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Navier-Stokes Flow for a Fluid Jet with a Free Surface

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NAVIER-STOKES FLOW FOR A FLUID JET WITH A FREE SURFACE

by

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Abstract

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The three-dimensional Navier-Stokes flow of a viscous fluid jet bounded by a moving free surface under isothermal conditions and without surface tension is considered. The fluid domain is assumed to be periodic in the axial direction and initially axisymmetric. A local-in-time existence and regularity result is proven for the full governing equations using a contraction argument in an appropriate function space. Here a Lagrangian specification of the flow field is employed in order to mitigate the difficulties involved in dealing with an evolving fluid domain. It is also shown that the associated linear problem gives rise to an analytic semigroup of contractions on the space of divergence-free Lebesgue-square-integrable vector fields.

Contents

1 Introduction

Central to the study of fluid dynamics are the Navier-Stokes equations (NSE) — nonlinear partial differential equations (PDE) which govern the motion of fluids under quite general conditions and which are used to model everything from the air flow around an airplane to the movement of stars inside galaxies. Despite being essentially the simplest equations which describe the motion of a fluid, the NSE are fundamentally difficult to study from a mathematical perspective. This is evidenced by the long-standing open question, now a Clay Millennium Prize problem, of global existence and smoothness for solutions of the NSE on all of $\mathbb{R}^3$ for initial data of arbitrary size. The situation is even more challenging when one considers that real-world applications require that the NSE be solved on a limitless range of fluid domains where they must be coupled with often nontrivial boundary conditions.

An important example of such boundary conditions arises in the study of "moving free-boundary" fluid flow — a rich and challenging class of problems dealing with flows which have an evolving interface, of *a priori* unknown shape and position, with another fluid (e.g., air). Since the space occupied by the fluid is constantly changing in response to the flow variables, the fluid domain is itself an unknown in free boundary problems (in this dissertation, the phrase "free boundary" always refers to a moving free-boundary). Modeling free boundary flow involves coupling the NSE with free surface boundary conditions which govern the interaction of the dominant forces shaping the interface between fluids. In its simplest form, the free surface boundary condition balances viscous forces in the fluid with the external pressure being applied to the fluid's interface. However, more general and physically accurate forms incorporate the often significant effects due to surface tension. Simple examples of free boundary flows include the coating of a wire as it is withdrawn from a bath of molten plastic, the breakup of a liquid jet into droplets, and the spreading of a viscous fluid (e.g., honey) as it is poured onto a rigid surface.

One particularly interesting area of investigation involves the important but not well understood multiple-scale problems of how free boundary models governing bulk flow (i.e., three-dimensional NSE) scale down to the simplified models which govern flow in "thin" fluid domains. In the cases of liquid sheets and jets, these models are often referred to as *thin-film* and *thin-filament approximations* and are obtained from the NSE by using the assumption of thinness to reduce the number of spatial dimensions required to describe the flow to two and one respectively; examples include Yeow's equations for film casting [45] and the Matovich-Pearson equations for fiber spinning [17, 22]. Such models play central roles in quantitatively describing the free surface when numerical computations using the full three-dimensional NSE are cost-prohibitive. Perhaps more significantly, these models can be used to identify stabilizing/destabilizing factors and can often capture the exact form of a solution as the fluid approaches *breakup* (Fig. 1) — a phenomenon, induced by cohesive properties (e.g., surface tension effects) of the liquid, which is encountered in nature as well as in various industrial applications. Problems involving free boundary flow in thin domains stand at the forefront of many cutting-edge scientific pursuits, such as the production of nanoscale fibers and films.

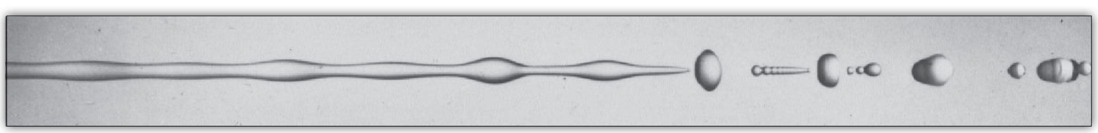

Fig. 1. The breakup of a liquid jet into droplets [11].

Experiments and simulations over the last several decades have shown that thin-filament approximations are often surprisingly accurate in describing the motion of fluid fibers and jets [5, 10, 35]. Despite the utility and longevity (elements date back over a hundred years [44]) of these one-dimensional models, there remains no rigorous mathematical link between thin-filament approximations and the three-dimensional NSE. Current derivations rely on the use of asymptotic expansions of flow variables with respect to a small parameter, ϵ, representing the ratio of a typical radial length scale to a typical axial length scale. While this approach provides a sophisticated and systematic method for obtaining thin-filament approximations from the NSE, it is also highly nonrigorous.

One method of establishing a rigorous connection between these two models is to show that the solutions of the NSE, averaged over the cross-sectional area of the fiber, converge to the solution of the corresponding thin-filament approximation as $\epsilon \rightarrow 0$ (i.e., as the fluid becomes more filament-like). Unfortunately, while great strides continue to be made in the questions of existence of solutions to thin-filament approximations (e.g., Hagen and Renardy's recent proof of global existence for the Matovich-Pearson equations [14]), currently there appear to be no existence results for a fluid fiber with a free surface using the full three-dimensional NSE. In this work, we aim to take a first step in this direction by proving the existence of local-in-time solutions of the NSE for a fluid jet which is assumed to be axially periodic.

1.1 A Survey of Existence Results for Free Boundary Problems

Free boundary flow modeled by the three-dimensional NSE has been studied most thoroughly in three settings: an isolated (and bounded) mass of fluid, multiple fluids contained in a bounded domain, and a semi-infinite "ocean" of fluid having free upper surfaces and fixed bottoms. The problem we consider in this paper lies somewhere between the isolated mass and infinite ocean cases. What follows is a very brief overview of the major work done in each, drawn heavily from the discussion provided in [25] (this article also reviews the history of multiple fluid problems which we omit).

In the first class of problem, the motion of an isolated mass of fluid bounded entirely by a free surface is examined. The seminal works here are due to Solonnikov who originally showed the existence of unique local-in-time solutions in Hölder spaces for the problem with external forces present and without surface tension [27]. Solonnikov subsequently extended this to include arbitrary initial data and, additionally, showed the global-in-time unique solvability in Sobolev spaces for the problem taken without external forces and sufficiently small initial data [31]. The latter result was subsequently extended to more general (anisotropic) Sobolev spaces by Shibata and Shimizu [25].

Local existence in the isolated mass setting with surface tension was first established (with no external forces and arbitrary initial data) by Solonnikov [29]. Solonnikov then extended this to global existence for small initial data and initial domain close to a sphere [30]. He later treated the addition of the self-gravitational force, obtaining first a local existence and uniqueness result [33] and eventually a global existence and uniqueness result [32]. Solonnikov's initial local existence result, obtained in Sobolev-Slobodetskii spaces, was ultimately brought to Hölder spaces by Moglilevskiĭ and Solonnikov [19]. An alternative approach to proving local existence and uniqueness for the problem, using semigroup theory, was later provided by Schweizer under the assumption of small initial data [24].

Work on the semi-infinite domain problem was pioneered by Beale who proved the local and small-data global existence of solutions when surface tension was not considered [7]. Beale, Allain, Sylvester, Tani, and Tanaka later extended these results to include surface tension effects, more general initial fluid domains, and higher regularity [3, 4, 8, 36, 37, 38]. Teramoto subsequently adapted Beale's techniques to gain similar results for a free surface problem involving axisymmetric flow down the exterior of a solid vertical column of sufficiently large radius [42]. More recently, Nishida, Teramoto, and Yoshihara obtained a global existence and uniqueness result (for sufficiently small data) when the fluid was taken to be horizontally periodic [20].

1.2 The Fluid Jet Free Boundary Problem

In this work, we discuss the local-in-time existence and regularity of solutions of the three-dimensional viscous flow of a fluid jet bounded by a free surface under isothermal conditions and without surface tension. The fluid is assumed to be viscous, Newtonian, and homogeneous (we assume unit density for simplicity). As in [42], the fluid domain is assumed to be periodic in the axial direction and initially axisymmetric. The periodic boundary condition is chosen because it leads to a simpler functional setting and avoids all axial boundary layer difficulties while retaining the primary mathematical challenges of the problem. In addition, it can be a natural assumption to make in the thin-filament setting, such as in the numerical simulation of drop dynamics on the beads-on-string structure for viscoelastic fluid jets [16].

We take as our general strategy the approach developed by Beale in [7] and summarized in Section 4.2. It is important, however, to note that while this scenario appears similar to the problem considered in [42], there are key differences which require novel ideas beyond Beale's techniques. In particular, unlike the fluid domains under consideration in [3, 4, 7, 8, 42], we do not have a stationary surface opposite the free surface to which we can assign a Dirichlet boundary condition (i.e., a condition fixing the value of the unknown function on a portion of the boundary). Foremost among the consequences of not having such a condition are the loss of general applicability of the Poincaré inequality, a fundamental tool in the analysis of PDE, and the loss of invertibility of the linear differential operator (analogous to the classical Stokes operator) central to the study of the full nonlinear problem. Moreover, where Teramoto is able to exploit axisymmetry and cylindrical coordinates to reduce his problem to two dimensions, the same approach introduces significant challenges in the fluid jet case since the NSE in cylindrical coordinates have singular coefficients when the axis at $r = 0$ is contained in the fluid domain.

2 Statement of the Problem

2.1 Initial Fluid Domain

We take as our initial fluid domain (the space occupied by the fluid at $t = 0$) the infinite cylinder along the a_3-axis,

$$\Omega_\infty = \left\{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1^2 + a_2^2 < \kappa^2\right\},$$

with free surface

$$\partial\Omega_\infty = \left\{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1^2 + a_2^2 = \kappa^2\right\}.$$

We restrict our attention to flow which is periodic in the a_3 direction, hence we are interested primarily in functions of the form

$$f = \sum_n \hat{f}_n(a_1, a_2) e^{2\pi i n a_3/\ell} \in H^k_{\text{loc}}(\Omega_\infty),$$

with $\hat{f}_n \in H^k(\mathbb{D})$, where $\mathbb{D}$ is the open disc of radius $\kappa < 1$ and ℓ is the period in the a_3 direction. Here H^k denotes the standard Sobolev space $W^{k,2}$ (see Appendix A.1). In practice however, we will find it more convenient to work with functions over a single period. It is natural then to interpret a_3-periodic functions on Ω_∞ as living on a solid torus. We take $\mathcal{T} \subset \mathbb{R}^3$ to be the toroid image of Ω_∞ under the transformation

$$\Phi : (a_1, a_2, a_3) \mapsto \left((a_1 + \kappa + 1)\cos\left(\frac{2\pi a_3}{\ell}\right), (a_1 + \kappa + 1)\sin\left(\frac{2\pi a_3}{\ell}\right), a_2\right).$$

It should be clear that $H^k(\mathcal{T})$ is isomorphic to the space of functions of interest. While $\mathcal{T}$ is a natural choice for the domain given the periodic setting, we prefer to work in the physical space occupied by Ω_∞. To this end, we notice that $\Phi|_{\mathbb{D}\times[0,\ell)}$ is a C^∞ diffeomorphism onto $\mathcal{T}$ (Fig. 2) and consider the bounded set

$$\Omega = \mathbb{D} \times (0, \ell)$$

with Lipschitz boundary

$$\partial\Omega = S_F \cup \Gamma_0 \cup \Gamma_\ell,$$

where $S_F = \partial\mathbb{D} \times (0, \ell), \Gamma_0 = \mathbb{D} \times \{0\}$, and $\Gamma_\ell = \mathbb{D} \times \{\ell\}$. Although $\Omega \neq \mathbb{D} \times [0, \ell)$, we choose to denote the diffeomorphism $\Phi|_{\mathbb{D}\times[0,\ell)}$ by Φ_Ω for the sake of simplicity. While the use of Ω in place of Ω_∞ does give rise to minor technical issues (as opposed to $\mathcal{T}$) concerning the regularity of functions as you approach the "artificial" corners in the boundaries, most of these problems can be dealt with by temporarily exchanging Ω for a larger subset of Ω_∞. As such we will occasionally find a use for the set

$$\Omega_n = \mathbb{D} \times (-n\ell, n\ell).$$

2.2 List of Quantities

We begin with a brief word about notation. Throughout the text, scalar and vectorial quantities will be designated using roman and bold typefaces respectively. Moreover, it is assumed that the i^{th} component of a vectorial quantity is denoted using the same letter as the vector, written in a roman typeface, with subscript i (e.g., $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)^T$). Unless otherwise specified, vectorial quantities denote

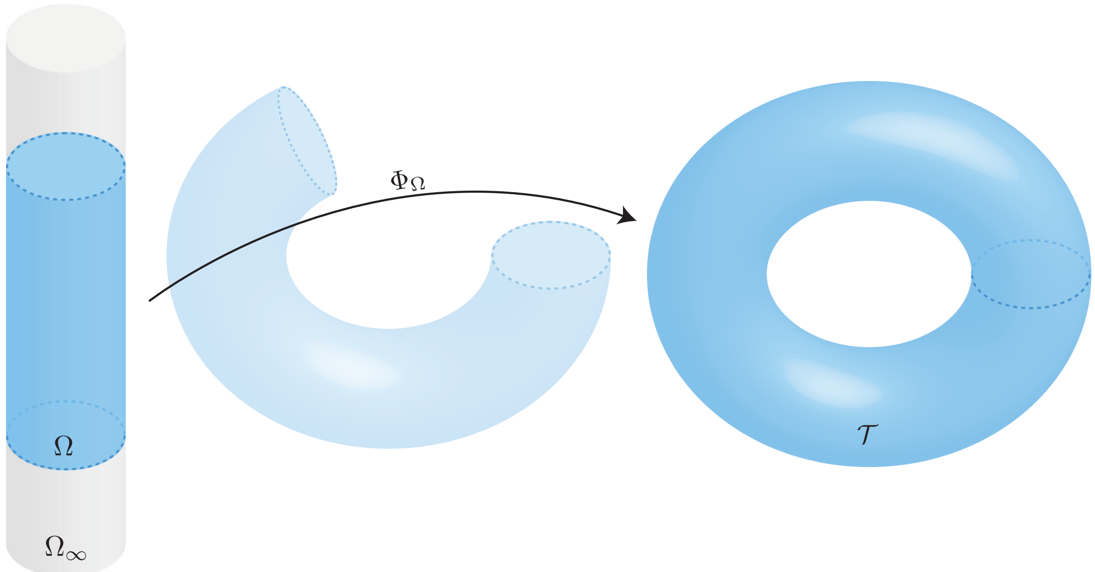

Fig. 2. The diffeomorphism Φ_Ω maps $\mathbb{D} \times [0, \ell)$ to $\mathcal{T}$.

column vectors. Partial derivatives with respect to time are generally denoted by

$$D_t = \frac{\partial}{\partial t}, \qquad D_t^k = \frac{\partial^k}{\partial t^k}$$

where $k \in \mathbb{N}$. In a slight abuse of notation, we also often employ the dot notation $\dot{u}$ in place of $D_t u$ to enhance readability. While this is typically reserved for denoting ordinary time derivatives, the distinction will rarely be important here. Partial derivatives taken with respect to a single spatial coordinate are denoted by

$$D_i = \frac{\partial}{\partial a_i}, \qquad D_i^k = \frac{\partial^k}{\partial a_i^k}$$

where $k \in \mathbb{N}$ and $i \in \{1, 2, 3\}$. For mixed partial derivatives involving spatial coordinates, we will find it useful to make use of the multi-index notation for partial derivatives: for a multi-index $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_i \in \mathbb{N}_0$, of order $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, we define

$$D_{\boldsymbol{\alpha}} = \frac{\partial^{\alpha_1}}{\partial a_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial a_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial a_n^{\alpha_n}}.$$

To mitigate the difficulties in dealing with flow in an (*a priori* unknown) evolving domain, we will find it useful to change from the usual Eulerian specification of the flow field (coordinate perspective) to the more convenient Lagrangian specification (fluid parcel perspective). The distinction is simple: in an Eulerian specification, flow variables (such as the velocity of a fluid parcel) are functions of time and *current* position whereas, in a Lagrangian specification, flow variables are functions of time and *original* position within the (fixed) initial domain. In other words, instead of focusing on a specific point $\mathbf{a} \in \mathbb{R}^3$ and finding the velocity of whatever fluid parcel (if any) is currently located there, a Lagrangian formulation allows one to focus on a specific point in the initial domain, $\mathbf{a}_0$, and track the velocity of the fluid parcel originating there as it follows its trajectory.

While the Eulerian formulation is generally preferred for fixed domains, it is problematic for domains with moving boundaries as the coordinates where fluid is present are subject to change as time progresses. In contrast, the Lagrangian formulation provides a means of obtaining a fixed domain for such problems. This conveniently avoids the problem of having to "locate" the *a priori* unknown evolving free surface since the fluid parcels present on the free surface at time t are the same parcels initially lying on the free surface. While new challenges are introduced when changing to a Lagrangian specification, the benefits of obtaining a fixed domain outweigh the consequences in this case.

To change specifications we make use of a "trajectory" map (*a priori* unknown) which yields the position of a fluid parcel at time t given the parcel's initial location in Ω. For some $T > 0$ and all $t \in (0, T)$, we have:

$\Omega(t)$, where $\Omega(0) = \Omega$, the domain occupied by the fluid at time t,

$S_F(t)$, where $S_F(0) = S_F$, the free surface of the fluid at time t,

$\mathbf{y}(t, \cdot) : \Omega \to \Omega(t)$, the fluid parcel trajectory, and

$\mathbf{x}(t, \cdot) : \Omega \to \mathbb{R}^3$, where $\mathbf{x}(t, \cdot) = \mathbf{y}(t, \cdot) - \mathbf{I}(\cdot)$, the fluid parcel displacement.

The following quantities are assumed constant and nonnegative:

P_0, the ambient pressure,

μ, the fluid viscosity, and

g, the acceleration due to gravity.

In addition, we have the Eulerian flow variables:

$\mathbf{u}(t, \cdot) : \Omega(t) \to \mathbb{R}^3$, the fluid velocity,

$p(t, \cdot) : \Omega(t) \to \mathbb{R}$, the fluid pressure, and

$\mathbf{n}(t, \cdot) : \partial\Omega(t) \to \mathbb{R}^3$, the outward unit normal vector.

Their respective Lagrangian counterparts are given by

$\mathbf{v}(t, \cdot) : \Omega \to \mathbb{R}^3$, where $\mathbf{v}(t, \cdot) = \mathbf{u}(t, \mathbf{y}(t, \cdot))$,

$q(t, \cdot) : \Omega \to \mathbb{R}$, where $q(t, \cdot) = p(t, \mathbf{y}(t, \cdot)) - P_0$, and

$\widetilde{\mathbf{n}}(t, \cdot) : \partial\Omega \to \mathbb{R}^3$, where $\widetilde{\mathbf{n}}(t, \cdot) = \mathbf{n}(t, \mathbf{y}(t, \cdot))$.

Note that q is not precisely the Lagrangian fluid pressure, but rather its difference with the ambient pressure. One consequence of converting the governing equations to the Lagrangian specification is the introduction of *a priori* unknown quantities involving derivatives of the trajectory map $\mathbf{y}$. We denote these by

$$\Lambda(t, \mathbf{a}) : \Omega \to \mathbb{R}, \text{ where } \Lambda = \big(\lambda_{i,j}(t, \mathbf{a})\big) = (\nabla \mathbf{y})^{-1} = \begin{pmatrix} D_1 y_1 & D_1 y_2 & D_1 y_3 \\ D_2 y_1 & D_2 y_2 & D_2 y_3 \\ D_3 y_1 & D_3 y_2 & D_3 y_3 \end{pmatrix}^{-1}.$$

These arise because of the relationship $D_j v_i = \sum_{k=1}^{3} D_j y_k D_{y_k} u_i$, from which it follows that $\nabla u_i = (\nabla \mathbf{y})^{-1} \nabla v_i$. Note that the Jacobian matrix of a vector (e.g., $\nabla \mathbf{y}$) is often defined as the transpose of the matrix used above. For convenience we also abbreviate the following sets:

$$G = (0, T) \times \Omega \text{ and}$$

$$\partial G_F = (0, T) \times S_F.$$

2.3 Governing Equations

Depending on the assumptions we make about certain inherent properties of the fluid, the equations governing fluid flow can take on a range of forms. In this work, we will restrict our consideration to fluids which are viscous, homogeneous, incompressible, and Newtonian. For the reader who is unfamiliar with fluid dynamics, we will now take a moment to briefly describe the meaning of these various properties.

A viscous fluid is one which displays resistance to stress and nearly all real fluids (except for matter in the so-called superfluid state) can be classified as such. Viscosity is defined as the ratio of stress to strain rate for a fluid; it can be thought of as a measure of the internal friction of a fluid, describing how resistant it is to flow. All other things being equal, where low viscosity fluids (e.g., water) are “thin” and flow quickly, high viscosity fluids (e.g., honey) are “thick” and flow slowly.

A homogeneous, incompressible fluid is simply one which has constant density. The distinction between Newtonian and non-Newtonian fluids, however, is a bit more technical: a Newtonian fluid is one which exhibits a strain rate which is proportional to stress. A Newtonian fluid can thus be understood as a fluid with constant viscosity whose strain rate vanishes with stress. The latter condition is required to exclude materials like Bingham plastic which exhibits constant viscosity yet behaves like a solid at low stresses.

Assuming that the fluid has unit density and that gravity is the only external body force acting on the fluid, the Navier-Stokes equations take the form

$$\dot{\mathbf{u}} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \mu\Delta\mathbf{u} + \nabla p = g\,\mathbf{e}_3 \qquad \text{on } \Omega(t) \tag{2.1}$$

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{on } \Omega(t) \tag{2.2}$$

in the Eulerian specification. Here $\mathbf{e}_3$ denotes the third standard basis vector in $\mathbb{R}^3$. Equations (2.1) and (2.2) are simply the descriptions of the conservation of momentum and mass, respectively, for an arbitrary fluid parcel in $\Omega(t)$. In order to properly correlate the evolution of the free surface to that of the fluid velocity, we require that, for each $t \in (0, T)$, the trajectory mapping satisfy

$$\Omega(t) = \mathbf{y}(t, \Omega) \tag{2.3}$$

$$\dot{\mathbf{y}}(t) = \mathbf{u}(t, \mathbf{y}) \qquad \text{on } \Omega. \tag{2.4}$$

To this we add the initial conditions

$$\mathbf{u}(0,\cdot) = \mathbf{u}_0(\cdot) \qquad \text{on } \Omega \tag{2.5}$$

$$\mathbf{y}(0,\cdot) = \mathbf{I}(\cdot) \qquad \text{on } \Omega \tag{2.6}$$

as well as free surface and periodic (in a_3) boundary conditions

$$pn_i - \mu \sum_{j=1}^{3} (D_j u_i + D_i u_j) n_j = P_0 n_i \qquad \text{for } i \in \{1,2,3\} \text{ on } S_F(t) \tag{2.7}$$

$$\left.\begin{aligned} p_{\mathrm{p}}(t,\cdot) &\in H^{s-1}_{\mathrm{loc}}(\Omega_\infty) \\ \mathbf{x}_{\mathrm{p}}(t,\cdot) &\in (H^{s}_{\mathrm{loc}}(\Omega_\infty))^3 \\ \mathbf{u}_{\mathrm{p}}(t,\cdot) &\in (H^{s}_{\mathrm{loc}}(\Omega_\infty(t)))^3 \end{aligned}\right\} \qquad \text{with } s \geq 2 \tag{2.8}$$

where $(\cdot)_{\mathrm{p}}$ denotes the a_3-periodic extension of $(\cdot)$. The free surface condition (2.7) assumes that the dominant forces governing the evolution of S_F are the viscous forces within the fluid jet. In particular, the cohesive effects due to surface tension are not considered. When working with the free surface condition, we will often find it useful to abbreviate the vector given by the left-hand side of (2.7), taken with $\mathbf{n} = \mathbf{n}(\mathbf{0},\cdot)$, by

$$\mathbf{S}(\mathbf{u}, p) = \Big(pn_i(0,\cdot) - \mu \sum_{j=1}^{3} (D_j u_i + D_i u_j) n_j(0,\cdot)\Big)_{i=1}^{3}.$$

We also define the *tangential part* of a vector field $\mathbf{f}$ on $\partial\Omega$, where $\mathbf{f} \in (L^2(\partial\Omega))^3$ for example, as

$$\mathbf{f}_{\tan} = \mathbf{f} - (\mathbf{f} \cdot \mathbf{n}(0,\cdot))\mathbf{n}(0,\cdot).$$

To avoid working in the unknown evolving domain $\Omega(t)$, we rewrite the Eulerian quantities using their Lagrangian counterparts and obtain the following reformulation of the NSE (2.1) and (2.2):

$$\dot{v}_i - \mu \sum_{j,k,m=1}^{3} \lambda_{j,k} D_k(\lambda_{j,m} D_m v_i) + \sum_{k=1}^{3} \lambda_{i,k} D_k q = g\delta_{3,i} \quad \text{for } i \in \{1,2,3\} \text{ on } G \tag{2.9}$$

$$\sum_{j,k=1}^{3} \lambda_{j,k} D_k v_j = 0 \qquad \text{on } G. \tag{2.10}$$

Here $\delta_{i,j}$ denotes the Kronecker delta. Notice that the nonlinear (in $\mathbf{u}$) term $(\mathbf{u}\cdot\nabla)\mathbf{u}$ from (2.1) is canceled out in the reformulation; this is an additional benefit of the Lagrangian specification. We also point out that where (2.3) is a condition that must be satisfied in the Eulerian specification, after the reformulation, it simply becomes a formula for recovering $\Omega(t)$ once the trajectory is known. Condition (2.4) can be restated more succinctly as

$$\dot{\mathbf{x}} = \mathbf{v} \qquad \text{on } G. \tag{2.11}$$

Next, we update the initial conditions

$$\mathbf{v}(0,\cdot) = \mathbf{u}_0(\cdot) \qquad \text{on } \Omega \tag{2.12}$$

$$\mathbf{x}(0,\cdot) = 0 \qquad \text{on } \Omega \tag{2.13}$$

and the free surface condition

$$q\widetilde{n}_i - \mu \sum_{j,k=1}^{3} (\lambda_{j,k} D_k v_i + \lambda_{i,k} D_k v_j)\widetilde{n}_j = 0 \qquad \text{for } i \in \{1,2,3\} \text{ on } \partial G_F.$$

Notice, however, that we can obtain an equivalent free surface boundary condition by replacing $\widetilde{\mathbf{n}}$ with any outward normal vector (i.e., not necessarily a unit vector) to $\mathbf{y}(t, S_F)$. In fact, this replacement corresponds to simply multiplying through the above equation by the magnitude of this outward normal vector. With this in mind, we now construct a normal vector which will significantly simplify our analysis of the nonlinear problem in Chapter 6. For orthogonal unit tangent vectors to S_F, we take $\boldsymbol{\tau}_1 = \mathbf{e}_3$ and $\boldsymbol{\tau}_2 = \kappa^{-1}(a_2, -a_1, 0)^T$. Now $(\nabla \mathbf{y})\boldsymbol{\tau}_1$ and $(\nabla \mathbf{y})\boldsymbol{\tau}_2$ are orthogonal tangent vectors to $\mathbf{y}(t, S_F)$ and hence

$$\mathbf{N} = \nabla \mathbf{y} \boldsymbol{\tau}_1 \times \nabla \mathbf{y} \boldsymbol{\tau}_2$$

is an outward normal vector to the surface. Moreover, $\mathbf{N}$ is such that $\mathbf{N}(0, \cdot) = \mathbf{n}(0, \cdot)$. We will ultimately discover that $\mathbf{N}$ (and hence $|\mathbf{N}|$) is continuous in time and space, so that multiplication by $|\mathbf{N}|$ does not alter the free surface condition in the L^2 setting. The free surface condition we consider is then given by

$$qN_i - \mu \sum_{j,k=1}^{3} (\lambda_{j,k} D_k v_i + \lambda_{i,k} D_k v_j) N_j = 0 \qquad \text{for } i \in \{1, 2, 3\} \text{ on } \partial G_F. \tag{2.14}$$

Finally, we obtain the updated periodic boundary conditions

$$\left.\begin{array}{c} q_{\mathrm{p}}(t, \cdot) \in H^{s-1}_{\mathrm{loc}}(\Omega_\infty) \\ \mathbf{x}_{\mathrm{p}}(t, \cdot), \mathbf{v}_{\mathrm{p}}(t, \cdot) \in (H^{s}_{\mathrm{loc}}(\Omega_\infty))^3 \end{array}\right\} \qquad \text{on } (0, T) \text{ with } s \geq 2. \tag{2.15}$$

The main result of this work, Theorem 4.1, demonstrates that the nonlinear problem (2.9)–(2.15) has a solution, $(\mathbf{v}, q)$, for any compatible initial data $\mathbf{u}_0$. Moreover, the length of time, T, that the solution is guaranteed to remain valid is dependent only on $\mathbf{u}_0$. Provided that such a solution is sufficiently regular, we will demonstrate that it can be transformed into a solution of the original problem (2.1)–(2.8).

3 Preliminary Topics

In this chapter, we introduce the fundamental objects relevant to this work and discuss some of their important properties. We also give an existence and uniqueness result for Laplace's equation with mixed Dirichlet-periodic boundary conditions that arises several times in later chapters. However, it is assumed that the reader is familiar with the standard Lebesgue and Sobolev spaces (a brief overview can be found in Appendix A.1) which will be used throughout.

3.1 Notation and Definitions

In this work, C and $C_0, C_1, C_2, \ldots$ denote generic positive constants where, in particular, C can change from instance to instance in a given proof. The dependence of these constants on any important quantities will be made explicit in each case, but they can always be assumed to be independent of T (this is particularly important in Chapter 6 where constants need to remain fixed when T is modified). Note that while the letter C will also be used to denote spaces of continuous functions, the intended meaning will always be obvious from the context.

Given a spatial domain $U \subset \mathbb{R}^3$, we strive to obey the following notational conventions for arbitrary function spaces $X(U)$.

Primary domain:	$X = X(\Omega)$
Vanishing on S_F:	${}^0X = \{u \in X : u = 0 \text{ on } S_F\}$
Vanishing up to S_F:	${}^{\mathrm{c}}X = \{u \in X : \mathrm{dist}(\mathrm{supp}\ u, S_F) > 0\}$
Vector field:	$\mathbf{X}(U) = (X(U))^3$
Tensor field:	$(X(U))^{m\times n} = \{(u_{i,j})_{1\le i\le m,\, 1\le j\le n} : u_{i,j} \in X(U)\}$
Divergence-free:	$\mathbf{X}_\sigma(U) = \{\mathbf{u} \in \mathbf{X}(U) : \nabla \cdot \mathbf{u} = 0\}$

Here the vector and tensor fields are equipped with the Euclidean and Frobenius norms respectively. Similarly, given a time interval $I \subset \mathbb{R}$ in addition to a spatial domain $U \subset \mathbb{R}^3$, we employ the following conventions for arbitrary function spaces $Y(I \times U)$.

Primary domain:	$Y = Y(G)$
Vanishing on S_F:	${}^0Y = \{u \in Y : u(t,\cdot) = 0 \text{ on } S_F\}$
Vanishing up to S_F:	${}^{\mathrm{c}}Y = \{u \in Y : \operatorname{dist}(\operatorname{supp} u(t,\cdot), S_F) > 0\}$
Vector field:	$\mathbf{Y}(I \times U) = (Y(I \times U))^3$
Tensor field:	$(Y(I \times U))^{m\times n} = \{(u_{i,j})_{1\le i\le m,\, 1\le j\le n} : u_{i,j} \in Y(I \times U)\}$
Divergence-free:	$\mathbf{Y}_\sigma(I \times U) = \{\mathbf{u} \in \mathbf{Y}(I \times U) : \nabla \cdot \mathbf{u} = 0\}$

Since we will be working with Hilbert spaces primarily, it is important to note that the subspaces created from Hilbert spaces in this way will be closed (and hence Hilbert spaces themselves). In particular, for subspaces of the form ${}^0X$ or X_σ, this can be shown easily using the fact that the kernels of continuous operators are closed. We also note that for ${}^0X$ to be well-defined in general, we require that $X \subset H^s$ where $s > 1/2$ (see Theorem B.2). To keep the notation simple, if a function space already has a subscript, its divergence-free subspace will be denoted by simply adding a σ to the existing subscript. For example, $\mathbf{C}_{\mathrm{c}\sigma}$ would be used in place of $(\mathbf{C}_{\mathrm{c}})_\sigma$. Spaces not following these conventions will be explicitly defined in each instance. We now introduce the spaces fundamental to this text; though each assumes Ω as its spatial domain, the extension to Ω_n is obvious. For the set of functions on Ω whose a_3-periodic extensions are continuously differentiable and bounded on Ω_∞ we simply take

$$C^k_{\mathrm{p}} = \left\{ u|_\Omega : u = \sum_{n=-\infty}^{\infty} \hat{u}_n(a_1, a_2) e^{2\pi i n a_3/\ell} \in C^k\left(\overline{\Omega_\infty}\right) \right\}$$

where $\hat{u}_n \in C^k(\mathbb{D})$ necessarily. Similarly, we define C^∞_{p} (or $C^{k,\alpha}_{\mathrm{p}}$) to be the set of all such functions which are bounded and smooth (or Hölder continuous with exponent α) on Ω_∞. Special care must be taken, however, in attempting to obtain a useful characterization of the set of functions on Ω whose a_3-periodic extensions are weakly differentiable on Ω_∞ (see Appendix A.1 to recall the definition of a weak derivative). It is not enough to take functions of the form $\sum_n \hat{u}_n e^{2\pi i n a_3/\ell} \in H^k$ since regularity inside Ω does not imply the same regularity for its periodic extension to Ω_∞; additional conditions must be established in order for regularity to be preserved across the artificial boundary at Γ_0/Γ_ℓ. It is straightforward to show (see Lemma 3.4) that the following space provides such a characterization:

$$H^k_{\mathrm{p}} = \left\{ u = \sum_{n=-\infty}^{\infty} \hat{u}_n(a_1, a_2) e^{2\pi i n a_3/\ell} \in H^k(\Omega) : \hat{u}_n \in H^k(\mathbb{D}) \text{ and } \|u\|_{H^k_{\mathrm{p}}} < \infty \right\}$$

for $k \in \mathbb{N}_0$, where the series converges with respect to $\|\cdot\|_{H^k}$ and

$$\|u\|_{H^k_{\mathrm{p}}} = \left(\sum_{n=-\infty}^{\infty} \sum_{m=0}^{k} \frac{(2\pi n)^{2m}}{\ell^{2m-1}} \|\hat{u}_n\|^2_{H^{k-m}(\mathbb{D})} \right)^{1/2}. \tag{3.16}$$

This is a Hilbert space (see Lemma 3.2) when equipped with the inner product

$$(u, v)_{H^k_{\mathrm{p}}} = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{k} \frac{(2\pi n)^{2m}}{\ell^{2m-1}} (\hat{u}_n, \hat{v}_n)_{H^{k-m}(\mathbb{D})}. \tag{3.17}$$

For $s \in \mathbb{R}^+$, we define H^s_{p} using complex interpolation (see Appendix A.2). For all $0 < \beta < 1$, $u \in H^{k+1}_{\mathrm{p}}$, the norms on the interpolation spaces satisfy

$$C_1 \|u\|_{H^k_{\mathrm{p}}} \leq \|u\|_{H^{k+\beta}_{\mathrm{p}}} \leq C_2 \|u\|_{H^{k+1}_{\mathrm{p}}} \quad \text{and} \quad \|u\|_{H^{k+\beta}_{\mathrm{p}}} \leq C_3 \|u\|^{1-\beta}_{H^k_{\mathrm{p}}} \|u\|^{\beta}_{H^{k+1}_{\mathrm{p}}}$$

where C_1, C_2, and C_3 are positive constants depending on β. Note that, throughout the text, we typically use r and s to denote non-integer regularity and k when we restrict ourselves to integer regularity.

Let us now briefly discuss an auxiliary space which will prove instrumental to our analysis. In variational approaches to solving the NSE (such as the one pursued in this work), a key result is the Helmholtz decomposition of $\mathbf{L}^2$ which allows any function in $\mathbf{L}^2$ to be uniquely decomposed into the sum of a divergence-free function and a gradient (see [26] for an excellent introduction). This provides us with an orthogonal projection (often referred to as either the Helmholtz or Leray projection) from $\mathbf{L}^2$ onto the subspace, $\{\mathbf{u} \in \mathbf{L}^2 : \nabla \cdot \mathbf{u} = 0, \mathbf{n} \cdot \mathbf{u}|_{\partial\Omega} = 0\}$, consisting (essentially) of divergence-free vectors. Note here that, as $\mathbf{n} \cdot \mathbf{u}|_{\partial\Omega}$ denotes a generalized trace, this space is particularly well-suited to problems coupled with Dirichlet boundary conditions. Applying this projection to (2.1) would allow us to both remove the troublesome pressure term (which, as a gradient, vanishes under this orthogonal projection) from (2.1) and also to incorporate the equation (2.2) into the underlying function space. It is standard practice to use techniques like this in an attempt to solve initially for the velocity independently of the pressure. The associated pressure is then determined in a second step.

