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Graph Colorings with Constraints

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Graph Colorings with Constraints

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GRAPH COLORINGS WITH CONSTRAINTS

by

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To my wife,

Thank you for your tremendous love and support these past years; you have been and continue to be such a blessing to me.

To my son,

In all that you do, may you do it with excellence and integrity for the glory of God.

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Abstract

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A graph is a collection of vertices and edges, often represented by points and connecting lines in the plane. A proper coloring of the graph assigns colors to the vertices, edges, or both so that proximal elements are assigned distinct colors. Here we examine results from three different coloring problems. First, adjacent vertex distinguishing total colorings are proper total colorings such that the set of colors appearing at each vertex is distinct for every pair of adjacent vertices. Next, vertex coloring total weightings are an assignment of weights to the vertices and edges of a graph so that every pair of adjacent vertices have distinct weight sums. Finally, edge list multi-colorings consider assignments of color lists and demands to edges; edges are colored with a subset of their color list of size equal to its color demand so that adjacent edges have disjoint sets. Here, color sets consisting of measurable sets are considered.

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1 Introduction

A common problem in graph theory involves partitioning the elements of a graph in such a way that certain conditions are met: the most common condition being that proximate elements are put in different subsets. A partition can be represented by assigning each element in a given subset a particular number or color; for this reason, such problems are often referred to as graph colorings. Here we will examine three such problems where additional restrictions are enforced.

1.1 Basic Terminology of Graphs

A graph G is an ordered pair (V, E) consisting of the vertices V, a set, and the edges E, a collection of pairs of vertices. Here we will consider only edges consisting of different unordered pairs of distinct vertices; such graphs are called simple graphs. Two graphs G_1 and G_2 are said to be isomorphic, or $G_1 \cong G_2$, if there exists a bijection between their vertex sets that preserves their edge sets. Here we consider isomorphic graphs to be the same graph.

For simplicity of notation, if the pair of vertices $\{u, v\}$ is an element of the edge set, we will write $uv \in E$ or equivalently $vu \in E$. In this case, we would say that u and v are adjacent vertices and that uv is incident to u and v; we say that two distinct edges are adjacent if they share a common vertex. For a vertex v, let $N(v)$ denote the set of neighbors of v; that is $N(v) = \{u \mid uv \in E\}$. The degree of a vertex, denoted $d(v)$, is the number of edges incident to v. Therefore in a simple graph $d(v) = |N(v)|$. We will occasionally refer to the maximum and minimum degrees of a graph, denoted by $\Delta(G)$ and $\delta(G)$ respectively. If $\Delta(G) = \delta(G) = k$, we say that G is k-regular.

A subgraph $H \subseteq G$ is an ordered pair (V', E') such that $V' \subseteq V$ and $E' \subseteq E$ where E' contains only edges incident to the vertices of V'; if E' contains every such edge, we say that H is an induced subgraph of G and write $H = G[V']$. We say that H spans G if $V' = V$. For some pair of vertices $u, v \in V$, a uv-path is a sequence of distinct vertices and incident edges starting with u and ending with v. If for every pair of vertices $u, v \in V$ there exists a uv-path, we say that G is connected. A maximal connected subgraph is called a component of G. A subset $C \subset V$ is called a cut set of a connected graph G if $G[V \setminus C]$ is disconnected; we say that G is k-connected if every cut set contains at least k vertices where $1 \leq k \leq |V| - 2$.

The graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{v_i v_{i+1} : 1 \leq i \leq n-1\}$ is called a path and is commonly denoted by P_n. If the additional edge $v_1 v_n$ is added, the graph is then called a cycle and is denoted by C_n. A cycle with an odd number of vertices is called an odd cycle, otherwise it is an even cycle. Together, paths and cycles make up the family of connected graphs with maximum degree two. The family of connected graphs containing no cycle as a subgraph are known as trees; equivalently, this is the family of graphs such that for every two vertices $u, v \in V$, there exists exactly one uv-path.

Another common family of graphs are those that contain no odd cycle as a subgraph; such graphs are called bipartite graphs. Bipartite graphs are thus called because their vertex sets can be partitioned into two independent sets: a set of vertices whose induced subgraph contains no edge. The complete graph on n vertices, denoted K_n, is the graph containing all possible $\binom{n}{2}$ edges. The complete bipartite graph denoted $K_{m,n}$ is the bipartite graph with m vertices in one independent set, n vertices in the other independent set, and all possible mn edges between them. This notion can be extended to a complete multipartite graph: a graph whose vertex set is partitioned into independent sets and has all possible edges between these sets.

1.2 Graph Coloring Foundations

Graph coloring problems investigate the ability to efficiently partition the elements of a graph such that adjacent or incident elements are put into different subsets; that is, the elements are partitioned into independent sets. There are three types of element sets of a graph that can be considered: vertices only, edges only, or both vertices and edges. In the following definitions, the use of $[k]$ denotes the set of positive integers $\{1, 2, 3, \ldots, k\}$

Definition 1.1. Let $G = (V, E)$ be a graph and $f : V \to [k]$ be an assignment of colors to its vertices. If f has the property that $f(u) \neq f(v)$ for every $uv \in E$, we say that f is a proper vertex coloring of G and that G is k-colorable. The smallest integer k such that G is k-colorable is called the chromatic number of G and is denoted $\chi(G)$.

Vertex coloring problems are the most well-known type of graph coloring. The most famous problem in graph coloring, if not all of graph theory, is the four-color theorem. It can be loosely stated as follows: every graph that can be drawn in the plane with no crossing edges is four-colorable. The problem gained renown and notoriety for remaining unsolved for over a century before being solved with computer assistance by Appel, Haken, and Koch in 1977 [2, 3].

Observe that a proper vertex coloring partitions the vertices of a graph into independent sets. Thus the family of graphs which are two-colorable are exactly the family of bipartite graphs; odd cycles require a third independent set, meaning they are three-colorable but not two-colorable. Note that a vertex coloring of any graph G restricted to some subgraph H induces a vertex coloring of that subgraph since independent sets of vertices remain independent under edge or vertex deletion. It suffices to find a coloring of each component of a disconnected graph, hence we always assume that a graph is connected.

For general graphs, it's easy to see that every graph G is $(\Delta(G) + 1)$-colorable: if we order the vertices and then sequentially assign to each vertex v the lowest color different from each of its previously colored neighbors, that number cannot be larger than $d(v) + 1$. The scheme thus described is known as a greedy coloring. The well-known theorem of Brooks shows that for most graphs this number can be reduced by one.

Theorem 1.2. *[12] Let G be a connected graph other than an odd cycle or a complete graph. Then $\chi(G) \leq \Delta(G)$.* $\square$

The maximum degree of a graph is not always a good upper bound for the chromatic number. For instance, a bipartite graph could have arbitrarily large maximum degree. However, there exist graphs for which this bound is sharp and, in two instances, is exceeded: the noted families of complete graphs and odd cycles each require $\Delta + 1$ colors. For our purposes here, Brooks' theorem will suffice for any required upper bound on the chromatic number.

Next we consider edge colorings of graphs.

Definition 1.3. Let $G = (V, E)$ be a graph and $f : E \to [k]$ be an assignment of colors to its edges. If f has the property that $f(uv) \neq f(wv)$ for every adjacent $uv, wv \in E$, we say that f is an edge k-coloring of G. The smallest integer k for which G has an edge k-coloring is called the chromatic index of G and is denoted $\chi'(G)$.

In other words, the edges of a graph must be partitioned into 1-regular subgraphs called matchings. It is clear that at least $\Delta(G)$ colors are needed for such a task, though this amount is not always sufficient. For instance, odd cycles once again require three colors. Vizing's theorem proves that these are the only two possibilities: $\chi'(G) = \Delta(G)$ or $\Delta(G) + 1$. In the former case the graph is said to be a class 1 graph and a class 2 graph in the latter.

Theorem 1.4. *[35] Let G be a graph. Then $\chi'(G) \leq \Delta(G) + 1$.* □

The proof of this statement involves a technique of using a maximal path consisting of edges that alternate between two different colors; this technique can be used to easily prove that $\Delta(G)$ is a bound for the chromatic index of bipartite graphs. Bipartite graphs were known to be class 1 graphs well before the result of Vizing; this result is attributed to Kőnig.

