p. 9) Assuming that $\varphi \notin \delta_2$, for all $L > 1$ there exists a decreasing sequence $(u_n)_{n=1}^\infty \subset (0, L^{-1}b)$, depending on L, such that $\varphi(Lu_n) > K_n\varphi(u_n)$ where the sequence $(K_n)_{n=1}^\infty$ can be chosen to have a property that $\sum_{n=1}^\infty K_n^{-1} < \infty$. Clearly $0 < \varphi(Lu_n) < \infty$ for every $n \in \mathbb{N}$. Hence

$$\sum_{n=1}^{\infty} \frac{\varphi(u_n)}{\varphi(Lu_n)} < \sum_{n=1}^{\infty} \frac{1}{K_n} < \infty. \tag{4.8}$$

Note that the above gives $u_n \to 0$ as $n \to \infty$. Indeed, otherwise we would be able to find a subsequence $(u_{n_k})_{k=1}^\infty$ such that for all $k \in \mathbb{N}$ and for some $\epsilon > 0$, $\varphi(u_{n_k}) \geqslant \epsilon$. Then for all $k \in \mathbb{N}$,

$$\frac{\varphi(u_{n_k})}{\varphi(Lu_{n_k})} \geqslant \frac{\epsilon}{\varphi(Lu_{n_1})} > 0,$$

which would contradict (4.8).

Let $(\epsilon_m)_{m=1}^\infty$ be any positive, decreasing sequence converging to zero. For any $m \in \mathbb{N}$ let $(K_n^{(m)})_{n=1}^\infty$ be a sequence of positive real numbers such that

$$\sum_{n=1}^{\infty} \frac{1}{K_n^{(m)}} \leqslant \frac{1}{2^{m+1}}.$$

Now by the first part, for any $m \in \mathbb{N}$ we can find a decreasing sequence $(u_n^{(m)})_{n=1}^\infty \subset (0, (1+\epsilon_m)^{-1}b)$ such that $u_n^{(m)} \to 0$ as $n \to \infty$, and

$$\varphi((1+\epsilon_m)u_n^{(m)}) > K_n^{(m)}\varphi(u_n^{(m)})$$

for all $m \in \mathbb{N}$. Thus for all $m \in \mathbb{N}$,

$$\sum_{n=1}^{\infty} \frac{\varphi(u_n^{(m)})}{\varphi((1+\epsilon_m)u_n^{(m)})} < \sum_{n=1}^{\infty} \frac{1}{K_n^{(m)}} \leqslant \frac{1}{2^{m+1}}.$$

In view of $u_n^{(m)} \to 0$ as $n \to \infty$, we can find a subsequence $(n_k) \subset \mathbb{N}$ such that $u_{n_1}^{(1)} > u_{n_2}^{(2)} > u_{n_3}^{(3)} > \ldots$. Hence without loss of generality we can assume that for $m > 1$, $u_m^{(m)} < u_{m-1}^{(m-1)}$. Let

$$c_n = \left\lceil \frac{1}{\varphi\left((1+\epsilon_n)u_n^{(n)}\right)} \right\rceil$$

for $n \in \mathbb{N}$. Note that $1 \leqslant c_n < \infty$ for all $n \in \mathbb{N}$. Define $x(i) = u_n^{(n)}$ whenever $i \in [c_1 + c_2 + \ldots + c_{n-1} + 1, \ldots, c_1 + c_2 + \ldots + c_n]$. It is clear that the sequence x is decreasing. We have

$$\begin{aligned} I_\varphi(x) &= \sum_{i=1}^{\infty} \varphi(|x(i)|) = \sum_{n=1}^{\infty} c_n \varphi(u_n^{(n)}) = \sum_{n=1}^{\infty} \left\lceil \frac{1}{\varphi\left((1+\epsilon_n)u_n^{(n)}\right)} \right\rceil \varphi\left(u_n^{(n)}\right) \\ &\leqslant \sum_{n=1}^{\infty} \frac{\varphi\left(u_n^{(n)}\right)}{\varphi\left((1+\epsilon_n)u_n^{(n)}\right)} + \varphi\left(u_n^{(n)}\right) \leqslant 2 \sum_{n=1}^{\infty} \frac{\varphi\left(u_n^{(n)}\right)}{\varphi\left((1+\epsilon_n)u_n^{(n)}\right)} \leqslant 2 \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} < \infty. \end{aligned}$$

