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On the local solvability of the initial-boundary value problem of fiber spinning of the upper convected Maxwell fluid

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ON THE LOCAL SOLVABILITY OF THE INITIAL-BOUNDARY VALUE PROBLEM OF FIBER SPINNING OF THE UPPER CONVECTED MAXWELL FLUID

by

Dias Kurmashev

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Dedicated to my family.

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ABSTRACT

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The fiber spinning process of a viscoelastic liquid modeled by the constitutive theory of the Maxwell fluid is analyzed. The governing equations are given by one-dimensional mass, momentum, and constitutive equations which arise in the slender body approximation by cross-sectional averaging of the two-dimensional axisymmetric Stokes equations with free boundary. Existence, uniqueness, and regularity results are proved by means of fixed point arguments, energy estimates, and weak/weak* convergence methods. The complexity in this problem lies with the constitutive model of the Maxwell fluid: when both the outflow velocity at the spinneret and the pulling velocity at take-up are prescribed, a boundary condition can be imposed for only one of the two elastic stress components at the inlet. The absence of the second stress boundary condition makes the mathematical analysis of the problem difficult.

Contents

Chapter 1

Introduction

1.1 Problem Description

In industrial applications such as fiber spinning and film casting, polymeric melts or solutions are extruded through dies to form synthetic fibers and films. Particularly, in the former case the polymeric liquid is withdrawn from a reservoir through a circular orifice (spinneret) and axially stretched. The resulting thin fiber is then wound up and passed on to post-processing. The rough sketch of the process is shown in Fig.1.

Molten fluid exits the orifice at $z = 0$ with radius R_0 and initial velocity V_0. It proceeds down while being stretched. The velocity V_L at the take up point $z = L$ is prescribed with the magnitude $V_L > V_0$.

Although the liquid gradually cools down during the stretching, the temperature gradient is negligible compared to the temperature loss in the cooling device, so the process is considered isothermal. The mass flow due to evaporation of the polymer solvent is not taken into account for simplicity reasons as well. Additional assumptions of vanishing inertia, surface tension, and gravity were suggested by the specifics of the industrial production. The radius of an orifice is typically around 1 mm in diameter with a wound up filament 10 μm thick [16]. The inertia, gravity, and the tension between air and fluid are small compared to the drawing forces.

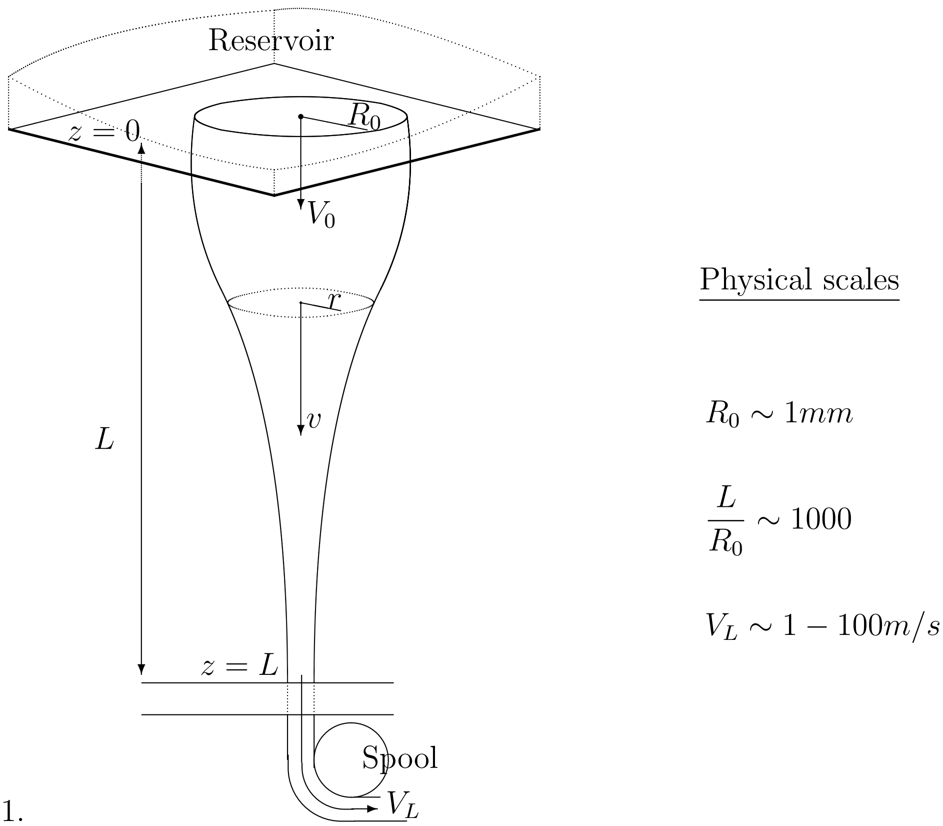

Figure 1.

1.2 Previous Work

Numerous studies are devoted to the subject of the fiber draw-down process. Many researchers have studied the dynamics, stability, and break up of viscous and visco-elastic fluids in extension. The wide range of physical parameters leads to different models and interpretations of the process. The difficulty arises not only in selecting the proper model, but also in assigning appropriate boundary conditions [19].

Several analytical results about the equations of fiber spinning were obtained by Hagen [5, 6, 7] and Hagen and Renardy [9]. In these works, either viscous stresses were included in the constitutive equations [7], thus allowing both stress boundary conditions at the inlet, or both elastic stress components were given at the inlet and one of the velocity components was dropped [5, 6, 9]. In either case the difficulty encountered with the absent stress boundary condition here was avoided.

The analytic approach chosen in [5, 6, 9] and for the earlier nonisothermal viscous case in [5, 8] was based on a contraction mapping argument in certain Sobolev-Bochner spaces. This approach, although being effective, is somewhat tedious and technically demanding. In [7] the contraction mapping argument was replaced by a compactness result and the Schauder fixed point theorem. In this work we will pursue a similar strategy. To this end we will make use of a crucial estimate for solutions of the linear transport equation in a certain regularity class borrowed from [5, 6, 8].

1.3 Upper Convected Maxwell Model

While the Newtonian fluid model, with the linear relation $\tau = \mu \dfrac{\partial v}{\partial y} = \mu\dot{\gamma}$ between the shear stress τ and the velocity gradient $\dot{\gamma}$, explains the behavior of many gases and liquids, there are numerous phenomena where this model fails. The peculiar reaction of a cornstarch suspension to a sudden stress and the Weissenberg effect (when paint or cream climb up a rotating rod) are examples that may be observed in everyday life. A broad set of even more interesting effects is described and commented on in [1].

The explanation of these phenomena requires an adjustment to the constitutive equation for the stress tensor and the introduction of different concepts, such as the concept of elasticity, which is normally a property of solid-state bodies. In addition to that, the linearity of the equation is also questionable. Nevertheless, even revised equations provide a good description for only some of the effects and fit poorly for others.

One of the first viscoelastic models (still linear) was based on the consideration of the so-called Maxwell element, which consists of a perfectly elastic body with the modulus G and a purely viscous unit with viscosity η_0 in sequence. The one-dimensional equation for this model is given by

$$\tau + \lambda \frac{\partial \tau}{\partial t} = \eta_0 \dot{\gamma}$$

where the new term $\lambda = \eta_0 / G$ is called the relaxation time.

The same viscous and elastic bodies joined in parallel will represent a Voigt (or Kelvin) element, and the combination of the Maxwell and Voigt elements leads to the so-called Jeffreys model, see [26]. The constitutive equations for the Voigt and Jeffreys models are given by

$$\tau = G\gamma + \eta_0 \dot{\gamma}$$

and

$$\tau + \lambda_1 \frac{\partial \tau}{\partial t} = \eta_0 \Big(\dot{\gamma} + \lambda_2 \frac{\partial \dot{\gamma}}{\partial t} \Big)$$

respectively. Here, $\lambda_1 = \lambda$ is still the relaxation parameter and λ_2 is the retardation time constant.

Although these three models provide a much better description of viscoelastic flows, they do not fit for the analysis of fluids that demonstrate non-linear effects. The rheological properties of the liquid undergoing the extension strongly determine its flow behavior. Various constitutive models derived from microstructural or phenomenological considerations [11] have been studied numerically in the hope to better understand the prevalent flow instabilities and other physical effects occurring during fiber- and film-forming flows of actual viscous and viscoelastic liquids. Among these, the constitutive theory of the upper convected Maxwell (UCM) fluid plays a special role, since it has a microstructural basis and is the principal representative for a large class of constitutive equations in differential form. Although its physical correctness and applicability to flows of real liquids is certainly questionable, more realistic fluid models (such as the Phan-Thien–Tanner and Giesekus fluids) can be obtained from the UCM fluid, see [1].

The constitutive equations in tensor form for the upper convected Maxwell model are given by

$$\mathbf{T} + \lambda \overset{\triangledown}{\mathbf{T}} = 2\eta_0 \mathbf{D}.$$

Here, $\mathbf{T}$ is the extra stress tensor, which is a real-valued matrix 3×3, and "$\triangledown$" is the so-called upper convected time derivative. For an arbitrary tensor $\mathbf{S}$

$$\overset{\triangledown}{\mathbf{S}} \overset{\text{def}}{=} \frac{\mathrm{D}\mathbf{S}}{\mathrm{D}t} - (\nabla \mathbf{u})^T \cdot \mathbf{S} - \mathbf{S} \cdot (\nabla \mathbf{u}),$$

where $\mathbf{u} \in \mathbb{R}^3$ denotes a fluid velocity field. The quantity $\mathbf{D} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$ is the rate of deformation tensor, η_0 is the polymer viscosity, and λ is the relaxation time. The operator $\dfrac{\mathrm{D}}{\mathrm{D}t}$ is the substantial (or material) derivative defined as

$$\frac{\mathrm{D}}{\mathrm{D}t} \overset{\text{def}}{=} \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla.$$

The main purpose of this study is to prove solvability of the particular problem, which arises from the fiber spinning process and to show uniqueness and regularity of such a solution.

1.4 Thesis Outline

In Chapter 2 we introduce the model as full three-dimensional mass and momentum conservation equations with free surface boundary conditions. These equations will be complemented with inflow-outflow conditions later. The axial symmetry of the fluid filament will allow for the description of the model in cylindrical coordinates. Non-dimensionalizing of those equations and the asymptotic analysis, due to the slenderness of the body, will yield a one-dimensional fiber model for which we shall formulate the main problem. The notations for spaces and norms, along with some definitions and used methods, will conclude the chapter.

In the beginning of Chapter 3, we will state the principal existence theorem for the linear transport equation and follow up with the solution estimation lemma. Both results were proved in [5] and [8]. The main result of this dissertation is substantially based on these two statements. Then we will discuss the compatibility requirements to be imposed onto the boundary and the initial data. The final section of the chapter will be the formulation of the main result.

Chapter 4 contains the discussion of the spaces and sets where we will be looking for solutions of the problem. We will introduce the bounded convex set $\mathbb{S}(t', L, M)$ which is a subset of $[L^2([0,t'] \times [0,1])]^3 \times L^2(0,t')$, and show that $\mathbb{S}(t', L, M)$ is sequentially compact in the latter, using the existence theorem and the estimation lemma. Next, rewriting governing equations in an implicit form, we will construct an operator Σ that, for the properly chosen constants $t', L,$ and M, continuously maps the set $\mathbb{S}(t', L, M)$ into itself. This provides the grounds to apply Schauder's fixed point theorem and prove the existence of a solution. Finally, using energy estimates, we will show that the solution is unique.

In Chapter 5 we provide a brief overview of some other viscoelastic models given in differential and integral forms and their relations. The overall conclusion of the thesis finishes the chapter.

The definitions of the functional spaces, a short overview of their properties, and related main theorems used in this work are given in the appendix for reference.

Chapter 2

Problem Setting

2.1 Derivation of the Governing Equations

The set of equations for the viscoelastic incompressible flow for an upper convected Maxwell model contains

Momentum and mass conservation equations:

$$\begin{aligned} \rho(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) = & -\nabla p + \nabla \cdot \mathbf{T} + \mathbf{f}, \\ \nabla \cdot \mathbf{u} = & \ 0. \end{aligned}$$

Constitutive law:

$$\mathbf{T} + \lambda \left(\frac{\partial}{\partial t} \mathbf{T} + \mathbf{u} \cdot \nabla \mathbf{T} - (\nabla \mathbf{u})^T \cdot \mathbf{T} - \mathbf{T} \cdot (\nabla \mathbf{u}) \right) = \eta_0 (\nabla \mathbf{u} + (\nabla \mathbf{u})^T),$$

(see [1], [11] or [26]).

Kinematic and dynamic conditions on the free surface:

$$\begin{aligned} (\mathbf{U} - \mathbf{u}) \cdot \mathbf{n} &= 0, \\ (\mathbf{T} - p\mathbf{I}) \cdot \mathbf{n} &= -k\sigma \mathbf{n} + S\mathbf{t}. \end{aligned}$$

The velocity field $\mathbf{u}$ and the stress tensor $\mathbf{T}$ were already introduced. The quantities p and ρ denote the pressure and the density of the fluid, respectively. The dynamic viscosity is denoted by η_0, and λ is the relaxation time. The body forces (per unit volume) acting on the fluid are represented by $\mathbf{f}$. We use $\mathbf{U}$ for the velocity of the fluid surface, the symbol $\mathbf{I}$ stands for the identity matrix, and $\mathbf{n}$, $\mathbf{t}$ are the normal and tangential vectors to the free fluid boundary. The surface tension coefficient is denoted by σ. The S and k coefficients are the shear stress and twice the mean curvature respectively [22].

Due to the axial symmetry, we use the cylindrical coordinate system, where

$$\mathbf{u} = \begin{pmatrix} u \\ w \\ v \end{pmatrix}, \qquad \mathbf{T} = \begin{pmatrix} T_{rr} & T_{r\theta} & T_{rz} \\ T_{\theta r} & T_{\theta\theta} & T_{\theta z} \\ T_{zr} & T_{z\theta} & T_{zz} \end{pmatrix}.$$

Each component of $\mathbf{u}$ and $\mathbf{T}$ is a function of time t, the radius r, angle θ, and the position along the axis z. Determining the filament surface through its distance $R(t,z)$ from the central axis we get the kinematic condition: $\mathbf{u}\cdot\mathbf{n} = \dfrac{\partial R}{\partial t}$.