Theorem 1.5. *[30] Let G be a bipartite graph. Then $\chi'(G) = \Delta(G)$.* □

The final type of coloring considered is known as a total coloring.

Definition 1.6. Let $G = (V, E)$ be a graph and $f : (V \cup E) \to [k]$ be an assignment of colors to its vertices and edges. We say that f is a total k-coloring of G if it has the following three properties: 1) $f(u) \neq f(v)$ for every $uv \in E$; 2) $f(uv) \neq f(wv)$ for every adjacent $uv, wv \in E$; 3) $f(u) \neq f(uv)$ for every u and incident edge $uv \in E$. The smallest k for which G has a total k-coloring is called the total chromatic number of G and is denoted $\chi''(G)$.

Observe that a total coloring restricted to the vertices gives a vertex coloring while restricting it to the edges gives an edge coloring; these are exactly properties 1 and 2 in the definition. Property 3 could easily be satisfied by using disjoint color sets on vertices and edges; this leads to the very trivial upper bound $\chi''(G) \leq \chi(G) + \chi'(G) \leq 2\Delta(G) + 1$. For a lower bound, observe that at least $\Delta(G) + 1$ colors are needed: at a vertex of maximum degree, $\Delta(G)$ colors are needed for its incident edges, and a brand new color is needed for the vertex itself.

Behzad [9] and Vizing [35, 36] both independently conjectured that graphs can be partitioned into two types based on their total chromatic number: those with total chromatic number $\Delta(G)+1$ and those with total chromatic number $\Delta(G)+2$. This conjecture is known as the total coloring conjecture.

Conjecture 1.7. *Let G be a graph. Then $\chi''(G) \leq \Delta(G) + 2$.*

Currently there is a huge gap between the conjectured and proven upper bounds. The best known upper bounds are $\chi''(G) \leq \Delta + 8 \log^8 \Delta$ found by Hind, Molloy, and Reed in [23] and $\chi''(G) \leq \Delta + 10^{26}$ found by Molloy and Reed in [31]. Both of these results are true for graphs with maximum degree Δ sufficiently large, and use probabilistic methods in their proofs.

I direct the reader to [10] for more information regarding general graph theory and [27] for information specifically related to graph coloring problems.

1.3 New Problems

In this dissertation I will examine three non-standard coloring problems. Each contains some flavor of one of these three traditional colorings, but require some sort of additional constraint. The first type of coloring to be considered is known as adjacent vertex distinguishing total colorings. Given a graph $G = (V, E)$ and a proper total coloring $f : (V \cup E) \to [k]$, for each vertex $v \in V$ define $C(v)$ to be the set of all colors appearing at v; that is, $C(v) = f(v) \cup \left(\bigcup_{uv\in E} f(uv)\right)$. f is called an adjacent vertex distinguishing total coloring if $C(u) \neq C(v)$ for every pair of adjacent vertices. The smallest k for which such a coloring exists is known at the adjacent vertex distinguishing chromatic number and is denoted $\chi_{at}(G)$. I give new and simple proofs of χ_{at} for bipartite and complete graphs in Corollary 2.4 and Proposition 2.6, in Theorem 2.11 I show that six colors are sufficient for graphs with maximum degree three, and I give evidence for Conjecture 2.18 which proposes that five colors are sufficient for graphs with maximum degree 3.

Next are vertex coloring total weightings. Given a graph $G = (V, E)$ and $S \subseteq \mathbb{R}$, let $w : (V \cup E) \to S$ be an assignment of weights to the elements of the graph and define $W(v) = w(v) + \sum_{uv \in E} w(uv)$. w is called a vertex coloring total weighting if $W(u) \neq W(v)$ for every pair of adjacent vertices, and is a vertex distinguishing total weighting of $W(u) \neq W(v)$ for every pair of vertices. It is known that the weight set $\{1, 2, 3\}$ suffices for all graphs. Here weight sets S of cardinality two are exclusively considered. I show that for any distinct real numbers a and b there exists a vertex coloring total weighting using weight set $\{a, b\}$ for bipartite graphs in Theorem 3.7 and complete multipartite graphs in Theorem 3.10; for graphs with chromatic number at most three, I show such a weighting exists for most pairs of real numbers in Theorem 3.12. Furthermore, I show that for weight set $\{0, 1\}$ graphs with maximum degree at most four have a vertex coloring total weighting in Theorem 3.14 and I classify graphs for which a vertex distinguishing total weighting exists in Theorem 3.21.

The last colorings considered are edge list multi-colorings with measurable sets. In a list coloring problem, the possible color options for an edge are restricted to different lists of colors assigned to each edge. Given a graph $G = (V, E)$ and a non-atomic measure space $(X, \mathcal{A}, \mu)$, let $L : E \to \mathcal{A}$ be an assignment of color lists to the edges of G, and $w : E \to \mathbb{R}^+$ be an assignment of color demands to the edges. A coloring $\phi : E \to \mathcal{A}$ is called an edge list multi-coloring if $\phi(e) \subseteq L(e)$, $\mu(\phi(e)) = w(e)$ for every edge $e \in E$, and $\mu(\phi(e) \cap \phi(f)) = 0$ for every pair of adjacent edges $e, f \in E$. In Lemma 4.4 I give a new proof of Hall's marriage theorem for measurable sets.

2 Adjacent Vertex Distinguishing Total Colorings

An increasingly popular restriction on traditional colorings are meta-colorings: proper graph colorings that themselves induce properties where vertices are distinguished from each other. For example, in any proper edge coloring of a triangle, note how the set of colors appearing at any vertex is distinct from each other vertex. This is an example of a vertex distinguishing edge coloring. This coloring was first investigated in the early 1990's by the independent teams of Burris and Schelp in [13], Černý, Horňák, and Soták as the observability of a graph in [34], and Aigner, Triesch, and Tuza in [1].

Numerous other papers have investigated vertex distinguishing edge colorings, including Favaron, Li, and Schelp in [17], Bazgan, Harkat-Benhamdine, Li, and Woźniak in [8], and Balister, Kostochka, Li, and Schelp in [7]. Balister, Bollobás, and Schelp in [5] found the minimum number of colors needed to color graphs with $\Delta(G) = 2$. Balister in [4] proved for random graphs $G(n, p)$, almost always $\Delta(G)$ colors are sufficient.

Zhang, Liu, and Wang in [40] extended this coloring to consider edge colorings where only adjacent vertices were required to have distinct color sets. Other papers addressing adjacent vertex distinguishing edge colorings include Balister, Győri, Lehel, and Schelp [6] and Hatami [20], who used probabilistic methods to show that $\Delta(G) + 300$ colors are sufficient to color all graphs.

Zhang et al. in [39] extended the coloring once more to consider total colorings, rather than edge colorings, such that adjacent vertices had distinct color sets. Such colorings are called adjacent vertex distinguishing total colorings.

2.1 Definitions and Initial Observations

Recall that a total coloring is an assignment of colors to the vertices and edges of a graph so that no adjacent or incident elements share the same color.

Definition 2.1. Let $G = (V, E)$ be a graph and $f : (V \cup E) \to [k]$ be a proper total k-coloring of it. For each $v \in V$, define $C(v) = \{f(v)\} \cup \{f(uv) \mid uv \in E\}$; we call $C(v)$ the color set of v. If for every $uv \in E$ we have $C(u) \neq C(v)$, then we say that f is an adjacent vertex distinguishing total k-coloring (k-AVDTC) of G. The minimum k for which G has a k-AVDTC is called the adjacent vertex distinguishing chromatic number, denoted $\chi_{at}(G)$.

An observation showing the difficulty when dealing with these colorings is that a k-AVDTC of a graph G restricted to some subgraph H does not necessarily give a k-AVDTC of H: certainly the coloring remains proper, but it is possible that the removal of one or more edges from G could cause two adjacent vertices in H to have the same color set. One consequence of this is that a standard procedure of considering only the case of regular graphs will not suffice for this problem. In the introduction of AVDTC's, Zhang, et al. proposed the following conjecture:

Conjecture 2.2. *[39]* $\chi_{at}(G) \leq \Delta(G) + 3$

They also noted the following trivial lower bound: if a graph G has two adjacent vertices of maximum degree, at least $\Delta(G) + 2$ colors are needed for an AVDTC. A trivial upper bound for $\chi_{at}(G)$ can be seen as follows:

Proposition 2.3. $\chi_{at}(G) \leq \chi(G) + \chi'(G)$.