Taking our lead from Beale in [7], we pursue a slightly different decomposition of $\mathbf{L}^2$ owing to the nonstandard boundary conditions under consideration in this work. In particular, to incorporate a_3-periodicity along with the divergence-free condition into the auxiliary space, we choose our decomposition so that $\mathbf{L}^2$ is projected onto $\mathbf{H}^0_{\mathrm{p}\sigma}$. For convenience, we set

$$\mathbf{P}^s = \mathbf{H}^s_{\mathrm{p}\sigma}.$$

Notice that these spaces neglect to include both the free surface condition and a generalized trace condition (as opposed to the standard decomposition). The generalized trace condition has been removed for its lack of relevance to our boundary conditions; its inclusion would unnecessarily restrict the space of functions under consideration. While we will introduce a space that incorporates the free surface condition momentarily, we do not want to include this condition in the projection space $\mathbf{P}^0$ itself as it would make finding a characterization for the orthogonal complement untenable.

The free surface boundary condition, as something which relates the value of the velocity to that of the unknown pressure along S_F, cannot be fully incorporated into the definition of any space of prospective velocity functions. However, since the pressure only enters the balance of forces across the free surface in the normal direction, the need for the tangential part of $\mathbf{S}(\mathbf{v}, q)$ to vanish on S_F is a condition which depends solely on the velocity. Therefore, we define

$$\mathbf{V}^s = \{\mathbf{v} \in \mathbf{P}^s : \mathbf{S}_{\tan}(\mathbf{v}) = 0 \text{ on } S_F\}.$$

We must also define spaces which ensure that our solutions have adequate regularity with respect to time. We do this using the so-called Lebesgue-Bochner spaces (see Appendix A.1 for a brief description) which treat functions of time and space as time-parameterized collections of functions of space which are parameterized by time. For a time interval I, the set of functions on $I \times \Omega$ whose a_3-periodic extensions are r-times weakly differentiable with respect to time and

s-times weakly differentiable with respect to space (on Ω_∞) is given by

$$H_{\mathrm{p}}^{r,s}(I \times \Omega) = H^r(I; H_{\mathrm{p}}^0) \cap H^0(I; H_{\mathrm{p}}^s).$$

Of particular interest, given the form of the NSE, are functions with half as much regularity in time as in space. We denote such spaces by

$$K_{\mathrm{p}}^s(I \times \Omega) = H_{\mathrm{p}}^{s/2,s}(I \times \Omega).$$

3.2 The Periodic Spaces

Most of the results in this section seek to relate, in various ways, H_{p}^k to H^k. This is certainly a worthwhile endeavor as it will frequently allow us to leverage the power of standard Sobolev theory in our analysis involving the a_3-periodic spaces. We begin by showing that $\|\cdot\|_{H_{\mathrm{p}}^k}$ and $\|\cdot\|_{H^k}$ are equivalent norms on H_{p}^k and are actually equal for $k \in \{0, 1\}$.

Lemma 3.1. *There exists $C > 0$, depending only on k, such that $C\|f\|_{H^k} \leq \|f\|_{H_p^k} \leq \|f\|_{H^k}$ for all $f \in H_p^k$. In particular, for $k \in \{0, 1\}$ we have:*

(i) $\|f\|_{H_p^k} = \|f\|_{H^k}$ *for all* $f \in H_p^k$.

(ii) $(f, \tilde{f})_{H_p^k} = (f, \tilde{f})_{H^k}$ *for all* $f, \tilde{f} \in H_p^k$.

Proof. First we consider the case where $k = 0$. Let $f \in H^0_{\mathrm{p}}$. Applying Fubini and the Lebesgue Monotone Convergence Theorem, we have

$$\begin{aligned}
\|f\|_{L^2}^2 &= \int_\Omega \left(\sum_n \hat{f}_n e^{2\pi i n a_3/\ell}\right)\left(\sum_m \overline{\hat{f}_m} e^{-2\pi i m a_3/\ell}\right) \\
&= \int_\Omega \sum_n \sum_m \hat{f}_n \overline{\hat{f}_m} e^{2\pi i (n-m) a_3/\ell} \\
&= \int_\Omega \sum_j \left(\sum_m \hat{f}_{j+m} \overline{\hat{f}_m}\right) e^{2\pi i j a_3/\ell} \\
&= \ell \int_{\mathbb{D}} \sum_m |\hat{f}_m|^2 \\
&= \ell \sum_m \int_{\mathbb{D}} |\hat{f}_m|^2 \\
&= \|f\|_{H^0_{\mathrm{p}}}^2.
\end{aligned}$$

Now let $f \in H^k_{\mathrm{p}}$ where $k \geq 1$. In this case,

$$\begin{aligned}
\|f\|_{H^k_{\mathrm{p}}}^2 &= \ell \sum_n \sum_{m=0}^k \left\| \left(\frac{2\pi i n}{\ell}\right)^m \hat{f}_n \right\|_{H^{k-m}(\mathbb{D})}^2 \\
&= \ell \sum_n \sum_{m=0}^k \sum_{\substack{|\boldsymbol{\alpha}| \leq k-m \\ \alpha_i \in \{1,2\}}} \left\| \left(\frac{2\pi i n}{\ell}\right)^m D_{\boldsymbol{\alpha}} \hat{f}_n \right\|_{L^2(\mathbb{D})}^2 \\
&= \sum_{m=0}^k \sum_{\substack{|\boldsymbol{\alpha}| \leq k-m \\ \alpha_i \in \{1,2\}}} \|D_3^m D_{\boldsymbol{\alpha}} f\|_{H^0_{\mathrm{p}}}^2 .
\end{aligned}$$

If $k = 1$, then this simplifies to

$$\|D_3 f\|_{L^2}^2 + \sum_{\substack{|\boldsymbol{\alpha}| \leq 1 \\ \alpha_i \in \{1,2\}}} \|D_{\boldsymbol{\alpha}} f\|_{L^2}^2 = \sum_{\substack{|\boldsymbol{\alpha}| \leq 1 \\ \alpha_i \in \{1,2,3\}}} \|D_{\boldsymbol{\alpha}} f\|_{L^2}^2 = \|f\|_{H^1}^2 .$$

Otherwise, we have (for some C depending on k) both

$$C\,\|f\|_{H^k}^2 = \sum_{\substack{|\boldsymbol{\beta}|\le k\\ \beta_i\in\{1,2,3\}}} C\,\|D_{\boldsymbol{\beta}} f\|_{L^2}^2 \le \sum_{m=0}^{k} \sum_{\substack{|\boldsymbol{\alpha}|\le k-m\\ \alpha_i\in\{1,2\}}} \|D_3^m D_{\boldsymbol{\alpha}} f\|_{H_{\mathrm{p}}^0}^2 = \|f\|_{H_p^k}^2$$

and

$$\|f\|_{H_p^k}^2 = \sum_{m=0}^{k} \sum_{\substack{|\boldsymbol{\alpha}|\le k-m\\ \alpha_i\in\{1,2\}}} \|D_3^m D_{\boldsymbol{\alpha}} f\|_{H_{\mathrm{p}}^0}^2 \le \sum_{\substack{|\boldsymbol{\beta}|\le k\\ \beta_i\in\{1,2,3\}}} \|D_{\boldsymbol{\beta}} f\|_{L^2}^2 = \|f\|_{H^k}^2 .$$

Finally, having shown (i), part (ii) follows immediately from the polarization identity. □

Our first application of this result justifies the earlier assertions that H_{p}^k equipped with the discussed norm and inner product forms a Hilbert space.

Lemma 3.2. *Let $k \ge 0$. Then H_p^k is a Hilbert space when equipped with* (3.16), (3.17). *In particular, $H_p^0 = L^2$.*

Proof. First we show that H_{p}^k is complete with respect to the $\|\cdot\|_{H_{\mathrm{p}}^k}$ norm. Let $(f_j) \in H_{\mathrm{p}}^k$ be Cauchy. It is straightforward to show that for each $n \in \mathbb{Z}, (\hat{f}_j)_n \in H^k(\mathbb{D})$ is Cauchy. Since $H^k(\mathbb{D})$ is complete, for each n there is $\hat{f}_n \in H^k(\mathbb{D})$ such that $(\hat{f}_j)_n \to \hat{f}_n$. We want to show that the construction $f = \sum_n \hat{f}_n e^{2\pi i n a_3/\ell} \in H_{\mathrm{p}}^k$ and $f_j \to f$ in the H_{p}^k norm. Since (f_j) is Cauchy, we can construct a subsequence (g_m) such that

$$\|g_{m+1} - g_m\|_{H_{\mathrm{p}}^k} < \frac{1}{2^m}.$$

Then we have both $(\hat{g}_m)_n \to \hat{f}_n$ in $H^k(\mathbb{D})$ and $\sum_m \|g_{m+1} - g_m\|_{H_{\mathrm{p}}^k} < \infty$. Consider the function $F = g_1 + \sum_m (g_{m+1} - g_m) = \lim_{m\to\infty} g_m$ from the $\|\cdot\|_{L^2}$-completion of

H^k_{p}. Now since $\Omega = [0, \kappa) \times [0, 2\pi) \times (0, \ell)$ in cylindrical coordinates, we have

$$L^2 = \left\{ \sum_{\mathbf{n} \in \mathbb{Z}^3} \hat{f}(\mathbf{n}) e^{(2\pi i n_1 r/\kappa) + (i n_2 \theta) + (2\pi i n_3 a_3/\ell)} : \sum_{\mathbf{n} \in \mathbb{Z}^3} |\hat{f}(\mathbf{n})|^2 < \infty \right\} \subset H^0_{\mathrm{p}}.$$

This means that we can set $F = \sum_n \hat{F}_n e^{2\pi i n a_3/\ell}$ and apply Lemma 3.1 to obtain

$$\|g_m - F\|^2_{L^2} = \|g_m - F\|^2_{H^0_{\mathrm{p}}} = \ell \sum_n \left\| (\hat{g}_m)_n - \hat{F}_n \right\|^2_{L^2(\mathbb{D})}.$$

Since $\|g_m - F\|^2_{L^2} \to 0$, this implies that for each n, $(\hat{g}_m)_n \to \hat{F}_n$ in $L^2(\mathbb{D})$ and hence $F = f$. It follows that $\hat{F}_n \in H^k(\mathbb{D})$, so that $\|F\|_{H^k_{\mathrm{p}}}$ is well-defined. Then

$$\|F\|_{H^k_{\mathrm{p}}} \leq \|g_1\|_{H^k_{\mathrm{p}}} + \sum_{m=1}^{\infty} \|g_{m+1} - g_m\|_{H^k_{\mathrm{p}}} < \infty.$$

Therefore $f = F \in H^k_{\mathrm{p}}$. That $f_j \to f$ in the H^k_{p} norm follows from choosing j, m large enough so that $\|f_j - f\|_{H^k_{\mathrm{p}}} \leq \|f_j - g_m\|_{H^k_{\mathrm{p}}} + \|g_m - f\|_{H^k_{\mathrm{p}}} < \varepsilon$. Thus H^k_{p} is complete with this norm. It is now not difficult to see that (3.17) is an inner product associated with the $\|\cdot\|_{H^k_{\mathrm{p}}}$ norm (making H^k_{p} a Hilbert space). □

Now that we have shown that H^k_{p} possesses all the structure we could have hoped for, we move on to a useful characterization for H^k_{p} in terms of H^k. Note that in this dissertation, restriction operators of the form $f|_X$ are generally meant in the sense of trace (see the discussion preceding Theorem B.2).

Lemma 3.3. *For $k \geq 1$, we have the characterization $H^k_p = \{f \in H^k : D^j_3 f|_{\Gamma_\ell} = D^j_3 f|_{\Gamma_0}$ for all $0 \leq j \leq k-1\}$.*

Proof. The equality $H^0_{\mathrm{p}} = L^2$ follows from the discussion in the proof of Lemma 3.2, so we let $k \geq 1$. Let $X_k = \{f \in H^k : D^j_3 f|_{\Gamma_\ell} = D^j_3 f|_{\Gamma_0}$ for all $0 \leq j \leq k-1\}$. That $H^k_{\mathrm{p}} \subset X_k$ is obvious so it suffices to show that the reverse inclusion holds. We proceed by induction in k. Let $k = 1$ and $f \in X_1$. Since $f, D_j f \in L^2$ for $1 \leq j \leq 3$, there exist $\hat{f}_n, (\hat{g}_j)_n \in L^2(\mathbb{D})$ such that

$$f = \sum_n \hat{f}_n e^{2\pi i n a_3/\ell}, \qquad D_j f = \sum_n (\hat{g}_j)_n e^{2\pi i n a_3/\ell}$$

with

$$\|f\|_{L^2}^2 = \ell \sum_n \|\hat{f}_n\|_{L^2(\mathbb{D})}^2 < \infty, \qquad \|D_j f\|_{L^2}^2 = \ell \sum_n \|(\hat{g}_j)_n\|_{L^2(\mathbb{D})}^2 < \infty.$$

For convenience, we set $\hat{g}_{j,n} = (\hat{g}_j)_n$. Let $\varphi_n \in C_{\mathrm{c}}^\infty(\mathbb{D})$ be chosen arbitrarily and consider $\varphi = \varphi_n e^{2\pi i n a_3/\ell} \in {}^{\mathrm{c}}C_{\mathrm{p}}^\infty$. For $j = 1, 2$ we have

$$\begin{aligned}
(D_j f, \varphi)_{L^2} &= \int_{S_F} f\varphi n_j + \int_{\Gamma_\ell} f\varphi n_j + \int_{\Gamma_0} f\varphi n_j - (f, D_j\varphi)_{L^2} \\
&= \int_{\Gamma_\ell} f\varphi \cdot 0 + \int_{\Gamma_0} f\varphi \cdot 0 - (f, D_j\varphi)_{L^2} \\
&= -(f, D_j\varphi)_{L^2} \\
(D_j f, \varphi)_{H_{\mathrm{p}}^0} &= -(f, D_j\varphi)_{H_{\mathrm{p}}^0} \\
\ell(\hat{g}_{j,n}, \varphi_n)_{L^2(\mathbb{D})} &= -\ell(\hat{f}_n, D_j\varphi_n)_{L^2(\mathbb{D})}.
\end{aligned}$$

Since $\varphi_n \in C_c^\infty(\mathbb{D})$ was arbitrary, this implies that $\hat{g}_{j,n} = D_j \hat{f}_n$. Moreover, since $\hat{g}_{j,n} \in L^2(\mathbb{D})$, we obtain $\hat{f}_n \in H^1(\mathbb{D})$. In the case that $j = 3$ we have

$$\begin{aligned}
(D_3 f, \varphi)_{L^2} &= \int_{S_F} f\varphi n_3 + \int_{\Gamma_\ell} f\varphi n_3 + \int_{\Gamma_0} f\varphi n_3 - (f, D_3\varphi)_{L^2} \\
&= \int_{\Gamma_\ell} f\varphi - \int_{\Gamma_0} f\varphi - (f, D_3\varphi)_{L^2} \\
&= (f|_{\Gamma_\ell} - f|_{\Gamma_0}, \varphi|_{\Gamma_0})_{L^2(\mathbb{D})} - (f, D_3\varphi)_{L^2} \\
&= -(f, D_3\varphi)_{L^2} \\
(D_3 f, \varphi)_{H^0_{\mathrm{p}}} &= -(f, D_3\varphi)_{H^0_{\mathrm{p}}} \\
\ell(\hat{g}_{3,n}, \varphi_n)_{L^2(\mathbb{D})} &= -\ell\left(\hat{f}_n, \frac{2\pi i n}{\ell}\varphi_n\right)_{L^2(\mathbb{D})} \\
(\hat{g}_{3,n}, \varphi_n)_{L^2(\mathbb{D})} &= \left(\frac{2\pi i n}{\ell}\hat{f}_n, \varphi_n\right)_{L^2(\mathbb{D})}.
\end{aligned}$$

Since $\varphi_n \in C_c^\infty(\mathbb{D})$ was arbitrary in a dense subset of $L^2(\mathbb{D})$, $\hat{g}_{3,n} = \frac{2\pi i n}{\ell}\hat{f}_n$. It now follows that $f \in H^1_{\mathrm{p}}$, since it is straightforward to verify that $\sum_n \hat{f}_n e^{2\pi i n a_3/\ell}$ converges to f in H^1 and

$$\begin{aligned}
\|f\|^2_{H^1_{\mathrm{p}}} &= \ell \sum_n \|\hat{f}_n\|^2_{H^1(\mathbb{D})} + \left(\frac{2\pi n}{\ell}\right)^2 \|\hat{f}_n\|^2_{L^2(\mathbb{D})} \\
&= \ell \sum_n \|\hat{f}_n\|^2_{L^2(\mathbb{D})} + \|D_1\hat{f}_n\|^2_{L^2(\mathbb{D})} + \|D_2\hat{f}_n\|^2_{L^2(\mathbb{D})} + \left\|\frac{2\pi i n}{\ell}\hat{f}_n\right\|^2_{L^2(\mathbb{D})} \\
&= \|f\|^2_{L^2} + \ell \sum_n \|\hat{g}_{1,n}\|^2_{L^2(\mathbb{D})} + \|\hat{g}_{2,n}\|^2_{L^2(\mathbb{D})} + \|\hat{g}_{3,n}\|^2_{L^2(\mathbb{D})} \\
&= \|f\|^2_{H^1} < \infty.
\end{aligned}$$

For the inductive step we now let $f \in X_{k+1}$. As we saw in the base case, it is sufficient to show that (i) $f_n \in H^{k+1}(\mathbb{D})$ and (ii) for all $|\boldsymbol{\alpha}| \leq k+1$, $\alpha_i \in \{1, 2, 3\}$,

$$\hat{g}_{\boldsymbol{\alpha},n} = \left(\frac{2\pi i n}{\ell}\right)^b D_{\boldsymbol{\beta}}\hat{f}_n$$

where $D_{\boldsymbol{\alpha}} f = \sum_n \hat{g}_{\boldsymbol{\alpha},n} e^{2\pi i n a_3/\ell}$ in L^2, $\beta_i \in \{1,2\}$, and $D_{\boldsymbol{\alpha}} = D_3^b D_{\boldsymbol{\beta}}$. However, since $f \in X_k = H_{\mathrm{p}}^k$ we have already that $f_n \in H^k(\mathbb{D})$ and (ii) holds for all $|\boldsymbol{\alpha}| \leq k$. Therefore we take $\boldsymbol{\alpha}$ such that $|\boldsymbol{\alpha}| = k+1$ and consider two cases:

Case 1: Suppose $D_{\boldsymbol{\alpha}} \neq D_3^{k+1}$. Then $D_{\boldsymbol{\beta}} = D_j D_{\boldsymbol{\gamma}}$ where $|\boldsymbol{\gamma}| \leq k$ and $j \in 1,2$. Taking φ as in the base case,

$$
\begin{aligned}
(D_{\boldsymbol{\alpha}} f, \varphi)_{L^2} &= -(D_3^b D_{\boldsymbol{\gamma}} f, D_j \varphi)_{L^2} \\
\ell(\hat{g}_{\boldsymbol{\alpha},n}, \varphi_n)_{L^2(\mathbb{D})} &= -\ell \left(\left(\frac{2\pi i n}{\ell} \right)^b D_{\boldsymbol{\gamma}} \hat{f}_n, D_j \varphi_n \right)_{L^2(\mathbb{D})}.
\end{aligned}
$$

Thus $\hat{g}_{\boldsymbol{\alpha},n} = D_j \left(\frac{2\pi i n}{\ell}\right)^b D_{\boldsymbol{\gamma}} \hat{f}_n = \left(\frac{2\pi i n}{\ell}\right)^b D_{\boldsymbol{\beta}} \hat{f}_n$ and $\hat{f}_n \in H^{k+1}(\mathbb{D})$.

Case 2: Suppose $D_{\boldsymbol{\alpha}} = D_3^{k+1}$. Taking φ as before,

$$
\begin{aligned}
(D_{\boldsymbol{\alpha}} f, \varphi)_{L^2} &= \int_{\Gamma_\ell} D_3^k f \varphi - \int_{\Gamma_0} D_3^k f \varphi - (D_3^k f, D_3 \varphi)_{L^2} \\
&= (D_3^k f|_{\Gamma_\ell} - D_3^k f|_{\Gamma_0}, \varphi|_{\Gamma_0})_{L^2(\mathbb{D})} - (D_3^k f, D_3 \varphi)_{L^2} \\
&= -(D_3^k f, D_3 \varphi)_{L^2} \\
\ell(\hat{g}_{\boldsymbol{\alpha},n}, \varphi_n)_{L^2(\mathbb{D})} &= -\ell \left(\left(\frac{2\pi i n}{\ell} \right)^k \hat{f}_n, \frac{2\pi i n}{\ell} \varphi_n \right)_{L^2(\mathbb{D})} \\
(\hat{g}_{\boldsymbol{\alpha},n}, \varphi_n)_{L^2(\mathbb{D})} &= \left(\left(\frac{2\pi i n}{\ell} \right)^{k+1} \hat{f}_n, \varphi_n \right)_{L^2(\mathbb{D})}.
\end{aligned}
$$

Thus $\hat{g}_{\boldsymbol{\alpha},n} = \left(\frac{2\pi i n}{\ell}\right)^{k+1} \hat{f}_n$ and the claim follows. □

Finally, we verify that H_{p}^k provides an appropriate setting for the a_3-periodic problem on Ω_∞. More precisely, we show that H_{p}^k is exactly the set of functions on Ω whose a_3-periodic extensions reside in $H_{\mathrm{loc}}^k(\Omega_\infty)$, as desired.

Lemma 3.4. *Let $\Psi : L^2(\mathcal{T}) \to L^2(\Omega)$ be defined by $\Psi f = f \circ \Phi$. Then $\Psi|_{H^k(\mathcal{T})}$ is an isomorphism from $H^k(\mathcal{T})$ onto $H_p^k(\Omega)$.*

Proof. Since Φ is C^∞ and periodicity is obviously preserved we immediately obtain $\Psi(H^k(\mathcal{T})) \subset H^k_{\mathrm{p}}$. It is then simple to show that Ψ is injective with surjectivity following from the fact that Φ_Ω^{-1} is C^∞. $\square$

3.3 Laplace's Equation

Several times in the course of this work, we will seek to show that certain quantities are uniquely determined. In following the general approach established by Beale, we will see that, just as in [7], many of these quantities can be cast as solutions of a particular problem involving Laplace's equation. Adapting the boundary conditions to reflect periodicity and the absence of a fixed bottom surface, the relevant problem in our setting takes the form

$$\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } S_F, \qquad D_3^k u|_{\Gamma_\ell} = D_3^k u|_{\Gamma_0} \quad \text{for } k \in \{0,1\} \tag{3.18}$$

where $f \in H^{s-2}_{\mathrm{p}}$ is given.

Before we move on to discussing the particulars of this problem, it is imperative that we understand the various types of "solutions" possible for boundary value problems when considered in a Sobolev setting. Let us consider the generic problem $Lu = f$, where L is some differential operator and f is a given function. To simplify our discussion and illustrate the main differences in solution types, we speak rather broadly now and ignore many details regarding the underlying domain, boundary conditions, and so forth. First, we consider a function u to be a *strong solution* of $Lu = f$ provided that it satisfies this equation in L^2_{loc}. Thus a strong solution u must have sufficient regularity to ensure that the appropriate weak derivatives all lie in L^2_{loc}. While the general notion of a strong solution is much broader than the one stated here (including, for instance, functions satisfying the equation in other L^p spaces), we do not concern ourselves with anything but the L^2 setting in this work. Note that we will usually refer to u as a solution, omitting the word "strong" except when we want to draw the reader's attention and enhance the distinction with solutions of a different type.

In contrast to a solution, a *classical solution* of the problem must satisfy $Lu = f$ in the sense of classical derivatives. It should be clear that any classical solution of the problem is also a solution. Additionally, we will often speak of *weak solutions* to a problem like $Lu = f$. The definition of a weak solution is specific to each problem, but generally involves using integration by parts on $Lu = f$ to obtain a weaker reformulation (often called a variational formulation) which makes sense for u with less regularity than would be required of a solution. Any function satisfying this variational formulation of the problem is then termed a weak solution of the original problem.

As an example, suppose that $u \in H^2(U)$ is a solution of the problem $\Delta u = f$ on U, $u = 0$ on ∂U, where U is a smooth, bounded domain and $f \in L^2(U)$ is given. To find a variational formulation of this problem, we begin by multiplying through the Laplace equation by a function v (which we will eventually restrict to the space of potential weak solutions) and integrating. Performing integration by parts on the left-hand side then yields

$$-(\nabla u, \nabla v)_{L^2(U)} + (\nabla u, v \cdot \mathbf{n})_{L^2(\partial U)} = (f, v)_{L^2(U)}$$

where $\mathbf{n}$ is the outward unit normal on ∂U. This equation makes sense for all $u, v \in H^1(U)$ but does not incorporate the fact that we want u to vanish on the boundary. We therefore seek to include this condition in the underlying space itself by considering $u, v \in H_0^1(U)$. Since v now vanishes on the boundary, the above equation simplifies further and we define a weak solution of the problem $\Delta u = 0$ on U, $u = 0$ on ∂U, to be any $u \in H_0^1(U)$ satisfying the variational formulation $(\nabla u, \nabla v)_{L^2(U)} = -(f, v)_{L^2(U)}$ for all $v \in H_0^1(U)$. Since most of the variational formulations of problems found in this work can be obtained through nearly identical means, we will generally refrain from defining a weak solution in each instance. When the variational formulation of a problem is not obvious (e.g., when the free boundary condition $\mathbf{S}(\mathbf{v}, q) = 0$ is included), we will derive it explicitly. While the above work implies that any solution is also a weak solution, the converse need not be true since the integration by parts generally requires that $u \in H^2(U)$.

The following result, analogous to Lemma 2.8 from [7], demonstrates that (3.18) has a unique solution and provides an estimate for it in terms of the inhomogeneity f. The proof given below, however, does not draw from the associated proof in [7].

Lemma 3.5. *For $f \in H_p^{s-2}, s \geq 2$, there is a unique solution $u \in {}^0H_p^s$ of*

$$\Delta u = f \qquad \text{on } \Omega.$$

Additionally, there exists $C > 0$, independent of f, such that

$$\|u\|_{H_p^s} \leq C\|f\|_{H_p^{s-2}}.$$

Proof. Let $f = \sum_n \hat{f}_n e^{2\pi i n a_3/\ell}$. We first consider the boundary-value problem, $L_n u = -\hat{f}_n$ on $\mathbb{D}$ with $u = 0$ on $\partial\mathbb{D}$, where L_n and its associated sesquilinear form ($B_n : H_0^1(\mathbb{D}) \times H_0^1(\mathbb{D}) \to \mathbb{C}$) are given by

$$L_n u = -\Delta u + \left(\frac{2\pi n}{\ell}\right)^2 u$$

$$B_n[u, v] = (\nabla u, \nabla v)_{\mathbf{L}^2(\mathbb{D})} + \left(\frac{2\pi n}{\ell}\right)^2 (u, v)_{L^2(\mathbb{D})}.$$

It is well-known that B_n is continuous and coercive on $H_0^1(\mathbb{D})$, thus we can apply Lax-Milgram (Theorem B.5) to obtain a unique weak solution, $\hat{u}_n \in H_0^1(\mathbb{D})$. The construction $u = \sum_n \hat{u}_n e^{2\pi i n a_3/\ell}$ is then our candidate for the solution of the boundary-value problem in Ω. We now restrict our discussion to the case when $s = k \in \mathbb{Z}$. Given the regularity of $\partial\mathbb{D}$ we can immediately conclude that each $\hat{u}_n \in H^k(\mathbb{D})$ is a strong solution. Our goal is to show that $u \in H_{\mathrm{p}}^k$. First we obtain some preliminary estimates for $\hat{u}_n$ where $n \neq 0$:

$$\begin{aligned} B_n[\hat{u}_n, \hat{u}_n] &= (-\hat{f}_n, \hat{u}_n)_{L^2(\mathbb{D})} \\ \|\nabla \hat{u}_n\|^2_{\mathbf{L}^2(\mathbb{D})} + \left(\frac{2\pi n}{\ell}\right)^2 \|\hat{u}_n\|^2_{L^2(\mathbb{D})} &\leq \|\hat{f}_n\|_{L^2(\mathbb{D})}\|\hat{u}_n\|_{L^2(\mathbb{D})} \\ \|\hat{u}_n\|_{L^2(\mathbb{D})} &\leq \left(\frac{\ell}{2\pi n}\right)^2 \|\hat{f}_n\|_{L^2(\mathbb{D})}. \end{aligned}$$

Notice that from this estimate we can conclude

$$\sum_n \left(\frac{2\pi n}{\ell}\right)^{2k} \|\hat{u}_n\|^2_{L^2(\mathbb{D})} \leq \sum_n \left(\frac{2\pi n}{\ell}\right)^{2(k-2)} \|\hat{f}_n\|^2_{L^2(\mathbb{D})} \leq \|f\|^2_{H_{\mathrm{p}}^{k-2}} < \infty.$$

This gives us an estimate on the lowest order terms in the H_{p}^k-norm. For the highest order terms, standard elliptic regularity theory (e.g., see [13], p. 323) provides an estimate of the form

$$\|\hat{u}_n\|_{H^k(\mathbb{D})} \leq C_1 \|\hat{f}_n\|_{H^{k-2}(\mathbb{D})},$$

though the constant C_1 here generally depends on the coefficients (and hence n) of L_n. However, upon closer inspection of the proof of this result (e.g., in [13]) we find that we can use our above estimates on $\|\hat{u}_n\|_{L^2(\mathbb{D})}$ in place of the usual L^∞ estimate on the coefficient $(2\pi n/\ell)^2$ of L_n. This ultimately allows C_1 to be chosen independently of n. Therefore

$$\sum_n \|\hat{u}_n\|^2_{H^k(\mathbb{D})} \leq C_1^2 \sum_n \|\hat{f}_n\|^2_{H^{k-2}(\mathbb{D})} \leq C_1^2 \|f\|^2_{H_{\mathrm{p}}^{k-2}(\Omega)} < \infty.$$

Finally, we must show that the intermediate order terms in the H_{p}^k-norm are also summable. Exploiting complex interpolation between $H^0(\mathbb{D})$ and $H^k(\mathbb{D})$ and Young's inequality we obtain for each $0 < m < k$

$$\begin{aligned}
\sum_n \left(\frac{2\pi n}{\ell}\right)^{2m} \|\hat{u}_n\|^2_{H^{k-m}(\mathbb{D})} &\leq \sum_n \left(\frac{2\pi n}{\ell}\right)^{2m} \left(\|\hat{u}_n\|^{m/k}_{L^2(\mathbb{D})} \|\hat{u}_n\|^{1-m/k}_{H^k(\mathbb{D})}\right)^2 \\
&\leq C_2 \sum_n \left(\frac{2\pi n}{\ell}\right)^{2m(k-2)/k} \|\hat{f}_n\|^{2m/k}_{L^2(\mathbb{D})} \|\hat{f}_n\|^{2(k-m)/k}_{H^{k-2}(\mathbb{D})} \\
&\leq C_2 \sum_n \left(\frac{m}{k}\left[\left(\frac{2\pi n}{\ell}\right)^{2m(k-2)/k} \|\hat{f}_n\|^{2m/k}_{L^2(\mathbb{D})}\right]^{k/m} \right. \\
&\qquad \left. + \frac{k-m}{k}\left[\|\hat{f}_n\|^{2(k-m)/k}_{H^{k-2}(\mathbb{D})}\right]^{k/(k-m)}\right) \\
&= C_2 \sum_n \frac{m}{k}\left(\frac{2\pi n}{\ell}\right)^{2(k-2)} \|\hat{f}_n\|^2_{L^2(\mathbb{D})} + \frac{k-m}{k}\|\hat{f}_n\|^2_{H^{k-2}(\mathbb{D})} \\
&\leq C_3 \|f\|^2_{H^{k-2}}.
\end{aligned}$$

Thus $u \in {}^0H_{\mathrm{p}}^k$ with $\|u\|^2_{H_{\mathrm{p}}^k} \leq C_4(k+1)\|f\|^2_{H_{\mathrm{p}}^{k-2}}$ which completes the proof for integer values of s. Interpolation then provides the remaining cases. □

3.4 The Modified Helmholtz Projection

We finish off this chapter by turning our attention to the projection space $\mathbf{P}^0$. Since $\mathbf{P}^0$ is a closed subspace of $\mathbf{L}^2$, $\mathbf{L}^2$ can be written as $\mathbf{L}^2 = \mathbf{P}^0 \oplus (\mathbf{P}^0)^\perp$. Thus, for any $\mathbf{f} \in \mathbf{L}^2$, $\mathbf{f}$ can be uniquely decomposed as $\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2$ where $\mathbf{f}_1 \in \mathbf{P}^0$ and $\mathbf{f}_2 \in (\mathbf{P}^0)^\perp$. This decomposition will be a cornerstone of our analysis of the linearized problem in Chapter 5. However, before we can take full advantage of this tool, we need to obtain a more useful description of the elements residing in $(\mathbf{P}^0)^\perp$. The following characterization provides us with the exact form of the decomposition.

Lemma 3.6. *The orthogonal complement of* $\mathbf{P}^0$ *in* $\mathbf{L}^2$ *has the characterization* $(\mathbf{P}^0)^\perp = \{\nabla q : q \in {}^0H^1_p\}$.

Proof. Let $\mathbf{Y} = \{\nabla q : q \in {}^0H^1_\mathrm{p}\}$. It is sufficient to show two things: (i) $\mathbf{Y}$ is closed in $\mathbf{L}^2$ so that $\mathbf{Y} = (\mathbf{Y}^\perp)^\perp$, and (ii) $\mathbf{P}^0 = \mathbf{Y}^\perp$. In order to prove (i), we will first need to show that the orthogonal complement of $\mathbf{X} = \overline{{}^0\mathbf{C}^\infty_{\mathrm{p}\sigma}}^{\|\cdot\|_{\mathbf{L}^2}}$ in $\mathbf{L}^2$ has the characterization $\mathbf{X}^\perp = \{\nabla q : q \in H^1_\mathrm{p}\}$. Let $q \in H^1_\mathrm{p}, \mathbf{u} \in \mathbf{X}$. There exist $\mathbf{u}_k \in {}^0\mathbf{C}^\infty_{\mathrm{p}\sigma}$ such that $\mathbf{u}_k \to \mathbf{u}$ in $\mathbf{L}^2$. Integration by parts yields

$$(\nabla q, \mathbf{u})_{\mathbf{L}^2} = \lim_{k\to\infty} (\nabla q, \mathbf{u}_k)_{\mathbf{L}^2} = \lim_{k\to\infty} \int_{\Gamma_\ell} q\overline{\mathbf{u}_k} \cdot \mathbf{e}_3 + \int_{\Gamma_0} q\overline{\mathbf{u}_k} \cdot (-\mathbf{e}_3) = 0.$$

Thus $\nabla q \in \mathbf{X}^\perp$. Conversely, let $\mathbf{w} \in \mathbf{X}^\perp$. Then, in particular, $(\mathbf{w}, \mathbf{u})_{\mathbf{L}^2} = 0$ for all $\mathbf{u} \in \mathbf{C}^\infty_{\mathrm{c}\sigma}$. Thus there exists $p \in H^1$ such that $\mathbf{w} = \nabla p$ by Theorem B.4(i). Now consider

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \\ u(a_1, a_2) \end{pmatrix} \in {}^0\mathbf{C}^\infty_{\mathrm{p}\sigma}$$

where $u \in C^\infty_\mathrm{c}(\mathbb{D})$ is arbitrary. Then, applying integration by parts, we obtain

$$\begin{aligned} 0 = (\mathbf{w}, \mathbf{u})_{\mathbf{L}^2} &= (\nabla p, \mathbf{u})_{\mathbf{L}^2} \\ &= \int_{\Gamma_\ell} p\overline{\mathbf{u}} \cdot \mathbf{e}_3 + \int_{\Gamma_0} p\overline{\mathbf{u}} \cdot (-\mathbf{e}_3) \\ &= \int_{\Gamma_\ell} p\overline{u} - \int_{\Gamma_0} p\overline{u} \\ &= \left(p|_{\Gamma_\ell} - p|_{\Gamma_0}, u\right)_{L^2(\mathbb{D})}. \end{aligned}$$

Since u is an arbitrary element of a dense subset of $L^2(\mathbb{D})$ by Theorem B.1, this implies that $p|_{\Gamma_\ell} = p|_{\Gamma_0}$ on $L^2(\mathbb{D})$. Hence $p \in H^1_{\mathrm{p}}$ by Lemma 3.3.

With this characterization in hand, we can now prove (i). Let $q_k \in {}^0H^1_{\mathrm{p}}$ such that $\nabla q_k \to \mathbf{f} \in \mathbf{L}^2$. Since $\nabla q_k \in \mathbf{X}^\perp$, we have

$$(\mathbf{f}, \mathbf{u})_{\mathbf{L}^2} = \lim_{k\to\infty} (\nabla q_k, \mathbf{u})_{\mathbf{L}^2} = 0$$

for all $\mathbf{u} \in \mathbf{X}$. Hence $\mathbf{f} \in \mathbf{X}^\perp$ and so there exists $p \in H^1_{\mathrm{p}}$ such that $\mathbf{f} = \nabla p$. Notice that for $n \neq 0$

$$\begin{aligned}\|(\hat{q}_k)_n - \hat{p}_n\|^2_{H^1(\mathbb{D})} &\leq \ell^2 \sum_n \left(\left(\frac{2\pi n}{\ell}\right)^2 \|(\hat{q}_k)_n - \hat{p}_n\|^2_{H^1(\mathbb{D})} \right. \\ &\qquad\qquad \left. + \sum_{j=1}^{2} \|D_j((\hat{q}_k)_n - \hat{p}_n)\|^2_{L^2(\mathbb{D})} \right) \\ &\leq \ell^2 \|\nabla(q_k - p)\|^2_{\mathbf{L}^2}.\end{aligned}$$

Thus $(\hat{q}_k)_n \to \hat{p}_n$ in $H^1(\mathbb{D})$. Since $(\hat{q}_k)_n \in H^1_0(\mathbb{D})$, a closed subspace of $H^1(\mathbb{D})$, we obtain $\hat{p}_n \in H^1_0(\mathbb{D})$ for $n \neq 0$. For $n = 0$, applying the standard Poincaré inequality yields a constant $C > 0$ such that

$$\|(\hat{q}_k)_0 - (\hat{q}_m)_0\|^2_{H^1(\mathbb{D})} \leq C\|\nabla((\hat{q}_k)_0 - (\hat{q}_m)_0)\|^2_{(L^2(\mathbb{D}))^2} \leq C\ell^2 \|\nabla(q_k - q_m)\|^2_{\mathbf{L}^2}$$

which implies that $(\hat{q}_k)_0$ converges in $H^1_0(\mathbb{D})$. Moreover, the limit is necessarily $\hat{p}_0 + \lambda$, for some $\lambda \in \mathbb{R}$, since it is readily seen that $(\hat{q}_k)_0$ converges to this in the weaker L^2 norm. Thus $\mathbf{f} = \nabla q$ where $q = p + \lambda \in {}^0H^1_{\mathrm{p}}$. Hence $\mathbf{Y}$ is closed in $\mathbf{L}^2$.

