4-20-2011

On Ramsey Theory and Slow Bootstrap Percolation

Fabricio Siqueira Benevides

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Béla Bollobás, Ph.D.
Major Professor

We have read this dissertation and recommend its acceptance:

Paul Balister, Ph.D.

Anna Kaminska, Ph.D.

Jenő Lehel, Ph.D.

Vladimir Nikiforov, Ph.D.

Accepted for the Graduate Council:

Karen D. Weddle-West, Ph.D.
Vice Provost for Graduate Programs

ON RAMSEY THEORY AND SLOW BOOTSTRAP PERCOLATION

by

Fabricio Siqueira Benevides

A Dissertation

Submitted in Partial Fulfillment of the

Requirements for the Degree of

Doctor of Philosophy

Major: Mathematical Sciences

The University of Memphis

May, 2011

To my wife Juliana.

ACKNOWLEDGEMENTS

This dissertation would not have been possible without the guidance and help of a host of people, senior and junior alike: it is a great pleasure to have an opportunity to thank them all.

I am especially grateful to my supervisor, Professor Dr. Béla Bollobás for all his support, constant encouragement and well-meant criticism. I would like to thank him for introducing me to many great mathematicians, taking me to many exciting places, and giving me many beautiful and challenging mathematical problems.

I am indebted to my examiners, Professors Paul Balister, Anna Kaminska, Jenő Lehel and Vladimir Nikiforov, for their comments and suggestions.

The memory of Professor Richard Schelp will stay with me forever: his deep admiration for mathematics inspired and motivated me. I am grateful to him for our many discussions, invaluable advice, and for telling me his conjectures.

I thank my co-authors Dr. Jozef Skokan and Michal Przykucki for working with me on part of the problems of my dissertation.

Mrs. Gabriella Bollobás has often greatly surprised me with her artistic view of the world: she showed me that the world is not made only for mathematics.
Mrs. Tricia Simmons has gone way beyond the call of duty to help me along and learn about American culture: if I had paid her a penny for every piece of advice she gave me, she would be a millionaire by now.

I am indebted to my many colleagues who supported me so many times: Alex, Ago, Andrew, Dominik, Karen, Neal, Tomas, Vivek, and many others whom I met in Memphis or around the world.

My family, especially my parents, elder brother and elder sister, supported me unconditionally along my journey; my nephews brought me joy whenever I was able to visit them. The one above all of us, the omnipresent God, led the right people into my way and gave me health and intellectual tools that I needed. For all of these I am grateful more than I can say.

I could not have written my Ph.D. without the financial support of the Capes Foundation, the Ministry of Education of Brazil, the Fulbright Commission, and the University of Memphis.

Last, but not least, I would like to thank with all my heart my wife, Juliana Gomes Varela: she has always been by my side, has always made me remember who I am and where I am going, and has provided a solid ground where I am sure we will always stand.

ABSTRACT

Fabricio Siqueira Benevides. Ph.D. The University of Memphis. May, 2010. Ramsey Theory and Slow Bootstrap Percolation. Major Professor: Béla Bollobás.

This dissertation concerns two sets of problems in extremal combinatorics. The major part, Chapters 1 to 4, is about Ramsey-type problems for cycles. The shorter second part, Chapter 5, is about a problem in bootstrap percolation. Next, we describe each topic more precisely.

Given three graphs G, L_1 and L_2, we say that G arrows (L_1, L_2) and write $G \to (L_1, L_2)$, if for every edge-coloring of G by two colors, say 1 and 2, there exists a color i whose color class contains L_i as a subgraph. The classical problem in Ramsey theory is the case where G, L_1 and L_2 are complete graphs; in this case the question is how large the order of G must be (in terms of the orders of L_1 and L_2) to guarantee that $G \to (L_1, L_2)$. Recently there has been much interest in the case where L_1 and L_2 are cycles and G is a graph whose minimum degree is large. In the past decade, numerous results have been proved about those problems. We will continue this work and prove two conjectures that have been left open. Our main weapon is Szemerédi's Regularity Lemma.

Our second topic is about a rather unusual aspect of the fast expanding theory of bootstrap percolation. *Bootstrap percolation* on a graph G with parameter r is a cellular automaton modeling the spread of an infection: starting with a set $A_0 \subseteq V(G)$ of *initially infected vertices*, define a nested sequence of sets, $A_0 \subseteq A_1 \subseteq \cdots \subseteq V(G)$, by the update rule that A_{t+1}, the *set of vertices infected at time* $t+1$, is obtained from A_t by adding to it all vertices with at least r neighbors in A_t. The initial set A_0 *percolates* if $A_t = V(G)$ for some t. The minimal such t is the *time* it takes for A_0 to percolate. We prove results about the *maximum* percolation time on the two-dimensional grid with parameter $r = 2$.

Contents

List of Figures

Chapter 1

Introduction

1.1 Introduction to Ramsey theory

The Ramsey-type problems, a particular type of problems in Extremal Combinatorics, have been much studied over the last decades. In 1928, the young mathematician Frank Plumpton Ramsey [34] wrote an article about an algorithm problem in propositional logic. In that article, Ramsey proved also a purely mathematical result, well-known nowadays as the Ramsey's Theorem. This theorem was originally only a tool in the original article but have turned out to be more acknowledged than the article itself. Before we state the theorem, let us introduce some notation.

Consider a graph G with vertex set V and edge set E. Given an integer k, a k-edge-coloring of G is any function $f : E \to S$ where S is any set with k elements. We say the G is colored by S and for each $s \in S$ the color class s is the set of edges e such that $f(e) = s$. It is sometimes convenient to take $S = \{1, \ldots, k\}$ and this is what we shall do in the current chapter. However, in most of this dissertation, we will have $k = 2$ in which case it will be convenient to take $S = \{\text{“red”}, \text{“blue”}\}$. So, for an edge $e \in E$ and a given coloring f, we say that e is colored red (or simply e is red) if

$f(e) =$ "red"; and similarly for "blue". Also, when $k = 3$ we shall use the set $S = \{$"red", "blue", "green"$\}$ as our standard set of colors. In this dissertation, whenever we talk about colorings we mean edge-colorings.

Given an integer s and graphs $G, L_1, \ldots, L_s$, we say that G arrows $(L_1, \ldots, L_s)$ and write $G \to (L_1, \ldots, L_s)$, if for every coloring of G by $\{1, 2, \ldots, s\}$, there exists a color i, with $1 \leq i \leq s$, such that the graph induced by the edges of color i contains L_i as a subgraph, (not necessarily as an induced subgraph). The classical problem in Ramsey theory is the case in which G and L_i, for all $1 \leq i \leq s$, are complete graphs; in this case the question is how large the order of G must be (in terms of the orders of L_i) to guarantee that $G \to (L_1, \ldots, L_s)$.

Next, we state the most commonly used version of Ramsey's Theorem for graphs, where we denote by K_n the complete graph on n vertices.

Theorem 1.1. *Given integers $\ell_1, \ldots, \ell_s$, there exists a number N such that K_N arrows $(K_{\ell_1}, \ldots, K_{\ell_s})$.*

In view of Theorem 1.1, for any fixed s, the (Ramsey) function $r : \mathbb{N}^s \to \mathbb{N}$ given by $r(\ell_1, \ldots, \ell_s) = \min\{N : K_N \to (K_{\ell_1}, \ldots, K_{\ell_s})\}$ is well defined. Computing the precise value of $r(\ell_1, \ldots, \ell_s)$ is considered an extremely hard problem, even in the case where $s = 2$ and $\ell_1 = \ell_2$. One can easily prove some bounds on $r(\ell, \ell)$ as shown by the next theorem, whose proof can be found on Chapter 6 of Bollobás [11]. But it is hard to provide any substantial improvement on these bounds.

Theorem 1.2. *We have that $2^{\ell/2} \leq r(\ell, \ell) \leq \frac{2^{2\ell-2}}{\sqrt{\ell}}$.*

The original theorem of Ramsey has been expanded and applied to a number of areas in Mathematics including areas outside Combinatorics. It involves a wide number of techniques which are now part of what is known as Ramsey theory. Notably in the past three decades, Ramsey theory has evolved from a collection of theorems to become a cohesive sub-area of Extremal Combinatorics. One can find full books on the topic, for example, the one by Graham [20]. Nevertheless, a number of the original problems are still unsolved.

1.2 Generalized Ramsey numbers

We consider the following generalization of the function $r(\ell_1, \ldots, \ell_s)$.

Definition 1.3. Let $R(L_1, \ldots, L_s) = \min\{N : K_N \to (L_1, \ldots, L_s)\}$ be a function whose domain is the set of s-tuples of graphs and co-domain is the set of natural numbers.

The number $R(L_1, \ldots, L_s)$ is called a generalized Ramsey number and has been studied by many authors for many classes of graphs. It is an immediate consequence of Theorem 1.1 that $R(L_1, \ldots, L_s)$ is indeed a function, that is, the set $\{N : K_N \to (L_1, \ldots, L_s)\}$ is non-empty. In order to prove this, one can simply select $N = r(\ell_1, \ldots, \ell_s)$, where ℓ_i is the number of vertices of L_i for every $1 \leq i \leq s$. Clearly, since $K_N \to (K_{\ell_1}, \ldots, K_{\ell_s})$, for any s-coloring of K_N there exists a color i whose color class contains K_{ℓ_i} as a subgraph. The result follows as L_i is a subgraph of K_{ℓ_i} and we do not require it to be an induced subgraph. This argument further implies that

$$R(L_1, \ldots, L_s) \leq r(\ell_1, \ldots, \ell_s). \tag{1.1}$$

A much more interesting fact, however, is that sometimes the left-hand side of inequality (1.1) is much smaller than its right hand side. In fact, it follows from Theorem 1.2 that $r(\ell_1, \ldots, \ell_s)$ is at least exponential in $\min\{\ell_1, \ldots, \ell_s\}$ while for some classes of graphs, as exemplified bellow, the number $R(L_1, \ldots, L_s)$ is linear in $\max\{\ell_1, \ldots, \ell_s\}$.

Here, we are particularly interested in the case where the graphs L_i are cycles. This is an example where $R(L_1, \ldots, L_s)$ is linear. The case where $s = 2$ and the graphs L_1, L_2 are cycles of length n, denoted C_n, was raised by Bondy and Erdős [13] and it was fully solved by Faudree and Schelp [18], and independently by Rosta [35]. (For a new short proof see Károlyi and Rosta [28]). They proved the following.

Theorem 1.4. *Given integers $n \geq 3$, we have*

$$R(C_n, C_n) = \begin{cases} 6, & \text{if } \ n = 3 \quad \text{or} \ \ 4 \\ 2n - 1, & \text{if } n \text{ is odd, } n \geq 5 \\ 3n/2 - 1, & \text{if } n \text{ is even, } n \geq 6. \end{cases}$$

Bondy and Erdős [13] conjectured that if $n > 3$ is odd then

$$R(C_n, C_n, C_n) = 4n - 3. \tag{1.2}$$

Kohayakawa, Simonovits and Skokan [26] proved that there exists an n_0 such that equation (1.2) holds for every n odd with $n > n_0$.

The case when n is even differs from the case when n is odd. Benevides and Skokan [9], proved that there exists an integer n_1 such that for every even $n > n_1$,

$$R(C_n, C_n, C_n) = 2n. \tag{1.3}$$

For a general number of colors s, one also has general (but not sharp) bounds on $R(\underbrace{C_n, \ldots, C_n}_{s \text{ times}})$ which are linear in n but exponential in s, by Bondy and Erdős [13] and recently improved by Łuczak, Simonovits and Skokan [31].

1.3 Ramsey-Turán problems

The main topics in this dissertation are the Ramsey-Turán problems recently popularized by Schelp [36], which in turn are different from those previously introduced by Simonovits and Sós [37]. To motivate the definition of this new kind of Ramsey-Turán problems, we first consider the notion of restricted size Ramsey number by Faudree and Sheehan [19]. For graphs G and H, denoting $R(H) = R(H, H)$, the *restricted size Ramsey number* of H is defined as the following quantity:

$$\min\{|E(G)| : G \subset K_{R(H)} \text{ and } G \to (H, H)\}.$$

Clearly, by the definition of $R(H)$, the graph $K_{R(H)}$ is the one with the smallest number of vertices that arrows H. However, we should expect that if the graph H above has few edges, for example, when H is a path or a cycle, many edges could be deleted from $K_{R(H)}$ to form a graph G that also arrows H. It turns out that these numbers are as hard to compute as the usual Ramsey numbers and very few of them are known exactly. There are two natural ways of weakening this problem, both being studied recently by quite a few authors.

The first one is to consider the case where G is a multi-partite subgraph of $K_{R(H)}$ whose partition classes are of approximately the same order. In Chapter 3, we solve a conjecture of Schelp about the multi-partite Ramsey number of a cycle C_n where n is any large enough odd integer.

The second way to weaken the definition of *restricted size Ramsey number* is one of Ramsey-Turán nature. It consists of finding the smallest possible constant c, with $0 < c < 1$ such that for any graph G with $R(H)$ vertices and minimum degree at least $c|V(G)|$, we have $G \to H$. In Chapter 4, we provide an exact result, as before, for the case where H is a large enough odd cycle. This result will actually generalize our main theorem of Chapter 3 and has an independent proof.

1.4 Notation

Our notation is mostly standard. Nevertheless, we emphasize some points here.

In most of our theorems/lemmas we use non-standard looking *subscripts for an absolute or relative constant* in its statement. We note that these subscripts are equal to the reference number of the theorem/lemma. This makes it much easier for the reader to find the place where a constant is defined.

We let $[n]$ denote the set $\{1, 2, \ldots, n\}$.

For graphs, unless otherwise stated, the first subscript indicates the number of vertices, e.g., K_n is the *complete graph*, C_n is the *cycle* and P_n is the *path* each with n vertices. The *complete k-partite* graph with partition sets of order $n_1, \ldots, n_k$ is denoted by $K_{n_1,\ldots,n_k}$.

The *length* of a path is the number of its edges and, if x is its first vertex and x' is its last vertex, then we call it an (x, x')-path. Given a set X of vertices of a graph G, $G[X]$ denotes the subgraph induced by the edges with both ends in X. Also, $G \setminus X$ denotes the subgraph obtained by deleting the vertices of X and the edges incident to the deleted vertices.

The *maximum degree* of the vertices of a graph G is denoted by $\Delta(G)$. Given two disjoint non-empty sets of vertices X and Y, $E(X, Y)$ denotes the set of all the edges with one end in X and the other one in Y. We also set $e(X, Y) = |E(X, Y)|$.

Define the *density* $d(X, Y)$ of the pair (X, Y) as

$$d(X, Y) = \frac{e(X, Y)}{|X||Y|} \ .$$

We denote the bipartite subgraph of G with bipartition $X \cup Y$ and the edge set $E(X, Y)$ by $G[X, Y]$, and in general for disjoint sets $X_1, X_2, \ldots, X_k$ we denote by

$G[X_1, X_2, \ldots, X_k]$ the multipartite graph induced by the edges of G from X_i to X_j for every $i \neq j$. Furthermore, when there is no risk of confusion, we use $\overline{G}$ to denote the *multipartite complement* of G which is defined as the graph we obtain from the usual complement of G by deleting all edges within the classes in the given vertex partition.

The subgraphs induced by the edges of a given color are indicated by superscripts: G^r is the red subgraph of G. But for the corresponding graph theoretical parameters such as number of edges or degrees we use subscripts: $e_r(X, Y)$ denotes the number of red edges joining X to Y in an edge-colored graph. If an edge xy of G is red, we say that y is a red neighbor of x (and vice-versa). For a vertex x, $N(x)$ denotes the set of all vertices adjacent to x and we set $\deg(x, Y) = |N(x) \cap Y|$ (the degree of x to Y) and $\deg_r(x, Y) = |N_r(x) \cap Y|$ (the red degree of x to Y).

A graph G_n is called γ-*dense* if it has at least $\gamma\binom{n}{2}$ edges. A bipartite graph with parts of order k and ℓ is γ-dense if it contains at least $\gamma k\ell$ edges.

We say that a graph G_n is q-*complete* if the maximum degree in its complement $\overline{G}$ is at most q. Note that a $\gamma(n-1)$-complete graph is $(1-\gamma)$-dense.

Chapter 2

The Regularity Lemma and Embeddings

In this chapter we introduce Szemerédi's seminal work, the Regularity Lemma. We define the so called reduced graphs and shall also discuss about a particular class of lemmas, the so called embedding lemmas. We shall give a concrete example of an embedding lemma along with its proof. Such a lemma together with Szemeredi's Lemma shall be our main tools for proving our main theorems of Chapter 3 and Chapter 4.

2.1 The Regularity Lemma for Graphs

Much of modern Extremal Graph Theory rests on a fundamental lemma by Szemerédi. Loosely put, Szemerédi's Regularity Lemma [38] asserts that every graph of positive edge-density can be approximated by the union of a *bounded* number of random-like bipartite graphs. Before we can present it in a formal and precise form, the concept of *ε-regular pair* needs to be defined.

Definition 2.1. Let $G = (V, E)$ be a graph and let $0 < \varepsilon \leq 1$. We say that a pair (A, B) of two disjoint subsets of V is ε-regular (with respect to G) if

$$|d(A', B') - d(A, B)| < \varepsilon$$

holds for any two subsets $A' \subset A$, $B' \subset B$ with $|A'| > \varepsilon|A|$, $|B'| > \varepsilon|B|$.

Thus, a pair of disjoint sets is regular if the distribution of the edges of the bipartite graph determined by them is close to uniform. In the next section, we shall implicitly make use of the following well-known facts about regular pairs. Both of them have very simple proofs. We prove them here for the sake of completeness.

Fact 2.2. *If (A, B) is an ε-regular pair with $0 < \varepsilon \leq 1/2$, then for any $A_0 \subset A$, $B_0 \subset B$ such that $|A_0| \geq |A|/2$ and $|B_0| \geq |B|/2$, the pair (A_0, B_0) is a 2ε-regular.*

Proof. Take $A' \subset A_0$ and $B' \subset B_0$ such that $|A'| > 2\varepsilon|A_0|$ and $|B'| > 2\varepsilon|B_0|$. This implies that $|A'| > \varepsilon|A|$ and $|B| > \varepsilon|B|$. Since (A, B) is an ε-regular pair, we have

$$|d(A', B') - d(A, B)| < \varepsilon.$$

Also, since $|A_0| \geq |A|/2 > \varepsilon|A|$ and $|B_0| \geq |B|/2 > \varepsilon|B|$, we have

$$|d(A_0, B_0) - d(A, B)| < \varepsilon.$$

Therefore

$$|d(A', B') - d(A_0, B_0)| \leq |d(A', B') - d(A, B)| + |d(A_0, B_0) - d(A, B)| < 2\varepsilon.$$

Hence, we conclude that (A_0, B_0) is a 2ε-regular pair. $\square$

Fact 2.3. *Let G be a bipartite graph with bipartition $V(G) = A \cup B$ such that the pair (A, B) is ε-regular with density $d = d(A, B)$. Then, for any $Y \subset B$ such that*

$|Y| > \varepsilon|B|$, we have

$$|\{x \in A : \deg(x, Y) < (d - \varepsilon)|Y|\}| \leq \varepsilon|A|.$$

In particular, all but at most $\varepsilon|A|$ vertices $v \in A$ satisfy $\deg(v) \geq (d - \varepsilon)|B|$.

Proof. Suppose, for a contradiction, that there exists a set $Y \subset B$ such that $|Y| > \varepsilon|B|$ and

$$|\{x \in A : \deg(x, B) < (d - \varepsilon)|Y|\}| > \varepsilon|A|.$$

Let $X = \{x \in A : \deg(x, Y) \leq (d - \varepsilon)|Y|\}$. Then

$$e(X, Y) = \sum_{x \in X} \deg(x, Y) < (d - \varepsilon)|X| \cdot |Y|,$$

and therefore

$$d(X, Y) < d - \varepsilon,$$

contradicting the fact that (A, B) is ε-regular. $\square$

The next lemma, concerning long paths in regular pairs, is a slightly stronger version of an assertion by Łuczak [30]. The original version treats the case where the density below γ is equal to 1/4. Although our proof is essentially the same as the original one, we exhibit it here for the sake of completeness. Recall that the subscript of an absolute or relative constant in the statement of the lemma is equal to its reference number. This make it easier for the reader to find the place where this constant is defined.

Lemma 2.4. *For every $0 < \gamma < 1$ and ε, with $0 < \varepsilon < \gamma/20$, there exists a constant $n_{2.4} = n_{2.4}(\gamma, \varepsilon)$ such that for every $n > n_{2.4}$ the following holds. Let G be a bipartite graph with bipartition $V(G) = V_1 \cup V_2$ such that $|V_1|, |V_2| = n$. Furthermore, let the pair (V_1, V_2) be ε-regular with density at least γ. Then, for every integer ℓ with*

$1 \leq \ell \leq n - 2\varepsilon n/\gamma$, *and for every pair of vertices* $v' \in V_1$, $v'' \in V_2$ *satisfying* $\deg(v'), \deg(v'') \geq \gamma n/2$, *the graph* G *contains a* (v', v'')*-path of length* $2\ell + 1$.

Proof. Given γ and ε as in the statement, let $n_{2.4}$ be such that $n_{2.4}\varepsilon > 1$. Let v' and v'' as in the statement of the lemma. The strategy for building our path depends (although only slightly) on range of the value of ℓ.

We first consider the case where $1 \leq \ell < \gamma n/3$.

For $i = 1, 2$, set

$$V_i^- = \{v \in V_i : \deg(v) < \gamma n/2\}.$$

Since $\gamma n/2 < (\gamma - \varepsilon)|V_{(3-i)}|$, by Fact 2.3, we have $|V_i^-| \leq \varepsilon |V_i|$.

Then, setting

$$V_i^+ = V_i \setminus V_i^-,$$

we have that $|V_i^+| \geq (1 - \varepsilon)n$. Take maximum size sets $\hat{V}_1 \subseteq V_1^+$ and $\hat{V}_2 \subseteq V_2^+$ satisfying $|\hat{V}_1| = |\hat{V}_2|$. It is easy to see that the bipartite subgraph $H = G[\hat{V}_1, \hat{V}_2]$ has minimum degree at least $\gamma n/2 - \varepsilon n > \gamma n/3$. Therefore, we can greedily construct a path of length $2\ell - 2$, say $P_{2\ell-2} = v_0 v_1 \ldots v_{2\ell-2}$, such that $v_0 = v'$ and $V(P_{2\ell-2}) \subseteq \hat{V}_1 \cup \hat{V}_2 \setminus \{v''\}$. In fact, first choose $v_0 = v'$ and, assuming that $v_0, \ldots, v_{i-1}$ were chosen, take v_i to be any of the neighbors of v_{i-1} in $V(H) \setminus \{v_0, \ldots, v_{i-1}\} \cup \{v''\}$. Such vertex v_i exists given that $\deg(v_{i-1}) > \ell$, and so $\deg(v_{i-1}) - V(P_{2\ell-2}) \geq 1$.
To show that we can extend $P_{2\ell-2}$ to a path of length $2\ell + 1$ ending at v'', it is enough to show that G contains an edge $\{v_{2\ell-1}, v_{2\ell}\}$ from $N_H(v_{2\ell-2}) \setminus (V(P_{2\ell-2}) \cup \{v''\})$ to $N_H(v'') \setminus V(P_{2\ell-2})$. More precisely, we would get a path $P_{2\ell+1} = P_{2\ell-2} v_{2\ell-1} v_{2\ell} v''$, i.e., $P_{2\ell+1} = v_0 \ldots v_{2\ell} v''$. Such an edge $\{v_{2\ell-1}, v_{2\ell}\}$ exists because

$$|N_H(v_{2\ell-2}) \setminus (V(P_{2\ell-2}) \cup \{v''\})| \geq \gamma n/2 - \varepsilon n - \gamma/3 - 1 > \varepsilon n$$

and, similarly,

$$|N_H(v'') \setminus V(P_{2\ell-2})| > \varepsilon n.$$

The ε-regularity of (V_1, V_2) implies that the density between these sets cannot be zero, and we note also that those sets are non-empty as $\varepsilon n > 1$.

In the range $\gamma n/3 \leq \ell \leq n - 2\varepsilon n/\gamma$, we use induction on ℓ. Assume that we have already constructed a path $P_{2\ell-1} = v_0 v_1 \dots v_{2\ell-1}$, such that $v_0 = v'$ and $v_{2\ell-1} = v''$. The strategy will be to replace one edge of this path by a path of length 3. We say that a vertex $v \in V(P_{2\ell-1})$ is 'good' if it has at least εn neighbors not in $V(P_{2\ell-1})$, that is, $|N_H(v) \setminus V(P_{2\ell-1})| \geq \varepsilon n$; otherwise we call v 'bad'.

If there exists an i, with $0 \leq i \leq 2\ell - 2$, such that the vertices $v_i \in V(P_{2\ell-1}) \cap V_1$ and $v_{i+1} \in V(P_{2\ell-1}) \cap V_2$ are good, then we can proceed as above: by the ε-regularity of (V_1, V_2), the density between $N(v_i) \setminus V(P_{2\ell-1})$ and $N(v_{i+1}) \setminus V(P_{2\ell-1})$ cannot be zero. In this case, there must be $w', w'' \notin V(P_{2\ell-1})$ such that $\{v_i, w'\}$, $\{w', w''\}$ and $\{w'', v_{i+1}\}$ are edges of G. Therefore, we have a path $v_0 v_1 \dots v_i w' w'' v_{i+1} \dots v_{2\ell-1}$ of length $2\ell + 1$ connecting v' to v''. It remains to prove that such an i exists.

Denote $Y = V_2 \setminus V(P_{2\ell-1})$. Recall that $|Y| \geq 2\varepsilon n/\gamma > \varepsilon n$. Let X be the set of vertices of V_1 which have degree at most $(\gamma - \varepsilon)|Y|$ in Y. By Fact 2.3, $|X| \leq \varepsilon n$. Since

$$(\gamma - \varepsilon)|Y| > (\gamma/2)|Y| \geq \varepsilon n,$$

all bad vertices of V_1 belong to X. Therefore there are at most εn bad vertices in V_1. Similarly, there are at most εn bad vertices in V_2. Since there are ℓ independent edges in $P_{2\ell-1}$ and at most $2\varepsilon n < \frac{\gamma n}{3} \leq \ell$ bad vertices, the bad vertices cannot cover all edges of $P_{2\ell-1}$. Hence, the desired i exists. $\square$

Given a graph G and a real number $0 < \varepsilon < 1$, suppose that we have a partition $V(G) = V_0 \cup V_1 \cup \cdots \cup V_t$ satisfying the following properties:

- $|V_0| \leq \varepsilon n; \quad |V_1| = |V_2| = \cdots = |V_t|$;
- and all but at most $\varepsilon\binom{t}{2}$ pairs (V_i, V_j), $1 \leq i < j \leq t$, are ε-regular with respect to G.

This means that most of the pairs of clusters (V_i, V_j) have the same order and satisfy Definition 2.1 with some uniform (small) ε. We call this partition *ε-regular with respect to G*. In his seminal work [38], Szemerédi proved that every sufficiently large graph has an ε-regular partition in which the number of clusters is bounded by a function of ε and is independent of the number of vertices of G. Its precise statement, extended to more than one graph, is as follows.

Lemma 2.5 (Regularity Lemma). *For every $\varepsilon > 0$ and $s, m \in \mathbb{N}$ there exist integers $N_{2.5} = N_{2.5}(\varepsilon, s, m)$ and $M_{2.5} = M_{2.5}(\varepsilon, s, m)$ such that: for all graphs $G_1, \ldots, G_s$ with the same vertex set V where $|V| \geq N_{2.5}$, there is a partition of V into $t + 1$ sets*

$$V = V_0 \cup V_1 \cup \ldots \cup V_t$$

which is ε-regular with respect to each G_k, $1 \leq k \leq s$, and such that $m \leq t \leq M_{2.5}$.

Remark. *The original regularity lemma refers to the case $s = 1$. The proof is essentially the same for an arbitrary but fixed number s of graphs. This version is used, for example, by Erdős, Hajnal, Sós, and Szemerédi [17], and formulated in a survey by Komlós and Simonovits [27].*

Remark. *The sets V_i in the partition given by this lemma are called* clusters*. When the lemma is applied to a multipartite graph, we can assume that each of those clusters is contained in one of the parts.*

The existence of the cluster V_0 above is only for technical reasons: it allows us to assume that all the other clusters have the same number of elements. Frequently, alternative formulations are sometimes used; for example, one may assume that $V_0 = \emptyset$ if we weaken the condition $|V_i| = |V_j|$ to $|V_i| - |V_j| \leq 1$ for all i, j.

Note that Lemma 2.5 is vacuously true unless the graph G to which it is applied has positive edge-density. Indeed, G is trivially "approximated" by a union of empty bipartite graphs.