Proof. Let f be a total coloring of G obtained by combining a proper vertex coloring of G using $\chi(G)$ colors and a proper edge coloring using $\chi'(G)$ new colors. Observe that f is a proper total coloring; furthermore $C(u) \neq C(v)$ for any $uv \in E(G)$ since $f(u) \neq f(v)$ and these two colors appear exclusively on vertices. Thus f is a $(\chi(G) + \chi'(G))$-AVDTC. □

This immediately gives the optimal upper bound for bipartite graphs.

Corollary 2.4. *Let G be a bipartite graph. Then $\chi_{at}(G) \leq \Delta + 2$.* □

Applying Brooks' theorem and Vizing's theorem to Proposition 2.3 gives this immediate corollary:

Corollary 2.5. *Let G be a graph that isn't a complete graph or an odd cycle. Then $\chi_{at}(G) \leq 2\Delta(G) + 1$.* □

These two basic exceptional cases can be handled separately. Zhang et al. found $\chi_{at}(G)$ for cycles and complete graphs in [39]; I presented independent and simpler proofs for these cases in [25]. This first proof is due to András Gyárfás (personal communication).

Proposition 2.6. $\chi_{at}(K_n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n+2 & \text{if } n \text{ is odd} \end{cases}$ *for $n \geq 2$.*

Proof. An optimal AVDTC for complete graphs can be easily obtained from the standard near-factorization of K_{2m+1}, defined as follows: on vertex set $v_0, v_1, \ldots, v_{2m}$, for every $i \in \{0, 1, 2, \ldots, 2m\}$ define $M_i = \{v_{i-x}v_{i+x} \mid 1 \leq x \leq m\}$ where $i \pm x$ here denotes addition modulo $2m + 1$. Observe M_i misses v_i; if the vertices are colored with their vertex labels and edges in M_i are colored with i, then we have a proper total coloring of K_{2m+1} with $2m + 1$ colors. Since every vertex has the same color set, if one vertex is removed an AVDTC of K_{2m} is obtained. We claim that if another vertex is removed, an AVDTC of K_{2m-1} is obtained (see Figure 2.1).

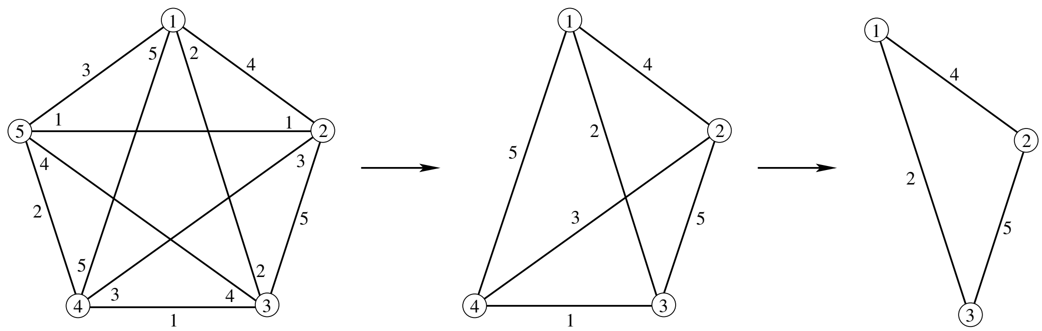

Figure 2.1: Coloring Sequence from K_5 to K_3

This follows from the fact that in the standard near-factorization above, there are no four-cycles in the union of two color classes and thus no two vertices lose the same two colors. Suppose there exists a four-cycle with edges colored a and b. This means the set of labels of vertices involved can be expressed as both $\{a-i, a+i, a-j, a+j\}$ and $\{b-k, b+k, b-\ell, b+\ell\}$ for some $i, j, k, \ell \in \{1, 2, \ldots, m\}$. Since these two sets must be equal modulo $2m+1$, their sums must also be equal. Therefore $4a \equiv 4b \mod (2m+1)$; but since $(4, 2m+1) = 1$, this implies $a = b$, a contradiction.

It is clear that $2m+1$ colors are needed for K_{2m}. To show that $2m+1$ colors are necessary for K_{2m-1}, suppose an AVDTC with $2m$ colors is possible. If a color is absent from the color set of one vertex, it must be present at every other vertex because all color sets must be distinct. Additionally, every vertex must be colored distinctly. By the pigeonhole principle there exists a color i missing from some vertex that colors another vertex. It follows that every remaining vertex must be incident to an edge colored by i. However, an odd number of vertices remain and obviously no perfect matching exists between them, a contradiction. $\square$

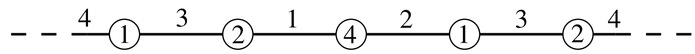

Figure 2.2: 4-AVDTC for Odd Cycles

Proposition 2.7. *$\chi_{at}(C_n) = 4$ for $n \geq 4$.*

Proof. If n is even, alternately color the vertices of the cycle 1 and 2, and alternately color the edges 3 and 4. This is an AVDTC because it is clearly proper and the color sets of adjacent vertices are distinguished by their vertex color. If n is odd, again alternately color the vertices and edges of the cycle as in the even case except for one vertex, say v_1, and its incident edges. Hence we assume v_2 is colored 1, v_n is colored 2, and both have an incident edge colored 3. Then color v_1 by 4, v_1v_2 by 2, and v_nv_1 by 1 (see Figure 2.2). $\square$

This in conjunction with Corollary 2.4 gives the following immediate corollary:

Corollary 2.8. *Let $G \neq K_3$ be a graph with $\Delta(G) = 2$. Then $\chi_{at}(G) \leq 4$.* $\square$

2.2 Graphs with Maximum Degree Three

Currently, there exist a small handful of proofs affirming Conjecture 2.2 in the first non-trivial case: $\Delta(G) = 3$. Wang provided an intricate case analysis to prove it in [38]; Chen provided another proof in [14]. I was able to show a much simpler proof of this case in [25]. The method used focuses on a restricted family of AVDTC's.

Definition 2.9. Let $G = (V, E)$ be a graph and $f : (V \cup E) \to [k]$ be a proper total coloring. Define $C_E \subseteq [k]$ to denote the set of colors that appear on the edges of G and $C_V \subseteq [k]$ to denote the set of colors that appear on the vertices of G. We say that f is an almost disjoint total coloring if $|C_E \cap C_V| \leq 1$.

Lemma 2.10. *An almost disjoint total coloring of a graph is an AVDTC.*

Proof. Observe that two adjacent vertices have identical color sets only if the color appearing on each vertex is used to color an incident edge of the other. However, this cannot happen since the colors used to color two adjacent vertices cannot both be used to color edges also. □

Theorem 2.11. *If G is a graph with $\Delta(G) = 3$, then $\chi_{at}(G) \leq 6$.*

Proof. If $G = K_4$, by Proposition 2.6 $\chi_{at}(K_4) = 5$. Now suppose $G \neq K_4$; we aim to show that G has an almost disjoint total coloring with 6 colors. We claim there exists a partial almost disjoint 6-coloring f of G with the following properties:

1. The vertices of G are colored 1, 2, and 3.
2. The edges incident to the 3 color class are colored 4, 5, and 6.
3. The edges between the 1 and 2 color classes are colored 3, 4, 5, 6, or remain uncolored.

The graph G has a proper vertex coloring with colors 1, 2, and 3 by Brooks' theorem. Consider the bipartite graph formed by all edges with one endpoint in color class 3, and the other endpoint in color classes 1 or 2; by König's theorem, these edges can be 3-colored with colors 4, 5, and 6. Therefore there exists a partial 6-coloring of G that satisfies Properties 1 and 2. Consider the collection F of all such colorings of G with these two properties. We claim there exists a coloring in F that satisfies Property 3 with no uncolored edges.

Suppose this is not the case; choose a coloring $f \in F$ with the fewest number of uncolored edges. Given such an f, we aim to create another partial 6-coloring $f' \in F$ such that the number of uncolored edges is one less, thus deriving a contradiction.

Consider an edge uv that is left uncolored by f. By property 3, uv must be incident to a 1 vertex and a 2 vertex. If $C(u)$ and $C(v)$ have a common color then at most three of the colors 3, 4, 5, and 6 are found at u or v, and so we may choose the fourth color with which to color uv in f'.