Finally, we show (ii). Let $\mathbf{u} \in \mathbf{Y}^\perp$ and $\varphi \in C^\infty_{\mathrm{c}}$. Then

$$0 = (\nabla\varphi, \mathbf{u})_{\mathbf{L}^2} = -\int_\Omega \varphi(\overline{\nabla \cdot \mathbf{u}}).$$

Hence $\nabla \cdot \mathbf{u}$ acts as a bounded linear functional on C_c^∞ and can be extended uniquely to one defined on all of L^2 by density. This unique operator must be the zero functional and thus $\mathbf{u} \in \mathbf{P}^0$. Conversely, let $\mathbf{v} \in \mathbf{P}^0$. Since $\mathbf{L}^2 = \mathbf{Y} \oplus \mathbf{Y}^\perp$, there are $q \in {}^0H_\mathrm{p}^1$ and $\tilde{\mathbf{v}} \in \mathbf{Y}^\perp$ such that $\mathbf{v} = \tilde{\mathbf{v}} + \nabla q$. Taking the divergence of both sides of this equation yields $\Delta q = 0$ and, by Lax-Milgram, q must be the unique solution of this equation in ${}^0H_\mathrm{p}^1$. Thus $q = 0$ and $\mathbf{v} = \tilde{\mathbf{v}} \in \mathbf{Y}^\perp$. Thus $\mathbf{P}^0 = \mathbf{Y}^\perp$ and the claim follows. □

Now that we know precisely how the decomposition works, we can consider the bounded orthogonal projection associated with it. The projection $P : \mathbf{H}_\mathrm{p}^0 \to \mathbf{P}^0$ is defined by $P\mathbf{f} = \mathbf{f}_1$ where $\mathbf{f}_1 \in \mathbf{P}^0$ is the divergence-free portion of $\mathbf{f}$ as described at the beginning of this section. In Chapter 5, we will apply this to a linearized version of (2.9) in an effort to solve this equation in a projection space where we are assured that the divergence-free and a_3-periodicity conditions are automatically satisfied. Before we can do this, however, we need to know whether or not the projection preserves regularity. The following lemma confirms that the projection does not affect a function's regularity.

Lemma 3.7. *Suppose $s \geq 0$. Then*

(i) $P\mathbf{H}_p^s = \mathbf{P}^s$ and $P|_{\mathbf{H}_p^s} : \mathbf{H}_p^s \to \mathbf{P}^s$ is bounded.

(ii) $P|_{\mathbf{K}_p^s} : \mathbf{K}_p^s \to \mathbf{K}_p^s$ is bounded with norm bounded independent of T.

Proof. (i) First we consider the case where $s \geq 1$. For $\mathbf{v} \in \mathbf{H}_\mathrm{p}^s$, we have $(I - P)\mathbf{v} = \nabla\phi$ for some $\phi \in {}^0H_\mathrm{p}^1$. Then for all $\psi \in {}^0H_\mathrm{p}^1$,

$$\int_\Omega \psi \nabla \cdot \mathbf{v} = \int_{\Gamma_0} \psi \mathbf{v} \cdot (-\mathbf{e}_3) + \int_{\Gamma_\ell} \psi \mathbf{v} \cdot \mathbf{e}_3 - \int_\Omega \nabla\psi \cdot \mathbf{v} = -\int_\Omega \nabla\psi \cdot \nabla\phi.$$

We also notice that for any $\phi' \in {}^0H^2_{\mathrm{p}}$ and $\psi \in {}^0H^1_{\mathrm{p}}$,

$$\int_\Omega \psi\Delta\phi' = -\int_{\Gamma_0} \psi\nabla\phi' \cdot (-\mathbf{e}_3) - \int_{\Gamma_\ell} \psi\nabla\phi' \cdot \mathbf{e}_3 + \int_\Omega \psi\Delta\phi' = -\int_\Omega \nabla\psi \cdot \nabla\phi'.$$

Thus ϕ is a weak solution of the problem

$$\begin{aligned} \Delta\phi &= \nabla \cdot \mathbf{v} && \text{on } \Omega \\ \phi &= 0 && \text{on } S_F \\ D_3^k\phi|_{\Gamma_\ell} &= D_3^k\phi|_{\Gamma_0} && \text{for } k \in \{0, 1\}. \end{aligned}$$

This weak solution is unique in ${}^0H^1_{\mathrm{p}}$ (which follows easily using Lax-Milgram (Theorem B.5)) and therefore, by Lemma 3.5, it must actually be a strong solution (in ${}^0H^{s+1}_{\mathrm{p}}$) satisfying

$$\|\phi\|_{H^{s+1}_{\mathrm{p}}} \leq C\|\nabla \cdot \mathbf{v}\|_{H^{s-1}_{\mathrm{p}}} \leq C\|\mathbf{v}\|_{\mathbf{H}^s_{\mathrm{p}}}.$$

Thus $(I-P)\mathbf{H}^s_{\mathrm{p}} \subset \mathbf{H}^s_{\mathrm{p}}$ and $(I-P)|_{\mathbf{H}^s_{\mathrm{p}}} : \mathbf{H}^s_{\mathrm{p}} \to \mathbf{H}^s_{\mathrm{p}}$ is bounded. Finally, we observe that $P\mathbf{v} = \mathbf{v} - (I-P)\mathbf{v} \in \mathbf{P}^s$ with the rest of the claim following from boundedness of $I - P$. The remaining cases are obtained by interpolation between $\mathbf{H}^0_{\mathrm{p}}$ and $\mathbf{H}^1_{\mathrm{p}}$.

(ii) We begin by demonstrating that P commutes with D_t. Let $\mathbf{f} \in \mathbf{H}^1((0,T);\mathbf{L}^2)$ and denote its decomposition by $\mathbf{f} = \mathbf{f}_\sigma + \nabla p$. Since D_t commutes with spatial derivatives, we have $D_t\mathbf{f}_\sigma \in \mathbf{P}^0$ and hence

$$D_tP\mathbf{f} - PD_t\mathbf{f} = D_t\mathbf{f}_\sigma - PD_t\mathbf{f}_\sigma - PD_t\nabla p = -P\nabla(D_tp) = 0.$$

Now take $\mathbf{f} \in \mathbf{K}^{2k}_{\mathrm{p}}$ where $k \in \mathbb{N}_0$. If $k = 0$, then we can use (i) to obtain

$$\|P\mathbf{f}\|^2_{\mathbf{K}^0_{\mathrm{p}}} = 2\|P\mathbf{f}\|^2_{\mathbf{L}^2((0,T);L^2)} = 2\int_0^T \|P\mathbf{f}\|^2_{\mathbf{L}^2} \leq 2C\int_0^T \|\mathbf{f}\|^2_{\mathbf{L}^2} = C\|\mathbf{f}\|^2_{\mathbf{K}^0_{\mathrm{p}}}.$$

Otherwise, using (i) and the commutativity of P and D_t yields

$$\begin{aligned}
\|P\mathbf{f}\|^2_{\mathbf{K}^{2k}_{\mathrm{p}}} &= \|P\mathbf{f}\|^2_{\mathbf{L}^2((0,T);H^{2k}_{\mathrm{p}})} + \|P\mathbf{f}\|^2_{\mathbf{H}^k((0,T);L^2)} \\
&= \int_0^T \|P\mathbf{f}\|^2_{\mathbf{H}^{2k}_{\mathrm{p}}} + \sum_{n\le k} \int_0^T \|D_t^n(P\mathbf{f})\|^2_{\mathbf{L}^2} \\
&\le C\left(\int_0^T \|\mathbf{f}\|^2_{\mathbf{H}^{2k}_{\mathrm{p}}} + \sum_{n\le k} \int_0^T \|D_t^n\mathbf{f}\|^2_{\mathbf{L}^2} \right) \\
&= C\|\mathbf{f}\|^2_{\mathbf{K}^{2k}_{\mathrm{p}}}.
\end{aligned}$$

The remaining cases follow by interpolation between the $\mathbf{K}^{2k}_{\mathrm{p}}$ spaces. □

Since we have chosen $\mathbf{P}^0$ to include the divergence-free functions whose generalized trace does not vanish on the boundary, we have modified the standard Helmholtz decomposition. In particular, we have increased the size of the underlying projection space and, as a consequence, reduced the size of its orthogonal complement. This means that applying P to the momentum equation will not necessarily remove the pressure term as p need not be constant on the free surface (excluding the possibility that ∇p can be written as ∇q for some $q \in {}^0H^1_{\mathrm{p}}$). However, though P does not fully remove the pressure gradient itself, it does remove the term's indeterminacy. That is, the projection of the pressure gradient will be a quantity whose value is determined completely by the velocity. We save the details for Chapter 5, but they will rely on the following characterization of the projections of gradients.

Lemma 3.8. *Suppose $s \ge 1$. If $f \in H^s_p$, then there is a unique $\tilde{f} \in H^s_p$ such that*

$$P(\nabla f) = \nabla \tilde{f}, \qquad f|_{S_F} = \tilde{f}|_{S_F}, \qquad \textit{and} \qquad \Delta \tilde{f} = 0.$$

Proof. Let us first consider the case when $s = 1$. Since $\nabla f \in L^2$ there exists a unique $q \in {}^0H^1_{\mathrm{p}}$ such that $\nabla f = P(\nabla f) + \nabla q$. Let $\tilde{f} = f - q \in H^1_{\mathrm{p}}$. Then

$$P(\nabla f) = \nabla(f - q) = \nabla \tilde{f},$$

$$\tilde{f}|_{S_F} = (f - q)|_{S_F} = f|_{S_F},$$

and

$$\Delta \tilde{f} = \nabla \cdot \nabla(f - q) = \nabla \cdot P(\nabla f) = 0.$$

Suppose that there were two such functions, $\tilde{f}_1$ and $\tilde{f}_2$, satisfying the desired equations. Then $\nabla(\tilde{f}_1 - \tilde{f}_2) = 0$ implies that $\tilde{f}_1 - \tilde{f}_2$ is constant. This constant must be zero, however, since $(\tilde{f}_1 - \tilde{f}_2)|_{S_F} = 0$. Hence $\tilde{f}$ is unique. Now we let $s \geq 2$. In the proof of Lemma 3.7(i), take $\mathbf{v} = \nabla f$ and let $\tilde{f} = f - \phi \in H^s_{\mathrm{p}}$. Then $\nabla f = P(\nabla f) + \nabla \phi$ implies $P(\nabla f) = \nabla \tilde{f}$, $\phi|_{S_F} = 0$ implies $\tilde{f}|_{S_F} = f|_{S_F}$, and $\Delta \phi = \nabla \cdot (\nabla f) = \Delta f$ implies $\Delta \tilde{f} = 0$ on Ω. Since the difference of any two such functions, $\tilde{f}_1$ and $\tilde{f}_2$, must be zero by Lemma 3.5, $\tilde{f}$ is unique. The remaining cases follow by interpolation. □

4 The Main Result

4.1 Statement

The main result of this work is stated as follows.

Theorem 4.1. *Suppose* $3 < s < \frac{7}{2}$. *For any* $\mathbf{u}_0 \in \mathbf{V}^{s-1}$ *there exists* $T > 0$, *depending on* $\|\mathbf{u}_0\|_{\mathbf{H}_p^{s-1}}$, *so that the problem (2.9)–(2.15) has a solution* $(\mathbf{v}, q)$ *with* $\mathbf{v} \in \mathbf{K}_p^s, q \in K_p^{s-3/2}(\partial G_F)$, *and* $\nabla q \in \mathbf{K}_p^{s-2}$.

There are a few remarks to be made here. First, observe that for $\mathbf{v} \in \mathbf{K}_{\mathrm{p}}^s$, Lemma C.3 implies that

$$\mathbf{v} \in \mathbf{H}^{\varepsilon+1/2}((0,T); H_{\mathrm{p}}^{s-1-2\varepsilon})$$

for all $0 < \varepsilon < (s-1)/2$. Localizing in t, it follows that $\mathbf{v}(0,\cdot) \in \bigcap_{\varepsilon>0} \mathbf{H}_{\mathrm{p}}^{s-1-2\varepsilon}$. Hence it is proper to take the initial data $\mathbf{u}_0$ in $\mathbf{H}_{\mathrm{p}}^{s-1}$. Second, notice that $\nabla q \in \mathbf{K}_{\mathrm{p}}^{s-2}$ does not imply $q \in K_{\mathrm{p}}^{s-1}$; although

$$q \in H^0((0,T); H_{\mathrm{p}}^{s-1}) \cap H^{(s-2)/2}((0,T); H_{\mathrm{p}}^1)$$

it is possible that $q \notin H^{(s-1)/2}((0,T); H_{\mathrm{p}}^0)$. Finally, the interval specified for s arises from several considerations:

1. The value of s needs to be large enough to define and estimate the appropriate nonlinear terms in Chapter 6 and also to transform the given solution into a solution of the original Eulerian problem (although $s > 5/2$ would suffice for this). Additionally, both the homogeneous and inhomogeneous linearized problems in Chapter 5 make use of the extra regularity.

2. The value of s needs to be small enough that additional compatibility conditions on $\mathbf{u}_0$ are not required (see the discussion at the beginning of Section 5.2).

The following corollary demonstrates how Theorem 4.1 can be used to obtain a solution to the original Eulerian problem (2.1)–(2.8), at least in the distributional sense (i.e., when integrated against smooth functions of compact support).

Corollary 4.2. *Suppose* $3 < s < \frac{7}{2}$. *For any* $\mathbf{u}_0 \in \mathbf{V}^{s-1}$ *there exists* $T > 0$, *depending on* $\|\mathbf{u}_0\|_{\mathbf{H}_p^{s-1}}$, *so that the problem* (2.1)–(2.8) *has a solution* $(\mathbf{u}, p)$, *in the distributional sense, on* $(0, T)$.

Proof. Let $(\mathbf{v}, q)$ be the solution of (2.9)–(2.15) provided by Theorem 4.1. Using (2.11) and (2.13), the associated displacement map is then given by $\mathbf{x} = \int_0^t \mathbf{v}$ so that $\mathbf{y} \in \mathbf{H}^1((0,T); H^s_{\mathrm{p}})$. Since $s > 3$, it now follows from the Sobolev Embedding Theorem (B.8) that $\mathbf{y} \in C^{0,1/2}([0,T]; H^s_{\mathrm{p}}) \subset C^{0,1/2}([0,T]; C^{1,1/2}_{\mathrm{p}})$. On the other hand, from Lemma C.3 we can conclude that $\mathbf{v} \in \mathbf{H}^{(s-2)/2}((0,T); H^2_{\mathrm{p}})$ so that $\mathbf{y} \in \mathbf{H}^{s/2}((0,T); H^2_{\mathrm{p}}) \subset C^{1,(s-3)/2}([0,T]; C^{0,1/2}_{\mathrm{p}})$. Hence $\mathbf{y}$ is a continuous function whose partial derivatives of first order (both with respect to time and space) are all Hölder-continuous. In other words, $\mathbf{y} \in C^1(\overline{G})$. Since $\nabla\mathbf{y}(0,\cdot) = I$, the 3×3 identity matrix, this implies that $\nabla\mathbf{y}(t,\cdot)$ remains invertible for sufficiently small t. Moreover, it is continuously dependent on t. Provided that $\dot{\mathbf{y}}(0,\cdot) = \mathbf{u}_0 \neq 0$, it follows from the Inverse Function Theorem that $\mathbf{y}$ is invertible for sufficiently small t. Now, for small enough t and $\mathbf{b} \in \Omega(t)$, we can define $\mathbf{u}(t,\mathbf{b}) = \mathbf{v}(t, \mathbf{y}^{-1}(t,\mathbf{b}))$ and $p(t,\mathbf{b}) = q(t, \mathbf{y}^{-1}(t,\mathbf{b})) + P_0$. It is now readily verified that $(\mathbf{u}, p)$ is a solution of (2.1)–(2.8) in the sense of distributions. Finally, note that in the case of vanishing initial velocity, $(\mathbf{u}, p) = (gt\mathbf{e}_3, P_0)$ is a solution of (2.9)–(2.15). $\square$

4.2 Overview of the Proof

Though many significant details differ, we closely follow the general approach developed by Beale in [7]. To motivate the chapters which follow, we now briefly outline the application of Beale's method to our problem.

As is often the case when studying difficult nonlinear problems, we begin by analyzing the closest linear approximation. Since our goal is to establish a local-in-time existence result for an initial value problem, we base our approximation around the nonlinear problem at $t = 0$. Heuristically, we can argue as follows: suppose that $\boldsymbol{\alpha} = (\mathbf{v}, q)$ is a strong solution of the nonlinear problem (2.9)–(2.15). Using (2.11) and (2.13), the associated displacement map is then given by $\mathbf{x} = \int_0^t \mathbf{v} \in \mathbf{H}^2((0,T); L^2)$. It now follows from the Sobolev Embedding Theorem (B.8) that $\mathbf{x}$ is continuous with respect to t, hence $\mathbf{x} \approx 0$ for small t. This implies that the matrix of conversion factors, $\Lambda = (\lambda_{i,j}(t, \mathbf{a})) = (\nabla \mathbf{y})^{-1}$, is approximately equal to the 3×3 identity matrix for small t. This reduces (2.9)–(2.15) to the linear problem

$$\dot{\mathbf{v}} - \mu \Delta \mathbf{v} + \nabla q = g\, \mathbf{e}_3 \qquad \text{on } G \tag{4.1}$$

$$\nabla \cdot \mathbf{v} = 0 \qquad \text{on } G \tag{4.2}$$

$$\mathbf{v}(0, \cdot) = \mathbf{u}_0 \qquad \text{on } \Omega \tag{4.3}$$

$$\mathbf{S}(\mathbf{v}, q) = 0 \qquad \text{on } \partial G_F \tag{4.4}$$

$$q_{\mathrm{p}}(t, \cdot) \in H^{s-1}_{\mathrm{loc}}(\Omega_\infty), \mathbf{v}_{\mathrm{p}}(t, \cdot) \in \mathbf{H}^s_{\mathrm{loc}}(\Omega_\infty) \qquad \text{on } (0,T) \text{ with } s \geq 2. \tag{4.5}$$

Let us denote the mapping

$$(\mathbf{v}, q) \mapsto (\dot{\mathbf{v}} - \mu \Delta \mathbf{v} + \nabla q,\ \nabla \cdot \mathbf{v},\ \mathbf{v}(0, \cdot),\ \mathbf{S}(\mathbf{v}, q))$$

by L (the periodic condition will be included in the underlying space). While we might be tempted to use the solution, $\boldsymbol{\alpha}_0 = (\mathbf{v}_0, q_0)$, of $L\boldsymbol{\alpha}_0 = (g\,\mathbf{e}_3, 0, \mathbf{u}_0, 0)$ to approximate $\boldsymbol{\alpha}$, this would only guarantee that $\mathbf{v}_0(0,\cdot) = \mathbf{v}(0,\cdot)$. A more useful approximation can be obtained by taking instead the solution of $L\boldsymbol{\alpha}_0 = (g\,\mathbf{e}_3, \sigma, \mathbf{u}_0, 0)$ where σ is constructed in such a way that $\sigma(0,\cdot) = 0$ while also ensuring that $\mathbf{v}_0(0,\cdot) = \mathbf{v}(0,\cdot)$, $\dot{\mathbf{v}}_0(0,\cdot) = \dot{\mathbf{v}}(0,\cdot)$, and $q_0(0,\cdot) = q(0,\cdot)$. Setting $\boldsymbol{\alpha}_1 = \boldsymbol{\alpha} - \boldsymbol{\alpha}_0$, the full nonlinear problem can now be rewritten in the form

$$(L + F)\boldsymbol{\alpha}_1 = \mathbf{g},$$

where F is a nonlinear operator (to be discussed in Chapter 6) and $\mathbf{g}$ depends only on known quantities (such as $\boldsymbol{\alpha}_0$). We can rearrange this equation and apply the inverse of L to get the reformulation $\boldsymbol{\alpha}_1 = L^{-1}(\mathbf{g} - F\boldsymbol{\alpha}_1)$. Finally, we define an operator R by $R\boldsymbol{\omega} = L^{-1}(\mathbf{g} - F\boldsymbol{\omega})$ and show that it is, when restricted to the proper subspace, a strict contraction. It then follows from the contraction mapping principle (Theorem B.9) that R has a fixed point (unique within this subspace), $\boldsymbol{\alpha}_1$, which yields our desired solution, $\boldsymbol{\alpha} = \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1$.

The remainder of this work is organized as follows: Chapter 5 is devoted to developing the understanding of the linear operator L and its invertibility which are key to this approach. Since L^{-1} must be applied to the unknown quantity $\mathbf{g} - F\boldsymbol{\alpha}_1$, the invertibility of L must be demonstrated for the fully inhomogeneous version of the problem (4.1)–(4.5). We begin by showing the unique solvability of the (mostly) homogeneous problem in Section 5.1 and deduce from this the solvability of the inhomogeneous problem in Section 5.2. In Chapter 6, the full nonlinear problem is treated. In Section 6.1, we implement the proof outlined above to obtain our main result and show, additionally, that any two solutions of (2.9)–(2.15) must agree for an initial period of time. In Section 6.2, we show that the solution provided by Theorem 4.1 is axisymmetric provided that the initial data is axisymmetric.

5 The Linearized Problem

In this chapter we study the properties of the linear differential operator L discussed in Chapter 4. The primary result of this chapter is Theorem 5.5 which demonstrates that the problem $L\boldsymbol{\alpha} = \boldsymbol{\beta}$ has a unique solution $\boldsymbol{\alpha}$ for general $\boldsymbol{\beta}$. We proceed in a fashion analogous to the one used by Beale in [7], reducing the fully inhomogeneous problem to the (mostly) homogeneous problem we now consider. For convenience, in this chapter we abbreviate $\mathbf{n}(0,\cdot)$, the outward unit normal on Ω, by $\mathbf{n}$.

5.1 The Homogeneous Case

The linear problem under consideration in this section is

$$\dot{\mathbf{v}} - \mu\Delta\mathbf{v} + \nabla q = \mathbf{f} \qquad \text{on } G \tag{5.1}$$

$$\nabla\cdot\mathbf{v} = 0 \qquad \text{on } G \tag{5.2}$$

$$\mathbf{v}(0,\cdot) = 0 \qquad \text{on } \Omega \tag{5.3}$$

$$\mathbf{S}(\mathbf{v},q) = 0 \qquad \text{on } \partial G_F \tag{5.4}$$

$$q_{\mathrm{p}}(t,\cdot) \in H^{s-1}_{\mathrm{loc}}(\Omega_\infty), \mathbf{v}_{\mathrm{p}}(t,\cdot) \in \mathbf{H}^s_{\mathrm{loc}}(\Omega_\infty) \qquad \text{on } (0,T) \text{ with } s \geq 2 \tag{5.5}$$

where $\mathbf{f} \in \mathbf{K}^s_{\mathrm{p}}$ is such that $P\mathbf{f}(0,\cdot) = 0$. Our first goal is to use the modified Helmholtz projection P to rewrite the problem (5.1)–(5.5) in a variational form which has the velocity as its only unknown. First we notice that for any solution $(\mathbf{v}, q)$ of the problem, (5.2) implies $\mathbf{v}(t) \in \mathbf{P}^0$ for each t. Thus, recalling from the proof of Lemma 3.7(ii) that P commutes with D_t, applying P to (5.1) yields

$$\dot{\mathbf{v}} - \mu P\Delta\mathbf{v} + \nabla q_1 = P\mathbf{f}$$

where $\nabla q_1 = P\nabla q$ (with $\Delta q_1 = 0$ on Ω and $q_1 = q$ on S_F) by Lemma 3.8. As mentioned in Section 3.4, this application of P removes the indeterminacy of the pressure term in the sense that the value of ∇q_1 is determined entirely by $\mathbf{v}$. In fact, more is true: the value of q_1 itself is uniquely determined by $\mathbf{v}$. We now formalize our earlier remarks.

Lemma 5.1. *Suppose $s \geq 2$ and $(\mathbf{v}, q) \in \mathbf{H}_p^s \times H_p^{s-1}$ satisfies (5.4). Then there exists a bounded linear operator $Q : \mathbf{H}_p^s \to H_p^{s-1}$ mapping $\mathbf{v} \mapsto q_1$ where q_1 is the function provided by Lemma 3.8 with $\nabla q_1 = P\nabla q$.*

Proof. To see this, we use the fact that q and q_1 agree on the free surface and observe that (5.4) implies that the normal component of $\mathbf{S}(\mathbf{v}, q)$ must vanish on S_F. Putting these together,

$$\begin{aligned} \mathbf{S}(\mathbf{v}, q_1) \cdot \mathbf{n} &= 0 \\ \sum_{i=1}^{3} (q_1 n_i^2 - \mu \sum_{j=1}^{3} (D_j v_i + D_i v_j) n_j n_i) &= 0 \\ q_1 &= 2\mu \sum_{i,j=1}^{3} n_i n_j D_j v_i \\ q_1 &= 2\mu\kappa^{-2} \sum_{i,j=1}^{2} a_i a_j D_j v_i \end{aligned}$$

on S_F. Recall here that κ is simply the radius of Ω. Given $\mathbf{v} \in \mathbf{H}_{\mathrm{p}}^s$, we note that $f = 2\mu\kappa^{-2} \sum_{i,j=1}^{2} a_i a_j D_j v_i \in H_{\mathrm{p}}^{s-1}$. For $s = 2$, we can apply Lax-Milgram (Theorem B.5) as in Lemma 3.5 to obtain the existence of a unique weak solution $q_1 \in H_{\mathrm{p}}^1$ of the problem

$$\begin{aligned} \Delta q_1 &= 0 && \text{on } \Omega \\ q_1 &= f && \text{on } S_F \\ D_3^k q_1|_{\Gamma_\ell} &= D_3^k q_1|_{\Gamma_0} && \text{for } k \in \{0,1\} \end{aligned}$$

with $\|q_1\|_{H^1_{\mathrm{p}}} \leq C_1 \|\mathbf{v}\|_{\mathbf{H}^2_{\mathrm{p}}}$ where $C_1 > 0$ is independent of $\mathbf{v}$. For $s \geq 3$, we consider the problem

$$\begin{aligned} \Delta \phi &= -\Delta f && \text{on } \Omega \\ \phi &= 0 && \text{on } S_F \\ D_3^k \phi|_{\Gamma_\ell} &= D_3^k \phi|_{\Gamma_0} && \text{for } k \in \{0,1\} \end{aligned}$$

which has a unique solution $\phi \in {}^0H^{s-1}_{\mathrm{p}}$, satisfying $\|\phi\|_{H^{s-1}_{\mathrm{p}}} \leq C_2 \|\Delta f\|_{H^{s-3}_{\mathrm{p}}}$ for some $C_2 > 0$ which is independent of $\mathbf{v}$, by Lemma 3.5. Finally, we set $q_1 = \phi + f \in H^{s-1}_{\mathrm{p}}$ and observe that $\|q_1\|_{H^{s-1}_{\mathrm{p}}} \leq C_2 \|\Delta f\|_{H^{s-3}_{\mathrm{p}}} + \|f\|_{H^{s-1}_{\mathrm{p}}} \leq C_3 \|\mathbf{v}\|_{\mathbf{H}^s_{\mathrm{p}}}$. Interpolation now yields the claim for the remaining values of s. It readily follows that the constructed operator is linear in $\mathbf{v}$. □

We now take the general approach used in semigroup theory by treating (5.1) as an ordinary differential equation with respect to time whose solution is, for each value of t, an element of the appropriate function space ($\mathbf{V}^s$) on Ω. If we define an operator $A : \mathbf{V}^s \to \mathbf{P}^{s-2}$ by

$$A\mathbf{v} = -\mu P \Delta \mathbf{v} + \nabla Q \mathbf{v},$$

the problem (5.1)–(5.5) takes on the form

$$\dot{\mathbf{v}} + A\mathbf{v} = P\mathbf{f} \qquad \text{on } G \tag{5.6}$$

$$\mathbf{v}(0,\cdot) = 0 \qquad \text{on } \Omega. \tag{5.7}$$

Recall that $\mathbf{S}_{\tan}(\mathbf{v})$ vanishes on S_F and both (5.2) and (5.5) are satisfied for all $\mathbf{v} \in \mathbf{V}^s$. Furthermore, (5.4) is satisfied since our construction of Q ensures that the normal component of $\mathbf{S}(\mathbf{v}, q)$ will also vanish on S_F. The operator A is a modification of the standard Stokes operator, an unbounded linear operator appearing frequently in partial differential equations in fluid dynamics and electromagnetics.

To successfully carry out the semigroup approach requires that we gain a thorough understanding of the operator $-A$. In particular, we want to know in which spaces $-A$ is densely defined, whether it is a dissipative and/or closed operator, and what its spectrum $\sigma(-A)$ looks like. We will tackle the matter of determining the spectrum of $-A$ first. Unfortunately, in contrast to the problems treated in [3, 4, 7, 8, 42], A is not invertible with our boundary conditions (implying that 0 lies in the spectrum of A); it is not injective since $A(\mathbf{v} + \mathbf{c}) = A\mathbf{v}$ for any constant vector $\mathbf{c}$ (where Dirichlet boundary conditions on $\mathbf{v}$ exclude this possibility, periodic boundary conditions do not). This, combined with the inability to apply the Poincaré inequality in general, makes the problem of determining the spectrum more challenging here than in the aforementioned cases. Restricting the spectrum of A to a sector in the right half of the plane and providing estimates on the resolvent operator (which immediately translate to similar results for $-A$), the following lemma is a key result of this dissertation.

Lemma 5.2. *Let $s \geq 2$. Then $\sigma(A) \subset \{\lambda \in \mathbb{C} : |Im(\lambda)| \leq Re(\lambda)\}$. Moreover, for λ with $|Im(\lambda)| > Re(\lambda)$ and $|\lambda| \geq \varepsilon > 0$ the resolvent operator $R(\lambda; A) = (A - \lambda I)^{-1}$ satisfies*

$$\|R(\lambda; A)\mathbf{f}\|_{\mathbf{H}_p^s} \leq C(\|\mathbf{f}\|_{\mathbf{H}_p^{s-2}} + (1 + \varepsilon^{-1})(|\lambda| + 1)^{(s-2)/2}\|\mathbf{f}\|_{\mathbf{L}^2}) \tag{5.8}$$

for all $\mathbf{f} \in \mathbf{P}^{s-2}$. Here $C > 0$ is a constant which is independent of λ, ε, and $\mathbf{f}$.

Proof. Our goal is to show that the resolvent set of A, $\rho(A)$, contains all λ such that $|\text{Im}(\lambda)| > \text{Re}(\lambda)$. This is accomplished by looking at an equivalent problem: given $\mathbf{f} \in \mathbf{P}^{s-2}$ and λ with $|\text{Im}(\lambda)| > \text{Re}(\lambda)$, find a unique solution $(\mathbf{v}, q) \in \mathbf{V}^s \times H_{\text{p}}^{s-1}$ of the problem given by

$$-\mu\Delta\mathbf{v} - \lambda\mathbf{v} + \nabla q = \mathbf{f} \tag{5.9}$$

along with (5.2), (5.4), and (5.5). To see that these are equivalent, suppose that there exists $\mathbf{v} \in \mathbf{V}^s$ such that $(A - \lambda I)\mathbf{v} = \mathbf{f}$. Using our decomposition of $\mathbf{L}^2$ there is a $q_0 \in {}^0H_{\text{p}}^1$ such that $\nabla q_0 = \mu(I - P)\Delta\mathbf{v}$. Setting $q = Q\mathbf{v} + q_0$, we obtain (5.9). It is now straightforward to verify that $(\mathbf{v}, q)$ also satisfies (5.2), (5.4), and (5.5). Conversely, given a solution $(\mathbf{v}, q)$ of the stationary problem we can apply P to (5.9) to obtain $(A - \lambda I)\mathbf{v} = \mathbf{f}$. Hence $(A - \lambda I)\mathbf{v} = \mathbf{f}$ has a unique solution $\mathbf{v}$ if and only if the problem (5.2), (5.4), (5.5), (5.9) has a unique solution $(\mathbf{v}, q)$.