2.2 Embeddings Lemmas

The Regularity Lemma has been applied to asymptotically solve a number of problems in extremal graph theory. Perhaps the most important classes of extremal problems are the Turán-type problems and the Ramsey-type problems. These problems involve finding large subgraphs with a particular property inside a larger graph G. An embedding of a graph H into G is a map from $V(H)$ to $V(G)$ that preserves adjacency. We loosely use the term 'embedding lemma' to refer to lemmas that guarantee the existence of a embedding of H onto G whenever H and G satisfy a certain property.

In this thesis, we are particularly interested in embeddings of paths and cycles. A common type of embedding lemma uses various properties about regular pairs to guarantee the existence of certain bipartite subgraphs in the graph determined by the pair. For example, in Lemma 2.4 above, for any fixed positive density γ, choosing ε small enough and n large enough, one can find 'very long' paths between the sets of an ε-regular pair of density γ. In a more general set up, if one aims to find a long path in a given graph G, it would be desirable to apply the Regularity Lemma to G so that we can find lots of regular pairs, then apply Lemma 2.4 to some of those pairs and finally try to 'glue' these paths together to find a longer path. We note, however, that the Regularity Lemma does not state anything about the density between the pairs of clusters. Such densities may differ significantly from one pair of clusters to another and may be zero for some pairs. Then, we may not be able to apply Lemma 2.4 to all pairs. On the other hand, if the original graph G is dense and large enough, many of the pairs of clusters shall also have positive density. Furthermore, since the number of clusters is bounded, each cluster should also have a relatively large number of vertices. It turns out that most of the difficulty comes from 'glueing' together those paths between regular pairs. Later, in Lemma 2.11, we prove that this strategy works under certain conditions on the connections between the clusters. This discussion motivates the following definition of a *reduced graph* which grasps the connections between clusters.

Definition 2.6. Given a graph G, two parameters $\varepsilon, d > 0$ and an ε-regular partition of $V(G)$ into $V_0, \ldots, V_t$ such that $|V_0| < \varepsilon n$, we define the *reduced graph* $R = R(\gamma, \varepsilon)$ as follows: the vertex set of R is $\mathcal{V} = \{1, \ldots, t\}$, and there is an edge from vertices i to j if and only if (V_i, V_j) is ε-regular and has density at least γ.

In most applications of the regularity lemma, one chooses the parameters ε and γ (along with many others) and construct such a reduced graph. One then uses the fact that many properties of the reduced graph are inherited by the original graph G. Proposition 2.8, bellow, whose proof can be found in Diestel [15], is probably the most well-known embedding property related to the regularity lemma. Though we will not use such proposition to prove our theorems, we believe it is relevant to mention it. We shall need the following definition in order to state it.

Definition 2.7. Given a graph R, the graph R^s is the graph obtained by replacing each vertex v of R by a set of s vertices and each edge of R by a complete bipartite graph between its two corresponding sets of s vertices. This is commonly known as a 'blow-up' of R.

Proposition 2.8. *For every $\gamma \in (0;1]$, $\Delta > 1$ and $s \geq 1$, there exists $\varepsilon_0 > 0$ and n_0 with the following property. Let G_n be a graph on $n \geq n_0$ vertices, and let $R = R(\gamma, \varepsilon)$ be a reduced graph of G_n with $\varepsilon \leq \varepsilon_0$ such that every cluster contains at least $2s/\gamma^{\Delta}$ vertices. Then any subgraph H of R^s whose maximum degree is $\Delta(H) \leq \Delta$, is also a subgraph of G_n.*

Next, we are going to state and prove an embedding lemma that we shall use in the proofs of our main results in Chapter 3 and 4. This lemma uses certain special matchings in the reduced graph to find long cycles in the original graph. This idea was first introduced by Łuczak [30]. In this version we combine implicit results from [9] (for even cycles) and from [6] (for odd cycles). In order to state this lemma, we need to introduce some notation.

Definition 2.9. A matching M in a graph G is a set of pairwise vertex-disjoint edges. The *size* of a matching is the number of edges that it contains and is denoted by $e(M)$.

Definition 2.10. A *connected matching* is a matching M such that all the edges of M are in the same connected component C of G. We say that M is an *odd connected matching*, if the component C is not bipartite.

Lemma 2.11. *Given $0 < \eta < 1/4$, there exists $c_{2.11} = c_{2.11}(\eta) > 0$, such that for any real numbers $0 < \gamma < 1$ and $0 < \varepsilon < 1$ satisfying $\varepsilon/\gamma \leq c_{2.11}$ and any natural number t, there exists $n_{2.11} = n_{2.11}(\eta, \gamma, \varepsilon, t)$ such that the following holds. Let G_n be a graph on $n > n_{2.11}$ vertices and let $R_t = R_t(\gamma, \varepsilon)$ be a reduced graph of G_n on t vertices. If R_t contains a connected matching M of size $t_1 \geq (1/4 + \eta)t$, then G_n contains an even cycle of order ℓ for any even ℓ such that $4t < \ell \leq (1/2 + \eta)n$. If, in addition, M is contained in an odd component, then G_n also contains also odd cycles of any order ℓ such that $4t < \ell \leq (1/2 + \eta)n$. Furthermore, $n_{2.11}(\eta, \gamma, \varepsilon, t)$ increases when $\eta, \gamma, \varepsilon$ are fixed and t increases.*

Proof. Let $0 < \eta < 1/4$ be given. Choose $c_{2.11} = \eta/20$ and note that such choice implies that for any reals $0 < \gamma < 1$, $0 < \varepsilon < 1$ satisfying $\varepsilon/\gamma < c_{2.11}$ we have

$$\left(\frac{1}{2} + 2\eta\right)\left(1 - \frac{8\varepsilon}{\gamma}\right)(1 - 2\varepsilon) \geq \left(\frac{1}{2} + \eta\right).$$

Fix such $\eta, \gamma, \varepsilon$, and let t be any natural number. We consider the constant $n_{2.4}(\gamma/2, 2\varepsilon)$ obtained when we input $\gamma/2$ and 2ε to Lemma 2.4. Let $n_{2.11}$ be such that

$$\frac{(1-\varepsilon)n_{2.11}}{t} > \max\{2t + 2n_{2.4}(\gamma/2, 2\varepsilon),\ 4t/\varepsilon,\ 32\gamma^{-3/2}\}.$$

Let G_n be any graph on $n > n_{2.11}$ vertices and let R_t be a reduced graph as in the statement of the lemma and let $V_0, V_1, \ldots, V_t$, with $|V_0| < \varepsilon n$, be the clusters of the ε-regular partition determining R_t. Note that for any $i \neq 0$, we have $|V_i| = m$ and the $m \geq (1-\varepsilon)n_{2.11}/t$ choice of $n_{2.11}$ implies that $m - 2t > m/2$.

Let $M = \{a_1b_1, \ldots, a_{t_1}b_{t_1}\}$ be a monochromatic connected matching in R_t of size $t_1 \geq (1/4 + \eta)t$. Let K be the monochromatic component of R_t containing M.

First, we show that K has a closed walk of even length which contains all edges of M. Let T be a spanning tree of K such that $E(T)$ contains all edges of M (this can be done via Kruskal's algorithm, i.e., starting with the edges of K and carefully adding new edges until we get a spanning tree). Let W_{even} be the minimal closed walk containing all the edges of T. Such a walk contains each edge of T exactly twice, therefore it has an even length. Also, its length must be at most $2t$.

In the case where K is an non-bipartite component, we can also find a closed walk of odd length containing all edges of M. In fact, consider some arbitrary vertex r of T and look at the levels of T as a rooted tree with root r. In this case, there must exist an edge $xy \notin E(T)$, such that x and y are in levels of same parity, i.e., the lengths of the unique paths from x to r and from y to r in T have the same parity.

Therefore, the unique path P_{xy} from x to y contained in W_{even} has even length. We can construct a walk W_{odd} by taking W_{even} and replacing P_{xy} by the edge xy. It is clear that W_{odd} is a closed walk, it has odd length and it contains every edge of M (at least once), as desired.

Now, consider any ℓ in the range $4t < \ell \leq (1/2 + \eta)n$. We aim to build a C_ℓ in G. We start by letting $L = W_{odd}$ in the case ℓ is odd and $L = W_{even}$ in the case ℓ is even. In particular, we can proceed with the case where ℓ is odd only when such W_{odd} exists, i.e., when the component K is non-bipartite. Denote $L = i_1 i_2 \dots i_s i_1$, which implies that s and ℓ have the same parity. Next we use standard regularity arguments and Lemma 2.4 to build the desired cycle in G_n.

For each j, with $0 \leq j \leq s$, we say that a vertex in V_{i_j} is 'good' if it has at least $(\gamma - \varepsilon)|V_{i_j}| = (\gamma - \varepsilon)m$ neighbors in each of $V_{i_{j-1}}$ and $V_{i_{j+1}}$, where we set $V_{i_0} = V_{i_s}$ and $V_{i_{s+1}} = V_{i_1}$; and we say that a vertex is 'bad' otherwise. Note that for any j, by Fact 2.3 applied to $(V_{i_j}, V_{i_{j+1}})$ and to $(V_{i_j}, V_{i_{j-1}})$, at most $2\varepsilon m$ vertices of V_{i_j} are bad. The next important step in the proof is to construct a (small) cycle $\tilde{C} = v_{i_1} v_{i_2} \dots v_{i_s}$ with $v_{i_j} \in V_{i_j}$ such that all its vertices are good. We emphasize that while we may have $V_{i_k} = V_{i_j}$, for some numbers k, j with $k \neq j$, the vertices v_{i_j} of C are chosen to be pairwise distinct. Let us construct such cycle step by step, adding one vertex at each step. At the first step, we let v_{i_1} be any good vertex in V_{i_1} (which exists since $(1 - 2\varepsilon)|V_{i_1}| \geq 1$). Suppose that for some j, with $1 \leq j \leq s-3$, we have constructed a path $P_j = v_{i_1} v_{i_2} \dots v_{i_j}$ in which all vertices are good. In particular, v_{i_j} has at least $(\gamma - \varepsilon)m$ neighbors in $V_{i_{j+1}}$. Among those, at most $2\varepsilon m$ are bad and less than j are in P_j. Therefore, v_{i_j} has at least $(\gamma - 3\varepsilon)m - j$ good neighbors not in P_j. Finally, since $j \leq s \leq t < \gamma m/2$ and $3\varepsilon < \gamma/4$, we have $(\gamma - 3\varepsilon)|V_{i_{j+1}}| - j \geq \gamma |V_{i_{j+1}}|/4 \geq 1$. So there exists $v_{i_{j+1}} \in |V_{i_{j+1}}|$ such that $v_{i_{j+1}}$ is good and $v_{i_1} v_{i_2} \dots v_{i_j} v_{i_{j+1}}$ is a path. At step $s-2$, we have contructed a path $P_{s-2} = v_{i_1} v_{i_2} \dots v_{i_{s-2}}$ in which all vertices are good. By the same argument as before, v_{s-2} has at least $|V_{i_{s-1}}|/4$ good neighbors in $V_{i_{s-1}}$ but not in P_{s-2}; let A be the set of such neighbors. Similarly, v_1 has at least

$\gamma|V_{i_s}|/4$ good neighbors in V_{i_s} but not in P_{s-2}; let B be set of such neighbors. Because the pair $(V_{i_{s-1}}, V_{i_s})$ is ε-regular and $|A|, |B| \geq \varepsilon m$, it follows that $G[A, B]$ has density at least $(\gamma - \varepsilon) > \gamma/2$. Therefore, the number of edges in $G[A, B]$ is at least $\gamma|A||B|/2 \geq \gamma^3 m^2/32 \geq 1$, where the last inequality follows by the choice of $n_{2.11}$. Letting $v_{i_{s-1}} v_{i_s}$ be any edge of $G[A, B]$, we have that $v_{i_1} v_{i_2} \ldots v_{i_{s-1}} v_{i_s}$ is a cycle as desired.

For each $a_k b_k \in M$, we take maximum size sets $V'_{a_k} \subset (V_{a_k} \setminus \tilde{C}) \cup \{v_{a_k}\}$, $V'_{b_k} \subset (V_{b_k} \setminus \tilde{C}) \cup \{v_{b_k}\}$ satisfying $|V'_{a_k}| = |V'_{b_k}|$ and notice that the assumptions of the lemma give

$$|V'_{a_k}| = |V'_{b_k}| \geq |V_{a_k}| - |\tilde{C}| \geq |V_{a_k}| - 2t > n_{2.4}(\frac{\gamma}{2}, 2\varepsilon) \tag{2.1}$$

We also note that

$$\deg(v_{a_k}, V'_{b_k}) \geq \deg(v_{a_k}, V_{b_k}) - t \geq (\gamma - \varepsilon)|V_{b_k}| - t \geq \gamma|V_{b_k}|/2, \tag{2.2}$$

where the last inequality follow from the fact that $\varepsilon < \gamma/4$ and $t/m < \gamma/4$ (by the definitions of ε and $n_{2.11}$ respectively). Of course, the analogous inequality holds for $\deg(v_{b_k}, V'_{a_k})$.

We can use Lemma 2.4 to replace the edges of $\tilde{C}$ corresponding to edges of M by long paths resulting in a larger cycle in G_n. Next, we give bound on how large such cycles can be.

It is clear that $|V'_{a_k}| \geq |V_{a_k}|/2$ and $|V'_{b_k}| \geq |V_{b_k}|/2$, which implies that $G[V'_{a_k}, V'_{b_k}]$ is (2ε)-regular by Fact 2.2. It is also easy to see that $G[V'_{a_k}, V'_{b_k}]$ has density at least $\gamma - \varepsilon > \gamma/2$. By Equations (2.1) and (2.2), together with the fact that $2\varepsilon < \frac{\gamma/2}{20}$, we are allowed to apply Lemma 2.4 to $G[V'_{a_k}, V'_{b_k}]$ with parameters $\gamma/2$ and 2ε: For each edge $a_k b_k$ of M, we choose a natural number ℓ_k satisfying

$$1 \leq \ell_k \leq (1 - 8\varepsilon/\gamma) \min\left\{|V_{a_k}| - 2t, |V_{b_k}| - 2t\right\} \leq (1 - 8\varepsilon/\gamma) \min\left\{|V'_{a_k}|, |V'_{b_k}|\right\},$$

and for any such choice there exists a path P_{a_k,b_k} of length $2\ell_k+1$ starting at v_{a_k}, ending at v_{b_k}, and consisting only of edges in $G[V'_{a_k}, V'_{b_k}]$. If we replace the edge $v_{a_k}v_{b_k}$ in $\tilde{C}$ by the path P_{a_k,b_k}, we get a cycle of order $s-t_1+\sum_{k=0}^{t_1-1}(2\ell_k+1) = s+\sum_{k=0}^{t_1-1} 2\ell_k$. So, the length of the expanded cycle can attain any value which has the same parity of s and is between $s+2t_1$ and

$$s+\sum_{i=0}^{t_1-1} 2\,(1-8\varepsilon/\gamma)\min\{|V_{a_k}|-2t, |V_{b_k}|-2t\}.$$

Furthemore, $s+2t_1<4t$ and

$$\begin{aligned}
s+\sum_{i=0}^{t_1-1} 2\,(1-8\varepsilon/\gamma)\min\{|V_{a_k}|-2t, |V_{b_k}|-2t\} \geq \\
&\geq 2t_1\,(1-8\varepsilon/\gamma)\left(\frac{(1-\varepsilon)n}{t}-2t\right)\\
&\geq \left(\frac{1}{2}+2\eta\right)t(1-8\varepsilon/\gamma)\frac{(1-2\varepsilon)n}{t} \geq \left(\frac{1}{2}+\eta\right)n.
\end{aligned}$$

Therefore, the expanded cycle can attain length ℓ as desired. $\square$

Corollary 2.12. *Let η, γ, ε, G_n and $R_t = R_t(\gamma,\varepsilon)$ be as in the statement of Lemma 2.11. Also, assume that $V_0, V_1, \ldots, V_t$ is the ε-regular partition of $V(G)$ which determines R_t and assume M is a matching of size $t_1 \geq (1/4+\eta)t$ contained in a monochromatic component K of R_t, as in the proof of the lemma. Then, there exists a set of vertices F, such that $|F| \leq 4\varepsilon n$ and for any two vertices*

$$u, v \in \left(\bigcup_{i\in K} V_i\right)\setminus F,$$

say $u\in V_i$ and $v \in V_j$, there exists a (u,v)-path of length ℓ in G_n for each ℓ in the range $4t<\ell\leq(1/2+\eta)n$ whose parity is the same as some walk from i to j in R_t.

Proof. As in the proof of Lemma 2.11, consider a spanning tree T of K containing all edges of M and let $W = i_1 i_2 \ldots i_s i_1$ be the closed walk which contains all edges of T

twice. As before, we consider the subscripts modulo s. For each $k \in \{1, 2, \ldots, s\}$, let F_k be the set of vertices of V_{i_k} with degree less than $(\gamma - \varepsilon)|V_{i_k}|$ in $V_{i_{k-1}}$ or $V_{i_{k+1}}$. Fact 2.3, applied to the pairs $(V_{i_{k-1}}, V_{i_k})$ and $(V_{i_k}, V_{i_{k+1}})$, implies that $|F_k| \leq 2\varepsilon|V_{i_k}|$. So, letting $F = \bigcup_{1 \leq k \leq s} F_k$, we have

$$|F| \leq s(2\varepsilon|V_{i_k}|) \leq 4t\varepsilon|V_{i_k}| \leq 4\varepsilon n.$$

We claim that F has the required properties. Let $u, v \in \left(\bigcup_{i \in K} V_i\right) \setminus F$ and assume that $u \in V_i$ and $v \in V_j$, for some $i, j \in V(T)$. It is easy to find a walk of length at most $2t$ using only edges of T, starting at i, ending at j and using each edge of M at least once. Let L be such a walk.

Because $u, v \notin F$, with the same argument of the proof of the lemma, we can greedily find a (u, v)-path of same length as L. We can also use this path and Lemma 2.4 to build (u, v)-path of any length ℓ, $4t < \ell \leq (1/2 + \eta)n$, as long as ℓ has the same parity of the length of L.

This completes the proof of the corollary. $\square$

Chapter 3

Multipartite Ramsey numbers of odd cycles

Recently, there has been much interest in seeing what happens to the Ramsey numbers when we allow fixed edge deletions from the complete graph K_N, in particular, if we delete the edges of a complete subgraphs K_r.

For example, a tripartite version of Gerencsér-Gyárfás's Theorem was given by Gyárfás, Ruszinkó, Sárközy and Szemerédi [22], i.e., it was proved that the Ramsey number for a path is about the same when two-colorings of a complete graph or a balanced complete tripartite graph are considered. In a paper of Nikiforov and Schelp [33], it was shown, among other things, that for any odd $n \geq 5$ if we delete the edges of a complete subgraph of order $(n-1)/2$ from the complete graph of order $2n-1$ and two-color the rest, we can still guarantee a monochromatic C_n. Furthermore, in a recent article of Gyárfás, Sárközy and Schelp [24], the following theorem in the same direction was proved.

Theorem 3.1. *For all* $0 < \eta < 1/2$ *there exists an* $n_{3.1} = n_{3.1}(\eta)$ *with the following properties. For any odd integer* $n > n_{3.1}$*, in any two-coloring of the edges of the complete* 5*-partite graph of order* $(2+\eta)n$ *with* 5 *parts of size* $g(1), g(2), g(3), g(4)$

and $g(5)$, where we have $n/2 \geq g(1) \geq g(2) \geq g(3) \geq g(4) \geq g(5) \geq \eta n$, there is a monochromatic C_n.

In this chapter, we prove, for sufficiently large n, that a similar result holds in a sharp form. This result was conjectured in the same article [24] in which Theorem 3.1 appeared. More precisely, we prove the following main theorem.

Theorem 3.2. *There exists $n_{3.2}$ such that, for any odd integer $n \geq n_{3.2}$, in any 2-coloring of the edges of the complete 5-partite graph $K_{(n-1)/2,(n-1)/2,(n-1)/2,(n-1)/2,1}$ there is a monochromatic C_n.*

Note that the graph we are coloring above is obtained from a K_{2n-1} by making four big 'holes' of order $(n-1)/2$ each. We are removing a total of $(n-1)(n-3)/2$ edges, i.e., almost $1/4$ of the total number of edges, and we are claiming that (for large odd n) the two-color Ramsey number for C_n does not change. This is somewhat surprising and sharp. It is sharp in two different ways:

- if we had made only a single hole of order $(n+1)/2$, instead of four holes of order $(n-1)/2$, there would be no guarantee that we could find a monochromatic C_n. In fact, let $A \subset V = V(K_{2n-1})$ with $|A| = (n+1)/2$ and consider the graph obtained by the removal of the edges spanned by A from K_{2n-1}. Split the vertices $V \setminus A$ into two sets B and C with $|B| = (n-1)/2$ and $|C| = n-1$. Color all the edges within B, within C and between A and B by red; and color the remaining edges, i.e., those between $A \cup B$ and C, by green. It is easy to see that there is no monochromatic C_n;

- there exists a 2-edge-coloring of K_{2n-2} without monochromatic C_n, as we recall from Theorem 1.4 that $R(C_n, C_n) = 2n-1$ for any odd $n > 3$.

It is also interesting to compare our result with the one from equation (1.2), where we 3-color the complete graph.

3.1 Extremal colorings and stability

In this chapter, we will use a variant of a stability theorem of Gyárfás, Ruszinkó, Sárközy, and Szemerédi [21, 23], stated by Benevides and Skokan [5, 9]. But before we can state this theorem we need to define particular (extremal) colorings. It is convenient, as we will notice later, to consider 3-multi-colorings instead of 3-colorings. In a 3-*multi-coloring* of a graph G, every edges get at least one color but some edges can be assigned more than one color. For $c \in \{\text{(r)ed,(g)reen, (b)lue}\}$, we say that c is the *exclusive color* of an edge if the edge is assigned only color c. We denote by G^{b^*} the subgraph induced by the edges exclusively colored blue ; and denote G^{r^*} and G^{g^*} the corresponding subgraph for red and green respectively.

Now we define the three types of coloring.

Coloring 3.3 ($EC_1(\alpha,\delta)$ type). *A 3-multi-coloring of a graph G is of type $EC_1(\alpha,\delta)$, where $0 \le \alpha, \delta < 1$, if there exists a partition $A \cup B \cup C \cup D$ of $V(G)$ such that*

(a) $|A|, |B|, |C|, |D| \ge (1-\alpha)|V(G)|/4$;

(b) The bipartite graphs $G^{r^*}[A,B], G^{r^*}[C,D], G^{g^*}[A,D], G^{g^*}[B,C], G^{b^*}[A,C]$ and $G^{b^*}[B,D]$ are $(1-\delta)$-dense.

Coloring 3.4 ($EC_2(\alpha,\delta)$ type). *A 3-multi-coloring of a graph G is of type $EC_2(\alpha,\delta)$, where $0 \le \alpha, \delta < 1$, if there exists a partition $A \cup B \cup C \cup D$ of $V(G)$ such that*

(a) $|A|, |B|, |C|, |D| \ge (1-\alpha)|V(G)|/4$;

(b) The bipartite graphs $G^{r^*}[A,B]$, $G^{g^*}[A\cup B, C]$ and $G^{b^*}[A\cup B, D]$ are $(1-\delta)$-dense.

Coloring 3.5 ($EC_3(\mu,c_1,c_2,\delta)$ type). *A 3-multi-coloring of a graph G is of type $EC_3(\mu,c_1,c_2,\delta)$, where $0 \le \mu$, c_1, c_2, $\delta < 1$, if there exists a partition $A \cup B \cup C \cup D$ of $V(G)$ such that* *(a)* $|A|, |B|, |C| \geq (1 - c_1\mu)|V(G)|/4, \quad |D| \geq \mu|V(G)|/4;$

(b) $|A| \geq \max\{|B|, |C|, |D|\} + \mu|V(G)|/4$, $|A \cup D| \leq (1 + c_2\mu)|V(G)|/2;$

(c) The bipartite graphs $G^{r^*}[A, B]$, $G^{r^*}[C, D]$, $G^{g^*}[A, D]$, $G^{g^*}[B, C]$, $G^{b^*}[A, C]$ *and* $G^{b^*}[B, D]$ *are* $(1 - \delta)$*-dense.*

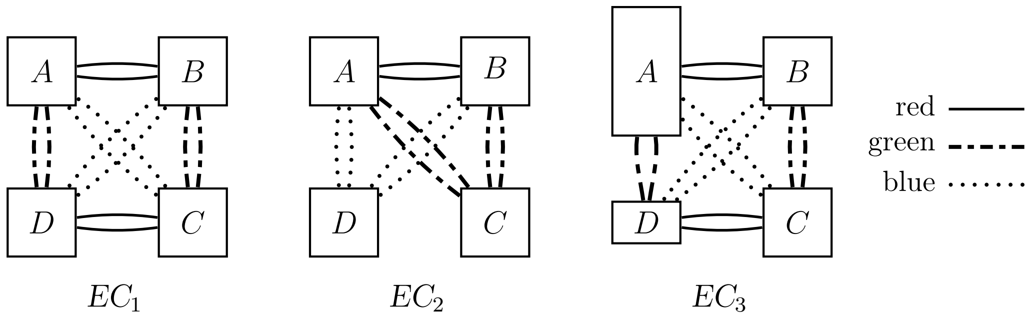

Figure 3.1: Three different types of colorings (EC_1, EC_2, EC_3).

Now we can state the variant [5, 9] of the stability lemma of Gyárfás, Ruszinkó, Sárközy and Szemerédi [21, 23].

Theorem 3.6. *Given* $\alpha_0 > 0$ *and* $\mu_0 > 0$*, there exist positive reals* $\eta_{3.6}$*,* $\beta_{3.6}$ *and* $\mu_{3.6}$*,* $\mu_{3.6} < \mu_0$*, such that for all* $\beta < \beta_{3.6}$ *there exists a positive integer* $n_{3.6} = n_{3.6}(\beta, \alpha_0, \mu_0)$ *such that the following holds. If* $n \geq n_{3.6}$ *and a* $(1 - \beta)$*-dense graph* G_n *of order* n *is 3-multi-colored, then one of the following cases occurs:*

a) G_n *contains a monochromatic connected matching of size at least* $(1/4 + \eta_{3.6})n$ *edges;*

b) *the coloring is of type* $EC_1(\alpha_0, \alpha_0)$*, or* $EC_2(\alpha_0, \alpha_0)$*, or* $EC_3(\mu_{3.6},\ 0.7,\ 0.2,\ \beta^{1/3})$.

Remark. *In a multi-coloring, we consider a set* E *of edges monochromatic if there is a color* c *such that all edges in* E *have been colored with* c*. However, note that we do not require the edges in* E *to be colored exclusively with* c*.*

The proof of Theorem 3.6 is essentially the same as the one by Gyárfás, Ruszinkó, Sárközy and Szemerédi [21, 23] and can be found in [5]. This theorem was used first to compute $R(P_n, P_n, P_n)$ and by Benevides and Skokan [9] to compute $R(C_n, C_n, C_n)$ when n is even. It basically says that either we find a large monochromatic connected matching or the coloring of the graph can be well described. Later in this chapter, we will use this theorem to prove Theorem 3.10 which, in turn, will be used in the proof of Theorem 3.2. Theorem 3.10 involves two other types of colorings, this time, 2-multi-colorings of a 4-partite graph. We define those colorings here, but we will state Theorem 3.10 only when needed, in Section 3.2.

Coloring 3.7 ($EC_A(\alpha, \delta)$ type)**.** *A 2-multi-coloring of a 4-partite graph G is of type $EC_A(\alpha, \delta)$, where $0 \le \alpha, \delta < 1$, if there exist disjoint sets of vertices A, B, C and D such that*

(a) $|A|, |B|, |C|, |D| \ge (1-\alpha)|V(G)|/4$ and each of A, B, C and D is an independent set;

(b) The bipartite graphs $\overline{G^{g^*}}[A, D]$, $\overline{G^{g^*}}[B, C]$ have maximum degree at most $\delta|V(G)|$;

(c) The bipartite graphs $\overline{G^{r^*}}[A, B]$, $\overline{G^{r^*}}[C, D]$ have maximum degree at most $\delta|V(G)|$.