Suppose $C(u)$ and $C(v)$ have no common color. Without loss of generality, suppose u has a 4 edge and a 3 edge, call it uw, and v has a 5 edge and a 6 edge. Now consider $C(w)$. If $C(w) \neq C(v)$, in f' we can color uv 3 and recolor uw with either 5 or 6, whichever is not present at w.

Suppose $C(w) = C(v)$. Consider the longest path P consisting of edges alternately colored 4 and 5 originating from u and switch the colors of each edge along it. If P does not terminate at v, we may now color uv 4 in f' since the color 4 no longer appears at u. If P does terminate at v, it obviously cannot terminate at w and so in f' we may color uv 3 and recolor uw 4. This exhausts all possibilities, and therefore there exists an almost disjoint total 6-coloring of G. □

2.3 Cubic Graphs

Although complete graphs provide a construction showing that the conjectured upper bound for χ_{at} is sharp for graphs with even maximum degree, no such construction exists for graphs with odd maximum degree. Chen in [14] constructed another example of a graph with even maximum degree for which $\chi_{at}(G) = \Delta(G) + 3$. In examining many smaller graphs by hand, common counterexamples with maximum degree three were found to need only five colors for an AVDTC. In particular, the Petersen graph needs only five colors. A simple proof of this fact can be extended to provide an AVDTC for a larger family of graphs.

Definition 2.12. For any given integer $n \geq 1$, consider the collection of subsets of $[2n-1]$ of size $n-1$. If we use a graph to represent this collection where each vertex corresponds to exactly one subset of size $n-1$ and two vertices are adjacent if and only if their two corresponding subsets are disjoint, we call this graph an odd graph and denote it O_n.

Odd graphs are a specific example of a larger family of graphs known as Kneser graphs. Note that under this definition, O_3 is the Petersen graph.

Proposition 2.13. *$\chi_{at}(O_n) \leq 2n-1$ for $n \geq 3$.*

Proof. For each vertex $v_i \in V(O_n)$, let A_i denote the subset corresponding to that vertex. Observe that for any $v_iv_j \in E(O_n)$, $|\overline{A_i} \cap \overline{A_j}| = |\overline{A_i \cup A_j}| = 1$ since $|A_i| = |A_j| = n-1$ and A_i and A_j are disjoint. Furthermore, note that $\overline{A_i} \cap \overline{A_j} \neq \overline{A_i} \cap \overline{A_k}$ for adjacent $v_iv_j, v_iv_k \in E(O_n)$, since this would imply $A_j = A_k$. Define a total coloring $f : (V(O_n) \cup E(O_n)) \to [2n-1]$ as follows: $f(v_i) = c$ where $c \in A_i$, and $f(v_iv_j) = \overline{A_i} \cap \overline{A_j}$. We see that this is a proper total coloring since each edge color appears uniquely at each vertex, the set of possible vertex colors is disjoint from the set of possible edge colors at each vertex, and the sets of possible vertex colors are disjoint for any two adjacent vertices, by definition of the odd graph. It suffices to show that the complement of the color sets are distinguished for adjacent vertices. Observe that for each vertex v_i, $\overline{C(v_i)} = A_i \setminus \{c\}$, where $c = f(v_i)$; call this set A'_i. Since $A_i \cap A_j = \emptyset$ for $v_iv_j \in E(O_n)$, it follows that $A'_i \cap A'_j = \emptyset$ as well. Therefore, the color sets of adjacent vertices are distinguished. $\square$

With the working conjecture that graphs with maximum degree 3 had $\chi_{at}(G) \leq 5$, we also obtained the following results for very specific families of 3-regular graphs. Note $\chi_{at}(G) \geq 5$ is immediately true in this situation since we have two adjacent vertices of maximum degree. Therefore, each of the following proofs demonstrate that five colors are sufficient for an AVDTC of the given graph.

Definition 2.14. Consider two disjoint n-cycles $i_1 i_2 \dots i_n$ and $o_1 o_2 \dots o_n$. Let π denote a permutation on n elements. Add to the set of edges $i_j o_{\pi(j)}$ for $1 \leq j \leq n$. We call the family of such graphs n-snares. If π is the identity permutation, we call the graph a drum and denote it D_n.

Proposition 2.15. $\chi_{at}(D_n) = 5$

Proof. Let f be a 4-AVDTC of C_n; let $f'(v) = f(v) + 1 \mod 4$ and similarly for edges. Define $g : V(D_n) \cup E(D_n) \to [5]$ as follows: $g(i_k) = f(i_k)$, $g(i_k i_{k+1}) = f(i_k i_{k+1})$, $g(o_k) = f'(o_k)$, $g(o_k o_{k+1}) = f'(o_k o_{k+1})$, $g(i_k o_k) = 5$. Since f is a proper total 4-coloring of C_n, it is clear that g is a proper total 5-coloring of D_n. Furthermore, since f is a 4-AVDTC of C_n, color sets of adjacent vertices in the same cycle are still distinguished, and color sets of adjacent vertices in different cycles are 1-shifts of each other, and therefore distinguished. □

Proposition 2.16. *Let G be an even snare. Then $\chi_{at}(G) = 5$*

Proof. Alternately color the vertices of the first cycle 1 and 2, and alternately color its edges 3 and 4. Similarly, alternately color the vertices of the second cycle 3 and 4, and alternately color its edges 1 and 2. Color the matching 5. It is clear that this is a proper total coloring. It is an AVDTC because any two adjacent vertices on the same cycle are distinguished by their vertex color, and any two adjacent vertices on different cycles are distinguished by their edge colors. □

Proposition 2.17. *Let G be an odd snare containing a C_4. Then $\chi_{at}(G) = 5$.*

Proof. Without loss of generality, assume the C_4 is induced by i_1, o_1, o_n, i_n in that order. Define a partial 5-coloring of G as follows: for $2 \leq j, k \leq n-1$, $f(o_j) = 1$

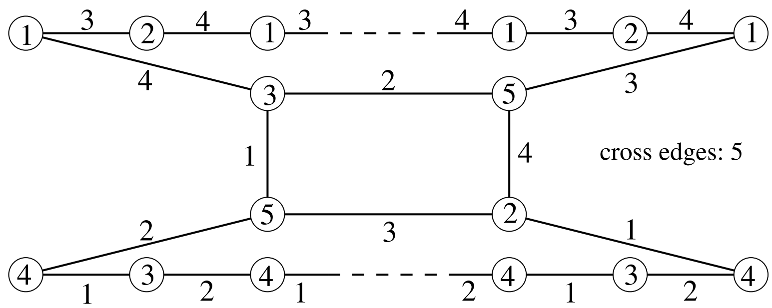

Figure 2.3: Odd Snare with a C_4

if j is even or 2 if odd, $f(i_k) = 4$ if k is even or 3 if odd, and $f(o_j i_k) = 5$; for $2 \leq j, k \leq n$, $f(o_j o_{j-1}) = 4$ if j is even or 3 if odd and $f(i_k i_{k-1}) = 2$ if k is even or 1 if odd. Observe that this partial coloring leaves only the vertices and edges of the C_4 uncolored. Additionally, f is a proper coloring and provides the colored vertices with color sets distinct from its neighbors; this distinguishing property can be seen in the fact that adjacent vertices in the same cycle are distinguished by their vertex color and adjacent vertices in different cycles are distinguished by their edge colors. We define a total 5-coloring $f' : V(G) \cup E(G) \to [5]$ as follows: $f'(x) = f(x)$ when $f(x)$ is defined; $f'(o_1) = 3$, $f'(i_1) = 5$, $f'(i_n) = 2$, $f'(o_n) = 5$, $f'(o_1 i_1) = 1$, $f'(o_1 o_n) = 2$, $f'(o_n i_n) = 4$, $f'(i_n i_1) = 3$ (see Figure 2.3). We observe that this total coloring is proper since f is proper and it can be seen that the coloring of the C_4 preserves this properness. Additionally, f' is adjacent vertex distinguishing: we have already shown that the vertices colored by f are distinguished; also observe that $C(o_1) = \{1, 2, 3, 4\}$, $C(o_n) = \{2, 3, 4, 5\}$, $C(i_1) = \{1, 2, 3, 5\}$, $C(i_n) = \{1, 2, 3, 4\}$, $C(o_2) = C(o_{n-1}) = \{1, 3, 4, 5\}$, and $C(i_2) = C(i_{n-1}) = \{1, 2, 4, 5\}$. Therefore, the coloring of the C_4 is also adjacent vertex distinguishing within itself and within the coloring of the rest of the graph given by f. $\square$