By the assumption we get $I_{ces_\varphi}(x) < \infty$. For any $\epsilon > 0$ and $M \in \mathbb{N}$ we obtain that

$$\begin{aligned} \sum_{i=M}^{\infty} \varphi((1+\epsilon)|x(i)|) &\geqslant \sum_{n=M'}^{\infty} c_n \varphi\left((1+\epsilon)u_n^{(n)}\right) \\ &\geqslant \sum_{n=M'}^{\infty} \frac{\varphi\left((1+\epsilon)u_n^{(n)}\right)}{\varphi\left((1+\epsilon_n)u_n^{(n)}\right)} \geqslant \sum_{n=M''}^{\infty} 1 = \infty \end{aligned} \tag{4.9}$$

for some $M'' \geqslant M' \geqslant M$. Denoting by $x|_{\{N,N+1,\ldots\}}$ the sequence which is equal to x

on $\{N, N+1, \ldots\}$ and 0 otherwise, we see that for N large enough

$$I_{ces_\varphi}(x|_{\{N,N+1,\ldots\}}) = \sum_{n=N}^{\infty} \varphi\left(\frac{1}{n}\sum_{i=N}^{n}|x(i)|\right) \leqslant \sum_{n=N}^{\infty} \varphi\left(\frac{1}{n}\sum_{i=1}^{n}|x(i)|\right) \leqslant 1,$$

since $I_{ces_\varphi}(x) < \infty$. Let $y = x|_{\{N,N+1,\ldots\}}$. Since x is decreasing, for any $\epsilon > 0$,

$$\begin{aligned}(1+\epsilon)\frac{1}{n}\sum_{i=N}^{n}|x(i)| &\geqslant (1+\epsilon)\left(\frac{n-N}{n}\right)|x(n)| \\ &= (1+\epsilon)\left(1-\frac{N}{n}\right)|x(n)| \geqslant \left(1+\frac{\epsilon}{2}\right)|x(n)|\end{aligned} \tag{4.10}$$

for n large enough. Now, by (4.9) and (4.10), for any $\epsilon > 0$ and N' large enough

$$I_{ces_\varphi}((1+\epsilon))y) \geqslant \sum_{n=N}^{\infty} \varphi\left(\frac{1+\epsilon}{n}\sum_{i=N}^{n}|x(i)|\right) \geqslant \sum_{n=N'}^{\infty} \varphi\left(\left(1+\frac{\epsilon}{2}\right)|x(n)|\right) = \infty.$$

We have constructed an element $y \in ces_\varphi$ such that $I_{ces_\varphi}(y) \leqslant 1$ and for every $\epsilon > 0$, $I_{ces_\varphi}((1+\epsilon)y) = \infty$. We observe that the subspace $(ces_\varphi)_a$ of all order continuous elements in ces_φ and the closure of the set of sequences with finite number of non-zero coordinates coincide (Theorem 2.3 in [21]). Now it is not difficult to see that the distance of y to $(ces_\varphi)_a$ is 1, since the above calculations hold true for arbitrary large N. By applying Theorem 2 from [31] we conclude that ces_φ contains an order linearly isometric copy of ℓ_∞.

Now assume there exists $a > 0$ such that $\varphi(u) = 0$ for $u \in [0, a]$ and $\varphi(u) > 0$ for $u > a$. Taking $x = (a, a, a, \ldots)$, it is easy to see that $I_{ces_\varphi}(x) = 0$. Moreover $I_{ces_\varphi}((1+\epsilon)(x-s)) = \infty$ for all $\epsilon > 0$ and all sequences $s \in \ell^0$ with finite support. Indeed, taking $n_0 = \max\{i \in \mathbb{N} : s(i) \neq 0\}$ and denoting $y = (1+\epsilon)(x-s)$ we see that

$$Gy(n) \geqslant \frac{n-n_0}{n}(1+\epsilon)a = \left(1-\frac{n_0}{n}\right)(1+\epsilon)a \geqslant \left(1+\frac{\epsilon}{2}\right)a,$$

for n large enough. Thus for some $N \in \mathbb{N}$,

$$I_{ces_\varphi}(y) \geqslant \sum_{n=N}^{\infty} \varphi(Gy(n)) \geqslant \sum_{n=N}^{\infty} \varphi\left(\left(1+\frac{\epsilon}{2}\right)a\right) = \infty.$$

It follows that $\|x-s\|_{ces_\varphi} = 1$ and so the distance of x to $(ces_\varphi)_a$ is 1. We finish the proof analogously as before by applying Theorem 2 from [31]. $\square$

Recall that the classical Hardy inequality for $p > 1$ reads [43]

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|x(i)|\right)^p \leqslant \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty}|x(n)|^p \quad \text{for all } x \in \ell^0.$$

The following proposition shows the connections between the sufficient condition under which ces_φ has an order isometric copy of ℓ_∞ and some other conditions which are easier to check.

Proposition 4.12. *Let φ be an Orlicz function. Consider the following conditions.*

(i) *There exist $p > 1$, a convex function γ and constants $A, B, u_0 > 0$ such that $\varphi(u_0) > 0$ and for $0 < u \leqslant u_0$,*

$$A\gamma(u) \leqslant \varphi(u)^{1/p} \leqslant B\gamma(u).$$

(ii) *There exist constants $C, u_0 > 0$ such that $\varphi(u_0) > 0$ and*

$$\sum_{n=1}^{\infty}\varphi\left(\frac{1}{n}\sum_{i=1}^{n}|x(i)|\right) \leqslant C\sum_{n=1}^{\infty}\varphi(|x(n)|)$$

for all $x \in \ell^0$ with $\|x\|_\infty = \sup_n |x(n)| \leqslant u_0$.

(iii) *The Orlicz class $\{x : I_\varphi(x) < \infty\}$ is closed under the averaging operator G.*

(iv) $\alpha_\varphi > 1$.