Remark. *Condition (a) implies that at most $\alpha|V(G)|$ vertices do not belong to $A \cup B \cup C \cup D$.*

Coloring 3.8 ($EC_B(\alpha, \delta)$ type)**.** *A 2-multi-coloring of a 4-partite graph G, whose vertex partition into independent sets is given, say $V(G) = U_1 \cup U_2 \cup U_3 \cup U_4$, is of type $EC_B(\alpha, \delta)$, where $0 \le \alpha, \delta < 1$, if there exist disjoint sets X, $Y \subseteq V(G)$ for which, letting $X_i = U_i \cap X$, $Y_i = U_i \cap Y$ for $1 \le i \le 4$, we have*

(a) $|X|, |Y| \ge (1-\alpha)|V(G)|/2$;

(b) For $1 \le i \le 4$, the bipartite graph $\overline{G^{r^*}}[X_i, \bigcup_{j \ne i} Y_j]$ has maximum degree at most $\delta|V(G)|$;

(c) For $1 \leq i \leq 4$, the bipartite graph $\overline{G^{r^*}}[Y_i, \bigcup_{j \neq i} X_j]$ has maximum degree at most $\delta|V(G)|$;

(d) The (multipartite) graphs $\overline{G^{g^*}}[X_1, X_2, X_3, X_4]$ and $\overline{G^{g^*}}[Y_1, Y_2, Y_3, Y_4]$ have maximum degree at most $\delta|V(G)|$.

Remark. *Condition (a) implies that at most $\alpha|V(G)|$ vertices do not belong to $X \cup Y$.*

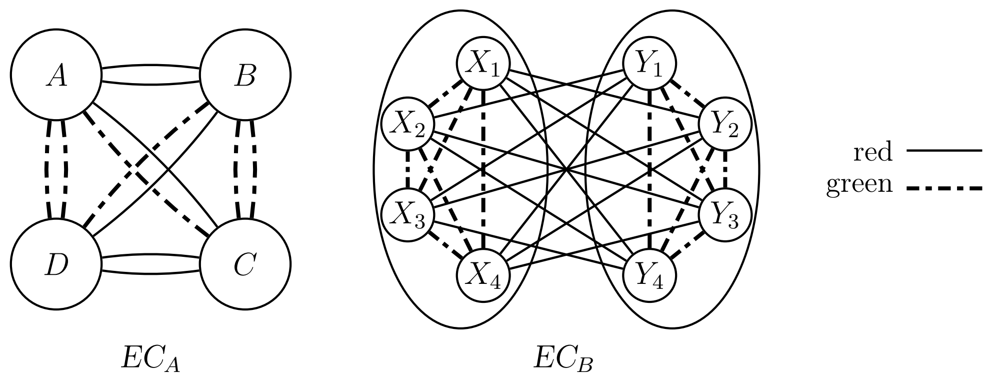

Figure 3.2: Two other types of colorings (EC_A, EC_B).

The remainder of this chapter is organized as follows: In Section 3.2 we state (without proofs) our main tools, one theorem and two lemmas, and use them to prove Theorem 3.2. In Sections 3.3 and 3.4, we give the missing proofs.

3.2 Main tools and proof Theorem 3.2

In the light of Lemma 2.11, if one aims to find large cycles in a graph G it is natural to search for a connected matching in a suitable reduced graph. In the case where we have a coloring of a graph G and want to find a monochromatic cycles, the following notion of a monochromatic connected matching will play a similar role

Definition 3.9. We say that M is a *monochromatic connected matching*, if all its edges have the same color and it is a connected matching within the graph induced by such this color. In addition, we say that M is *odd* if this component is non-bipartite.

Our main tool is the following theorem, which shall eventually be used to find a monochromatic connected matching in a suitable reduced graph. We postpone its proof to Section 3.3.

Theorem 3.10. *Given α_1, there exist strictly positive real numbers $\eta_{3.10} = \eta_{3.10}(\alpha_1)$, $\beta_{3.10} = \beta_{3.10}(\alpha_1)$ and also $n_{3.10} = n_{3.10}(\beta_{3.10}, \eta_{3.10})$ such that for any $n > n_{3.10}$ the following holds: if G is a 4-partite graph on n vertices such that each part has at least $(1/4 - \beta)n$ vertices and its multipartite complement $\overline{G}$ satisfies $\Delta(\overline{G}) \leq \beta n$, then for any 2-multi-coloring of G, either we find an odd connected monochromatic matching of size at least $(1/4 + \eta_{3.10})n$ edges or the coloring is of type $EC_A(\alpha_1, \alpha_1)$ or $EC_B(\alpha_1, \alpha_1)$.*

We will also need the following two lemmas, whose proofs we also postpone.

Lemma 3.11. *For n odd, let $G = K_{(n-1)/2,(n-1)/2,(n-1)/2,(n-1)/2,1}$, let u be its only vertex of degree $2n - 2$ and let $H = G \setminus \{u\}$. There exists $\alpha_{3.11} > 0$ such that, for all $\alpha \leq \alpha_{3.11}$ and $\delta \leq \alpha$, there is a positive integer $n_{3.11} = n_{3.11}(\alpha, \delta)$ with the following property: for every odd $n \geq n_{3.11}$, every 2-coloring of G, such that the induced coloring in H is of type $EC_A(\alpha, \delta)$, contains a monochromatic C_n.*

Lemma 3.12. *For n odd, let $G = K_{(n-1)/2,(n-1)/2,(n-1)/2,(n-1)/2,1}$, let u be its only vertex of degree $2n - 2$ and let $H = G \setminus \{u\}$. There exists $\alpha_{3.12} > 0$ such that, for all $\alpha \leq \alpha_{3.12}$ and $\delta \leq \alpha$, there is a positive integer $n_{3.12} = n_{3.12}(\alpha, \delta)$ with the following property: for every odd $n \geq n_{3.12}$, every 2-coloring of G, such that the induced coloring in H is of type $EC_B(\alpha, \delta)$, contains a monochromatic C_n.*

We restate Theorem 3.2 for easy reference. Afterward we give a concise sketch of its proof, which is then immediately followed by the full proof.

Theorem 3.2. *There exists $n_{3.2}$ such that, for any odd integer $n \geq n_{3.2}$, in any 2-coloring of the edges of the complete 5-partite graph $K_{(n-1)/2,(n-1)/2,(n-1)/2,(n-1)/2,1}$ there is a monochromatic C_n.*

We consider a 2-coloring of the graph $G = K_{(n-1)/2,(n-1)/2,(n-1)/2,(n-1)/2,1}$, say (G^r, G^g), where n is odd and $n > n_0$. Let u be the (only) vertex of G of degree $2n-2$. We apply the Regularity Lemma (Lemma 2.5) with carefully chosen ε (see equation (3.1) below) to the graphs $G^r \setminus \{u\}, G^g \setminus \{u\}$ $(s = 2)$ and obtain a partition $V_0 \cup V_1 \cup \cdots \cup V_t$ of $V(G) \setminus \{u\}$ satisfying conditions (a)-(c) in Lemma 2.5. Using this partition we define a reduced graph R, as well as an appropriate 2-multi-coloring of its edges: the vertex set of R is $\{1, \ldots, t\}$, we have an edge between i and j if and only if (V_i, V_j) has positive density and is an ε-regular pair with respect to G^r and G^g, and an edge ij is colored red (resp. green) if $G^r[V_i, V_j]$ (resp. $G^g[V_i, V_j]$) has edge density at least $\varepsilon^{1/3}$.

By Remark 2.1, we can assume that the reduced graph R is 4-partite. Then, we apply Theorem 3.10 to R, which will lead us to one of three cases: either R has a monochromatic connected odd matching of a certain size or its 2-multi-coloring is of type EC_A or of type EC_B. In the first case, we use Lemma 2.11, the embedding lemma, to find a C_n in G as the same color of the matching. In the other two cases, we prove that the original coloring of G must be of the same type as the one of R. In this case, we apply Lemma 3.11 or Lemma 3.12 to G to find a monochromatic C_n.

Proof of Theorem 3.2. We start by choosing some parameters.

Let $\alpha_1 = \min\{(\alpha_{3.11}/10)^2, (\alpha_{3.12}/10)^2, 1/20\}$ so that, in particular, we can input $\delta = \alpha = 10\sqrt{\alpha_1}$ to Lemmas 3.11 and 3.12 and get $n_{3.11} = n_{3.11}(10\sqrt{\alpha_1}, 10\sqrt{\alpha_1})$ and $n_{3.12} = n_{3.12}(10\sqrt{\alpha_1}, 10\sqrt{\alpha_1})$. Passing α_1 to Theorem 3.10, we obtain $\eta_{3.10} = \eta_{3.10}(\alpha_1)$ and $\beta_{3.10} = \beta_{3.10}(\alpha_1)$.

Let $\eta = \eta_{3.10}/2$. Now Lemma 2.11 give us $c_{2.11}(\eta)$ and we can finally define ε as follows:

$$\varepsilon = \frac{1}{2} \min\left\{(\beta_{3.10}/2)^2, 1/10^6, \frac{\alpha_1^3}{1000}, \frac{\eta_{3.10}^2}{2000}, c_{2.11}^{3/2}\right\}. \tag{3.1}$$

Let $\beta = 2\sqrt{\varepsilon}$ and notice that $\beta < \beta_{3.10}$. With this β, Theorem 3.10 yields $n_{3.10} = n_{3.10}(\beta, \eta_{3.10})$. We also set $m = \max\{2n_{3.10}, 1/\varepsilon\}$ and from Lemma 2.5 we obtain $N_{2.5} = N_{2.5}(\varepsilon, 2, m)$ and $M_{2.5} = M_{2.5}(\varepsilon, 2, m)$. Because $\varepsilon/\varepsilon^{1/3} \leq c_{2.11}$, it is legal to apply Lemma 2.11 to get $n_{2.11} = n_{2.11}(\eta, \varepsilon^{1/3}, \varepsilon, M_{2.5})$. Then we may finally choose

$$n_{3.2} = \max\left\{N_{2.5},\ 2M_{2.5}n_{2.11},\ n_{3.11},\ n_{3.12},\ \frac{2}{\eta}\right\}. \tag{3.2}$$

Consider any 2-coloring (G^r, G^g) of $G = K_{(n-1)/2,(n-1)/2,(n-1)/2,(n-1)/2,1}$ with n odd and $n > n_{3.2}$. We denote $V(G) = U_1 \cup U_2 \cup U_3 \cup U_4 \cup \{u\}$, where U_1, U_2, U_3, U_4 are the independent sets of order $(n-1)/2$ and u is the (only) vertex of degree $2n-2$. We apply the Regularity Lemma (Lemma 2.5) to the pair of graphs $G^r \setminus \{u\}$ and $G^g \setminus \{u\}$, with parameters ε and m chosen as above (and $s = 2$).

Let $V = V(G) = V_0 \cup V_1 \cup \ldots \cup V_t$ be the partition guaranteed by this lemma, thus satisfying

(a) $m \leq t \leq M_{2.5}$,

(b) $|V_0| \leq \varepsilon(2n-2)$, $|V_1| = \ldots = |V_t|$, and

(c) all but at most $\varepsilon\binom{t}{2}$ pairs (V_i, V_j), $1 \leq i < j \leq t$, are ε-regular with respect to both G^r and G^g.

By Remark 2.1, we can assume that each of these clusters (V_k) lies inside one of the sets U_i, $1 \leq i \leq 4$.

Now we define a *reduced graph* $R = R(0, \varepsilon)$ in the following way: the vertex set of R is $\{1, \ldots, t\}$ and we have an edge between vertices i and j if and only if V_i and V_j are contained in different sets of the partition $\{U_1, U_2, U_3, U_4\}$ and (V_i, V_j) is an ε-regular pair with respect to both G^r and G^g. By definition, R is a 4-partite graph, say $V(R) = W_1 \cup W_2 \cup W_3 \cup W_4$, with $W_i = \{k : V_k \subset U_i, 1 \leq k \leq t\}$. It easy to see that all sets W_i have approximately the same order. More precisely, if we denote $t_i = |W_i|$, then $t_i \geq (1/4 - \varepsilon)t$, for $1 \leq i \leq 4$. In fact, for any $1 \leq i \leq 4$ and for arbitrary $k \neq 0$, the above property (b) implies that

$$t_i \frac{2n-2}{t} \geq t_i |V_k| = |U_i| - |U_i \cap V_0| \geq \left(\frac{1}{4} - \varepsilon\right)(2n-2),$$

and the previous statement follows.

We also define a 2-multi-coloring (R^r, R^g) of R as follows: for $c \in \{r, g\}$, and $ij \in E(R)$ we put ij into H^c if $e_c(V_i, V_j) \geq \varepsilon^{1/3}|V_i||V_j|$. Note that, whenever $ij \in E(R)$, that is, i and j are in different sets of the partition $\{W_1, W_2, W_3, W_4\}$, we have that $G[V_i, V_j]$ is a complete bipartite graph. So, at least one of $G^r[V_i, V_j]$ and $G^b[V_i, V_j]$ has density at least $1/2$. Since $1/2 > \varepsilon^{1/3}$, all edges of R receive at least one of the colors.

Remark. *We note that the graph R^r defined above is a reduced graph of G^r with parameters $\varepsilon^{1/3}$ and ε; and R^b is a reduced graph of G^b also with parameters $\varepsilon^{1/3}$ and ε. One could start by defining R^r and R^b directly in an attempt to shorten the proof and skip the definition of R. But later in the proof, we will need the fact that $R = R^r \cup R^b$ is an $(1-\varepsilon)$-dense graph.*

It is convenient here to work on graphs with high degree (rather than simply on dense graphs). So, we start by cleaning up R: We throw away the (small) set of vertices that do not have high degree. Let $F = \{v \in V(\overline{R}) : \deg_{\overline{R}}(v) \geq \sqrt{\varepsilon}t\}$ where $\overline{R}$ is the multipartite complement of R. We have $|F|\sqrt{\varepsilon}t \leq 2e(\overline{R}) \leq 2\varepsilon\binom{t}{2}$, where the second inequality follows from property (c) above. Then, $|F| \leq \sqrt{\varepsilon}(t-1) < \sqrt{\varepsilon}t$. We consider the graph H induced by $V(R) \setminus F$ and denote $t' = |V(H)|$ and $W_i' = W_i \setminus F$. Clearly, $t' \geq (1 - \sqrt{\varepsilon})t$.

Therefore,

$$\Delta(\overline{H}) \le \sqrt{\varepsilon}t \le \frac{\sqrt{\varepsilon}}{1-\sqrt{\varepsilon}}t' \le 2\sqrt{\varepsilon}t' = \beta t'$$

and

$$|W_i'| \ge (1/4-\varepsilon)t - \sqrt{\varepsilon}t \ge (1/4-2\sqrt{\varepsilon})t \ge (1/4-2\sqrt{\varepsilon})t' = (1/4-\beta)t'.$$

We also consider the induced coloring (H^r, H^g) of H where $H^r = H \cap R^r$ and $H^b = H \cap R^b$. Because $t' \ge (1-\sqrt{\varepsilon})t \ge (1-\sqrt{\varepsilon})m \ge m/2 \ge n_{3.10}$, by the above conditions on $|W_i'|$ and $\Delta(\overline{H})$ and since $\beta < \beta_{3.10}$, we can apply Theorem 3.10 (with parameters α_1, $\eta_{3.10}$, β) to H so that either we find an odd monochromatic connected matching M of size t_1 at least $(1/4+\eta_{3.10})t'$ or we conclude that the coloring of H is of type $EC_A(\alpha_1,\alpha_1)$ or of type $EC_B(\alpha_1,\alpha_1)$. We analyze each of these three cases now.

Case 1: *There is an odd monochromatic connected matching M of size t_1 in H, $t_1 \ge (1/4+\eta_{3.10})t'$.*

Note that

$$\left(\frac{1}{4}+\eta_{3.10}\right)t' \ge \left(\frac{1}{4}+\eta_{3.10}\right)(1-\sqrt{\varepsilon})t \ge \left(\frac{1}{4}+\frac{\eta_{3.10}}{2}\right)t = \left(\frac{1}{4}+\eta\right)t.$$

Without loss of generality assume that M is red and let a_ib_i, $0 \le i < t_1$, be all the edges of M.

Now, by Lemma 2.11, such an (odd connected) matching in $R^r = R_t^r(\varepsilon^{1/3},\varepsilon)$ implies that we can find in G^r any cycle of length between $4t$ and $(1/2+\eta)(2n-2)$. In particular, we can find a C_n.

Case 2: *(H^r, H^g) is a coloring of type $EC_A(\alpha_1,\alpha_1)$.*

We will show that this implies that $(G^r \setminus \{u\}, G^g \setminus \{u\})$ is of type $EC_A(10\sqrt{\alpha_1}, 10\sqrt{\alpha_1})$. Let A, B, C, D be subsets of $V(H)$ satisfying conditions (a)-(c) of $EC_A(\alpha_1, \alpha_1)$. It is natural to consider the collection $\{f(A), f(B), f(C), f(D)\}$ of subsets of $V(G)$ given by $f(S) = \bigcup_{j \in S} V_j$ for $S \in \{A, B, C, D\}$. Note that

$$
\begin{aligned}
|f(A)| &\geq |A| \frac{(1-\varepsilon)(2n-2)}{t} \\
&\geq (1-\alpha_1)\, \frac{t'}{4} \frac{(1-\varepsilon)(2n-2)}{t} \\
&\geq (1-\alpha_1)\, (1-\sqrt{\varepsilon})(1-\varepsilon) \frac{2n-2}{4} \\
&\geq (1-2\alpha_1) \frac{2n-2}{4}.
\end{aligned}
$$

Similarly, we obtain that $|f(B)|, |f(C)|, |f(D)| \geq (1-2\alpha_1)(2n-2)/4$. Therefore, condition (a) of $EC_A(10\sqrt{\alpha_1}, 10\sqrt{\alpha_1})$ is satisfied with room to spare. Unfortunately, the partition $\{f(A), f(B), f(C), f(D)\}$ might not satisfy conditions (b) and (c) of $EC_A(10\sqrt{\alpha_1}, 10\sqrt{\alpha_1})$. But we shall prove that we can remove a few (bad) vertices from each $f(S)$, $S \in \{A, B, C, D\}$, so that the resulting sets continue to satisfy (a) and also satisfy (b) and (c).

So, we count how many vertices do not have low degree in one of the bipartite graphs $\overline{G^{g^*}}[f(A), f(D)]$, $\overline{G^{g^*}}[f(B), f(C)]$, $\overline{G^{r^*}}[f(A), f(B)]$ or $\overline{G^{r^*}}[f(C), f(D)]$: we say that a vertex *bad* if its induced degree in any of the above graphs is larger than $2\sqrt{\alpha_1}|V(G) \setminus \{u\}| = 2\sqrt{\alpha_1}(2n-2)$. We claim that at most $2\sqrt{\alpha_1}(2n-2)$ vertices of G are bad.

Fix a vertex $i \in V(H)$ and assume without loss of generality that $i \in A$. We bound the number of red edges from V_i to $f(D)$ in the following way. Recalling that $f(D) = \bigcup_{j \in D} V_j$, it is enough to bound $e_r(V_i, V_j)$ for each $j \in D$. When $ij \notin H^{g^*}$, we use the trivial bound $|V_i||V_j|$ for $e_r(V_i, V_j)$, but we note that condition (b) implies that there are at most $\alpha_1 t'$ such j's. However, for $ij \in H^{g^*}$ we can conclude that $ij \notin H^r$, thus, from the definition of H^r, $e_r(V_i, V_j) \leq \varepsilon^{1/3}|V_i||V_j|$. This implies the following.

$$
\begin{aligned}
e_r(V_i, f(D)) &\le \sum_{\substack{j\in D\\ ij\notin H^{g^*}}} |V_i||V_j| + \sum_{\substack{j\in D\\ ij\in H^{g^*}}} \varepsilon^{1/3}|V_i||V_j| \\
&\le \alpha_1 t'|V_i||V_i| + |D|\varepsilon^{1/3}|V_i||V_i| \\
&\le \alpha_1 t|V_i||V_i| + \varepsilon^{1/3} t|V_i||V_i| \\
&\le 2\alpha_1|V_i|(2n-2),
\end{aligned}
$$

where we have used that $|V_i| = |V_j|$ for any $i, j \ge 1$, $t|V_j| \le 2n-2$ and $\varepsilon^{1/3} \le \alpha_1$.

Therefore, at most $\sqrt{\alpha_1}|V_i|$ vertices of V_i can have more than $2\sqrt{\alpha_1}(2n-2)$ red neighbors in $f(D)$. Similarly, at most $\sqrt{\alpha_1}|V_i|$ vertices of V_i can have more than $2\sqrt{\alpha_1}(2n-2)$ green neighbors in $f(B)$. Hence at most $2\sqrt{\alpha_1}|V_i|$ vertices of V_i are bad. Now, if we vary i over all vertices of $V(H)$, we conclude that at most $2\sqrt{\alpha_1}|f(A) \cup f(B) \cup f(C) \cup f(D)| \le 2\sqrt{\alpha_1}(2n-2)$ vertices are bad.

Finally, we define $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ as the sets obtained from $f(A)$, $f(B)$, $f(C)$, $f(D)$ by removing the bad vertices. We have that

$$
|\tilde{A}| \ge |f(A)| - 2\sqrt{\alpha_1}(2n-2) \ge (1 - 10\sqrt{\alpha_1})(2n-2)/4.
$$

The same holds for $|\tilde{B}|$, $|\tilde{C}|$ and $|\tilde{D}|$, that is, condition (a) of $EC_A(10\sqrt{\alpha_1}, 10\sqrt{\alpha_1})$ is satisfied. Clearly, conditions (b) and (c) are satisfied by $\{\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}\}$ as well. So, the original 2-coloring of $G \setminus \{u\}$ is of type $EC_A(10\sqrt{\alpha_1}, 10\sqrt{\alpha_1})$.

Now, as $10\sqrt{\alpha_1} \le \alpha_{3.11}$ and $n > n_{3.11}(10\sqrt{\alpha_1}, 10\sqrt{\alpha_1})$, we can use Lemma 3.11 to conclude that there is a monochromatic C_n in G.

Case 3: *(H^r, H^g) is a coloring of type $EC_B(\alpha_1, \alpha_1)$.*

Similarly to the previous case, we can show that the coloring $(G^r \setminus \{u\}, G^g \setminus \{u\})$ of $V(G) \setminus \{u\}$ is also of type $EC_B(10\sqrt{\alpha_1}, 10\sqrt{\alpha_1})$. We omit some of the technical details here, but we still give a sketch of the argument to prove this. Let X, Y be subsets of $V(H)$ satisfying the conditions (a)-(d) of $EC_B(\alpha_1, \alpha_1)$ when we consider $X_i = X \cap W_i$ and $Y_i = Y \cap W_i$.

As in the previous case, we consider the collection $\{f(X), f(Y)\}$ of subsets of $V(G)$, where we denote $f(S) = \bigcup_{j \in S} V_j$ for any $S \subset V(H)$. We also observe that $f(X_i) = f(X) \cap U_i$. Much as before, we have that

$$|f(X)|, |f(Y)| \geq (1 - 2\alpha_1)\frac{2n-2}{2}.$$

Therefore, condition (a) of $EC_B(10\sqrt{\alpha_1}, 10\sqrt{\alpha_1})$ is satisfied with room to spare. Similarly to Case 2, conditions (b)-(d) may not be satisfied by $f(X), f(Y)$. But, again, we can give an upper bound for the number of vertices that do not have low degree in one of the bipartite graphs: $\overline{G^{r^*}}[f(X_i), \bigcup_{j \neq i} f(Y_j)]$, $\overline{G^{r^*}}[f(Y_i), \bigcup_{j \neq i} f(X_j)]$, for $1 \leq i \leq 4$, $\overline{G^{g^*}}[f(X_1), f(X_2), f(X_3), f(X_4)]$ and $\overline{G^{g^*}}[f(Y_1), f(Y_2), f(Y_3), f(Y_4)]$. We call a vertex *bad* if its induced degree in any of the above graphs is larger than $2\sqrt{\alpha_1}|V(G) \setminus \{u\}| = 2\sqrt{\alpha_1}(2n-2)$. The same argument from Case 2 shows that there are at most $2\sqrt{\alpha_1}(2n-2)$ bad vertices. By removing the bad vertices from $f(X)$ and $f(Y)$, we obtain sets which satisfy all the conditions of $EC_B(10\sqrt{\alpha_1}, 10\sqrt{\alpha_1})$. Therefore, the original 2-coloring of $G \setminus \{u\}$ is of type $EC_B(10\sqrt{\alpha_1}, 10\sqrt{\alpha_1})$.

Finally, since $10\sqrt{\alpha_1} \leq \alpha_{3.12}$ and $n > n_{3.12}(10\sqrt{\alpha_1}, 10\sqrt{\alpha_1})$, we can use Lemma 3.12 to conclude that there is a monochromatic C_n in G. $\square$

3.3 Proof of Theorem 3.10

We will need the following two easy lemmas which are variants of lemmas by Gyárfás, Sárközy and Schelp [24]. The first lemma is rather trivial but since it is used so many times we rather state it formally and prove it.

Lemma 3.13. *Let H be a bipartite graph with part A and B so that every vertex in one part is not adjacent to at most m vertices in the other part. If $2m < |A| \leq |B|$, then H is a connected and contains a matching of size at least $|A| - m$.*

Proof. Two vertices in A (resp. B) have a common neighbor in B (resp. A). Also, if $a \in A$, $b \in B$ then b and any neighbor of a have a common neighbor in A. Thus H is a connected subgraph. Moreover any maximum matching M misses fewer than m vertices of A, otherwise we could select any unmatched vertex of B and such vertex would need to have a neighbor among the (at least) $m + 1$ unmatched vertices of A. $\square$

Lemma 3.14. *Assume that G is an r-partite graph with N vertices such that $r \geq 2$, and $\Delta(\overline{G}) < m$. Suppose that the largest class in the partition of $V(G)$ has at most as many vertices as the sum of the orders of the others. Then G has a matching covering all but at most rm vertices.*

Proof. We prove the lemma by induction on the order of the graph G. If $|G| \leq rm$, there is nothing to do, since an empty matching suffices. Let $V(G) = V_1 \cup \ldots V_r$ where $|G| > rm$ and assume that $|V_1| \leq \ldots \leq |V_r|$ where $|V_r| \leq |V_1 \cup \ldots \cup V_{r-1}|$. Clearly, $|V_r| > m$ and therefore $|V_1 \cup \ldots \cup V_{r-1}| > m$. In particular $V_{r-1} \neq \emptyset$. Then we can find an edge xy from V_{r-1} to V_r.

The hypothesis that the largest partite class is at most as large as the sum of the others still holds on the graph $G' = G \setminus \{u, v\}$, though the relative order for the size of the sets $V_i' = V_i \setminus \{u, v\}$ might change. Now, G' is r'-partite, with $r' \leq r$ and, by induction, we can find a matching M' that covers all but $r'm \leq rm$ vertices of G'. Finally, $M = M' \cup \{xy\}$ is the matching that we are looking for. $\square$

Remark. *With just a little more care, one can prove that there is a matching that covers all but at most $2m$ vertices of G. But here, we will only use the lemma with $r = 4$ and omit unnecessary details.*

Corollary 3.15. *Let G be an r-partite graph with N vertices, say with vertex partition $V(G) = V_1 \cup \ldots \cup V_r$, with $r \geq 2$. Assume that V_r is its largest class and let $k = \max\{|V_r| - \sum_{i=1}^{r-1} |V_i|, 0\}$. Suppose that $\Delta(\overline{G}) < m$. Then we can find a matching covering all but at most $k + rm$ vertices.*

Proof. Simply remove any k vertices from V_r and use the previous lemma in the resulting graph. □

Now we are ready to prove Theorem 3.10.

Proof of Theorem 3.10. Let $\alpha_1 > 0$ be given. We define two extra parameters by $\alpha_0 = \mu_0 = 1/20$ that will eventually be used as input to Theorem 3.6 which, in turn, outputs $\eta_{3.6} = \eta_{3.6}(\alpha_0, \mu_0)$, $\beta_{3.6} = \beta_{3.6}(\alpha_0, \mu_0)$ and $\mu_{3.6} = \mu_{3.6}(\alpha_0, \mu_0) < \mu_0 = 1/20$. We also define $\eta_{3.10} = \min\{\eta_{3.6}/5, \alpha_1/10\}$ and

$$\beta = \beta_{3.10} = \min\left\{\beta_{3.6}/4, 10^{-5}, \eta_{3.10}/10\right\}.$$

By Theorem 3.6 there exists a constant $n_{3.6} = n_{3.6}(2\beta, \mu_{3.6}, \eta_{3.6}, \alpha_0)$. Finally, define

$$n_{3.10} = \max\{n_{3.6}, (2\beta)^{-1}\}.$$

Suppose we are given a 4-partite graph G of order n, with $n > n_{3.10}$, and a partition $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$ into independent sets that satisfies the conditions in the statement of the lemma, i.e., $|V_i| \geq (1/4 - \beta)n$ $(1 \leq i \leq 4)$ and $\Delta(\overline{G}) \leq \beta n$. Take any 2-multi-coloring of its edges, say with red and green.