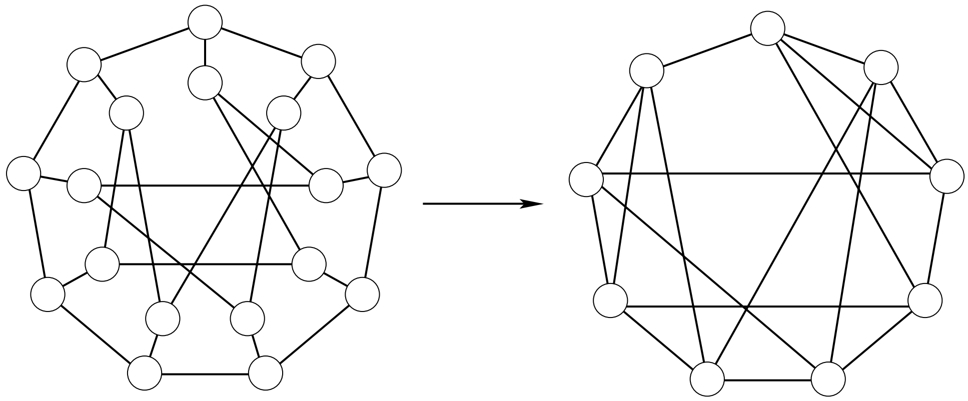

Figure 2.4: Contracting the Edges of a Snare

The final case, namely odd snares with no C_4, have proven to be much more elusive. The first unsolved case is for 7-snares, since every 3-snare is a drum and the only 5-snare with no C_4 is the Petersen graph. In practice, every small case considered has had a 5-AVDTC that could be found without extreme difficulty. No general proof for this case has been found, but I feel it almost certainly is true.

This case can be reduced to an edge coloring problem in the following manner. Observe that for n-snares with no C_4, by contracting the matching between the two cycles a 4-regular graph with vertices $v_1, v_2, \ldots, v_n$ is created (see Figure 2.4). Refer to the edges corresponding to the outside cycle as outside edges, and those corresponding to the inside cycle as inside edges. The goal is to find a proper edge 5-coloring of this graph with the following conditions: there exist a pair of adjacent vertices with disjoint sets of colors appearing on their inside edges and a pair of adjacent vertices with disjoint sets of colors appearing on their outside edges.

If such an edge coloring of this new graph exists, then there exists a corresponding 5-AVDTC of the original snare. Convert this graph back to an n-snare, preserving all edge colorings, and coloring each new edge of the matching with the only available color. Without loss of generality, say that o_1 and o_n correspond to the adjacent

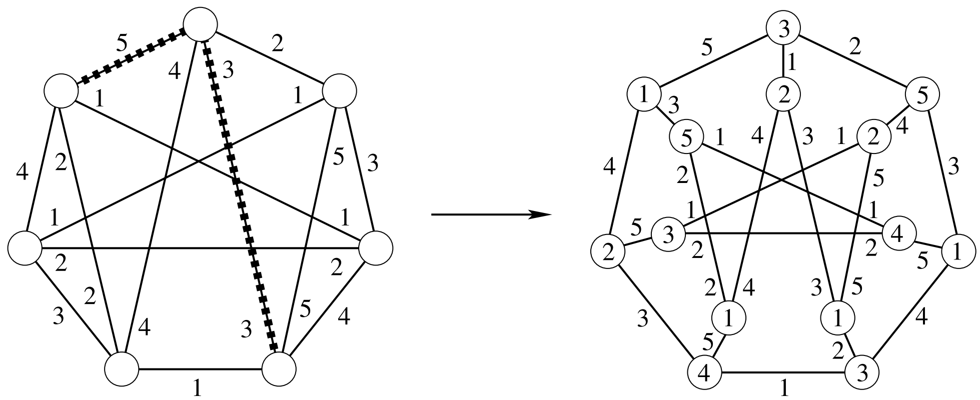

Figure 2.5: Extending an Edge-Coloring to an AVDTC

vertices with disjoint colors on their inside edges and i_1 and i_n correspond to the adjacent vertices with disjoint colors on their outside edges.

Choose a color for each vertex in the order $o_1, o_2, \ldots, o_n$ obeying the following rules: the color of o_i must be chosen from the set of colors appearing on the inside edges of v_i; if c was chosen for o_{i-1}, it cannot be chosen for o_i; and if c is a possible color for both o_{i-1} and o_i but was not chosen for o_{i-1}, then it must be chosen for o_i. Note that o_1 and o_n have disjoint color options, so there will be no chance for the coloring to be improper. Similarly, color $i_1, i_2, \ldots, i_n$ in that order following these rules: the color of i_j must be chosen from the set of colors appearing on the outside edges of v_j; if c was chosen for i_{j-1}, it cannot be chosen for i_j; and if c is a possible color for both i_{j-1} and i_j but was not chosen for i_{j-1}, then it must be chosen for i_j. Again, i_1 and i_n have disjoint color options, so this coloring will be proper (see Figure 2.5).

The conditions of choosing vertex colors ensure that adjacent vertices on the same cycle will not avoid the same color, thus their color sets must be distinct. Adjacent vertices on different cycles are distinguished since the colors used on the inside edges are disjoint from the colors used on the outside edges.

In light of these results with this particular class of 3-regular graphs, I propose the following conjecture.

Conjecture 2.18. *For every graph G with maximum degree 3, $\chi_{at}(G) \leq 5$.*

3 Vertex Coloring Total Weightings

A graph weighting is an assignment of values, usually real numbers, to the elements of the graph; it can be thought of as a graph coloring with the removal of the condition that adjacent and incident graph elements have distinct colors. Coupling these values with a commutative binary operation, addition for instance, induces a new assignment of values to the vertices of a graph by taking the sum of all values on or incident to the vertex. If adjacent vertices have distinct induced values, we say that the graph weighting is vertex coloring; if every vertex has a distinct value, we say the weighting is vertex distinguishing.

In 2004, Karoński, Łuczak, and Thomason introduced vertex coloring edge weightings in [29]: the edges of a graph are assigned values; vertices are then assigned the sum of the values appearing on its incident edges. It is clear in this instance that any graph with an isolated edge cannot have a vertex coloring edge weighting, as the two vertices on that edge will always have the same induced value. The authors considered the following question: what restrictions can be placed on the set of weights used so that it is still possible to find a vertex coloring edge weighting for any graph without an isolated edge. They conjectured that it is always possible to do so with the weight set $\{1, 2, 3\}$. The triangle shows simply that three different weights are necessary. Kalkowski, Karoński, and Pfender recently showed in [28] that the weight set $\{1, 2, 3, 4, 5\}$ is sufficient for all such graphs.

In 2007, Przybyło and Woźniak considered the related question where vertices were also weighted in addition to the edges [32, 33]. These are known as vertex coloring total weightings. In this extension, the requirement that graphs have no isolated edge can be removed since individual vertices can be weighted. Furthermore, it appears that at most two weights, not three, are necessary to induce a proper vertex coloring. This led Przybyło and Woźniak to propose a conjecture analogous to that made for vertex coloring edge weightings: every graph has a vertex coloring total weighting using weight set $\{1, 2\}$.

Independent of Przybyło and Woźniak, this problem was investigated by Lehel, Ozeki, Yoshimoto, and myself. However, our motivation was different: we considered this problem as a generalization of adjacent vertex distinguishing total colorings where the assignment of colors did not need to be proper. We proposed that every graph had a vertex coloring total weighting with weight set $\{0, 1\}$ and $\{1, 2\}$; later, we strengthened this to say any set of two distinct real values $\{a, b\}$ would suffice. Here I will examine progress on this guiding question as found in [26].