To interpret the kinematic and dynamic free boundary conditions, we need to find the unit normal and tangential vectors **n** and **t** to the fluid surface (Fig. 2):

$$\mathbf{n} = (m, 0, n)^T \qquad \mathbf{t}\cdot\mathbf{n} = 0$$

$$\mathbf{t} = (s, 0, q)^T \qquad s^2 + q^2 = 1$$

$$\frac{q}{s} = \frac{\partial R}{\partial z}$$

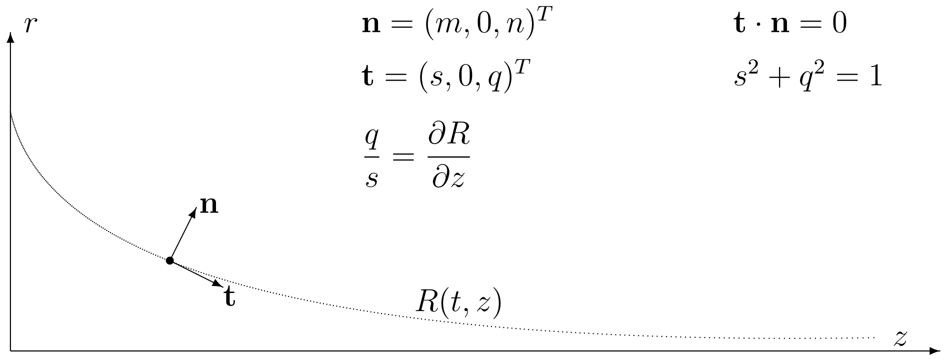

Figure 2.

We have

$$\mathbf{t} = \frac{1}{(1+(\partial R/\partial z)^2)^{1/2}}\begin{pmatrix} 1 \\ 0 \\ \partial R/\partial z \end{pmatrix}, \qquad \mathbf{n} = \frac{1}{(1+(\partial R/\partial z)^2)^{1/2}}\begin{pmatrix} -\partial R/\partial z \\ 0 \\ 1 \end{pmatrix}$$

Following [6], the main set of equations will be adjusted in accordance with the assumptions that reflect the physical realities of the process.

- The inertial and gravitational forces are negligible, which makes the terms $\rho(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u})$ and $\mathbf{f}$ disappear.
- All quantities are independent of θ, so $\dfrac{\partial}{\partial\theta} = 0$. No angular motion presented, therefore the velocity component w vanish. The components $T_{r\theta}, T_{z\theta}, T_{\theta r}$ and $T_{\theta z}$ of the stress tensor all dropped out.
- Surface tension and shear stress vanish on the free surface, and thus the normal and tangential components of the stress are zero on the boundary R :

$$[(\mathbf{T} - p\mathbf{I})\cdot\mathbf{n}]\cdot\mathbf{t} = 0 = [(\mathbf{T} - p\mathbf{I})\cdot\mathbf{n}]\cdot\mathbf{n}.$$

The simplified equations will now be non-dimensionalized. Done properly, this will allow us to employ further simplification based on the comparison of the relative sizes of corresponding terms [13]. We rescale the radial length by $1/R_0$, the axial length by $1/L$, and time by L/V_L. The axial velocity will be divided by V_L and the radial velocity by R_0V_L/L. The rescaling coefficient for the stress components and the pressure will be η_0V_L/L.

The acquired system will contain the following non-dimensional constants:

$$\begin{aligned} &\text{Reynolds number} && \mathrm{Re} = \frac{\rho L V_L}{\eta} \\ &\text{Weissenberg number} && \mathrm{We} = \frac{\lambda V_L}{L} \\ &\text{The ``slenderness'' parameter} && \varepsilon = \frac{R_0}{L} \end{aligned}$$

The system of equations is identical to the one in [6] and consists of

Momentum and mass conservation equations

$$\frac{\partial u}{\partial r} + \frac{1}{r}u + \frac{\partial v}{\partial z} = 0 \tag{2.1}$$

$$-\frac{\partial p}{\partial r} + \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}T_{rr} + \varepsilon\frac{\partial T_{rz}}{\partial z} - \frac{1}{r}T_{\theta\theta} = 0 \tag{2.2}$$

$$-\varepsilon\frac{\partial p}{\partial z} + \frac{\partial T_{rz}}{\partial r} + \frac{1}{r}T_{rz} + \varepsilon\frac{\partial T_{zz}}{\partial z} = 0 \tag{2.3}$$

Constitutive law:

$$\mathrm{We}\left(\frac{\partial T_{rr}}{\partial t} + u\frac{\partial T_{rr}}{\partial r} + v\frac{\partial T_{rr}}{\partial z} - 2T_{rr}\frac{\partial u}{\partial r} - 2\varepsilon T_{rz}\frac{\partial u}{\partial z}\right) + T_{rr} = 2\frac{\partial u}{\partial r} \tag{2.4}$$

$$\mathrm{We}\left(\frac{\partial T_{zz}}{\partial t} + u\frac{\partial T_{zz}}{\partial r} + v\frac{\partial T_{zz}}{\partial z} - \frac{2}{\varepsilon}T_{rz}\frac{\partial v}{\partial r} - 2T_{zz}\frac{\partial v}{\partial z}\right) + T_{zz} = 2\frac{\partial v}{\partial z} \tag{2.5}$$

$$\mathrm{We}\left(\frac{\partial T_{\theta\theta}}{\partial t} + u\frac{\partial T_{\theta\theta}}{\partial r} + v\frac{\partial T_{\theta\theta}}{\partial z} - 2T_{\theta\theta}\frac{u}{r}\right) + T_{\theta\theta} = 2\frac{u}{r} \tag{2.6}$$

$$\begin{aligned} \mathrm{We}\left(\frac{\partial T_{rz}}{\partial t} + u\frac{\partial T_{rz}}{\partial r} + v\frac{\partial T_{rz}}{\partial z} - \frac{1}{\varepsilon}T_{rr}\frac{\partial v}{\partial r} - T_{rz}\left(\frac{\partial v}{\partial z} + \frac{\partial u}{\partial r}\right) - \varepsilon T_{zz}\frac{\partial u}{\partial z}\right)& \\ + T_{rz} = \varepsilon\frac{\partial u}{\partial z} + \frac{1}{\varepsilon}\frac{\partial v}{\partial r}& \end{aligned} \tag{2.7}$$

Kinematic surface condition:

$$\frac{\partial R}{\partial t} + v\frac{\partial R}{\partial z} = u. \tag{2.8}$$

Normal and tangential stress conditions:

$$\varepsilon\Big(T_{rr} - T_{zz}\Big)\frac{\partial R}{\partial r} + T_{rz}\Big(1 - \Big(\varepsilon\frac{\partial R}{\partial z}\Big)^2\Big) = 0, \tag{2.9}$$

$$T_{rr} - 2\varepsilon T_{rz}\frac{\partial R}{\partial z} + T_{zz}\Big(\varepsilon\frac{\partial R}{\partial z}\Big)^2 - p\Big(1 + \Big(\varepsilon\frac{\partial R}{\partial z}\Big)^2\Big) = 0. \tag{2.10}$$

The magnitude of the aspect ratio ε is of order 10^{-3} [16]. This gives rise to the "order of magnitude analysis" [1], which is a commonly used procedure of reducing the complete set of governing equations by deriving their asymptotic counterparts for the small parameter ε. To attain an asymptotic equation, one formally expands each flow variable in a "pseudo-Taylor" series:

$$g = g^{[0]} + \varepsilon g^{[1]} + \varepsilon^2 g^{[2]} + \text{"higher order terms"} \tag{2.11}$$

and then compares, in regards to powers of ε, each term's contribution to the equation. We refer to [6] for the details. In the aftermath, one gets the following:

- The tensor component $T_{rz}^{[0]}$ vanishes and the equations for $T_{rr}^{[0]}$ and $T_{\theta\theta}^{[0]}$ are identical. Moreover, all the leading order stress components are radially independent, so we have $u^{[0]}\dfrac{\partial T_{zz}^{[0]}}{\partial r} = u^{[0]}\dfrac{\partial T_{\theta\theta}^{[0]}}{\partial r} = u^{[0]}\dfrac{\partial T_{rr}^{[0]}}{\partial r} = 0.$
- Six equations drop out, leaving only four for consideration:

$$\begin{aligned}
&\frac{\partial}{\partial t}(R^{[0]})^2 + \frac{\partial}{\partial z}(v^{[0]}(R^{[0]})^2) = 0,\\
&\frac{\partial}{\partial z}\Big((R^{[0]})^2(T_{rr}^{[0]} - T_{zz}^{[0]})\Big) = 0,\\
&T_{rr}^{[0]} + \mathrm{We}\left(\frac{\partial T_{rr}^{[0]}}{\partial t} + v^{[0]}\frac{\partial T_{rr}^{[0]}}{\partial z} + \frac{\partial v^{[0]}}{\partial z}T_{rr}^{[0]}\right) = -\frac{\partial v^{[0]}}{\partial z},\\
&T_{zz}^{[0]} + \mathrm{We}\left(\frac{\partial T_{zz}^{[0]}}{\partial t} + v^{[0]}\frac{\partial T_{zz}^{[0]}}{\partial z} - 2\frac{\partial v^{[0]}}{\partial z}T_{zz}^{[0]}\right) = 2\frac{\partial v^{[0]}}{\partial z}.
\end{aligned}$$

Several investigators [1],[6],[13], [19] have used this technique, and it is believed to be an adequate tool for studying elongational flows like fiber spinning.

In the resulting asymptotic equations we rename $v^{[0]}$ as v and $R^{[0]}$ as r. We also denote the components $T_{rr}^{[0]}$ and $T_{zz}^{[0]}$ of the stress tensor as $S = S(t,z)$ and $T = T(t,z)$. Doing so will allow us to use subscripts for partial derivatives.

2.2 Formulation of the Problem

The dominant balances of the governing equations from the previous section, written with new notations, will consist of the equations for mass and momentum conservation

$$\left(r^2\right)_t + \left(v\, r^2\right)_z = 0, \tag{2.12}$$

$$\left(r^2\ (S - T)\right)_z = 0, \tag{2.13}$$

and the constitutive equations for the upper convected Maxwell fluid

$$\mathrm{We}\ (S_t + v\, S_z + S\, v_z) + S = -v_z, \tag{2.14}$$

$$\mathrm{We}\ (T_t + v\, T_z - 2\, T\, v_z) + T = 2\, v_z. \tag{2.15}$$

To close the formulation of the problem, we impose the boundary conditions

$$r(t,0) = 1, \tag{2.16}$$

$$v(t,0) = 1, \tag{2.17}$$

$$v(t,1) = D > 1, \tag{2.18}$$

$$S(t,0) = 0, \tag{2.19}$$

and initial conditions of the form

$$r(0,z) = r^0(z), \tag{2.20}$$

$$S(0,z) = S^0(z), \tag{2.21}$$

$$T(0,z) = T^0(z). \tag{2.22}$$

The equations are stated on the normalized domain $0 \leqslant z \leqslant 1$, $t \geq 0$. Here the inlet/spinneret and take-up point are assumed at $z = 0$ and $z = 1$, respectively. The quantity We is a dimensionless (positive) relaxation time, called the Weissenberg number, which is a measure of the elasticity of the fluid. The quantity $D > 1$, referred to as "draw ratio," is a dimensionless velocity at the take-up point $z = 1$. The governing equations, as we have shown, arise in the slender body approximation of the axisymmetric Stokes equations with moving boundary. In the purely viscous case, the governing equations are essentially due to Matovich and Pearson [13]. Further details are given in [16].

The boundary conditions discussed here are the ones considered by Forest and Wang in [4]. They are motivated by the desire to control the outflow and take-up velocities, as well as the flow rate at the spinneret. Condition (2.19) is based on the observation that the second normal stress difference of the upper convected Maxwell fluid vanishes inside the spinneret, and that the radial elastic stress is expected to be small compared to the axial stress - at least for large Weissenberg numbers. A discussion of these boundary conditions is given in [4, 19]. We emphasize specifically that imposing an additional boundary condition for the axial elastic stress would render the governing equations overdetermined. This observation will be rigorously shown to follow from the results presented in this work. We also note that there is little mathematical difference in prescribing the radial or axial elastic stress at the inlet, or a ratio of the two as long as not both elastic stress components are given. For the sake of presentation, we have chosen the boundary values in (2.16)–(2.19) constant. All our results, however, will hold true (with minor modifications) for more general right-hand sides.

As was pointed out in [19], the boundary conditions chosen here are an idealization of the physical reality. There is no consensus on which conditions are physically most appropriate and enforceable in actual spinning applications of viscoelastic fluids. The boundary conditions listed above have, however, been commonly used in the literature.

Several authors have commented on the difficulties present in the governing equations due to the absence of one stress boundary condition, see e.g. [14, 15, 19]. This difficulty becomes apparent when one attempts to solve the governing equations numerically [14, 15]. For viscoelastic fluids with constitutive theory in integral form or for purely viscous flow, this problem does not arise: in the former case a stress history condition is imposed [14, 15], while in the latter case the stresses are given directly in terms of the velocity gradient.

In this work we address the solvability of the boundary-initial value problem given by Eqs. (2.12)–(2.22). We establish a (local-in-time) existence and uniqueness result of rather smooth solutions in suitably chosen function spaces. Our objective is to show the existence of solutions which allow the interpretation of the governing equations in the sense of classical derivatives.

2.3 Norms and Notations

The theory of Lebesgue and Bochner spaces will be involved in our study to some extent. Here we introduce notations that will be used in these spaces.

Let $a < b$, $t' > 0$ and m, $n \in \mathbb{N}_0$. Throughout this work we adopt the following abbreviations:

- $||\cdot||_p$ for the norm on the Lebesgue space $L^p(a,b)$, $1 \leqslant p \leqslant \infty$,
- $||\cdot||_{L^p}$ for the norm on the Lebesgue space $L^p([0,t'] \times [a,b])$,
- $||\cdot||_{H^n}$ for the norm on the Sobolev space $H^n(a,b)$,
- $||\cdot||_{m,n}$ for the norm on the Sobolev-Bochner space $W^{m,\infty}([0,t'];H^n(a,b))$.

Moreover, for functions $h \in L^\infty([0,t'];H^2(a,b)) \cap W^{1,\infty}([0,t'];H^1(a,b))$ we define the norm $|||\cdot|||$ by

$$|||h|||^2 = \|h\|_{0,2}^2 + \|h\|_{1,1}^2.$$

It will be clear in each situation what the concrete values of a, b, and t' are.