Now we consider the graph K obtained from G by adding all edges inside the sets V_i. We color those new edges exclusively with blue and let all other edges of K keep the same colors they have in G. Notice that now we have a 3-multi-coloring of an *almost* complete graph on n vertices. In particular,

$$\Delta(\overline{K}) \leq \beta n$$

implies that K is a $(1-2\beta)$-dense graph. As $n \geq n_{3.6}$ and $2\beta < \beta_{3.6}$, we can apply Theorem 3.6 to K in order to find either a monochromatic matching of size at least $(1/4+\eta_{3.6})n \geq (1/4+5\eta_{3.10})n$ (edges), or an $EC_1(\alpha_0,\alpha_0)$, or an $EC_2(\alpha_0,\alpha_0)$, or an $EC_3(\mu_{3.6}, 0.7, 0.2, (2\beta)^{1/3})$.

Note, however, that our coloring of K is not of any of these types. In fact, first note that all color classes defined by these three types of colorings contain a monochromatic bipartite subgraph where each set in the bipartition has order at least $(1-\max\{\alpha_0, 0.7\mu_{3.6}\})n/4 > n/5$ which are $(1-\max\{\alpha_0,(2\beta)^{1/3}\})$-dense. In particular, those bipartite graphs are at least 19/20-dense. However, the graph induced by the blue edges in K does not have this property, beeing a union of four cliques of order close to $n/4$ with no edges connecting them. Therefore, there must exist a monochromatic connected matching M of size at least $(1/4+5\eta_{3.10})n$.

Since there exists no blue edge from V_i to V_j, where $i \neq j$, every blue connected component has order at most $(1/4+3\beta)n$. As $\beta < \eta_{3.10}$ and M is connected, M cannot be blue. Therefore, M is a monochromatic connected matching in the original coloring of G. Assume, without loss of generality, that M is red. From this point on, we will return to work on the original multipartite graph G, i.e., we will ignore the blue edges. Let C be the (maximal) connected component of G^r containing M. Recall that this means that all edges of C are colored red but that they are not necessarily exclusively red. If C is non-bipartite we are done. Therefore, we can assume C is bipartite.

Let $V(C) = X \cup Y$ be an arbitrary bipartition of C and let $Z = V \setminus C$. From the definition of C and the choice of X and Y, no edge inside X, inside Y or from Z to

$X \cup Y$ is colored red. Therefore, these edges are exclusively colored green. Note that $e(M) \geq (1/4 + 5\eta_{3.10})n$ implies

$$|Z| \leq \left(\frac{1}{2} - 10\eta_{3.10}\right) n.$$

For $1 \leq i \leq 4$, denote $X_i = V_i \cap X$, $Y_i = V_i \cap Y$ and $Z_i = V_i \cap Z$. Since $|X| \geq e(M) \geq (1/4 + 5\eta_{3.10})n$ and $|X_i| \leq |V_i| \leq (1/4 + 3\beta)n \leq (1/4 + 3\eta_{3.10})n$, at least two of the sets $X_i's$ are larger than $2\eta_{3.10}n > 2\beta n$. By Lemma 3.13, these two X_i's induce a (green) connected graph. Also, all other vertices in X and in Z have at least one neighbor in the union of those two sets. Therefore, $G^g[X \cup Z]$ is connected. Similarly, $G^g[Y \cup Z]$ is connected. So if $Z \neq \emptyset$, then $G^g[X \cup Y \cup Z]$ is connected. In the next cases, we will prove that this (green) component is odd and has a large matching, unless many of the sets X_i, Y_i, Z_i are very small, in which case we will prove that the coloring has the desired structure.

Case 1: $|Z| > \eta_{3.10}n$.

We claim that we can find a large enough odd connected green matching. Because $Z \neq \emptyset$, we have that $G^g[X \cup Y \cup Z]$ is connected. To verify that $G^g[X \cup Y \cup Z]$ is not bipartite, we can easily check that it contains a triangle. In fact, we can assume, without loss of generality, $|Z_1| > \eta_{3.10}n/4$, which implies $|Z_1| > 2\beta n$. Look at the orders of the sets X_i and Y_j. If there is any edge uv in $G^g[X_2, X_3, X_4]$, since $\Delta(\bar(G)) \leq \beta n$, we can find a common neighbor of u and v in Z_1 and we are done. But we already know that at least one of X_2, X_3, X_4, say X_2, is larger than $2\beta n$. If either X_3 or X_4 is nonempty, we can find an edge in $G^g[X_2, X_3 \cup X_4]$ and we are done. Therefore we can assume that X_3 and X_4 are empty. Similarly, either we have a triangle or two of the sets Y_2, Y_3, Y_4 are empty, which means that at least one of Y_3 or Y_4 is empty. Call it Y_i ($i = 3$ or 4). Notice now that $Z_i = U_i$ and, in particular, $|Z_i| \geq 2\beta n$ and we can find a triangle in $G^g[X_1, X_2, Z_i]$.

Now, we only need to find a large matching in the green component. The basic idea is to use Hall's Theorem to find a matching M_1 in $G[Z, X \cup Y]$ that covers the all vertices in Z and afterward use Corollary 3.15 to prove that there are large matchings M_2 in $V(X) \setminus V(M_1)$ and M_3 in $V(Y) \setminus V(M_1)$. But in order to use Corollary 3.15 effectively, we want the difference between the largest part in $V(X) \setminus V(M_1)$ and the sum of the others to be small. So, the matching M_1 needs to be chosen with some care.

We select a set $L \subset X \cup Y$ that shall be avoided by M_1. Let L be a subset of $X \cup Y$ of order $4\lfloor 2\eta_{3.10} n \rfloor$ containing $\lfloor 2\eta_{3.10} n \rfloor$ vertices from each of two different X_i's and two different Y_i's, and otherwise arbitrary.

We check that Hall's condition works to find a matching M_1, among the (green) edges from Z to $(X \cup Y) \setminus L$, that covers all vertices of Z. In fact, a single vertex in Z, say $z \in Z_1$, has degree at least $|(X \cup Y) \setminus L| - |X_1 \cup Y_1| - \beta n >$ $2(1/4 + 5\eta_{3.10})n - (8\eta_{3.10} n) - (1/4 + 3\beta)n - \beta n > (1/4 + \eta_{3.10})n$. Then, for any $S \subset Z$, denoting by $N(S)$ the set of neighbors of S in $(X \cup Y) \setminus L$, we have: if $|S| < (1/4 + \eta_{3.10})n$ then $|N(S)| \geq |S|$; and if $|S| \geq (1/4 + \eta_{3.10})n$ then S intersects at least two of the sets Z_i's, in which case we have

$$\begin{aligned} |N(S)| &\geq |(X \cup Y) \setminus L| - 2\beta n \\ &> 2(1/4 + 5\eta_{3.10})n - (8\eta_{3.10} n) - 2\beta n > (1/2 + \eta_{3.10})n > |Z| \geq |S|. \end{aligned}$$

Therefore, there exists a green matching M_1 that covers all vertices of Z. Denote $X' = X \setminus V(M_1)$, $X_i' = X_i \setminus V(M_1)$ and assume, without loss of generality, that X_1' is the largest among X_1', X_2', X_3' and X_4'. Let

$$k = \max\{|X_1'| - (|X_2'| + |X_3'| + |X_4'|), 0\}.$$

Since $|X_1'| \le |V_1| \le (1/4 + 3\beta)n$ and because at least one of the sets X_2', X_3', X_4' contains $\lfloor 2\eta_{3.10} n \rfloor$ vertices from L, we have $k \le (1/4 + 3\beta - \lfloor 2\eta_{3.10} \rfloor)n$. By Corollary 3.15, applied to $G^g[X_1', X_2', X_3', X_4']$ with $m = \beta n$ and $r = 4$, there is a matching M_2 that covers all vertices in X' except for at most

$$k + 4\beta n \le (1/4 + 7\beta - \lfloor 2\eta_{3.10} \rfloor)n$$

vertices. The analogous statement holds replacing X_i' by Y_i'.

The conclusion is that $M_1 \cup M_2 \cup M_3$ leaves uncovered at most

$$2(1/4 + 7\beta - \lfloor 2\eta_{3.10} \rfloor)n$$

vertices. Therefore,

$$|V(M_1) \cup V(M_2) \cup V(M_3)| \ge |V(G)| - (1/2 - 2\lfloor 2\eta_{3.10} \rfloor + 14\beta)n \ge (1/2 + 2\eta_{3.10})n,$$

as desired.

Case 2: $|Z| \le \eta_{3.10} n$.

We claim that if $|X| > (1/2 + 2\eta_{3.10})n$, we can find a large monochromatic odd connected (green) matching in $G^g[X]$. In fact, if $|X| > (1/2 + 2\eta_{3.10})n$, then at least three of the sets X_i's are larger than $\eta_{3.10} n > 2\beta n$. Therefore $G^g[X]$ contains a triangle and, in particular, is not bipartite. Also remember that $G^g[X]$ is connected. Finally, we check that Lemma 3.14 gives us a large matching inside X: since $|X_i| < (1/4 + 3\beta)n < |X|/2$, for $1 \le i \le 4$, no X_i can be larger than the sum of the others, so we apply the lemma and conclude that there exists a matching of order at least $|X| - 4\beta n > (1/2 + \eta_{3.10})n$, i.e., the orders of X and Y are close to each other.

Now, we can assume that $|X|, |Y| \le (1/2 + 2\eta_{3.10})n$. Since $Z \le \eta_{3.10} n$, we have $|X|, |Y| \ge (1/2 - 3\eta_{3.10})n = (1 - 6\eta_{3.10})n/2$. If there is no green edge from X to Y, then we have an $EC_B(6\eta_{3.10}, \beta)$ which in particular is an $EC_B(\alpha_1, \alpha_1)$. Now, assume that there is a green edge uv from X to Y. Since $G^g[X]$ and $G^g[Y]$ are connected, we conclude that $G^g[X \cup Y]$ is connected. Using Corollary 3.15 twice, we can find large green matchings inside each of X and Y. In fact, as $|X| > (1/2 - 3\eta_{3.10})n$ and $\max\{|X_i|\} \leq (1/4 + 3\beta)n$, the difference between the largest $|X_i|$ and the sum of the others is at most $3\eta_{3.10} + 6\beta$. This implies that there is a matching in $G^g[X]$ that misses at most $((3\eta_{3.10} + 6\beta) + 4\beta)n$ vertices of X. Similarly, there is a matching in $G^g[Y]$ that misses at most $(3\eta_{3.10} + 10\beta)n$ vertices of Y. The union of those matchings is a (very) large green connected matching M: it covers almost all vertices of G and we only need to cover $(1/2 + 2\eta_{3.10})n$ vertices.

If either X or Y has at least three non-empty parts, then we can find a triangle, as in the beginning of the previous case, in which case M is an odd matching and we are done. Otherwise, at least two of X_i's and two of Y_i's are empty. We can assume, without loss of generality, that the sets X_3 and X_4 are empty. This implies that $|X_1|, |X_2| \geq ((1/4 - \beta) - \eta_{3.10})n \geq (1 - 5\eta_{3.10})n/4$. Therefore, $|Y_1|, |Y_2| \leq 5\eta_{3.10}n$ and, as $|Y| \geq (1/2 + 2\eta_{3.10})n$ and $|Y_i| \leq n/4$ for all i, we have that $|Y_3|$ and $|Y_4|$ are non-empty. It follows that Y_1 and Y_2 must be empty, which implies $|Y_3|, |Y_4| \geq (1 - 5\eta_{3.10})n/4$.

We are getting closer to prove that the coloring of G must be an $EC_A(5\eta_{3.10}, \beta)$. In fact, we already know that there is no red edge in $G[X_1, X_2]$ or $G[Y_3, Y_4]$. We can assume, without loss of generality, that the green edge uv from X to Y is such that $u \in X_1$ and $v \in Y_3$. If there is any green edge in $G[X_1, Y_4]$ we can greedily construct an odd green cycle, in which case M will be odd. Therefore we can assume that there is no green edge in $G[X_1, Y_4]$. Similarly, we can assume that there is no green edge in $G[X_2, Y_3]$. Then, we conclude that our coloring is of type $EC_A(5\eta_{3.10}, \beta)$ which in particular is an $EC_A(\alpha_1, \alpha_1)$. $\square$

3.4 Paths and cycles in bipartite graphs and in the extremal colorings

The aim of this section is to prove Lemmas 3.11 and 3.12. To this end, we will need the following fact which appears as Theorem 15 of Chapter 10 of Berge [10].

Lemma 3.16. *Let $G = (A, B)$ be a bipartite graph with $|A| = |B| = n \geq 2$, $\delta(G) \geq 2$ such that for each j, $2 \leq j \leq \frac{n+1}{2}$, in each of the sets A, B, the number of vertices of degree at most j is smaller than $j - 1$. Then G is Hamilton-connected, i.e., each pair of vertices v, w with $v \in A$ and $w \in B$ can be connected by a Hamiltonian path.*

The next easy lemma, originally from [5] (in Portuguese), state that we can find long paths in bipartite graph with large minimum degree. The idea of the proof is to build such paths in a greedy fashion. We give a full proof here for easy reference.

Lemma 3.17. *Let H be a bipartite graph with bipartition $X \cup Y$, $|X|, |Y| \geq 4$, and let p and q be integers such that $0 \leq p < |X|/3$ and $0 \leq q < |Y|/3$. Assume that for every $x \in X$, $\deg(x, Y) \geq |Y| - q$ and for every $y \in Y$, $\deg(y, X) \geq |X| - p$. Then*

(a) for any two vertices $x, x' \in X$ there exists an (x, x')-path of length $2k - 2$ for every k, $2 \leq k \leq \min\{|X|, |Y| - 2q\}$; the analogous statement, obtained by exchanging the two vertex classes, also holds;

(b) for any two vertices $x \in X$, $y \in Y$ there exists an (x, y)-path of length $2k - 1$ for every k odd, $2 \leq k \leq \min\{|X| - 2p, |Y| - 2q\}$.

Proof. In order to prove (a), we first select k distinct vertices $x_1, \ldots, x_k \in X$ (recall $k \leq |X|$) such that $x_1 = x$, $x_k = x'$. It is easy to build a path $P_k = x_1y_1x_2y_2 \ldots y_{k-1}x_k$, with $y_i \in Y$ for all i, $1 \leq i \leq k - 1$. Assuming that for a given ℓ, $1 \leq \ell \leq k - 1$, we have built $P_\ell = x_1y_1 \ldots y_{\ell-1}x_\ell$, let y_ℓ be any vertex in the common neighborhood of x_ℓ

and $x_{\ell+1}$ which is not in $V(P_\ell)$. Then set $P_{\ell+1} = P_\ell y_\ell x_{\ell+1}$. Such a vertex exists as

$$|(N(x_{\ell-1}) \cap N(x_\ell)) \setminus V(P_l)| \geq (|Y| - 2q) - (l-1) \geq 2 > 1,$$

since

$$l \leq k - 1 \leq |Y| - 2q - 1.$$

The proof of (b) is similar: first take a neighbor x' of y such that $x' \neq x$, and then apply the previous construction to find a path of length $2k$ from x to x', while making sure that this path also avoids y. □

Lemma 3.18. *Let $r \geq 3$ and let G be an r-partite graph of order $n \geq 3$, with parts V_i such that $|V_i| \leq \lfloor n/2 \rfloor$, $1 \leq i \leq r$. Assume that each V_i is partitioned into $X_i \cup W_i$ where $|\bigcup_{i=1}^r W_i| < n/(2r)$ and that for every $i \neq j$ the graphs $G[X_i, X_j]$ and $G[X_i, W_j]$ are complete. Then G has a Hamiltonian cycle.*

Proof. In this proof, contrary to our standard notation, we write P_k for a path with $2k$ vertices. We also set $V_i^k = V_i \setminus V(P_k)$, $X_i^k = X_i \setminus V(P_k)$, $W_i^k = W_i \setminus V(P_k)$, $V^k = \bigcup_{i=1}^r V_i^k$, $W^k = \bigcup_{i=1}^r W_i^k$ and $n_k = |V^k| = n - 2k$.

We say that a path P_k in G is *good* if it is such that $|V_i^k| \leq \lfloor n_k/2 \rfloor$ for every $1 \leq i \leq r$ and that either $|W^k| \leq 1$ or $|W^k| < n_k/r$ whenever k is odd and $|W^k| < n_k/(2r)$ whenever k is even. We prove by induction on k that, for $k \leq \lfloor (n-2)/2 \rfloor$, there exists a good path P_k.

For $k = 1$, we let $P_k = x_1 y_1$, where x_1 is a vertex belonging to a largest class V_i and y_1 a vertex belonging to the second largest class. One can easily check that this is a good path. Now, assume that $P_k = x_k x_{k-1} \ldots x_1 y_1 \ldots y_{k-1} y_k$ is a good path for some $k \leq \lfloor (n-2)/2 \rfloor - 1$.

We claim that we can extend P_k to a good path P_{k+1} by adding a new neighbor to each endpoint of P_k. Let i_k be such that $|V_{i_k}^k|$ is maximum among $|V_1^k|, \ldots, |V_r^k|$.

Select two vertices u, v such that $u \in V^k_{i_k}$, $v \in V^k \setminus V^k_{i_k}$, u is adjacent to one of x_k, y_k and v is adjacent to the other. Notice that $|W^k| < n_k/r$ implies that $X^k_{i_k} = V^k_{i_k} \setminus W^k$ and $X^k \setminus X^k_{i_k}$ are nonempty, therefore we have no trouble with the existence of u and v (even if $x_k, y_k \in W^k$). But we require extra care while choosing v. In the case where $|V^k_{i_k}| = (n_k - 1)/2$, two things can happen: either all other classes V^k_i have order strictly less than $(n_k - 1)/2$ or there are only three nonempty classes, two of order $(n_k - 1)/2$ and one of order 1. In the latter case, we require v to be chosen from the large class not containing u. We also assume that u and v are chosen from W^k whenever this is possible. Finally, we let $\{x_{k+1}, y_{k+1}\} = \{u, v\}$ and $P_{k+1} = y_{k+1}y_k \dots y_1x_1 \dots x_kx_{k+1}$.

We claim that for the choice of u, v as above the path P_{k+1} is good. The fact that $|V^{k+1}_i| \leq \lfloor n_{k+1}/2 \rfloor$ is straightforward. One also verify that for every i, with $1 \leq i \leq k$, either $|W^i| \leq 1$ or at least one among the vertices $x_i, y_i, x_{i+1}, y_{i+1}$ is chosen from W. In fact, if both x_i, y_i are not in W, then x_{i+1} or y_{i+1} can be chosen from W except in the particular case where there are only three nonempty classes, two of order $(n_i - 1)/2$ and one of order 1 and in which the only vertex of W is that in the class of order 1. If $k+1$ is even, then the facts that $n_{k+1} = n_{k-1} - 4$, one $x_k, y_k, x_{k+1}, y_{k+1}$ is in W and $|W^{k-1}| \leq n_{k-1}/(2r)$ implies that $|W^{k+1}| \leq n_{k-1}/(2r) - 1 \leq n_{k+1}/(2r)$. If $k+1$ is odd, the fact that $|W^k| \leq n_k/(2r)$ implies that $|W^{k+1}| \leq n_{k+1}/r$. Therefore P_{k+1} is good. Next treat the case whether n is even or n is odd separately.

First, we assume that n is odd. Let $k = (n-3)/2$. We conclude that there exists a good path $P_k = y_ky_{k-1} \dots y_1x_1 \dots x_{k-1}x_k$ (of order $2k = n - 3$), such that $P_{k-1} = y_{k-1} \dots y_1x_1 \dots x_{k-1}$ is also good. Let $V^k = V \setminus V(P_k) = \{a, b, c\}$. The fact that P_{k-1} is good implies that at most one of x_k, y_k, a, b, c is in W. And the fact that P_k is good implies that a, b and c belong to different partition classes. Therefore a, b, c are adjacent to each other. Also, two of them, say a, b, are such that a is adjacent to x_k and b is adjacent to y_k. Therefore, we have a Hamiltonian cycle $C_n = cby_{(n-3)/2} \dots y_1x_1 \dots x_{(n-3)/2}ac$.

Finally, assume that n is even. Let $k = (n-2)/2$. As in the previous case, we consider a good path denoted by $P_k = y_k y_{k-1} \dots y_1 x_1 \dots x_{k-1} x_k$ (of order $2k = n-2$), and so that P_{k-1} is also good and we let $V^k = V \setminus V(P_k) = \{a, b\}$. Using that P_k and P_{k-1} are good we conclude that at most one among x_k, y_y, a, b is in W and that a and b are in different partition classes. Therefore, we have a Hamiltonian cycle $C_n = by_{(n-2)/2} \dots y_1 x_1 \dots x_{(n-2/2} ab$. □

We are ready to prove Lemma 3.11, which we shall restate for easy reference.

Lemma 3.11. *For n odd, let $G = K_{(n-1)/2,(n-1)/2,(n-1)/2,(n-1)/2,1}$, let u be the only vertex of degree $2n-2$ and let $H = G \setminus \{u\}$. There exists $\alpha_{3.11} > 0$ such that, for all $\alpha \leq \alpha_{3.11}$ and $\delta \leq \alpha$, there is a positive integer $n_{3.11}$ with the following property: for every odd $n \geq n_{3.11}$, every 2-coloring of G such that the induced coloring in H is of type $EC_A(\alpha, \delta)$ contains a monochromatic C_n.*

Proof. We set

$$\alpha_{3.11} = 10^{-4}$$

and consider any $\alpha \leq \alpha_{3.11}$. Note that, for every $\delta \leq \alpha$, any coloring of type $EC_A(\alpha, \delta)$ is also of type $EC_A(\alpha, \alpha)$, hence, we may assume that $\delta = \alpha$. Take

$$n_{3.11} = \lfloor \alpha^{-4} \rfloor .$$

Select n odd, with $n \geq n_{3.11}$. We let $V(G) = U_1 \cup U_2 \cup U_3 \cup U_4 \cup \{u\}$, where U_1, U_2, U_3, U_4 are independent sets of order $(n-1)/2$ and u is the (only) vertex of degree $2n-2$. We also let $H = G \setminus \{u\}$. Consider any 2-coloring of G such that the coloring restricted to H is of type $EC_A(\alpha, \alpha)$. We aim to find a monochromatic C_n in this coloring. Let A, B, C, D be sets satisfying conditions (a), (b) and (c) of $EC_A(\alpha, \alpha)$ and notice that we must have $A \subset U_1$, $B \subset U_2$, $C \subset U_3$, $D \subset U_4$ (without loss of generality on the ordering of the sets U_i). Also, let $Z = V(H) \setminus (A \cup B \cup C \cup D)$.

Now, consider the vertex u with full degree and look at the color of the edges from u to $A \cup B \cup C \cup D$.

Claim 3.19. *If u has* red *neighbors in both A and B, we can find a monochromatic C_n. Similarly, if either u has* red *neighbors in both C and D or* green *neighbors in both B and C or* green *neighbors in both A and D, then we can find find a monochromatic C_n.*

Proof. Suppose that there exist $a \in A$ and $b \in B$ such that ua and ub are red. We show how to find a C_n in this case; the other cases can be dealt with similarly.

We show that if there exists a pair of vertex-disjoint red edges between $A \setminus \{a\}$ and C, say a_1c_1 and a_2c_2, with $a_i \in A \setminus \{a\}$ and $c_i \in C$, $i = 1, 2$, one can find a red C_n. In fact, we can find such a path by applying Lemma 3.17 a few times with $p = q = \alpha 2n$. More precisely, there exists a (b, a_1)-path P in $G^r[A \setminus \{a\}, B]$ of length 3. Also, there is a (c_1, c_2)-path Q in $G^r[C, D]$ of any even length between 2 and $2(\min\{|C|, |D| - 2\alpha(2n)\}) - 2$, and a (a_2, a)-path R in $G^r[A \setminus V(P), B \setminus V(P)]$ for any even length between 2 and $2\min\{|A \setminus V(P)|, |B \setminus V(P)| - 2\alpha(2n)\} - 2$.

Then for any even number k between 4 and

$$2\big(\min\{|A \setminus V(P)|, |B \setminus V(P)| - 2\alpha(2n)\} + \min\{|C|, |D| - 2\alpha(2n)\}\big) - 4, \qquad (3.3)$$

we can choose Q and R so that $e(Q) + e(R) = k$. Clearly, $P \cup Q \cup R \cup \{au, ub, a_1c_1, a_2c_2\}$ is a copy of C_{k+7}. Notice from the above expression that we can take $k = n - 7$ with room to spare. In fact, by condition (a) of $EC_A(\alpha, \delta)$ we have

$$|A \setminus V(P)|, |B \setminus V(P)|, |C|, |D| \geq \frac{(1-\alpha)(n-1)}{2} - 2.$$

Together with the bound (3.3), we have that k can be any even number between 4 and $2\big((1-\alpha)(n-1) - 8\alpha n\big) - 4 = 2n - 18\alpha n - 6 + \alpha$, which is much bigger than $n - 7$.

This means that we can assume that there is no red edge in $E(A \setminus \{a\}, C)$, with the exception of at most one red star. This implies that all red edges in $E(A, C)$ are contained in at most two stars. By the same argument, there are no red edges in $E(B, D)$ with the exception of at most two red stars. So, almost all edges in $E(A \cup B, C \cup D)$ are green.

Again by Lemma 3.17 with $p = q = \alpha(2n) \geq \alpha(2n-2) + 4$, this time applied to $G^g[A \cup B, C \cup D]$, for any $x, y \in A \cup B$, we can find a (x, y)-path of any given even length between 2 and $2(\min\{|A \cup B|, |C \cup D|\} - 2\alpha(2n)) - 2$. We remark that when $x = a$ or when x is the center of a red star, we cannot apply the lemma directly (as a might not satisfy the condition $\deg(a, C \cup D) \geq |C \cup D| - \alpha(2n)$). However, we still can select one of its green neighbors in D, say d, and use the lemma to find a long (d, y)-path. Again, the upper estimate on the order of our path is close to $2n$ and is clearly larger than $n - 1$. Therefore, if there is any green edge xy with $x \in A$ and $y \in B$, we can find a green C_n.

Now, we can assume that *all* edges in $G[A, B]$ are red. Similarly, we can assume that *all* edges in $G[C, D]$ are red. Once more, by applying Lemma 3.17 to $G^g[A \cup B, C \cup D]$, for any $x \in A \cup B$ and $y \in C \cup D$, we can find a (x, y)-path of any odd length up to almost $2n$ and in particular we can find a (x, y)-path of length $n - 2$. Therefore, if there is any vertex in $Z \cup \{u\}$ that has green neighbors in both $A \cup D$ and $B \cup C$ we can find a green C_n. So, we can assume that this does not happen, which means that we can partition the set $Z \cup \{u\}$ into sets S and T such that the vertices in S have only red neighbors in $A \cup B$ and the vertices in T have only red neighbors in $C \cup D$. Since we have $2n - 1$ vertices in total (in G), either $A \cup B \cup S$ or $C \cup D \cup T$ has at least n vertices. Without loss of generality, we can assume that $|A \cup B \cup S| \geq n$. Let W be any subset of S such that $|A \cup B \cup W| = n$.

Notice that now we can apply Lemma 3.18 to find a red C_n in $G[A \cup B \cup W]$ as follows: denote $X_1 = A$, $X_2 = B$, $X_3 = X_4 = X_5 = \emptyset$, $W_i = W \cap U_i$, for $1 \leq i \leq 4$ and

$W_5 = W \cap \{u\}$. Clearly, $|X_i \cup W_i| \subset |U_i| \leq \lfloor n/2 \rfloor$ and $|W| \leq |Z \cup \{u\}| \leq \alpha(2n-2)$, so the conditions of the lemma are satisfied. Therefore, we can find a red C_n. This finishes the proof of the claim. □

Continuing with the proof of Lemma 3.11, select any edge from u to A. From the symmetry of the coloring, we can assume that such an edge is red. Applying Claim 3.19 repeatedly, either we find a C_n, or we can assume that all edges from u to B are green, all edges from u to C are red, all from u to D are green and all from u to A are red.

Consider any edge $xy \in E(A, C)$. Either if xy is red or green we can use an argument similar to the one in proof of Claim 3.19 to find a monochromatic C_n. More precisely, if xy is red take $a \in A$, $c \in C$ with $a \neq x$ and $c \neq y$. So, we have that au and cu are red. We can use Lemma 3.17 to find an even length (a, x)-path P in $G^r[A, B]$ and an even length (c, y)-path Q in $G^r[C, D]$ so that $P \cup Q \cup \{au, uc, xy\}$ is a red C_n. Similarly, if xy is green we consider any $b \in B$ and $d \in D$. So, we have bu and du are green and by Lemma 3.17 we can find odd length (x, d)-path P in $G^g[A, D]$ and an odd length (y, b)-path Q in $G^g[B, C]$ such that $P \cup Q \cup \{xy, bu, ud\}$ is a green C_n.