It should be noted that the use of natural numbers and addition in the cases discussed here appears to be arbitrary. In fact, the same question could be posed using any commutative semigroup. In general, it is not clear that the existence of a vertex coloring weighting with one weight set should imply the existence of a vertex coloring weighting with a different weight set; far be it that such results for one semigroup should imply the same for another in general. For the sake of simplicity, real numbers and addition will be used here.

3.1 Definitions and Basic Results

Unlike a proper coloring, here we are considering an assignment of numbers to the vertices and edges of a graph where adjacent or incident elements are allowed to be the same. Rather, we require that the sum of all numbers appearing at each vertex be distinct from that of its neighbors.

Definition 3.1. Let $G = (V, E)$ be a graph and $w : (V \cup E) \to S$, $S \subseteq \mathbb{R}$, be an assignment of real numbers, called weights, to the edges and vertices of G. For every $v \in V$, define the total weight sum to be $W(v) = w(v) + \sum_{uv \in E} w(uv)$, which we call the color of v. If $W(u) \neq W(v)$ for all $uv \in E$, then we say that w is a vertex coloring total S-weighting, in short an S-VCTW of G. If $|S| = k$, we call w a k-VCTW.

In this language, the 1,2 conjecture can be stated as follows:

Conjecture 3.2. *[32] Every graph G has a $\{1, 2\}$-VCTW.*

In this same paper of Przybyło and Woźniak, the conjecture was confirmed for bipartite graphs, complete graphs, and graphs with maximum degree at most three. In the subsequent paper [33], the conjecture was confirmed for 3-colorable and 4-regular graphs. We will show more general results for each of these families. The following is the best known result for general graphs and was found by Kalkowski; it was mentioned in [28].

Theorem 3.3. *Every graph G has a $\{1, 2, 3\}$-VCTW.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and initially weight all vertices and edges of the graph 2. For any vertex refer to the set of neighbors with lower index as left neighbors, and neighbors with higher index as right neighbors. It suffices to show that for every $k \geq 2$, we can adjust the weight sum of v_k so that it is distinct from each left neighbor without changing the weight sum of any of those neighbors. We describe such a procedure with the additional restriction that when adjusting the weight sum of a vertex, its right neighbors do not have any incident weights changed. This condition means that when adjusting the weight sum of v_k, every incident edge $v_k v_j$ is still initially weighted 2.

If v_k has ℓ left neighbors with vertex weight 2 or 3, there are ℓ possible larger weight sums of v_k that do not change the weight sum of any left neighbor: if $v_k v_j$

describes such an edge, increase the weight of v_kv_j from 2 to 3 and decrease the weight of v_j by 1; this can be done ℓ times, resulting in as many distinct weight sums of v_k. Similarly, if v_k has m left neighbors with vertex weight 1, there are m possible smaller weight sums of v_k that do not change the weight sum of any left neighbor: if v_j is such a neighbor, decrease the weight of v_kv_j from 2 to 1 and increase the weight of v_j from 1 to 2. Combined with the original unchanged weight sum, this gives $\ell + m + 1$ distinct possibilities for the weight sum of v_k. Since v_k needs only to distinguish itself from $\ell + m$ neighbors, by the pigeonhole principle there must be an option distinct from the weight sums of each of its left neighbors.

□

Most results on VCTW's have used weight sets of the form $\{1, 2, \ldots, k\}$ for some positive integer k, with particular interest in proving Conjecture 3.2. Here we consider more general two element weight sets. It is not clear that the existence of $\{1, 2\}$-VCTW's should imply the existence of some other 2-VCTW; a simple bijection between two different weight sets will not, in general, produce a new VCTW. We introduce the notion of vertex coloring total labelings (VCTL's): instead of summing the weights appearing at a vertex v, the multiset of weights are considered; let $C(v)$ denote this multiset. It is clear that a bijection between weight sets would produce new VCTL's. These two notions of VCTW's and VCTL's are identical for regular graphs, and thus bijections between weight sets do produce new VCTW's.

Proposition 3.4. *Let G be a k-regular graph. Then for distinct $a, b \in \mathbb{R}$, G has an $\{a, b\}$-VCTW if and only if it has an $\{a, b\}$-VCTL.*

Proof. Let $w : (V \cup E) \to \{a, b\}$. We show that w is not a VCTL if and only if it is not a VCTW. It is clear that for $uv \in E$, if $C(u) = C(v)$, then $W(u) = W(v)$. Suppose $W(u) = W(v)$. Assume u has m elements weighted a and n elements weighted b; similarly, assume v has m' elements weighted a and n' weighted b. Observe that $m+n = m'+n' = k+1$ and $am+bn = am'+bn'$, since $W(u) = W(v)$. It follows from these two observations that $(m-m')+(n-n') = 0$ and $(m-m')a+(n-n')b = 0$; we conclude that $c(a-b) = 0$, where $c = m-m' = n'-n$, and so $m = m'$ and $n = n'$ since a and b are distinct. Therefore $C(u) = C(v)$. $\square$

Corollary 3.5. *Let G be a k-regular graph. Then for $a, a', b, b' \in \mathbb{R}$, where $a \neq b$ and $a' \neq b'$, G has an $\{a, b\}$-VCTW if and only if it has an $\{a', b'\}$-VCTW.* $\square$

In approaching this subject, our working hypothesis is stronger than Conjecture 3.2: for any distinct real values a and b, every graph has an $\{a, b\}$-VCTW. We have just shown that this is equivalent to the conjecture in the case of regular graphs. For general graphs, it suffices to prove the hypothesis for distinct integers a and b by means of the following lemma.

Lemma 3.6. *Let G be a graph and $a, b, c \in \mathbb{R}$, $c \neq 0$. Then G has an $\{a, b\}$-VCTW if and only if it has an $\{ac, bc\}$-VCTW.*

Proof. We claim an $\{ac, bc\}$-VCTW can be obtained from an $\{a, b\}$-VCTW by replacing a weights with ac weights and b weights with bc weights. Let $W(v)$ denote the weight sum of v from the $\{a, b\}$-VCTW and $W'(v)$ denote the weight sum of v from the $\{ac, bc\}$-weighting obtained in the manner described. Suppose this process results in two adjacent vertices u and v with $W'(u) = W'(v)$. Observe that $W'(u) = cW(u)$ and $W'(v) = cW(v)$; this implies $W(u) = W(v)$, a contradiction. $\square$

Our claim follows from the lemma in this way: suppose $\frac{a}{b} \neq 1$ is rational; then there exists some real number c such that ac and bc are distinct integers. Thus if there exists an $\{ac, bc\}$-VCTW, there exists an $\{a, b\}$-VCTW. Suppose $\frac{a}{b} = \gamma$ is irrational; considering some $\{0, 1\}$-VCTW, we may obtain a $\{\gamma, 1\}$-VCTW by replacing every 0 weight with γ. Multiplying all weights by b gives us an $\{a,b\}$-VCTW. Because of this, for many proofs we will consider only the case where a and b are integers with $(|a|,|b|)=1$.

We begin by proving the hypothesis for several simple families of graphs using relatively simple arguments.

Theorem 3.7. *Let G be a bipartite graph and a,b be distinct real numbers. Then G has an $\{a,b\}$-VCTW.*

Proof. Assume a and b are integers. Let V_1, V_2 denote the partite sets of G and assume without loss of generality $|a| \leq |b|$. If $a=0$, weight the vertices of V_1 with b and all other elements with 0; since for $v_1 \in V_1$ and $v_2 \in V_2$ we have $W(v_1)=b$ and $W(v_2)=0$, this results in a proper coloring. Assume a and b are nonzero and suppose $|a|<|b|$. Weight the vertices of V_1 with a and all other elements with b. Observe that for $v_1 \in V_1$, $W(v_1) \equiv a \mod b$ and for $v_2 \in V_2$, $W(v_2) \equiv 0 \mod b$. Thus for $uv \in E$, $W(u) \neq W(v)$. Now suppose $|a|=|b|$; that is, $a=-b$. By Lemma 3.6, it suffices to prove the existence of a $\{-1,1\}$-VCTW. Weight all edges with 1 and weight the vertices as follows:

$$w(v)=\begin{cases} 1 & \text{if } v \in V_1 \text{ and } d(v) \equiv 0 \text{ or } 1 \mod 4 \\ -1 & \text{if } v \in V_1 \text{ and } d(v) \equiv 2 \text{ or } 3 \mod 4 \\ -1 & \text{if } v \in V_2 \text{ and } d(v) \equiv 0 \text{ or } 1 \mod 4 \\ 1 & \text{if } v \in V_2 \text{ and } d(v) \equiv 2 \text{ or } 3 \mod 4 \end{cases}$$