(v) *There exists* $n_0 \in \mathbb{N}$ *such that* $\sum_{n=n_0}^{\infty} \varphi\left(\frac{1}{n}\right) < \infty$.

We have the implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v).

Proof. If $\varphi(u) = 0$ for $u \in [0, a]$ and $\varphi(u) = \infty$ for $u > a$, where $a > 0$ then all conditions (i)-(v) are satisfied. In the remaining case, without loss of generality we can assume that the constant u_0 which appears in (i) and (ii) is such that $0 < \varphi(u_0) < \infty$, and that the function γ is finite on $[0, \infty)$. Now, the implication (i) $\Longrightarrow$ (ii) follows by Jensen's inequality and Hardy's inequality for $p > 1$. In fact

$$\begin{aligned}
\sum_{n=1}^{\infty} \varphi\left(\frac{1}{n}\sum_{i=1}^{n}|x(i)|\right) &\leqslant B^p \sum_{n=1}^{\infty}\left(\gamma\left(\frac{1}{n}\sum_{i=1}^{n}|x(i)|\right)\right)^p \\
&\leqslant B^p \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\gamma(|x(i)|)\right)^p \leqslant \frac{B^p}{A^p}\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\varphi(|x(i)|)^{1/p}\right)^p \\
&\leqslant \left(\frac{pB}{(p-1)A}\right)^p \sum_{n=1}^{\infty}\varphi(|x(n)|).
\end{aligned}$$

Now we show implication (ii) $\Longrightarrow$ (iii). Let $a = \sup\{u \geqslant 0 : \varphi(u) = 0\}$, and let $x \in \ell^0$ be such that $I_\varphi(x) < \infty$. For every $\epsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that $|x(n)| \leqslant a + \epsilon$, for $n \geqslant N_1$. Taking $\epsilon = (u_0 - a)/2$ and $n \geqslant N_1$ we get that $|x(n)| \leqslant u_0$ and

$$Gx(n) \leqslant \frac{1}{n}\sum_{i=1}^{N_1}|x(i)| + \left(\frac{n-N_1}{n}\right)(a+\epsilon) = \frac{a}{2} + \frac{u_0}{2} + \frac{1}{n}\left(\sum_{i=1}^{N_1}|x(i)| - N_1\frac{a+u_0}{2}\right).$$

Thus there exists $N \in \mathbb{N}$ such that $Gx(n) \leqslant u_0$ and $|x(n)| \leqslant u_0$ for all $n \geqslant N$. Let $y = (\underbrace{Gx(N), \ldots, Gx(N)}_{N \text{ times}}, x(N+1), x(N+2), \ldots)$. Clearly $\|y\|_\infty \leqslant u_0$ and $Gy(n) =$

$Gx(n)$ for all $n \geqslant N$. By (ii) applied to y we get that

$$\sum_{n=N+1}^{\infty} \varphi(Gx(n)) \leqslant \sum_{n=1}^{\infty} \varphi(Gy(n)) \leqslant C \sum_{n=1}^{\infty} \varphi(y(n))$$
$$= C\left(N\varphi(Gx(N)) + \sum_{n=N+1}^{\infty} \varphi(x(n))\right) < \infty.$$

Since $\|Gx\|_\infty \leqslant \|x\|_\infty$ we get $I_\varphi(Gx) < \infty$.

(iii) $\Longrightarrow$ (iv). Let $x \in \ell_\varphi$, that is $I_\varphi(\lambda x) < \infty$ for some $\lambda > 0$. By (iii) we get that $\infty > I_\varphi(G(\lambda x)) = I_{ces_\varphi}(\lambda x)$ and so $x \in ces_\varphi$. Hence $\ell_\varphi \subset ces_\varphi$ which is equivalent to $\alpha_\varphi > 1$ by Corollary 4.2.

(iv) $\Longrightarrow$ (v). Follows from Corollary 4.2. $\square$

An immediate consequence of Theorem 4.11 and Proposition 4.12 is the following corollary.

Corollary 4.13. *If $\varphi \notin \delta_2$ and for some $p > 1$ the function $\varphi^{1/p}$ is strongly equivalent to a convex function, then the Cesàro sequence space ces_φ contains an order isometric copy of ℓ_∞.*

Remark 4.14.

1. In the case when $1 < \alpha(\varphi) \leqslant \beta(\varphi) < \infty$, the condition (i) of Proposition 4.12 is satisfied for $1 < p < \alpha(\varphi)$. It follows from [35, Theorem 1.7] applied to function $\varphi^{1/p}$, since $\alpha(\varphi^{1/p}) = \alpha(\varphi)/p > 1$ and $\beta(\varphi^{1/p}) = \beta(\varphi)/p < \infty$.