This completes the proof of Lemma 3.11. □

To finish this section, we give a proof for Lemma 3.12, which we also restate for easy reference.

Lemma 3.12. *For n odd, let $G = K_{(n-1)/2,(n-1)/2,(n-1)/2,(n-1)/2,1}$, let u be its only vertex of degree $2n-2$ and let $H = G \setminus \{u\}$. There exists $\alpha_{3.12} > 0$ such that, for all $\alpha \leq \alpha_{3.12}$ and $\delta \leq \alpha$, there is a positive integer $n_{3.12}$ with the following property: for every odd $n \geq n_{3.12}$, every 2-coloring of G, such that the induced coloring in H is of type $EC_B(\alpha, \delta)$, contains a monochromatic C_n.*

Proof. Similarly to the proof of Lemma 3.11, we set

$$\alpha_{3.12} = 10^{-4}$$

and consider any $\alpha \leq \alpha_{3.12}$. Again, note that for every $\delta \leq \alpha$, any coloring of type $EC_B(\alpha, \delta)$ is also of type $EC_B(\alpha, \alpha)$, hence, we may assume that $\delta = \alpha$. Take

$$n_{3.12} = \lfloor \alpha^{-4} \rfloor .$$

Let n be odd, with $n \geq n_{3.12}$. We let $V(G) = U_1 \cup U_2 \cup U_3 \cup U_4 \cup \{u\}$, where U_1, U_2, U_3, U_4 are independent sets of order $(n-1)/2$ and u is the (only) vertex of degree $2n-2$. We also let $H = G \setminus \{u\}$. Consider any 2-coloring of G such that the coloring restricted to H is of type $EC_B(\alpha, \alpha)$. We aim to find a monochromatic C_n in this coloring.

Let $X \cup Y \cup Z$ be a partition of $V(H)$ where X and Y satisfy conditions (a)-(d) of $EC_B(\alpha, \delta)$. Let $X_i = X \cap U_i$, $Y_i = Y \cap U_i$. In particular, $|X|, |Y| \geq (1-\alpha)(n-1)$ which implies that $|Z| \leq \alpha(2n-2)$.

We claim that if there is any red edge inside X we can find a red C_n. To see that, assume that wx is such an edge. Let y be any red neighbor of x in Y. We claim that we can construct a (w, y)-path P of length $n-2$ in $G^r[X \setminus \{x\}, Y]$. We choose subsets $X' \subset X$ and $Y' \subset Y$ such that:

(a) $w \in X'$, $x \notin X'$, $y \in Y'$,

(b) $|X'| = |Y'| = (n-1)/2$ and

(c) $|X'_i \cup Y'_i| \leq (n+1)/4 + \alpha n$, where $X'_i = X' \cap U_i$ and $Y'_i = Y' \cap U_i$.

This can be done because $(1+\alpha)(n-1) \geq |X|, |Y| \geq (1-\alpha)(n-1)$ and $|X_i \cup Y_i| \leq |U_i| = (n-1)/2$. In fact, for example, one can start taking half of the elements of each set X_i and Y_i (rounded to the closest integer), so that property (c) will be true with some room to spare, and then add or subtract at most $\alpha n/2$ vertices to each X' and Y', so that properties (a) and (b) are satisfied.

Let us check that the graph $G^r[X', Y']$ satisfies the conditions of Lemma 3.16. Let $2 \le j \le (|Y'|+1)/2$ and write $j = (|Y'|+1)/2 - k = (n+1)/4 - k$, for some $0 \le k \le (|Y'|+1)/2 - 2$. Let $R_j = \{v \in X' : \deg(v, Y') \le j\}$. We need to check that $|R_j| < j - 1$.

We claim that for $k > 3\alpha n$ we have $R_j = \emptyset$ and for $k \le 3\alpha n$ we have $|R_j \cap X_i'| \le 3\alpha n - (k-1)$. To see this, assume that $R_j \cap X_i' \ne \emptyset$, for some $1 \le i \le 4$, and let $v \in R_j \cap X_i'$. Since v is adjacent to all but at most $\alpha(2n-2)$ vertices in $\bigcup_{t \ne i} Y_t$, we have that

$$\sum_{t \ne i} |Y_t'| - \alpha(2n-2) \le \deg(v, Y') \le j = \frac{|Y'|+1}{2} - k.$$

Therefore,

$$|Y_i'| = |Y'| - \left(\sum_{t \ne i} |Y_t'|\right) \ge \frac{|Y'|-1}{2} + k - \alpha(2n-2) \ge \frac{n-3}{4} + k - 2\alpha n.$$

This and condition (c) above imply that

$$|R_j \cap X_i'| \le |X_i'| = |X_i' \cup Y_i'| - |Y_i'| \le 3\alpha n - (k-1).$$

In particular, whenever $R_j \ne \emptyset$ we have $k \le 3\alpha n$, proving the claim.

We conclude that $|R_j| \le 12\alpha n - 4(k-1) < \frac{n+1}{4} - k - 1 = j - 1$. Therefore, we can use Lemma 3.16 to find a (red) Hamiltonian (w, y)-path in $G^g[X', Y']$. Appending the edges wx and xy to this path we get a red C_n.

We can assume now that all edges of $G[X]$ are green, i.e., $G[X]$ is a complete green multipartite graph. Similarly, we can assume that all edges in $G[Y]$ are also green. Furthermore, if there is any vertex z in Z such that z has red neighbors x, y with $x \in X$ and $y \in Y$, we can use the same argument as above to find a (x, y)-path P in $G^r[X, Y]$ such that $P \cup \{xz, zy\}$ is a (red) C_n. Finally, if this does not happen, the set $Z \cup \{u\}$ can be partitioned into $S \cup T$ such that all edges from S to X and all edges from T to Y are green. Since the total number of vertices in G is $2n - 1$, we have that either $|X \cup S| \geq n$ or $|Y \cup T| \geq n$. Assume, without loss of generality, that the first inequality holds. Letting W be any subset of S such that $|X \cup W| = n$, one can apply Lemma 3.18 to find a green C_n in $G[X \cup W]$. In fact, the conditions of Lemma 3.18 are satisfied by the sets $V_i = X_i \cup W_i$ where $W_i = W \cap U_i$, for $1 \leq i \leq 4$, $W_5 = W \cap \{u\}$ and $X_5 = \emptyset$. This completes the proof. $\square$

We remark that our main theorem of Chapter 4 (Theorem 4.2) shall generalizes Theorem 3.2 as follows: if the graph $K_{(n-1)/2,(n-1)/2,(n-1)/2,(n-1)/2,1}$ in the statement of Theorem 3.2 is replaced by any graph G on $2n - 1$ vertices and large minimum degree, then any 2-coloring G must still contain a monochromatic C_n. We note, however, that the proofs in Chapter 4 do not rely on any theorem of Chapter 3. Indeed, what we shall do is to prove a more general version of Theorem 3.10 whose proof is self-contained; then we use this more general version to prove Theorem 4.2.

Chapter 4

Ramsey numbers of cycles in graphs with large degree

In a recent article, Li, Nikiforov and Schelp [29] conjectured that the following generalization of Theorem 3.2 holds.

Conjecture 4.1. *Let $N \geq 4$ and let G be a graph of order N and minimum degree bigger than $3N/4$. For any 2-coloring of the edges of G and any k, $4 \leq k \leq \lceil N/2 \rceil$, G contains a monochromatic C_k.*

They proved [29] that, for any $\varepsilon > 0$ and n large enough, the same assumptions imply that we can find a monochromatic C_k for every k between 4 and $\lfloor (1/8 - \varepsilon)N \rfloor$.

Compare Conjecture 4.1 with Theorem 3.2: given a natural number n, letting $G = K_{\frac{n-1}{2},\frac{n-1}{2},\frac{n-1}{2},\frac{n-1}{2},1}$ and $N = 2n - 1$, so that N is the number of vertices of G, we have that all but one vertex of G has degree exactly $\lceil 3N/4 \rceil$. We remark, however, that denoting by u the vertex of G which has full degree, if we remove a few edges incident to u in a way that u has degree $\lceil 3N/4 \rceil$, our proof of Theorem 3.2 shall still work. In fact, one would only need make slight changes in the proofs of Lemmas 3.11 and 3.12 to make them work with this subgraph of G, and those lemmas are the only places where u plays a crucial role. This means that Theorem 3.2 is a tight situation in which the above conjecture holds.

Together with Bollobás and Skokan [7], who independently thought about the same conjecture, we attacked Conjecture 4.1 in the opposite direction that Li, Nikiforov and Schelp did, i.e., we considered the case where $k = \lceil N/2 \rceil$. In this chapter, we give a proof of the following theorem which generalizes Theorem 3.2 and is also sharp.

Theorem 4.2. *There exists an integer n_0 with the following property: If $n > n_0$ is an odd integer and G is a graph on $2n-1$ vertices such that its complement, $\overline{G}$, has maximum degree at most $(n-3)/2$, we have that G arrows C_n.*

Remark. *We note that in Theorem 4.2 the minimum degree of G is at least $\frac{3n-1}{2} = \left\lceil \frac{3(2n-1)}{4} \right\rceil = \left\lceil \frac{3N}{4} \right\rceil$, where $N = 2n-1 = |V(G)|$.*

4.1 Tools for finding large paths and cycles

We shall make use of a series of well-known results. The first one is due to Erdős and Gallai [16].

Theorem 4.3. *Given integers n and ℓ, with $\ell \geq 3$, every graph G on n vertices and at least $(\ell-1)(n-1)/2+1$ edges contains a cycle of length at least ℓ. In particular, if G has at least $\ell n/2$ edges, then it contains a connected matching of size at least $\ell/2$ (edges).*

When a graph has large minimum degree we can say a little more. The following theorem is a consequence of the well-known result of Bondy [12].

Theorem 4.4. *Suppose that G is a graph with minimum degree $\delta(G) > |V(G)|/2$. Then G contains the cycle C_k for each $k = 3, \ldots, |V(G)|$.*

We will also use the following well know construction and theorem by Bondy and Chvátal.

Definition 4.5. The Hamilton closure of a graph on n vertices is obtained by recursively joining any two non-adjacent vertices whose sum of degrees is at least n.

Theorem 4.6. *A graph is Hamiltonian if and only if it Hamilton closure also is.*

To finish this section, we state and prove another lemma which is a simple consequence of Theorem 4.6.

Lemma 4.7. *Let G be a graph on k vertices. Suppose that there is a partition of the vertex set $V(G)$ into $X \cup W$ so that every vertex in X has at most $(k-3)/2$ non-neighbors in $X \cup W$, every vertex in W has at most $(k-3)/2$ non-neighbors in X and $|W| \leq (k+1)/4$. Then G is hamiltonian.*

Proof. Let H be the hamiltonian closure of G. Any vertex in X has degree at least $k - 1 - \lfloor (k-3)/2 \rfloor = \lceil (k+1)/2 \rceil$. Therefore, any two vertices in X are connected by an edge in H, i.e., $H[X]$ is a complete graph. Knowing this, we also conclude that, in H, every vertex of X has degree at least $k - 1 - |W|$. We also knew from start that every vertex of W has degree at least $|X| - (k-3)/2$. Also, we trivially have $|X| - |W| = k - 2|W| \geq (k-1)/2$. Hence, if we choose a vertex from X and a vertex from W, the sum of their degrees in H is at least $k - 1 - |W| + |X| - (k-3)/2 \geq k$. Therefore, $H[X, W]$ is a complete bipartite graph.

It is easy to see that H is Hamiltonian. Indeed, first we take a path in $H[X, W]$ starting and ending at X and saturates all the vertices of W. Then, in the complete graph $H[X]$, we complete this path to a Hamiltonian cycle. By Theorem 4.6, G is also Hamiltonian. □

Remark. *Lemma 4.7 would not be true if $|W| \geq (k+1)/4 + 1$. For example, when k is congruent to 3 modulo 4, we can consider sets X_1, X_2, W so that $|W| = (k+5)/4$,*

$|X_1| = (k+1)/4$, $|X_2| = (k-3)/2$. Denoting $X = X_1 \cup X_2$, we consider the graph on $X \cup W$ containing all edges inside X, all edges from W to X_1 and nothing else. Such graph is not hamiltonian as we can find no path covering all vertices of W.

4.2 Graphs with large minimum degrees arrow large connected matchings

In this section, we shall prove a self-contained stability theorem concerning large monochromatic connected matchings in a 2-multi-coloring of a graph with large minimum degree. Such theorem will be our main tool to prove Theorem 4.2. As in the previous chapter, before we can state our stability theorem we need to introduce some notation and define two particular (extremal) colorings.

A bipartite graph H with bipartition $V(H) = A \cup B$ is said to be bi-q-complete if the maximum degree in its multipartite complement $\overline{H}$ is at most q, that is, a vertex in A misses at most q vertices in B and vice-versa. We shall omit the prefix "bi-" when there is no risk of confusion. Also, note that if for some n and γ we have that $|A| = |B| = n$ and $A \cup B$ is bi-γn-complete, then H is $(1-\gamma)$-dense.

Coloring 4.8 ($EC_1(\alpha, \delta, \gamma)$-type)**.** *Let G be a graph with $|V(G)| = n$. A 2-multi-coloring of a graph G is of type $EC_1(\alpha, \delta, \gamma)$, where $0 \le \alpha, \delta, \gamma < 1$, if there exist disjoint sets of vertices A, B, C, D such that*

(a) $|A|, |B|, |C|, |D| \ge (1/4 - \alpha)n$;

(b) the graphs $G[A]$, $G[B]$, $G[C]$, $G[D]$ are δn-complete;

(c) the bipartite graphs $G^{r^*}[A, B], G^{r^*}[C, D], G^{b^*}[A, D], G^{b^*}[B, C]$, are δn-complete.

(d) the bipartite graphs $\overline{G}[A, C], \overline{G}[B, D]$ are γn-complete.

Remark. *We do not require $A \cup B \cup C \cup D$ to contains all vertices of $V(G)$, but condition (a) implies that at most $2\alpha|V(G)|$ vertices do not belong to $A \cup B \cup C \cup D$.*

Coloring 4.9 ($EC_2(\alpha, \delta)$-type)**.** *Let G be a graph with $|V(G)| = n$. A 2-multi-coloring of a graph G is of type $EC_2(\alpha, \delta)$, where $0 \le \alpha, \delta < 1$, if there exist disjoint set of vertices A and B such that*

(a) $|A|, |B| \ge (1/2 - \alpha)n$;

(b) *in one color, say* red, *the graphs $G^{r^*}[A]$, $G^{r^*}[B]$ are $(1/4 + \delta)n$-complete;*

(c) *in the other color, say* blue, *the bipartite graph $G^b[A, B]$ is connected and contains a matching of size $(1/4 + \delta)n$.*

Remark. *We do not require $A \cup B$ to contains all vertices of $V(G)$, but condition (a) implies that at most $4\alpha|V(G)|$ vertices do not belong to $A \cup B$.*

One should also recall Definition 3.9, of a monochromatic connected matching in a multi-coloring, to understand the next lemma.

Lemma 4.10. *For every η with $0 < \eta < 10^{-4}$ there exists an integer $t_{4.10} = t_{4.10}(\eta)$ with the following property: For every $t > t_{4.10}$ and for every 2-multi-coloring of a graph G on t vertices such that its complement $\overline{G}$ has maximum degree at most $(1/4 + \eta)t$, either G has a monochromatic connected matching of size (strictly) bigger than $(1/4 + \eta)t$ or the coloring of G is of type $EC_1(4\eta, 4\eta, 0)$.*

Proof. Assume that we are given $0 < \eta < 10^{-4}$ and let $t_{4.10} = 2/\eta$. Also, define $s = \lfloor (1/4 + \eta)t \rfloor$, let G be a graph on t vertices such that $\Delta(\overline{G}) \le s$ and consider any 2-multi-coloring of G. Let M be the largest monochromatic connected matching and assume that M is red. Let C be the connected component of G^r containing M.

We assume that

$$|M| \le (1/4 + \eta)t,$$

aiming to prove that the coloring of G is of type $EC_1(4\eta, 4\eta, 0)$. Let $Z = C \setminus V(M)$ and observe that no edge in Z is colored red because the size of M is maximal. Moreover, the maximality of M also implies that we can write $M = \{x_1y_1, \ldots, x_my_m\}$, where every x_i has at most one red neighbor in $C \setminus V(M)$. Let $X = \{x_i \,:\, i \in [1, \ldots, m]\}$ and $Y = \{y_i \,:\, i \in [1, \ldots, m]\}$. Put $C' := V(G) \setminus C$ and note that, by the maximality of C, no edge between C and C' is colored red.

We distinguish two cases according to the number of vertices in $|C'|$.

Case 1: $|C'| \leq 5\eta t$ *(including* $|C'| = \emptyset$*).*

If $|Z| \geq 2s + 3$, then the blue graph induced on Z has minimun degree at least $|Z| - 1 - s > |Z|/2$. So, it satisfies the assumptions of Theorem 4.4 and, therefore, it contains a blue cycle of length $|Z| > 2s + 2 > 2|M|$. Thus it has a (monochromatic connected) matching bigger than M, a contradiction.

Hence, assume that $|Z| \leq 2s + 2$. From this we obtain that

$$\left(\frac{1}{4} - 4\eta\right) t \leq \frac{t}{2} - s - 1 - 2.5\eta t \leq \frac{t - |Z| - |C'|}{2} = |X| = |Y| \leq \left(\frac{1}{4} + \eta\right) t$$

and

$$\left(\frac{1}{2} - 7\eta\right) t \leq t - |X| - |Y| - |C'| = |Z| \leq 2s + 2 \leq \left(\frac{1}{2} + 3\eta\right) t.$$

Claim 4.11. *$X \cup Z$ is contained in one* blue *component.*

Proof of Claim 4.11. Suppose, for a contradiction, that Z has non-empty intersections with at least two blue components and let $Z = Z_1 \cup Z_2$ be a partition such that there are no blue edges between Z_1 and Z_2. There cannot be any red edges between Z_1 and Z_2 as well because there are no red edges in Z. Therefore there are no edges between Z_1 and Z_2 at all. We immediately have that $|Z_1|, |Z_2| \leq s$.

Consequently, for $i = 1, 2$,

$$s - 9\eta t \leq (1/4 - 7\eta - \eta)t \leq |Z| - s \leq |Z_i| \leq s = (1/4 + \eta)t.$$

Now, every vertex of Z_i is non-adjacent to all vertices of Z_{3-i}, then it has at most another $9\eta t$ non-neighbors in X. Each vertex of X has at most one red neighbor in Z, in particular, the number of red edges from X to Z_i is at most $|X|$. Since for $i = 1$ and $i = 2$, we have $2|Z_i| > |X|$, we can find vertices $z_i \in Z_i$ such that z_i has at most one red neighbor in X. But now, as $9\eta t + 1 < |X|/2$, z_1 and z_2 must have a common blue neighbor in X. This contradicts the assumption that Z_1 and Z_2 are contained in different blue components.

Therefore, Z is contained in one blue component. Now, as $|Z| > s + 1$, every vertex of X has a blue neighbor in Z. Hence, X is contained in the same blue component. This finishes the proof of the claim. □

Remark. *It is also worth noting that such a component is non-bipartite, although we do not need to use this immediately . In fact, as $|X \cup Z| > t - |C'| - |Y| > 2s + 2$, we can choose any edge in Z (which exists because $|Z| \geq s + 1$) and find a common neighbor for its endpoints, yielding a triangle. This will be useful in the proof of the next lemma.*

Now we shall build a blue matching in $X \cup Z$. First, we select a maximal blue matching M_1 between X and Z. Such a matching has size at least

$$\min\{|X|, |Z| - s - 1\} > (1/4 - 9\eta)t,$$

as this is the size of a matching build by greedily choosing vertices of X and matching them to an unsaturated vertex in Z. As $|X| \geq (1/4 + \eta)t$, M_1 covers all but at most $10\eta t$ vertices of X. Let M_2 be the largest matching in $Z \setminus V(M_1)$.

If $|M_1| + |M_2| \geq s+1$, then we are done. Otherwise, $|M_1| + |M_2| \leq s$, and we have that $|M_2| \leq 10\eta t$.

We consider the sets $Z_1 = Z \setminus (V(M_1) \cup V(M_2))$ and $Z_2 = Z \setminus Z_1$. By the maximality of M_2, we have that Z_1 is an independent set. And clearly, it has order at least $|Z| - (|M_1| + |M_2|) - |M_2| \geq (1/4 - 18\eta)t$. Therefore, a vertex in Z_1 has at least $s - 19\eta t - 1$ non-neighbors in Z_1 itself, so it has at most another $20\eta t$ non-neighbors $Z_2 \cup X$. This means that there are at most $20\eta t|Z_1| < 20\eta t^2$ missing edges in $G[Z_1, Z_2 \cup X]$. We say that a vertex in $Z_2 \cup X$ is bad if it misses more than $t/8$ vertices of Z_1; and it is good otherwise. So, there are at most $160\eta t$ bad vertices.

Because $160\eta t < |M_1| - 10\eta$, we can find a subset M^* of the edges of M_1 such that $|M^*| = 10\eta t$ and all endpoints of the edges in M^* are good vertices. And since all vertices of $Z_2 \cup X$ have at most one red neighbor in Z_1, each of them must have at least $|Z_1| - t/8 - 1 > 20\eta t$ blue neighbors in Z_1. Then we can remove M^* from M_1 and use its $2|M^*|$ vertices to construct (greedily) a blue matching M' in $G[Z_1, Z_2 \cup X]$ of size $2|M^*|$. Clearly, $(M_1 \cup M_2 \cup M') \setminus M^*$ is a blue matching of size $|M_1| + |M_2| - |M^*| + |M'| = |M_1| + |M_2| + |M^*| \geq s+1$.

Case 2: $|C'| > 5\eta t$.

We treat two subcases according to the order of C.

Subcase 2.1: $|C| \geq 2s+1$.

In this subcase, any two vertices of C' have a common (blue) neighbor in C. This implies that C' is contained in one blue component. Also, note that we can find a matching from C' to C covering $\min\{|C'|, |C| - s\} \geq \min\{|C'|, s+1\}$ vertices of C'. Indeed, one can do that simply by greedily choosing vertices from C' and finding an unsaturated vertex in C which is its neighbor. If $|C'| > |X|$ then such a matching is larger than M and we have a contradiction. Hence, we have

$$|C'| \leq |X| \leq \lfloor (1/4 + \eta)\, t \rfloor = s,$$

which implies that

$$|Z| \geq t - 3|X| \geq (1/4 - 3\eta)\, t.$$

We will also use that

$$|Z| + |C'| = t - |X| - |Y| \geq (1/2 - 2\eta)t.$$

Claim 4.12. *$X \cup Z \cup C'$ is contained in the same* blue *component.*

Proof of Claim 4.12. We already know that $C' \neq \emptyset$ and C' is in one blue component. Let $X_1 \subset X$ and $Z_1 \subset Z$ be such that $X_1 \cup Z_1 \cup C'$ is the intersection of the largest blue connected component containing C' with $X \cup Z \cup C'$. Also, let $X_2 = X \setminus X_1$ and $Z_2 = Z \setminus Z_1$. Assume for a contradiction that $X_2 \cup Z_2 \neq \emptyset$. In this case, there are no blue edges from $X_1 \cup Z_1 \cup C'$ to $X_2 \cup Z_2$. Every vertex in $X_2 \cup Z_2$ is such that it has no (blue or red) neighbor in C', and it has no blue and at most one red neighbor in Z_1. Therefore, $|Z_1| + |C'| \leq s + 1$. Since $|Z_2| \geq t - |X| - |Y| - (s + 1)$, it follows that $|Z_2| \geq (1/4 - 4\eta)t$. In particular, the set Z_2 is non-empty.

Now, any vertex in Z_2 has no (blue or red) neighbor in $C' \cup Z_1$ and no blue neighbor in X_1. Additionally, since there are at most $|X_1|$ red edges from X_1 to Z_2 and $|X_1| < 2|Z_2|$, there must be a vertex in Z_2 that has at most one red neighbor in X_1. Therefore, $|X_1 \cup C' \cup Z_1| \leq s + 1$. Similarly, since no vertex in C' has a (blue or red) neighbor in $X_2 \cup Z_2$, we have $|X_2 \cup Z_2| \leq s$. This is impossible because
$|X| + |Z| + |C'| = t - |Y| > 2s + 1$.

This finishes the proof of Claim 4.12. $\square$

To continue with Subcase 2.1, our next goal is to find a blue matching M' of size $|X| - |C'| + 1$ in $G[X \cup Z]$. Assuming that one has such M', observe that we can greedily match all vertices from C' to vertices in $(X \cup Y \cup Z) \setminus V(M')$, yielding a matching larger than M. Indeed we can cover C', as all edges from C' to $(X \cup Y \cup Z) \setminus V(M')$ are blue and $|C'| \leq (|X \cup Y \cup Z| - 2|M'|) - s$ (because $|X \cup Y \cup Z| - 2|M'| - s = |Z| + 2|C'| - s - 2 \geq |C'| + (1/2 - 2\eta)t - s - 2 \geq |C'|$). Finally, by the previous claim, the resulting matching is connected, contradicting the fact that M is maximal.

To prove the existence of a matching M' as above, consider the largest matching L from X to Z. Assume, without loss of generality, that $L = \{x_i z_i \ : \ i \in [\ell]\}$. If $\ell \geq |X| - |C'| + 1$, there is nothing to prove, so we may assume that $\ell \leq |X| - |C'|$.

Let $X' = X \setminus V(L)$ and $Z' = Z \setminus V(L)$. By the maximality of L, there are no blue edges from X' to Z' and, by the choice of X, there are at most $|X'|$ red edges from X' to Z'. Now, every vertex of Z' has at least $|X'| + |Z'| - s - 1$ (red or blue) neighbors in $X' \cup Z'$. Discounting the edges from Z' to X', we conclude that the number of edges inside Z' is at least $\left(|Z'|(|X'| + |Z'| - s - 1) - |X'|\right)/2$. This number is positive and all those edges are blue as there are no red edges inside Z'. By Theorem 4.3, the set Z' contains a matching L' of size at least $\frac{|X'|+|Z'|-s-1}{2} - \frac{|X'|}{2|Z'|}$. Therefore, $L \cup L'$ is a matching of size

$$|L \cup L'| \geq \ell + \frac{|X'| + |Z'| - s - 1}{2} - \frac{|X'|}{2|Z'|} \tag{4.1}$$

Clearly, $2|Z'| = 2(|Z| - \ell) \geq 2(|Z| - |X| + |C'|) \geq (1/2 - 6\eta)t \geq |X| \geq |X'|$, and so $\frac{|X'|}{2|Z'|} \leq 1$.

Since $|X'| = |X| - \ell$ and $|Z'| = |Z| - \ell$, inequality (4.1) implies that that

$$|L \cup L'| \geq \frac{|X| + |Z| - s - 1}{2} - 1.$$

To conclude Subcase 2.1, we only need to check that

$$\frac{|X| + |Z| - s - 1}{2} - 1 \geq |X| - |C'| + 1.$$

This inequality is equivalent to

$$|X| + |Z| - s - 1 \geq 2|X| - 2|C'| + 4.$$

Replacing $|Z|$ by $t - 2|X| - |C'|$, we see that the inequality above is equivalent to

$$|C'| + t \geq 3|X| + s + 5.$$

This does hold since $|C'| > 5\eta t$, $|X| \leq (1/4 + \eta)t$ and $s \leq (1/4 + \eta)t$.

Subcase 2.2: $|C| \leq 2s$.

If $|C'| \geq 2s + 1$, then every two vertices of C have a common blue neighbor, what implies that C is contained in one blue component. It easy to see that one can greedily find a blue matching saturating all the vertices of X and one vertex from Y in the blue bipartite graph $G[C, C']$. This contradicts the choice of M. Hence, we have $|C'| \leq 2s$ and this implies

$$(1/2 - 2\eta)t = t - 2s \leq |C| \leq 2s,$$
$$(1/2 - 2\eta)t = t - 2s \leq |C'| \leq 2s.$$

Suppose that one of C and C' is contained in one blue component. Let M' be[1] the largest matching in the blue graph $G[C, C']$. Since M' must be connected, we must have $|M'| < (1/4 + \eta)t$, so there must be vertices $u \in C$ and $v \in C'$ not saturated

[1]Although we used the same letter, M', for a matching in the previous subcase, these two matchings are unrelated.

by M'. Notice, however, that we can greedily construct a blue matching in $G[C, C']$ of size at least $(1/2 - 2\eta)t - s = (1/4 - 3\eta)t$. Therefore, $|M'| \geq (1/4 - 3\eta)t$. All the vertices of $C' \setminus V(M')$ are non-neighbors of u and all the vertices of $C \setminus V(M')$ are non-neighbors of v. Hence, the number of non-neighbors of u in $C' \cap V(M')$ is at most $s - (1/4 - 3\eta)t = 4\eta t < |M'|/2$. Similarly, v has at most $|M'|/2$ non-neighbors in $C \cap V(M')$. Hence, there are $u'v' \in M'$ such that $u'v$ and $v'u$ are blue. Consequently, $M' \cup \{u'v, v''u\} \setminus \{u'v'\}$ is a larger blue matching than M', a contradiction.