Observe $W(v)=d(v)\pm 1$ for every vertex v. Suppose there exist adjacent vertices $v_1 \in V_1$ and $v_2 \in V_2$ such that $W(v_1)=W(v_2)$. Thus either $d(v_1)=d(v_2)$ or $d(v_1)=d(v_2)\pm 2$. The former case implies $w(v_1)\neq w(v_2)$ and thus $W(v_1)\neq W(v_2)$, a contradiction; the latter case implies $w(v_1)=w(v_2)$ and thus again $W(v_1) \neq W(v_2)$, a contradiction. Therefore no such adjacent vertices exist and w is a $\{-1,1\}$-VCTW. $\square$

Corollary 3.8. *Trees have an $\{a,b\}$-VCTW for distinct real weights a and b.* $\square$

Corollary 3.9. *Every graph with maximum degree 2 has an $\{a,b\}$-VCTW for any two distinct real weights a and b.*

Proof. By Theorem 3.7, the result is true for paths and even cycles. By Corollary 3.5, it suffices to show odd cycles have a $\{1,2\}$-VCTW. Suppose G is an odd cycle $(v_1, v_2, \ldots, v_{2k+1})$. Let $w(v_1) = w(v_1v_2) = w(v_{2j}) = 1$ for $2 \leq j \leq k$, and weight all other elements with 2. This results in a weighting of G satisfying $W(v_1) = 4$, $W(v_{2j}) = 5$ and $W(v_{2j+1}) = 6$, for $1 \leq j \leq k$. $\square$

Theorem 3.10. *Let G be a complete multipartite graph and let $a < b$ be real numbers. Then G has an $\{a,b\}$-VCTW.*

Proof. Let $V_1, V_2, \ldots, V_\ell$ denote the maximal independent sets of G. Define G_i to be the subgraph of G induced by $\bigcup_{m=1}^{i} V_m$; furthermore set $k_i = |V_i|$ and $n_i = |G_i|$. We may assume the V_i's are ordered so that that $k_2 \leq k_1$, and furthermore that $k_{i-2} \leq k_i$ for i odd and $k_{i-2} \geq k_i$ for i even, where $3 \leq i \leq \ell$. We incrementally weight the elements of G as follows: at the ith step, weight V_i and its incident edges in G_i with a or b for i odd or even, respectively. Let $W_i(v)$ denote the weight sum of v at step i.

Here let v_i denote a vertex of V_i. Observe that $W_2(v_1) < W_2(v_2)$ since $a + bk_2 < b + bk_1$. For $i \geq 3$ odd, we note that $W_i(v_i) < W_i(v_{i-2})$:

$$\begin{aligned}
ak_{i-1} + ak_{i-2} &< bk_{i-1} + ak_i \\
ak_{i-1} + ak_{i-2} + an_{i-3} + a &< bk_{i-1} + ak_i + an_{i-3} + a \\
a(n_{i-1} + 1) &< a(n_{i-3} + 1) + bk_{i-1} + ak_i \\
W_i(v_i) &< W_{i-2}(v_{i-2}) + bk_{i-1} + ak_i \\
W_i(v_i) &< W_i(v_{i-2}).
\end{aligned}$$

Similarly, for $i \geq 4$ even, $W_i(v_i) > W_i(v_{i-2})$:

$$\begin{aligned}
bk_{i-1} + bk_{i-2} &> ak_{i-1} + bk_i \\
bk_{i-1} + bk_{i-2} + bn_{i-3} + b &> ak_{i-1} + bk_i + bn_{i-3} + b \\
b(n_{i-1} + 1) &> b(n_{i-3} + 1) + ak_{i-1} + bk_i \\
W_i(v_i) &> W_{i-2}(v_{i-2}) + ak_{i-1} + bk_i \\
W_i(v_i) &> W_i(v_{i-2}).
\end{aligned}$$

At each step i, the weight sum of each $v \in V(G_{i-1})$ increases by the same amount; with our previous observations this implies for j odd and m even,

$$W_i(v_j) < W_i(v_{j-2}) < W_i(v_1) < W_i(v_2) < W_i(v_{m-2}) < W_i(v_m)$$

where $5 \leq j, m \leq i$. Therefore G_i has an $\{a, b\}$-VCTW for $1 \leq i \leq \ell$; since $G_\ell = G$, we have the desired result. $\square$

Theorem 3.10 proves the existence of $\{a, b\}$-VCTW's for many classic families of graphs, including complete graphs, complete graphs with a matching removed, and Turán graphs. Note also that in this proof we do not need to assume that a and b are integers.

3.2 Three-colorable Graphs

We next consider the case of graphs with chromatic number three. We show that if we choose a and b to be integers such that $a \not\equiv b \mod 3$, then there exists a total $\{a, b\}$-weighting of any three-colorable graph so that the weight sums of adjacent vertices are distinct modulo 3; in other words, the weighting induces a three-coloring.

Lemma 3.11. *Let G be a three-colorable graph and $a, b \in \{1, 2, 3\}$ be distinct. Then G has an $\{a, b\}$-weighting w such that $W(u) \not\equiv W(v) \mod 3$ for every $uv \in E$.*

Proof. Let $x \in V(G)$ be a non-cut vertex of G and f be a three-coloring of G. Let S_1, S_2, and S_3 denote the color classes of f where $x \in S_1$; let v_i denote an arbitrary vertex contained in S_i. We claim there exists a total $\{a, b\}$-weighting of G where $W(v_i) \equiv i \mod 3$ for every $v_i \in V(G) \setminus \{x\}$. Suppose not; let w be a total $\{a, b\}$-weighting with the fewest number of vertices such that $W(v_i) \not\equiv i \mod 3$. Observe if $W(v_i) \not\equiv i \mod 3$, then either $W(v_i) \equiv i+a-b \mod 3$ or $W(v_i) \equiv i+b-a \mod 3$; furthermore, observe that in the first case $w(v_i) = b$ and in the second $w(v_i) = a$, otherwise a correct weighting could be obtained simply by changing the weight of v_i. Since these two cases are symmetric, we may focus only on the first.

Consider the following scheme for reducing the number of incorrectly weighted vertices or moving an incorrect weighting from v_i to a neighbor v_j. If $w(v_iv_j) = a$, change its weight to b; if $w(v_iv_j) = b$, change its weight to a and change $w(v_i)$ to a. Observe in both of these cases, this change results in a correct weighting for v_i and only affects the weighting of v_j. By using this process, we may "push" an incorrect weighting of u along a uv-path by first moving the incorrect weighting of u to its neighbor in the path and then iterating this process, starting at the next incorrectly weighted vertex on the path.

When this process completes, there can be at most one incorrectly weighted vertex in this path, namely v. If w has more than one incorrectly weighted vertex, this number can be reduced by one by pushing a bad weighting along a path between two of these vertices. Since w contains the fewest number of such vertices, we conclude it can have at most one incorrectly weighted vertex, and furthermore we may push it to any vertex in the graph; by pushing the bad vertex to x, we prove the claim.

If $W(x) \equiv 1 \mod 3$, we are done. Suppose not; without loss of generality, let $W(x) \equiv 1 + a - b \mod 3$. If $w(x) = a$, then a correct weighting could be obtained by changing $w(x)$ to b; assume $w(x) = b$. Suppose x has two edges xy and xz with the same weight. Since x is not a cut vertex, there exists a cycle C containing x, y, and z. If we push the incorrect weighting of x along C so that at some point no incorrect weighting is induced, we are done; otherwise, suppose we push the weighting back to x, that is, all the way around C. Call these new weights w'. If $w(xy) = w(xz) = a$, then $w'(xy) = w'(xz) = b$; by letting $w'(x) = a$, we have $W'(x) = W(x) + 2(b - a) + (a - b) \equiv 1 \mod 3$. If $w(xy) = w(xz) = b$, then $w'(xy) = w'(xz) = a$; by keeping $w'(x) = b$, we have $W'(x) = W(x) + 2(a - b) \equiv 1 \mod 3$.