2. Levinson showed in [46] that the composition $\varphi^{1/p}$ is convex for $p > 1$ if an Orlicz function φ is twice differentiable and

$$\varphi(u)\varphi^{''}(u) \geqslant (1 - 1/p)(\varphi^{'}(u))^2. \tag{4.11}$$

Clearly, it is sufficient that condition (4.11) is satisfied in a neighborhood of zero in order to get condition (i) of Proposition 4.12. This condition is satisfied for example by the following functions ($a > 0$),

$$\varphi_a(u) = \begin{cases} 0 & \text{if } \ u = 0, \\ e^{-u^{-a}} & \text{if } \ 0 < u \leqslant (a(a+1)^{-1})^{1/a}, \\ \left(ea(a+1)^{-1}\right)^{-(a+1)a^{-1}} u^{a+1} & \text{if } \ u > (a(a+1)^{-1})^{1/a}, \end{cases}$$

whenever $p > 1$. Indeed, for u small enough

$$\varphi_a(u)\varphi_a^{''}(u) - \left(1 - \frac{1}{p}\right)(\varphi_a^{'}(u))^2 = \left(\frac{ae^{-u^{-a}}}{u^{a+1}}\right)^2 \left(\frac{1}{p} - \frac{a+1}{a}u^a\right) \geqslant 0.$$

Note that $\beta(\varphi_a) = \infty$ and $\alpha(\varphi_a) > 1$ [51].

Similarly, condition (4.11) in a neighborhood of zero for any $p > 1$ is satisfied by functions $\psi_a(u) = u^{1+a}\ln(1 + e^{-u^{-a}})$, $a > 0$. Indeed, for $u > 0$ small enough

$$\begin{aligned} \psi_a(u)\psi_a^{''}(u) - \left(1 - \frac{1}{p}\right)(\psi_a^{'}(u))^2 = \left(\frac{a}{1 + e^{u^{-a}}}\right)^2 \times \\ \times \Bigg[\frac{(a+1)}{a^2}u^{2a}\left(1 + e^{u^{-a}}\ln(1 + e^{-u^{-a}})\right)^2\left(\frac{a+1}{p} - 1\right) \\ + (a+1)u^a \ln(1 + e^{-u^{-a}})\frac{1 + e^{u^{-a}}}{a}\left(\frac{2}{p} - 1\right) + e^{u^{-a}}\ln(1 + e^{-u^{-a}}) - 1 + \frac{1}{p}\Bigg] \geqslant 0, \end{aligned}$$

since $e^{u^{-a}}\ln(1 + e^{-u^{-a}}) \to 1$ as $u \to 0^+$. We also have $\beta(\psi_a) = \infty$ and $\alpha(\psi_a) > 1$ [51].

Thus by Corollary 4.13, the Orlicz functions φ_a and ψ_a generate the Cesàro spaces ces_{φ_a} and ces_{ψ_a} such that both contain order isometric copies of ℓ_∞.

The following example shows that in general, condition (iv) of Proposition 4.12, ($\alpha_\varphi > 1$), is not necessary for the existence of order isometric copy of ℓ_∞ in ces_φ.

Example 4.15. Let function φ be defined as in Example 4.6. We have that $\varphi, \varphi^* \notin \delta_2$, that is $\beta_\varphi = \infty$ and $\alpha_\varphi = 1$. Also $ces_\varphi \neq \{0\}$.

Recall that $\varphi(u) = \int_0^u p(t)dt$, where

$$p(t) = \begin{cases} 0 & \text{if } t = 0, \\ \frac{1}{n!} & \text{if } t \in \left[\frac{1}{(n+1)!}, \frac{1}{n!}\right) \text{ for } n \in \mathbb{N}, \\ t & \text{if } t \geqslant 1. \end{cases}$$

Now we will make preparation to define $x \in ces_\varphi$ such that $I_{ces_\varphi}(x) \leqslant 1$ and $I_{ces_\varphi}((1+\epsilon)x) = \infty$ for all $\epsilon > 0$. Let for $m = 4, 5, \ldots$,

$$c_m = 3! + \sum_{k=4}^{m} (k-3)!(k! - (k-1)!),$$

and let for $m = 4, 5, \ldots$ and $n = 0, 1, 2, \ldots, (m+1)! - m! - 1$,

$$E_{m,n} = \{c_m + (m-2)!n + 1, c_m + (m-2)!n + 2, \ldots, c_m + (m-2)!(n+1)\}.$$

The sets $E_{m,n}$ are pairwise disjoint and their union and the set $\{1, 2, \ldots, c_4\}$ gives the whole $\mathbb{N}$. Note that there are exactly $(m-2)!$ integers in each $E_{m,n}$, and that for every integer $r > 24 = c_4$, there exists a unique triple (m, n, j) of non-negative integers satisfying $m \geqslant 4$, $0 \leqslant n \leqslant (m+1)! - m! - 1$, and $1 \leqslant j \leqslant (m-2)!$ such that $r = c_m + (m-2)!n + j$.

Now we will construct $x \in \ell^0$ such that the sequence $(Gx(n))_{n=1}^\infty$ is decreasing for large n and

$$Gx(c_m + (m-2)!n) = \frac{1}{m! + n} \tag{4.12}$$

for $m = 4, 5, \ldots$ and $n = 0, 1, \ldots, (m+1)! - m! - 1$.