We have learned that, each of C and C' intersects at least two blue components of G. Let $C = C_1 \cup C_2$ be such that $C_1 \neq \emptyset$ and $C_2 \neq \emptyset$ are in different blue components. Clearly, we can assume that C_1 is contained in one blue component. Let C_1' be the set of all vertices in C' with a blue neighbor in C_1. Set $C_2' = C' \setminus C_1'$. From the previous paragraph, C' is not contained in a single blue component, therefore $C_2' \neq \emptyset$.

By the definition of C_1', no vertex in C_1 can have any blue neighbors in C_2'. So, $G[C_1, C_2']$ is empty which implies $|C_1|, |C_2'| \leq s$. Now $|C_1'| \geq (t - 2s) - s \geq (1/4 - 3\eta)t$. In particular, we also have $C_1' \neq \emptyset$. As no vertex of C_1' has any blue or red neighbors in C_2, we have $|C_1'|, |C_2| \leq s$ and $G[C_1', C_2]$ is empty. We conclude that

$$\min\{|C_1|, |C_1'|, |C_2|, |C_2'|\} \geq (t - 2s) - s \geq (1/4 - 3\eta)t. \tag{4.2}$$

It follows that every vertex of C_1 has at most $4\eta t$ non-neighbors in $C_1 \cup C_2 \cup C_1'$, since it has no neighbor in C_2'. We have similar statements for vertices in C_2, C_1' and C_2'. So, we obtain that $G[C_1, C_1']$ and $G[C_2, C_2']$ are blue $4\eta t$-complete bipartite graphs, $G[C_1, C_2]$ and $G[C_1', C_2']$ are red $4\eta t$-complete bipartite graphs, and $G[C_1]$, $G[C_2]$, $G[C_1']$ and $G[C_2']$ are $4\eta t$-complete graphs in which both colors are possible. Therefore, we have a $EC_1(4\eta, 4\eta, 0)$ coloring.

This completes the proof of Subcase 2.2 and so Lemma 4.10 is proved. $\square$

Lemma 4.13. *For any $0 < \eta < 10^{-4}$ there is an integer $t_{4.13} = t_{4.13}(\eta)$ such that:*

For any $t \geq t_{4.13}$ and every two-coloring of a graph G on t vertices such that its complement $\overline{G}$ has maximum degree at most $(1/4+\eta)t$, if G has a monochromatic connected matching of size bigger than $(1/4+\eta)t$ then either it must contain a monochromatic connected matching of size at least $(1/4+\eta)t$ in a non-bipartite component or the coloring of G is of type $EC_2(3\eta, \eta)$.

Proof. Given $0 < \eta < 10^{-4}$, set $t_{4.13} := 1/\eta^2$ and consider G as in the statement of the lemma. Also, let $s = \lfloor (1/4+\eta)t \rfloor$ and consider any two-coloring of G containing a monochromatic connected matching of size bigger than s, say in a red component C.

If C is not bipartite, there is nothing to prove, so assume it is. Let X and Y be a bipartition of the red bipartite component C and let $C' = V(G) \setminus C$. We distinguish several cases according to the order of C'.

Case 1: $|C'| \leq s/2$ *(includes $C' = \emptyset$).*

Suppose that one of $X \cup C'$ or $Y \cup C'$, say $X \cup C'$, has order at least $2s+3$. Choose $W \subset C'$ such that $|X \cup W| = 2s+3$.

Since the missing degree of each vertex is at most s and all edges inside X and from X to W are blue, the graph $G^b[X \cup W]$ satisfy the conditions of Lemma 4.7 (with $k = 2s+3$). Hence, $G^b[X \cup W]$ is hamiltonian and we have a blue cycle of order $2s+3$. In particular, we have a matching of size $s+1$ in a blue non-bipartite component.

Therefore, we may assume that $|X \cup C'| \leq 2s+2$ and $|Y \cup C'| \leq 2s+2$. Hence, we have $|Y| = t - |X \cup C'| \geq t - 2s - 2 \leq (1/2 - 3\eta)t$ and, similarly, $|X| \geq (1/2 - 3\eta)t$. Consequently, $|C'| \leq 6\eta s$. The graphs $G[X]$ and $G[Y]$ are s-complete and all their edges are blue, $G^r[X, Y]$ is connected and contains a matching of size at least s. Thus, the coloring of G is of type $EC_2(3\eta, \eta)$.

Case 2: $|C'| \geq s+1$.

Recall that $|X|, |Y| > s$. Since all edges from C' to $X \cup Y$ are blue and $|X \cup Y| > 2s$, we can find a blue matching from C' to $X \cup Y$ of size at least $s + 1s$, by greedily choosing vertices of C' together with an unsaturated neighbor of it in $X \cup Y$. Next, we prove that, unless we have an $EC_1(4\eta, 4\eta)$, the whole blue graph, G^b, is connected. But if G^b is connected, we are back to the situation of Case 1 with the roles of blue and red interchanged and therefore done with this case.

We have that $|X \cup C'| \geq 2s+1$ and $|Y \cup C'| \geq 2s+1$. Since the missing degree of each vertex is at most s, and because all edges inside X and from X to C' are blue, any pair of non-adjacent vertices in X have a common (blue) neighbor in $X \cup C'$. So, X is contained in one blue component. Similarly, Y is contained in one blue component. Furthermore, as $|X|, |Y| > s$, every vertex in C' has a (blue) neighbor in both X and Y. Since C' is non-empty, the blue component containing X is the same as the one containing Y. Therefore, G^b is connected.

Case 3: $s/2 \leq |C'| \leq s+1$.

First, if $|X| + |C'| \leq 2s+1$, then $|Y| \geq (1/2 - 3\eta)t$ and all edges inside Y are blue. Since $|C'| \geq s/2 \geq 6\eta t$, we can take a subset W of C' so that $|W| + |Y| = 2s+3$. This time, because $|W| \leq 6\eta n < s/2$, the graph $G^b[Y, W]$ satisfies the conditions of Lemma 4.7 with room to spare. Hence, it must be hamiltonian and we obtain an odd blue cycle of length $2s+3$, which, in particular, give us a matching of size s in an odd component. The analogous argument holds if $|Y| + |C'| \leq 2s+1$.

This shows that we can assume $|X| + |C'| \geq 2s+2$ and $|Y| + |C'| \geq 2s+2$. Consequently, any two vertices in X have a common blue neighbor (in $X \cup C'$), and so X is contained in a blue component. Similarly, any two vertices in Y have a common blue neighbor (in $Y \cup C'$), so Y is contained in a blue component. In addition, we have $|X|, |Y| \geq s+1$, hence each vertex of C' has a blue neighbor in both X and in Y. Hence, the component containing X and the component containing Y are the same and it also contains C'. This means that the graph G^b is connected.

Let M be the largest blue matching in G. If M has size at least $s+1$, we are again in Case 1 with the roles of red and blue reversed. Then assume that $|M| \leq s$. Now, one should realize that we are in the same situation as in Case 1 of the proof of Lemma 4.10 (with the roles of red and blue reversed). Using the exact same steps of such case, one can prove that there is a large red connected matching and check that such matching is odd by the remark following Claim 4.11. For clarity, we include here the full details on how to finish this case. Luckily, here we already have some extra information, more precisely, we already know that there exists a red matching of size $(1/4+\eta)t$, and this makes the proof shorter.

Set $Z = V(G) \setminus V(M)$. By the maximality of M, all the edges inside Z are red. If $|Z| \geq 2s+3$, by Dirac's theorem, any subgraph of $G^r[Z]$ with $2s+3$ vertices is Hamiltonian. So there must be a red cycle on $2s+3$ vertices. In particular, we have an odd connected monochromatic matching of size bigger than s. So, we can assume that $|Z| \leq 2s+2$, which implies that $|M| \geq (1/4-\eta)t - 1$.

Now, suppose that $(1/4-\eta)t - 1 \leq |M| \leq s$, so that $(1/2-2\eta)t \leq |Z| \leq 2s+2$. By the maximality of M, we can write $M = \{a_1b_1, \ldots, a_\ell b_\ell\}$, where b_i has at most one blue neighbor in Z. Let $A = \{a_1, \ldots, a_\ell\}$ and $B = \{b_1, \ldots, b_\ell\}$. By Claim 4.11 and the remark following it, $B \cup Z$ is contained in a red component which is non-bipartite. Such a component contains at least $|B \cup Z| = t - |A| \geq (3/4-\eta)t$ vertices. On the other hand, we know that G^r has a connected matching of size $(1/4+\eta)t$, which is therefore in a component with at least $(1/2+2\eta)t$ vertices. Since $(3/4-\eta)t + (1/2+2\eta)t > t$, these two components must be the same. Therefore, the component containing the red matching is non-bipartite. $\square$

4.3 The proof of Theorem 4.2

We are ready to prove Theorem 4.2. We restate it for easy reference.

Theorem 4.2. *There exists an integer n_0 with the following property: If $n > n_0$ is an odd integer and G is a graph on $2n-1$ vertices such that its complement, $\overline{G}$, has maximum degree at most $(n-3)/2$, then G arrows C_n.*

Proof. Take $\eta = 10^{-8}$. For such η, Lemma 4.10 and Lemma 4.13 give us numbers $t_{4.10}(\eta)$ and $t_{4.13}(\eta)$ respectively; and Lemma 2.11, our embedding lemma, gives us the constant $c_{2.11} = c_{2.11}(\eta/2)$.

Let $\varepsilon = \min\{\eta^2/16, c_{2.11}^2\}$, in order that we have $\eta \geq 4\varepsilon^{1/2}$ and $\varepsilon^{1/2} \leq c_{2.11}(\eta)$. Define $m_0 = 2\max\{t_{4.10}(\eta), t_{4.13}(\eta)\}$. The Regularity Lemma (Lemma 2.5), with parameters ε, m_0 and $s = 2$, gives constants $N_0 = N_0(\varepsilon, 2, m_0)$ and $M_0 = M_0(\varepsilon, 2, m_0)$. We also consider the number $n_{2.11}(\eta/2, \varepsilon^{1/2}, \varepsilon, M_0)$ obtained from Lemma 2.11.

Define $n_0 = \max\{N_0,\ n_{2.11}(\eta/2, \varepsilon^{1/2}, \varepsilon, M_0),\ 1/(4\varepsilon^{1/2})\}$.

Let n be an odd integer with $n \geq n_0$ and let G be a graph on $2n-1$ vertices such that $\Delta(\overline{G}) \leq (n-3)/2$. Consider any 2-coloring of the edges of G, say by red and blue. We aim to prove that there exists a monochromatic C_n. We apply the Regularity Lemma (Lemma 2.5) with parameters ε, m_0 and s to the graphs G^r and G^b. The Regularity Lemma yields a partition $V_0 \cup V_1 \cup \ldots \cup V_t$ of $V(G)$ satisfying:

(a) $m_0 \leq t \leq M_0$,

(b) $|V_0| \leq \varepsilon(2n-1)$, $\quad |V_1| = \ldots = |V_t|$, and

(c) all but at most $\varepsilon\binom{t}{2}$ pairs (V_i, V_j), $1 \leq i < j \leq t$, are ε-regular with respect to both G^b and G^r.

In particular, letting $\ell = |V_i|$, where $1 \le i \le t$, we have

$$\ell = \frac{|V(G) \setminus V_0|}{t} \ge \frac{(1-\varepsilon)(2n-1)}{t}.$$

As in the previous chapter, we also consider a *reduced graph*. Let $R = R(2\varepsilon^{1/2}, \varepsilon)$ be the graph whose vertex set is $\{1, \ldots, t\}$ and in which there is an edge between vertices i and j if and only if the following conditions hold:

(I) (V_i, V_j) is an ε-regular pair with respect to both G^r and G^b;

(II) $G[V_i, V_j]$ has density at least $2\varepsilon^{1/2}$.

We define a 2-multi-coloring (R^r, R^b) of the edges of R in the following way: for $i, j \in V(R)$ and $c \in \{r, b\}$, we let ij be an edge of R^c if and only if $G^c[V_i, V_j]$ has density at least $\varepsilon^{1/2}$. Note that condition (II) implies that either $G^r[V_i, V_j]$ or $G^b[V_i, V_j]$ has density at least $\varepsilon^{1/2}$, so every edge of R receives at least one of the colors. We remark that $R^b \subseteq R^b(\varepsilon^{1/2}, \varepsilon)$, where $R^b(\varepsilon^{1/2}, \varepsilon)$ is a reduced graph for G^b representing the ε-regular pairs which have density at least $\varepsilon^{1/2}$ (in fact, R^b represents the ε-regular pair which are also regular with respect to G^r); and similarly $R^r \subseteq R^r(\varepsilon^{1/2}, \varepsilon)$ is a reduced graph for G^r.

We claim that all vertices of R, except for at most $\varepsilon^{1/2}t$, have at most $(\frac{1}{4} + 2\varepsilon^{1/2})t$ non-neighbors (in R). This is easy to show, as follows.

Firstly, we consider the set F of vertices $i \in V(R)$ such that there are at least $\varepsilon^{1/2}t$ vertices j for which (V_i, V_j) is not an ε-regular pair in G^r or G^b, i.e., (V_i, V_j) does not satisfy (I). Clearly, property (c) above implies that condition (I) is not satisfied by at most $\varepsilon\binom{t}{2}$ pairs (i, j). This implies that $\frac{|F|\varepsilon^{1/2}t}{2} \le \varepsilon\binom{t}{2}$. Then we have that

$$|F| \le \varepsilon^{1/2}t.$$

Secondly, we note that for any vertex $i \in V(R)$ there are at most $(\frac{1}{4} + \varepsilon^{1/2})t$ vertices j for which (V_i, V_j) does not satisfy (II). Let

$$S_i = \{j \in V(R) : j \neq i,\ (V_i, V_j) \text{ does not satisfy (II)}\}$$

and let $s_i = |S_i|$. For each $j \in S_i$ the graph $G[V_i, V_j]$ has at most $2\varepsilon^{1/2}\ell^2$ edges, or equivalently, $\overline{G}[V_i, V_j]$ has at least $(1 - 2\varepsilon^{1/2})\ell^2$ edges. Therefore, $\overline{G}[V_i, V \setminus V_i]$ has at least $s_i(1 - 2\varepsilon^{1/2})\ell^2$ edges. However, since $\Delta(\overline{G}) \leq (n-3)/2$, the number of edges in $\overline{G}[V_i, V \setminus V_i]$ is at most $\ell(n-3)/2$. Hence we have

$$s_i(1 - 2\varepsilon^{1/2})\ell^2 \leq \frac{\ell(n-3)}{2}.$$

As $\ell \geq (1-\varepsilon)(2n-1)/t$, this implies

$$s_i(1 - 2\varepsilon^{1/2})\frac{(1-\varepsilon)(2n-1)}{t} \leq \frac{n-3}{2},$$

which implies

$$s_i \leq \frac{t}{4(1 - 2\varepsilon^{1/2})(1-\varepsilon)} \leq \left(\frac{1}{4} + \varepsilon^{1/2}\right)t.$$

Remember that, for any i, j, the edge ij is in the graph R when (V_i, V_j) satisfy conditions (I) and (II) simultaneously. Summarizing the above we have that: for every $i \notin F$ there are at most $\varepsilon^{1/2}$ vertices j so that (V_i, V_j) does not satisfy (I) and at most $(1/4 + \varepsilon^{1/2})t$ for which (V_i, V_j) does not satisfy (II). So, in total, at most $(1/4 + 2\varepsilon^{1/2})t$ vertices are non-adjacent to a vertex i which is not in F. This proves our claim.

Now, if we consider the subgraph of R induced by $V(R) \setminus F$, say H, and define $t' = |V(H)|$ we have that $t' \geq (1 - \varepsilon^{1/2})t$. Furthermore,

$$\Delta(\overline{H}) \leq \max\{s_i : i \in [t]\} \leq \left(\frac{1}{4} + 2\varepsilon^{1/2}\right)t \leq \left(\frac{1}{4} + 3\varepsilon^{1/2}\right)t' \leq \left(\frac{1}{4} + \eta\right)t'.$$

Finally, we consider the 2-multi-coloring (H^r, H^b) of the edges of H induced by the 2-multi-coloring of R.

We apply Lemma 4.10 to H with parameter $\eta = 4\varepsilon^{1/2}$. Note that the conditions to apply Lemma 4.10 are indeed satisfied as $\eta < 10^{-4}$ and

$$t' \geq (1-\varepsilon^{1/2})t \geq (1-\varepsilon^{1/2})m_0 \geq t_{4.10}(\eta). \tag{4.3}$$

As a result we have two possibilities: either H contains a connected monochromatic matching of size $k \geq (1/4 + 4\varepsilon^{1/2})t'$ or the coloring of H is of type $EC_1(4\eta, 4\eta, 0)$. We treat two cases accordingly.

Case 1: *H contains a monochromatic connected matching, of size $k \geq (1/4+\eta)t'$.*

Now, similarly to equation (4.3), we also have $t' \geq t_{4.13}(\eta)$. Hence, we can apply Lemma 4.13 to H in order to show that: either H contains an odd connected monochromatic matching M, say blue, of size $k \geq (1/4+\eta)t' \geq (1/4+\eta/2)t$ or the coloring of H is of type $EC_2(3\eta, \eta)$.

In the first subcase, we only need to check that the conditions to apply Lemma 2.11 to the graph G^b with its reduced graph $R^b(\varepsilon^{1/2}, \varepsilon)$, (which contains R^b and hence contains the odd connected matching M), are satisfied. This is clear, as $|V(G)| > n_{2.11}(\eta/2, \varepsilon^{1/2}, \varepsilon, M_0)$ and $\varepsilon/\varepsilon^{1/2} \leq c_{2.11}(\eta/2)$. And because M is an odd matching, Lemma 2.11 tell us that for any integer ℓ satisfying $4t < \ell < (1/2+\eta/2)|V(G)|$, the graph G contains a monochromatic cycle of length ℓ. In particular, G contains a monochromatic C_n.

In the second subcase, the coloring of H is of type $EC_2(3\eta, \eta)$. This means that there are sets $\mathcal{A}, \mathcal{B} \subset V(H)$ each of order at least $(1/2 - 3\eta)t' > (1/2 - 4\eta)t$, and such that the subgraph $H[\mathcal{A}, \mathcal{B}]$ contains a monochromatic, say blue, matching M of size $k \geq (1/4+\eta)t' \geq (1/4+\eta/2)t$. Also we have that $H^b[\mathcal{A}, \mathcal{B}]$ is connected. Letting $A = \bigcup_{i\in\mathcal{A}} V_i$ and $B = \bigcup_{i \in \mathcal{B}} V_i$, we have that $|A \cup B| \geq (1-9\eta)(2n)$.

We note that the conditions to apply Corollary 2.11 are the same as those to apply Lemma 2.12, hence they are satisfied by G^b together with its reduced graph $R^b(\varepsilon^{1/2}, \varepsilon)$ and the matching M. Therefore, there exists a set $F \subset V(G)$ such that $|F| \leq 4\varepsilon n$ and for any $u, v \in A \setminus F$ and any even number ℓ satisfying $4t < \ell < (1/2 + \eta/2)|V(G)|$, there is a blue (u, v)-path of length ℓ. In particular, we can find a blue (u, v)-path of length $n - 1$. Hence, if there are $u, v \in A \setminus F$ such that uv is a blue edge, then we can find a blue C_n. Therefore, we can assume that all edges in $A \setminus F$ are red. Similarly, we can assume that all edges in $B \setminus F$ are red.

Again by Corollary 2.12, for any vertices $u \in A \setminus F$ and $v \in B \setminus F$ and any odd number ℓ, with $4t < \ell < (1/2 + \eta/2)|V(G)|$, there is a blue (u, v)-path of length ℓ. In particular, there is a blue path of length $(n - 2)$ between any such u, v. Consider the set X defined as the union of F and all clusters not in $\mathcal{A} \cup \mathcal{B}$ (including V_0). If a vertex in X has a blue neighbor in A and B, again, we can find a blue C_n. Otherwise, we can partition $X = X_A \cup X_B$ so that there are only red edges between A and X_A and between B and X_B.

Now, we note that $|X| \leq 10\eta n$. We also have that $|A \cup B \cup X| = 2n - 1$, so one of the sets $A \cup X_A$ or $B \cup X_B$, say $A \cup X_A$, must have size at least n. Choose $X'_A \subset X_A$ such that $|A \cup X'_A| = n$. Since the missing degree of each vertex is at most $(n - 3)/2$, all edges inside A and from A to X_A are red, and $|X'_A| < 10\eta n$, the conditions of Lemma 4.7 are satisfied with room to spare. Therefore, the graph $G^r[A \cup X_A]$ is also Hamiltonian and we have a red C_n.

Case 2: *The coloring of H is of type $EC_1(4\eta, 4\eta, 0)$.*

This means that there are sets $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4 \subset V(H)$, each of size $(1/4 - 4\eta)t'$, such that

- $H^{b^*}[\mathcal{C}_1, \mathcal{C}_3]$ and $H^{b^*}[\mathcal{C}_2, \mathcal{C}_4]$ are $4\eta t'$-complete bipartite graphs,
- $H^{r^*}[\mathcal{C}_1, \mathcal{C}_2]$ and $H^{r^*}[\mathcal{C}_3, \mathcal{C}_4]$ are $4\eta t'$-complete bipartite graphs,

- $H[\mathcal{C}_1]$, $H[\mathcal{C}_2]$, $H[\mathcal{C}_3]$ and $H[\mathcal{C}_4]$ are $4\eta t'$-complete graphs in which both colors are allowed, and

- $\overline{H}[\mathcal{C}_1, \mathcal{C}_4]$ and $\overline{H}[\mathcal{C}_3, \mathcal{C}_2]$ are complete bipartite graphs.

Now, we shall use the same type of argument from Cases 2 and 3 of the proof of Theorem 3.2 to conclude that the coloring of G is of type $EC_1(15\eta^{1/2}, 12\eta^{1/2}, 12\eta^{1/2})$. We remark that for our original graph, G, all edges receive only one color, so we can use G^b and G^{b^*} interchangeably (as well as G^r and G^{r^*}).

For $1 \le j \le 4$, let $Z_j = \bigcup_{i \in \mathcal{C}_j} V_i$. We would like to say that Z_1, Z_2, Z_3, Z_4 satisfy the conditions (a)-(d) of $EC_1(5\eta^{1/2}, 12\eta^{1/2}, 12\eta^{1/2})$. Unfortunately, they may not satisfy (b) and (c). Nevertheless, we prove that they do satisfy (a) and (d) with room to spare, and use this to help us to prove that we can remove a few vertices from each Z_i so that the resulting sets do satisfy conditions (b) and (c) and still satisfy conditions (a) and (d).

First, we note that $|Z_j| > (1/4 - 5\eta)(2n-1)$. In fact,

$$|Z_1| \ge |\mathcal{C}_1| \frac{(1-\varepsilon)(2n-1)}{t} \ge \left(\frac{1}{4} - 4\eta\right) t' \frac{(1-\varepsilon)(2n-1)}{t} \ge \left(\frac{1}{4} - 5\eta\right)(2n-1),$$

and similarly, we obtain that $|Z_2|, |Z_3|, |Z_4| \ge (1/4 - 5\eta)(2n-1)$. Therefore, condition (a) of $EC_1(15\eta^{1/2}, 12\eta^{1/2}, 12\eta^{1/2})$ is satisfied by $\{Z_1, Z_2, Z_3, Z_4\}$ with room to spare.

Now, we estimate the number of edges in $\overline{G}[Z_1, Z_4]$. Since $\overline{H}[\mathcal{C}_1, \mathcal{C}_4]$ is complete, it has at least $|\mathcal{C}_1||\mathcal{C}_4| \ge \left((\frac{1}{4} - 4\eta)t'\right)^2 \ge (\frac{1}{16} - 3\eta)t^2$ edges. Each of them is a pair $\{i, j\}$ for which (V_i, V_j) does not satisfy condition (I) or condition (II) above. Recall that at most $\varepsilon\binom{t}{2} < \varepsilon t^2$ pairs $\{i, j\}$ do not satisfy (I). Therefore, at least $(\frac{1}{16} - 3\eta - \varepsilon)t^2$ of the pairs do not satisfy (II). Now, for each pair (V_i, V_j) that does not satisfy condition (II), we have that $\overline{G}(V_i, V_j)$ has density at least $(1 - 2\varepsilon^{1/2}) = (1 - \eta/2)$, that is, it has at least $(1 - \eta/2)\ell^2$ edges. Summing this bound over all those pairs

(V_i, V_j) (not satisfying (II)) we obtain that $\overline{G}[Z_1, Z_4]$ has at least

$$\left(1-\frac{\eta}{2}\right)\ell^2 \cdot \left(\frac{1}{16} - 3\eta - \varepsilon\right) t^2 \geq \left(\frac{1}{16} - 5\eta\right)(2n-1)^2$$

edges.

Similarly, $\overline{G}[Z_2, Z_3]$ has at least $(\frac{1}{16} - 5\eta)(2n-1)^2$ edges. But, $\Delta(\overline{G}) \leq (n-3)/2$ implies that the number of edges of $\overline{G}$ is less than $(2n-1)(n-3)/4 \leq (2n-1)^2/8$. This implies that there are at most $10\eta(2n-1)^2$ edges all together in the bipartite graphs $\overline{G}[Z_i, Z_j]$ and $\overline{G}[Z_i]$, where $1 \leq i, j \leq 4$ and $\{i, j\} \notin \{\{1, 4\}, \{2, 3\}\}$.

Now, we give a bound on the number of edges with 'wrong color' in $G[Z_1, Z_3]$, $G[Z_2, Z_4]$, $G[Z_1, Z_2]$ and $G[Z_3, Z_4]$. For example, let us show that there are few red edges in $G[Z_1, Z_3]$.

Fix a vertex $i \in \mathcal{C}_1$. We bound the number of red edges from V_i to Z_3 as follows. Recalling that $Z_3 = \bigcup_{j \in \mathcal{C}_3} V_j$, it is enough to bound $e_r(V_i, V_j)$ for each $j \in \mathcal{C}_3$. When $ij \notin H^{b^*}$, we use the trivial bound $|V_i||V_j|$ for $e_r(V_i, V_j)$, but notice that, as $H^{b^*}[\mathcal{C}_1, \mathcal{C}_3]$ is $4\eta t'$-complete, there are at most $4\eta t'$ such j. While for $ij \in H^{b^*}$ we can conclude that $ij \notin H^r$, thus, from the definition of H^r, $e_r(V_i, V_j) \leq \varepsilon^{1/2}|V_i||V_j|$. Hence,

$$\begin{aligned} e_r(V_i, Z_3) &\leq \sum_{\substack{j \in \mathcal{C}_3 \\ ij \notin H^{b^*}}} |V_i||V_j| + \sum_{\substack{j \in \mathcal{C}_3 \\ ij \in H^{b^*}}} \varepsilon^{1/2}|V_i||V_j| \\ &\leq 4\eta t'|V_i||V_i| + |\mathcal{C}_3|(\varepsilon^{1/2})|V_i||V_i| \\ &\leq 4\eta t|V_i||V_i| + \varepsilon^{1/2} t|V_i||V_i| \\ &\leq 5\eta|V_i|(2n-1), \end{aligned}$$

where we have used that for any $i, j \geq 1$, we have $|V_i| = |V_j|$, $t'|V_j| \leq t|V_j| \leq 2n-1$ and $\varepsilon^{1/2} < \eta$. Summing the previous equation for all possible values of $i \in \mathcal{C}_1$, we have that

$$e_r(Z_1, Z_3) \leq 5\eta(2n-1)^2.$$

We conclude that the complement of the blue bipartite graph $G^b[Z_1, Z_3]$ has at most $15\eta(2n-1)^2$ edges. Similarly, the same bound holds for the number of edges in each of the bipartite graphs $\overline{G^b}[Z_2, Z_4]$, $\overline{G^r}[Z_1, Z_2]$ and $\overline{G^r}[Z_3, Z_4]$.

Now, we are able to prove that only a few vertices do not have very low degree in each of the bipartite graphs $\overline{G^b}[Z_1, Z_3]$, $\overline{G^b}[Z_2, Z_4]$, $\overline{G^r}[Z_1, Z_2]$, $\overline{G^r}[Z_3, Z_4]$ and $\overline{G}[Z_i]$ for $1 \le i \le 4$. We call a vertex bad if its induced degree in any of those graphs is larger than $(15\eta)^{1/2}(2n-1)$. The bound on the number of edges for those graphs imply that each Z_i has most $3(15\eta)^{1/2}(2n-1)$ bad vertices.