We begin by trying to find a weak solution of the problem (5.2), (5.4), (5.5), (5.9). As usual, the general approach is to find a way to solve for $\mathbf{v}$ first and then derive the associated q in a second step. To do this, we want to find a weak formulation of the problem which does not explicitly involve q. Since a solution of the original problem would lie in $\mathbf{V}^2$, we might naively assume that $\mathbf{V}^1$ was the appropriate setting for the variational formulation; unfortunately, no such space exists (recall that $\mathbf{V}^s$ is only defined for $s \geq 2$) as there is simply not enough regularity to make sense of the free boundary condition. However, neglecting the free surface condition in $\mathbf{V}^2$ yields $\mathbf{P}^2$ whose regularity can be relaxed to obtain $\mathbf{P}^1$; let us take this as the underlying space for our variational formulation. Notice that for any $\mathbf{v} \in \mathbf{P}^1$, $\mathbf{v}$ satisfies (5.2) and the portion of (5.5) referring to the velocity. The free surface condition (5.4) is not necessarily satisfied though and will need to be incorporated into the variational formulation directly. To that end, let us consider the sesquilinear form $\langle \cdot, \cdot \rangle : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbb{C}$ defined by

$$\langle \mathbf{v}, \mathbf{u} \rangle = -\lambda(\mathbf{v}, \mathbf{u})_{\mathbf{L}^2} + \frac{\mu}{2} \sum_{i,j=1}^{3} \int_{\Omega} (D_j v_i + D_i v_j)(D_j \bar{u}_i + D_i \bar{u}_j). \tag{5.10}$$

Now suppose $\mathbf{u} \in \mathbf{H}^1_{\mathrm{p}}, \mathbf{v} \in \mathbf{P}^2$, $q \in H^1_{\mathrm{p}}$ and observe that

$$\begin{aligned}
\int_\Omega (-\mu\Delta\mathbf{v} - \lambda\mathbf{v} + \nabla q)\cdot\overline{\mathbf{u}} &= -\mu\left(\int_\Omega \sum_i \Delta v_i \overline{u}_i\right) - \lambda(\mathbf{v},\mathbf{u})_{\mathbf{L}^2} \\
&\qquad + \int_{\partial\Omega} q(\overline{\mathbf{u}}\cdot\mathbf{n}) - \int_\Omega q\nabla\cdot\overline{\mathbf{u}} \\
&= -\mu\left(\sum_i \int_{\partial\Omega} \overline{u}_i(\nabla v_i\cdot\mathbf{n}) - \int_\Omega \nabla\overline{u}_i\cdot\nabla v_i\right) \\
&\qquad - \lambda(\mathbf{v},\mathbf{u})_{\mathbf{L}^2} + \int_{\partial\Omega} q(\overline{\mathbf{u}}\cdot\mathbf{n}) - \int_\Omega q\nabla\cdot\overline{\mathbf{u}} \\
&= \mu\sum_{i,j}\int_\Omega D_j\overline{u}_i D_j v_i - \lambda(\mathbf{v},\mathbf{u})_{\mathbf{L}^2} - \int_\Omega q\nabla\cdot\overline{\mathbf{u}} \\
&\qquad + \sum_i \int_{\partial\Omega} \overline{u}_i(qn_i - \mu\sum_j D_j v_i n_j) \\
&= \langle\mathbf{v},\mathbf{u}\rangle + \int_{\partial\Omega}\mathbf{S}(\mathbf{v},q)\cdot\overline{\mathbf{u}} - \int_\Omega q\nabla\cdot\overline{\mathbf{u}} \\
&\qquad + \mu\sum_{i,j}\int_{\partial\Omega}\overline{u}_i D_i v_j n_j - \int_\Omega D_j\overline{u}_i D_i v_j \\
&= \langle\mathbf{v},\mathbf{u}\rangle + \int_{\partial\Omega}\mathbf{S}(\mathbf{v},q)\cdot\overline{\mathbf{u}} - \int_\Omega q\nabla\cdot\overline{\mathbf{u}} \\
&\qquad + \mu\sum_{i,j}\int_\Omega \overline{u}_i D_i(D_j v_j) \\
&= \langle\mathbf{v},\mathbf{u}\rangle + \int_{\partial\Omega}\mathbf{S}(\mathbf{v},q)\cdot\overline{\mathbf{u}} - \int_\Omega q\nabla\cdot\overline{\mathbf{u}} \\
&\qquad + \mu\sum_i\int_\Omega \overline{u}_i D_i(\nabla\cdot\mathbf{v}) \\
&= \langle\mathbf{v},\mathbf{u}\rangle + \int_{\partial\Omega}\mathbf{S}(\mathbf{v},q)\cdot\overline{\mathbf{u}} - \int_\Omega q\nabla\cdot\overline{\mathbf{u}}. \qquad (5.11)
\end{aligned}$$

Here $\langle\mathbf{v},\mathbf{u}\rangle$ is understood to be the expression given in (5.10) which, of course, remains perfectly well-defined for $\mathbf{u} \in \mathbf{H}^1_{\mathrm{p}}$. Notice that the pair $(\mathbf{v}, q)$ currently satisfies (5.2) and (5.5). If we suppose that $(\mathbf{v}, q)$ additionally satisfies (5.9) and

(5.4), then we obtain

$$
\begin{aligned}
(\mathbf{f}, \mathbf{u})_{\mathbf{L}^2} &= \int_\Omega (-\mu \Delta \mathbf{v} - \lambda \mathbf{v} + \nabla q) \cdot \overline{\mathbf{u}} \\
&= \langle \mathbf{v}, \mathbf{u} \rangle + \int_{\Gamma_\ell} \mathbf{S}(\mathbf{v}, q) \cdot \overline{\mathbf{u}} + \int_{\Gamma_0} \mathbf{S}(\mathbf{v}, q) \cdot \overline{\mathbf{u}} \\
&= \langle \mathbf{v}, \mathbf{u} \rangle + \int_{\Gamma_\ell} (\overline{u}_3 q - \mu \sum_i \overline{u}_i (D_3 v_i + D_i v_3)) \\
&\qquad\qquad - \int_{\Gamma_0} (\overline{u}_3 q - \mu \sum_i \overline{u}_i (D_3 v_i + D_i v_3)) \\
&= \langle \mathbf{v}, \mathbf{u} \rangle.
\end{aligned}
$$

for all $\mathbf{u} \in \mathbf{P}^1$. Thus $\langle \mathbf{v}, \mathbf{u} \rangle = (\mathbf{f}, \mathbf{u})_{\mathbf{L}^2}$ can be seen as a weak formulation of the full problem which does not involve q. In an effort to apply Lax-Milgram (Theorem B.5), we verify that the sesquilinear form is both continuous and coercive. Applying Hölder,

$$
\begin{aligned}
|\langle \mathbf{v}, \mathbf{u} \rangle| &\leq |\lambda| \cdot \|\mathbf{v}\|_{\mathbf{L}^2} \|\mathbf{u}\|_{\mathbf{L}^2} + \frac{\mu}{2} \sum_{i,j} \int_\Omega (|D_j v_i| + |D_i v_j|)(|D_j u_i| + |D_i u_j|) \\
&\leq |\lambda| \cdot \|\mathbf{v}\|_{\mathbf{H}^1_{\mathrm{p}}} \|\mathbf{u}\|_{\mathbf{H}^1_{\mathrm{p}}} + \mu \sum_{i,j} \|D_j v_i\|_{L^2} \|D_j u_i\|_{L^2} + \|D_j v_i\|_{L^2} \|D_i u_j\|_{L^2} \\
&\leq |\lambda| \cdot \|\mathbf{v}\|_{\mathbf{H}^1_{\mathrm{p}}} \|\mathbf{u}\|_{\mathbf{H}^1_{\mathrm{p}}} + 2\mu \sum_{i,j} \|\mathbf{v}\|_{\mathbf{H}^1_{\mathrm{p}}} \|\mathbf{u}\|_{\mathbf{H}^1_{\mathrm{p}}} \\
&\leq C \|\mathbf{v}\|_{\mathbf{H}^1_{\mathrm{p}}} \|\mathbf{u}\|_{\mathbf{H}^1_{\mathrm{p}}}
\end{aligned}
$$

where $C > 0$ depends on μ and λ. Hence the sesquilinear form is continuous. That it is also coercive follows from Korn's inequality (see Section B.2):

$$
\begin{aligned}
|\langle \mathbf{v}, \mathbf{v} \rangle|^2 &= \left| -\lambda \|\mathbf{v}\|^2_{\mathbf{L}^2} + \frac{\mu}{2} \sum_{i,j=1}^{3} \int_\Omega |D_j v_i + D_i v_j|^2 \right|^2 \\
&= \left(\frac{\mu}{2} \sum_{i,j=1}^{3} \int_\Omega |D_j v_i + D_i v_j|^2 - \mathrm{Re}(\lambda) \|\mathbf{v}\|^2_{\mathbf{L}^2} \right)^2 + \left(\mathrm{Im}(\lambda) \|\mathbf{v}\|^2_{\mathbf{L}^2} \right)^2 \qquad (5.12) \\
&\geq \frac{1}{2} \left(\frac{\mu}{2} \sum_{i,j=1}^{3} \int_\Omega |D_j v_i + D_i v_j|^2 - \mathrm{Re}(\lambda) \|\mathbf{v}\|^2_{\mathbf{L}^2} + |\mathrm{Im}(\lambda)| \cdot \|\mathbf{v}\|^2_{\mathbf{L}^2} \right)^2 \\
&\geq C \left(\int_\Omega \sum_{i,j=1}^{3} |v_i|^2 + \frac{1}{4} |D_j v_i + D_i v_j|^2 \right)^2 \\
&\geq C \|\mathbf{v}\|^4_{\mathbf{H}^1_{\mathrm{p}}}.
\end{aligned}
$$

It is also noteworthy that, for $\mathrm{Re}(\lambda) \geq 0$, line (5.12) implies

$$
\begin{aligned}
|\langle \mathbf{v}, \mathbf{v} \rangle|^2 &\geq \mathrm{Im}(\lambda)^2 \|\mathbf{v}\|^4_{\mathbf{L}^2} \\
&\geq \frac{1}{2} (\mathrm{Im}(\lambda)^2 + \mathrm{Re}(\lambda)^2) \|\mathbf{v}\|^4_{\mathbf{L}^2} \\
&\geq \frac{1}{2} |\lambda|^2 \|\mathbf{v}\|^4_{\mathbf{L}^2}. \qquad (5.13)
\end{aligned}
$$

Moreover, the same estimate can be obtained for $\mathrm{Re}(\lambda) < 0$ since line (5.12) then expands to something of the form $\phi + |\lambda|^2 \|\mathbf{v}\|^4_{\mathbf{L}^2}$ where $\phi \geq 0$. (5.13) will prove to be a crucial inequality when establishing the resolvent estimate (5.8) later in the proof. Since the sesquilinear form satisfies the conditions of Lax-Milgram, we obtain a unique weak solution $\mathbf{v} \in \mathbf{P}^1$ of (5.2), (5.4), (5.5), (5.9) satisfying

$$\|\mathbf{v}\|_{\mathbf{H}^1_{\mathrm{p}}} \leq C \|\mathbf{f}\|_{\mathbf{L}^2}.$$

We now seek an associated pressure, q, of $\mathbf{v}$. Recall that an associated pressure need only satisfy (5.9) in the sense of distributions (i.e., when tested against arbitrary $\mathbf{u} \in \mathbf{C}_c^\infty$). As with the velocity, we begin by finding a weak formulation for the pressure. Notice that, for $q \in H^1_\mathrm{p}$ with $(\mathbf{v}, q)$ satisfying (5.4), we obtain from (5.11) that

$$\int_\Omega q \nabla \cdot \overline{\mathbf{u}} = \langle \mathbf{v}, \mathbf{u} \rangle - (\mathbf{f}, \mathbf{u})_{\mathbf{L}^2} \tag{5.14}$$

must be satisfied for all $\mathbf{u} \in \mathbf{H}^1_\mathrm{p}$. Using continuity of the sesquilinear form we obtain immediately that the right-hand side is a bounded linear functional in $\overline{\mathbf{u}}$, $\mathbf{F} : \mathbf{C}_\mathrm{c}^\infty \to \mathbb{C}$, which vanishes when $\nabla \cdot \overline{\mathbf{u}} = 0$. By Theorem B.4(ii), there is a unique $\tilde{q} \in L^2$ such that

$$\mathbf{F} = \nabla \tilde{q} \qquad \text{and} \qquad \int_\Omega \tilde{q} = 0. \tag{5.15}$$

It is now straightforward to verify that $q = -\tilde{q}$ satisfies (5.9) in the distributional sense and hence is an associated pressure of $\mathbf{v}$. It is uniquely determined under the additional condition $\int_\Omega q = 0$, but otherwise is unique only up to a constant. Having found a weak solution of the problem, we would now like to demonstrate that it can, in fact, be made into a strong solution. There are two tasks involved here: showing that $\mathbf{v}$ and q have the additional regularity required, and proving that $\mathbf{v}$ and q actually do solve the original formulation of the problem. Unfortunately, the "artificial" corners in our domain become problematic at this point because the standard results used to obtain additional regularity up to the boundary generally require that the domain in question be smooth.

We can sidestep this technical issue by taking advantage of the fact that we could similarly find a weak solution $(\mathbf{v}_1, q_1)$ of the problem (5.2), (5.4), (5.5), (5.9) on the larger domain Ω_1. Moreover, by choosing q_1 such that $\int_{\Omega_1} q_1 = 0$ we can ensure that $(\mathbf{v}_1, q_1)$ is simply the periodic extension of $(\mathbf{v}, q)$ to Ω_1. By Theorem B.6, $(\mathbf{v}_1, q_1)$ has the additional regularity we seek on compactly contained subsets of Ω_1. Moreover, Solonnikov's local method of proof of Theorem B.7 can be applied to $(\mathbf{v}_1, q_1)$ in order to obtain the desired regularity near S_F up to and including the intersections with Γ_0/Γ_ℓ since these regions occur on a smooth portion of the free surface on Ω_1 (Fig. 3). It follows that $\mathbf{v}_{\mathrm{p}} \in \mathbf{H}^2_{\mathrm{loc}}(\Omega_\infty)$ and $q_{\mathrm{p}} \in H^1_{\mathrm{loc}}(\Omega_\infty)$, hence $\mathbf{v} \in \mathbf{P}^2$ and $q \in H^1_{\mathrm{p}}$.

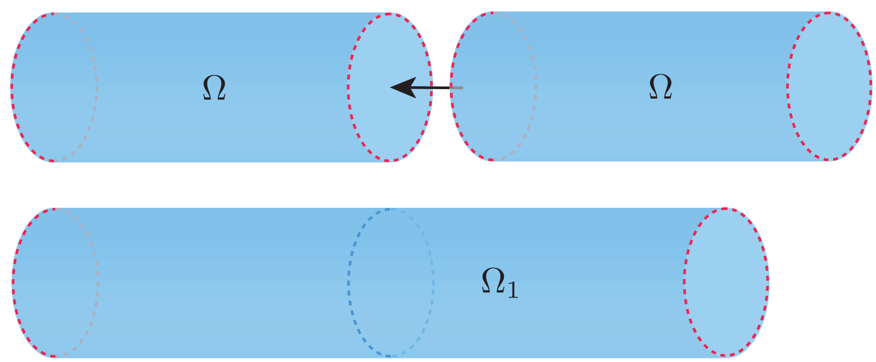

Fig. 3. Regularity up to the boundary is difficult to obtain near any sharp edges on a domain's boundary (traced in red). Examining the same functions over two periods allows us to obtain regularity near both of the problematic edges of Ω since they now occur on a smooth region of the boundary of Ω_1 (traced in blue).

To see that $(\mathbf{v}, q)$ provides us with a strong solution of our problem, we only need to verify that (5.4) and (5.9) are satisfied. Using (5.11), for all $\mathbf{u} \in \mathbf{P}^1$ we have

$$(-\mu\Delta\mathbf{v} - \lambda\mathbf{v} + \nabla q - \mathbf{f}, \mathbf{u})_{\mathbf{L}^2} = \int_{\partial\Omega} \mathbf{S}(\mathbf{v}, q) \cdot \overline{\mathbf{u}} = \int_{S_F} \mathbf{S}(\mathbf{v}, q) \cdot \overline{\mathbf{u}}. \tag{5.16}$$

Taking $\mathbf{u} \in {}^0\mathbf{C}^\infty_{\mathrm{p}\sigma}$ implies that $-\mu\Delta\mathbf{v} - \lambda\mathbf{v} + \nabla q - \mathbf{f}$ lies in the orthogonal complement of $\overline{{}^0\mathbf{C}^\infty_{\mathrm{p}\sigma}}^{\|\cdot\|_{\mathbf{L}^2}}$, so that $-\mu\Delta\mathbf{v} - \lambda\mathbf{v} + \nabla q - \mathbf{f} = \nabla p$ for some $p \in H^1_{\mathrm{p}}$ (see the proof of

Lemma 3.6). However, (5.11) now yields

$$\begin{aligned}(\mathbf{f},\mathbf{u})_{\mathbf{L}^2} &= (-\mu\Delta\mathbf{v} - \lambda\mathbf{v} + \nabla(q-p), \mathbf{u})_{\mathbf{L}^2} \\ &= \langle \mathbf{v}, \mathbf{u}\rangle + \int_{\partial\Omega} \mathbf{S}(\mathbf{v}, q-p)\cdot\overline{\mathbf{u}} - \int_{\Omega}(q-p)\nabla\cdot\overline{\mathbf{u}}\end{aligned}$$

for all $\mathbf{u} \in \mathbf{H}^1_{\mathrm{p}}$. Restricting $\mathbf{u}$ to $\mathbf{C}^\infty_{\mathrm{c}}$ and exploiting (5.15) reduces this to $\int_\Omega p\nabla\cdot\overline{\mathbf{u}} = 0$. Integrating by parts, we see that $\int_\Omega \nabla p\cdot\overline{\mathbf{u}}$ vanishes for arbitrary $\mathbf{u} \in \mathbf{C}^\infty_{\mathrm{c}}$. Since this is a dense subset of $\mathbf{L}^2$, $\nabla p = 0$ and q satisfies (5.9). All that remains is to show that (5.4), the free surface boundary condition, is also satisfied. From (5.16) we now immediately obtain

$$\int_{S_F} \mathbf{S}(\mathbf{v}, q)\cdot\overline{\mathbf{u}} = 0$$

for all $\mathbf{u} \in \mathbf{P}^1$. Following the lead of Solonnikov and Ščadilov in [34], we localize to a neighborhood $\Sigma \subset S_F$ and construct $\mathbf{u} \in \mathbf{P}^1$ such that $\mathbf{u}|_{S_F} = (\mathbf{S}(\mathbf{v}, q) - (\mathbf{S}(\mathbf{v}, q)\cdot\mathbf{n})\mathbf{n})\phi$ where ϕ is a smooth nonnegative function vanishing outside Σ. Then

$$\begin{aligned}\int_{S_F} \mathbf{S}(\mathbf{v}, q)\cdot\overline{\mathbf{u}} &= \int_\Sigma |\mathbf{S}(\mathbf{v}, q) - (\mathbf{S}(\mathbf{v}, q)\cdot\mathbf{n})\mathbf{n}|^2\overline{\phi} \\ &\qquad + (\mathbf{S}(\mathbf{v}, q)\cdot\mathbf{n})\mathbf{n}\cdot\overline{(\mathbf{S}(\mathbf{v}, q) - (\mathbf{S}(\mathbf{v}, q)\cdot\mathbf{n})\mathbf{n})\phi} \\ &= \int_\Sigma |\mathbf{S}_{\tan}(\mathbf{v})|^2\overline{\phi} \\ &= 0\end{aligned}$$

implies that $\mathbf{S}_{\tan}(\mathbf{v}) = 0$ on Σ. Since Σ was chosen arbitrarily, we obtain $\mathbf{S}(\mathbf{v}, q) = (\mathbf{S}(\mathbf{v}, q)\cdot\mathbf{n})\mathbf{n}$ on S_F. Let $\theta(\mathbf{v}, q) = q - 2\mu\kappa^{-2}\sum a_i a_j D_j v_i \in H^1_{\mathrm{p}}$. Since $\theta(\mathbf{v}, q)|_{S_F} = \mathbf{S}(\mathbf{v}, q)\cdot\mathbf{n}$, (5.16) yields

$$\int_{\partial\Omega} \theta(\mathbf{v}, q)\mathbf{n}\cdot\overline{\mathbf{u}} = \int_\Omega \nabla\theta(\mathbf{v}, q)\cdot\overline{\mathbf{u}} = 0$$

for all $\mathbf{u} \in \mathbf{P}^1$. By density, $\nabla\theta(\mathbf{v}, q) \in (\mathbf{P}^0)^\perp$ and $\theta(\mathbf{v}, q) = p + \omega$ for some $p \in {}^0H^1_\mathrm{p}$ and $\omega \in \mathbb{R}$. Since this implies $\mathbf{S}(\mathbf{v}, q) \cdot \mathbf{n} = q - 2\mu\kappa^{-2} \sum a_i a_j D_j v_i = \omega$ on S_F, we take $q^* = q - \omega$ and obtain a unique strong solution $(\mathbf{v}, q^*) \in \mathbf{V}^2 \times H^1_\mathrm{p}$ of the problem (5.2), (5.4), (5.5), (5.9). It is important to realize that this final step fully specifies the pressure q^*; it is now unique in the full sense of the word and no longer just unique up to a constant.

To further increase regularity, we turn to the standard *a priori* estimates of Agmon, Douglis, and Nirenberg (ADN) [2] (see Appendix D of [9] for an introduction). Since these estimates require that the problem be set on a smooth domain, we consider the boundary value problem corresponding to (5.2), (5.4), (5.5), (5.9) which has been remapped to the toroid $\mathcal{T}$ using the isomorphism Ψ^{-1} (see Lemma 3.4):

$$-\mu\dot{\Delta}\mathbf{w} - \lambda\mathbf{w} + \dot{\nabla}p = \mathbf{g} \qquad \text{on } \mathcal{T} \tag{5.17}$$

$$\dot{\nabla} \cdot \mathbf{w} = 0 \qquad \text{on } \mathcal{T} \tag{5.18}$$

$$pm_1 - \mu(2m_1\dot{D}w_1 + m_2D_zw_1 + m_2\dot{D}w_2) = 0 \qquad \text{on } \partial\mathcal{T} \tag{5.19}$$

$$pm_2 - \mu(2m_2D_zw_2 + m_1\dot{D}w_2 + m_1D_zw_1) = 0 \qquad \text{on } \partial\mathcal{T} \tag{5.20}$$

$$m_1\dot{D}w_3 + m_2D_zw_3 + m_1\ddot{D}w_1 + m_2\ddot{D}w_2 = 0 \qquad \text{on } \partial\mathcal{T} \tag{5.21}$$

where

$$\dot{D} = \frac{xD_x + yD_y}{\sqrt{x^2 + y^2}}, \qquad \ddot{D} = \left(\frac{2\pi}{\ell}\right)(xD_y - yD_x), \qquad \dot{\nabla} = \begin{pmatrix} \dot{D} \\ D_z \\ \ddot{D} \end{pmatrix},$$

$$\dot{\Delta} = \dot{\nabla}^2, \qquad m_j = \Psi^{-1}n_j, \qquad \text{and} \qquad g_j = \Psi^{-1}f_j.$$

Note that the periodic boundary condition is absent here; this property is intrinsic to the problem on $\mathcal{T}$. Evidently, the transformed quantities $(\Psi^{-1}\mathbf{v}, \Psi^{-1}q^*)$ provide us with a unique solution to the problem (5.17)–(5.21). The system (5.17)–(5.18) is also readily seen to be uniformly elliptic in the sense of ADN (see Appendix D of [9]), but verifying that the boundary conditions (5.19)–(5.21) satisfy the complementing condition, a technical condition required by the ADN theory, is a tedious affair. Recall that the complementing condition holds if at each point $\mathbf{X}_0 \in \partial\mathcal{T}$ an associated constant coefficient problem has no nontrivial solutions of the form

$$\mathbf{w}(\mathbf{X}) = e^{i\boldsymbol{\alpha}\cdot(\mathbf{X}-\mathbf{X}_0)}\mathbf{v}((\mathbf{X}-\mathbf{X}_0)\cdot\boldsymbol{\nu}) \tag{5.22}$$

where $\boldsymbol{\nu}$ is the outward normal to $\mathcal{T}$ at $\mathbf{X}_0$, $\boldsymbol{\alpha}$ is a nonzero real vector perpendicular to $\boldsymbol{\nu}$, and $\mathbf{v}$ tends to zero exponentially as $(\mathbf{X}-\mathbf{X}_0)\cdot\boldsymbol{\nu} \to \infty$ (see [23]). The constant coefficient problem under consideration is the homogeneous problem on the half-space $(\mathbf{X}-\mathbf{X}_0)\cdot\boldsymbol{\nu} < 0$ (with boundary $(\mathbf{X}-\mathbf{X}_0)\cdot\boldsymbol{\nu} = 0$) obtained by evaluating the coefficients of the principal parts of (5.17)–(5.21) at $\mathbf{X}_0$. For convenience, we set

$$\begin{aligned} b_0 &= -\mu(b_1^2+b_2^2+b_3^2), & b_1 &= (\alpha_1+\alpha_2)i/\sqrt{2}, & b_2 &= \alpha_3 i, \\ b_3 &= 2\pi x_0(\alpha_2-\alpha_1)i/\ell, & c_1 &= \sqrt{2}\nu_1, & c_2 &= \nu_3. \end{aligned}$$

Then, without loss of generality we assume that $x_0 = y_0 > 0$ and make the substitutions

$$\tilde{v}_j = \frac{b_0}{\mu}v_j \quad \text{for } j=1,2,3, \quad \tilde{v}_4 = \frac{1}{\mu}v_4, \quad \tilde{v}_5 = v_1' - c_1 v_4, \quad \tilde{v}_6 = v_2' - c_2 v_4, \quad \tilde{v}_7 = v_3'.$$

This reduces the problem to a 7×7 homogeneous system of first-order linear ODEs with coefficient matrix

$$\begin{pmatrix} 0 & 0 & 0 & b_0c_1/\mu & b_0/\mu & 0 & 0 \\ 0 & 0 & 0 & b_0c_2/\mu & 0 & b_0/\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_0/\mu \\ -c_1 & -c_2 & 0 & 0 & -b_1 & -b_2 & -b_3 \\ 1 & 0 & 0 & b_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & b_3 & 0 & 0 & 0 \end{pmatrix}.$$

Given the constraints on $\mathbf{v}$, we restrict ourselves to the eigenvalues with positive real part, of which there is only one, $\sqrt{b_0/\mu}$, of algebraic multiplicity 3 and geometric multiplicity 2; the two linearly independent eigenvectors and one generalized eigenvector corresponding to this eigenvalue are given by $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3$

respectively if $b_1 = c_1 = 0$ and $\widetilde{\boldsymbol{\beta}}_1, \widetilde{\boldsymbol{\beta}}_2, \widetilde{\boldsymbol{\beta}}_3$ otherwise, where

$$\boldsymbol{\beta}_1 = \begin{pmatrix} 0 \\ -b_3\sqrt{b_0\mu} \\ c_2 b_0 \\ 0 \\ 0 \\ -b_3\mu \\ c_2\sqrt{b_0\mu} \end{pmatrix}, \qquad \boldsymbol{\beta}_2 = \begin{pmatrix} \sqrt{b_0} \\ 0 \\ 0 \\ 0 \\ \sqrt{\mu} \\ 0 \\ 0 \end{pmatrix}, \qquad \boldsymbol{\beta}_3 = \begin{pmatrix} 0 \\ 2b_3\mu \\ -c_2\sqrt{b_0\mu} \\ \frac{-2b_3\mu^{3/2}}{c_2\sqrt{b_0}} \\ 0 \\ \frac{3b_3\mu^{3/2}}{\sqrt{b_0}} \\ 0 \end{pmatrix},$$

$$\widetilde{\boldsymbol{\beta}}_1 = \begin{pmatrix} -b_3\sqrt{b_0\mu} \\ 0 \\ c_1 b_0 + b_1\sqrt{b_0\mu} \\ 0 \\ -b_3\mu \\ 0 \\ c_1\sqrt{b_0\mu} + b_1\mu \end{pmatrix}, \qquad \widetilde{\boldsymbol{\beta}}_2 = \begin{pmatrix} -(c_2 b_0 + b_2\sqrt{b_0\mu}) \\ c_1 b_0 + b_1\sqrt{b_0\mu} \\ 0 \\ 0 \\ -(c_2\sqrt{b_0\mu} + b_2\mu) \\ c_1\sqrt{b_0\mu} + b_1\mu \\ 0 \end{pmatrix}, \qquad \text{and}$$

$$\widetilde{\boldsymbol{\beta}}_3 = \begin{pmatrix} \sqrt{\mu}(c_1^2 b_0 - b_1^2\mu - 3b_0) \\ \sqrt{\mu}(c_1\sqrt{b_0} + b_1\sqrt{\mu})(c_2\sqrt{b_0} - b_2\sqrt{\mu}) \\ -b_3\mu(c_1\sqrt{b_0} + b_1\sqrt{\mu}) \\ 2\mu(c_1\sqrt{b_0} + b_1\sqrt{\mu}) \\ -3\mu\sqrt{b_0} \\ 0 \\ 0 \end{pmatrix}.$$

Now, with eigenvalues and eigenvectors in hand, we possess all solutions of the form (5.22) to the aforementioned constant coefficient equations. When we plug these solutions into the corresponding boundary conditions, we obtain $\mathbf{w} = 0$ after careful examination. Having verified that the complementing condition holds, we are finally able to apply the *a priori* estimates of ADN in [2] yielding $\Psi^{-1}v_j \in H^s(\mathcal{T})$, $\Psi^{-1}q^* \in H^{s-1}(\mathcal{T})$ and

$$\|\Psi^{-1}q^*\|_{H^{s-1}(\mathcal{T})} + \sum_{j=1}^{3} \|\Psi^{-1}v_j\|_{H^s(\mathcal{T})} \leq C_\lambda \sum_{j=1}^{3} \|\Psi^{-1}f_j\|_{H^{s-2}(\mathcal{T})}$$

where $C_\lambda > 0$ is a constant which depends on λ. Since a quick analysis yields the existence of positive constants C_1 and C_2 such that

$$C_1\|g\|_{H^k} \leq \|\Psi^{-1}g\|_{H^k(\mathcal{T})} \leq C_2\|g\|_{H^k}$$

for all $g \in H^k_{\mathrm{p}}$, we obtain the following estimate in Ω:

$$\|q^*\|_{H^{s-1}_{\mathrm{p}}} + \sum_{j=1}^{3} \|v_j\|_{H^s_{\mathrm{p}}} \leq C_3 C_\lambda \sum_{j=1}^{3} \|f_j\|_{H^{s-2}_{\mathrm{p}}}. \tag{5.23}$$

Thus $(\mathbf{v}, q^*) \in \mathbf{V}^s \times H^{s-1}_{\mathrm{p}}$ is the unique solution of (5.2), (5.4), (5.5), (5.9) and $\sigma(A) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \geq 0\}$. Now all that remains is to show that the resolvent estimate (5.8) is satisfied. From (5.23) we obtain the estimate

$$\|\mathbf{v}\|^2_{\mathbf{H}^s_{\mathrm{p}}} \leq \left(\sum_{j=1}^{3} \|v_j\|_{H^s_{\mathrm{p}}}\right)^2 \leq 3C_3^2 C_\lambda^2 \|\mathbf{f}\|^2_{\mathbf{H}^{s-2}_{\mathrm{p}}}.$$

Thus we have

$$
\begin{aligned}
\|\mathbf{v}\|_{\mathbf{H}_{\mathrm{p}}^s} &\le C_4 \|(A+I)\mathbf{v}\|_{\mathbf{H}_{\mathrm{p}}^{s-2}} \\
&\le C_4 \left(\|(A-\lambda I)\mathbf{v}\|_{\mathbf{H}_{\mathrm{p}}^{s-2}} + (|\lambda|+1)\|\mathbf{v}\|_{\mathbf{H}_{\mathrm{p}}^{s-2}} \right) \\
&\le C_5 \left(\|\mathbf{f}\|_{\mathbf{H}_{\mathrm{p}}^{s-2}} + (|\lambda|+1)\|\mathbf{v}\|_{\mathbf{H}_{\mathrm{p}}^{s}}^{(s-2)/s} \|\mathbf{v}\|_{\mathbf{L}^2}^{2/s} \right) \qquad (5.24)
\end{aligned}
$$

where C_4 and C_5 are positive constants which do not depend on λ. Here we have used complex interpolation between $\mathbf{L}^2$ and $\mathbf{H}_p^s$. Finally, we apply Hölder to (5.13) which yields

$$
\|\mathbf{f}\|_{\mathbf{L}^2} \ge \frac{|\lambda|}{\sqrt{2}} \|\mathbf{v}\|_{\mathbf{L}^2}. \qquad (5.25)
$$

Now let us restrict ourselves to $|\lambda| > \varepsilon$ for arbitrary $\varepsilon > 0$. If $s = 2$, then (5.24) and (5.25) yield (5.8) directly. Otherwise, we can apply Young's inequality to (5.24) obtain

$$
\begin{aligned}
\|\mathbf{v}\|_{\mathbf{H}_{\mathrm{p}}^s} &\le C_5 \left(\|\mathbf{f}\|_{\mathbf{H}_{\mathrm{p}}^{s-2}} + C_6(|\lambda|+1)^{s/2}\|\mathbf{v}\|_{\mathbf{L}^2} + \frac{1}{2C_5}\|\mathbf{v}\|_{\mathbf{H}_{\mathrm{p}}^s} \right) \\
&\le C_7 \left(\|\mathbf{f}\|_{\mathbf{H}_{\mathrm{p}}^{s-2}} + (|\lambda|+1)^{s/2}\|\mathbf{v}\|_{\mathbf{L}^2} \right) \\
&\le C_8 \left(\|\mathbf{f}\|_{\mathbf{H}_{\mathrm{p}}^{s-2}} + (1+\varepsilon^{-1})(|\lambda|+1)^{(s-2)/2}\|\mathbf{f}\|_{\mathbf{L}^2} \right)
\end{aligned}
$$

where C_6, C_7, and C_8 are positive constants which do not depend on λ. Since $\mathbf{v} = R(\lambda; A)\mathbf{f}$, this completes the proof. □

Having completed the lion's share of the necessary work in Lemma 5.2, we can now show (with comparative ease) that $-A$ is the infinitesimal generator of an analytic semigroup of contractions. This will allow us to easily obtain a unique solution to the abstract Cauchy problem given by (5.6), (5.7). We refer the reader to [21] for an introduction to semigroup theory.

Lemma 5.3. *The operator $-A$, with domain $\mathbf{V}^2$, generates an analytic semigroup of contractions, $J(t)$, on $\mathbf{P}^0$ with $\|J(t)\| = 1$.*

Proof. As we seek to apply Lumer-Phillips (Theorem B.10), we begin by showing that $-A$ is a dissipative. To do this, we must improve (slightly) upon the estimate provided by (5.25). For $\lambda < 0$, we obtain

$$|(\mathbf{f}, \mathbf{v})_{\mathbf{L}^2}| = |\langle \mathbf{v}, \mathbf{v} \rangle| = -\lambda \|\mathbf{v}\|_{\mathbf{L}^2}^2 + \frac{\mu}{2} \sum_{i,j=1}^{3} \int_\Omega |D_j v_i + D_i v_j|^2 \geq -\lambda \|\mathbf{v}\|_{\mathbf{L}^2}^2. \tag{5.26}$$

Dissipativity now follows using the Hölder inequality. Since $A + I$ is surjective by Lemma 5.2 and $\mathbf{P}^0$ is reflexive (as a Hilbert space), we can apply Lumer-Phillips to obtain that $\mathbf{V}^2$ is dense in $\mathbf{P}^0$ and $-A$ generates a C_0 semigroup of contractions, $J(t)$, on $\mathbf{P}^0$. As the generator of a C_0 semigroup of contractions, $-A$ is closed (see Theorem II.1.4 in [12], for example) and using (5.25) together with Theorem 12.31 from [23] we see that $J(t)$ is actually an analytic semigroup on $\mathbf{P}^0$. Now, since $J(t)$ is a semigroup of contractions, we have $\|J(t)\| \leq 1$. However, 0 is contained in the point spectrum of $-A$ (see the discussion preceding Lemma 5.2) which implies that 1 is contained in the point spectrum of $J(t)$ by the Spectral Mapping Theorem (Theorem B.11). It then follows from Corollary IV.3.8 in [12] that, for any constant vector $\mathbf{c} \neq 0$, we have $J(t)\mathbf{c} = \mathbf{c}$ for all t. Hence $\|J(t)\| \geq 1$ and thus $\|J(t)\| = 1$ as required. □

With this semigroup result in hand, we are finally ready to solve the homogeneous linear problem (5.1)–(5.5).

Theorem 5.4. *Let $3 < s \leq 4$, $T > 0$, and $\mathbf{f} \in \mathbf{K}_p^{s-2}$ such that $P\mathbf{f}(0,\cdot) = 0$. Then the problem (5.1)–(5.5) has a unique solution $(\mathbf{v}, q)$ such that $\mathbf{v} \in \mathbf{K}_p^s$, $\nabla q \in \mathbf{K}_p^{s-2}$, and $q|_{S_F} \in K_p^{s-3/2}(\partial G_F)$. Moreover, this solution satisfies*

$$\|\mathbf{v}\|_{\mathbf{K}_p^s} + \|\nabla q\|_{\mathbf{K}_p^{s-2}} + \|q|_{S_F}\|_{K_p^{s-3/2}(\partial G_F)} \leq C\|\mathbf{f}\|_{\mathbf{K}_p^{s-2}} \tag{5.27}$$

where C is a positive constant which is independent of T and $\mathbf{f}$.

Proof. First we notice that $P\mathbf{f} \in C^{0,(s-3)/2}([0,T]; \mathbf{P}^0)$ by the Sobolev Embedding Theorem (Theorem B.8). Combining Corollary 4.3.3 and Theorem 4.3.5(iii) from [21], the abstract Cauchy problem

$$\dot{\mathbf{v}} + A\mathbf{v} = P\mathbf{f}$$

$$\mathbf{v}(0,\cdot) = 0$$

has a unique strong solution $\mathbf{v} \in C^{1,(s-3)/2}([0,T]; \mathbf{P}^0)$, with $\mathbf{v}(t) \in \mathbf{V}^2$ for each $t \in [0,T]$. Here we are exploiting the fact that $-A$ is the generator of an analytic semigroup on $\mathbf{P}^0$. Note that $\mathbf{v}$ is a strong solution in the sense of semigroups; that is, $\mathbf{v}$ is differentiable almost everywhere on $[0,T]$, with $\dot{\mathbf{v}} \in L^1((0,T); \mathbf{P}^0)$, such that $\mathbf{v}(0,\cdot) = 0$ and $\dot{\mathbf{v}}(t) = -A\mathbf{v}(t) + P\mathbf{f}(t)$ almost everywhere on $[0,T]$. In fact, $\mathbf{v}$ is a classical solution in the semigroup sense since it is continuously differentiable with respect to time (this differs from the general definition of a classical solution since the spatial derivatives are still taken in the distributional sense).