Occasionally the Bochner spaces $L^m(0,T;H^k(0,1))$ and $W^{m,n}(0,T;H^k(0,1))$ will be abbreviated to $L^m(H^k)$ and $W^{m,n}(H^k)$ respectively to simplify the exposition.

Let H be a Hilbert space with the inner product $(\cdot,\cdot)_H$. The pairing between elements of the Sobolev-Bochner spaces $f \in L^\infty(0,T;H)$ and $g \in L^1(0,T;H)$ will be denoted as

$$\langle f, g\rangle_{(L^\infty,L^1)} = \int_0^T \big(f(s), g(s)\big)_H ds.$$

The inner product in $L^2(0,T;H)$ will be written as

$$(f,g)_{L^2(H)} = \int_0^T \big(f(s), g(s)\big)_H ds.$$

Note that for $T < \infty$ we have inclusion

$$L^\infty(0,T;H^k(0,1)) \subset L^2(0,T;H^k(0,1)) \subset L^1(0,T;H^k(0,1)).$$

Therefore if $f \in L^\infty(0,T;H)$ and $g \in L^2(0,T;H)$ (and hence $g \in L^1(0,T;H)$), the pairing $\langle f, g\rangle_{(L^\infty, L^1)}$ coincides with the inner product $(f,g)_{L^2(H)}$.

2.4 Methods

Several proofs in this work use the same standard methods and inequalities. Since the next section contains such a proof, we introduce them here rather than in the appendix.

Grönwall's Inequality (differential form). See [3]

(i) For an absolutely continuous function $\psi :\ [0,T] \to \mathbb{R}_0^+$ that satisfies

$$\frac{\partial}{\partial t}\psi(t) \ \leqslant\ \alpha(t)\psi(t) + \beta(t), \qquad \text{where } 0 \leqslant \alpha(t), \beta(t) \in L^1[0,T],$$

the following is true:

$$\psi(t) \ \leqslant\ e^{\int_0^t \alpha(s)ds}\left[\psi(0) + \int_0^t \beta(s)ds\right] \qquad \text{for all } 0 \leqslant t \leqslant T.,$$

(ii) In particular, if $\frac{\partial}{\partial t}\psi \ \leqslant\ \alpha\psi$ on $[0,T]$ and $\psi(0) = 0$, then

$$\psi \ \equiv\ 0 \quad \text{on } [0,T].$$

Cauchy's Inequality: for a, b -positive

$$ab \leqslant \frac{a^2}{2} + \frac{b^2}{2}.$$

Hölder's Inequality: for $1 < p, q < \infty$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$

$$\int_0^1 \big|f(x)g(x)\big| \,\mathrm{d}x \quad \leqslant \quad \left(\int_0^1 |f(x)|^p \,\mathrm{d}x\right)^{1/p} \left(\int_0^1 |g(x)|^q \,\mathrm{d}x\right)^{1/q}.$$

We also have

$$\int_0^1 \big|f(x)g(x)\big| \,\mathrm{d}x \leqslant \|g\|_\infty \int_0^1 \big|f(x)\big| \,\mathrm{d}x$$

for $p = 1,\ q = \infty$.

Energy Method. This method was used in [6] and [7] to estimate the difference of two solutions of the following transport equation:

$$u_t(t,x) + p(t,x)u_x(t,x) = f(t,x).$$

The application of the same method for classical heat and wave equations is provided in [3]. In the following arguments we assume the required smoothness of all the relevant terms a priori.

Let $p, f_i, u_i,\ i = 1, 2$ be the functions defined on the domain $[0,T] \times [0,1]$ such that $p(t,x) > 0$ and $u_i(t,x)$ are solutions of the following initial-boundary problem:

$$(u_i)_t + p(u_i)_x = f_i,$$

$$u_i(0,x) = u^0(x),$$

$$u_i(t,0) = u^*(t).$$

Taking the difference of the two transport equations, we get another initial-boundary problem, which, after denoting $\bar{u} = u_1 - u_2$, $\bar{f} = f_1 - f_2$, receives the following form:

$$\bar{u}_t + p\bar{u}_x = \bar{f},$$
$$\bar{u}(0, x) = 0,$$
$$\bar{u}(t, 0) = 0.$$

Multiplication by $\bar{u}$ and integration from 0 to 1 with respect to x provides:

$$\int_0^1 \bar{u}\bar{u}_t \, dx = -\int_0^1 p\bar{u}\bar{u}_x \, dx + \int_0^1 \bar{u}\bar{f} \, dx,$$
$$\frac{1}{2}\int_0^1 (\bar{u}^2)_t \, dx \leqslant \frac{1}{2}\int_0^1 p(\bar{u}^2)_x \, dx + \int_0^1 |\bar{u}\bar{f}| \, dx. \tag{2.23}$$

Given that we have continuous differentiability of $\bar{u}$ with respect to time, the term on the left side of (2.23) can be rewritten as

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \bar{u}^2 \, dx = \frac{1}{2}\frac{d}{dt}\|\bar{u}(t, \cdot)\|_2^2. \tag{2.24}$$

The last integral in Eq.(2.23) shows, using Cauchy's inequality

$$\int_0^1 |\bar{u}\bar{f}| \, dx \leqslant \|\bar{u}(t, \cdot)\|_2 \|\bar{f}(t, \cdot)\|_2 \leqslant \frac{1}{2}\|\bar{u}(t, \cdot)\|_2^2 + \frac{1}{2}\|\bar{f}(t, \cdot)\|_2^2. \tag{2.25}$$

We evaluate the middle term in Eq.(2.23) by parts:

$$-\int_0^1 p(\bar{u}^2)x \, dx = -p\bar{u}^2\Big|_{x=0}^{x=1} + \int_0^1 p_x\bar{u}^2 \, dx. \tag{2.26}$$

Since $-p(t,1)\bar{u}^2(t,1) \leqslant 0$ and $\bar{u}(t,0) = 0$, combining (2.24-2.26) transforms (2.23) into

$$\frac{d}{dt}\|\bar{u}(t,\cdot)\|_2^2 \leqslant \|p_x(t,\cdot)\|_\infty \|\bar{u}(t,\cdot)\|_2^2 + \|\bar{u}(t,\cdot)\|_2^2 + \|\bar{f}(t,\cdot)\|_2^2. \tag{2.27}$$

Note that according to the Sobolev embedding

$$\|p_x(s,\cdot)\|_\infty \leqslant \|p_x(s,\cdot)\|_{H^1(0,1)} \leqslant \|p(s,\cdot)\|_{H^2(0,1)}.$$

Finally, Grönwall's inequality gives

$$\|\bar{u}(t,\cdot)\|_2^2 \leqslant e^{\displaystyle\int_0^t (\|p(s,\cdot)\|_{H^2(0,1)} + 1)ds} \int_0^t \|\bar{f}(s,\cdot)\|_2^2 ds. \tag{2.28}$$

Chapter 3

The Linear Transport Equation

3.1 Existence Theorem

Definition 3.1. *The space* $\mathbb{BR}(0,t';0,1)$ *of boundary-regular functions consists of all functions* $g = g(t,x)$ *on* $[0,t'] \times [0,1]$ *such that*

$$g \in W^{1,\infty}([0,t'];H^1(0,1)) \cap L^\infty([0,t'];H^2(0,1)), \tag{3.1}$$

$$g_x(\cdot,0), g_x(\cdot,1) \in H^1(0,t'). \tag{3.2}$$

The space $\mathbb{BR}(0,t';0,1)$ *is endowed with the energy norm*

$$\mathcal{E}(g) \stackrel{\text{def}}{=} \left(||g||_{0,2}^2 + ||g||_{1,1}^2 + ||g_x(\cdot,0)||_{H^1}^2 + ||g_x(\cdot,1)||_{H^1}^2\right)^{\frac{1}{2}}. \tag{3.3}$$

The importance of the notion of "boundary-regularity" lies in the following theorem and its corollary.

Theorem 3.2. *Let* f *,* p *,* u^0*, and* u^* *be functions such that*

$$p, f \in \mathbb{BR}(0, t'; 0, 1) \tag{3.4}$$

$$p > 0 \text{ on } [0, t'] \times [0, 1], \tag{3.5}$$

$$u^0 \in H^2(0, 1), \tag{3.6}$$

$$u^* \in H^2(0, t'), \tag{3.7}$$

$$u^0(0) = u^*(0), \tag{3.8}$$

$$u_t^*(0) + p(0, 0)\, u_x^0(0) = f(0, 0). \tag{3.9}$$

Then the boundary-initial value problem on $[0, t'] \times [0, 1]$

$$u_t(t, x) + p(t, x)\, u_x(t, x) = f(t, x), \tag{3.10}$$

$$u(0, x) = u^0(x), \tag{3.11}$$

$$u(t, 0) = u^*(t) \tag{3.12}$$

has a solution u *such that*

$$u \in C^1([0, t']; H^1(0, 1)) \cap C([0, t']; H^2(0, 1)) \cap \mathbb{BR}(0, t'; 0, 1), \tag{3.13}$$

$$u \text{ is unique in } W^{1,\infty}([0, t']; L^2(0, 1)) \cap L^\infty([0, t']; H^1(0, 1)). \tag{3.14}$$

The main argument in the proof of this theorem is based on quasidissipativity in $L^2(0, 1)$ of family of operators

$$[A(t)u](x) = p(t, x)u_x(x) + q(t, x)u(x), \qquad t \in [0, t'].$$

Applying semigroup theory provides the result. We refer to [5, 6, 8] for the details.

The proof of the theorem contains an important estimate, which we state here in the form most useful for the following developments.

Corollary 3.3. *For* $t_0 > 0$ *there exists a polynomial* P *and a constant* $C > 0$ *such that, for each* $0 < t' \leqslant t_0$ *and each solution* u *of the boundary-initial value problem* (3.10)–(3.12) *on* $[0,t'] \times [0,1]$ *with functions* f , p , u^0, *and* u^* *satisfying the conditions* (3.4)–(3.9)*, the following holds true:*

$$\begin{aligned}
\|u\|_{0,2}^2 \leqslant{}& P(\mu^{-1}) \Big(\|u^0\|_{H^2}^2 + \big(1 + \|p\|_{0,2}^2\big)^3 \, \|u^*\|_{H^2}^2 + \\
& t' \, (1 + |||p|||)^3 \, \big(\|u^*\|_{H^2}^2 + |||f|||^2\big)\Big) \, e^{(C\,|||p||| + 1)\, t'}, \qquad (3.15)\\
\mathcal{E}(u)^2 \leqslant{}& P(\mu^{-1}) \Big(\|u^0\|_{H^2}^2 + \|p(0,\cdot)\, u_x^0 + f(0,\cdot)\|_{H^1}^2 + \big(1 + \|p\|_{0,2}^2\big)^3 \, \|u^*\|_{H^2}^2 + \\
& t' \, (1 + |||p|||)^3 \, \big(\|u^*\|_{H^2}^2 + |||f|||^2\big) + t' \, |||p|||^2 \, \|u\|_{0,2}\Big) \times \qquad (3.16)\\
& e^{C\,(|||p||| + \|u\|_{0,2} + 1)\, t'},
\end{aligned}$$

where μ *is the minimum value of* p *on* $[0,t'] \times [0,1]$.

The estimates above are obtained by using the energy method that was introduced in the previous section. While Lemma 3.15 and Corollary 3.17 and their corresponding proofs in [5, 8] contain the details, we give a brief indication of how the argument proceeds.

From (3.10) it is apparent that

$$u_x(t,0) = \frac{f(t,0) - u_t(t,0)}{p(t,0)} = \frac{f(t,0) - u_t^*(t)}{p(t,0)}, \qquad (3.17)$$

$$u_{xx}(t,0) = \frac{f_x(t,0) - u_{tx}(t,0) - p_x(t,0) u_x(t,0)}{p(t,0)}, \qquad (3.18)$$

$$u_{xt}(t,0) = \frac{f_t(t,0) - u_{tt}^*(t) - p_t(t,0) u_t^*(t)}{p(t,0)}. \qquad (3.19)$$

The formal differentiation of Eq. (3.10) yields:

$$u_{tx} + pu_{xx} + p_x u_x = f_x, \tag{3.20}$$

$$u_{txx} + pu_{xxx} + 2p_x u_{xx} + p_{xx} u_x = f_{xx}, \tag{3.21}$$

$$u_{tt} + pu_{tx} + p_t u_x = f_t, \tag{3.22}$$

$$u_{ttx} + pu_{txx} + p_x u_{tx} + p_t u_{xx} + p_{tx} u_x = f_{tx}. \tag{3.23}$$

Next, we multiply Eqs. (3.10),(3.20),(3.21) by u, u_x, u_{xx} respectively, add the three together, and integrate over $[0,1]$ with respect to z to obtain:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^1 (u^2 + u_x^2 + u_{xx}^2)dx = -\frac{1}{2}\left[p(u^2 + u_x^2 + u_{xx}^2)\right]\Big|_0^1 + \frac{1}{2}\int_0^1 p_x(u^2 + u_x^2 + u_{xx}^2)dx \\ - \int_0^1 p_x u_x^2 dx - 2\int_0^1 p_x u_{xx}^2 dx - \int_0^1 p_{xx} u_x u_{xx} dx + \int_0^1 (fu + f_x u_x + f_{xx} u_{xx})dx. \end{aligned} \tag{3.24}$$

To get rid of p_{xx} we use integration by parts:

$$-\int_0^1 p_{xx}\, u_x\, u_{xx}\, dx = -p_x\, u_x\, u_{xx}\Big|_0^1 + \int_0^1 p_x\, u_{xx}^2\, dx$$

Substituting Eqs. (3.17-3.19) into $-\frac{1}{2}\left[p(u^2 + u_x^2 + u_{xx}^2)\right]\Big|_0^1$ and $-p_x\, u_x\, u_{xx}\Big|_0^1$ evaluated at the boundaries, letting C be a generic constant that absorbs all the Sobolev embedding constants and others, and applying Grönwall's lemma, we obtain (3.15). In a similar fashion, we estimate the other norms of the solution u and that will result in (3.16).

3.2 Compatibility Conditions

In this section we state our principal existence/uniqueness result and set the stage for the proofs in later sections. Compatibility conditions for the boundary/initial values are discussed.