Let $\tilde{s}_{m,n}$ be the solution of

$$\frac{(c_m + (m-2)!n)Gx(c_m + (m-2)!n)}{c_m + (m-2)!n + \tilde{s}_{m,n}} = \frac{1}{m! + n + 1},$$

that is

$$\tilde{s}_{m,n} = (c_m + (m-2)!n)(m! + n)^{-1}$$

for $m = 4, 5, \ldots$ and $n = 0, 1, \ldots, (m+1)! - m! - 1$. Letting

$$s_{m,n} = \lceil \tilde{s}_{m,n} \rceil,$$

the number $s_{m,n}$ is the smallest integer such that

$$\frac{(c_m + (m-2)!n)Gx(c_m + (m-2)!n)}{c_m + (m-2)!n + s_{m,n}} \leqslant \frac{1}{m! + n + 1}.$$

Moreover, we can show by induction that $s_{m,n} \leqslant (m-2)!$ for $n = 0, 1, \ldots, (m+1)! - m! - 1$, since $c_m \leqslant m!(m-2)!$ for $m = 4, 5, \ldots$.

We now define x as $x(r) = 0$ for $r = 1, 2, \ldots, c_4 - 1$, $x(c_4) = 1$, and

$$x(r) = \begin{cases} 0, & \text{if } r \in E_{m,n} \text{ and } r < c_m + (m-2)!n + s_{m,n}, \\ (c_m + (m-2)!n + s_{m,n})(m! + n + 1)^{-1} & \\ \quad -(c_m + (m-2)!n)(m! + n)^{-1}, & \text{if } r = c_m + (m-2)!n + s_{m,n}, \\ (m! + n + 1)^{-1}, & \text{if } r \in E_{m,n} \text{ and } r > c_m + (m-2)!n + s_{m,n}. \end{cases}$$

We will show that x satisfies (4.12). We proceed by induction. Observe that $Gx(c_4) = \frac{1}{4!}$ and that the condition $Gx(c_m + (m-2)!n) = \frac{1}{m!+n}$ implies that $Gx(c_m + (m-2)!(n+1)) = \frac{1}{m!+n+1}$ for all $m = 4, 5, \ldots$ and $n = 0, 1, 2, \ldots, (m+1)! - m! - 1$.

Indeed,

$$
\begin{aligned}
&Gx(c_m+(m-2)!(n+1)) = \frac{1}{c_m+(m-2)!(n+1)}\times \\
&\times\left((c_m+(m-2)!n)Gx(c_m+(m-2)!n)+\frac{c_m+(m-2)!n+s_{m,n}}{m!+n+1}-\frac{c_m+(m-2)!n}{m!+n}\right. \\
&\left.+\frac{c_m+(m-2)!(n+1)-(c_m+(m-2)!n+s_{m,n})}{m!+n+1}\right)=\frac{1}{c_m+(m-2)!(n+1)}\times \\
&\times\left(\frac{c_m+(m-2)!n}{m!+n}-\frac{c_m+(m-2)!n}{m!+n}+\frac{c_m+(m-2)!(n+1)}{m!+n+1}\right)=\frac{1}{m!+n+1}.
\end{aligned}
$$

It follows, that for $n=(m+1)!-m!-1$ we have $Gx(c_{m+1})=\frac{1}{(m+1)!}$, which gives (4.12).

Clearly, Gx is decreasing on $\{c_4, c_4+1,\ldots\}$ by the choice of $s_{m,n}$. Moreover, the sequence x has the properties that $y\leqslant Gx\leqslant y+z$, where

$$
\begin{aligned}
&y(n)=Gx(n),\ \text{and}\ z(n)=0 \text{ for } n=1,2,\ldots,c_4,\\
&y(r)=\frac{1}{m!+n+1}\ \text{and}\\
&z(r)=\frac{1}{m!+n}-\frac{1}{m!+n+1}\ \text{for all } m=4,5,\ldots,\ n=0,1,\ldots,(m+1)!-m!-1\\
&\qquad\text{whenever } r\in E_{m,n}.
\end{aligned}
$$