To show that $\mathbf{v} \in \mathbf{K}_{\mathrm{p}}^s$, we reconsider the abstract Cauchy problem (now with a new unknown variable $\tilde{\mathbf{v}}$) from another perspective. We begin by applying Lemma C.2(ii) in order to extend $P\mathbf{f}$ to $\mathbf{K}_{\mathrm{p}}^{s-2}(\mathbb{R} \times \Omega)$ in such a way that the extension is bounded independent of T and vanishes for $t < 0$. Multiplying through the abstract Cauchy problem by the weight $w(t) = e^{-t}$ and taking Fourier transforms in t, we obtain

$$\mathcal{F}_w(\tilde{\mathbf{v}})(\xi) = (A + (1+i\xi)I)^{-1}\mathcal{F}_w(P\mathbf{f})(\xi).$$

Since it is clear that $\mathcal{F}_w(P\mathbf{f})(\xi) \in \mathbf{P}^{s-2}$, this uniquely defines $\mathcal{F}_w(\tilde{\mathbf{v}})(\xi) \in \mathbf{V}^s$ by Lemma 5.2. Making use of the Fourier transform characterization of H^s-spaces for $s \in \mathbb{R}^+$ (e.g., see [1]) and the fact that Fourier transforms are unitary transformations, we have

$$\begin{aligned}
\|\tilde{\mathbf{v}}\|^2_{\mathbf{K}^s_{\mathrm{p}}(\mathbb{R}\times\Omega)} &\le 2\left(\|\tilde{\mathbf{v}}\|^2_{\mathbf{L}^2(\mathbb{R};H^s_{\mathrm{p}})} + \|\tilde{\mathbf{v}}\|^2_{\mathbf{H}^{s/2}(\mathbb{R};L^2)}\right)\\
&= 2\left(\|\mathcal{F}_w(\tilde{\mathbf{v}})(\xi+1)\|^2_{\mathbf{L}^2(\mathbb{R};H^s_{\mathrm{p}})} + \|(1+\xi^2)^{s/4}\mathcal{F}_w(\tilde{\mathbf{v}})(\xi+1)\|^2_{\mathbf{L}^2(\mathbb{R};L^2)}\right)\\
&= 2\int_{\mathbb{R}}\left(\|\mathcal{F}_w(\tilde{\mathbf{v}})(\xi+1)\|^2_{\mathbf{H}^s_{\mathrm{p}}} + (1+\xi^2)^{s/2}\|\mathcal{F}_w(\tilde{\mathbf{v}})(\xi+1)\|^2_{\mathbf{L}^2}\right)d\xi.
\end{aligned}$$

Applying the resolvent estimate (5.8) to the first term of the integral, we obtain

$$\begin{aligned}
\|\mathcal{F}_w(\tilde{\mathbf{v}})(\xi+1)\|^2_{\mathbf{H}^s_{\mathrm{p}}} &\le C_1\Big(\|\mathcal{F}_w(P\mathbf{f})(\xi+1)\|_{\mathbf{H}^{s-2}_{\mathrm{p}}}\\
&\qquad +2(|1+i(\xi+1)|+1)^{(s-2)/2}\|\mathcal{F}_w(P\mathbf{f})(\xi+1)\|_{\mathbf{L}^2}\Big)^2\\
&\le C_2\Big(\|\mathcal{F}_w(P\mathbf{f})(\xi+1)\|^2_{\mathbf{H}^{s-2}_{\mathrm{p}}}\\
&\qquad +(\sqrt{1+(\xi+1)^2}+1)^{s-2}\|\mathcal{F}_w(P\mathbf{f})(\xi+1)\|^2_{\mathbf{L}^2}\Big)\\
&\le C_2\Big(\|\mathcal{F}_w(P\mathbf{f})(\xi+1)\|^2_{\mathbf{H}^{s-2}_{\mathrm{p}}}\\
&\qquad +\left(3\sqrt{1+\xi^2}\right)^{s-2}\|\mathcal{F}_w(P\mathbf{f})(\xi+1)\|^2_{\mathbf{L}^2}\Big)\\
&\le C_3\Big(\|\mathcal{F}_w(P\mathbf{f})(\xi+1)\|^2_{\mathbf{H}^{s-2}_{\mathrm{p}}}\\
&\qquad +\left(1+\xi^2\right)^{(s-2)/2}\|\mathcal{F}_w(P\mathbf{f})(\xi+1)\|^2_{\mathbf{L}^2}\Big)
\end{aligned}$$

where C_1, C_2, and C_3 are positive constants which are independent of ξ and $\mathbf{f}$. Similarly, we can apply estimate (5.25) to the second term of the integral to get

$$\begin{aligned}(1+\xi^2)^{s/2}\|\mathcal{F}_w(\tilde{\mathbf{v}})(\xi+1)\|_{\mathbf{L}^2}^2 &\le 2(1+\xi^2)^{s/2}|1+i(\xi+1)|^{-2}\|\mathcal{F}_w(P\mathbf{f})(\xi+1)\|_{\mathbf{L}^2}^2 \\ &= 2\left(\frac{1+\xi^2}{1+(1+\xi)^2}\right)(1+\xi^2)^{(s-2)/2}\|\mathcal{F}_w(P\mathbf{f})(\xi+1)\|_{\mathbf{L}^2}^2 \\ &\le 6(1+\xi^2)^{(s-2)/2}\|\mathcal{F}_w(P\mathbf{f})(\xi+1)\|_{\mathbf{L}^2}^2.\end{aligned}$$

Combining these estimates yields

$$\begin{aligned}\|\tilde{\mathbf{v}}\|_{\mathbf{K}_p^s(\mathbb{R}\times\Omega)}^2 &\le C_4\int_{\mathbb{R}}\Big(\|\mathcal{F}_w(P\mathbf{f})(\xi+1)\|_{\mathbf{H}_\mathrm{p}^{s-2}}^2 \\ &\qquad + \left(1+\xi^2\right)^{(s-2)/2}\|\mathcal{F}_w(P\mathbf{f})(\xi+1)\|_{\mathbf{L}^2}^2\Big)\,d\xi \\ &= C_4\Big(\|\mathcal{F}_w(P\mathbf{f})(\xi+1)\|_{\mathbf{L}^2(\mathbb{R};H_\mathrm{p}^{s-2})}^2 \\ &\qquad + \|(1+\xi^2)^{(s-2)/4}\mathcal{F}_w(P\mathbf{f})(\xi+1)\|_{\mathbf{L}^2(\mathbb{R};L^2)}^2\Big) \\ &= C_4\left(\|P\mathbf{f}\|_{\mathbf{L}^2(\mathbb{R};H_\mathrm{p}^{s-2})}^2 + \|P\mathbf{f}\|_{\mathbf{H}^{(s-2)/2}(\mathbb{R};L^2)}^2\right) \\ &\le C_4\|P\mathbf{f}\|_{\mathbf{K}_\mathrm{p}^{s-2}(\mathbb{R}\times\Omega)}^2\end{aligned}$$

where $C_4 > 0$ is a constant which is independent of ξ and $\mathbf{f}$. By uniqueness, we must have $\mathbf{v} = \tilde{\mathbf{v}}|_G \in \mathbf{K}_\mathrm{p}^s$. We now seek a suitable q so that $(\mathbf{v}, q)$ is the unique solution of (5.1)–(5.5). For fixed t, this amounts to finding a unique $q \in H_\mathrm{p}^{s-1}$ such that

$$\begin{aligned}\nabla q &= \mu\Delta\mathbf{v} + A\mathbf{v} + \mathbf{f} - P\mathbf{f} && \text{on } \Omega \\ q &= Q\mathbf{v} && \text{on } S_F.\end{aligned}$$

Since $s > 3$, this is easily accomplished by taking the divergence of the first equation and applying Lemma 3.5. All that remains is to show that $(\mathbf{v}, q)$ satisfies

(5.27). To estimate q we first notice that

$$\nabla q = \mu(I - P)\Delta \mathbf{v} + \nabla Q\mathbf{v} + (I - P)\mathbf{f}.$$

The only term which we do not yet know how to estimate is $\nabla Q\mathbf{v}$. However, since $\Delta Q\mathbf{v} = 0$ on Ω and $Q\mathbf{v} = \phi$ on S_F where $\phi = 2\mu\kappa^{-2}\sum_{i,j=1}^{2} a_i a_j D_j v_i \in H_{\mathrm{p}}^{s-1}$, it follows from Lemma 3.8 that $\nabla Q\mathbf{v} = P(\nabla\phi)$. Then by Lemma 3.7,

$$\|\nabla Q\mathbf{v}\|_{\mathbf{K}_{\mathrm{p}}^{s-2}} = \|P(\nabla\phi)\|_{\mathbf{K}_{\mathrm{p}}^{s-2}} \leq C_5\|\nabla\phi\|_{\mathbf{K}_{\mathrm{p}}^{s-2}} \leq C_6\|\mathbf{v}\|_{\mathbf{K}_{\mathrm{p}}^{s}}$$

where C_5 and C_6 are positive constants. Similarly, since Q was constructed so that $q = Q\mathbf{v}$ on S_F,

$$\|q|_{S_F}\|_{K_{\mathrm{p}}^{s-3/2}(\partial G_F)} = \|Q\mathbf{v}|_{S_F}\|_{K_{\mathrm{p}}^{s-3/2}(\partial G_F)} \leq C_7\|Q\mathbf{v}\|_{K_{\mathrm{p}}^{s-1}} \leq C_8\|\mathbf{v}\|_{\mathbf{K}_{\mathrm{p}}^{s}}$$

where C_7 and C_8 are positive constants. Thus, combining estimates, we obtain

$$\begin{aligned}
\|\mathbf{v}\|_{\mathbf{K}_{\mathrm{p}}^{s}} + \|\nabla q\|_{\mathbf{K}_{\mathrm{p}}^{s-2}} + \|q|_{S_F}\|_{K_{\mathrm{p}}^{s-3/2}(\partial G_F)} &\leq C_9\left(\|\mathbf{v}\|_{\mathbf{K}_{\mathrm{p}}^{s}} + \|\mathbf{f}\|_{\mathbf{K}_{\mathrm{p}}^{s-2}}\right) \\
&\leq C_9\left(\|\tilde{\mathbf{v}}\|_{\mathbf{K}_{\mathrm{p}}^{s}(\mathbb{R}\times\Omega)} + \|\mathbf{f}\|_{\mathbf{K}_{\mathrm{p}}^{s-2}}\right) \\
&\leq C_{10}\left(\|\mathbf{h}\|_{\mathbf{K}_{\mathrm{p}}^{s-2}(\mathbb{R}\times\Omega)} + \|\mathbf{f}\|_{\mathbf{K}_{\mathrm{p}}^{s-2}}\right) \\
&\leq C_{11}\|\mathbf{f}\|_{\mathbf{K}_{\mathrm{p}}^{s-2}}
\end{aligned}$$

where C_9, C_{10}, and C_{11} are positive constants which do not depend on $\mathbf{f}$ (or T). □

5.2 The Inhomogeneous Case

We now attempt to find a solution $(\mathbf{v}, q)$ of the fully inhomogeneous problem

$$\dot{\mathbf{v}} - \mu\Delta\mathbf{v} + \nabla q = \mathbf{f} \qquad \text{on } G \tag{5.28}$$

$$\nabla \cdot \mathbf{v} = \sigma \qquad \text{on } G \tag{5.29}$$

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0(\cdot) \qquad \text{on } \Omega \tag{5.30}$$

$$\mathbf{S}(\mathbf{v}, q) = \mathbf{h} \qquad \text{on } \partial G_F \tag{5.31}$$

$$q_{\mathrm{p}}(t,\cdot) \in H^{s-1}_{\mathrm{loc}}(\Omega_\infty), \mathbf{v}_{\mathrm{p}}(t,\cdot) \in \mathbf{H}^{s}_{\mathrm{loc}}(\Omega_\infty) \qquad \text{on } (0,T) \text{ with } s \geq 2 \tag{5.32}$$

by reducing it to the homogeneous problem discussed in the previous section. The details regarding the allowable inhomogeneities $(\mathbf{f}, \sigma, \mathbf{v}_0, \mathbf{h})$ require some motivation so we have left them temporarily unspecified. Let us suppose that, in agreement with the homogeneous case, any solution of the problem (5.28)–(5.32) will be such that $\mathbf{v} \in \mathbf{K}^s_{\mathrm{p}}$, $\nabla q \in \mathbf{K}^{s-2}_{\mathrm{p}}$, and $q|_{S_F} \in K^{s-3/2}_{\mathrm{p}}(\partial G_F)$. Using Lemma C.3, it seems clear that we should be selecting $\mathbf{f} \in \mathbf{K}^{s-2}_{\mathrm{p}}$, $\sigma \in K^{s-1}_{\mathrm{p}}$, $\mathbf{v}_0 \in \mathbf{H}^{s-1}_{\mathrm{p}}$, and $\mathbf{h} \in \mathbf{K}^{s-3/2}_{\mathrm{p}}(\partial G_F)$. However, shortly we will want the quantity $\dot{\sigma}(0,\cdot)$ to be well-defined and taking $\sigma \in K^{s-1}_{\mathrm{p}}$ would not provide sufficient regularity with respect to time. To see what can be done, we take a closer look at the divergence operator. Let $j \leq \frac{s}{2}$ where $\frac{s}{2} \in \mathbb{N}$ initially. Then, for $\mathbf{v} \in \mathbf{C}^\infty([0,T]; H^s_{\mathrm{p}})$ and $\phi \in {}^0H^1_{\mathrm{p}}$, we have

$$\left(D^j_t(\nabla \cdot \mathbf{v}), \phi\right)_{L^2} = \left(\nabla \cdot (D^j_t \mathbf{v}), \phi\right)_{L^2} = -\left(D^j_t \mathbf{v}, \nabla\phi\right)_{\mathbf{L}^2}.$$

Thus, for each t, we can apply Hölder to obtain

$$\left|\left(D^j_t(\nabla \cdot \mathbf{v}), \phi\right)_{L^2}\right| \leq \|D^j_t \mathbf{v}\|_{\mathbf{L}^2} \|\phi\|_{H^1_{\mathrm{p}}},$$

so that

$$\left\| D_t^j (\nabla \cdot \mathbf{v}) \right\|_{{}^0H_{\mathrm{p}}^{-1}} \leq \| D_t^j \mathbf{v} \|_{\mathbf{L}^2},$$

where ${}^0H_{\mathrm{p}}^{-1}$ is the dual space of ${}^0H_{\mathrm{p}}^{1}$. Now squaring, integrating in time, and summing over j yields

$$\| \nabla \cdot \mathbf{v} \|^2_{H^{s/2}((0,T);{}^0H_{\mathrm{p}}^{-1})} \leq \| \mathbf{v} \|^2_{\mathbf{H}^{s/2}((0,T);L^2)} \leq \| \mathbf{v} \|^2_{\mathbf{K}_{\mathrm{p}}^s}. \tag{5.33}$$

That this inequality actually holds for all $\mathbf{v} \in \mathbf{K}_{\mathrm{p}}^s$ follows from density and extends to arbitrary $s \geq 2$ through interpolation. Thus we see that the divergence operator is bounded from $\mathbf{K}_{\mathrm{p}}^s$ to

$$\widetilde{K}_{\mathrm{p}}^s = H^{s/2}((0,T);{}^0H_{\mathrm{p}}^{-1}) \cap L^2((0,T);H_{\mathrm{p}}^{s-1}).$$

Observe that taking $\sigma \in \widetilde{K}_{\mathrm{p}}^s$ means that $\dot{\sigma}(0,\cdot)$ is now a well-defined quantity in ${}^0H_{\mathrm{p}}^{-1}$. Additionally, we must make sure that the inhomogeneities satisfy certain compatibility conditions at $t = 0$. It is important to note that the number of compatibility conditions required increases with the value of s. As discussed in Chapter 4, certain considerations require that we take $s > 3$. Hence the following compatibility conditions must certainly be imposed on the initial data:

$$\nabla \cdot \mathbf{v}_0 = \sigma(0,\cdot) \qquad \text{on } \Omega, \qquad\qquad \mathbf{S}_{\tan}(\mathbf{v}_0) = \mathbf{h}_{\tan}(0,\cdot) \qquad \text{on } S_F. \tag{5.34}$$

However, notice that if $s \geq 7/2$, the situation become a little more complicated. It is readily seen that the value of q at $t = 0$ on S_F is determined by $\mathbf{v}_0$ and $\mathbf{h}(0,\cdot)$ using (5.4). Consequently, the value of $\dot{\mathbf{v}}(0,\cdot)$ is determined at $t = 0$ on S_F by $\mathbf{v}_0$, $\mathbf{h}(0,\cdot)$, and $\mathbf{f}(0,\cdot)$ using (5.1). Now, since $\mathbf{S}(\mathbf{v},q)$ and $\mathbf{h}$ lie in $\mathbf{H}^{(2s-3)/4}((0,T);L^2)$ generally, $s \geq 7/2$ would imply that they were both in $\mathbf{H}^1((0,T);L^2)$. This would mean that the tangential component of (5.4) could be differentiated with respect to time and evaluated at $t = 0$ to yield the condition

$$D_t(\mathbf{S}_{\tan}(\mathbf{v}))(0,\cdot) = D_t(\mathbf{h}_{\tan})(0,\cdot) \qquad \text{on } S_F. \tag{5.35}$$

Since this involves only quantities whose values are determined by the inhomogeneities (i.e., $\mathbf{v}_0$, $\dot{\mathbf{v}}(0,\cdot)$, and $D_t(\mathbf{h}_{\tan})(0,\cdot)$), (5.35) would have to be included as an additional compatibility condition. To minimize the number of compatibility conditions required, we therefore restrict s to the interval $(3, \frac{7}{2})$ and define

$$\begin{aligned}
\mathbf{X}^s &= \left\{(\mathbf{v}, q) : \mathbf{v} \in \mathbf{K}^s_{\mathrm{p}},\ \nabla q \in \mathbf{K}^{s-2}_{\mathrm{p}},\ q|_{S_F} \in K^{s-3/2}_{\mathrm{p}}(\partial G_F)\right\} \\
\mathbf{Y}^s &= \Big\{(\mathbf{f}, \sigma, \mathbf{v}_0, \mathbf{h}) : \mathbf{f} \in \mathbf{K}^{s-2}_{\mathrm{p}},\ \sigma \in \widetilde{K}^s_{\mathrm{p}}, \\
&\qquad\qquad \mathbf{v}_0 \in \mathbf{H}^{s-1}_{\mathrm{p}} \text{ satisfying (5.34)},\ \mathbf{h} \in \mathbf{K}^{s-3/2}_{\mathrm{p}}(\partial G_F)\Big\}\,.
\end{aligned}$$

Theorem 5.5. *The map $L : \mathbf{X}^s \to \mathbf{Y}^s$ has a bounded inverse for $3 < s < \frac{7}{2}$.*

Proof. Let $\boldsymbol{\beta} = (\mathbf{f}, \sigma, \mathbf{v}_0, \mathbf{h}) \in \mathbf{Y}^s$. We proceed by constructing (in several steps) an approximation $\widetilde{\boldsymbol{\alpha}} \in \mathbf{X}^s$ to the desired solution $\boldsymbol{\alpha}$ of $L\boldsymbol{\alpha} = \boldsymbol{\beta}$ in such a way that $\boldsymbol{\alpha} - \widetilde{\boldsymbol{\alpha}}$ must solve a homogeneous problem of the form (5.1)–(5.5). By Theorem 5.4, this homogeneous problem has a unique solution $\boldsymbol{\alpha}^*$, implying that $\boldsymbol{\alpha} = \widetilde{\boldsymbol{\alpha}} + \boldsymbol{\alpha}^*$ solves $L\boldsymbol{\alpha} = \boldsymbol{\beta}$. Uniqueness easily follows: since the difference of any two such solutions would solve the fully homogeneous problem $L\boldsymbol{\alpha} = 0$, their difference must be zero by Theorem 5.4.

We first attempt to find $(\mathbf{v}_1, q_1) \in \mathbf{X}^s$ such that $\boldsymbol{\beta}_1 = L(\mathbf{v}_1, q_1)$ agrees with $\boldsymbol{\beta}$ at time $t = 0$. That is, we seek $(\mathbf{v}_1, q_1)$ such that

$$\begin{aligned} \mathbf{f}_1(0,\cdot) &= \mathbf{f}(0,\cdot) \\ \sigma_1(0,\cdot) &= \sigma(0,\cdot) \\ (\mathbf{v}_0)_1 &= \mathbf{v}_0 \\ \mathbf{h}_1(0,\cdot) &= \mathbf{h}(0,\cdot) \end{aligned}$$

where $L(\mathbf{v}_1, q_1) = (\mathbf{f}_1, \sigma_1, (\mathbf{v}_0)_1, \mathbf{h}_1)$. We begin by observing that $q_1(0,\cdot) \in H_{\mathrm{p}}^{s-5/2}(S_F)$ can be defined using the equation

$$\mathbf{S}(\mathbf{v}_0, q_1(0,\cdot)) \cdot \mathbf{n} = \mathbf{h}(0,\cdot) \cdot \mathbf{n}$$

since one can write

$$q_1(0,\cdot) = \mu \sum_{i,j=1}^{3} (D_j(v_0)_i + D_i(v_0)_j) n_j n_i + \mathbf{h}(0,\cdot) \cdot \mathbf{n}.$$

We extend $q_1(0,\cdot)$ to H_{p}^{s-2} using surjectivity of the trace operator (e.g., on $\mathcal{T}$) and then again to K_{p}^{s-2}. We now want to select $\mathbf{v}_1 \in \mathbf{K}_{\mathrm{p}}^s$ such that

$$\begin{aligned} \dot{\mathbf{v}}_1(0,\cdot) &= \mu \Delta \mathbf{v}_0 - \nabla q_1(0,\cdot) + \mathbf{f}(0,\cdot) \\ \mathbf{v}_1(0,\cdot) &= \mathbf{v}_0. \end{aligned}$$

That such a $\mathbf{v}_1$ exists follows from Lemma C.1(iii). It is now straightforward to check that $\boldsymbol{\beta}_1(0,\cdot) = \boldsymbol{\beta}(0,\cdot)$.

We now seek to improve our selection by finding $\mathbf{v}_2 \in \mathbf{K}^s_{\mathrm{p}}$ such that $L(\mathbf{v}_2, q_1) = (\mathbf{f}_2, \sigma_2, (\mathbf{v}_0)_2, \mathbf{h}_2)$ satisfies

$$\begin{aligned} P\mathbf{f}_2(0,\cdot) &= P\mathbf{f}(0,\cdot) \\ \sigma_2 &= \sigma \\ (\mathbf{v}_0)_2 &= \mathbf{v}_0 \\ \mathbf{h}_2(0,\cdot) &= \mathbf{h}(0,\cdot). \end{aligned} \tag{5.36}$$

Consider an extension of $\tilde{\sigma} = \sigma - \sigma_1$ to $\widetilde{K}^s_{\mathrm{p}}(\mathbb{R} \times \Omega)$. Taking Fourier transforms, we have

$$\left\| |\xi|^{s/2} \mathcal{F}(\tilde{\sigma})(\xi) \right\|^2_{L^2(\mathbb{R}; {}^0H^{-1}_{\mathrm{p}})} \leq \left\| (1+\xi^2)^{s/4} \mathcal{F}(\tilde{\sigma})(\xi) \right\|^2_{L^2(\mathbb{R}; {}^0H^{-1}_{\mathrm{p}})} = \|\tilde{\sigma}\|^2_{H^{s/2}(\mathbb{R}; {}^0H^{-1}_{\mathrm{p}})}$$

so that $\mathcal{F}(\tilde{\sigma})(\xi) \in L^2(\mathbb{R}; H^{s-1}_{\mathrm{p}})$ and $|\xi|^{s/2}\mathcal{F}(\tilde{\sigma})(\xi) \in L^2(\mathbb{R}; {}^0H^{-1}_{\mathrm{p}})$. For each $\xi \in \mathbb{R}$, we take $\mathcal{F}(\phi)(\xi) \in {}^0H^{s+1}_{\mathrm{p}}$ to be the unique solution (provided by Lemma 3.5) of

$$\begin{aligned} \Delta \mathcal{F}(\phi)(\xi) &= \mathcal{F}(\tilde{\sigma})(\xi) && \text{on } \Omega \\ \mathcal{F}(\phi)(\xi) &= 0 && \text{on } S_F \\ D_3^k \mathcal{F}(\phi)(\xi)|_{\Gamma_\ell} &= D_3^k \mathcal{F}(\phi)(\xi)|_{\Gamma_0} && \text{for } k \in \{0,1\} \end{aligned}$$

which satisfies $\|\mathcal{F}(\phi)(\xi)\|_{H^{s+1}_{\mathrm{p}}} \leq C \|\mathcal{F}(\tilde{\sigma})(\xi)\|_{H^{s-1}_{\mathrm{p}}}$. It is also noteworthy here that, in evaluating at $t = 0$ the corresponding problem obtained by taking inverse Fourier transforms, $\tilde{\sigma}(0,\cdot) = 0$ implies $\phi(0,\cdot) = 0$ (using Lemma 3.5 once again). Integrating by parts, we obtain the identity

$$(\nabla \mathcal{F}(\phi)(\xi), \nabla \psi)_{\mathbf{L}^2} = -\left(\mathcal{F}(\tilde{\sigma})(\xi), \psi\right)_{L^2}$$

for all $\psi \in {}^0H^1_{\mathrm{p}}$. Letting $\psi = \mathcal{F}(\phi)(\xi)$ and applying Poincaré yields

$$\|\nabla\mathcal{F}(\phi)(\xi)\|^2_{\mathbf{L}^2} \le \|\mathcal{F}(\tilde{\sigma})(\xi)\|_{{}^0H^{-1}_{\mathrm{p}}}\|\mathcal{F}(\phi)(\xi)\|_{H^1_{\mathrm{p}}} \le C\|\mathcal{F}(\tilde{\sigma})(\xi)\|_{{}^0H^{-1}_{\mathrm{p}}}\|\nabla\mathcal{F}(\phi)(\xi)\|_{\mathbf{L}^2},$$

so that $\|\nabla\mathcal{F}(\phi)(\xi)\|_{\mathbf{L}^2} \le C\|\mathcal{F}(\tilde{\sigma})(\xi)\|_{{}^0H^{-1}_{\mathrm{p}}}$. Then

$$\begin{aligned}
\|\nabla\phi\|^2_{\mathbf{K}^s_{\mathrm{p}}} &\le \|\nabla\phi\|^2_{\mathbf{K}^s_{\mathrm{p}}(\mathbb{R}\times\Omega)} \\
&\le 2\left(\|\nabla\phi\|^2_{\mathbf{L}^2(\mathbb{R};H^s_{\mathrm{p}})} + \|\nabla\phi\|^2_{\mathbf{H}^{s/2}(\mathbb{R};L^2)}\right) \\
&= 2\left(\sum_{i=1}^{3}\|D_i\phi\|^2_{L^2(\mathbb{R};H^s_{\mathrm{p}})} + \|D_i\phi\|^2_{H^{s/2}(\mathbb{R};L^2)}\right) \\
&= 2\left(\sum_{i=1}^{3}\|\mathcal{F}(D_i\phi)(\xi)\|^2_{L^2(\mathbb{R};H^s_{\mathrm{p}})} + \|(1+\xi^2)^{s/4}\mathcal{F}(D_i\phi)(\xi)\|^2_{L^2(\mathbb{R};L^2)}\right) \\
&= 2\left(\int_{\mathbb{R}}\|\nabla\mathcal{F}(\phi)(\xi)\|^2_{\mathbf{H}^s_{\mathrm{p}}} + (1+\xi^2)^{s/2}\|\nabla\mathcal{F}(\phi)(\xi)\|^2_{\mathbf{L}^2}\right) \\
&\le C\left(\int_{\mathbb{R}}\|\mathcal{F}(\tilde{\sigma})(\xi)\|^2_{H^{s-1}_{\mathrm{p}}} + (1+\xi^2)^{s/2}\|\mathcal{F}(\tilde{\sigma})(\xi)\|^2_{{}^0H^{-1}_{\mathrm{p}}}\right) \\
&= C\left(\|\tilde{\sigma}\|^2_{L^2(\mathbb{R};H^{s-1}_{\mathrm{p}})} + \|\tilde{\sigma}\|^2_{H^{s/2}(\mathbb{R};{}^0H^{-1}_{\mathrm{p}})}\right) \\
&\le C\|\tilde{\sigma}\|^2_{\tilde{K}^s_{\mathrm{p}}}.
\end{aligned}$$

Setting $\mathbf{v}_2 = \mathbf{v}_1 + \nabla\phi$, it is easy to verify that all four conditions in (5.36) are satisfied.

Next, we aim to modify the velocity expression in such a way that its divergence remains unaffected, while allowing the tangential component of the stress on the free surface to be compatible with $\mathbf{h}$. That is, we seek $\mathbf{v}_3 \in \mathbf{K}^s_{\mathrm{p}}$ such that

$L(\mathbf{v}_3, q_1) = (\mathbf{f}_3, \sigma_3, (\mathbf{v}_0)_3, \mathbf{h}_3)$ satisfies

$$P\mathbf{f}_3(0,\cdot) = P\mathbf{f}(0,\cdot)$$
$$\sigma_3 = \sigma$$
$$(\mathbf{v}_0)_3 = \mathbf{v}_0$$
$$\mathbf{h}_3(0,\cdot) = \mathbf{h}(0,\cdot)$$
$$(\mathbf{h}_3)_{\tan} = \mathbf{h}_{\tan}.$$

This is accomplished by setting $\mathbf{v}_3 = \mathbf{v}_2 + \mathbf{w}$ where $\mathbf{w}$ is given by Lemma C.7 with $\mathbf{b} = (\mathbf{h} - \mathbf{h}_2)_{\tan}$.

Lastly, we seek $q_2 \in K_{\mathrm{p}}^{s-1}$ such that $L(\mathbf{v}_3, q_2) = (\mathbf{f}_4, \sigma, \mathbf{v}_0, \mathbf{h})$ where $P\mathbf{f}_4(0,\cdot) = P\mathbf{f}(0,\cdot)$. This can be done by simply taking $q_2 = q_1 + \tilde{q}$ where $\tilde{q} \in K_{\mathrm{p}}^{s-1}$ is chosen so that

$$\tilde{q} = (\mathbf{h} - \mathbf{h}_3) \cdot \mathbf{n} \quad \text{on } \partial G_F, \qquad\qquad \tilde{q}(0,\cdot) = 0 \quad \text{on } \Omega.$$

Such a $\tilde{q}$ exists by Lemma C.1(iii). It is straightforward to see that, at each stage, $\mathbf{v}_i$ and q_j could be chosen so that they depended on $\boldsymbol{\beta}$ in a bounded fashion. In particular,

$$\|(\mathbf{v}_3, q_2)\|_{\mathbf{X}^s} \leq C\|\boldsymbol{\beta}\|_{\mathbf{Y}^s}.$$

The final step is to notice that $(\mathbf{v}, q) \in \mathbf{X}^s$ satisfies $L(\mathbf{v}, q) = \boldsymbol{\beta}$ if and only if $(\mathbf{v} - \mathbf{v}_3, q - q_2)$ solves $L\boldsymbol{\alpha} = (\mathbf{f} - \mathbf{f}_4, 0, 0, 0)$. The claim now follows from Theorem 5.4 as discussed at the beginning of this proof. □

Recall from our discussion in Chapter 4, that the full nonlinear problem can be reduced to solving an equation of the form $(L + F)\boldsymbol{\alpha}_1 = \mathbf{g}$ where F is a certain nonlinear operator, $\mathbf{g}$ is a given function, and $\boldsymbol{\alpha}_1 \in \mathbf{X}_0^s$ where

$$\mathbf{X}_0^s = \{(\mathbf{v}, q) \in \mathbf{X}^s : \mathbf{v}(0, \cdot) = \dot{\mathbf{v}}(0, \cdot) = q(0, \cdot) = 0\}.$$

The ultimate goal is to show that the operator, R, defined by $R\boldsymbol{\alpha}_1 = L^{-1}(\mathbf{g} - F\boldsymbol{\alpha}_1)$ is a contraction mapping (on some subspace of $\mathbf{X}_0^s$) for small enough T. As such, it is crucial that we understand the dependence of L^{-1} (and F) on T. The following result demonstrates that the image of $\mathbf{X}_0^s$ under L is given by

$$\mathbf{Y}_0^s = \{(\mathbf{f}, \sigma, 0, \mathbf{h}) \in \mathbf{Y}^s : \mathbf{f}(0, \cdot) = \sigma(0, \cdot) = \dot{\sigma}(0, \cdot) = \mathbf{h}(0, \cdot) = 0\},$$

the restriction of L to $\mathbf{X}_0^s$ is invertible, and the norms of L and L^{-1} on these spaces remain bounded as $T \to 0$. This last point is vital and follows largely from the absence of an initial velocity field (relating the norms of $\mathbf{v}_0$ and $\mathbf{v}$ would generally depend on T otherwise).

Theorem 5.6. *If $3 < s < \frac{7}{2}$, then $L\mathbf{X}_0^s = \mathbf{Y}_0^s$ and $L|_{\mathbf{X}_0^s}$ is invertible. Both $L|_{\mathbf{X}_0^s}$ and $L^{-1}|_{\mathbf{Y}_0^s}$ are bounded independent of T.*

Proof. That $L\mathbf{X}_0^s \subset \mathbf{Y}_0^s$ is immediate. Since L is invertible, it suffices to show that $\mathbf{Y}_0^s \subset L\mathbf{X}_0^s$. Let $(\mathbf{f}, \sigma, 0, \mathbf{h}) \in \mathbf{Y}_0^s$ and $(\mathbf{v}, q) = L^{-1}(\mathbf{f}, \sigma, 0, \mathbf{h}) \in \mathbf{X}^s$. Since $\mathbf{v}(0, \cdot) = 0$, it follows that $q(0, \cdot) = \mathbf{S}(0, q(0, \cdot)) \cdot \mathbf{n} = 0$ on S_F. This implies that $\dot{\mathbf{v}}(0, \cdot) = -\nabla q(0, \cdot)$ so that $\dot{\mathbf{v}}(0, \cdot) \in (\mathbf{P}^0)^\perp$. However, since $(I - P)\dot{\mathbf{v}}(0, \cdot) = -\nabla q(0, \cdot)$, the identity

$$(\nabla q(0, \cdot), \nabla \psi)_{\mathbf{L}^2} = (\nabla \cdot \dot{\mathbf{v}}(0, \cdot), \psi)_{L^2} = (\dot{\sigma}(0, \cdot), \psi)_{L^2} = 0$$

must be satisfied for all $\psi \in {}^0H^1_{\mathrm{p}}$ (see the proof of Lemma 3.7(i)). Hence $\nabla q(0, \cdot) \in \mathbf{P}^0$, which implies that $\nabla q(0, \cdot) = \dot{\mathbf{v}}(0, \cdot) = 0$. Since $q(0, \cdot)$ vanishes on S_F, it follows that $q(0, \cdot) = 0$ everywhere in Ω. Thus $(\mathbf{v}, q) \in \mathbf{X}_0^s$ and $\mathbf{Y}_0^s \subset L\mathbf{X}_0^s$.

That L, restricted to $\mathbf{X}_0^s$, is bounded independent of T is simple to verify directly. As for the restriction of L^{-1} to $\mathbf{Y}_0^s$, to see that it is bounded independent of T we must examine the proof of Theorem 5.5. The construction of $(\mathbf{v}_1, q_1)$ is irrelevant since we can take both components to be identically zero for $\boldsymbol{\beta} \in \mathbf{Y}_0^s$. Next, we note that σ can be extended to $\tilde{\sigma}$, in the construction of $\mathbf{v}_2$, with bound independent of T using Lemma C.2. Finally, the estimates obtained from Theorem 5.4 were previously shown to be independent of T. $\square$

6 The Full Nonlinear Problem

With the linearized problem complete, we are finally able to follow through with the proof of Theorem 4.1. We follow the method employed by Beale in [7]: (i) we construct an approximation, $\boldsymbol{\alpha}_0 = (\mathbf{v}_0, q_0)$, to the desired solution, $\boldsymbol{\alpha} = (\mathbf{v}, q)$, in such a way that their difference, $\boldsymbol{\alpha}_1 = (\mathbf{v}_1, q_1) = (\mathbf{v}, q) - (\mathbf{v}_0, q_0)$, lies in $\mathbf{X}_0^s$; (ii) we rewrite the nonlinear problem in the form $(L + F)\boldsymbol{\alpha}_1 = \mathbf{g}$, where L is the (linear) differential operator discussed in Section 5.2 and F is a nonlinear operator; (iii) we utilize Theorem 5.6 and show that the operator R, defined by $R\boldsymbol{\omega} = L^{-1}(\mathbf{g} - F\boldsymbol{\omega})$, is a strict contraction on a subspace of $\mathbf{X}_0^s$. It then follows from the contraction mapping principle that R has a unique fixed point (in that subspace) which yields our desired solution.

6.1 Proof of the Main Result

Proof. Fix $T_0 > 0$ arbitrarily and set $G_0 = (0, T_0) \times \Omega$. Note that while the generic constants $(C, C_0, C_1, \dots)$ used in this proof will frequently depend on T_0, they will always be independent of T; the dependence of estimates on T will be made explicit in each case. We begin by constructing $\boldsymbol{\beta}_0 \in \mathbf{Y}^s$ so that if $\boldsymbol{\alpha}_0$ is the solution of $L\boldsymbol{\alpha}_0 = \boldsymbol{\beta}_0$, then it follows that $\boldsymbol{\alpha}_1 \in \mathbf{X}_0^s$. Here again, $\boldsymbol{\alpha}_1$ is taken to be the difference between $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}_0$, where $\boldsymbol{\alpha}$ is the desired solution of the full nonlinear problem. The difficulty in this construction lies in ensuring that $q_0(0,\cdot) = q(0,\cdot)$ and $\dot{\mathbf{v}}_0(0,\cdot) = \dot{\mathbf{v}}(0,\cdot)$. To see how this can be accomplished, we first notice that

$$\nabla q_1(0,\cdot) = -\dot{\mathbf{v}}_1(0,\cdot) \tag{6.1}$$

follows from (2.9) and (5.28) provided that $(\boldsymbol{\beta}_0)_1 = g\,\mathbf{e}_3$ and $(\boldsymbol{\beta}_0)_3 = \mathbf{u}_0$. Taking the divergence of both sides, (6.1) implies

$$\Delta q_1(0,\cdot) = -\nabla\cdot\dot{\mathbf{v}}_1(0,\cdot). \tag{6.2}$$

This equation is useful because the quantity $\nabla\cdot\dot{\mathbf{v}}(0,\cdot)$ is determined entirely by $\mathbf{u}_0$. We can verify this by differentiating the matrix $\Lambda = (\lambda_{i,j}(t,\mathbf{a}))$ with respect to t and evaluating at $t=0$ which yields

$$\begin{aligned} D_t\Lambda(0,\cdot) &= D_t(\nabla\mathbf{y})^{-1}(0,\cdot) \\ &= -(\nabla\mathbf{y})^{-1}D_t(\nabla\mathbf{y})(\nabla\mathbf{y})^{-1}(0,\cdot) \\ &= -(\nabla\mathbf{a})^{-1}\nabla(\dot{\mathbf{y}}(0,\cdot))(\nabla\mathbf{a})^{-1} \\ &= -\nabla\mathbf{v}(0,\cdot) \\ &= -\nabla\mathbf{u}_0. \end{aligned} \tag{6.3}$$

Hence $\dot{\lambda}_{i,j}(0,\cdot) = -D_i(\mathbf{u}_0)_j$. Now, differentiating (2.10) with respect to t and evaluating at $t=0$,

$$\begin{aligned} \sum_{k,j=1}^{3}(\dot{\lambda}_{j,k}D_k v_j + \lambda_{j,k}D_k\dot{v}_j)(0,\cdot) = 0 \\ \sum_{k=1}^{3} D_k\dot{v}_k(0,\cdot) = \sum_{k,j=1}^{3} D_j(\mathbf{u}_0)_k D_k(\mathbf{u}_0)_j \\ \nabla\cdot\dot{\mathbf{v}}(0,\cdot) = \sum_{k,j=1}^{3} D_j(\mathbf{u}_0)_k D_k(\mathbf{u}_0)_j. \end{aligned}$$

Suppose that $\mathbf{v}_0$ is chosen so that $\nabla\cdot\dot{\mathbf{v}}_0(0,\cdot)=\sigma$ where $\sigma=\sum_{k,j=1}^{3}D_j(\mathbf{u}_0)_kD_k(\mathbf{u}_0)_j$. Then (6.2) implies $\Delta q_1(0,\cdot)=0$. If we take $(\boldsymbol{\beta}_0)_4=0$, then

$$q_0(0,\cdot)=2\mu\kappa^{-2}\sum_{i,j=1}^{2}a_ia_jD_j(\mathbf{u}_0)_i=q(0,\cdot)$$

on S_F (see the beginning of the proof of Lemma 5.1) so that $q_1(0,\cdot)\;=\;0$ on S_F. Exploiting the uniqueness of solutions, it now follows from Lemma 3.5 that $q_1(0,\cdot)=0$ and hence $\dot{\mathbf{v}}_1(0,\cdot)=0$ by (6.1). Thus, to show that $\boldsymbol{\alpha}_1\in\mathbf{X}_0^s$, it suffices to prove that $(\boldsymbol{\beta}_0)_2\in\widetilde{K}_{\mathrm{p}}^s$ can be chosen such that it is a solution of

$$\begin{aligned}\phi(0,\cdot)&=0\\ \dot{\phi}(0,\cdot)&=\sigma.\end{aligned}$$

The second equation ensures that $\mathbf{v}_0$ will satisfy $\nabla\cdot\dot{\mathbf{v}}_0(0,\cdot)\;=\;\sigma$ while the first keeps the problem $L\boldsymbol{\alpha}_0=\boldsymbol{\beta}_0$ in agreement with (4.2) at $t=0$. Lemma C.1 would seem to be the obvious way to find ϕ satisfying these conditions, but observe that it requires $\sigma\;\in\;H_{\mathrm{p}}^{s-3}$. At first glance this appears to be a problem since a direct application of Lemma C.5(ii) only gives $\sigma\in H_{\mathrm{p}}^0$. However, it is straightforward to verify that $\sigma=\nabla\cdot(\mathbf{u}_0\cdot\nabla\mathbf{u}_0)$ since $\nabla\cdot\mathbf{u}_0=0$. It then follows from Lemma C.5(i) that $\mathbf{u}_0\cdot\nabla\mathbf{u}_0\in\mathbf{H}_{\mathrm{p}}^{s-2}$ so that $\sigma\in H_{\mathrm{p}}^{s-3}$. Lemma C.1 now provides a $\phi\in K_{\mathrm{p}}^s\subset\widetilde{K}_{\mathrm{p}}^s$ satisfying the desired conditions.