Definition 3.4. *A vector field* (r, v, S, T), *defined on* $[0, t'] \times [0, 1]$ *for some* $t' > 0$, *is a solution of the boundary-initial value problem* (2.12)–(2.22) *if*

$$r, v, S, T \in C^1([0, t']; H^1(0,1)) \cap C([0, t']; H^2(0,1)), \tag{3.25}$$

$$r,\ v,\ S,\ T \textit{ satisfy Eqs. } (2.12)\text{–}(2.15), \tag{3.26}$$

$$r \textit{ satisfies Eqs. } (2.16),\ (2.20) \textit{ and } r > 0, \tag{3.27}$$

$$S \textit{ satisfies Eqs. } (2.19),\ (2.21), \tag{3.28}$$

$$T \textit{ satisfies Eq. } (2.22), \tag{3.29}$$

$$v \textit{ satisfies Eqs. } (2.17),\ (2.18) \textit{ and } v > 0. \tag{3.30}$$

The velocity and radius are required to be positive for physical reasons. Observe that the regularity of the solution is strong enough to guarantee continuous differentiability on $[0, t'] \times [0, 1]$. We tacitly assume that the boundary and initial data are regular enough (the regularity requirements will be made clear later on).

The velocity can be expressed in terms of other functions. First, we note that Eq. (2.13) implies

$$(S - T)_z = -2\frac{r_z(S - T)}{r}. \tag{3.31}$$

Integrating the same equation and evaluating it at $z = 0$ provides

$$r^2(S - T) = -T(t, 0) \qquad \text{for all } (t, z). \tag{3.32}$$

Next, subtraction of Eq. (2.15) from Eq. (2.14) and multiplication by r^2 yields

$$\mathrm{We}\big(r^2(S-T)_t + vr^2(S-T)_z + v_z r^2(S+2T)\big) + r^2(S-T) = -3v_z r^2. \qquad (3.33)$$

Applying Eqs. (3.31)–(3.32) to the latter, we obtain v_z, which, after integration with respect to z while taking into account boundary conditions for the velocity, results in

$$v(t,z) = D + \int_1^z \frac{T(t,0) + \mathrm{We}\, T_t(t,0)}{-2\,\mathrm{We}\, T(t,0) + 3\,\left(r(t,x)\right)^2\,\left(1 + \mathrm{We}\, T(t,x)\right)}\, dx. \qquad (3.34)$$

Here, the constant $D > 1$ is the draw ratio.

The smoothness required for solutions can only hold true for initial and boundary data which satisfy certain compatibility conditions that are in agreement with Eqs. (2.12)–(2.22). Specifically, for consistency, we need

$$r^0(z) > 0 \quad \text{for } 0 \leqslant z \leqslant 1 \qquad (3.35)$$

and

$$\left(r^0(z)\right)^2\,\left(S^0(z) - T^0(z)\right) = \text{const.} \qquad (3.36)$$

However, the regularity of a solution also requires that the boundary/initial data and their respective derivatives satisfy additional conditions. In particular, to match initial and boundary data at $t = 0$, $z = 0$, we impose the constraints

$$r^0(0) = 1 \quad \text{and} \quad S^0(0) = 0. \qquad (3.37)$$

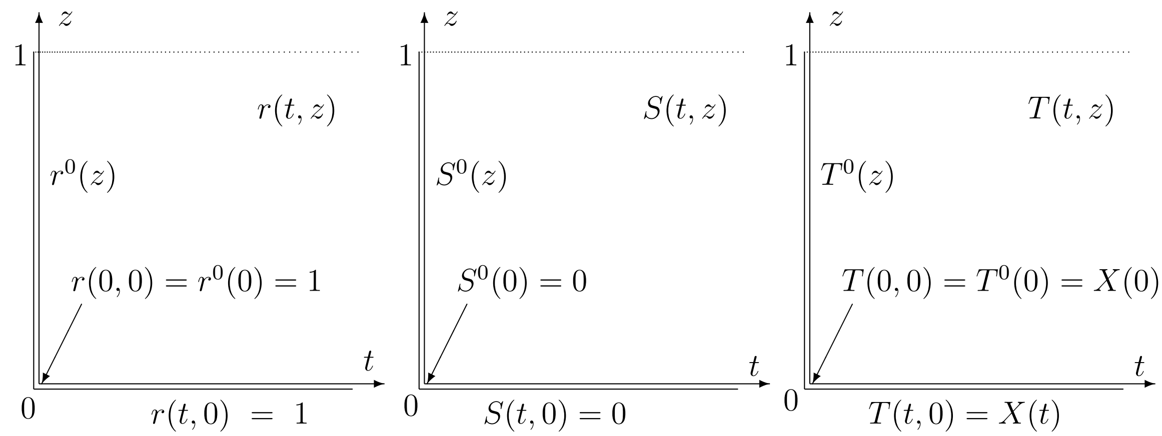

Figure 3. Compatibility Conditions

To match the first derivatives, we demand

$$r_z^0(0) = -\frac{1}{2}\, v_z(0,0) \quad \text{and} \quad S_z^0(0) = -\frac{v_z(0,0)}{\mathrm{We}}, \tag{3.38}$$

and

$$v_z(0,0) = \frac{T^0(0) + \mathrm{We}\, T_t(0,0)}{3 + \mathrm{We}\, T^0(0)}. \tag{3.39}$$

Notice that the dependence of the velocity on the unknown boundary data of T is the core problem that we will have to address. To this end, we introduce the unknown boundary function

$$X(t) = T(t,0) \tag{3.40}$$

and note that Eq. (3.34), evaluated at $z=0$, gives a first-order ordinary differential equation for X

$$\mathrm{We}\, X_t(t) + X(t) = \frac{D-1}{\displaystyle\int_0^1 \frac{1}{-2\,\mathrm{We}\, X(t) + 3\,\left(r(t,x)\right)^2\,\left(1 + \mathrm{We}\, T(t,x)\right)}\, dx}. \tag{3.41}$$

Together with the initial condition

$$X(0) = T^0(0) \tag{3.42}$$

this equation is, in principle, solvable when all the other quantities are known and division by zero is avoided. Observe that X solving Eq. (3.41) implies that $v = 1$ at $z = 0$ in Eq. (3.34). For such a solution X we still have to make sure that its first derivative is correctly related to the initial value T^0. First, Eq. (3.41) imposes the condition

$$X_t(0) = -\frac{1}{\mathrm{We}}\, T^0(0) + \frac{D-1}{\displaystyle\int_0^1 \frac{\mathrm{We}}{-2\,\mathrm{We}\,T^0(0) + 3\,\left(r^0(x)\right)^2\,\left(1 + \mathrm{We}\,T^0(x)\right)}\,dx}. \tag{3.43}$$

Second, Eq. (2.15) requires

$$\begin{aligned} X_t(0) &= -T^0_z(0) + \left(2\,v_z(0,0) - \frac{1}{\mathrm{We}}\right) T^0(0) + \frac{2}{\mathrm{We}}\,v_z(0,0) \\ &= -T^0_z(0) + 2v_z(0,0)\left(T^0(0) + \frac{1}{\mathrm{We}}\right) - \frac{1}{\mathrm{We}} T^0(0). \end{aligned} \tag{3.44}$$

Using Eq. (3.39) in (3.44), we obtain

$$X_t(0) = -T^0_z(0) + 2\frac{T^0(0) + \mathrm{We}\,X_t(0,0)}{3 + \mathrm{We}\,T^0(0)}\left(T^0(0) + \frac{1}{\mathrm{We}}\right) - \frac{1}{\mathrm{We}} T^0(0),$$

which, after solving for $X_t(0)$, provides

$$X_t(0) = -\frac{1}{\mathrm{We}}\,T^0(0) + \frac{\mathrm{We}\,T^0(0) + 3}{\mathrm{We}\,T^0(0) - 1}\,T^0_z(0). \tag{3.45}$$

Hence Eqs. (3.43), (3.45) give the condition

$$\frac{D-1}{\displaystyle\int_0^1 \frac{\mathrm{We}}{-2\,\mathrm{We}\,T^0(0)+3\,\left(r^0(x)\right)^2\,\left(1+\mathrm{We}\,T^0(x)\right)}\,dx}=\frac{\mathrm{We}\,T^0(0)+3}{\mathrm{We}\,T^0(0)-1}\,T_z^0(0). \tag{3.46}$$

In summary, in addition to the compatibility conditions (3.35)–(3.37) and (3.46), we have Eq. (3.38) which assumes the form

$$r_z^0(0)=-\frac{1}{2}\,\frac{\mathrm{We}}{\mathrm{We}\,T^0(0)-1}\,T_z^0(0), \tag{3.47}$$

$$S_z^0(0)=-\frac{1}{\mathrm{We}\,T^0(0)-1}\,T_z^0(0). \tag{3.48}$$

Here we have used the identity

$$v_z(0,0)=\frac{\mathrm{We}}{\mathrm{We}\,T^0(0)-1}\,T_z^0(0). \tag{3.49}$$

Since the velocity has to be positive initially, we demand that for $0\leqslant z\leqslant 1$

$$\begin{aligned} v(0,z)=v^0(z)=D+ & \\ \int_1^z \frac{\mathrm{We}\,(\mathrm{We}\,T^0(0)+3)\,T_z^0(0)}{(\mathrm{We}\,T^0(0)-1)\,\left(3\,\left(r^0(x)\right)^2\,\left(1+\mathrm{We}\,T^0(x)\right)-2\,\mathrm{We}\,T^0(0)\right)}\,dx>0.\end{aligned} \tag{3.50}$$

Of course, it is clear that all expressions appearing in denominators have to be nonzero and that the velocity cannot be constant. Hence we have to ensure that

$$T^0(0)\notin\left\{-\frac{3}{\mathrm{We}},\frac{1}{\mathrm{We}}\right\}, \tag{3.51}$$

$$T_z^0(0)\neq 0, \tag{3.52}$$

$$3\,\left(r^0(z)\right)^2\,\left(1+\mathrm{We}\,T^0(z)\right)-2\,\mathrm{We}\,T^0(0)\neq 0,\quad 0\leqslant z\leqslant 1. \tag{3.53}$$

To demonstrate that the set of initial conditions satisfying the compatibility requirements above is nonempty, we give one mathematically possible choice of data.

Example 3.5 Let $m = \frac{1}{4}\ln(2\,D - 1)$ and set

$$r^0(z) = e^{-mz}, \quad S^0(z) = -\frac{1}{\mathrm{We}} + \frac{1}{\mathrm{We}}\, e^{-2mz} = T^0(z). \tag{3.54}$$

With these functions, it follows readily that conditions (3.35)–(3.37), (3.46)–(3.48), (3.50)–(3.53) hold true.

Throughout the remainder of this work we will tacitly assume that all initial data considered satisfy the compatibility conditions (3.35)–(3.37), (3.46)–(3.48), (3.50)–(3.53).

3.3 Statement of the Main Result

We are now in a position to state the central result of this work.

Theorem 3.6. *Suppose the initial values* r^0, S^0, T^0 *are given in* $H^2(0,1)$ *(and are compatible). Then the boundary-initial value problem* (2.12)–(2.22) *has a solution* (r, v, S, T) *on* $[0, t'] \times [0, 1]$ *for some* $t' > 0$. *This solution has the properties*

$$r, S, T \in \bigcap_{k=0}^{2} C^k([0, t']; H^{2-k}(0, 1)), \tag{3.55}$$

$$r, S, T \in \mathbb{BR}(0, t'; 0, 1), \tag{3.56}$$

$$v \in \bigcap_{k=0}^{2} C^k([0, t']; H^{3-k}(0, 1)). \tag{3.57}$$

Moreover, (r, v, S, T) *is the unique solution of the boundary-initial value problem* (2.12)–(2.22) *in*

$$\left(\mathbb{BR}(0, t'; 0, 1)\right)^4 . \tag{3.58}$$

The proof of Theorem 3.6 will be split up in several steps. The idea is to use the Schauder fixed point theorem on a suitable compact set. A similar strategy was applied in the much simpler case of the Jeffreys fluid in [7], where both elastic stress components were prescribed at the inlet.

Chapter 4

The Solution Map

4.1 Set of Potential Solutions

We begin by introducing the set in which we will search for a solution.

Definition 4.7. *For* L *,* M *,* $t' > 0$ *, let* $\mathbb{S}(t', L, M)$ *be the set of all functions* (q, U, V, Y) *such that*

$$q, U, V \in \mathbb{BR}(0, t'; 0, 1) \quad \textit{and} \quad Y \in H^2(0, t'), \tag{4.1}$$

$$\mathcal{E}(q)^2 + \mathcal{E}(U)^2 + \mathcal{E}(V)^2 \leqslant L^2 \quad \textit{and} \quad \|Y\|_{H^2} \leqslant M, \tag{4.2}$$

$$q(0, z) = r^0(z) \quad \textit{and} \quad q(t, 0) = 1, \tag{4.3}$$

$$U(0, z) = S^0(z) \quad \textit{and} \quad U(t, 0) = 0, \tag{4.4}$$

$$V(0, z) = T^0(z), \tag{4.5}$$

$$Y(0) = T^0(0). \tag{4.6}$$

If the constants L and M are sufficiently large, then $(r^0, S^0, T^0, T^0(0))$ belongs to the set $\mathbb{S}(t', L, M)$ for any $t' > 0$. Since q_t, $V_t \in L^\infty((0, t'); H^1(0, 1))$ then, in particular, the following estimations hold for q_t :

$$|q(t,z)-r^0(z)| \leqslant \int_0^t |q_t(s,z)|\,ds \leqslant \int_0^t \sup_{z\in[0,1]} |q_t(s,z)|ds = \int_0^t \|q_t(s,\cdot)\|_{L^\infty(0,1)}ds.$$

By the Sobolev embedding we have $\|f\|_{L^\infty(\alpha,\beta)} \leqslant C_E\|f\|_{H^1(\alpha,\beta)}$ where the embedding constant C_E depends only on the interval (α,β), therefore

$$\begin{aligned} |q(t,z)-r^0(z)| &\leqslant C_E \int_0^t \|q_t(s,\cdot)\|_{H^1(0,1)}ds \leqslant t\,C_E\|q_t\|_{L^\infty(H^1)} \\ &\leqslant t\,C_E\|q\|_{W^{1,\infty}(H^1)} \leqslant t'\,C_E\|q\|_{1,1} \\ &\leqslant t'C_E\mathcal{E}(q). \end{aligned} \tag{4.7}$$

The norm of function $Y_t \in H^1(0,t')$ is bounded by M by the construction of $\mathbb{S}(t',L,M)$, so, as a consequence, we can find constants $C=C(L)>0$ and $c=c(M)>0$ such that

$$|q(t,z)-r^0(z)| \leqslant C\,t', \tag{4.8}$$

$$|V(t,z)-T^0(z)| \leqslant \int_0^t |V_t(s,z))|\,ds \leqslant C\,t', \tag{4.9}$$

$$|Y(t)-T^0(0)| \leqslant \int_0^t |Y_t(s)|\,ds \leqslant c\sqrt{t'}. \tag{4.10}$$

for all (q,U,V,Y) in $\mathbb{S}(t',L,M)$. Since condition (3.53) is assumed to hold, we obtain the following result.