Finally, we define W_i, $1 \le i \le 4$, as the set obtained from Z_i by removing its bad vertices. We have that

$$|W_i| \ge |Z_i| - 3(15\eta)^{1/2}(2n-1) \ge (1/4 - 15\eta^{1/2})(2n-1),$$

that is, condition (a) of $EC_1(15\eta^{1/2}, 15\eta^{1/2}, 15\eta^{1/2})$ is satisfied. Clearly, by the definition of a bad vertex, conditions (b), (c) and (d) of $EC_1(15\eta^{1/2}, 15\eta^{1/2}, 15\eta^{1/2})$ are satisfied by $\{W_1, W_2, W_3, W_4\}$ as well. So, the original 2-coloring of G is of type $EC_1(15\eta^{1/2}, 15\eta^{1/2}, 15\eta^{1/2})$.

Denote by X the union of the set of bad vertices with V_0 and with the clusters not in $\bigcup_{1\le i\le 4} \mathcal{C}_i$. There are at most $12(15\eta)^{1/2}(2n-1) \le 95\eta^{1/2}n$ bad vertices, at most $\varepsilon(2n-1) \le \eta^{1/2}n$ vertices in V_0 and at most $16\eta t'\ell \le \eta^{1/2}n$ vertices not in $\bigcup_{1\le i\le 4} \mathcal{C}_i$. Therefore, $|X| < 100\eta^{1/2}n$. Clearly, as $\Delta(\overline{G}) \le (n-3)/2$, each $u \in X$ has at least $n/10$ (in fact, close to $n/4$) neighbors in each of at least three of $W_1, \ldots, W_4$. In particular, each $u \in X$ has at least $n/20$ neighbors of the same color in each of at least three of the sets $W_1, \ldots, W_4$. Using this fact, we classify all vertices of X into at least one of the following types (see Figure 4.3).

v **is $\boldsymbol{W_1'}$-type** if $\deg_R(v, W_2) \ge n/20$ and $\deg_B(v, W_3) \ge n/20$.

v **is $\boldsymbol{W_2'}$-type** if $\deg_R(v, W_1) \ge n/20$ and $\deg_B(v, W_4) \ge n/20$.

v **is W_3'-type** if $\deg_R(v, W_4) \geq n/20$ and $\deg_B(v, W_1) \geq n/20$.

v **is W_4'-type** if $\deg_R(v, W_3) \geq n/20$ and $\deg_B(v, W_2) \geq n/20$.

v **is $R1$-type** if either $\deg_R(v, W_1), \deg_R(v, W_3) \geq n/20$

or $\deg_R(v, W_2), \deg_R(v, W_4) \geq n/20$;

v **is $R2$-type** if either $\deg_R(v, W_1), \deg_R(v, W_4) \geq n/20$

or $\deg_R(v, W_2), \deg_R(v, W_3) \geq n/20$;

v **is $B1$-type** if either $\deg_B(v, W_1), \deg_B(v, W_2) \geq n/20$

or $\deg_B(v, W_3), \deg_B(v, W_4) \geq n/20$;

v **is $B2$-type** if either $\deg_B(v, W_1), \deg_B(v, W_4) \geq n/20$

or $\deg_B(v, W_2), \deg_B(v, W_3) \geq n/20$;

Note that those classes of vertices are not necessarily disjoint, but one can check that every vertex in X belongs to at least one of them. We also say that v is **W_*-type** if it is W_i'-type for some i. We define $R*$-type and $B*$-type vertices similarly.

We remark that vertices of W_i'-type are those who could be added to W_i partially preserving the global structure of the coloring of G. With that in mind, we define W_i' be the set of vertices of W_i'-type and let $\tilde{W}_i = W_i \cup W_i'$.

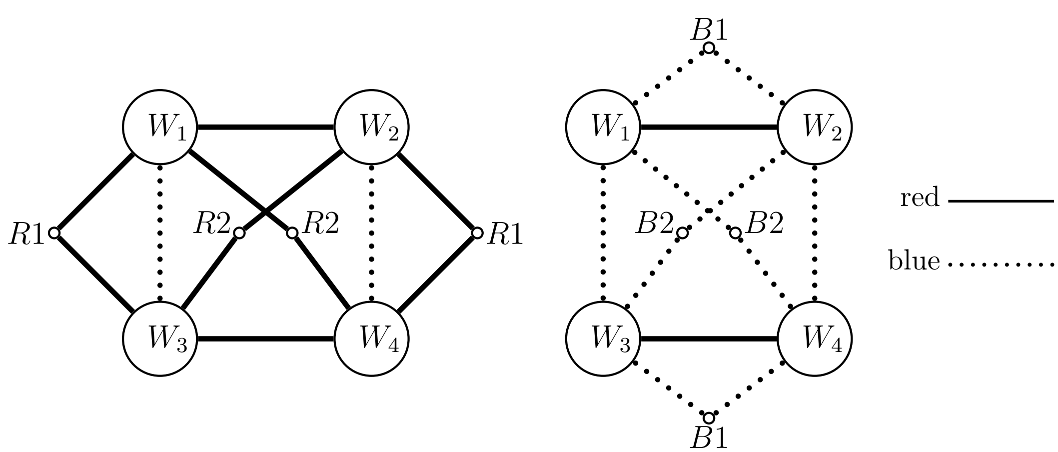

Figure 4.1: Vertices of $R*$-type on the left and of $B*$-type on the right.

Now, although there are few (possibly no) edges in $G[\tilde{W}_1, \tilde{W}_4]$ and $G[\tilde{W}_2, \tilde{W}_3]$, if there is such an edge we say that it is an edge of Type 1. The reason for this name is that one edge of Type 1 has a similar effect as a vertex of Type 1 (of the same color as the edge) toward finding a monochromatic C_n in our next claim.

Claim 4.14. *Either G has a monochromatic C_n or all the following facts must hold.*

(a) There are no distinct vertices $u, v \in X$ such that u is ***R1****-type and v is* ***R2****-type. Also, there is no* red *edge e and vertex v such that is e is of Type 1 and v is* ***R2****-type. The analogous statement for* blue *types also holds.*

(b) If there are two vertices v_1, v_2 such that both are of type ***R1*** *or both of type* ***R2****, then, for all i, all edges inside $\tilde{W}_i \setminus \{v_1, v_2\}$ are* blue*. Similarly, if there is a* red *edge of Type 1, say $e = ab$, and a vertex of type* ***R∗****, say v_1, then all edges inside $\tilde{W}_i \setminus \{a, b, v_1\}$ must be* blue*. Finally, if there are two* independent red *edges of Type 1, say $e_1 = ab$ and $e_2 = cd$, then all edges inside $\tilde{W}_i \setminus \{a, b, c, d\}$ must be* blue*. The analogous statements for* blue *also holds.*

Proof. The idea of the proof of Claim 4.14 is to use Lemma 3.17 to find paths of appropriate lengths in $G^b[W_1, W_3]$, $G^b[W_2, W_4]$, $G^r[W_1, W_2]$ and $G^r[W_3, W_4]$, and use vertices of suitable types to glue those paths together.

We give the full details for the first case in (a), that is, assuming that there are distinct $u, v \in X$ such that u is ***R1***-type and v is ***R2***-type we aim to find a red C_n in G. Without loss of generality, assume that $\deg_R(u, W_1), \deg_R(u, W_3) \geq n/20$ and $\deg_R(v, W_1), \deg_R(v, W_4) \geq n/20$. Clearly, there are red neighbors $u' \in W_1$ and $u'' \in W_3$ of u and red neighbors $v' \in W_1$, $v'' \in W_4$ of v such that u', u'', v', v'' are pairwise distinct.

It follows from Lemma 3.17, applied to $G^r[W_1, W_2]$ and $G^r[W_3, W_4]$, that for any even number k_1 and odd number k_2 such that

$$2 \le k_1, k_2 \le 2(\min\{|W_1|, |W_2|, |W_3|, |W_4|\} - 24\eta^{1/2}|V(G)|) - 2,$$

there exists a (u', v')-path P in $G^r[W_1, W_2]$ of length k_1 and a (v', v'')-path Q in $G^r[W_3, W_4]$ of length k_2.

Clearly, the union $P \cup Q \cup \{uu', uu'', vv', vv''\}$ form a red copy of $C_{k_1+k_2+4}$. Since $2(\min\{|W_1|, |W_2|, |W_3|, |W_4|\} - 24\eta^{1/2}|V(G)|) - 2 \ge 2((1/4 - 40\eta^{1/2})|V(G)|) = (1/2 - 80\eta^{1/2})|V(G)|$, and as n is odd, we can choose k_1 and k_2 so that $k_1 + k_2 + 4 = n$.

The proofs of the other statements in (a) and in (b) are analogous. For the above argument to work for each statement involving vertices of $\boldsymbol{R*}$-type or red edges of Type 1, one only needs to check the following: in the auxiliary graph of Figure 4.3 there is a red closed walk of odd length containing both edges W_1W_2 and W_3W_4. $\square$

Now, we consider a few cases according to the type of the vertices of X.

Subcase 3.1: at least three vertices of X are not $\boldsymbol{W_*}$-type.

This implies that either there are two vertices u and v such that both are $\boldsymbol{R*}$-type or both are $\boldsymbol{B*}$-type. Assume, without loss of generality, that the former happens.

By part (b) of Claim 4.14 we can assume that most edges inside the sets $\tilde{W}_i$ are blue. We claim that either there is a red C_n or for every vertex x in X (including u and v) there exists i_x, $1 \le i_x \le 4$, such that x has at least $n/20$ blue neighbors in W_{i_x}. This claim is true, because there are three W_i in which x has $n/20$ neighbors of the same color. If such color were red to all three of them, we would have a vertex w which is both $R1$ and $R2$. But there are at least two vertices of $\boldsymbol{R*}$-type, so we would have distinct vertices (say w and one of u or v) such that one is $\boldsymbol{R1}$-type and the other is $\boldsymbol{R2}$-type. This yields a monochromatic C_n by Part (a) of the Claim 4.14.

Now we simply aim to find a blue C_n in G. Looking at the indices $1 \leq i \leq 4$ modulo four, we define W_i'' to be the set of vertices $v \in X$ which have $n/20$ blue neighbors in W_{i+2}. By the previous discussion, we can assume that $X \subset \bigcup_{1\leq i\leq 4} W_i''$. Hence, either $|W_1 \cup W_3 \cup W_1'' \cup W_3''| \geq n$ or $|W_2 \cup W_4 \cup W_2'' \cup W_4''| \geq n$. Say the former holds. So, $G[W_1 \cup W_3 \cup W_1'' \cup W_3'']$ is an almost complete blue graph. By Theorem 4.6, applied to any subgraph of $G[W_1 \cup W_3 \cup W_1'' \cup W_3'']$ of order n, it must contain a blue C_n.

The case that there are at least two vertices of $\boldsymbol{B}*$-type is analogous.

Subcase 3.2: exactly two vertices are not of $\boldsymbol{W}_*$-type.

Let u and v be two vertices of X which are not $\boldsymbol{W}_*$-type. If both u and v are $\boldsymbol{R}*$-type or both are $\boldsymbol{B}*$-type, we are done by the same argument as in Subcase 3.1. So, assume that v is $\boldsymbol{R}*$-type and u is $\boldsymbol{B}*$-type.

Notice that we have $|\tilde{W}_1 \cup \tilde{W}_2 \cup \tilde{W}_3 \cup \tilde{W}_4| = |V(G) \setminus \{u, v\}| = 2n - 3$. So there exists i such that $|\tilde{W}_i| \geq (n-1)/2$. Since $\Delta(\overline{G}) \leq (n-3)/2$, either $G[\tilde{W}_1, \tilde{W}_3]$ or $G[\tilde{W}_2, \tilde{W}_4]$ must contain an edge. This means that there is an edge e of Type 1. Assume, without loss of generality, that e is red. Hence, by Part (b) of the previous claim, applied to the vertex v and the edge e, we can assume that most edges induced by the sets $\tilde{W}_i$ are blue. Now we can finish using the steps of Subcase 3.1.

Subcase 3.3: exactly one vertex, say $x \in X$, is not of $\boldsymbol{W}_*$-type.

Assume, again without loss of generality, that x is $\boldsymbol{R}*$-type. Now, if there is any red edge of Type 1, part (b) of Claim 4.14 implies that we can assume that most edges induced by the sets $\tilde{W}_i$ are blue and we can proceed as in Subcase 3.1.

Therefore we can assume that all Type-1 edges are blue. We claim that there are at least two independent edges of Type 1. If such claim is true, by part (b) of Claim 4.14, have that most edges inside the sets $\tilde{W}_i$ are red and once more we can proceed as in Subcase 3.1, this time with the roles of red and blue reversed. So, it only remains to prove that there are such independent edges of Type 1.

Since $|\tilde{W}_1 \cup \tilde{W}_2 \cup \tilde{W}_3 \cup \tilde{W}_4| \geq 2n-2$, either there exists i such that $|\tilde{W}_i| \geq (n+1)/2$ or for all i we have $|\tilde{W}_i| = (n-1)/2$. In the first case, if $|\tilde{W}_1| \geq (n+1)/2$ or $|\tilde{W}_4| \geq (n+1)/2$, we can easily find two independent edges in $G[\tilde{W}_1, \tilde{W}_4]$; if $|\tilde{W}_2| \geq (n+1)/2$ or $|\tilde{W}_3| \geq (n+1)/2$, we can find two independent edges in $G[\tilde{W}_2, \tilde{W}_3]$. Finally, in the latter case, where $|\tilde{W}_i| = (n-1)/2$, there must be at least one edge in $G[\tilde{W}_1, \tilde{W}_4]$ and another one in $G[\tilde{W}_2, \tilde{W}_3]$.

Subcase 3.4: every vertex of X is $\boldsymbol{W_*}$-type.

Once again, our goal is to find at least two independent edges of Type 1, in which case we are done. Assume, without loss of generality, that $\tilde{W}_1 = \max\{\tilde{W}_1, \ldots, \tilde{W}_4\}$. Clearly $\bigcup_{1 \leq i \leq 4} \tilde{W}_i = 2n-1$, so $|\tilde{W}_1| \geq (n+1)/2$.

Consider first the case where $|\tilde{W}_1| \geq (n+3)/2$. Since $\Delta(\overline{G}) \leq (n-3)/2$, there are at least three independent edges from $\tilde{W}_1$ to $\tilde{W}_4$, two of which must be of the same color and we are done. The other possibility is that $|\tilde{W}_1| = (n+1)/2$. Here we must have at least two independent edges in $G[\tilde{W}_1, \tilde{W}_4]$ (not necessarily of the same color). But, since $|\tilde{W}_4| \leq |\tilde{W}_1| = (n+1)/2$, we must have that either $\tilde{W}_2$ or $\tilde{W}_3$ has at least $(n-1)/2$ vertices. This implies that $G[\tilde{W}_2, \tilde{W}_3]$ has at least one edge. We conclude that we have at least three edges of Type 1. Therefore, two of them must be of the same color and we are done. □

4.4 Open problems

There are many natural and interesting open problem, the first one being to solve Conjecture 4.1 completely. We refer to a recent survey article of Schelp [36] for a list of conjectures related to the following problem. Given a graph H and a constant c, with $0 < c \leq 1$, consider the property $P(H, c)$ that "if G is a graph of order equal to the Ramsey number $R(H)$ and minimum degree bigger than $c|V(G)|$, then any 2-coloring of G contains a monochromatic copy of H". Then define $c(H) = \inf\{c : P(H, c) \text{ holds}\}$.

By Theorem 4.2, we have that $c(C_n) = 3/4$ when n is odd. So, the most natural question is to determine the value of $c(C_n)$ for n even. We conjecture that this value is approximately equal to $2/3$.

One should also consider the analogous questions related to the multi-colored Ramsey numbers. For example, given n, is there a constant $0 < c < 1$, such that if G is a graph of order equal to $R(C_n, C_n, C_n)$ and minimum degree at least $c|V(G)|$ then any 3-coloring of G must contain a monochromatic C_n?

Chapter 5

Slowly percolating sets

5.1 Introduction

In this chapter we study the slowly growing 2-neighbor bootstrap percolating sets in the grid $[n]^2$, a concept that we shall soon make precise. Bootstrap percolation is a particular type of cellular automaton, a concept studied, for example, by von Neumann [32].

Given a (finite) graph G, bootstrap percolation on G is a particular class of models that describe an 'infection' spreading over the set of vertices of G. In the context of percolation, vertices of G are commonly called sites and edges of G are called bonds. For each $v \in V(G)$ we consider the set of neighbors of v, denoted $N(v)$, and let $\mathcal{S}_v$ be the family of all subsets of $N(v)$. For each site $v \in V(G)$, we select one of two *initial states* for v, say '*infected*' or '*healthy*', and we let A be the set of sites whose initial state is 'infected'. We are also given an update function $f_v : \mathcal{S}_v \to \{\text{'safe', 'susceptible'}\}$. The infection process is defined as follows: set $A_0 = A$ and, for $t \in \mathbb{N}$, set

$$A_t = A_{t-1} \cup \{v \in V(G) : f_v(A_{t-1} \cap N(v)) = \text{'susceptible'}\}.$$

In this process, we think of t as time and A_t as the set of sites whose state at time t is 'infected', so that $A_t \cap N(v)$ is the set of neighbors of v which are infected at time t, and f_v determines if v will becomes infected at time t based on which of its neighbors are infected at time $t-1$. We call A the set of 'initially infected sites'. We note that, in bootstrap percolation, once a site is infected it never becomes healthy.

The closure of $A \subset V(G)$ is the set $\langle A \rangle = \bigcup_{t=0}^{\infty} A_t$ of all sites that are eventually infected. We say that the set A percolates if eventually all sites are infected, that is, if $\langle A \rangle = V(G)$. Furthermore, we say that A takes time T to percolate if $\langle A \rangle = V(G)$ and T is the smallest natural number such that $A_T = V(G)$.

The r-neighbor bootstrap percolation on G is the particular case where we have $f_v(S) =$ 'susceptible' if and only if $|S| \geq r$. This means that sites of G become infected if they have at least r infected neighbors. Hence,

$$A_t = A_{t-1} \cup \{v \in V(G) : |N(v) \cap A_{t-1}| \geq r\}.$$

We are interested in a particular case where, for some natural number n, the graph G above is the grid $[n]^2$ defined as follows: the set of sites of G is $V(G) = \{(i,j) : 1 \leq i,j \leq n\}$, which we represent by an n by n square-grid where each site is a unit square whose center has coordinates (i,j) in the Cartesian plane; and two sites are adjacent if the corresponding squares share an edge. This particular model was introduced in 1979 by Chalupa, Leith and Reich [14], and rediscovered by many authors. Aizenman and Lebowitz [1] considered the problem where the set of originally infected sites is chosen by selecting sites independently at random with uniform probability p. They tried to determine for what values of p the set A percolates with high probability. The first sharp result was given by Holroyd [25] in 2003. Many sharper results were obtained by Balogh, Bollobás and Morris [2, 3, 4] for the same problem and also for various other graphs G and values of the threshold r.

However, here instead of choosing A at random, we consider the (deterministic) extremal problem of finding a set A for which the percolation time is the largest possible, assuming that A does percolate. We shall make this question more precise later. Throughout this chapter all the results concern 2-neighbor bootstrap percolation on $[n]^2$. All the results are in collaboration with Michal Przykucki [8].

5.2 Preliminaries

Given integers k, ℓ and n with $1 \leq k, \ell \leq n$, a k by ℓ rectangle is a subset of $\mathbb{N}^2$ of the form $\{a, a+1, \ldots, a+k-1\} \times \{b, b+1, \ldots, b+\ell-1\}$ for some choice of a and b. Given a subset R of $[n]^2$, we will write $R = \text{Rec}(k, \ell)$ to say that R is a k by ℓ rectangle. We say that a rectangle R is *internally spanned* by a given set of infected sites A if $\langle A \cap R \rangle = R$.

Definition 5.1. Given a finite set $A \subset \mathbb{N}^2$, we represent a site $(i, j) \in A$ as a shaded unit square on the grid, (say so that the center has coordinates (i, j) in the Cartesian Plane). The boundary of A is the set of bonds of $\mathbb{N}^2$ such that exactly one of its endpoints is in A, which in our pictures shall be represented by the sides shared between a shaded and a non-shaded unit square. The perimeter of A is the number of bonds in its boundary. Its semi-perimeter is half of the perimeter and is denoted by $\Phi(A)$. In particular, if $R = \text{Rec}(k, \ell) \subset \mathbb{N}^2$ is a k by ℓ rectangle, its semi-perimeter is $\Phi(R) = k + \ell$.

Now, let us define the distance between sites and rectangles.

Definition 5.2. The distance between a pair of sites, (i_1, j_1) and (i_2, j_2), is given by $|i_1 - i_2| + |j_1 - j_2|$. The distance between two rectangles R' and R'' is the minimal distance between a site $(i_1, j_1) \in R'$ and a site $(i_2, j_2) \in R''$ and is denoted by $\text{dist}(R', R'')$.

Remark. *This coincides with the length of the shortest path from A to B when viewing $[n]^2$ as a graph. Two sets are at distance zero from each other if they intersect and at distance one if their boundaries share at least one edge.*

Fact 5.3. *For any two finite sets $A, B \subset \mathbb{N}^2$, we have $\Phi(A) + \Phi(B) \geq \Phi(A \cup B)$ and equality occurs if and only if $\operatorname{dist}(A, B) \geq 2$, that is, A and B have disjoint boundaries.*

Proposition 5.4. *Let K be a set of infected sites and let $\langle K \rangle$ be its closure. Then*

$$\Phi(\langle K \rangle) \leq \Phi(K).$$

Proof. Let $K_0 = K$ and let K_t be the set of infected sites at time t. A healthy site becomes infected at time $t+1$ if at least two of its neighbors are in K_t. As a result, for every $v \in K_{t+1} \setminus K_t$, there are at least two bonds adjacent to v which are in the boundary of K_t but not in the boundary of K_{t+1}. In addition, for any two sites $v, w \in K_{t+1} \setminus K_t$ the pairs of bonds determined by each of them are disjoint. Furthermore, for each $v \in K_{t+1} \setminus K_t$ at most two new edges are in the boundary of K_{t+1} but not in the boundary of K_t. Thus the perimeter cannot grow during the infection process. $\square$

Corollary 5.5. *Given $k, \ell \in \mathbb{N}$ and a rectangle $R = \operatorname{Rec}(k, \ell)$, if $A \subset R$ is a set that internally spans R then $|A| \geq \lceil \Phi(R)/2 \rceil = \left\lceil \frac{k+\ell}{2} \right\rceil$. In particular, if $n \in \mathbb{N}$ and $A \subset [n]^2$ percolates, then $|A| \geq n$.*

As we mentioned before, we are interested in finding sets that do percolate but do so in the maximum possible time. Now, we define specific functions to make this notion precise.

Definition 5.6. Given a natural number n, for $0 \leq s \leq n^2 - n$, we define $T_s(n)$ to be the *maximum time* t for which there exists a set $A \subseteq [n]^2$ of order $|A| = n + s$ which percolates in time t.

It is worth remarking that for a fixed n, the sequence $T_0(n), T_1(n), \ldots, T_{n^2-n}(n)$ is not be monotone (though we do not give a proof for that here). In this chapter we determine the exact value of $T_0(n)$. The idea of the proof is simple and relies on building a family of set that percolate on a particular way, proving that one of the sets in this family maximizes the percolation time and then determining such set. In order to do so, we shall need to use induction. Then it is natural to extend the definition of $T_0(n)$ for percolation on rectangles.

Definition 5.7. Given natural numbers k and ℓ, for $0 \leq s \leq k\ell - \lceil \frac{k+\ell}{2} \rceil$, we define $T_s(k, \ell)$ to be the maximal time t for which there exists a set $A \subseteq [k] \times [\ell]$ of order $|A| = \lceil \frac{k+\ell}{2} \rceil + s$ which percolates the rectangle $[k] \times [\ell]$ in time t. For a rectangle $R = \mathrm{Rec}(k, \ell)$ we shall let $T_s(R)$ be the maximum time in which a set internally spans R. Of course, $T_s(R)$ is just another notation for $T_s(k, \ell)$.

Before trying to compute bounds for $T_0(n)$, we should also understand how the infection happens on a broader scale. The first simple but important observation is the following.

Fact 5.8. *Given any set K of infected sites, $\langle K \rangle$ is a union of rectangles such that any distinct pair of them are at distance at least* 3.

This fact is clearly true by the following argument. The set K can be viewed as a union of 1 by 1 rectangles and any two fully infected rectangles within distance at most 2 do span a larger rectangle containing both. The next proposition from Holroyd [25] is a much more precise result in this direction.

Proposition 5.9. *Let R be a rectangle with area at least* 2. *Suppose that R is internally spanned by a set of sites K. Then there exist disjoint subsets of K, say K' and K'', and rectangles R' and R'' such that:*

(a) *the strict inclusions $R' \subsetneq R$ and $R'' \subsetneq R$ hold,* *(b) R' is internally spanned by sites in K' and*

R'' is internally spanned by sites in K'',

(c) $\langle R' \cup R'' \rangle = R$. In particular, $\mathrm{dist}(R', R'') \le 2$.

Remark. *Note that in Proposition 5.9 we cannot require the rectangles R' and R'' to be disjoint (see Figure 5.1).*

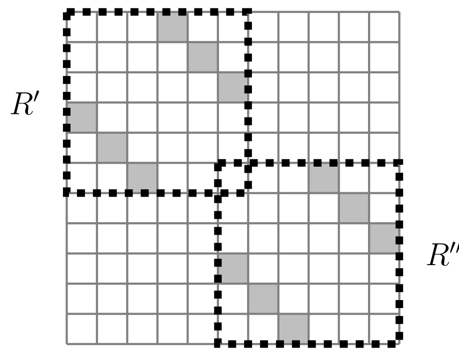

Figure 5.1: An example where rectangles R' and R'' are uniquely determined by the initially infected sites and do overlap.

Remark. *Although Proposition 5.9 is sharp, it does not describe the percolation process in a step by step fashion (i.e., as the time t increases by one). In fact, it may happen that for a particular time t some sites in $R \setminus (R' \cup R'')$ become infected while some of $R' \cup R''$ are still healthy. Even though the problem we study is intrinsically time related, we are able to make heavy use of Proposition 5.9.*

5.3 Slowly percolating sets with the minimum number of sites

In this section our aim is to compute the exact value of $T_0(n)$ for every $n \in \mathbb{N}$. We start by defining a family which percolates rectangles in a particular way.

Definition 5.10. Given positive integers k, ℓ, let $\mathcal{R}^{k,\ell}$ be the family of sets $A \subset [k] \times [\ell]$ where $|A| = \lceil (k + \ell)/2 \rceil$ and such that A percolates $[k] \times [\ell]$ in the following way. There exists an integer r and a nested sequence of rectangles $R_0 \subset R_1 \subset \ldots \subset R_r = [k] \times [\ell]$ such that denoting $R_i = \mathrm{Rec}(s_i, t_i)$ the following conditions hold:

(a) either $s_0 \leq 2$ or $t_0 \leq 2$ or $s_0 = t_0 = 3$; and $s_1, t_1 \geq 3$ and $(s_1, t_1) \neq (3,3)$;

(b) among the sites in R_0 the last one to be infected is one of its corners;

(c) for every $0 \leq i \leq r-1$ we have $\Phi(R_{i+1}) = \Phi(R_i) + 2$;

(d) R_i is internally spanned and there exists a site $v_i \in A$ such that $R_i \cup \{v_i\}$ internally spans R_{i+1}; and v_i is at distance exactly two from the last site to become infected in R_i (as in Figure 5.2 or in Figure 5.3).

We remark that for every $A \in \mathcal{R}^{k,\ell}$, the last site to become infected is one of the sites $(1,1)$, $(1,\ell)$, $(k,1)$, (k,ℓ).

Definition 5.11. For $A \in \mathcal{R}^{k,\ell}$, we say that the sequence $R_0 \subset R_1 \subset \ldots \subset R_r$ satisfying the conditions above is the configuration associated with A. We also say that we have used *Option A* at *moment* i (to construct R_{i+1}) if $R_{i+1} = \mathrm{Rec}(s_i + 1, t_i + 1)$ and we have used *Option B* at *moment* i if either $R_{i+1} = \mathrm{Rec}(s_i + 2, t_i)$ or $R_{i+1} = \mathrm{Rec}(s_i, t_i + 2)$. Finally, for a natural number n, we let $\mathcal{R}^n = \mathcal{R}^{n,n}$.

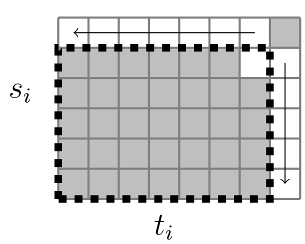

Figure 5.2: Option A at moment i.