To help bridge the gap between the nonlinear problem (2.9)–(2.15) and the linear problem solved by $\boldsymbol{\alpha}_0$, we introduce the approximate trajectory/displacement maps "predicted" by $\boldsymbol{\alpha}_0$. We set

$$\mathbf{x}_0(t,\mathbf{a}) = \int_0^t \mathbf{v}_0(\tau,\mathbf{a})\,d\tau, \qquad \mathbf{x}_1 = \mathbf{x} - \mathbf{x}_0 = \int_0^t \mathbf{v}_1(\tau,\cdot)\,d\tau,$$
$$\mathbf{y}_0(t,\mathbf{a}) = \mathbf{x}_0(t,\mathbf{a}) + \mathbf{a}, \qquad \mathbf{N}_0 = \nabla\mathbf{y}_0\boldsymbol{\tau}_1 \times \nabla\mathbf{y}_0\boldsymbol{\tau}_2,$$
$$\mathbf{N}_1 = \mathbf{N} - \mathbf{N}_0, \qquad \Lambda_0(t,\mathbf{a}) = (\nabla\mathbf{y}_0(t,\mathbf{a}))^{-1} = (I + \nabla\mathbf{x}_0(t,\mathbf{a}))^{-1},$$
$$\Lambda_1 = \Lambda - \Lambda_0, \qquad \Pi_0(t,\mathbf{a}) = ((\pi_0)_{i,j}(t,\mathbf{a})) = ((\lambda_0)_{i,j} - \delta_{i,j}) = \Lambda_0(t,\mathbf{a}) - I,$$

where I is the 3×3 identity matrix. There is no benefit to defining Π_1 since we already have $\Lambda_1 = (\Lambda - I) - (\Lambda_0 - I)$. Notice that both $\Lambda_1(0,\cdot) = I - I = 0$ and, using (6.3),

$$\begin{aligned} D_t\Lambda_1(0,\cdot) &= -\nabla\mathbf{u}_0 + \left((I + \nabla\mathbf{x}_0)^{-1} D_t(I + \nabla\mathbf{x}_0)(I + \nabla\mathbf{x}_0)^{-1}\right)(0,\cdot) \\ &= -\nabla\mathbf{u}_0 + \nabla\dot{\mathbf{x}}_0(0,\cdot) \\ &= -\nabla\mathbf{u}_0 + \nabla\mathbf{v}_0(0,\cdot) \\ &= 0. \end{aligned}$$

Hence our choice of $\boldsymbol{\alpha}_0$ also provides excellent agreement between Λ and Λ_0 at $t = 0$. We denote by M the linear operator (i.e., acting on $\mathbf{X}^s$ with Λ fixed), analogous to L, formed by (2.9), (2.10), (2.12), and (2.14). Similarly, we use M_0 to denote the linear operator formed using the same equations but with $\Lambda, \mathbf{N}$ replaced by $\Lambda_0, \mathbf{N}_0$, respectively. It is important to note that although M and M_0 are linear as operators, they themselves depend nonlinearly on $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}_0$ respectively. Setting $L_0 = M_0 - L$ and $L_1 = M - M_0$, we see that the full nonlinear problem can be rewritten as

$$
\begin{aligned}
M\boldsymbol{\alpha} &= (g\,\mathbf{e}_3, 0, \mathbf{u}_0, 0) \\
(L + L_0 + L_1)\boldsymbol{\alpha} &= (g\,\mathbf{e}_3, 0, \mathbf{u}_0, 0) \\
L\boldsymbol{\alpha}_0 + L\boldsymbol{\alpha}_1 + L_0\boldsymbol{\alpha}_0 + L_0\boldsymbol{\alpha}_1 + L_1\boldsymbol{\alpha}_0 + L_1\boldsymbol{\alpha}_1 &= (g\,\mathbf{e}_3, 0, \mathbf{u}_0, 0) \\
L\boldsymbol{\alpha}_1 + L_0\boldsymbol{\alpha}_1 + L_1\boldsymbol{\alpha}_0 + L_1\boldsymbol{\alpha}_1 &= (0, -\phi, 0, 0) - L_0\boldsymbol{\alpha}_0 \\
(L + F)\boldsymbol{\alpha}_1 &= \mathbf{g}
\end{aligned}
$$

where $F : \mathbf{X}_0^s \to \mathbf{Y}_0^s$ is given by $F\boldsymbol{\alpha}_1 = L_0\boldsymbol{\alpha}_1 + L_1\boldsymbol{\alpha}_0 + L_1\boldsymbol{\alpha}_1$ and $\mathbf{g} = (0, -\phi, 0, 0) - L_0\boldsymbol{\alpha}_0$ depends only on known terms. The bulk of our remaining work will lie in showing that each term in F is actually Lipschitz continuous with respect to $\boldsymbol{\alpha}_1$ provided that $\boldsymbol{\alpha}_1$ is taken from a fixed, bounded subset of $\mathbf{X}_0^s$. Moreover, we will prove that each forms a contraction mapping in $\boldsymbol{\alpha}_1$ for sufficiently small T. In particular, we will see that there exists a constant $C > 0$, independent of T, such that

$$
\|F\boldsymbol{\alpha}_1 - F\widetilde{\boldsymbol{\alpha}}_1\|_{\mathbf{Y}^s} \le CT^\delta \|\boldsymbol{\alpha}_1 - \widetilde{\boldsymbol{\alpha}}_1\|_{\mathbf{X}^s} \tag{6.4}
$$

for some $\delta > 0$ (still assuming that $\boldsymbol{\alpha}_1$ and $\widetilde{\boldsymbol{\alpha}}_1$ are taken from a fixed, bounded subset of $\mathbf{X}_0^s$).

Suppose that F satisfies (6.4) and that we take the closed set $\mathbf{B} = \{\boldsymbol{\alpha}_1 \in \mathbf{X}_0^s : \|\boldsymbol{\alpha}_1 - L^{-1}\mathbf{g}\|_{\mathbf{X}^s} \le \|L^{-1}\mathbf{g}\|_{\mathbf{X}^s}\}$ as the relevant subset of $\mathbf{X}_0^s$. Note that this is a fixed and bounded subset of $\mathbf{X}_0^s$ because we can exploit the fact that $\boldsymbol{\alpha}_0 \in \mathbf{X}_0^s(G_0)$ to bound $\|L^{-1}\mathbf{g}\|_{\mathbf{X}^s}$ above by a constant which is independent of T (using techniques explored in detail over the next several pages). Observe that if we define the map $R : \mathbf{X}_0^s \to \mathbf{X}_0^s$ by $R\boldsymbol{\alpha}_1 = L^{-1}(\mathbf{g} - F\boldsymbol{\alpha}_1)$, then the existence of a fixed point of R would yield a solution of the full nonlinear problem (2.9)–(2.15). Taking $\boldsymbol{\alpha}_1 \in \mathbf{B}$

and applying Theorem 5.6,

$$\begin{aligned}\|R\boldsymbol{\alpha}_1 - L^{-1}\mathbf{g}\|_{\mathbf{X}^s} &= \|L^{-1}F\boldsymbol{\alpha}_1\|_{\mathbf{X}^s}\\ &\le C_1\|F\boldsymbol{\alpha}_1\|_{\mathbf{Y}^s}\\ &\le C_2T^{\delta}\|\boldsymbol{\alpha}_1\|_{\mathbf{X}^s}\\ &\le C_2T^{\delta}(\|\boldsymbol{\alpha}_1 - L^{-1}\mathbf{g}\|_{\mathbf{X}^s} + \|L^{-1}\mathbf{g}\|_{\mathbf{X}^s})\\ &\le C_3T^{\delta}\|L^{-1}\mathbf{g}\|_{\mathbf{X}^s}\end{aligned}$$

so that $R(\mathbf{B}) \subset \mathbf{B}$ for sufficiently small T. Here C_1, C_2, and C_3 are positive constants which do not depend on T. Moreover, R is a contraction on $\mathbf{B}$ for the same T since

$$\begin{aligned}\|R\boldsymbol{\alpha}_1 - R\widetilde{\boldsymbol{\alpha}}_1\|_{\mathbf{X}^s} &= \|L^{-1}(F\boldsymbol{\alpha}_1 - F\widetilde{\boldsymbol{\alpha}}_1)\|_{\mathbf{X}^s}\\ &\le C_1\|F\boldsymbol{\alpha}_1 - F\widetilde{\boldsymbol{\alpha}}_1\|_{\mathbf{Y}^s}\\ &\le C_2T^{\delta}\|\boldsymbol{\alpha}_1 - \widetilde{\boldsymbol{\alpha}}_1\|_{\mathbf{X}^s}.\end{aligned}$$

Thus we can apply the contraction mapping principle to obtain a unique fixed point of R in $\mathbf{B}$ which yields the desired solution of our nonlinear problem.

All that remains is to show that F indeed satisfies the estimate $\|F\boldsymbol{\alpha}_1 - F\widetilde{\boldsymbol{\alpha}}_1\|_{\mathbf{Y}^s} \le CT^{\delta}\|\boldsymbol{\alpha}_1 - \widetilde{\boldsymbol{\alpha}}_1\|_{\mathbf{X}^s}$ for some $\delta > 0$ and for all $\boldsymbol{\alpha}_1, \widetilde{\boldsymbol{\alpha}}_1$ taken from a fixed, bounded subset of $\mathbf{X}_0^s$. Setting $s = 3 + 2\delta$ where $0 < \delta < \frac{1}{4}$, we now estimate the terms arising in the various components of F individually. It should be noted here that while the operator L_1 depends on both $\boldsymbol{\alpha}_0$ and $\boldsymbol{\alpha}_1$, L_0 is independent of $\boldsymbol{\alpha}_1$ (depending only on $\boldsymbol{\alpha}_0$). To avoid repetitive comments, we also note that in the following claims $C, C_0, C_1, \ldots,$ denote positive constants which may depend on

T_0, but are always independent of T. In particular, C denotes a generic positive constant which can change from instance to instance.

Claim 1. $(L_0)_1$ is a contraction for sufficiently small T, with $\|(L_0\boldsymbol{\alpha}_1)_1\|_{\mathbf{K}_{\mathrm{p}}^{s-2}} \leq CT^{\delta}\|\boldsymbol{\alpha}_1\|_{\mathbf{X}^s}$.

Proof of Claim 1. First we notice that the i^{th} component of $(L_0\boldsymbol{\alpha}_1)_1$ can be rewritten, using the entires of Π_0, as

$$\begin{aligned}((L_0\boldsymbol{\alpha}_1)_1)_i &= -\mu\sum_{j,k,m}(\lambda_0)_{j,k}D_k((\lambda_0)_{j,m}D_m(\mathbf{v}_1)_i) + \mu\sum_k D_k^2(\mathbf{v}_1)_i \\ &\qquad + \sum_k(\lambda_0)_{i,k}D_kq_1 - D_iq_1 \\ &= -\mu\sum_{j,k,m}(\pi_0)_{j,k}D_k((\pi_0)_{j,m}D_m(\mathbf{v}_1)_i) - \mu\sum_{j,m}D_j((\pi_0)_{j,m}D_m(\mathbf{v}_1)_i) \\ &\qquad - \mu\sum_{j,k}(\pi_0)_{j,k}D_jD_k(\mathbf{v}_1)_i + \sum_k(\pi_0)_{i,k}D_kq_1.\end{aligned}$$

The reason that we opt to write terms using Π_0 instead of Λ_0 has to do with our need to make repeated use of Lemma C.6(ii). This result is used to split products, such as $(\pi_0)_{j,m}D_m(\mathbf{v}_1)_i$, into pieces that can be estimated individually without introducing an unknown dependence on T and relies on both pieces (and potentially their time derivatives) vanishing at $t = 0$. Since $(L_0\boldsymbol{\alpha}_1)_1$ can be written so that every term contains entries from Π_0 and these entries all vanish at $t = 0$ (unlike those of Λ_0), it is only natural to rewrite things in this form.

Our first goal is to show that $(\pi_0)_{j,m}D_m(\mathbf{v}_1)_i \in K_{\mathrm{p}}^{2+2\delta}$. To obtain estimates for Π_0, we begin by examining the derivatives of $\mathbf{v}_0$ (and subsequently $\mathbf{x}_0$). Since $D_m(\mathbf{v}_0)_i \in K_{\mathrm{p}}^{2+2\delta}(G_0)$, we have $D_m(\mathbf{v}_0)_i \in H^{2\delta}((0,T_0);H_{\mathrm{p}}^{2-2\delta})$ by Lemma C.3. It then follows from Lemma C.4 that $D_m(\mathbf{x}_0)_i \in H^{1+\delta}((0,T);H^{2-2\delta}_{\mathrm{p}})$ and

$$\begin{aligned}\|D_m(\mathbf{x}_0)_i\|_{H^{1+\delta}((0,T);H^{2-2\delta}_{\mathrm{p}})} &\leq C_1T^\delta\|D_m(\mathbf{v}_0)_i\|_{H^{2\delta}((0,T);H^{2-2\delta}_{\mathrm{p}})}\\ &\leq C_1T^\delta\|D_m(\mathbf{v}_0)_i\|_{H^{2\delta}((0,T_0);H^{2-2\delta}_{\mathrm{p}})}\\ &= C_2T^\delta \qquad (6.5)\end{aligned}$$

for $T \leq T_0$. This implies that $\Pi_0 = -I + \sum_n (-1)^n(\nabla\mathbf{x}_0)^n$ is well-defined and

$$\begin{aligned}\|\Pi_0\|_{(H^{1+\delta}((0,T);H^{2-2\delta}_{\mathrm{p}}))^{3\times3}} &\leq \sum_{n=1}^{\infty} C_0^{n-1}\,\|\nabla\mathbf{x}_0\|^n_{(H^{1+\delta}((0,T);H^{2-2\delta}_{\mathrm{p}}))^{3\times3}}\\ &= \frac{\|\nabla\mathbf{x}_0\|_{(H^{1+\delta}((0,T);H^{2-2\delta}_{\mathrm{p}}))^{3\times3}}}{1-C_0\|\nabla\mathbf{x}_0\|_{(H^{1+\delta}((0,T);H^{2-2\delta}_{\mathrm{p}}))^{3\times3}}}\\ &\leq C_3T^\delta, \qquad (6.6)\end{aligned}$$

provided that we take T small enough that $\|\nabla\mathbf{x}_0\|_{(H^{1+\delta}((0,T);H^{2-2\delta}_{\mathrm{p}}))^{3\times3}} < \min\{\frac{1}{2}, \frac{1}{2C_0}\}$, where C_0 is the appropriate constant taken from Lemma C.6(ii). Finally, we observe that by Lemma C.3(ii), $\mathbf{v}_1 \in \mathbf{H}^{1+\delta}((0,T);H^1_{\mathrm{p}})$ with

$$\|\mathbf{v}_1\|_{\mathbf{H}^{1+\delta}((0,T);H^1_{\mathrm{p}})} \leq C\|\mathbf{v}_1\|_{\mathbf{K}^{3+2\delta}_{\mathrm{p}}},$$

where C is independent of T. Thus, applying Lemmas C.6(ii) and C.5(i), we obtain

$$\begin{aligned}\|(\pi_0)_{j,m}D_m(\mathbf{v}_1)_i\|_{H^{1+\delta}((0,T);L^2)} &\leq C_4\|(\pi_0)_{j,m}\|_{H^{1+\delta}((0,T);H^{2-2\delta}_{\mathrm{p}})}\|D_m(\mathbf{v}_1)_i\|_{H^{1+\delta}((0,T);L^2)}\\ &\leq C_4\|\Pi_0\|_{(H^{1+\delta}((0,T);H^{2-2\delta}_{\mathrm{p}}))^{3\times3}}\|\mathbf{v}_1\|_{\mathbf{H}^{1+\delta}((0,T);H^1_{\mathrm{p}})}\\ &\leq C_5T^\delta\|\mathbf{v}_1\|_{\mathbf{K}^{3+2\delta}_{\mathrm{p}}}. \qquad (6.7)\end{aligned}$$

To conclude that $(\pi_0)_{j,m}D_m(\mathbf{v}_1)_i \in K^{2+2\delta}_{\mathrm{p}}$, we now need to estimate its spatial derivatives. Since we have $D_m(\mathbf{v}_0)_i \in H^0((0,T_0);H^{2+2\delta}_{\mathrm{p}})$, it follows that $D_m(\mathbf{x}_0)_i \in$

$H^1((0,T);H^{2+2\delta}_{\mathrm{p}})$ by Lemma C.4. In fact, the Sobolev Embedding Theorem (B.8) implies that $D_m(\mathbf{x}_0)_i \in C([0,T];H^{2+2\delta}_{\mathrm{p}})$. To obtain our desired estimate we will now need to exploit complex interpolation between H^2_{p} and H^3_{p} (see Appendix A.2 for details regarding complex interpolation). First, we notice that

$$\|D_m(\mathbf{x}_0)_i\|_{C([0,T];H^{2+2\delta}_{\mathrm{p}})} = \sup_{t\in[0,T]} \left\| \int_0^t D_m(\mathbf{v}_0)_i \right\|_{H^{2+2\delta}_{\mathrm{p}}}.$$

Now, we suppose temporarily that $\int_0^t D_m(\mathbf{v}_0)_i \in H^k_{\mathrm{p}}$ where $k \in \{2,3\}$. Applying Hölder yields

$$\begin{aligned}
\left\| \int_0^t D_m(\mathbf{v}_0)_i \right\|^2_{H^k_{\mathrm{p}}} &\le \left\| \int_0^t D_m(\mathbf{v}_0)_i \right\|^2_{H^k} \\
&= \sum_{|\boldsymbol{\alpha}|\le k} \left\| \int_0^t D_{\boldsymbol{\alpha}} D_m(\mathbf{v}_0)_i \right\|^2_{L^2} \\
&= \sum_{|\boldsymbol{\alpha}|\le k} \left\| \int_0^T \chi_{[0,t]} D_{\boldsymbol{\alpha}} D_m(\mathbf{v}_0)_i \right\|^2_{L^2} \\
&\le \sum_{|\boldsymbol{\alpha}|\le k} \left\| \|\chi_{[0,t]}\|_{L^2(0,T)} \|D_{\boldsymbol{\alpha}} D_m(\mathbf{v}_0)_i\|_{L^2(0,T)} \right\|^2_{L^2} \\
&= \|\chi_{[0,t]}\|^2_{L^2(0,T)} \left\| \sum_{|\boldsymbol{\alpha}|\le k} \|D_{\boldsymbol{\alpha}} D_m(\mathbf{v}_0)_i\|_{L^2} \right\|^2_{L^2(0,T)} \\
&\le C_1 t \, \|D_m(\mathbf{v}_0)_i\|^2_{\mathbf{L}^2((0,T);H^k_{\mathrm{p}})} \\
&\le C_1 t \, \|D_m(\mathbf{v}_0)_i\|^2_{\mathbf{L}^2((0,T_0);H^k_{\mathrm{p}})} \\
&\le C_2 t
\end{aligned}$$

where C_1 and C_2 are positive constants depending on k. If we take $D_m(\mathbf{x}_0)_i \in C([0,T]; H^3_\mathrm{p})$, then applying Theorem A.1(iv) yields

$$\begin{aligned}
\|D_m(\mathbf{x}_0)_i\|_{C([0,T];H^{2+2\delta}_\mathrm{p})} &= \sup_{t\in[0,T]} \left\| \int_0^t D_m(\mathbf{v}_0)_i \right\|_{H^{2+2\delta}_\mathrm{p}} \\
&\le \sup_{t\in[0,T]} \left\| \int_0^t D_m(\mathbf{v}_0)_i \right\|^{1-2\delta}_{H^2_\mathrm{p}} \left\| \int_0^t D_m(\mathbf{v}_0)_i \right\|^{2\delta}_{H^3_\mathrm{p}} \\
&\le C \sup_{t\in[0,T]} t^{1/2} \\
&= CT^{1/2}.
\end{aligned} \tag{6.8}$$

Estimate (6.8) remains valid for all $D_m(\mathbf{x}_0)_i \in C([0,T]; H^{2+2\delta}_\mathrm{p})$, since H^3_p is dense in $H^{2+2\delta}_\mathrm{p}$. Now the approach used to obtain (6.6) can be applied again, this time with the constant $C_0 > 0$ coming from Lemma C.5(i), to obtain

$$\|\Pi_0\|_{(C([0,T];H^{2+2\delta}_\mathrm{p}))^{3\times 3}} \le C_6 T^{1/2}.$$

Since $D_m(\mathbf{v}_1)_i \in H^0((0,T); H^{2+2\delta}_\mathrm{p})$, Lemma C.5(i) yields

$$\begin{aligned}
\|(\pi_0)_{j,m} D_m(\mathbf{v}_1)_i\|_{H^0((0,T);H^{2+2\delta}_\mathrm{p})} &= \left(\int_0^T \|(\pi_0)_{j,m} D_m(\mathbf{v}_1)_i\|^2_{H^{2+2\delta}_\mathrm{p}} \right)^{1/2} \\
&\le C \left(\int_0^T \|(\pi_0)_{j,m}\|^2_{H^{2+2\delta}_\mathrm{p}} \|D_m(\mathbf{v}_1)_i\|^2_{H^{2+2\delta}_\mathrm{p}} \right)^{1/2} \\
&\le CT^{1/2} \|D_m(\mathbf{v}_1)_i\|_{H^0((0,T);H^{2+2\delta}_\mathrm{p})} \\
&\le CT^{1/2} \|\mathbf{v}_1\|_{\mathbf{K}^{3+2\delta}_\mathrm{p}}.
\end{aligned} \tag{6.9}$$

Combining (6.7) and (6.9), we obtain $(\pi_0)_{j,m} D_m(\mathbf{v}_1)_i \in K^{2+2\delta}_\mathrm{p}$ with

$$\|(\pi_0)_{j,m} D_m(\mathbf{v}_1)_i\|_{K^{2+2\delta}_\mathrm{p}} \le CT^{\delta} \|\mathbf{v}_1\|_{\mathbf{K}^{3+2\delta}_\mathrm{p}}. \tag{6.10}$$

The same series of steps can be applied to show that the terms

$$(\pi_0)_{j,k} D_j D_k(\mathbf{v}_1)_i, \qquad (\pi_0)_{j,k} D_k((\pi_0)_{j,m} D_m(\mathbf{v}_1)_i), \qquad \text{and} \qquad (\pi_0)_{i,k} D_k q_1,$$

all belong to $K_{\mathrm{p}}^{1+2\delta}$ and satisfy the same type of estimate as (6.10). Specifically, we obtain

$$\|(L_0\boldsymbol{\alpha}_1)_1\|_{\mathbf{K}_{\mathrm{p}}^{1+2\delta}} \leq CT^\delta \|\mathbf{v}_1\|_{\mathbf{K}_{\mathrm{p}}^{3+2\delta}} + CT^\delta \|\nabla q_1\|_{\mathbf{K}_{\mathrm{p}}^{1+2\delta}}.$$

Since L_0 is linear in $\boldsymbol{\alpha}_1$, the first component is obviously a contraction mapping for sufficiently small T. ∎

Claim 2. For any fixed constant $K > 0$, $(L_1\boldsymbol{\alpha}_0)_1$ is a contraction in $\boldsymbol{\alpha}_1$ for sufficiently small T and $\|\boldsymbol{\alpha}_1\|_{\mathbf{X}^s} \leq K$. In particular, $\|(L_1\boldsymbol{\alpha}_0)_1 - (\widetilde{L}_1\boldsymbol{\alpha}_0)_1\|_{\mathbf{K}_{\mathrm{p}}^{s-2}} \leq CT^\delta \|\boldsymbol{\alpha}_1 - \widetilde{\boldsymbol{\alpha}}_1\|_{\mathbf{X}^s}$ for all $\boldsymbol{\alpha}_1, \widetilde{\boldsymbol{\alpha}}_1$ bounded as above, where $\widetilde{L}_1$ depends on $\widetilde{\boldsymbol{\alpha}}_1$ in precisely the same fashion that L_1 depends on $\boldsymbol{\alpha}_1$. Here the constant $C > 0$ depends on K.

Proof of Claim 2. The i^{th} component of $(L_1\boldsymbol{\alpha}_0)_1$ can be rewritten as

$$\begin{aligned}
((L_1\boldsymbol{\alpha}_0)_1)_i &= -\mu \sum_{j,k,m} \lambda_{j,k} D_k(\lambda_{j,m} D_m(\mathbf{v}_0)_i) + \mu \sum_{j,k,m} (\lambda_0)_{j,k} D_k((\lambda_0)_{j,m} D_m(\mathbf{v}_0)_i) \\
&\qquad + \sum_k \lambda_{i,k} D_k q_0 - \sum_k (\lambda_0)_{i,k} D_k q_0 \\
&= -\mu \sum_{j,k,m} (\lambda_1)_{j,k} D_k((\lambda_1)_{j,m} D_m(\mathbf{v}_0)_i) - \mu \sum_{j,k,m} (\lambda_1)_{j,k} D_k((\pi_0)_{j,m} D_m(\mathbf{v}_0)_i) \\
&\qquad - \mu \sum_{j,k,m} (\pi_0)_{j,k} D_k((\lambda_1)_{j,m} D_m(\mathbf{v}_0)_i) - \mu \sum_{j,k} (\lambda_1)_{j,k} D_j D_k(\mathbf{v}_0)_i) \\
&\qquad - \mu \sum_{j,m} D_j((\lambda_1)_{j,m} D_m(\mathbf{v}_0)_i) + \sum_k (\lambda_1)_{i,k} D_k q_0
\end{aligned}$$

The approach which led to (6.5) and (6.8) can be used again to obtain

$$\|D_m(\mathbf{x}_1)_i\|_{H^{1+\delta}((0,T);H_{\rm p}^{2-2\delta})} \le CT^{\delta}\|\mathbf{v}_1\|_{\mathbf{K}_{\rm p}^{3+2\delta}},$$
$$\|D_m(\mathbf{x}_1)_i\|_{C([0,T];H_{\rm p}^{2+2\delta})} \le CT^{1/2}\|\mathbf{v}_1\|_{\mathbf{K}_{\rm p}^{3+2\delta}}.$$

It is important to note here that since $\boldsymbol{\alpha}_1$ will eventually be restricted to a fixed bounded set (i.e., $\|\boldsymbol{\alpha}_1\|_{\mathbf{X}^s} \le K$ for fixed K), we will be able to take T small enough to ensure that both $\|D_m(\mathbf{x}_0)_i\| < \epsilon$ and $\|D_m(\mathbf{x}_1)_i\| < \epsilon$, in the appropriate norms, for any $\epsilon > 0$. Temporarily denoting the space $(H^{1+\delta}((0,T);H_{\rm p}^{2-2\delta}))^{3\times 3}$ by $\widetilde{H}$, we apply Lemmas C.6(ii) and C.5(i) to get

$$\begin{aligned}
\|\Lambda_1\|_{(H^{1+\delta}((0,T);H_{\rm p}^{2-2\delta}))^{3\times 3}} &= \|(\nabla\mathbf{y})^{-1} - (\nabla\mathbf{y}_0)^{-1}\|_{\widetilde{H}} \\
&= \|(\nabla\mathbf{y})^{-1}(\nabla\mathbf{y}_0 - \nabla\mathbf{y})(\nabla\mathbf{y}_0)^{-1}\|_{\widetilde{H}} \\
&= \left\|\left(\sum_n (-1)^n(\nabla\mathbf{x})^n\right)\nabla\mathbf{x}_1\left(\sum_m (-1)^m(\nabla\mathbf{x}_0)^m\right)\right\|_{\widetilde{H}} \\
&= \left\|\sum_{m,n}(-1)^{n+m}(\nabla\mathbf{x})^n\nabla\mathbf{x}_1(\nabla\mathbf{x}_0)^m\right\|_{\widetilde{H}} \\
&\le \sum_{m,n} C_0^{m+n}\|\nabla\mathbf{x}\|_{\widetilde{H}}^n\|\nabla\mathbf{x}_1\|_{\widetilde{H}}\|\nabla\mathbf{x}_0\|_{\widetilde{H}}^m \\
&\le \|\nabla\mathbf{x}_1\|_{\widetilde{H}}\sum_{m,n} C_0^{m+n}(\|\nabla\mathbf{x}_1\|_{\widetilde{H}} + \|\nabla\mathbf{x}_0\|_{\widetilde{H}})^n\|\nabla\mathbf{x}_0\|_{\widetilde{H}}^m \\
&\le \|\nabla\mathbf{x}_1\|_{\widetilde{H}}\sum_{m,n} C_0^{m+n}2^n(\|\nabla\mathbf{x}_1\|_{\widetilde{H}}^n + \|\nabla\mathbf{x}_0\|_{\widetilde{H}}^n)\|\nabla\mathbf{x}_0\|_{\widetilde{H}}^m \\
&= \frac{\|\nabla\mathbf{x}_1\|_{\widetilde{H}}}{1 - C_0\|\nabla\mathbf{x}_0\|_{\widetilde{H}}}\left(\frac{1}{1-2C_0\|\nabla\mathbf{x}_1\|_{\widetilde{H}}} + \frac{1}{1-2C_0\|\nabla\mathbf{x}_0\|_{\widetilde{H}}}\right) \\
&\le C\|\nabla\mathbf{x}_1\|_{\widetilde{H}} \\
&\le CT^{\delta}\|\mathbf{v}_1\|_{\mathbf{K}_{\rm p}^{3+2\delta}}.
\end{aligned} \tag{6.11}$$

Here we have taken T small enough that all of the geometric series converge and are easily estimated (e.g., $\epsilon = \min\{\frac{1}{2}, \frac{1}{4C_0}\}$). We have also employed the trivial estimate $(a+b)^n \leq 2^{n-1}(a^n + b^n)$ for $a, b \geq 0$. Since $D_m(\mathbf{v}_0)_i \in H^{1+\delta}((0,T);L^2)$ by Lemma C.3(i), the triangle inequality yields

$$\|(\lambda_1)_{j,m} D_m(\mathbf{v}_0)_i\|_{H^{1+\delta}((0,T);L^2)} \leq \|(\lambda_1)_{j,m} D_m((\mathbf{v}_0)_i - (\mathbf{u}_0)_i)\|_{H^{1+\delta}((0,T);L^2)} + \|(\lambda_1)_{j,m} D_m(\mathbf{u}_0)_i\|_{H^{1+\delta}((0,T);L^2)}. \tag{6.12}$$

We subtract the initial velocity $\mathbf{u}_0$ from $\mathbf{v}_0$ here so that we will be able to apply Lemma C.6(ii) in a later step. While this subtraction introduces the second term on the right side of (6.12), this term is not problematic since $\mathbf{u}_0$ is itself independent of time. In fact, we can find a constant $C > 0$ such that

$$\|(\lambda_1)_{j,m} D_m(\mathbf{u}_0)_i\|_{H^{1+\delta}((0,T);L^2)} \leq C \|D_m(\mathbf{u}_0)_i\|_{L^2} \|(\lambda_1)_{j,m}\|_{H^{1+\delta}((0,T);H_{\mathrm{p}}^{2-2\delta})} \tag{6.13}$$

using the interpolation property for linear operators given in Theorem A.2. Similar interpolations will be required in the proofs of subsequent claims; here we describe the manner in which the interpolation property is used, but in later instances we will leave most of the details to the reader. Using Lemmas C.5(i) and C.6(i), we can define the linear multiplication operator $T : H^1((0,T); H_{\mathrm{p}}^{2-2\delta}) \to H^1((0,T); L^2)$ by $Tf = f D_m(\mathbf{u}_0)_i$. Now, for $k \in \{1, 2\}$ and $f \in H^k((0,T); H_{\mathrm{p}}^{2-2\delta})$, we apply

Lemma C.5(i) to obtain

$$\begin{aligned}
\|fD_m(\mathbf{u}_0)_i\|^2_{H^k((0,T);L^2)} &= \sum_{j\le k}\|D_t^j(fD_m(\mathbf{u}_0)_i)\|^2_{L^2((0,T);L^2)} \\
&= \sum_{j\le k}\int_0^T\int_\Omega |D_m(\mathbf{u}_0)_i D_t^j f|^2 \\
&= \sum_{j\le k}\int_\Omega |D_m(\mathbf{u}_0)_i|^2\int_0^T |D_t^j f|^2 \\
&= \Big\| D_m(\mathbf{u}_0)_i \|f\|_{H^k(0,T)}\Big\|^2_{L^2} \\
&\le C_k\|D_m(\mathbf{u}_0)_i\|^2_{L^2}\Big\| \, \|f\|_{H^k(0,T)}\Big\|^2_{H^{2-2\delta}_{\mathrm{p}}} \\
&= C_k\|D_m(\mathbf{u}_0)_i\|^2_{L^2}\|f\|^2_{H^k((0,T);H^{2-2\delta}_{\mathrm{p}})},
\end{aligned}$$

where $C_k > 0$ is a constant depending on k. Hence we immediately obtain that T and $T|_{H^2((0,T);H^{2-2\delta}_{\mathrm{p}})}$ are continuous into $H^1((0,T);L^2)$ and $H^2((0,T);L^2)$, respectively, and have operator norms bounded above by a positive constant depending on $\mathbf{u}_0$. Let us temporarily set $X_0 = H^2((0,T);H^{2-2\delta}_{\mathrm{p}})$, $X_1 = H^1((0,T);H^{2-2\delta}_{\mathrm{p}})$, $Y_0 = H^2((0,T);L^2)$, and $Y_1 = H^1((0,T);L^2)$. Theorem A.2 then implies that $T|_{H^{1+\delta}((0,T);H^{2-2\delta}_{\mathrm{p}})}$ is continuous into $H^{1+\delta}((0,T);L^2)$ and

$$\begin{aligned}
\|fD_m(\mathbf{u}_0)_i\|_{H^{1+\delta}((0,T);L^2)} &= \|Tf\|_{H^{1+\delta}((0,T);L^2)} \\
&\le \|T\|_{\mathcal{L}(H^{1+\delta}((0,T);H^{2-2\delta}_{\mathrm{p}});H^{1+\delta}((0,T);L^2))}\|f\|_{H^{1+\delta}((0,T);L^2)} \\
&\le C\|T\|^{1-\delta}_{\mathcal{L}(X_1;Y_1)}\|T\|^{\delta}_{\mathcal{L}(X_0;Y_0)}\|f\|_{H^{1+\delta}((0,T);L^2)} \\
&\le C\|D_m(\mathbf{u}_0)_i\|_{L^2}\|f\|_{H^{1+\delta}((0,T);H^{2-2\delta}_{\mathrm{p}})}
\end{aligned}$$

for all $f \in H^{1+\delta}((0,T);H^{2-2\delta}_{\mathrm{p}})$, where we have neglected to denote restrictions in order to enhance readability. Hence (6.13) holds. We can now combine Lemma

C.6(ii) with (6.13) and (6.11) to obtain

$$\begin{aligned}\|(\lambda_1)_{j,m}D_m(\mathbf{v}_0)_i\|_{H^{1+\delta}((0,T);L^2)} &\le C_1\|(\lambda_1)_{j,m}\|_{\widetilde{H}}\|D_m((\mathbf{v}_0)_i-(\mathbf{u}_0)_i)\|_{H^{1+\delta}((0,T);L^2)}\\ &\qquad + C_2\|(\lambda_1)_{j,m}\|_{\widetilde{H}}\\ &\le C_3T^\delta\|\mathbf{v}_1\|_{\mathbf{K}_\mathrm{p}^{3+2\delta}}\end{aligned}$$

from (6.12). Here we have temporarily denoted the space $H^{1+\delta}((0,T);H_\mathrm{p}^{2-2\delta})$ by $\widetilde{H}$ for convenience. The same general approach used to obtain (6.11) is now applied again to get

$$\|\Lambda_1\|_{(C([0,T];H_\mathrm{p}^{2+2\delta}))^{3\times 3}} \le CT^{1/2}\|\mathbf{v}_1\|_{\mathbf{K}_\mathrm{p}^{3+2\delta}}.$$

The estimate analogous to (6.9) can also be found as before and from this it follows that

$$\|(\lambda_1)_{j,m}D_m(\mathbf{v}_0)_i\|_{K_\mathrm{p}^{2+2\delta}} \le CT^\delta\|\mathbf{v}_1\|_{\mathbf{K}_\mathrm{p}^{3+2\delta}}. \tag{6.14}$$

The same line of reasoning can now be used to show that all but the first term in $((L_1\boldsymbol{\alpha}_0)_1)_i$ belong to $K_\mathrm{p}^{1+2\delta}$ and satisfy an estimate of the form (6.14). The first term, on the other hand, satisfies

$$\|(\lambda_1)_{j,k}D_k((\lambda_1)_{j,m}D_m(\mathbf{v}_0)_i)\|_{K_\mathrm{p}^{1+2\delta}} \le CT^\delta\|\mathbf{v}_1\|_{\mathbf{K}_\mathrm{p}^{3+2\delta}}(1+\|\mathbf{v}_1\|_{\mathbf{K}_\mathrm{p}^{3+2\delta}}).$$

Here the two $\mathbf{v}_1$ norms arise as a result of the estimation of the two λ_1 terms individually. However, since we will eventually restrict ourselves to $\boldsymbol{\alpha}_1$ lying in a fixed bounded set, the above estimate is equivalent to one of the form (6.14). To see that the $(L_1\boldsymbol{\alpha}_0)_1$ is a contraction in $\boldsymbol{\alpha}_1$ in this setting, we take $\boldsymbol{\alpha}_1$ and $\widetilde{\boldsymbol{\alpha}}_1$ from a fixed bounded set and observe that

$$\begin{aligned}\Lambda_1 - \widetilde{\Lambda}_1 &= (\Lambda - \Lambda_0) - (\widetilde{\Lambda} - \Lambda_0) \\ &= \Lambda - \widetilde{\Lambda} \\ &= (\nabla \mathbf{y})^{-1} - (\nabla \widetilde{\mathbf{y}})^{-1} \\ &= -(\nabla \mathbf{y})^{-1}(\nabla \mathbf{y} - \nabla \widetilde{\mathbf{y}})(\nabla \widetilde{\mathbf{y}})^{-1} \\ &= -(\nabla \mathbf{y})^{-1}\nabla(\mathbf{y}_1 - \widetilde{\mathbf{y}}_1)(\nabla \widetilde{\mathbf{y}})^{-1}\end{aligned}$$

where the quantities with the tilde $\sim$ correspond to $\widetilde{\boldsymbol{\alpha}}_1$ in the obvious way. This allows us to estimate differences of terms involving Λ_1 and $\widetilde{\Lambda}_1$ (in the various norms) by $\|\Lambda_1 - \widetilde{\Lambda}_1\| \leq CT^\delta \|\mathbf{v}_1 - \widetilde{\mathbf{v}}_1\|_{\mathbf{K}_\mathrm{p}^{3+2\delta}}$. Adding and subtracting to compare terms from Λ_1 and $\widetilde{\Lambda}_1$, we can conclude

$$\|(L_1\alpha_0)_1 - (\widetilde{L}_1\alpha_0)_1\|_{\mathbf{K}_\mathrm{p}^{1+2\delta}} \leq CT^\delta \|\mathbf{v}_1 - \widetilde{\mathbf{v}}_1\|_{\mathbf{K}_\mathrm{p}^{3+2\delta}}$$

so that $(L_1\boldsymbol{\alpha}_0)_1$ is a contraction in $\boldsymbol{\alpha}_1$ for sufficiently small T and $\|\boldsymbol{\alpha}_1\|_{\mathbf{X}^s} \leq K$. ■

It readily follows from the techniques/estimates used in the proofs of Claims 1 and 2 that $(L_1\boldsymbol{\alpha}_1)_1$ also satisfies Claim 2 under the same conditions. We now move on to the second component of F which deals with the conservation of mass equations.