Hence for all $m=4,5,\ldots$ we have that

$$z(r)\leqslant\frac{1}{m!(m!+1)}$$

and

$$\frac{1}{m!+n+1}+z(r)\leqslant\frac{1}{m!}$$

whenever $r\in E_{m,n}, n=0,1,\ldots,(m+1)!-m!-1.$

Applying the above relations and the definition of φ, we obtain that

$$\begin{aligned}
I_{ces_\varphi}(x) &= I_\varphi(Gx) \leqslant I_\varphi(y+z) \\
&= \varphi\left(\frac{1}{4!}\right) + \sum_{m=4}^{\infty} \sum_{n=0}^{(m+1)!-m!-1} \sum_{r \in E_{m,n}} \varphi\left(\frac{1}{m!+n+1} + z(r)\right) \\
&= \varphi\left(\frac{1}{4!}\right) + \sum_{m=4}^{\infty} \sum_{n=0}^{(m+1)!-m!-1} \sum_{r \in E_{m,n}} \left[\varphi\left(\frac{1}{m!+n+1}\right) + \int_{(m!+n+1)^{-1}}^{(m!+n+1)^{-1}+z(r)} p(t)\,dt\right] \\
&= \varphi\left(\frac{1}{4!}\right) + \sum_{m=4}^{\infty} \sum_{n=0}^{(m+1)!-m!-1} \sum_{r \in E_{m,n}} \varphi\left(\frac{1}{m!+n+1}\right) + \sum_{m=4}^{\infty} \sum_{n=0}^{(m+1)!-m!-1} \sum_{r \in E_{m,n}} \frac{z(r)}{m!} \\
&\leqslant I_\varphi(y) + \sum_{m=4}^{\infty} ((m+1)!-m!)(m-2)! \frac{1}{m!^2(m!+1)} \\
&= I_\varphi(y) + \sum_{m=4}^{\infty} \frac{1}{(m-1)(m!+1)} \\
&\leqslant I_\varphi(y) + \frac{1}{3}\left(e-2-\frac{2}{3}\right) = I_\varphi(y) + \frac{e}{3} - \frac{8}{9} \leqslant I_\varphi(y) + 0.018,
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=m!+1}^{(m+1)!} \varphi\left(\frac{1}{n}\right) &= \sum_{i=1}^{(m+1)!-m!} \left[\varphi\left(\frac{1}{(m+1)!}\right) + \frac{1}{m!}\left(\frac{1}{m!+i} - \frac{1}{(m+1)!}\right)\right] \\
&= ((m+1)!-m!)\,\varphi\left(\frac{1}{(m+1)!}\right) - ((m+1)!-m!-1)\frac{1}{m!}\frac{1}{(m+1)!} \\
&\quad + \frac{1}{m!} \sum_{i=1}^{(m+1)!-m!-1} \frac{1}{m!+i} \\
&= mm!\varphi\left(\frac{1}{(m+1)!}\right) - mm!\frac{1}{m!}\frac{1}{(m+1)!} + \frac{1}{m!}\left(H_{(m+1)!} - H_{m!}\right) \\
&\leqslant \frac{m}{(m+1)(m+1)!} - \frac{m}{(m+1)!} + \frac{\ln(m+1)}{m!}.
\end{aligned}$$

By the latter inequality and by

$$\ln(m+1) \leqslant \sqrt{m-1} \text{ for } m \geqslant 3,$$

we get that

$$\begin{aligned}
I_\varphi(y) &= \varphi\left(\frac{1}{4!}\right) + \sum_{m=4}^{\infty} \sum_{n=0}^{(m+1)!-m!-1} \sum_{r\in E_{m,n}} \varphi\left(\frac{1}{m!+n+1}\right) \\
&= \varphi\left(\frac{1}{4!}\right) + \sum_{m=4}^{\infty}(m-2)! \sum_{n=m!+1}^{(m+1)!} \varphi\left(\frac{1}{n}\right) \\
&\leqslant \frac{1}{4!^2} + \sum_{m=4}^{\infty} \frac{1}{(m+1)^2(m-1)} - \sum_{m=4}^{\infty} \frac{1}{(m-1)(m+1)} + \sum_{m=4}^{\infty} \frac{\ln(m+1)}{m(m-1)} \\
&\leqslant \frac{1}{576} + \left(\frac{247}{288} - \frac{\pi^2}{12}\right) - \frac{7}{24} + \left(\frac{\ln(5)}{12} + \zeta\left(\frac{3}{2}\right) - 1 - \frac{\sqrt{2}}{4} - \frac{\sqrt{3}}{9}\right) \\
&= -\frac{83}{192} - \frac{\pi^2 + \ln 5}{12} + \zeta\left(\frac{3}{2}\right) - \frac{\sqrt{2}}{4} - \frac{\sqrt{3}}{9} \leqslant 0.95,
\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta function. Combining all the above calculations, we obtain that $I_{ces_\varphi}(x) \leqslant 1$.

Let now $\epsilon > 0$ and s be a positive integer such that $\epsilon > s^{-1}$. Since

$$(s+1)\ln\left(1+s^{-1}\right) > 1 \text{ for any } s \geqslant 1,$$

we can take $N > s$ large enough such that for integers $m \geqslant N$, we have

$$\begin{aligned}
H_{m!+s^{-1}m!} - H_{m!} &\geqslant \int_{m!+1}^{m!+s^{-1}m!} \frac{1}{t}\,dt \\
&= \ln\left(m!\left(1+\frac{1}{s}\right)\right) - \ln(m!+1) \\
&= \ln\left(1+\frac{1}{s}\right) + \ln\left(\frac{m!}{m!+1}\right).
\end{aligned}$$