Proposition 4.8. *There exist* L, $M>0$ *such that the set* $\mathbb{S}(t,L,M)$ *is nonempty for all* $t>0$. *Moreover, if* $\mathbb{S}(t_0,L,M)\neq\varnothing$, *then there is* $0<t^*\leqslant t_0$ *such that, for any* $0<t'\leqslant t^*$ *and* $(q,U,V,Y)\in\mathbb{S}(t',L,M)$,

$$3\,(q(t,z))^2\,(1+\mathrm{We}\,V(t,z))-2\,\mathrm{We}\,Y(t)\neq 0 \quad \textit{for } 0\leqslant t\leqslant t',\ 0\leqslant z\leqslant 1. \tag{4.11}$$

From now on we will tacitly assume that the conclusions of Proposition 4.8 hold true for our choices of t', L, $M > 0$.

Theorem 4.9. *$\mathbb{S}(t', L, M)$ is convex and compact in* $(L^2((0,t')\times(0,1)))^3 \times L^2(0,t')$.

Proof The set $\mathbb{S}(t', L, M)$ is convex by construction. It is precompact in the space $(L^2((0,t')\times(0,1)))^3 \times L^2(0,t')$ since it is bounded in $(H^1((0,t')\times(0,1)))^3 \times H^1(0,t')$, the latter space being compactly embedded in the former.

Suppose we have a sequence $(q_n, U_n, V_n, Y_n)_{n\in\mathbb{N}}$ in $\mathbb{S}(t', L, M)$ which is a Cauchy sequence in $(L^2((0,t')\times(0,1)))^3 \times L^2(0,t')$ with limit (q_0, U_0, V_0, Y_0). We will show that (q_0, U_0, V_0, Y_0) belongs to $\mathbb{S}(t', L, M)$.

First, since $(Y_n)_n$ is bounded in $H^2(0,t')$, $(Y_n)_n$ has a weakly convergent subsequence with weak limit Y^* in $H^2(0,t')$. The space $H^2(0,t')$ is compactly embedded into $L^2(0,t')$, and thus weak convergence in $H^2(0,t')$ implies strong convergence in $L^2(0,t')$. So we conclude $Y^* = Y_0$, hence $Y_0 \in H^2(0,t')$ and $\|Y_0\|_{H^2} \leqslant M$.

Next, let $(Z_n)_n$ be one of the sequences $(q_n)_n$, $(U_n)_n$, or $(V_n)_n$ and denote its limit in $L^2((0,t')\times(0,1))$ by Z_0. Since $(Z_n)_n$ is bounded in $L^\infty([0,t']; H^2(0,1))$ and since $L^\infty([0,t']; H^2(0,1))$ by definition is the conjugate of the separable Banach space $L^1([0,t']; H^2(0,1))$, applying the sequential Banach-Alaoglu theorem [2, 21], we can extract a weak* convergent subsequence with weak* limit Z^* in the space $L^\infty\big(0,t'; H^2(0,1)\big)$. Due to the boundedness of interval $[0,t']$, the sequence and its weak* limit both belong to $L^2\big(0,t'; H^2(0,1)\big)$, and the latter is a subset of $L^1\big(0,t'; H^2(0,1)\big)$.

For an arbitrary function $h \in L^2\big([0,t']; H^2(0,1)\big)$ (and hence in $L^1\big([0,t']; H^2(0,1)\big)$), consider the following sequence:

$$\int_0^{t'} \big(Z_n(t), h(t)\big)_{H^2(0,1)} dt. \tag{4.12}$$

Each Z_n, acting through integration, represents a bounded linear functional over the Lebesgue-Bochner space $L^1(0,t';H^2(0,1))$, thus the sequence (4.12) written as $\langle Z_n,h\rangle_{(L^\infty,L^1)}$ has to converge to $\langle Z^*,h\rangle_{(L^\infty,L^1)}$. On the other hand, the sequence $(Z_n)_n$ and the function h - both belong to $L^2(0,t';H^2(0,1))$; therefore, we may interpret the convergence of $\langle Z_n,h\rangle_{(L^\infty,L^1)}$ to $\langle Z^*,h\rangle_{(L^\infty,L^1)}$ in the sense of the inner product and this will confirm the convergence of $(Z_n)_n$ to Z^* in $L^2(0,t';H^2(0,1))$ weakly. This argument can be cast in the form that we will use repeatedly:

$$(Z_n,h)_{L^2(H^2)} = \langle Z_n\,,\,h\rangle_{(L^\infty,L^1)} \to \langle Z^*\,,\,h\rangle_{(L^\infty,L^1)} = (Z^*,h)_{L^2(H^2)}.$$

To show that $Z^*=Z_0$, we take an arbitrary $\psi \in L^2\big([0,t']\times[0,1]\big)$ and construct a functional M_ψ by setting

$$M_\psi(h) \overset{\text{def}}{=} \int_0^{t'}\!\!\int_0^1 \psi(t,z)\,h(t,z)\,dt\,dz \qquad\qquad \text{for any } h\in L^2\big(0,t';H^2(0,1)\big).$$

While M_ψ is being defined through the inner product in $L^2((0,t')\times(0,1))$, it is definitely a bounded linear functional acting on $L^2\big(0,t';H^2(0,1)\big)$. Therefore, the following is true:

$$(Z_n,\psi)_{L^2([0,t']\times[0,1])} = M_\psi(Z_n) \to M_\psi(Z^*) = (Z^*,\psi)_{L^2([0,t']\times[0,1])}.$$

This proves weak convergence $(Z_n)_n$ in $L^2((0,t')\times(0,1))$ to element Z^*. Now,

$$\begin{aligned} &(Z_n)_n \to Z_0 && \text{strongly in } L^2((0,t')\times(0,1)),\\ &(Z_n)_n \rightharpoonup Z^* && \text{weakly in } L^2((0,t')\times(0,1)) \end{aligned}$$

and this implies $Z^*=Z_0$, hence $Z_0\in L^\infty([0,t'];H^2(0,1))$.

Because $(Z_n)_n$ is also bounded in $W^{1,\infty}([0,t'];H^1(0,1))$, and thus in both spaces $H^1([0,t'];H^1(0,1))$ and $H^1([0,t']\times[0,1])$, we can extract yet another subsequence – call it $(Z_n)_n$ for simplicity – such that

$$(\partial Z_n/\partial t)_n \text{ converges weak}^* \text{ in } L^\infty([0,t'];H^1(0,1)) \text{ with weak}^* \text{ limit } z^*, \tag{4.13}$$

$$(Z_n)_n \text{ converges weakly to } \tilde{Z} \text{ in } H^1([0,t'];H^1(0,1)), \tag{4.14}$$

$$(Z_n)_n \text{ converges weakly in } H^1([0,t']\times[0,1]). \tag{4.15}$$

The previous argument of boundedness of the time interval can be applied again, so $(\partial Z_n/\partial t)_n$ and z^*, both have to be in $L^2(0,t';H^1(0,1))$. Meanwhile, any function g from $L^2(0,t;H^1(0,1))$ is also an element of $L^1(0,t';H^1(0,1))$ and satisfies:

$$\left(\frac{\partial Z_n}{\partial t}, g\right)_{L^2(H^1)} = \left\langle \frac{\partial Z_n}{\partial t}, g \right\rangle_{(L^\infty,L^1)} \to \langle z^*, g\rangle_{(L^\infty,L^1)} = (z^*, g)_{L^2(H^1)}.$$

Therefore $\partial Z_n/\partial t$ converges weakly to z^* in $L^2(0,t';H^1(0,1))$.

Given $\phi \in L^2(0,t';H^1(0,1)) \cap L^1(0,t';H^1(0,1))$ we consider the bounded linear functional

$$L_\phi(u) \stackrel{\text{def}}{=} \int_0^{t'} \left(\frac{\partial}{\partial t}u(t)\,,\,\phi(t)\right)_{H^1(0,1)} dt \qquad \text{for any } u \in H^1(0,t';H^1(0,1)).$$

By definition this functional L_ϕ belongs to the space $\left(H^1(0,t';H^1(0,1))\right)^*$, and as a result we have

$$\left(\frac{\partial Z_n}{\partial t}\,,\,\phi\right)_{L^2(H^1)} = L_\varphi(Z_n) \to L_\varphi(\tilde{Z}) = \left(\frac{\partial \tilde{Z}}{\partial t}\,,\,\phi\right)_{L^2(H^1)}.$$

Consequently, $z^* = \partial\tilde{Z}/\partial t$.

Let χ be an arbitrary function in $H^1((0,t') \times (0,1))$. We treat the inner product $(\chi\,,\,\cdot)_{H^1((0,t')\times(0,1))}$ as a bounded linear functional over the Hilbert space $H^1((0,t');H^1(0,1))$. Then $(\chi, Z_n)_{H^1((0,t')\times(0,1))}$ has to converge to $(\chi, \tilde{Z})_{H^1((0,t')\times(0,1))}$, and element $\tilde{Z}$ is the weak limit of $(Z_n)_n$ in $H^1((0,t') \times (0,1))$ as well.

Again, by compact embedding the weak convergence in $H^1((0,t') \times (0,1))$ implies strong convergence in $L^2((0,t') \times (0,1))$. So we conclude that $\tilde{Z} = Z_0$ in $H^1((0,t') \times (0,1))$. Thus, Z_0 belongs to $H^1((0,t');H^1(0,1))$.

Hence we have shown that the sequence $\left(\frac{\partial}{\partial t} Z_n\right)_n$ converges weakly to z^* in the space $L^2([0,t'];H^1(0,1))$ and

$$z^* = \frac{\partial}{\partial t}\tilde{Z} = \frac{\partial}{\partial t} Z_0 \qquad \text{in } L^2([0,t'];H^1(0,1)). \tag{4.16}$$

The last equation implies, however, that

$$Z_0(t) = Z_0(0) + \int_0^t z^*(s)ds \tag{4.17}$$

where the integral is taken in the Bochner sense. Since z^* was initially assumed from $L^\infty([0,t'];H^1(0,1))$, then Z_0 belongs to $W^{1,\infty}([0,t'];H^1(0,1))$ as well.

Next, we note that the sequences $\left(\frac{\partial}{\partial x} Z_n(\cdot,0)\right)_n$ and $\left(\frac{\partial}{\partial x} Z_n(\cdot,1)\right)_n$ are bounded in $H^1(0,t')$. We may pass again to a subsequence of $(Z_n)_n$, called $(Z_n)_n$ as well, such that $\left(\frac{\partial}{\partial x} Z_n(\cdot,0)\right)_n$ and $\left(\frac{\partial}{\partial x} Z_n(\cdot,1)\right)_n$ converge weakly in $H^1(0,t')$. Since weak convergence in $H^1(0,t')$ implies strong convergence in $L^2(0,t')$, we deduce that the sequences have the strong limits $\frac{\partial}{\partial x} Z_0(\cdot,0)$ and $\frac{\partial}{\partial x} Z_0(\cdot,1)$ in $L^2(0,t')$ respectively, and that $\frac{\partial}{\partial x} Z_0(\cdot,0)$ and $\frac{\partial}{\partial x} Z_0(\cdot,1)$ belong to $H^1(0,t')$.

From standard norm estimates for the weak and weak* convergent sequences [27] we have $\|Z_0\| \leqslant \liminf_{n\to\infty} \|Z_n\|$ with norms taken in the corresponding spaces.

Therefore

$$\mathcal{E}(q_0)^2 + \mathcal{E}(U_0)^2 + \mathcal{E}(V_0)^2 \leqslant L^2, \tag{4.18}$$

and it remains to show that Z_0 satisfies the boundary conditions. This requirement follows from the next lemma and the Sobolev imbedding theorem.

Lemma 4.10. *Let V and H be Hilbert spaces such that V is continuously and densely embedded in H. Assume that $u \in L^\infty([0,t'];V) \cap C([0,t'];H)$. Then $u(t) \in V$ for every $t \in [0,t']$ and $u(t)$ is weakly continuous; i.e. $(\psi, u(t))$ is a continuous function of t for every $\psi \in V^*$*

Proof See [20], p. 392.

Since the inclusion $H^2(0,1) \subset H^1(0,1)$ is dense, the theorem applies to q_0 in particular, and we can evaluate $q_0(t,z)|_{t=0} \in H^2(0,1)$. By Sobolev embedding $q(0,z)$ is a continuous function.

In summary, we conclude that the set $\mathbb{S}(t',L,M)$ is precompact and closed in $\left(L^2((0,t') \times (0,1))\right)^3 \times L^2(0,t')$, and therefore compact. $\square$

4.2 The Schauder Map

In this section we construct a map Σ on a suitable set $\mathbb{S}(t',L,M)$ and show that the Schauder fixed point theorem applies.

First, we reformulate Eqs. (2.12), (2.14), and (2.15) implicitly:

$$\begin{aligned} r_t + v r_z &= -\frac{1}{2} v_z, \\ S_t + v S_z &= -\Big(v_z + \frac{1}{\mathrm{We}}\Big) S - \frac{1}{\mathrm{We}} v_z, \\ T_t + v T_z &= \Big(2v_z - \frac{1}{\mathrm{We}}\Big) T + \frac{2}{\mathrm{We}} v_z. \end{aligned}$$

Next, we note that all these implicit equations have a common form:

$$u_t(t,z) + v(t,z)u_z(t,z) = f(t,z,u,v_z). \tag{4.19}$$

which is, in general, a transport equation. If a function u^e is given and suitable initial and boundary conditions posed, we may consider an abstract correspondence $u^e \to \tilde{u}$ where $\tilde{u}$ is a solution (if there is any) of

$$\tilde{u}_t(t,z) + v(t,z)\tilde{u}_z(t,z) = f(t,z,u,v_z).$$

By the existence Theorem 3.2 the solution $\tilde{u}$ exists and is boundary regular if f and v are sufficiently smooth and $v > 0$.