Claim 3. $(L_0)_2$ is a contraction for sufficiently small T, with $\|(L_0\boldsymbol{\alpha}_1)_2\|_{\widetilde{K}_\mathrm{p}^s} \leq CT^\delta \|\mathbf{v}_1\|_{\mathbf{K}_\mathrm{p}^s}$.

Proof of Claim 3. First we observe that

$$(L_0\boldsymbol{\alpha}_1)_2 = \sum_{j,k}(\lambda_0)_{j,k} D_k(\mathbf{v}_1)_j - \nabla \cdot \mathbf{v}_1 = \sum_{j,k}(\pi_0)_{j,k} D_k(\mathbf{v}_1)_j$$

and note that the estimate (6.9) directly applies. All that remains is to estimate the product $(\pi_0)_{j,k} D_k(\mathbf{v}_1)_j$ in $H^{3/2+\delta}((0,T);{}^0H_\mathrm{p}^{-1})$.

Since $\mathbf{v}_1(0,\cdot) = \dot{\mathbf{v}}_1(0,\cdot) = 0$, we can use Lemma C.2(ii) to extend $\mathbf{v}_1$ to $(0,T_0)$ with norm bounded independent of T. Combining this with (5.33), we find

$$\begin{aligned}
\|D_k(\mathbf{v}_1)_j\|_{H^{3/2+\delta}((0,T),{}^0H_\mathrm{p}^{-1})} &\leq \|D_k(\mathbf{v}_1)_j\|_{H^{3/2+\delta}((0,T_0),{}^0H_\mathrm{p}^{-1})} \\
&\leq C\|(\mathbf{v}_1)_j\|_{H^{s/2}((0,T_0);H_\mathrm{p}^0)\cap H^0((0,T_0);H_\mathrm{p}^s)} \\
&\leq C\|\mathbf{v}_1\|_{\mathbf{K}_\mathrm{p}^s}.
\end{aligned}$$

By Lemma C.3(ii), we have $D_k(\mathbf{v}_1)_j \in H^1((0,T);H_\mathrm{p}^{2\delta})$ with $\|D_k(\mathbf{v}_1)_j\|_{H^1((0,T);H_\mathrm{p}^{2\delta})} \leq C\|\mathbf{v}_1\|_{\mathbf{K}_\mathrm{p}^s}$. Hence we can apply (6.6) and Lemmas C.6(ii) and C.5(i) to obtain

$$\|(\pi_0)_{j,k} D_k(\mathbf{v}_1)_j\|_{H^1((0,T);H_\mathrm{p}^{2\delta})} \leq CT^\delta \|\mathbf{v}_1\|_{\mathbf{K}_\mathrm{p}^s}.$$

We now estimate the terms arising from differentiating this product with respect to time. Since $D_k(\dot{\mathbf{v}}_1)_j \in H^{1/2+\delta}((0,T);{}^0H_\mathrm{p}^{-1})$, the same approach used above, modified slightly to use Lemma C.5(iii), yields

$$\|(\pi_0)_{j,k} D_k(\dot{\mathbf{v}}_1)_j\|_{H^{1/2+\delta}((0,T);{}^0H_\mathrm{p}^{-1})} \leq CT^\delta \|\mathbf{v}_1\|_{\mathbf{K}_\mathrm{p}^s}.$$

Notice that the condition $\dot{\mathbf{v}}_1(0,\cdot) = 0$ is necessary to obtain this estimate. Next we try to estimate terms of the form $(\dot{\pi}_0)_{j,k} D_k(\mathbf{v}_1)_j$ in the same space. Since $D_k(\dot{\mathbf{x}}_0)_j = D_k(\mathbf{v}_0)_j \in K_\mathrm{p}^{2+2\delta}(G_0)$, we have both $D_k(\dot{\mathbf{x}}_0)_j \in H^{1/2+\delta}((0,T_0);H_\mathrm{p}^1)$ by Lemma C.3(ii) and $D_k(\mathbf{x}_0)_j \in H^{1/2+\delta}((0,T_0);H_\mathrm{p}^{2+2\delta})$ by Lemma C.4. Since $\dot{\Pi}_0 = -\Lambda_0 \nabla \dot{\mathbf{x}}_0 \Lambda_0$, it follows from Lemmas C.5(i) and C.6(i) that $\dot{\Pi}_0 \in (H^{1/2+\delta}((0,T_0);H_\mathrm{p}^1))^{3\times 3}$.

While it would seem that we are now ready to estimate $(\dot{\pi}_0)_{j,k} D_k(\mathbf{v}_1)_j$ in the usual fashion, the fact that $\dot{\Pi}_0(0,\cdot) = -\nabla \dot{\mathbf{x}}_0(0,\cdot) = -\nabla \mathbf{u}_0$ means that we cannot apply Lemma C.6(ii) directly. However, with the appropriate addition and subtraction, Lemmas C.6(ii) and C.5(iv) can be applied to split the product up as desired:

$$\begin{aligned}\|(\dot{\pi}_0)_{j,k} D_k(\mathbf{v}_1)_j\|_{\widetilde{H}} &\le \|((\dot{\pi}_0)_{j,k} + D_j(\mathbf{u}_0)_k) D_k(\mathbf{v}_1)_j\|_{\widetilde{H}} + \|D_j(\mathbf{u}_0)_k D_k(\mathbf{v}_1)_j\|_{\widetilde{H}} \\ &\le C_1 \|(\dot{\pi}_0)_{j,k} + D_j(\mathbf{u}_0)_k\|_{H^{1/2}((0,T);H^1_{\mathrm{p}})} \|D_k(\mathbf{v}_1)_j\|_{H^{1/2}((0,T);H^{2\delta}_{\mathrm{p}})} \\ &\quad + C_2 \|D_j(\mathbf{u}_0)_k\|_{H^1_{\mathrm{p}}} \|D_k(\mathbf{v}_1)_j\|_{H^{1/2+\delta}((0,T);L^2)}. \end{aligned} \tag{6.15}$$

Here we have temporarily denoted the space $H^{1/2+\delta}((0,T); {}^0H^{-1}_{\mathrm{p}})$ by $\widetilde{H}$ for convenience. We have also used the estimate

$$\|f D_j(\mathbf{u}_0)_k\|_{H^{1/2+\delta}((0,T);{}^0H^{-1}_{\mathrm{p}})} \le C \|D_j(\mathbf{u}_0)_k\|_{H^1_{\mathrm{p}}} \|f\|_{H^{1/2+\delta}((0,T);L^2)}, \tag{6.16}$$

which holds for all $f \in H^{1/2+\delta}((0,T); L^2)$ and can be obtained using Theorem A.2 in a similar fashion to the one detailed in the proof of Claim 2. Unfortunately, the second term in (6.15) can not be estimated in the usual way; ordinarily we are able to obtain our estimates' dependence on T from whichever function is multiplied onto $\mathbf{v}_1$, yet here $\mathbf{u}_0$ is completely independent of T. To get around this, we observe that since $D_k(\dot{\mathbf{v}}_1)_j \in H^0((0,T); H^{2\delta}_{\mathrm{p}})$ by Lemma C.3(ii), we can rewrite $D_k(\mathbf{v}_1)_j$ as the integral (in time) of $D_k(\dot{\mathbf{v}}_1)_j$ and apply Lemma C.4 to get

$$\begin{aligned}\|D_k(\mathbf{v}_1)_j\|_{H^{1/2+\delta}((0,T);H^{2\delta}_{\mathrm{p}})} &= \|D_k(\mathbf{v}_1)_j\|_{H^{1-(1/2-\delta)}((0,T);H^{2\delta}_{\mathrm{p}})} \\ &\le C T^{1/2-\delta} \|D_k(\dot{\mathbf{v}}_1)_j\|_{H^0((0,T);H^{2\delta}_{\mathrm{p}})} \\ &\le C T^{1/2-\delta} \|\mathbf{v}_1\|_{\mathbf{K}^s_{\mathrm{p}}}. \end{aligned} \tag{6.17}$$

Now both terms in (6.15) can be estimated easily, yielding

$$\|(\dot{\pi}_0)_{j,k} D_k(\mathbf{v}_1)_j\|_{H^{1/2+\delta}((0,T);{}^0H_{\mathrm{p}}^{-1})} \le CT^{1/2-\delta}\|\mathbf{v}_1\|_{\mathbf{K}_{\mathrm{p}}^s}.$$

The inequality in the claim follows trivially. As with the first component, linearity of L_0 in $\boldsymbol{\alpha}_1$ now implies that the second component is a contraction mapping for sufficiently small T. ∎

Claim 4. Under the hypotheses of Claim 2, $(L_1\boldsymbol{\alpha}_0)_2$ is a contraction in $\boldsymbol{\alpha}_1$ for sufficiently small T, with $\|(L_1\boldsymbol{\alpha}_0)_2 - (\widetilde{L}_1\boldsymbol{\alpha}_0)_2\|_{\widetilde{K}_{\mathrm{p}}^s} \le CT^\delta\|\boldsymbol{\alpha}_1 - \widetilde{\boldsymbol{\alpha}}_1\|_{\mathbf{K}_{\mathrm{p}}^s}$ where $\boldsymbol{\alpha}_1, \widetilde{\boldsymbol{\alpha}}_1$ are taken as in Claim 2.

Proof of Claim 4. As before, we note that

$$(L_1\boldsymbol{\alpha}_0)_2 = \sum_{j,k}(\lambda_1)_{j,k} D_k(\mathbf{v}_0)_j.$$

Since the estimate analogous to (6.9) directly applies, we need only estimate the product in $H^{3/2+\delta}((0,T);{}^0H_{\mathrm{p}}^{-1})$. Then, adding and subtracting so as to exploit Lemma C.6(ii),

$$\begin{aligned}\|(\lambda_1)_{j,k}D_k(\mathbf{v}_0)_j\|_{H^1((0,T);L^2)} &\le \|(\lambda_1)_{j,k}D_k((\mathbf{v}_0)_j - (\mathbf{u}_0)_j)\|_{\widetilde{H}} + \|(\lambda_1)_{j,k}D_k(\mathbf{u}_0)_j\|_{\widetilde{H}} \\ &\le C_1\|(\lambda_1)_{j,k}\|_{H^1((0,T);H_{\mathrm{p}}^{2-2\delta})}\|D_k((\mathbf{v}_0)_j - (\mathbf{u}_0)_j)\|_{\widetilde{H}} \\ &\qquad + C_2\|D_k(\mathbf{u}_0)_j\|_{H_{\mathrm{p}}^{2\delta}}\|(\lambda_1)_{j,k}\|_{H^1((0,T);H_{\mathrm{p}}^{2-2\delta})} \\ &\le C_3T^\delta\|\mathbf{v}_1\|_{\mathbf{K}_{\mathrm{p}}^s}.\end{aligned}$$

Here we have temporarily denoted the space $H^1((0,T);L^2)$ by $\widetilde{H}$ for convenience. We have also used the estimate

$$\|fD_k(\mathbf{u}_0)_j\|_{H^1((0,T);L^2)} \leq C_2\|D_k(\mathbf{u}_0)_j\|_{H_\mathrm{p}^{2\delta}}\|f\|_{H^1((0,T);H_\mathrm{p}^{2-2\delta})},$$

valid for all $f \in H^1((0,T);H_\mathrm{p}^{2-2\delta})$, which is a simple application of Lemma C.5(i). Applying the same approach as that leading to (6.17), we see that

$$\|D_k(\mathbf{x}_1)_j\|_{H^{1/2+\delta}((0,T);H_\mathrm{p}^{2+2\delta})} \leq CT^{1/2-\delta}\|\mathbf{v}_1\|_{\mathbf{K}_\mathrm{p}^s}.$$

Similarly, the steps that led to (6.11) can now be used to show that

$$\|\Lambda_1\|_{(H^{1/2+\delta}((0,T);H_\mathrm{p}^{2+2\delta}))^{3\times 3}} \leq CT^{1/2-\delta}\|\mathbf{v}_1\|_{\mathbf{K}_\mathrm{p}^s}.$$

Since $D_k(\dot{\mathbf{v}}_0)_j \in H^{1/2+\delta}((0,T);{}^0H_\mathrm{p}^{-1})$ (see the similar discussion in Claim 3), we can use Lemmas C.6(ii) and C.5(iii) to obtain

$$\begin{aligned}
\|(\lambda_1)_{j,k}D_k(\dot{\mathbf{v}}_0)_j\|_{\widetilde{H}} &\leq \|(\lambda_1)_{j,k}D_k((\dot{\mathbf{v}}_0)_j - (\dot{\mathbf{v}}_0)_j(0,\cdot))\|_{\widetilde{H}} \\
&\qquad + \|(\lambda_1)_{j,k}D_k(\dot{\mathbf{v}}_0)_j(0,\cdot)\|_{\widetilde{H}} \\
&\leq C_4\|(\lambda_1)_{j,k}\|_{H^{1/2+\delta}((0,T);H_\mathrm{p}^{2+2\delta})}\|D_k((\dot{\mathbf{v}}_0)_j - (\dot{\mathbf{v}}_0)_j(0,\cdot))\|_{\widetilde{H}} \\
&\qquad + C_5\|(\lambda_1)_{j,k}\|_{H^{1/2+\delta}((0,T);H_\mathrm{p}^{2+2\delta})}\|D_k(\dot{\mathbf{v}}_0)_j(0,\cdot)\|_{{}^0H_\mathrm{p}^{-1}} \\
&\leq C_6T^{1/2-\delta}\|\mathbf{v}_1\|_{\mathbf{K}_\mathrm{p}^s}.
\end{aligned}$$

Here we have temporarily denoted the space $H^{1/2+\delta}((0,T);{}^0H_\mathrm{p}^{-1})$ by $\widetilde{H}$ for convenience. We have also used the estimate

$$\|fD_k(\dot{\mathbf{v}}_0)_j(0,\cdot)\|_{H^{1/2+\delta}((0,T);{}^0H_\mathrm{p}^{-1})} \leq C\|D_k(\dot{\mathbf{v}}_0)_j(0,\cdot)\|_{{}^0H_\mathrm{p}^{-1}}\|f\|_{H^{1/2+\delta}((0,T);H_\mathrm{p}^{2+2\delta})},$$

which holds for all $f \in H^{1/2+\delta}((0,T);H_{\mathrm{p}}^{2+2\delta})$ and can be obtained using Theorem A.2 in a similar fashion to the one detailed in the proof of Claim 2. All that remains is to estimate terms of the form $(\dot{\lambda}_1)_{j,k}D_k(\mathbf{v}_0)_j$ in the same space. Lemma C.4 implies

$$\|D_k(\mathbf{x}_1)_j\|_{H^{1/2+\delta}((0,T);H_{\mathrm{p}}^{2+2\delta})} \leq CT^{1/2-\delta}\|\mathbf{v}_1\|_{\mathbf{K}_{\mathrm{p}}^s}.$$

Adding and subtracting terms, it is straightforward to verify that

$$\dot{\Lambda}_1 = -(\Lambda_1+\Pi_0+I)\nabla\dot{\mathbf{x}}_1(\Lambda_1+\Pi_0+I) - (\Lambda_1+\Pi_0+I)\nabla\dot{\mathbf{x}}_0\Lambda_1 - \Lambda_1\nabla\dot{\mathbf{x}}_0(\Pi_0+I).$$

Since every term contains either Λ_1 or $\nabla\dot{\mathbf{x}}_1$, with all other quantities bounded (for $\boldsymbol{\alpha}_1$ taken from a fixed, bounded set), we can apply Lemmas C.5(i) and C.6(ii) repeatedly to get the estimate

$$\|\dot{\Lambda}_1\|_{(H^{1/2+\delta}((0,T);H_{\mathrm{p}}^1))^{3\times 3}} \leq CT^{1/2-\delta}\|\mathbf{v}_1\|_{\mathbf{K}_{\mathrm{p}}^s}.$$

Finally, since $D_k(\mathbf{v}_0)_j \in H^{1/2+\delta}((0,T);H_{\mathrm{p}}^1)$, we can apply Lemmas C.6(ii) and C.5(iv) to obtain

$$\begin{aligned}
\|(\dot{\lambda}_1)_{j,k}D_k(\mathbf{v}_0)_j\|_{H^{1/2+\delta}((0,T);{}^0H_{\mathrm{p}}^{-1})} &\leq \|(\dot{\lambda}_1)_{j,k}(D_k(\mathbf{v}_0)_j - D_k(\mathbf{u}_0)_j)\|_{\widetilde{H}} \\
&\qquad + \|(\dot{\lambda}_1)_{j,k}D_k(\mathbf{u}_0)_j\|_{H^{1/2+\delta}((0,T);{}^0H_{\mathrm{p}}^{-1})} \\
&\leq C_7\|(\dot{\lambda}_1)_{j,k}\|_{\widetilde{H}}\|D_k(\mathbf{v}_0)_j - D_k(\mathbf{u}_0)_j\|_{\widetilde{H}} \\
&\qquad + C_8\|D_j(\mathbf{u}_0)_k\|_{H_{\mathrm{p}}^1}\|D_k(\mathbf{v}_1)_j\|_{H^{1/2+\delta}((0,T);L^2)} \\
&\leq C_9T^{1/2-\delta}\|\mathbf{v}_1\|_{\mathbf{K}_{\mathrm{p}}^s}.
\end{aligned}$$

Here we have temporarily denoted the space $H^{1/2+\delta}((0,T);H_{\mathrm{p}}^1)$ by $\widetilde{H}$ for convenience and have also made use of the estimate (6.16). That $(L_1\boldsymbol{\alpha}_0)_2$ is a contraction in $\boldsymbol{\alpha}_1$ can now be shown using the techniques described at the end of Claim 2. ∎

As before, $(L_1\boldsymbol{\alpha}_1)_2$ can be shown to satisfy Claim 4 (under the same conditions) using a combination of the techniques/estimates described in the proofs of Claims 3 and 4. Since it is clear that $(F\boldsymbol{\alpha}_1)_3 = (\mathbf{g})_3 = 0$, all that remains to be shown is that the fourth component of F is also contraction in $\boldsymbol{\alpha}_1$.

Claim 5. $(L_0)_4$ is a contraction for sufficiently small T, with $\|(L_0\boldsymbol{\alpha}_1)_4\|_{\mathbf{K}_{\mathrm{p}}^{s-3/2}(\partial G_F)} \leq CT^\delta \|\boldsymbol{\alpha}_1\|_{\mathbf{X}^s}$.

Proof of Claim 5. As usual, we begin by rewriting the i^{th} component in terms of the entries of Π_0:

$$\begin{aligned}((L_0\boldsymbol{\alpha}_1)_4)_i = q_1(\mathbf{N}_0 - \mathbf{n})_i &- \mu \sum_{j,k} ((\pi_0)_{j,k} D_k(\mathbf{v}_1)_i + (\pi_0)_{i,k} D_k(\mathbf{v}_1)_j)(\mathbf{N}_0 - \mathbf{n})_j \\ &- \mu \sum_j (D_j(\mathbf{v}_1)_i + D_i(\mathbf{v}_1)_j)(\mathbf{N}_0 - \mathbf{n})_j \\ &- \mu \sum_{j,k} ((\pi_0)_{j,k} D_k(\mathbf{v}_1)_i + (\pi_0)_{i,k} D_k(\mathbf{v}_1)_j) n_j \qquad (6.18)\end{aligned}$$

where $\mathbf{n} = \mathbf{n}(0,\cdot) = \mathbf{N}(0,\cdot) = \mathbf{N}_0(0,\cdot)$. It will be most convenient for us to estimate terms in $\mathbf{K}_{\mathrm{p}}^{2+2\delta}(G_0)$ and then restrict them to ∂G_F. In keeping with this, we can use Lemmas C.2(ii) and C.1 to extend q_1 first to $K_{\mathrm{p}}^{3/2+2\delta}((0,T_0) \times S_F))$ and then to $\mathbf{K}_{\mathrm{p}}^{2+2\delta}(G_0)$ in such a way that it remains bounded independent of T. Since n_j is smooth and, like $\mathbf{u}_0$, independent of time, the last term in (6.18) is straightforward to estimate using the general techniques discussed in the proofs of the previous claims. For the remaining terms, the only expression which has not already been estimated is $\mathbf{N}_0 - \mathbf{n}$.

The tangent vectors $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ are both $\mathbf{C}^\infty(\Omega)$, so $\mathbf{N}_0$ is easily extended into Ω. Moreover, Lemma C.3 and the Sobolev Embedding Theorem imply that $\mathbf{N}_0 \in$

$\mathbf{H}^{1+\delta}((0,T);H_{\mathrm{p}}^{2-2\delta}) \cap \mathbf{C}([0,T];H_{\mathrm{p}}^{2+2\delta})$ which is contained in $\mathbf{C}([0,T];C_{\mathrm{p}}^{0})$. Now, since

$$\mathbf{N}_0 - \mathbf{n} = (\nabla \mathbf{x}_0 \boldsymbol{\tau}_1 \times \nabla \mathbf{x}_0 \boldsymbol{\tau}_2) + (\nabla \mathbf{x}_0 \boldsymbol{\tau}_1 \times \boldsymbol{\tau}_2) + (\boldsymbol{\tau}_1 \times \nabla \mathbf{x}_0 \boldsymbol{\tau}_2),$$

it readily follows from (6.5) and (6.8) that both

$$\|\mathbf{N}_0 - \mathbf{n}\|_{\mathbf{H}^{1+\delta}((0,T);H_{\mathrm{p}}^{2-2\delta})} \leq CT^{\delta} \qquad \text{and} \qquad \|\mathbf{N}_0 - \mathbf{n}\|_{\mathbf{C}([0,T];H_{\mathrm{p}}^{2+2\delta})} \leq CT^{1/2}.$$

With these new estimates in hand, the claimed inequality now follows from the usual techniques. As always, linearity of $(L_0)_4$ now implies that it is a contraction.

■

Claim 6. Under the hypotheses of Claim 2, $(L_1\boldsymbol{\alpha}_0)_4$ is a contraction in $\boldsymbol{\alpha}_1$ for sufficiently small T, with $\|(L_1\boldsymbol{\alpha}_0)_4 - (\widetilde{L}_1\boldsymbol{\alpha}_0)_4\|_{\mathbf{K}_{\mathrm{p}}^{s-3/2}(\partial G_F)} \leq CT^{\delta}\|\boldsymbol{\alpha}_1 - \widetilde{\boldsymbol{\alpha}}_1\|_{\mathbf{K}_{\mathrm{p}}^{s}}$ where $\boldsymbol{\alpha}_1, \widetilde{\boldsymbol{\alpha}}_1$ are taken as in Claim 2.

Proof of Claim 6. The i^{th} component of $(L_1\boldsymbol{\alpha}_0)_4$ can be rewritten as

$$\begin{aligned}((L_1\boldsymbol{\alpha}_0)_4)_i = &-\mu\sum_{j,k}((\lambda_1)_{j,k}D_k(\mathbf{v}_0)_i + (\lambda_1)_{i,k}D_k(\mathbf{v}_0)_j)((\mathbf{N}_1)_j + (\mathbf{N}_0 - \mathbf{n})_j + n_j)\\ &- \mu\sum_{j,k}((\pi_0)_{k,j}D_k(\mathbf{v}_0)_i + (\pi_0)_{k,i}D_k(\mathbf{v}_0)_j)(\mathbf{N}_1)_j\\ &- \mu\sum_{j,k}((\lambda_1)_{j,k}D_k(\mathbf{v}_0)_i + (\lambda_1)_{i,k}D_k(\mathbf{v}_0)_j)((\mathbf{N}_0 - \mathbf{n})_j + n_j)\\ &+ q_0(\mathbf{N}_1)_i - \mu\sum_{j}(D_j(\mathbf{v}_0)_i + D_i(\mathbf{v}_0)_j)(\mathbf{N}_1)_j.\end{aligned}$$

Here we seek appropriate estimates for $\mathbf{N}_1$. This can be done in roughly the same way as in Claim 5 (with $\mathbf{N}_0$) using the corresponding estimates for $D_m(\mathbf{x}_1)_i$. Here the condition that $\boldsymbol{\alpha}_1$ be taken from a fixed, bounded set is required in order to obtain

$$\|\mathbf{N}_1\|_{\mathbf{H}^{1+\delta}((0,T);H^{2-2\delta}_{\mathrm{p}})} \le CT^{\delta}\|\mathbf{v}_1\|_{\mathbf{K}^s_{\mathrm{p}}} \qquad \text{and} \qquad \|\mathbf{N}_1\|_{\mathbf{C}([0,T];H^{2+2\delta}_{\mathrm{p}})} \le CT^{1/2}\|\mathbf{v}_1\|_{\mathbf{K}^s_{\mathrm{p}}}.$$

Now $(L_1\boldsymbol{\alpha}_0)_4$ is readily seen to be a contraction in $\boldsymbol{\alpha}_1$ using the same technique outlined in Claim 2. ■

Finally, we can combine the approaches in the proofs of Claims 5 and 6 to demonstrate that $(L_1\boldsymbol{\alpha}_1)_4$ similarly satisfies Claim 6. Thus F satisfies (6.4), completing the proof of Theorem 4.1. □

Since $\boldsymbol{\alpha}_0$ and $\boldsymbol{\alpha}_1$ are uniquely determined, it is tempting to assume that the solution provided by Theorem 4.1 is the only one. Unfortunately, however, though we obtain that $\boldsymbol{\alpha}_1$ is the unique fixed of $R|_{\mathbf{B}}$ using the contraction mapping principle, this says nothing of whether it is also the unique fixed point of R (on $\mathbf{X}^s_0$). The following result shows that any other solution of the nonlinear problem (2.9)–(2.15) must agree with the one provided by Theorem 4.1 for some initial period of time. Here we will denote changes to the underlying time interval of a space by appending this interval onto the name of that space.

Lemma 6.1. *Let $3 < s < \frac{7}{2}$. Suppose that $\boldsymbol{\alpha} \in \mathbf{X}^s(0, T_{\boldsymbol{\alpha}})$ is the solution of* (2.9)–(2.15) *provided by Theorem 4.1 and let $\boldsymbol{\beta} \in \mathbf{X}^s(0, T_{\boldsymbol{\alpha}})$ be any other (strong) solution to the same problem on $(0, T_{\boldsymbol{\alpha}})$. There exists $T_{\boldsymbol{\beta}} > 0$ such that $\boldsymbol{\alpha} = \boldsymbol{\beta}$ in $\mathbf{X}^s(0, T_{\boldsymbol{\beta}})$.*

Proof. Let $\boldsymbol{\alpha} \in \mathbf{X}^s(0, T_{\boldsymbol{\alpha}})$ be the solution discussed in Theorem 4.1 where $T_{\boldsymbol{\alpha}}$ denotes the fixed upper limit of the time interval for this solution. Now suppose that $\boldsymbol{\beta} \in \mathbf{X}^s(0, T_{\boldsymbol{\alpha}})$ is another solution. If $\boldsymbol{\beta} - \boldsymbol{\alpha}_0 \in \mathbf{B}(0, T_{\boldsymbol{\alpha}})$, then $\boldsymbol{\alpha} = \boldsymbol{\beta}$ by uniqueness of the fixed point of R in $\mathbf{B}(0, T_{\boldsymbol{\alpha}})$. However, if $\boldsymbol{\beta} - \boldsymbol{\alpha}_0 \notin \mathbf{B}(0, T_{\boldsymbol{\alpha}})$, then there exists an n such that $\|\boldsymbol{\beta} - \boldsymbol{\alpha}_0 - L^{-1}\mathbf{g}\|_{\mathbf{X}^s(0,T_{\boldsymbol{\alpha}})} \leq n\|L^{-1}\mathbf{g}\|_{\mathbf{X}^s(0,T_{\boldsymbol{\alpha}})}$. Define

$$\mathbf{B}_n(0,T) = \{\boldsymbol{\gamma} \in \mathbf{X}_0^s(0,T) : \|\boldsymbol{\gamma} - L^{-1}\mathbf{g}\|_{\mathbf{X}^s(0,T)} \leq n\|L^{-1}\mathbf{g}\|_{\mathbf{X}^s(0,T_{\boldsymbol{\alpha}})}\}$$

and notice that $\boldsymbol{\alpha} - \boldsymbol{\alpha}_0, \boldsymbol{\beta} - \boldsymbol{\alpha}_0 \in \mathbf{B}_n(0,T)$ for all $T \leq T_{\boldsymbol{\alpha}}$. We estimate F exactly as before to obtain (6.4) where the constant now depends on n. Taking $\boldsymbol{\gamma} \in \mathbf{B}_n$ yields

$$\begin{aligned}
\|R\boldsymbol{\gamma} - L^{-1}\mathbf{g}\|_{\mathbf{X}^s(0,T)} &= \|L^{-1}F\boldsymbol{\gamma}\|_{\mathbf{X}^s(0,T)} \\
&\leq C\|F\boldsymbol{\gamma}\|_{\mathbf{Y}^s(0,T)} \\
&\leq C(n)T^{\delta}\|\boldsymbol{\gamma}\|_{\mathbf{X}^s(0,T)} \\
&\leq C(n)T^{\delta}(\|\boldsymbol{\gamma} - L^{-1}\mathbf{g}\|_{\mathbf{X}^s(0,T)} + \|L^{-1}\mathbf{g}\|_{\mathbf{X}^s(0,T)}) \\
&\leq C(n)T^{\delta}(n\|L^{-1}\mathbf{g}\|_{\mathbf{X}^s(0,T_{\boldsymbol{\alpha}})})
\end{aligned}$$

which implies that $R(\mathbf{B}_n(0,T)) \subset \mathbf{B}_n(0,T)$ for sufficiently small $T = T(n)$. As before, it is easily verified that R is a contraction on $\mathbf{B}_n(0,T)$ for the same T. Applying the contraction mapping principle and exploiting uniqueness of the fixed point, we have $\boldsymbol{\alpha} = \boldsymbol{\beta}$ in $\mathbf{X}^s(0,T)$. □

Note that if Theorem 4.1 could be proven for displacements from Ω which are initially nonzero (i.e., replace (2.13) with $\mathbf{x}(0,\cdot) = \mathbf{f}$ for sufficiently general $\mathbf{f}$), then uniqueness on $(0,T_\alpha)$ could be obtained using Lemma 6.1 in the following way: for $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$ in $\mathbf{X}^s(0,T_{\boldsymbol{\alpha}})$, there exists $0 < T_{\max} < T_{\boldsymbol{\alpha}}$ such that $\boldsymbol{\alpha} = \boldsymbol{\beta}$ in $\mathbf{X}^s(0,T_{\max})$ and $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$ in $\mathbf{X}^s(0,T)$ for $T > T_{\max}$. Making the change of variable $\tau = t - T_{\max}$, both $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are solutions of the nonlinear problem given by (2.9)–(2.15), with (2.12) replaced by $\mathbf{v}_\tau(0,\cdot) = \mathbf{v}(T_{\max},\cdot)$ and (2.13) replaced by $\mathbf{x}_\tau(0,\cdot) = \mathbf{x}(T_{\max},\cdot)$, for $\tau \in (0, T_{\boldsymbol{\alpha}} - T_{\max})$. The argument contained in the proof of Lemma 6.1 could then be applied again to contradict the maximality of $T_{\max}$ and prove uniqueness on $(0,T_\alpha)$.