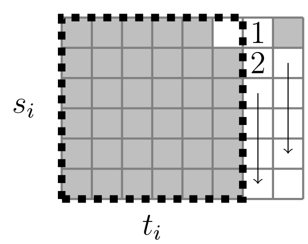

Figure 5.3: Option B at moment i.

We shall prove a recursion formula for $T_0(k,\ell)$ that works for all values k and ℓ such that $k+\ell$ is even. Furthermore, when $\min\{k,\ell\} \geq 2$ we prove that there is an element of $\mathcal{R}^{k,\ell}$ whose time to percolate $[k] \times [\ell]$ is $T_0(k,\ell)$ (but this is not necessarily true for all elements of $\mathcal{R}^{k,\ell}$). In the next lemma, we compute $T_0(2,\ell)$ for all values of ℓ and later we use that lemma as one of the base cases for the recursion.

Lemma 5.12. *For any ℓ even we have that $T_0(2,\ell) = \frac{3\ell-4}{2}$. Furthemore, there is a set $A^0(2,\ell)$ which percolates $[2] \times [\ell]$ in time $T_0(2,\ell)$ in a way that one of the four corners of $T_0(2,\ell)$ gets infected last.*

Proof. We define $A^0(2,\ell)$ as the set of shaded sites in Figure 5.4. Clearly $A^0(2,\ell)$ percolates $[2] \times [\ell]$ in time $\frac{3\ell-4}{2}$ and the last infected site in the infection process initiated by $A^0(2,\ell)$ is either $(2,\ell)$ or $(1,\ell)$.

We have that $T_0(2,\ell) \geq \frac{3\ell-4}{2}$ for any ℓ even. Now we prove by induction on ℓ that for any ℓ even we have $T_0(2,\ell) \leq \frac{3\ell-4}{2}$. Clearly, $T_0(2,2) = 1$. Assume that we are given $\ell \geq 4$, ℓ even, and suppose that $T_0(2,\ell-2) = \frac{3\ell-6}{2}$. Let A be any set that percolates $[2] \times [\ell]$. Since A percolates, any two consecutive columns of $[2] \times [\ell]$ contain at least one site of A. In particular, each of the 2 by 2 squares of the form $\{1,2\} \times \{2i-1, 2i\}$, $1 \leq i \leq \ell/2$, must contain at least one site of A. But $|A| = (\ell/2) + 1$, so only one of such squares can contain two sites of A. Therefore, either $\{1,2\} \times \{1,2\}$ or $\{1,2\} \times \{\ell-1,\ell\}$ contains exactly one site of A. Assume without loss of generality that the later holds. Since A percolates, either $(1,\ell)$ or $(2,\ell)$ must be the originally infected site. Again without loss of generality we may assume that the later happens. In this setting, it is trivial to check that A must internally span $[2] \times [\ell-2]$. Therefore, A takes time at most $T_0(2,\ell-2) + 3 = \frac{3\ell-4}{2}$ to percolate. $\square$

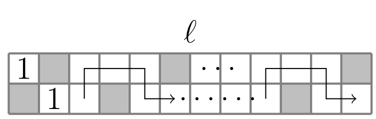

Figure 5.4: A set of initially infected sites which gives the maximum percolation time on $[2] \times [\ell]$ when ℓ is even.

Since $T_s(k,\ell) = T_s(\ell,k)$, in the statement of the next lemma we omit some cases where $k > \ell$.

Lemma 5.13. *We have $T_0(1,1) = 0$; $T_0(1,\ell) = 1$ for all odd $\ell \geq 3$; $T_0(2,\ell) = \frac{3\ell-4}{2}$ for all even $\ell \geq 2$; and $T_0(3,3) = 4$. For $k,\ell \geq 3$ such that $(k,\ell) \neq (3,3)$ and $k+\ell$ is even, we have*

$$T_0(k,\ell) = \max \begin{cases} T_0(k-1,\ell-1) + \max\{k-1,\ell-1\}, \\ T_0(k,\ell-2) + k + 1, \\ T_0(k-2,\ell) + \ell + 1. \end{cases} \tag{5.1}$$

Furthermore, if $\min\{k,\ell\} \geq 2$, then there exists a set $A^0(k,\ell) \in \mathcal{R}^{k,\ell}$ that percolates in time $T_0(k,\ell)$.

Proof. We prove Lemma 5.13 by induction on $k+\ell$. Our aim is to define a set $A^0(k,\ell) \subset [k] \times [\ell]$ satisfying the conditions of the Lemma 5.13.

We leave the trivial cases where $k = 1$ or $k = \ell = 3$ to the reader. The case where $k = 2$ and ℓ even follows from Lemma 5.12. We note also that the set $A^0(2,\ell)$ of shaded sites in Figure 5.4 trivially satisfies $A^0(2,\ell) \in \mathcal{R}^{k,\ell}$.

Now, assume that we are given $k,\ell \geq 3$ such that $(k,\ell) \neq (3,3)$ and $k+\ell$ is even. Our induction hypothesis is that for any k',ℓ' such that $k'+\ell'$ is even, $k'+\ell' < k+\ell$ and $\min\{k',\ell'\} \geq 2$, there exists $A^0(k',\ell') \in \mathcal{R}^{k',\ell'}$ which percolates in time $T_0(k',\ell')$, as in the statement of Lemma 5.13. We can further assume, by considering symmetries of $A^0(k',\ell')$, that in the infection started by $A^0(k',\ell')$ the site (k',ℓ') is infected the latest, that is, at time $T_0(k',\ell')$.

Assume without loss of generality that $k \leq \ell$. We shall first prove that the following holds.

$$T_0(k,\ell) \geq \max \begin{cases} T_0(k-1,\ell-1)+\ell-1, \\ T_0(k,\ell-2)+k+1, \\ T_0(k-2,\ell)+\ell+1. \end{cases} \tag{5.2}$$

Consider the following three particular ways of infecting $[k]\times[\ell]$ (see Figures 5.2 and 5.3).

(a) Let $A^0(k-1,\ell-1) \in \mathcal{R}^{k-1,\ell-1}$, such that it spans $[k-1]\times[\ell-1]$ in time $T_0(k-1,\ell-1)$ and $|A^0(k-1,\ell-1)| = (k+\ell-2)/2$. Also, assume that the site $(k-1,\ell-1)$ becomes infected at time $T_0(k-1,\ell-1)$. Note that such $A^0(k-1,\ell-1)$ exists by induction hypothesis. Let $A' = A^0(k-1,\ell-1)\cup\{(k,\ell)\}$. We have that A' takes time $T_0(k-1,\ell-1)+\ell-1$ to percolate. In addition, the corner site $(k,1)$ becomes infected only at the last time step.

(b) Let $A^0(k,\ell-2) \in \mathcal{R}^{k,\ell-2}$, such that it spans $[k]\times[\ell-2]$ in time $T_0(k,\ell-2)$ and $|A^0(k,\ell-2)| = (k+\ell-2)/2$. Also, assume that the site $(k,\ell-2)$ becomes infected at time $T_0(k,\ell-2)$. Note that such $A^0(k,\ell-2)$ exists by induction hypothesis. Let $A'' = A^0(k,\ell-2)\cup\{(k,\ell)\}$. We have that A'' takes time $T_0(k,\ell-2)+k+1$ to percolate. In addition, the corner site $(1,\ell)$ becomes infected only at the last time step.

(c) When $k\geq 4$, so that $k-2,\ell\geq 2$, we can also select $A^0(k-2,\ell)\in\mathcal{R}^{k-2,\ell}$, such that it spans $[k-2]\times[\ell]$ in time $T_0(k-2,\ell)$ and $|A^0(k-2,\ell)| = (k+\ell-2)/2$. Also, assume that the site $(k-2,\ell)$ becomes infected at time $T_0(k-2,\ell)$. Note that $A^0(k-2,\ell)$ exists by induction hypothesis. Let $A''' = A^0(k-2,\ell)\cup\{(k,\ell)\}$. We have that A''' takes time $T_0(k-2,\ell)+\ell+1$ to percolate. In addition, the corner site $(k,1)$ becomes infected only at the last time step.

Note that, for $k, \ell \geq 4$, all three sets A', A'' and A''' above are well defined. Hence, inequality (5.2) holds in this case. For $k = 3$ and $\ell \geq 4$ only A' and A'' are well defined. However, for $k = 3$ and $\ell \geq 4$, the condition that $3 + \ell$ is even imply that $\ell \geq 5$. So we have $T_0(2, \ell - 1) + \ell - 1 \geq T_0(1, \ell) + \ell + 1$. Hence, inequality (5.2) also holds in this case. So, the lower bound on $T_0(k, \ell)$ is proved and is attained by a set in $\mathcal{R}^{k,\ell}$.

Now, we only need to give a analogous upper bound on $T_0(k, \ell)$, that is,

$$T_0(k,\ell) \leq \max \begin{cases} T_0(k-1, \ell-1) + \ell - 1, \\ T_0(k, \ell-2) + k + 1, \\ T_0(k-2, \ell) + \ell + 1. \end{cases} \tag{5.3}$$

Consider any set K which internally spans the rectangle $R = [k] \times [\ell]$ in time $T_0(k, \ell)$, and is such that $|K| = (k + \ell)/2$. By Proposition 5.9, there exist disjoint subsets of K, say K' and K'', and two rectangles R' and R'' satisfying conditions (a)-(c) of Proposition 5.9. By Proposition 5.4 and condition (c), we have that

$$\Phi(R' \cup R'') \geq \Phi(\langle R' \cup R'' \rangle) = \Phi(R) = k + \ell.$$

By Fact 5.3, condition (b) and Corolary 5.5,

$$\Phi(R' \cup R'') \leq \Phi(R') + \Phi(R'') \leq 2|K'| + 2|K''| \leq 2|K| = k + \ell.$$

Therefore, each of the above inequalities must be an equality. In particular, $\Phi(R' \cup R'') = \Phi(R') + \Phi(R'')$. Fact 5.3 implies that $\operatorname{dist}(R', R'') \geq 2$, which together with condition (c) gives that R' and R'' must be at distance exactly 2. Also, we must have $\Phi(R') = 2|K'|$ and $\Phi(R'') = 2|K''|$, therefore, both $\Phi(R')$ and $\Phi(R'')$ are even.

Let s_1, t_1, s_2, t_2 be such that $R' = \mathrm{Rec}(s_1, t_1)$ and $R'' = \mathrm{Rec}(s_2, t_2)$. We have $\Phi(R') + \Phi(R'') = \Phi(R)$, so $s_1 + s_2 + t_1 + t_2 = k + \ell$. Since R' and R'' must be at distance exactly 2, the values for s_1, t_1, s_2, t_2 and the positions of R' and R'' inside R, must satisfy exactly one of the following conditions.

Condition A: Either $s_1 + s_2 = k + 1$, $t_1 + t_2 = \ell - 1$ and the rectangles align like in Figure 5.5 (A), or $s_1 + s_2 = k - 1$ and $t_1 + t_2 = \ell + 1$ and we have an analogous picture.

Condition B: We have $s_1 + s_2 = k$, $t_1 + t_2 = \ell$ and the rectangles align like in Figure 5.5 (B).

Condition C: Either $s_1 = k$, $s_2 = 1$, $t_1 + t_2 = \ell - 1$ and there is an $0 \leq m \leq k - 1$ so that the rectangles align as in Figure 5.5 (C), or $s_1 + s_2 = k - 1$, $t_1 = \ell$ and $t_2 = 1$ and we have an analogous picture.

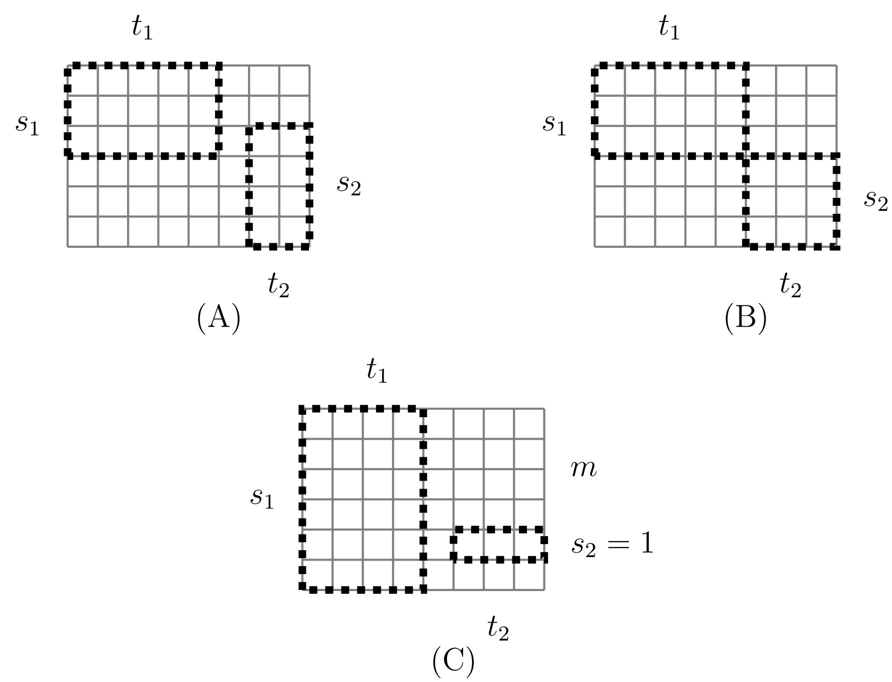

Figure 5.5: Three possible Rectangles alignments.

Additionally, the rectangles R' and R'' are non-degenerate and must be internally spanned by $\frac{s_1+t_1}{2}$ and $\frac{s_2+t_2}{2}$ sites respectively.

Note that, if Condition A or Condition B holds, we can assume without loss of generality that $T_0(R') \geq T_0(R'')$. If Condition C holds, then the roles of R' and R'' are not interchangeable, but we have $T_0(R'') \leq 1$, so we also have $T_0(R') \geq T_0(R'')$. Later it will be convenient to assume that $T(R') \geq 2$, so we consider now the case where $T_0(R') = T_0(R'') = 1$. If this happens, both R' must have a side of length one. Considering that $\min\{k, \ell\} \geq 3$ and $\max\{k, \ell\} \geq 4$, a small case analysis shows that if $T_0(R') = T_0(R'') = 1$, the percolation time for K is at most equal to the lower bound given by inequality (5.2). From now on, we assume that $s_1, t_1 \geq 2$.

We can bound from above the time that K takes to percolate $[k] \times [\ell]$ by the maximum time to internally span R' plus the time to grow from R' to R, that is, to infect all sites in $R \setminus (R' \cup R'')$ given that all sites in R' and R'' are infected. Therefore, the time that K takes to percolate is at most

$$\begin{cases} T_0(R') + \max\{s_1 + t_2, s_2 + t_1\}, & \text{if Condition A holds,} \\ T_0(R') + \max\{s_1 + t_2 - 1, s_2 + t_1 - 1\}, & \text{if Condition B holds,} \\ T_0(R') + \max\{m + t_2 + 1, s_1 - m - s_2 + t_2 + 1\}, & \text{if Condition C holds.} \end{cases} \tag{5.4}$$

Now, fix $0 \leq i, j \leq 2$ such that $i + j = 2$, $s_1 + i \leq k$, $t_1 + j \leq \ell$, $s_2 - i > 0$ and $t_2 - j > 0$. Next, we show that each of the above bounds does not decrease when we replace (s_1, t_1, s_2, t_2) by $(s_1 + i, t_1 + j, s_2 - i, t_2 - j)$ and $T_0(R')$ by $T_0(s_1 + i, t_1 + j)$. This implies that the weakest, i.e., largest, upper bound on the percolation time of K is attained when Rec' has semi-perimeter $k + \ell - 2$ and Rec'' is a single site.

Firstly, define

$$\begin{aligned} M^A_{s_1,t_1,s_2,t_2} &= \max\{s_1 + t_2, s_2 + t_1\}, \\ M^B_{s_1,t_1,s_2,t_2} &= \max\{s_1 + t_2 - 1, s_2 + t_1 - 1\}, \\ M^C_{m,s_1,t_1,s_2,t_2} &= \max\{m + t_2 + 1, s_1 - m - s_2 + t_2 + 1\}. \end{aligned}$$

Note that for any $0 \leq m \leq s_1 - s_2$ we have $M^C_{m,s_1,t_1,s_2,t_2} \leq M^C_{0,s_1,t_1,s_2,t_2}$ so let $M^C_{s_1,t_1,s_2,t_2} = M^C_{0,s_1,t_1,s_2,t_2}$. Therefore for any $Q \in \{A, B, C\}$ we have

$$M^Q_{s_1+i,t_1+j,s_2-i,t_2-j} \geq M^Q_{s_1,t_1,s_2,t_2} - 2. \tag{5.5}$$

Secondly, we also give a lower bound on the growth of $T(s_1, t_1)$ as follows.

Claim 5.14. *We have that $T_0(s_1 + i, t_1 + j) \geq T_0(s_1, t_1) + 2$.*

Proof of Claim 5.14. We consider the three possible values for (i, j).

Case 1: $(i, j) = (2, 0)$. We have that $s_1 + 2 \leq k$ and $t_1 \leq \ell$. So, if in addition we had $\min\{s_1 + 2, t_1\} \geq 3$ and $\max\{s_1 + 2, t_1\} \geq 4$, we could use inequality (5.2) to obtain

$$T_0(s_1 + 2, t_1) - T_0(s_1, t_1) \geq t_1 + 1 \geq 2.$$

Since $s_1 \geq 2$, if $t_1 \geq 3$ we are done as above. If $t_1 = 2$, as $s_1 + t_1$ is even, we have that $T_0(s_1 + 2, 2) - T_0(s_1, 2) = 3$ follows from Lemma 5.12.

Case 2: $(i, j) = (0, 2)$. An analogous argument to the previous case works.

Case 3: $(i, j) = (1, 1)$. We have that $s_1 + 1 \leq k$ and $t_1 + 1 \leq \ell$. So, if in addition we had $\min\{s_1 + 1, t_1 + 1\} \geq 3$ and $\max\{s_1 + 1, t_1 + 1\} \geq 4$, we could use inequality (5.2) to obtain

$$T_0(s_1 + 1, t_1 + 1) - T_0(s_1, t_1) \geq \max\{t_1, s_1\} \geq 2.$$

Assume, without loss of generality, that $t_1 \leq s_1$. If $t_1 \geq 3$ or if $t_1 = 2$ and $s_1 \geq 3$, we are done. If $t_1 = 2$ and $s_1 = 2$, we just need to check that
$T_0(3, 3) - T_0(2, 2) = 3 > 2$. $\square$

Applying Claim 5.14 together with inequality (5.5) several times we conclude the following. If R', R'' satisfies either Condition A or C, then $\max\{T_0(k, \ell-2)+k+1, T_0(k-2, \ell)+\ell+1\}$ is an upper bound on the time that the K takes to percolate. If R', R'' satisfies Condition B then $T_0(k-1, \ell-1)+\ell-1$ is an upper bound for the time that K takes to percolate. Since one of the three conditions must hold, we have that

$$\max\{T_0(k-1, \ell-1)+\ell-1, T_0(k, \ell-2)+k+1, T_0(k-2, \ell)+\ell+1\}$$

is a general upper bound for the percolation time of K. Since K was arbitrary, it is also an upper bound for $T_0(k, \ell)$. This completes the proof. $\square$

In the next theorem we shall give the precise value of $T_0(n)$ for $n \geq 4$. In its statement we use $_{\{a|b\}}$ to denote the indicator

$$_{\{a|b\}} = \begin{cases} 1, & \text{if } b \text{ is a multiple of } a \\ 0, & \text{otherwise.} \end{cases} \tag{5.6}$$

Theorem 5.15. *Let $n \geq 4$ and let $m = \left\lfloor \frac{n}{2} - \frac{5}{2} \right\rfloor + {}_{\{4|n-1\}} + {}_{\{4|n\}}$. Then*

$$T_0(n) = \frac{n^2 + n(m+2) - (m^2+5m+6)}{2}. \tag{5.7}$$

Proof. Let, $n \geq 4$ be given. By Lemma 5.13, there exists a set $A^0(n, n) \in \mathcal{R}^{n,n}$ which percolates $[n]^2$ in the maximum time $T_0(n)$. So, it is enough to determine which set in $\mathcal{R}^{n,n}$ takes the longest to percolate and compute how long it takes to do so. Assume that $K \in \mathcal{R}^n$ is a set that percolates in time $T_0(n)$ and let $R_0 \subset R_1 \subset \ldots \subset R_r = [n]^2$ be the configuration associated with K. It is easy to check that for every i, with $1 \leq i \leq r$, the sites $K \cap R_i$ must internally span R_i in the maximum possible time, i.e., in time $T_0(R_i)$.

First, we treat a number of small cases to exclude some, a priori possible, values for the numbers s_0 and t_0.

Suppose, for a contradiction, that $R_0 = \mathrm{Rec}(1, t)$. Since $R_1 = \mathrm{Rec}(s_1, t_1)$ where $s_1, t_1 \geq 3$ and $\max\{s_1, t_1\} \geq 4$, we must have $R_1 = \mathrm{Rec}(3, t)$ with $t \geq 5$. Since we have $T_0(2, t-1) \geq 4$, we obtain $T_0(3, t) \geq t - 1 + 4 = t + 3$. However, $R_0 = \mathrm{Rec}(1, t)$ and $R_1 = \mathrm{Rec}(3, t)$, so, in the infection process defined by K, it takes time at most $t + 1$ to infect all sites of R_1. This contradicts the fact the time that K takes to percolates in maximum.

Suppose now that $R_0 = \mathrm{Rec}(3, 3)$. Note that either $R_1 = \mathrm{Rec}(4, 4)$ or $R = \mathrm{Rec}(3, 5)$. In the first case, it takes time 3 to infect R_1 after R_0 has been fully infected. Since $T_0(3) = 4$, this procedure takes time at most $4 + 3 = 7$ to infect R_1. However, $T_0(4) = T_0(2, 4) + 4 + 1 = 9$. So, we have a contradiction like is the previous paragraph. In the second case, where $R_1 = \mathrm{Rec}(3, 5)$, it takes at most time 4 to grow from R_0 to R_1, resulting in R_1 being fully infected at time at most $T_0(3) + 4 = 8$. However, $T_0(3, 5) = T_0(2, 4) + 4 = 8$. Although, this does not contradict the maximality of K, we can replace K by a set K' whose infection process starts with a $\mathrm{Rec}(2, 4)$ and expands to R_1, so that K' takes the same time to percolate $[n]^2$ as K. Because of that, we may as well assume that $R_0 \neq \mathrm{Rec}(3, 3)$. Therefore we assume that $R_0 = \mathrm{Rec}(2, t)$ for some even $t \geq 4$.

The following two observations are crucial to determine the precise value of $T_0(n)$. In fact, with those observations and equation (5.1), we shall be able to find a percolating set which takes time exactly $T_0(n)$ to percolate.

Observation 5.16. *For any $i \geq 1$, no matter weather one uses Option A or Option B at moment i (to infect the rectangle R_{i+1}), at each time step after R_i is fully infected and until all sites of R_{i+1} are infected we have that at most two sites become infected.*

Observation 5.17. *For any $i \geq 1$, the following statements hold.*

(a) If we use Option A at moment i, there are exactly $|s_i - t_i|$ time steps after R_i is fully infected and until all sites of R_{i+1} are infected where only one new site becomes infected.

(b) If $s_i, t_i \geq 2$ and we use Option B at moment i, then there are exactly 3 time steps after R_i is fully infected and until all sites of R_{i+1} are infected where only one new site becomes infected.

By Observation 5.16 and because the number of initially infected sites is constant, a set from $\mathcal{R}^n$ that maximizes the percolation time, must also maximize the number of time steps in which only one new site becomes infected. Let $\mathcal{S}^n_m \subset \mathcal{R}^n$ be the subfamily of sets for which in its infection process the Option B is used exactly m times. (Note that when n and m have opposite parities we have $\mathcal{S}^n_m = \emptyset$).

By Observation 5.17, for a fixed m, the configuration associated with a set in $\mathcal{S}^n_m$ which maximizes the percolation time among those in $\mathcal{S}^n_m$, can be described as follows:

(a) Phase 1: start with $R_0 = \mathrm{Rec}(2, n-m)$, where $n - m \geq 4$.

(b) Phase 2: use Option A m times in order to get a rectangle $R_m = \mathrm{Rec}(2+m, n)$.

(c) Phase 3: use Option B $\frac{n-2-m}{2}$ times, finally percolating the whole $[n]^2$ grid.

Let the configuration satisfying the above description be denoted by $\mathcal{C}^n_m$. For example, Figure 5.6 shows the set of initially infected sites whose associated configuration is $\mathcal{C}^{12}_4$.

Now, we notice that for every $n \geq 4$ and $0 \leq m \leq n-4$ for which m and n have the same parity, the percolation time for $\mathcal{C}^n_m$ can be given explicitly as follows:

(a) Phase 1 takes time $T_0(2, n-m) = \left\lfloor \frac{3(n-m-1)}{2} \right\rfloor = \frac{3(n-m)}{2} - 2$;

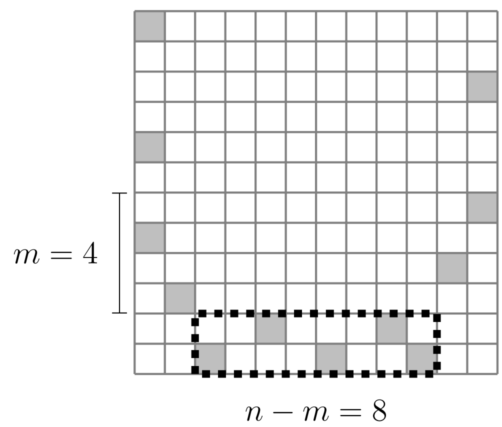

Figure 5.6: Configuration $\mathcal{C}_4^{12}$.

(b) Phase 2 takes time $\sum_{i=0}^{m-1}(n-m+i) = mn - m^2 + \frac{m(m-1)}{2} = mn - \frac{m(m+1)}{2}$;

(c) Phase 3 takes time $\frac{n-m-2}{2}(n+1) = \frac{n^2-n-mn-m-2}{2}$.

Letting $f(n,m)$ denote the percolation time for $\mathcal{C}_m^n$, by the above calculations we have

$$f(n,m) = \frac{n^2 + n(m+2) - (m^2+5m+6)}{2}.$$

For a given n, the function $f_n(m) = f(n,m)$ is a quadratic function in m with maximum value at $m = \frac{n-5}{2}$. As we are interested in maximizing $f_n(m)$ subject to $m \in \mathbb{N}$ and m having the same parity of n, its maximum value is obtained for

$$m = \left\lfloor \frac{n}{2} - \frac{5}{2} \right\rfloor + {}_{\{4|n-1\}} + {}_{\{4|n\}},$$

That ends the proof. □

From (5.7) we obtain the following corollary.

Corollary 5.18. *We have*

$$\lim_{n\to\infty} \frac{T_0(n)}{n^2} = \frac{5}{8}.$$

The most natural open problem would be to compute $T_m(n)$ for all suitable values of m. First, we generalize the definition of the family $\mathcal{R}^{k,\ell}$.

Definition 5.19. Given positive integers m, k, ℓ, let $\mathcal{R}_m^{k,\ell}$ be the family of sets $A \subset [k] \times [\ell]$ where $|A| = \lceil (k+\ell)/2 \rceil + m$ and such that A percolates $[k] \times [\ell]$ in the following way. There exists an integer r and a nested sequence of rectangles $R_0 \subset R_1 \subset \ldots \subset R_r = [k] \times [\ell]$ such that denoting $R_i = \mathrm{Rec}(s_i, t_i)$ the following conditions hold:

(a) either $s_0 \leq 2$ or $t_0 \leq 2$ or $s_0 = t_0 = 3$; and $s_1, t_1 \geq 3$ and $(s_1, t_1) \neq (3, 3)$;

(b) For at most $2m$ possible values of i, with $0 \leq i \leq r-1$, we have $\Phi(R_{i+1}) = \Phi(R_i) + 1$; for the remaining values of i we have $\Phi(R_{i+1}) = \Phi(R_i) + 2$.

(c) R_i is internally spanned and there exists a site $v_i \in A$ such that $R_i \cup \{v_i\}$ internally spans R_{i+1}.

We remark that $\mathcal{R}_0^{k,\ell} = \mathcal{R}^{k,\ell}$. We conjecture that there is a set A in $\mathcal{R}_m^{k,\ell}$ which percolates in time $T_m(n)$. One can also aim to compute directly the quantity $M(n) = \max\{T_s(n) : 0 \leq s \leq n^2 - n\}$. We have an example which comes from solving a recursion for a lower bound on $M(n)$ and that percolates in time approximately $13n^2/18$. We hope to prove in a short-coming article that this example is optimal, that is, $M(n)$ is approximately $13n^2/18$.

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