It follows for $m \geqslant N$ that

$$\sum_{n=m!+1}^{(m+1)!} \varphi\left(\frac{1+\frac{1}{s}}{n}\right) \geqslant \sum_{n=m!+1}^{(1+s^{-1})m!} \varphi\left(\frac{1+\frac{1}{s}}{n}\right) = \left(\varphi\left(\frac{1}{m!}\right) + \frac{1}{(m-1)!}\left(\frac{1+\frac{1}{s}}{m!+1} - \frac{1}{m!}\right)\right) +$$

$$\left(\varphi\left(\frac{1}{m!}\right) + \frac{1}{(m-1)!}\left(\frac{1+\frac{1}{s}}{m!+2} - \frac{1}{m!}\right)\right) + \ldots + \left(\varphi\left(\frac{1}{m!}\right) + 0\right)$$

$$= \left(\left(1+\frac{1}{s}\right)m! - m!\right)\varphi\left(\frac{1}{m!}\right) - \left(\left(1+\frac{1}{s}\right)m! - m! - 1\right)\frac{1}{(m-1)!m!} +$$

$$\frac{1}{(m-1)!}\left(1+\frac{1}{s}\right)\left(H_{m!+s^{-1}m!-1} - H_{m!}\right) = \frac{1}{s}m!\varphi\left(\frac{1}{m!}\right) - \frac{1}{s}\frac{1}{(m-1)!} +$$

$$\frac{1}{(m-1)!}\left(1+\frac{1}{s}\right)\left(H_{m!+s^{-1}m!} - H_{m!}\right)$$

$$\geqslant \frac{1}{s(m-1)!}\left[(s+1)\left(H_{m!+s^{-1}m!} - H_{m!}\right) - 1\right]$$

$$\geqslant \frac{1}{s(m-1)!}\left[(s+1)\ln\left(1+\frac{1}{s}\right) - 1 + (s+1)\ln\left(\frac{m!}{m!+1}\right)\right] \geqslant \frac{C}{s(m-1)!}$$

for some $C > 0$ not dependent on m. By the above inequality, we obtain

$$\begin{aligned}
I_{ces_\varphi}((1+\epsilon)x) &= I_\varphi((1+\epsilon)Gx) \geqslant I_\varphi((1+\epsilon)y) \geqslant I_\varphi\left(\left(1+\frac{1}{s}\right)y\right) \\
&\geqslant \sum_{m=4}^{\infty} \sum_{n=0}^{(m+1)!-m!-1} \sum_{r\in E_{m,n}} \varphi\left(\frac{1+\frac{1}{s}}{m!+n+1}\right) \\
&= \sum_{m=4}^{\infty}(m-2)! \sum_{n=m!+1}^{(m+1)!} \varphi\left(\frac{1+\frac{1}{s}}{n}\right) \\
&\geqslant \sum_{m=N}^{\infty}(m-2)! \sum_{n=m!+1}^{(1+s^{-1})m!} \varphi\left(\frac{1+\frac{1}{s}}{n}\right) \geqslant \frac{1}{s}\sum_{m=N}^{\infty}\frac{C}{m-1} = \infty.
\end{aligned}$$

The existence of an order linearly isometric copy of ℓ_∞ follows now exactly in the same way as at the end of the proof of Theorem 4.11.

Note that in [29] there is another explicit example of an Orlicz function φ for which the space ces_φ contains an order isometric copy of ℓ_∞. However in that case it can be easily checked that $\alpha(\varphi) > 1$. Also, it is not immediately clear whether the condition (iii) of Proposition 4.12 is satisfied or not by this function φ.

4.5 On B-convexity of Cesàro-Orlicz sequence spaces

Throughout this section we adopt the notation $\|\cdot\|_b$ for either $\|\cdot\|_{ces_\varphi}$ or $\|\cdot\|^0_{ces_\varphi}$. In [21] it has been shown that the n-th (strong) James constant in ces_φ for the Luxemburg or the Orlicz norm satisfies

$$J^s_n(ces_\varphi) := \sup\left\{\min_{\epsilon_k=\pm1}\left\|\sum_{k=1}^n \epsilon_k x_k\right\|_b : \|x_k\|_b = 1, k = 1, 2, \ldots, n\right\} = n, \quad (n \geqslant 2) \tag{4.13}$$

under some additional assumption on the function φ. We show that this extra assumption is in fact not necessary. Taking in the definiton of J^s_n the supremum over the whole unit ball we obtain constants J_n. A Banach space X is said to be B-convex if $J_n(X) < n$ for some $n \geqslant 2$. For more details on James constant(s) and B-convexity we refer to [21, 24] and references given therein.

Theorem 4.16. *Let φ be an Orlicz function and the space ces_φ be equipped with either the Luxemburg or Orlicz norm. If $ces_\varphi \neq \{0\}$ then $J^s_n(ces_\varphi) = n$ for $n = 2, 3, \ldots$.*

Proof. Let $n \geqslant 2$ be a fixed integer and $x_{k,m} = e_{m+k-1}/\|e_{m+k-1}\|_b$ for $k = 1, 2, \ldots, n$, and $m \in \mathbb{N}$. For both Luxemburg and Orlicz norm we have

$$\|x_{k,m}\|_b = 1 \quad \text{and} \quad \min_{\epsilon_k=\pm1}\left\|\sum_{k=1}^n \epsilon_k x_{k,m}\right\|_b = \left\|\sum_{k=1}^n x_{k,m}|\right\|_b.$$

Denoting by $a_{m,i} = \sum_{j=m}^{m+i} \|e_j\|_b^{-1}$ for $i = 0, 1, \ldots, n-1$, we see that $G\left(\sum_{k=1}^n x_{k,m}\right) \geqslant a_{m,n-1}G(e_{m+n-1})$, and hence

$$n \geqslant \left\|\sum_{k=1}^n x_{k,m}\right\|_b \geqslant a_{m,n-1}\|e_{m+n-1}\|_b = \sum_{k=m}^{m+n-1} \frac{\|e_{m+n-1}\|_b}{\|e_k\|_b} \geqslant n\frac{\|e_{m+n-1}\|_b}{\|e_m\|_b}.$$

We will show that $\|e_{m+n-1}\|_b/\|e_m\|_b \to 1$ as $m \to \infty$ for both norms, which in view of the above inequality proves that $J_n^s(ces_\varphi) = n$ for $n = 2, 3, \ldots$.