So, for $(q, U, V, Y) \in \mathbb{S}(t', L, M)$ we define the operators w and w_z as

$$w(q,V,Y)(t,z) = D + \frac{\displaystyle\int_1^z \frac{(D-1)\,dx}{-2\,\mathrm{We}\,Y(t) + 3\,\left(q(t,x)\right)^2\,\left(1 + \mathrm{We}\,V(t,x)\right)}}{\displaystyle\int_0^1 \frac{dx}{-2\,\mathrm{We}\,Y(t) + 3\,\left(q(t,x)\right)^2\,\left(1 + \mathrm{We}\,V(t,x)\right)}} \tag{4.20}$$

and

$$w_z(q,V,Y)(t,z) = \frac{\dfrac{(D-1)}{-2\,\mathrm{We}\,Y(t) + 3\,\left(q(t,z)\right)^2\,\left(1 + \mathrm{We}\,V(t,z)\right)}}{\displaystyle\int_0^1 \frac{dx}{-2\,\mathrm{We}\,Y(t) + 3\,\left(q(t,x)\right)^2\,\left(1 + \mathrm{We}\,V(t,x)\right)}}. \tag{4.21}$$

By Proposition 4.8 we have

$$-2\mathrm{We}\,Y(t) + 3\,\left(q(t,x)\right)^2\,\left(1 + \mathrm{We}\,V(t,x)\right) \neq 0 \quad \text{ for all } (t,z) \in [0,t'] \times [0,1],$$

hence $w(q,V,Y)$ and $w_z(q,V,Y)$ are well-defined, and $w(q,V,Y)$ takes the minimum value 1. Moreover, the regularity of q, V, Y implies that

$$-2\mathrm{We}\, Y(t) + 3\ (q(t,x))^2\ (1 + \mathrm{We}\, V(t,x)) \in \mathbb{BR}(0,t';0,1).$$

As a consequence both w, w_z are boundary regular and

$$w_z(q,V,Y) \in L^\infty(0,t';H^2(0,1)) \cap W^{1,\infty}(0,t';H^1(0,1)),$$
$$w(q,V,Y) \in L^\infty(0,t';H^3(0,1)) \cap W^{1,\infty}(0,t';H^2(0,1)).$$

Let $\tilde{q}$, $\tilde{U}$, and $\tilde{V}$ be the solutions to the following boundary-initial value problems on $[0,t'] \times [0,1]$

$$\tilde{q}_t + w(q,V,Y)\,\tilde{q}_z = -\frac{1}{2}\,w_z(q,V,Y)\,q, \tag{4.22}$$

$$\tilde{q}(t,0) = 1, \quad \tilde{q}(0,z) = r^0(z), \tag{4.23}$$

$$\tilde{U}_t + w(q,V,Y)\,\tilde{U}_z = -\left(w_z(q,V,Y) + \frac{1}{\mathrm{We}}\right) U - \frac{1}{\mathrm{We}}\,w_z(q,V,Y), \tag{4.24}$$

$$\tilde{U}(t,0) = 0, \quad \tilde{U}(0,z) = S^0(z), \tag{4.25}$$

$$\tilde{V}_t + w(q,V,Y)\,\tilde{V}_z = \left(2\,w_z(q,V,Y) - \frac{1}{\mathrm{We}}\right) V + \frac{2}{\mathrm{We}}\,w_z(q,V,Y), \tag{4.26}$$

$$\tilde{V}(t,0) = Y(t), \quad \tilde{V}(0,z) = T^0(z). \tag{4.27}$$

Since the data are assumed compatible in the sense of Section 3.2 and the coefficients have the required regularity, Theorem 3.2 applies. Hence $\tilde{q}$, $\tilde{U}$, and $\tilde{V}$ are well-defined and belong to

$$C^1([0,t'];H^1(0,1)) \cap C([0,t'];H^2(0,1)) \cap \mathbb{BR}(0,t';0,1). \tag{4.28}$$

Finally we let $\tilde{Y}$ be the solution of the initial value problem

$$\tilde{Y}_t + \frac{1}{\mathrm{We}}\,\tilde{Y} = \frac{D-1}{\mathrm{We}}\,\frac{1}{\displaystyle\int_0^1 \frac{dx}{-2\,\mathrm{We}\,Y(t) + 3\,\left(q(t,x)\right)^2\,\left(1+\mathrm{We}\,V(t,x)\right)}}, \tag{4.29}$$

$$\tilde{Y}(0) = T^0(0). \tag{4.30}$$

The regularity of the right-hand side in Eq. (4.29) proves that

$$\tilde{Y} \in H^2(0,t'). \tag{4.31}$$

Note that for $\tilde{Y}$ given by (4.29)–(4.30)

$$w(q,V,Y)(t,z) = D + \int_1^z \frac{\tilde{Y}(t) + \mathrm{We}\,\tilde{Y}_t(t)}{-2\,\mathrm{We}\,Y(t) + 3\,\left(q(t,x)\right)^2\,\left(1+\mathrm{We}\,V(t,x)\right)}\,dx \tag{4.32}$$

The Schauder map Σ is now defined by

$$\sum : \begin{cases} \mathbb{S}(t',L,M) & \longrightarrow \quad L^2((0,t')\times(0,1))^3 \times L^2(0,t') \\ (q,U,V,Y) & \longmapsto \qquad\qquad (\tilde{q},\tilde{U},\tilde{V},\tilde{Y}) \end{cases}. \tag{4.33}$$

As noted above, we have the following conclusion.

Proposition 4.11. *The operator* Σ *is well-defined on* $\mathbb{S}(t',L,M)$ *.*

Lemma 4.12. *There are* $L>0$ *,* $M>0$*, and* $t'>0$ *such that the operator* Σ *maps* $\mathbb{S}(t',L,M)$ *into* $\mathbb{S}(t',L,M)$ *.*

Proof. Throughout we may assume that $t' \leqslant 1$. We will make use of the estimate (3.16), applied to each of the boundary-initial value problems (4.22)–(4.23), (4.24)–(4.25) and (4.26)–(4.27).

First, for arbitrary $(q, U, V, Y) \in \mathbb{S}(t', L, M)$, we note that using an argument along the lines of (4.8)–(4.10) will give us

$$\Big|3(q(t,z))^2(1+\mathrm{We}\,V(t,z)) - 2\mathrm{We}\,Y(t) - 3\big(r^0(z)\big)^2\big(1+\mathrm{We}\,T^0(z)\big) + 2\mathrm{We}\,T^0(0)\Big| \\ \leqslant C(L,M)\sqrt{t'}$$

for some constant $C(L,M)$ which depends on L and M only (here we have used $t' \leqslant 1$). From this we may assume that t' is chosen sufficiently small compared to $C(L,M)$ such that, for $0 \leqslant t \leqslant t'$, $0 \leqslant z \leqslant 1$

$$\Big|3\,(q(t,z))^2\,(1+\mathrm{We}\,V(t,z)) - 2\,\mathrm{We}\,Y(t) - 3\,\big(r^0(z)\big)^2\,\big(1+\mathrm{We}\,T^0(z)\big) + \\ 2\,\mathrm{We}\,T^0(0)\Big| \leqslant \frac{1}{2}\,\Big|3\,\big(r^0(z)\big)^2\,\big(1+\mathrm{We}\,T^0(z)\big) - 2\,\mathrm{We}\,T^0(0)\Big|. \tag{4.34}$$

Hence, we can bound the term

$$\big|-2\,\mathrm{We}\,Y(t) + 3\,(q(t,z))^2\,(1+\mathrm{We}\,V(t,z))\big|, \tag{4.35}$$

its inverse, and integrals thereof, from both above and below in terms of the initial data. Therefore the only term left to consider when taking the first and second derivatives of $w(q,V,Y)$ with respect to z is

$$6\,q\,q_z\,(1+\mathrm{We}\,V) + 3\,q^2\,\mathrm{We}\,V_z. \tag{4.36}$$

Here all terms can be estimated by an expression in L. However, it is advantageous to note that the regularity properties of q and V imply that there exists a constant

$c(L)$, depending on L, such that

$$\int_0^1 \left(6\, q\, q_z\, (1 + \mathrm{We}\, V) + 3\, q^2\, \mathrm{We}\, V_z - 6\, r^0\, r_z^0\, (1 + \mathrm{We}\, T^0) - \right.$$
$$\left. 3\, \left(r^0\right)^2\, \mathrm{We}\, T_z^0\right)^2\, dz \leqslant c(L)\, t'. \tag{4.37}$$

It follows that $\|w(q, V, Y)\|_{0,2}$ can be estimated by the initial data up to a term involving L that can be made arbitrarily small if t' is chosen sufficiently small.

In order to apply the estimate (3.16) of Corollary 3.3, we note that almost all terms on the right side of (3.16) involve only initial data or terms which are bounded by expressions in L and M and multiplied by t'. Therefore terms of the latter form can be made small or, in the case of the exponential, close to 1. Only the polynomial involving the minimum value μ of the flux coefficient p and the third term in the parenthesis on the right of the estimate (3.16) are left to be discussed. In the situation here we have $p = w(q, V, Y)$. The polynomial term is trivial since the relevant minimum value μ assumed by $w(q, V, Y)$ is 1. The third term involves, however, the boundary data and $\|w(q, V, Y)\|_{0,2}$. As seen above, the latter quantity can be bounded in terms of the initial data plus a term multiplied by t'. Therefore, after having taken L and M sufficiently large to accommodate initial and boundary data, we can make t' so small in estimate (3.16) that

$$\mathcal{E}(\tilde{q})^2 + \mathcal{E}(\tilde{U})^2 + \mathcal{E}(\tilde{V})^2 \leqslant L^2. \tag{4.38}$$

It remains to estimate the solution of the initial value problem (4.29)-(4.30).

To this end, we note that

$$\tilde{Y}(t) = e^{-\dfrac{t}{\mathrm{We}}}\, T^0(0) + \int_0^t \frac{(D-1)\, e^{\dfrac{s-t}{\mathrm{We}}}}{\displaystyle\int_0^1 \frac{\mathrm{We}\, dx}{-2\,\mathrm{We}\, Y(s) + 3\, \left(q(s,x)\right)^2\, \left(1 + \mathrm{We}\, V(s,x)\right)}}\, ds. \quad (4.39)$$

Hence, using the estimate (4.34), $|\tilde{Y}|$ can be bounded in terms of the initial data. Due to Eq. (4.29) the same applies to $|\tilde{Y}_t|$. As we differentiate Eq. (4.29) with respect to t, we note that quantities like $|q|$, $|q_t|$, $|V|$, and $|V_t|$ are bounded in terms of L. Consequently, the integral

$$\int_0^{t'} \tilde{Y}_{tt}^2\, dt$$

can be estimated by the initial data and an expression of the form $c(L)\, t'$, where $c(L)$ is a constant depending on L. However, this result implies that for M chosen large enough to take care of the initial data, t' can be taken small enough to enforce

$$\|\tilde{Y}\|_{H^2} \leqslant M. \quad (4.40)$$

This concludes the proof. $\square$

Hereafter we may assume that t', L, M are selected in such a way that

$$\sum \left(\mathbb{S}(t', L, M)\right) \subseteq \mathbb{S}(t', L, M).$$

Lemma 4.13. *The operator Σ is continuous on $\mathbb{S}(t', L, M)$ with respect to the topology of $\left(L^2([0,t'] \times [0,1])\right)^3 \times L^2(0,t')$.*

Proof. Let (q, U, V, Y) and $(\bar{q}, \bar{U}, \bar{V}, \bar{Y})$ be in $\mathbb{S}(t', L, M)$ and set

$$(\kappa, \Omega, \Phi, \Psi) = \Sigma(q, U, V, Y), \tag{4.41}$$

$$(\bar{\kappa}, \bar{\Omega}, \bar{\Phi}, \bar{\Psi}) = \Sigma(\bar{q}, \bar{U}, \bar{V}, \bar{Y}). \tag{4.42}$$

In the following, we let C be a generic constant which is allowed to depend on L, M, and t'.

We use the short notation $G = -2\mathrm{We}\, Y(t) + 3(q(t,z))^2(1 + \mathrm{We}\, T(t,z))$, and $\overline{G}$ will similarly denote the expression with variables $\bar{Y}, \bar{q}, \bar{T}$ in place of Y, q, T respectively. Then the difference of the derivatives of velocities can be cast in the form

$$w_z(q, V, Y) - w_z(\bar{q}, \bar{V}, \bar{Y}) = (D-1) \frac{(\overline{G} - G)\displaystyle\int_0^1 \frac{dz}{\overline{G}} + G \int_0^1 \left(\frac{G - \overline{G}}{G\overline{G}} \right) dz}{G\overline{G} \displaystyle\int_0^1 \frac{dz}{\overline{G}} \int_0^1 \frac{dz}{G}}. \tag{4.43}$$

As was stated before, the quantities $|G|, |\overline{G}|$, their inverses, and integrals thereof are bounded in terms of the initial data. Therefore, using those bounds appropriately, we get:

$$|w_z(q, V, Y) - w_z(\bar{q}, \bar{V}, \bar{Y})| \leqslant C\Big(|\overline{G} - G| + \int_0^1 |\overline{G} - G| \Big). \tag{4.44}$$

The expansion of $|\overline{G} - G|$ provides

$$\begin{aligned} |\overline{G} - G| &\leqslant 2\mathrm{We}|\bar{Y} - Y| + 3|\bar{q}^2 - q^2| + 3\mathrm{We}|\bar{q}^2 \bar{V} - q^2 V| \\ &\leqslant 2\mathrm{We}|\bar{Y} - Y| + 3|\bar{q}^2 - q^2| + 3\mathrm{We}(\|\bar{q}^2\|_{L^\infty} |\bar{V} - V| + \|V\|_{L^\infty} |\bar{q}^2 - q^2|). \end{aligned}$$

Due to the continuous embedding $L^\infty(0,t';H^2(0,1)) \hookrightarrow L^\infty([0,t']\times[0,1])$, the norms $\|\cdot\|_{L^\infty}$ may be replaced by $C_E\|\cdot\|_{0,2}$, where C_E is an embedding constant. The variables $\bar{q}$, q, and V are elements of $\mathbb{S}(t',L,M)$, therefore $\|\bar{q}^2\|_{0,2}, \|V\|_{0,2}$ are bounded by a constant C, which may depend on L and M, and the term $|q^2-\bar{q}^2|$ can be estimated as follows:

$$|\bar{q}^2 - q^2| \leqslant |\bar{q}-q|\ \|\bar{q}+q\|_{L^\infty} \leqslant |q-\bar{q}|\, C_E(\|q\|_{0,2}+\|\bar{q}\|_{0,2}) \leqslant C|\bar{q}-q|.$$

Now, we rewrite Eq. (4.44) as

$$\begin{aligned}|w_z(q,V,Y) - w_z(\bar{q},\bar{V},\bar{Y})| \leqslant C\Big(&|\overline{Y}-Y| + |\overline{q}-q| + |\overline{V}-V| \\ &+ \int_0^1 \Big(|\overline{Y}-Y| + |\overline{q}-q| + |\overline{V}-V|\Big)dz\Big)\end{aligned} \tag{4.45}$$

and square both sides of (4.45). On the right-hand side we obtain square terms and composite products. The Cauchy's inequality allows us to split them as follows:

$$|\bar{Y}-Y|\int_0^1 |\bar{q}-q|dz \leqslant \frac{|\bar{Y}-Y|^2}{2} + \frac{1}{2}\Big(\int_0^1 |\bar{q}-q|dz\Big)^2.$$

The latter integral $\Big(\int_0^1 |\bar{q}-q|dz\Big)^2$ may be estimated by $\int_0^1 |\bar{q}-q|^2 dz$ in accordance with the Hölder's inequality.