6.2 The Axisymmetric Case

Given that our initial domain is a cylinder, a natural question to ask is whether axisymmetric initial conditions will necessarily yield axisymmetric solutions. This is especially important if one hopes to draw a connection to solutions of corresponding thin-filament approximations since these arise from the axisymmetric Navier-Stokes equations. To examine this, we rewrite the original nonlinear problem (2.1)–(2.2) in cylindrical coordinates:

$$\begin{aligned}
\dot{u}_r &= -u_r D_r u_r - \frac{1}{r} u_\theta D_\theta u_r - u_z D_z u_r + \frac{1}{r} u_\theta^2 - D_r p \\
&\qquad + \mu \left(\frac{1}{r} D_r (r D_r u_r) + \frac{1}{r^2} D_\theta^2 u_r + D_z^2 u_r - \frac{1}{r^2} u_r - \frac{2}{r^2} D_\theta u_\theta \right) \\
\dot{u}_\theta &= -u_r D_r u_\theta - \frac{1}{r} u_\theta D_\theta u_\theta - u_z D_z u_\theta - \frac{1}{r} u_\theta u_r - \frac{1}{r} D_\theta p \\
&\qquad + \mu \left(\frac{1}{r} D_r (r D_r u_\theta) + \frac{1}{r^2} D_\theta^2 u_\theta + D_z^2 u_\theta - \frac{1}{r^2} u_\theta + \frac{2}{r^2} D_\theta u_r \right) \\
\dot{u}_z &= -u_r D_r u_z - \frac{1}{r} u_\theta D_\theta u_z - u_z D_z u_z - D_z p \\
&\qquad + \mu \left(\frac{1}{r} D_r (r D_r u_z) + \frac{1}{r^2} D_\theta^2 u_z + D_z^2 u_z \right) + g \\
0 &= \frac{1}{r} D_r (r u_r) + \frac{1}{r} D_\theta u_\theta + D_z u_z.
\end{aligned}$$

Similarly, in cylindrical coordinates, (2.7) becomes

$$\begin{aligned}
(p - P_0) n_r &= \mu (2 D_r u_r n_r + D_r u_\theta n_\theta + \frac{1}{r} D_\theta u_r n_\theta - \frac{1}{r} u_\theta n_\theta + D_z u_r n_z + D_r u_z n_z) \\
(p - P_0) n_\theta &= \mu (D_r u_\theta n_r + \frac{1}{r} D_\theta u_r n_r + \frac{2}{r} u_r n_\theta + \frac{2}{r} D_\theta u_\theta n_\theta \\
&\qquad - \frac{1}{r} u_\theta n_r + D_z u_\theta n_z + \frac{1}{r} D_\theta u_z n_z) \\
(p - P_0) n_z &= \mu (D_z u_r n_r + D_r u_z n_r + D_z u_\theta n_\theta + \frac{1}{r} D_\theta u_z n_\theta + \frac{1}{r} D_\theta u_r + 2 D_z u_z n_z).
\end{aligned}$$

Lemma 6.2. *The solution* $(\mathbf{u}, p)$ *of the problem* (2.1)–(2.8) *established in Corollary 4.2 (via Theorem 4.1) is axisymmetric provided that* $\mathbf{u}_0$ *is axisymmetric.*

Proof. For the purposes of this proof, all functions are assumed to be given in cylindrical coordinates. Suppose that $\mathbf{u}_0 = \mathbf{u}_0(t, r, z)$ and let $(\mathbf{u}, p)$ be the solution of (2.1)–(2.8) established in Corollary 4.2. It should be clear from the above equations that $(\mathbf{u}(t, r, \theta+c, z), p(t, r, \theta+c, z))$ must also be a solution of this problem for any $c \in \mathbb{R}$. Let $\boldsymbol{\alpha} = (\mathbf{v}, q)$ denote the solution of the associated Lagrangian problem (2.9)–(2.15) provided by Theorem 4.1 and further let $\boldsymbol{\alpha} = \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1$ be the decomposition of $\boldsymbol{\alpha}$ described in that proof. In what follows, we will only need to keep track of differences in quantities' angular (θ) arguments so we now abbreviate any function of the form $f(t, r, \theta, z)$ by $f(\theta)$.

Taking ϕ as in Section 6.1, it is readily seen that, in addition to $\boldsymbol{\alpha}_0(\theta)$, $\boldsymbol{\alpha}_0(\theta + c)$ is also a solution of $L\boldsymbol{\alpha} = (g\,\mathbf{e}_3, \phi, \mathbf{u}_0, 0)$. It then follows from Theorem 5.6 that $\boldsymbol{\alpha}_0(\theta + c) = \boldsymbol{\alpha}_0(\theta)$ by uniqueness of solutions. This implies that $\boldsymbol{\alpha}_1(\theta + c) \in \mathbf{B}$ is a fixed point of R and hence $\boldsymbol{\alpha}_1(\theta + c) = \boldsymbol{\alpha}_1(\theta)$ since this fixed point is unique by the contraction mapping principle. Hence the solution to the Lagrangian formulation of the problem, $\boldsymbol{\alpha}$, is axisymmetric. To see that this translates into axisymmetry for $(\mathbf{u}, p)$, we first observe that

$$y(\theta) + c = x(\theta) + (\theta + c) = \int_0^t v(\theta) + (\theta + c) = \int_0^t v(\theta + c) + (\theta + c) = y(\theta + c)$$

where $y(\theta), x(\theta)$, and $v(\theta)$ denote the angular components of the trajectory, displacement, and velocity maps, respectively, corresponding to $\boldsymbol{\alpha}(\theta)$. Similarly denoting the Eulerian solution $(\mathbf{u}, p)$ by $\boldsymbol{\beta}$, it follows that

$$\boldsymbol{\beta}(y+c) = \boldsymbol{\beta}(y(\theta)+c) = \boldsymbol{\beta}(y(\theta+c)) = \boldsymbol{\alpha}(\theta+c) = \boldsymbol{\alpha}(\theta) = \boldsymbol{\beta}(y(\theta)) = \boldsymbol{\beta}(y).$$

Thus $(\mathbf{u}, p)$ is axisymmetric. □

6.3 Concluding Remarks

In this work, we have established the local-in-time existence and regularity of solutions (Theorem 4.1) to the three-dimensional Navier-Stokes flow of a viscous fluid jet assumed to be periodic in the axial direction and everywhere else bounded by a moving free surface. This was accomplished using a functional analytic approach which revolved around a fixed point argument employing the contraction mapping principle. A Lagrangian specification of the flow field was utilized in place of the typical Eulerian specification in order to mitigate the difficulties involved in having an *a priori* unknown domain. In addition to the existence result, we have shown that the associated linear problem gives rise to an analytic semigroup of contractions on $\mathbf{P}^0$ (Theorem 5.3) whose generator has its spectrum contained in the sector $\{\lambda \in \mathbb{C} : \frac{3\pi}{4} \leq \arg(\lambda) \leq \frac{5\pi}{4}\}$.

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A Function Spaces

A.1 Notation for Standard Function Spaces

While we define many of the spaces relevant to this work in the text, we assume that the reader is already familiar with the standard Hölder, Lebesgue, and Sobolev function spaces. These spaces are summarized below and we refer the reader to [1, 13, 43] for further details. For the following, let $k \in \mathbb{N}_0$, $U \subset \mathbb{R}^n$ is nonempty and open, $S \subset \mathbb{R}^n$ is nonempty and of positive measure, and X is a complex separable Hilbert space with norm $\|\cdot\|_X$.

(i) Continuous and continuously differentiable functions:

$$C(U) = \{u : U \to \mathbb{C} \mid u \text{ is continuous}\}.$$

$$C(\overline{U}) = \{u : \overline{U} \to \mathbb{C} \mid \text{there exists } v \in C(U), \text{ uniformly continuous on bounded subsets of } U, \text{ such that } u \text{ is the unique bounded, continuous extension of } v \text{ to } \overline{U}\}.$$

$$C^k(U) = \{u : U \to \mathbb{C} \mid D_{\boldsymbol{\alpha}} u \in C(U) \text{ for all } |\boldsymbol{\alpha}| \leq k\}.$$

$$C^k(\overline{U}) = \{u : \overline{U} \to \mathbb{C} \mid D_{\boldsymbol{\alpha}} u \in C(\overline{U}) \text{ for all } |\boldsymbol{\alpha}| \leq k\}.$$

(ii) Smooth functions:

$$C^\infty(U) = \textstyle\bigcap_{k=0}^\infty C^k(U).$$

$$C^\infty(\overline{U}) = \textstyle\bigcap_{k=0}^\infty C^k(\overline{U}).$$

(iii) Hölder continuous and Hölder continuously differentiable functions with exponent $0 < \lambda \leq 1$:

$$C^{0,\lambda}(S) = \{u : S \to \mathbb{C} \mid \text{there exists } C \geq 0 \text{ such that } |u(x) - u(y)| \leq C|x-y|^\lambda \text{ for all } x, y \in S\}.$$

$$C^{k,\lambda}(S) = \{u : S \to \mathbb{C} \mid D_{\boldsymbol{\alpha}} u \in C^{0,\lambda}(S) \text{ for all } |\boldsymbol{\alpha}| \leq k\}.$$

(iv) Compactly supported continuous functions:

$$C_c(S) = \{u \in C(\overline{S}) \mid \text{supp } u \subset V \subset S \text{ where } V \text{ is compact}\}.$$

$$C_c^k(S) = \{u \in C^k(\overline{S}) \mid \text{supp } u \subset V \subset S \text{ where } V \text{ is compact}\}.$$

$$C_c^\infty(S) = \{u \in C^\infty(\overline{S}) \mid \text{supp } u \subset V \subset S \text{ where } V \text{ is compact}\}.$$

$$C_c^{k,\lambda}(S) = \{u \in C^{k,\lambda}(S) \mid \text{supp } u \subset V \subset S \text{ where } V \text{ is compact}\}.$$

(v) Lebesgue p-integrable functions:

$$L^p(S) = \{u : S \to \mathbb{C} \mid u \text{ is Lebesgue measurable and } \|u\|_{L^p(S)} < \infty\},$$

where $1 < p < \infty$ and

$$\|u\|_{L^p(S)}^p = \int_S |u|^p.$$

$$L^\infty(S) = \{u : S \to \mathbb{C} \mid u \text{ is Lebesgue measurable and } \|u\|_{L^\infty(S)} < \infty\},$$

where

$$\|u\|_{L^\infty(S)} = \text{ess } \sup_S |u|.$$

(vi) Locally L^p functions:

$$L^p_{\text{loc}}(S) = \{u : S \to \mathbb{C} \mid u \in L^p(V) \text{ for each open } V \subset \overline{V} \subset S \text{ with } \overline{V} \text{ compact}\}.$$

(vii) Weakly differentiable square-integrable functions (Sobolev spaces):

$$H^k(S) = \{u \in L^2(S) \mid D_{\boldsymbol{\alpha}} u \in L^2(S) \text{ for all } |\boldsymbol{\alpha}| \leq k \text{ and } \|u\|_{H^k(S)} < \infty\},$$

where

$$\|u\|_{H^k(S)}^2 = \sum_{|\boldsymbol{\alpha}| \leq k} \|D_{\boldsymbol{\alpha}} u\|_{L^2(S)}^2.$$

Here $D_{\boldsymbol{\alpha}} u$ is a *weak derivative* of u. For $u \in L^2(S)$, we define $D_{\boldsymbol{\alpha}} u = v$ where v satisfies

$$\int_S u D_{\boldsymbol{\alpha}} \phi = (-1)^{|\boldsymbol{\alpha}|} \int_S v\phi$$

for all *test functions* $\phi \in C_c^\infty(S)$.

(viii) H^k functions with vanishing trace (see the discussion preceding Theorem B.2):

$$H_0^k(S) = \{u \in H^k(S) \mid u|_{\partial S} = 0 \text{ in the sense of trace}\}.$$

(ix) Locally H^k functions:

$H^k_{\text{loc}}(S) = \{u : S \to \mathbb{C} \mid u \in H^k(V)$ for each open $V \subset \overline{V} \subset S$ with $\overline{V}$ compact$\}$.

(x) L^p and H^k functions with values in a separable Hilbert space (Lebesgue-Bochner and Sobolev-Bochner spaces):

$L^p(S;X) = \{u : S \to X \mid u$ is measurable and $\|u\|_{L^p(S;X)} < \infty\}$, where $1 < p < \infty$ and

$$\|u\|^p_{L^p(S;X)} = \int_S \|u\|^p_X.$$

$L^\infty(S;X) = \{u : S \to X \mid u$ is measurable and $\|u\|_{L^\infty(S;X)} < \infty\}$, where

$$\|u\|_{L^\infty(S;X)} = \operatorname{ess\,sup}_S \|u\|_X.$$

$H^k(S;X) = \{u \in L^2(S;X) \mid D_{\boldsymbol{\alpha}} u \in L^2(S;X)$ for all $|\boldsymbol{\alpha}| \leq k$ and $\|u\|_{H^k(S;X)} < \infty\}$, where

$$\|u\|^2_{H^k(S;X)} = \sum_{|\boldsymbol{\alpha}| \leq k} \|D_{\boldsymbol{\alpha}} u\|^2_{L^2(S)}.$$

As before, we define $D_{\boldsymbol{\alpha}} u = v$ (for $u \in L^2(S; X)$) where v satisfies

$$\int_S u D_{\boldsymbol{\alpha}} \phi = (-1)^{|\boldsymbol{\alpha}|} \int_S v \phi$$

for all $\phi \in C_c^\infty(S)$.

A.2 Interpolation Spaces

A crucial role in this work is played by the Sobolev and Sobolev-Bochner interpolation spaces,

$$H^{k+\beta}(S) = [H^k(S), H^{k+1}(S)]_\beta, \quad \text{and} \quad H^{k+\beta}(S; X) = [H^k(S; X), H^{k+1}(S; X)]_\beta,$$

respectively (where $k \in \mathbb{N}_0$, $0 < \beta < 1$, and S and X are as in Appendix A.1). These interpolation spaces give meaning to the notion of non-integer regularity and provide a spectrum of spaces which are intermediate to (and consistent with) the standard integer-regularity spaces. While there are several methods of interpolation, the Sobolev and Sobolev-Bochner interpolation spaces are generally obtained using the method of complex interpolation. We do not outline this method here and refer the reader instead to [39, 40, 43] for further details. Explicit characterizations of these spaces are unnecessary as we interact with them primarily through the use of the interpolation properties which now follow.

Theorem A.1. *Let* $X_0 \subset X_1$ *be Hilbert spaces such that* X_0 *is dense and continuously embedded in* X_1*. Complex interpolation provides a family of Hilbert spaces denoted by* $[X_0, X_1]_\beta$, $0 \le \beta \le 1$, *which satisfy*

(i) $[X_0, X_1]_0 = X_0$.

(ii) $[X_0, X_1]_1 = X_1$.

(iii) $X \subset [X_0, X_1]_\alpha \subset [X_0, X_1]_\beta \subset X_1$ *for all* $0 \le \alpha \le \beta \le 1$, *where each embedding is continuous.*

(iv) $\|u\|_{[X_0,X_1]_\beta} \le C\|u\|_{X_0}^{1-\beta}\|u\|_{X_1}^{\beta}$ *for all* $u \in X_1$ *and* $0 \le \beta \le 1$, *where* $C > 0$ *is a constant depending on* β.

In fact, a much more general version of property (iv) is true and we will often find it very useful in the proof of the main result in Chapter 6.

Theorem A.2. *Let* $X_0 \subset X_1$ *and* $Y_0 \subset Y_1$ *be pairs of Hilbert spaces which satisfy the conditions of Theorem A.1. Suppose* $T \in \mathcal{L}(X_1; Y_1)$ *is such that* $T|_{X_0} \in \mathcal{L}(X_0; Y_0)$, *where* $\mathcal{L}(A; B)$ *denotes the space of bounded linear maps from* A *to* B. *For* $0 \le \beta \le 1$, $T|_{[X_0,X_1]_\beta} \in \mathcal{L}([X_0, X_1]_\beta; [Y_0, Y_1]_\beta)$ *and satisfies*

$$\|T\|_{\mathcal{L}([X_0,X_1]_\beta;[Y_0,Y_1]_\beta)} \le C_\beta \|T\|_{\mathcal{L}(X_0;Y_0)}^{1-\beta} \|T\|_{\mathcal{L}(X_1;Y_1)}^{\beta}$$

where we have omitted the restriction notation for readability. The constant $C_\beta > 0$ *depends on* β.

B Referenced Results

This work utilizes standard results taken from a wide array of topics including linear and nonlinear functional analysis, calculus of variations, spectral theory, semigroup theory, and fluid dynamics. The most important of these are collected in Appendices B.1, B.2, and B.3 for the reader's convenience. Since we follow the general overarching approach outlined by Beale in [7], we find many of the technical lemmas from that work to be useful here. The majority of these require adaptation in order to be compatible with the a_3-periodic function spaces underlying our work. These (modified) technical lemmas are collected in Appendix C along with the necessary proofs.

B.1 Elementary Inequalities

The following inequalities are used throughout the text.

(i) For $a, b \geq 0$ and $1 \leq p < \infty$,

$$(a+b)^p \leq 2^{p-1}(a^p + b^p).$$

(ii) *Young's inequality.* For $a, b \geq 0$ and $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

B.2 Integral Inequalities

For the following, suppose $U \subset \mathbb{R}^n$.

(i) *Hölder's inequality.* For $u \in L^p(U), v \in L^q(U)$, and $1 \leq p, q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$\int_U |uv| \leq \|u\|_{L^p(U)} \|v\|_{L^q(U)}.$$

(ii) *Poincaré's inequality.* Let U be bounded with a Lipschitz boundary. There exists $C > 0$ such that for all $u \in H_0^1(U)$ (see Appendix A.1),

$$\|u\|_{L^2(U)} \leq C \|\nabla u\|_{\mathbf{L}^2(U)}.$$

(iii) *Korn's inequality.* Let U be an open, connected domain in $\mathbb{R}^n, n \geq 2$. There exists $C > 0$ such that for all $\mathbf{u} \in (H^1(U))^n$,

$$\|\mathbf{u}\|^2_{(H^1(U))^n} \leq C \int_U \left(\sum_{i,j=1}^{n} |u_i|^2 + \frac{1}{4} |(D_i u_j + D_j u_i)|^2 \right).$$

B.3 Standard Results

We begin with a simple density result that allows us to approximate functions in $H^k(\Omega)$ by smooth functions whose derivatives are all uniformly continuous on Ω. It bears mentioning that a cylinder is easily verified to be "star-shaped with respect to a point," but as there will be no further need for to discuss this geometric condition we omit its definition.

Theorem B.1. *If U is a bounded domain, star-shaped with respect to a point, then $C^\infty(\overline{U})$ is dense in $H^k(U)$.*

Source. [18, p. 13].

It is of fundamental importance that we be able to make sense of a function's value along boundaries, whether a boundary is with respect to the time interval (e.g., at $t = 0$) or with respect to the spatial domain (e.g., on S_F). However, since most of the functions we deal with are only well-defined on their domain up to a set of measure zero, it is not immediately obvious whether such functions can have well-defined values along a boundary (i.e., boundaries necessarily having measure zero within their domains). The following result demonstrates that with sufficient regularity, an $L^2(U)$ function has a well-defined trace (of decreased regularity) on the boundary ∂U (or any subset of positive measure within ∂U). Moreover, we learn that this trace operator is bounded, linear, and surjective; in particular, surjectivity is crucial since we will frequently need to construct functions with a given trace.

Theorem B.2. *Let $U \subset \mathbb{R}^n$ have a Lipschitz boundary. Then the trace operator $T_0 : C(\overline{U}) \to C(\partial U)$ defined by $T_0 u = u|_{\partial U}$ extends to a surjective and bounded linear map $T : H^s(U) \to H^{s-1/2}(\partial U)$ for any $s > 1/2$.*

Source. [6, p. 201]

The next result verifies that integration-by-parts can be performed on a domain with a Lipschitz boundary provided that the functions involved have sufficient regularity.

Theorem B.3. *(Integration-by-parts) Let $U \subset \mathbb{R}^n$ be open and bounded with a Lipschitz boundary. Then*

$$\int_U p(\nabla \cdot \mathbf{u}) = \int_{\partial U} p(\mathbf{u} \cdot \mathbf{n}) - \int_U \nabla p \cdot \mathbf{u}$$

holds for all $p \in H^1(U)$ and $\mathbf{u} \in (H^1(U))^n$, where $\mathbf{n}$ is the outward unit normal defined on ∂U.

Source. [6, p. 207]

As discussed in Sections 3.1 and 3.4, in the study of the Navier-Stokes equations there is great utility in being able to decompose L^2 into its orthogonal divergence-free and gradient parts. The following result provides sufficient "orthogonality" conditions for a function to be recognizable as a gradient.

Theorem B.4. *Let $U \subset \mathbb{R}^n, n \geq 2$, be open and bounded with a Lipschitz boundary. Define $\mathbf{Z} = \{\mathbf{u} \in (C_c^\infty(U))^n : \nabla \cdot \mathbf{u} = 0\}$.*

(i) Let $\mathbf{f} \in (L^2(U))^n$. If $\int_U \mathbf{f} \cdot \mathbf{u} = 0$ for all $\mathbf{u} \in \mathbf{Z}$, then there exists $p \in H^1(U)$ such that $\mathbf{f} = \nabla p$.

(ii) Let $\mathbf{f} \in (H^{-1}(U))^n$ where $H^{-1}(U)$ denotes the dual space of $H_0^1(U)$. If $\mathbf{f}(\mathbf{v}) = 0$ for all $\mathbf{v} \in \mathbf{Z}$, then there is a unique $p \in L^2(U)$ satisfying

$$\mathbf{f} = \nabla p, \qquad \int_U p = 0.$$

Moreover, there are constants $C_1, C_2 > 0$ such that

$$\|p\|_{L^2(U)} \leq C_1 \|\mathbf{f}\|_{(H^{-1}(U))^n} \leq C_1 C_2 \|p\|_{L^2(U)}.$$

Source. (i) [41, pp. 10–11], (ii) [26, p. 75].

The Lax-Milgram Theorem is one of the most powerful tools available for obtaining weak solutions to partial differential equations and we make frequent use of it. Below is a version of the classical result which has been adapted for a complex setting.

Theorem B.5. *(Lax-Milgram) Let X be a complex Hilbert space with closed subspace H. Let $B : X \times X \to \mathbb{C}$ be a sesquilinear functional which is both*

(i) continuous *on X, i.e. there is $M > 0$ such that $|B(x, y)| \leq M\|x\|_X\|y\|_X$ for all $x, y \in X$*

(ii) coercive *on H, i.e. there is $\gamma > 0$ such that $|B(x, x)| \geq \gamma\|x\|_X^2$ for all $x \in H$.*

If $u_0 \in X$ and $F \in H^*$, there is a unique $u \in (H+u_0) \subset X$ such that $B(u, v) = F(v)$ for all $v \in H$ and

$$\|u\|_X \leq \frac{1}{\gamma}\|F\|_{H^*} + \left(\frac{M}{\gamma} + 1\right)\|u_0\|_X.$$

Source. [6, p. 218] (i.e., the complex analog).

The next two theorems are concerned with gaining additional regularity for weak solutions of the Stokes equations (a significant simplification of the full Navier-Stokes equations). The first is only able to gain the desired regularity away from the boundary of the spatial domain, but has the benefit of not requiring that the domain be smooth. The second yields regularity all the way up to the boundary, but demands that the domain have at least a C^2 boundary. The proof of this result is done locally, however, so regularity can be gained up to the boundary wherever it is locally C^2. Both results suppose the existence of weak solutions to nontrivial problems; the proper variational formulations for these problems are made clear in the proof of Lemma 5.2, where both of these results are exploited.

Theorem B.6. *Let $U \subset \mathbb{R}^3$ be bounded and open. Suppose $(\mathbf{v}, p)$ is a weak solution of*

$$-\mu\Delta\mathbf{v} + \nabla p = \mathbf{f}$$
$$\nabla \cdot \mathbf{v} = 0$$

where $\mathbf{f} \in \mathbf{L}^2(U)$. Then $\mathbf{v} \in \mathbf{H}^2(V)$ and $p \in H^1(V)$ for any open $V \subset \overline{V} \subset U$.

Source. [15, p. 38].

Theorem B.7. *Let $U \subset \mathbb{R}^3$ be bounded and open such that ∂U is C^2. Suppose $(\mathbf{v}, p)$ is a weak solution of*

$$-\mu\Delta\mathbf{v} + \nabla p = \mathbf{f}$$
$$\nabla \cdot \mathbf{v} = 0$$
$$\mathbf{S}(\mathbf{v}, p) = 0 \qquad \text{on } \partial U$$

where $\mathbf{f} \in \mathbf{L}^2(U)$. Then $\mathbf{v} \in \mathbf{H}^2(U), p \in H^1(U)$, and $(\mathbf{v}, q)$ satisfies the given boundary value problem.

Source. [28, p. 144].

The following is a simplified version of the second part of what is collectively referred to as the Sobolev Embedding Theorem. It describes when functions in Sobolev spaces have sufficient regularity to be identified with Hölder continuously differentiable functions. Notice that the regularity required to ensure continuity increases as the dimension of the underlying domain U increases.

Theorem B.8. *(Sobolev Embedding Theorem) Let U be a bounded Lipschitz domain in $\mathbb{R}^n$, $j \in \mathbb{N}_0$, and $m \in \mathbb{N}$. If $m > \frac{n}{2} > m - 1$, then*

$$H^{j+m}(U) \subset C^{j,\lambda}(U)$$

for $0 < \lambda \leq m - \frac{n}{2}$.

Source. [1, pp. 85–86].

Also known as the Banach Fixed Point Theorem, the Contraction Mapping Principle forms the foundation of the approach taken in this work. Below we only detail the portion of the theorem which will be of interest to us in the current work—proving the existence and uniqueness of a fixed point of a contraction mapping—but it is worthwhile to note that the full result is much stronger and addresses nearly every relevant mathematical concern (existence, uniqueness, construction, approximation, and error estimation).

Theorem B.9. *(Contraction Mapping Principle) Let (X, d) be a complete metric space with $M \subset X$, a closed nonempty subset. If $T : M \to M$ is an operator for which there exists $0 \leq k < 1$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in M$ (called a* contraction mapping*), then T has exactly one fixed point on M.*

Source. [46, p. 17].

We now detail the Lumer-Phillips Theorem, a key result in semigroup theory which provides a very useful characterization of the infinitesimal generators of contraction semigroups that does not require explicit knowledge of the resolvent operator. The following is a variation of the classical result which reduces the number of sufficient conditions on the proposed generator given that its domain lies in a reflexive space.

Theorem B.10. *(Lumer-Phillips) Suppose X is a reflexive Banach space and $D(B) \subset X$. Let $B : D(B) \to X$ be a linear operator satisfying both*

(i) $\|(\lambda I - B)x\|_X \geq \lambda \|x\|_X$ *for all* $x \in D(B)$ *and* $\lambda > 0$ *(in which case we say that B is* dissipative*),*

(ii) $\lambda_0 I - B$ *is surjective for some* $\lambda_0 > 0$.

Then $D(B)$ is dense in X and B is the infinitesimal generator of a C_0 semigroup of contractions on X.

Source. [12, p. 86].

We conclude this section with a portion of the Spectral Mapping Theorem for C_0 semigroups. In general, a spectral mapping theorem is one which relates the spectrum of a semigroup to that of its generator. We restrict ourselves to the identity relating their respective point spectrums since this will be sufficient for our needs.

Theorem B.11. *(Spectral Mapping Theorem) If $(B, D(B))$ generates a C_0 semigroup, $T(t)$, on a Banach space X, then*

$$\sigma_p(T(t)) - \{0\} = e^{t\sigma_p(B)}$$

where $\sigma_p(\cdot)$ denotes the point spectrum (i.e., the set of all eigenvalues) of the enclosed operator.

Source. [12, p. 277].

C Technical Lemmas Adapted for the Periodic Setting

Since this dissertation follows the general approach due to Beale in [7], we require analogous versions of the technical lemmas used in that article. While most of Beale's lemmas require modification in order to match the periodic setting used in this work, those not requiring adaptation have also been included for the reader's convenience. In the results that follow, note the attention paid to being able to obtain bounds which are independent of the length of the underlying time interval. These estimates will be used repeatedly in the proof of the main result to ensure that we obtain a contraction mapping for sufficiently small T.

We begin with a trace theorem that allows us to find functions in K_{p}^s (see Section 3.1) for prescribed initial conditions (i.e. trace with respect to time) and/or normal derivative conditions on the free surface (i.e. trace with respect to space). (i) and (ii) describe the traces individually and (iii) brings them together along with a compatibility condition to ensure surjectivity of the combined trace operator.

Lemma C.1. *Suppose* $\frac{1}{2} < s \leq 5$.

(i) *The mapping* $v \mapsto D_{\mathbf{n}}^{j} v$ *extends to a bounded linear operator from* K_p^s *to* $K_p^{s-j-1/2}(\partial G_F)$, *where* $j \in \mathbb{Z}$ *such that* $0 \leq j < s - \frac{1}{2}$.

(ii) *If* $s > 1$, *then the mapping* $v \mapsto D_t^k v(0, \cdot)$ *also extends to a bounded linear operator from* K_p^s *to* H_p^{s-2k-1}, *where* $k \in \mathbb{Z}$ *such that* $k < \frac{1}{2}(s-1)$.

(iii) *Suppose* $s > \frac{3}{2}$ *such that* $s \neq 3, 5$ *and* $s - \frac{1}{2} \notin \mathbb{Z}$. *Define*

$$\mathbf{W}^s = \prod_{0 \leq j < s - \frac{1}{2}} K_p^{s-j-1/2}(\partial G_F) \times \prod_{0 \leq k < (s-1)/2} H_p^{s-2k-1}$$

and let $\mathbf{W}_0^s$ *be the subspace consisting of* $\{\mathbf{b}, \mathbf{w}\}$ *such that, whenever* $j + 2k < s - \frac{3}{2}$,

$$D_t^k b_j(0, \cdot) = D_{\mathbf{n}}^j w_k(\cdot).$$

The traces of (i) and (ii) form a bounded linear operator from K_p^s onto $\mathbf{W}_0^s$ *(so that this operator has a bounded right inverse).*

Proof. Transforming first to $\mathcal{T}$, this can be obtained exactly as in [7].
Source of the original result: [7, Lemma 2.1, pp. 364–365]. □

We will often seek to extend functions to larger time intervals in a bounded way. This is usually done either to pass to a fixed time interval $(0, T_0)$, where $T_0 \geq T$, in order to gain estimates which are independent of T, or to extend the function to all of $\mathbb{R}$ in preparation for techniques involving Fourier transforms.

Lemma C.2. *Let X be a Hilbert space and $s \geq 0$.*

(i) There exists a bounded extension operator $J : H^s((0,T);X) \to H^s(\mathbb{R};X)$.

(ii) Provided $s \leq 2$ and $s - \frac{1}{2} \notin \mathbb{Z}$, there exists an extension operator from $\{v \in H^s((0,T);X) : D_t^j v(0,\cdot) = 0$ for $j < \frac{1}{2}(s-1)\}$ to $H^s(\mathbb{R};X)$ which is bounded independent of T. The extension of such a $v \in H^s((0,T);X)$ vanishes for $t < 0$.

(iii) Analogous statements hold for extending from K_p^{2s} to $K_p^{2s}((0,\infty) \times \Omega)$.

Proof. (i), (ii) require no modification and (iii) follows exactly as in [7].
Source of the original result: [7, Lemma 2.2, p. 365]. □

When regarding the definition of K_{p}^s, one might question how well it corresponds with our expectations of regularity with respect to separate variables. For example, given $f \in K_{\mathrm{p}}^s$ for sufficiently large s, what can we say about the spatial regularity of $\dot{f}$ or the temporal regularity of ∇f? It is clear that $\dot{f} \in H^{(s-2)/2}((0,T);H_{\mathrm{p}}^0)$ and $\nabla f \in H^0((0,T);H_{\mathrm{p}}^{s-1})$, but that does not answer our question. The following result provides us with a way to exchange temporal for spatial regularity (and vice versa) in order to obtain more optimal information. Returning to our example, it would imply that $f \in H^1((0,T);H_{\mathrm{p}}^{s-2}) \cap H^{(s-1)/2}((0,T);H_{\mathrm{p}}^1)$, so that we actually have $\dot{f} \in H^0((0,T);H_{\mathrm{p}}^{s-2})$ and $\nabla f \in H^{(s-1)/2}((0,T);H_{\mathrm{p}}^0)$.

Lemma C.3. *Suppose $0 \leq s \leq 4$.*

(i) For $r \leq \frac{s}{2}$, the identity operator extends to a bounded operator $I : K_p^s \to H^r((0,T);H_p^{s-2r})$.

(ii) Provided s is not an odd integer, the restriction of this operator to $\{v \in K_p^s : D_t^j v(0,\cdot) = 0$ for $j < \frac{1}{2}(s-1)\}$ is bounded independent of T.

Proof. Transforming first to $\mathcal{T}$, this can be obtained exactly as in [7].

Source of the original result: [7, Lemma 2.3, p. 365]. □

The following lemma provides us with our chief tool for introducing an explicit dependence on T into our estimates in the proof of the main result. The power of T present in these estimates ultimately allows us to obtain a contraction mapping by taking T small enough to balance whatever constants may show up. In most cases, we will exploit the fact that $\mathbf{x} = \int_0^t \mathbf{v}$ by (2.13) and take $V = \mathbf{x}$.

Lemma C.4. *Fix $T_0 > 0$ arbitrarily and let $T \leq T_0$. For $v \in H^0((0,T);X)$, we define $V \in H^1((0,T);X)$ by*

$$V(t) = \int_0^t v(\tau) d\tau.$$

For all $0 \leq \varepsilon \leq 1$, the function V satisfies

$$\|V\|_{H^{1-\varepsilon}((0,T);X)} \leq C_1 T^{\varepsilon} \|v\|_{H^0((0,T);X)}.$$

If $v \in H^s((0,T);X)$ where $0 \leq s < \frac{1}{2}$, then $V \in H^{s+1-\varepsilon}((0,T);X)$ for $0 \leq \varepsilon < s$ and satisfies

$$\|V\|_{H^{s+1-\varepsilon}((0,T);X)} \leq C_2 T^{\varepsilon} \|v\|_{H^s((0,T);X)}.$$

In both cases, the constants C_1 and C_2 are positive and independent of T.

Source: [7, Lemma 2.4, pp. 365–366]. □

The next two results are variations on the standard "multiplication" results in Sobolev spaces which seek to determine the regularity of products of functions and estimate them by their factors. They are especially important in studying the full nonlinear problem where most terms involve products of $\mathbf{v}$ or q with entries of Λ. Recall here that ${}^0H_{\mathrm{p}}^{-1}$ is defined to be the dual space of ${}^0H_{\mathrm{p}}^1$ (see the discussion beginning Section 5.2).

Lemma C.5. *Suppose $r > \frac{3}{2}$ and $r \geq s \geq 0$. There exist positive constants $C_1, C_2, C_3,$ and C_4, such that*

(i) If $v \in H_p^r$ and $w \in H_p^s$, then $vw \in H_p^s$ and $\|vw\|_{H_p^s} \leq C_1\|v\|_{H_p^r}\|w\|_{H_p^s}$.

(ii) If $v \in H_p^1$ and $w \in H_p^1$, then $vw \in H_p^0$ and $\|vw\|_{H_p^0} \leq C_2\|v\|_{H_p^1}\|w\|_{H_p^1}$.

(iii) If $v \in H_p^r$ and $w \in {}^0H_p^{-1}$, then $vw \in {}^0H_p^{-1}$ and $\|vw\|_{{}^0H_p^{-1}} \leq C_3\|v\|_{H_p^1}\|w\|_{{}^0H_p^{-1}}$.

(iv) If $v \in H_p^1$ and $w \in H_p^0$, then $vw \in {}^0H_p^{-1}$ and $\|vw\|_{{}^0H_p^{-1}} \leq C_4\|v\|_{H_p^1}\|w\|_{H_p^0}$.

Proof. (i) Take $s = k \in \mathbb{N}_0$. We immediately obtain $vw \in H^k$ using the original result in [7]. Now it should be obvious from the characterization of H_{p}^k given in Lemma 3.3 that $vw \in H_{\mathrm{p}}^k$. The inequality now follows from Lemma 3.1 with interpolation providing the remaining cases. (ii) Since $H_{\mathrm{p}}^0 = H^0$, the inequality is the only distinction from the original result in [7] and it follows from Lemma 3.1. (iii) and (iv) both follow exactly as in [7].
Source of the original result: Lemma 2.5 from [7], p. 365. □

Lemma C.6. *Suppose X, Y, and Z are Hilbert spaces and $M : X \times Y \to Z$ is a bounded, bilinear operator (called multiplication).*

(i) Suppose $v \in H^s((0,T);X)$ and $w \in H^s((0,T);Y)$ where $s > \frac{1}{2}$. If vw is defined by $(vw)(t) = M(v(t), w(t))$, then $vw \in H^s((0,T);Z)$ and $\|vw\|_{H^s((0,T);Z)} \leq C\|v\|_{H^s((0,T);X)}\|w\|_{H^s((0,T);Y)}$.

(ii) If additionally $s \leq 2$, where $s - \frac{1}{2} \notin \mathbb{Z}$, and v, w satisfy $D_t^k v(0,\cdot) = D_t^k w(0,\cdot) = 0$ for all $k < s - \frac{1}{2}$, then the constant C above can be chosen independently of T.

Source: Lemma 2.6 from [7], p. 365. □

The final lemma is a technical result only used once in the text; it is required during the reduction of the inhomogeneous linear problem in Section 5.2 to the homogeneous one considered in Section 5.1.

Lemma C.7. *Suppose* $3 < s < \frac{7}{2}$. *Given* $\mathbf{b} \in \mathbf{K}_p^{s-3/2}(\partial G_F)$ *with* $\mathbf{b} \cdot \mathbf{n} = 0$ *and* $\mathbf{b}(0, \cdot) = 0$, *there exists* $\mathbf{w} \in \mathbf{K}_p^s$ *such that* $\mathbf{w}(0, \cdot) = 0, \mathbf{w}_t = 0, \nabla \cdot \mathbf{w} = 0, \mathbf{S}_{tan}(\mathbf{w}) = \mathbf{b}$, *and* $\|\mathbf{w}\|_{\mathbf{K}_p^s} \leq C \|\mathbf{b}\|_{\mathbf{K}_p^{s-3/2}(\partial G_F)}$.

Proof. Using Lemma C.1(iii), choose $\mathbf{u} \in \mathbf{K}_{\mathrm{p}}^{s+1}$ such that

$$\begin{aligned} \mathbf{u}(0, \cdot) = \dot{\mathbf{u}}(0, \cdot) &= 0 && \text{on } \Omega \\ \mathbf{u} = D_{\mathbf{n}}\mathbf{u} = 0, \quad D_{\mathbf{n}}^2\mathbf{u} &= \mu \mathbf{n} \times \mathbf{b} && \text{on } S_F. \end{aligned}$$

It can now be verified that $\mathbf{w} = \nabla \times \mathbf{u}$ satisfies the claim. In particular, that the boundary condition is satisfied is most easily seen by first transforming an arbitrary point on S_F to the origin such that the transformed normal vector, evaluated at the origin, is parallel to one of the coordinate axes (see the proof of Lemma 4.2 in [7]). *Source of the original result*: Lemma 4.2 from [7], p. 377. □