Consider the Luxemburg norm now. For any $m \in \mathbb{N}$ we get

$$\begin{aligned}\|me_m\|_{ces_\varphi} = \|G(me_m)\|_\varphi &= \left\|\left(0, \ldots, 0, 1, \frac{m}{m+1}, \frac{m}{m+2}, \ldots\right)\right\|_\varphi \\ &= \left\|\left(1, \frac{m}{m+1}, \frac{m}{m+2}, \ldots\right)\right\|_\varphi = \|[G(me_m)]^*\|_\varphi,\end{aligned}$$

where x^* denotes the decreasing rearrangement of x for $x \in \ell^0$. So $[G(me_m)]^*$ is an increasing sequence converging to $(1, 1, 1, \ldots)$ coordinatewise. If $\varphi(u) > 0$ for $u > 0$, we have

$$\sup_{m\in\mathbb{N}} \|me_m\|_{ces_\varphi} = \lim_{m\to\infty} \|me_m\|_{ces_\varphi} = \lim_{m\to\infty} \|G(me_m)\|_\varphi = \infty$$

by the Fatou property of ℓ_φ, because otherwise we would get $(1, 1, 1, \ldots) \in \ell_\varphi$ which is not the case since $\varphi(u) > 0$ for any $u > 0$. Hence

$$\begin{aligned}1 \geqslant &\frac{\|e_{m+n-1}\|_{ces_\varphi}}{\|e_m\|_{ces_\varphi}} = \frac{\|Ge_{m+n-1}\|_\varphi}{\|Ge_m\|_\varphi} = \frac{\|Ge_m - \sum_{i=0}^{n-2}(m+i)^{-1}e_{m+i}\|_\varphi}{\|Ge_m\|_\varphi} \\ \geqslant &1 - \frac{\|\sum_{i=0}^{n-2}(m+i)^{-1}e_{m+i}\|_\varphi}{\|Ge_m\|_\varphi} = 1 - \sum_{i=0}^{n-2} \frac{\|e_1\|_\varphi}{(m+i)\|Ge_m\|_\varphi} \to 1\end{aligned}$$

as $m \to \infty$, since $\|Ge_m\|_\varphi \geqslant \|\sum_{i=0}^{n-2}(m+i)^{-1}e_{m+i}\|_\varphi$, and for $i = 0, 1, \ldots, n-2$,

$$0 \leqslant \frac{\|e_1\|_\varphi}{(m+i)\|Ge_m\|_\varphi} = \frac{\|e_1\|_\varphi}{\|(m+i)e_m\|_{ces_\varphi}} \leqslant \frac{\|e_1\|_\varphi}{\|me_m\|_{ces_\varphi}} \to 0 \quad \text{as } m \to \infty.$$

If $\varphi(u) = 0$ on $[0, a]$ for $a > 0$, and $\varphi(u) > 0$ for $u > a$, then $\lim_{m\to\infty} \|me_m\|_{ces_\varphi} = \|(1, 1, 1, \ldots)\|_\varphi = a^{-1}$ by the Fatou property of ℓ_φ. Since

$$1 \geqslant \frac{(m+n-1)\|e_{m+n-1}\|_{ces_\varphi}}{m\|e_m\|_{ces_\varphi}} \to 1 \quad \text{as } m \to \infty,$$

we get $\|e_m\|_{ces_\varphi}^{-1}\|e_{m+n-1}\|_{ces_\varphi} \to 1$ as $m \to \infty$.

Now we will show the similar equality for the Orlicz norm in the Amemyia form. Since this norm is equivalent to the Luxemburg norm it is easy to see that in the case when $\varphi(u) > 0$ for $u > 0$ we also have

$$\lim_{m\to\infty} \|me_m\|_{ces_\varphi}^0 = \lim_{m\to\infty} \|G(me_m)\|_\varphi^0 = \infty.$$

Now we can repeat the reasoning which we used for the Luxemburg norm, taking into account that $\|x\|_{ces_\varphi}^0 = \|Gx\|_\varphi^0$. In the case when φ is equal to 0 on some interval then we also proceed similarly as in the case of the Luxemburg norm. $\square$

Corollary 4.17. *For any Orlicz function φ, if $ces_\varphi \neq \{0\}$ then ces_φ is not B-convex.*

We finish with the immediate consequence of the above result in view of the well known fact that if a Banach space is uniformly non-square then it is B-convex [51].

Corollary 4.18. *For any Orlicz function φ, if $ces_\varphi \neq \{0\}$ then ces_φ is not uniformly non-square for both Luxemburg and Orlicz norm.*

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