Then (4.45) transforms into

$$\begin{aligned}|w_z(q,V,Y) - w_z(\bar{q},\bar{V},\bar{Y})| \leqslant C\Big(&|\bar{Y}-Y|^2 + |\bar{q}-q|^2 + |\bar{V}-V|^2 \\ &+ (\bar{Y}-Y)^2 + \|\bar{q}-q\|_2^2 + \|\bar{V}-V\|_2^2\Big).\end{aligned} \tag{4.46}$$

Integration of both sides of (4.46) over the interval [0,1] with respect to z provides:

$$\begin{aligned}
&\|w_z(q,V,Y)(t,\cdot)-w_z(\bar q,\bar V,\bar Y)(t,\cdot)\|_2^2 \leqslant C\,(\|q(t,\cdot)-\bar q(t,\cdot)\|_2^2+\\
&\quad \|V(t,\cdot)-\bar V(t,\cdot)\|_2^2+\big(Y(t)-\bar Y(t)\big)^2), && (4.47)\\
&\|w(q,V,Y)(t,\cdot)-w(\bar q,\bar V,\bar Y)(t,\cdot)\|_2^2 \leqslant C\,(\|q(t,\cdot)-\bar q(t,\cdot)\|_2^2+\\
&\quad \|V(t,\cdot)-\bar V(t,\cdot)\|_2^2+\big(Y(t)-\bar Y(t)\big)^2). && (4.48)
\end{aligned}$$

Estimation of the difference of $\bar\Phi-\Phi$ goes in a similar way, and goes along the lines of the energy method, but there are some details that need to be addressed. Since $\bar\Phi,\Phi$ are images of $\bar V,V$ under the Σ mapping, they are solutions of the transport equations

$$\bar\Phi_t+w(\bar q,\bar V,\bar Y)\bar\Phi_z=\Big(2w_z(\bar q,\bar V,\bar Y)-\frac{1}{\mathrm{We}}\Big)\bar V+\frac{2}{\mathrm{We}}w_z(\bar q,\bar V,\bar Y), \tag{4.49}$$

$$\Phi_t+w(q,V,Y)\Phi_z=\Big(2w_z(q,V,Y)-\frac{1}{\mathrm{We}}\Big)V+\frac{2}{\mathrm{We}}w_z(q,V,Y) \tag{4.50}$$

equipped with the initial/boundary conditions

$$\bar\Phi(t,0)=Y(t),\quad \bar\Phi(0,z)=T^0(z)$$
$$\Phi(t,0)=Y(t),\quad \Phi(0,z)=T^0(z).$$

From now on, we use $\bar w$ and w to denote operators $\bar w(\bar q,\bar V,\bar Y)$ and $w(q,V,Y)$ respectively. Subtracting Eq.(4.50) from Eq.(4.49) provides

$$\begin{aligned}
&(\bar\Phi-\Phi)_t+\bar w(\bar\Phi-\Phi)_z+\Phi_z(\bar w-w)\\
&\quad =2(\bar w_z(\bar V-V)+V(\bar w_z-w_z))-\frac{1}{\mathrm{We}}(\bar V-V)+\frac{2}{\mathrm{We}}(\bar w_z-w_z). \qquad (4.51)
\end{aligned}$$

We multiply Eq.(4.51) by $2(\bar{\Phi}-\Phi)$, isolate $2(\bar{\Phi}-\Phi)(\bar{\Phi}-\Phi)_t$, and integrate from 0 to 1 with respect to z, which provides, on the left-hand side

$$\int_0^1 2(\bar{\Phi}-\Phi)(\bar{\Phi}-\Phi)_t dz = \int_0^1 \frac{\partial}{\partial t}(\bar{\Phi}-\Phi)^2 dz = \frac{d}{dt}\int_0^1 (\bar{\Phi}-\Phi)^2 dz = \frac{d}{dt}\|\bar{\Phi}-\Phi\|_2^2.$$

On the right-hand side, we have the term containing $(\bar{\Phi}-\Phi)_z$. Next, we take the integral by parts:

$$-\int_0^1 2\bar{w}(\bar{\Phi}-\Phi)(\bar{\Phi}-\Phi)_z dz = -\bar{w}(\bar{\Phi}-\Phi)^2\Big|_0^1 + \int_0^1 \bar{w}_z(\bar{\Phi}-\Phi)^2 dz.$$

Since $w(0)=1$, $w(1)=D>1$, and $\bar{\Phi}(0)-\Phi(0)=\bar{Y}-Y$, the evaluation from 1 to 0 results in

$$-D\big(\bar{\Phi}(t,1)-\Phi(t,1)\big)^2 + \big(\bar{Y}(t)-Y(t)\big)^2$$

and the integral will be estimated as follows:

$$\Big|\int_0^1 \bar{w}_z(\bar{\Phi}-\Phi)^2 dz\Big| \leqslant \|\bar{w}_z\|_{L^\infty}\int_0^1 |(\bar{\Phi}-\Phi)^2| dz \leqslant \|\bar{w}_z\|_{0,2}\|\bar{\Phi}-\Phi\|_2^2$$

The norm $||\bar{w}_z||_{0,2}$, is bounded in terms of initial data. Proceeding further, we get the resulting estimation:

$$\frac{d}{dt}\|\Phi(t,\cdot)-\bar{\Phi}(t,\cdot)\|_2^2 \leqslant C\,(\|\Phi(t,\cdot)-\bar{\Phi}(t,\cdot)\|_2^2 + \|q(t,\cdot)-\bar{q}(t,\cdot)\|_2^2 + \\ \|V(t,\cdot)-\bar{V}(t,\cdot)\|_2^2 + \big(Y(t)-\bar{Y}(t)\big)^2). \qquad (4.52)$$

We take the differences of the equations corresponding to Eqs. (4.22)–(4.23) for κ and $\bar{\kappa}$, multiply by $\kappa-\bar{\kappa}$, and integrate in z from 0 to 1.

The resulting inequality reads:

$$\frac{d}{dt}\|\kappa(t,\cdot)-\bar{\kappa}(t,\cdot)\|_2^2 \leqslant C\,(\|\kappa(t,\cdot)-\bar{\kappa}(t,\cdot)\|_2^2+\|q(t,\cdot)-\bar{q}(t,\cdot)\|_2^2+$$
$$\|V(t,\cdot)-\bar{V}(t,\cdot)\|_2^2+\big(Y(t)-\bar{Y}(t)\big)^2). \tag{4.53}$$

In an analogous way we obtain

$$\frac{d}{dt}\|\Omega(t,\cdot)-\bar{\Omega}(t,\cdot)\|_2^2 \leqslant C\,(\|\Omega(t,\cdot)-\bar{\Omega}(t,\cdot)\|_2^2+\|q(t,\cdot)-\bar{q}(t,\cdot)\|_2^2+$$
$$\|U(t,\cdot)-\bar{U}(t,\cdot)\|_2^2+\|V(t,\cdot)-\bar{V}(t,\cdot)\|_2^2+\big(Y(t)-\bar{Y}(t)\big)^2). \tag{4.54}$$

As we take the difference of the equations determining Ψ and $\bar{\Psi}$ and multiply by $\Psi-\bar{\Psi}$, we obtain the estimate

$$\frac{d}{dt}\big(\Psi(t)-\bar{\Psi}(t)\big)^2 \leqslant C\,(\big(\Psi(t)-\bar{\Psi}(t)\big)^2+\|q(t,\cdot)-\bar{q}(t,\cdot)\|_2^2+$$
$$\|V(t,\cdot)-\bar{V}(t,\cdot)\|_2^2+\big(Y(t)-\bar{Y}(t)\big)^2). \tag{4.55}$$

We set

$$\rho(t)=\|\kappa(t,\cdot)-\bar{\kappa}(t,\cdot)\|_2^2+\|\Omega(t,\cdot)-\bar{\Omega}(t,\cdot)\|_2^2+\|\Phi(t,\cdot)-\bar{\Phi}(t,\cdot)\|_2^2+$$
$$\big(\Psi(t)-\bar{\Psi}(t)\big)^2, \tag{4.56}$$
$$\sigma(t)=\|q(t,\cdot)-\bar{q}(t,\cdot)\|_2^2+\|U(t,\cdot)-\bar{U}(t,\cdot)\|_2^2+\|V(t,\cdot)-\bar{V}(t,\cdot)\|_2^2+$$
$$\big(Y(t)-\bar{Y}(t)\big)^2. \tag{4.57}$$

Then estimates (4.53)–(4.55) can be combined to read

$$\frac{d}{dt}\,\rho(t) \leqslant C\,\left(\rho(t)+\sigma(t)\right). \tag{4.58}$$

Finally, for the function ρ we have by definition $\rho(0) = 0$, thus applying Grönwall's inequality we get

$$\rho(t) \leqslant C \int_0^t e^{C\,(t-s)}\,\sigma(s)\,ds. \tag{4.59}$$

This implies the claim. □

Suppose the map Σ has two fixed points, say (q, U, V, Y) and $(\bar{q}, \bar{U}, \bar{V}, \bar{Y})$ in $\mathbb{S}(t', L, M)$. Then proceeding as in the proof above, we obtain in (4.58)

$$\frac{d}{dt}\,\rho(t) \leqslant C\,\rho(t) \tag{4.60}$$

with some constant C. Consequently $\rho(t) \leqslant 0$. We have shown the following result.

Proposition 4.14. *The operator Σ has at most one fixed point in $\mathbb{S}(t', L, M)$.*

Finally we can give the proof of Theorem 3.6.

Proof of Theorem 3.6 According to Lemmas 4.9 and 4.13, the Schauder fixed point theorem applies to the operator Σ on $\mathbb{S}(t', L, M)$ for appropriate choices of L, M and t', i.e. Σ has a fixed point (r, S, T, X) in $\mathbb{S}(t', L, M)$. The regularity conclusions of Theorem 3.2 applied to this fixed point show immediately that

$$r, S, T \in C^1([0, t']; H^1(0, 1)) \cap C([0, t']; H^2(0, 1)) \cap \mathbb{BR}(0, t'; 0, 1). \tag{4.61}$$

Moreover, by Proposition 4.14, this is the only fixed point in $\mathbb{S}(t', L, M)$. When we define the velocity v by

$$v = w(r, T, X), \tag{4.62}$$

then we readily obtain

$$v \in C^1([0, t']; H^2(0, 1)) \cap C([0, t']; H^3(0, 1)). \tag{4.63}$$

It is clear that r, S, T, v satisfy Eqs. (2.12), (2.14) and (2.15) together with the boundary/initial conditions. The structure of these equations and the regularity properties of r, S, T, v imply then also that

$$r, S, T \in C^2([0,t']; L^2(0,1)) \quad \text{and} \quad X \in C^2[0,t']. \tag{4.64}$$

(Actually, we even have $X \in C^3[0,t']$.) Consequently,

$$v \in C^2([0,t']; H^1(0,1)). \tag{4.65}$$

To show that Eq. (2.13) holds true, we note that taking the difference between Eq.(2.14) and Eq.(2.15), multiplying it by r^2, and adding/subtracting extra terms we receive

$$(r^2(S-T))_t + v(r^2(S-T))_z + \Big(2v_z + \frac{1}{\mathrm{We}}\Big) r^2(S-T) = -3v_z r^2\Big(T + \frac{1}{\mathrm{We}}\Big),$$

thus $u = r^2\,(S-T)$ is a solution of the boundary-initial value problem

$$u_t + v\,u_z + \left(2\,v_z + \frac{1}{\mathrm{W}e}\right) u = -3\,v_z\,r^2 \left(T + \frac{1}{\mathrm{W}e}\right), \tag{4.66}$$

$$u(t,0) = -X(t), \quad u(0,z) = -X(0) = \left(r^0(z)\right)^2 \left(S^0(z) - T^0(z)\right). \tag{4.67}$$

However, since v_z satisfies

$$v_z = \frac{\mathrm{We}\,X_t + X}{-2\,\mathrm{We}\,X + 3\,r^2\,\left(1 + \mathrm{We}\,T\right)}, \tag{4.68}$$

$\tilde{u}(t,z) = -X(t)$ is readily seen to be a solution of problem (4.66)–(4.67) as well. Obviously, $(-X(t))_z = 0$.

Now, suppose we have two different solutions $u_1,\ u_2$ to (4.66)–(4.67). For the difference function $\bar{u} = u_1 - u_2$ we get

$$\begin{aligned}
&\bar{u}_t + v\bar{u}_z + \Big(2v_z + \frac{1}{\mathrm{We}}\Big)\bar{u} = 0,\\
&\bar{u}(t,0) = 0,\\
&\bar{u}(0,z) = 0.
\end{aligned}$$

Respectively, the standard energy estimation argument yields

$$\begin{aligned}
\frac{d}{dt}\|\bar{u}(t)\|_2^2 &\leqslant \|v_z\|_\infty\|\bar{u}(t)\|_2^2 + \|4v_z + \frac{2}{\mathrm{We}}\|_\infty\|\bar{u}(t)\|_2^2\\
&\leqslant C\|\bar{u}(t)\|_2^2.
\end{aligned}$$

We immediately conclude then, that solutions of this boundary-initial value problem are unique. Hence Eq. (2.13) holds.

Finally, if (r, v, S, T) is a solution of Eqs. (2.12)–(2.22) with the regularity required in (3.58), then v satisfies Eq. (4.62) with $X(t) = T(t, 0)$. Consequently, (r, S, T, X) is a fixed point of the Schauder map Σ on some set $\mathbb{S}(t', L, M)$. Because of Proposition 4.14, uniqueness is established. This concludes the proof. $\square$

Chapter 5

Discussions

5.1 Other Fluid Models

Depending on the flow type (extensional, planar, shear, almost steady) and the fluid material (polymer solution or melt, molecular structure), one has to choose a model correspondingly, because “no single choice of constitutive equation is best for all purposes” [11]. Here we will briefly overview several standard models and show how they are relevant to the UCM.

The discussed UCM model represents the family of models described by nonlinear differential constitutive laws and provides a good description for molten polymer flows. Another one, the upper convected Jeffreys model, is also known as Oldroyd B (see [12], [11]):

$$\mathbf{T} + \lambda_1 \overset{\triangledown}{\mathbf{T}} = 2\eta_0(\mathbf{D} + \lambda_2 \overset{\triangledown}{\mathbf{D}})$$

Here, the new term λ_2 is called the “retardation time”. The Jeffreys model takes a solvent’s contribution to the stress tensor into account [12].

To get a more accurate description of viscoelastic flow, the further improvements may be achieved by introducing additional terms related to the rheological properties of the fluid. Some models distinguish between the polymer and the solvent stress contribution - $\mathbf{T}_p,\ \mathbf{T}_s$ respectively, and among them is the Giesekus equation, which takes the following form:

$$
\begin{aligned}
&\mathbf{T} = \mathbf{T}_p + \mathbf{T}_s, \\
&\mathbf{T}_s = 2\eta_s \mathbf{D}, \\
&\mathbf{T}_p + \lambda_1 \overset{\triangledown}{\mathbf{T}}_p - \alpha \frac{\lambda_1}{\eta_p} (\mathbf{T}_p \cdot \mathbf{T}_p) = 2\eta_p \mathbf{D}.
\end{aligned}
$$

The solvent and polymer components of the viscosity are denoted as $\eta_s,\ \eta_p$. The "mobility factor" α was obtained "from a molecular theory associated with anisotropic hydrodynamic drag on the constituent polymer molecules" [26]. The parameter varies from 0 to 1 and measures the degree of such anisotropy.

Setting $\mathbf{T}_p = \mathbf{T} - \mathbf{T}_s = \mathbf{T} - 2\eta_s \mathbf{D},$ the model equations can be rewritten as a single constitutive law:

$$
\mathbf{T} + \lambda_1 \overset{\triangledown}{\mathbf{T}} - a \frac{\lambda_1}{\eta_0} (\mathbf{T} \cdot \mathbf{T}) - 2a\lambda_2 (\mathbf{D} \cdot \mathbf{T} + \mathbf{T} \cdot \mathbf{D}) = 2\eta_0 \Big(\mathbf{D} + \lambda_2 \overset{\triangledown}{\mathbf{D}} - a \frac{\lambda_2^2}{\lambda_1} (\mathbf{D} \cdot \mathbf{D}) \Big)
$$

with the constants defined as

$$
\begin{aligned}
&\eta_0 = \eta_s + \eta_p \\
&\lambda_2 = \lambda_1 \eta_s / \eta_p \\
&a = \frac{\alpha}{1 - (\lambda_1/\lambda_2)}
\end{aligned}
$$

Letting $\alpha = 0$, we immediately receive the Oldroyd B equation, and if, in addition, we assume $\lambda_2 = 0$, then the UCM model is obtained.

Some analytical and numerical methods may restrain the choice of constitutive equations in favor of integral rather than differential form [11]. The consideration of the macromolecular structure of melt polymers as temporarily cross-linked chains with "equal probabilities of breaking and reforming" junctions between polymer molecules leads to the so-called Lodge *rubber like liquid* model (see [23], p.124). In such fluid the stress tensor depends on both the rate of deformation and the time t' when deformation occurred, accumulated to the present time t:

$$\boldsymbol{\tau} = \int_{-\infty}^{t} M(t-t')\mathbf{C}^{-1}(t,t')dt'.$$

Here, $M(t-t')$ is called the memory function and $\mathbf{C}^{-1}(t,t')$ is the Finger deformation tensor, which arises from the following considerations [18]:

Let $\mathbf{X}$ *denote the position of a fluid particle before deformation at time* $t = t'$, *and* $\boldsymbol{x} = \boldsymbol{x}(\mathbf{X}, t)$ *- the position of the same particle after deformation occurred at time* t. *Then*

- *the relative deformation gradient tensor* $\mathbf{F}(t,t')$ *is given by*

$$\mathbf{F}(t,t') \stackrel{\text{def}}{=} \left(\frac{\partial \boldsymbol{x}(t)}{\partial \mathbf{X}(t')}\right)^T = \begin{bmatrix} \dfrac{\partial x_1}{\partial X_1} & \dfrac{\partial x_1}{\partial X_2} & \dfrac{\partial x_1}{\partial X_3} \\ \dfrac{\partial x_2}{\partial X_1} & \dfrac{\partial x_2}{\partial X_2} & \dfrac{\partial x_2}{\partial X_3} \\ \dfrac{\partial x_3}{\partial X_1} & \dfrac{\partial x_3}{\partial X_2} & \dfrac{\partial x_3}{\partial X_3} \end{bmatrix}.$$

- *the Finger tensor* $\mathbf{C}^{-1}(t,t')$ *is defined by*

$$\mathbf{C}^{-1}(t,t') \stackrel{\text{def}}{=} (\mathbf{F}^{-1}(t,t'))^T \cdot \mathbf{F}^{-1}(t,t').$$

Depending on the type of the memory function $M(t-t')$ one obtains various differential models, including UCM and Oldroyd B.

While reformulating the equations in the integral form provides (in general) a more accurate description of viscoelastic fluids, this approach does not work for some nonlinear constitutive laws.

Despite its relative simplicity, the UCM model is able to "predict qualitatively the phenomena of rod-climbing, extrudate swell, and spinning flows such as the tubeless syphon" [11].

5.2 Conclusion

In this work we have given an existence, uniqueness and regularity result for the equations of isothermal fiber spinning for a viscoelastic liquid modeled by the constitutive theory of the upper convected Maxwell fluid. The proofs were based on energy estimates, a compactness argument and the Schauder fixed point theorem. The main difficulty in the existence proof was due to the fact that only one boundary condition was given for the elastic stresses at the inlet. This issue was addressed by introducing the undetermined boundary stress as an unknown of the problem and as a variable in the solution map. The resultant solution of the governing equations had sufficient smoothness to allow classical derivatives both in time and space.

It is easily seen that, instead of prescribing the radial stress component, we could have imposed an axial stress boundary condition or a condition involving both stress components (such as the ratio of the two) as long as not both stresses are given explicitly at the inlet. For such changes or for nonconstant boundary data the existence and uniqueness results remain correct with minor modifications of the proofs.

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Appendix A

A.1 Banach Space-Valued Functions

We refer to [10, 17, 20] and [25] for an introductory review of Banach space-valued functions. The Lebesgue measure on $[0, T]$ will be denoted as μ.

Definition *Let X be a Banach space with norm $\| \|_X$ and let $[0, T] \subseteq \mathbb{R}$ be a finite interval. The map $u : [0, T] \to X$ is called*

- *a finitely (countably) valued function, if there exists a finite (countable) sequence $(u_k) \subset X$ and a sequence $(B_k) \subseteq [0, T]$ of mutually disjoint subintervals such that $[0, T] = \cup_k B_k$ and*

$$u(t) = \sum_k \chi_k u_k$$

  where χ_k is the characteristic function of B_k.

- *almost separably valued, if there exists a Lebesgue null-set $\Omega_0 \subset [0, T]$ such that $u([0, T] \setminus \Omega_0)$ is separable.*

- *measurable if there exists a sequence $u_n : [0, T] \to X$ of countably valued functions such that $\lim_{n\to\infty} \|u_n(t) - u(t)\|_X \to 0$ almost everywhere in $[0, T]$.*

- *weakly measurable (on $[0,T]$) if $\langle x^*, u(t)\rangle$ is a measurable scalar valued function on the interval $[0,T]$ for each $x^* \in X^*$.*

Theorem (Pettis) *The function $u(t)$ is measurable if and only if $u(t)$ is weakly measurable and almost separably valued.*

Lemma *Let $u, v : [0,T] \to X$, $w : [0,T] \to X^*$ be measurable. Then*

$$\langle w(t), u(t)\rangle_{(X^*,X)} : [0,T] \to \mathbb{R}$$

is a measurable scalar function.

The space $C([0,T];X)$ is the set of bounded continuous functions $u : [0,T] \to X$ equipped with the norm

$$\|u\| = \sup_{t\in[0,T]} \|u(t)\|_X.$$

The space $C^n([0,T];X)$ consists of all functions $u : [0,T] \to X$ whose derivatives in the classical sense up to order n are in $C([0,T];X)$. Similar to the scalar case, the space $C^\infty([0,T];X)$ is defined as

$$C^\infty([0,T];X) = \cap_{k=0}^{\infty} C^k([0,T];X)$$

For a bounded *open* interval $(0,T) \subset \mathbb{R}$, the space $\mathscr{D}(0,T;X)$ is defined as the set of all C^∞-functions mapping $(0,T)$ into X, with compact support in $(0,T)$.

We say that a function $u : [0,T] \to X$ is integrable if u is measurable and the positive function $\|u(t)\|_X : [0,T] \to \mathbb{R}$ is Lebesgue integrable. For an integrable finitely valued function $u(t)$ the Bochner integral is defined by

$$\int_0^T u(t)\ dt \stackrel{\text{def}}{=} \sum_k \mu(B_k) u_k.$$

For any integrable function $u : [0, T] \to X$ we define the (Bochner) integral

$$\int_0^T u(t)\,dt \overset{\text{def}}{=} \lim_{n\to\infty} \int_0^T u_n(t)\,dt,$$

where (u_n) is any sequence of finitely valued functions mapping $[0, T]$ into X such that $\|u_n\|_X \leqslant \|u\|_X$ and $u_n \to u$ pointwise almost everywhere.

Let $1 \leqslant p \leqslant \infty$. The Lebesgue-Bochner space $L^p(0, T; X)$ consists of all measurable X-valued functions u on $[0, T]$ such that

- $\displaystyle\int_0^T \big\|u(t)\big\|_X^p dt < \infty \qquad 1 \leqslant p < \infty,$
- $\displaystyle\operatorname*{ess\,sup}_{0\leqslant t\leqslant T} \big\|u(t)\big\|_X < \infty \qquad p = \infty.$

$L^p(0, T; X)$ is a Banach space with respect to the norm

- $\displaystyle\big\|u\big\|_{L^p(0,T;X)} \overset{\text{def}}{=} \Big(\int_0^T \big\|u(t)\big\|_X^p dt\Big)^{1/p} \qquad 1 \leqslant p < \infty,$
- $\displaystyle\big\|u\big\|_{L^\infty(0,T;X)} \overset{\text{def}}{=} \operatorname*{ess\,sup}_{0\leqslant t\leqslant T} \big\|u(t)\big\|_X \qquad p = \infty.$

If X is a reflexive Banach space with separable dual space X^* and $1 < q, p < \infty$ are Hölder conjugates ($i.e.$ $\dfrac{1}{q} + \dfrac{1}{s} = 1$) then

- $L^q(0, T; X^*)$ is the dual of $L^p(0, T; X)$
- $L^\infty(0, T; X^*)$ is the dual of $L^1(0, T; X)$.

In particular, for a Hilbert space H with the inner product $(\cdot, \cdot)_H$ $L^2(0, T; H)$ is also a Hilbert space with respect to the inner product

$$(u,v)_{L^2(H)} \stackrel{\text{def}}{=} \int_0^T (u(s),v(s))_H ds \qquad\qquad u,v \in L^2(0,T;H)$$

Let $u \in L^p(0,T;X)$. We define the distributional (weak) derivative of u as a function $Du : [0,T] \to X$ such that

$$\int_0^T \psi(t)\, Du(t)\, dt = -\int_0^T u(t)\, \psi'(t)\, dt \qquad\qquad \text{for all } \psi \in \mathscr{D}(0,T;\mathbb{R})$$

where $\psi' = d\psi/dt$ - is the classical time derivative. Inductively, this definition is used to define the higher derivatives $D^m u$.

For a natural number $m \geqslant 1$ the Sobolev-Bochner space $W^{m,p}(0,T;X)$ is defined as the set of all functions $u \in L^p(0,T;X)$ such that $D^m u \in L^p(0,T;X)$. It is a Banach space with respect to the norm

- $\|u\|_{W^{m,p}(X)} \stackrel{\text{def}}{=} \Big(\int_0^T \sum_{k=0}^{m} \|D^k u(t)\|_X^p dt\Big)^{1/p} \qquad\qquad 1 \leqslant p < \infty,$
- $\|u\|_{W^{m,\infty}(X)} \stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{0 \leqslant t \leqslant T} (\sum_{k=0}^{m} \|D^k u(t)\|_X) \qquad\qquad p = \infty.$

The space $H^n(H) \stackrel{\text{def}}{=} W^{n,2}(0,T;H)$ is a Hilbert space with the inner product

$$(u,v)_{H^1(H)} \stackrel{\text{def}}{=} \int_0^T \sum_{k=0}^{m} \Big(D^k u(t), D^k v(t)\Big)_H dt.$$

Theorem. There is a continuous embedding of $W^{1,2}(0,T;H)$ into $C([0,T];H)$ with

$$\sup_{t \in [0,T]} \|u(t)\|_H \leqslant c \|u\|_{W^{1,2}(0,T;H)}$$

Proof: see [25], Theorem 25.5