If no $xy, xz \in E(G)$ exist with the same weight, then either $d(x) = 1$ or $d(x) = 2$ and x has exactly one edge of each weight. If $d(x) = 1$, it has a unique neighbor y; by choosing $w(x)$ appropriately, we may conclude $W(x) \not\equiv W(y)$, thus proving the result. If $d(x) = 2$, then $W(x) = a + 2b \equiv 1 + a - b \mod 3$; but this implies $3b \equiv 1 \mod 3$, a contradiction. $\square$

Most remaining values of a and b can be handled in a slightly simpler manner.

Theorem 3.12. *Let G be a three-colorable graph and a and b be distinct real numbers. Assuming without loss of generality $|a| \leq |b|$, if $\frac{a}{b} \neq -\frac{1}{2}$ then G has an $\{a, b\}$-VCTW.*

Proof. It suffices to prove the result for integers. By Lemma 3.6 we may assume $(|a|, |b|) = 1$. By Lemma 3.11, all that remains are a and b where $a \equiv b \mod 3$; in particular, note that the cases where $|a| = |b|$ and $|a| = 0$ follow as a consequence of this lemma. Thus we may assume $0 < |a| < |b|$. Suppose $|b| \geq 4$. Let S_1, S_2, and S_3 denote the color classes for a greedy 3-coloring of G (that is, every vertex in S_i has a neighbor in S_j where $j < i$). In S_3, weight all vertices and incident edges with b. In S_2, weight each vertex a and exactly one edge connecting the vertex to a neighbor in S_1 with a; weight all other incident edges with b. In S_1, all incident edges should have been previously weighted. Weight each vertex of S_1 so that $W(v) \not\equiv 0$ or $2a$ modulo $|b|$: if the weight sum of the incident edges is congruent to 0 or $2a$ mod $|b|$, then weight the vertex a, otherwise weight it b. By our assumption that $|b| \geq 4$ and $(a, b) = 1$, it follows that $0, a, 2a, 3a \mod b$ are all distinct values. Thus $W(v_i) \neq W(v_j)$ for $v_i \in S_i$ and $v_j \in S_j$ where $1 \leq i < j \leq 3$. Now suppose $|b| < 4$; since $0 < |a| < |b| \leq 3$ and $a \equiv b \mod 3$, we must have $a = \pm 1$ and $b = \mp 2$, but then $\frac{a}{b} = -\frac{1}{2}$. $\square$

I feel that the same result for the last case where the weight set used is $\{\pm 1, \mp 2\}$ is almost certainly true, but obviously requires a different approach.

3.3 Graphs with Small Maximum Degree

By Theorems 3.10 and 3.12, we may conclude that every graph with maximum degree at most three has an $\{a, b\}$-VCTW for most pairs of real numbers a and b. In examining the next case of graphs with maximum degree four, techniques quite different from those previously used were needed. We restricted our investigation to the specific, though not previously investigated, weight set $\{0, 1\}$. We use the existence of a restricted $\{0, 1\}$-VCTW's of graphs with maximum degree three to draw conclusions about graphs with maximum degree four.

Lemma 3.13. *If G is a graph with $\Delta(G) \leq 3$ then it has a $\{0, 1\}$-VCTW such that $W(v) \geq 1$ for every $v \in V(G)$.*

Proof. If $G = K_4$, by Theorem 3.10, there exists a $\{0, 1\}$-VCTW of K_4. If there exists a vertex v with $W(v) = 0$, then every vertex has an incident edge weighted 0. By swapping all 0 and 1 weights, we obtain a new VCTW of K_4 such that all vertices have strictly positive color.

Now suppose $G \neq K_4$. Consider a greedy 3-coloring of G into color classes S_1, S_2, and S_3. We will consider two types of vertices in S_3: define T_1 to be the set of vertices in S_3 which have exactly one neighbor in S_1, and define T_2 to be the set of vertices in S_3 with two neighbors in S_1. Additionally, define Y to be the set of vertices in S_2 with two neighbors in S_3, at least one of which is in T_2. We aim to weight G so that vertices in S_1 and T_1 have odd color while those in S_2 and T_2 have even color; furthermore, we will distinguish adjacent vertices that have the same color parity. We will incrementally weight the vertices and edges of G in three steps. At each step, let $L(v)$ denote the sum of all incident edge weights up to that step.

Step 1. For each $v \in S_2 \setminus Y$ weight exactly one edge incident to a vertex in S_1 by 1. Weight all edges between S_1 and T_2 by 1.

Step 2. Let $v \in S_1$ have exactly one neighbor $x \in T_1$. Note that $L(v) \leq 2$ since v can have at most two neighbors in $(S_2 \setminus Y) \cup T_2$. If $L(v) = 0$, let $u \in S_2$ be a neighbor of x and weight vx, ux, and x by 1. If $L(v) = 1$, weight v and vx by 1; if $L(v) = 2$, weight vx by 1.

Now let $v \in S_1$ have exactly two neighbors $x_1, x_2 \in T_1$. Note that $L(v) \leq 1$ since v can have at most one neighbor in $(S_2 \setminus Y) \cup T_2$. If $L(v) = 0$, weight vx_1, vx_2, and v by 1. If $L(v) = 1$, weight vx_1 and vx_2 by 1. If v has three neighbors $x_1, x_2, x_3 \in T_1$ then $L(v) = 0$ and we weight vx_1, vx_2, and vx_3 by 1. If $v \in S_1$ has no neighbors in T_1, choose $w(v)$ so that $W(v)$ is odd; that is, if $L(v) = 0$ or 2, let $w(v) = 1$, otherwise let $w(v) = 0$.Weight all other vertices and incident edges in S_1 and T_1 with 0.

Note that at the end of Step 2, vertices in S_1 and T_1 have an odd color. Furthermore, adjacent vertices in S_1 and T_1 have distinct colors.

Step 3. Let $v \in S_2 \setminus Y$. If v has exactly one neighbor $x \in T_2$, then $L(v) = 1$, otherwise a higher value would imply v has a neighbor in T_1, and thus $v \in Y$; weight vx and x by 1. If v has no such neighbor in T_2, choose $w(v)$ so that $W(v)$ is even; that is, if $L(v) = 1$ or 3, let $w(v) = 1$, otherwise let $w(v) = 0$. Note that $L(v) \geq 1$ from Step 1, so $W(v) > 0$. Suppose $v \in Y$ has exactly one neighbor $x \in T_2$; note that $0 \leq L(v) \leq 1$. If $L(v) = 0$, weight v, x, and vx by 1; if $L(v) = 1$, weight x and vx by 1. Suppose v has two neighbors $x_1, x_2 \in T_2$. In this case $L(v) = 0$ and we weight vx_1, vx_2, x_1, and x_2 by 1. Weight all remaining vertices and edges with 0.

In Step 3 we have weighted only vertices in S_2 and T_2 and edges between them, so adjacent vertex pairs in S_1 and T_1 that were distinguished in Step 2 remain distinguished. Furthermore, vertices in S_2 and T_2 have even color such that adjacent pairs in these two sets have distinct colors. Since vertices in S_1 and T_1 have odd color while vertices in S_2 and T_2 have even color, adjacent pairs of vertices between these two collections are distinguished. Therefore we have a $\{0,1\}$-VCTW of G. $\square$

Theorem 3.14. *If G is a graph with $\Delta(G) \leq 4$ then it has a $\{0,1\}$-VCTW.*

Proof. Let G be a graph with $\Delta(G) \leq 4$. We take a maximum independent set I of G and let H be the subgraph of G induced by $V(G) \setminus I$. Since every vertex in $V(H)$ has a neighbor in I, we have $\Delta(H) \leq 3$. By Lemma 3.13, H has a $\{0,1\}$-VCTW such that no vertex is colored by 0. Extend this to a weighting of G by weighting every other element with 0. Every vertex in I has color 0, and every vertex in H has strictly positive color distinct from that of its neighbors in H. Thus adjacent vertices have distinct colors. $\square$

Applying Corollary 3.5 gives the following immediate corollary:

Corollary 3.15. *Let G be a 4-regular graph and a, b be distinct real numbers. Then G has an $\{a,b\}$-VCTW.* $\square$

3.4 Vertex Distinguishing Total Weightings

Allowing the additive identity in the weight set allows for an additional degree of flexibility in finding VCTW's. Removing some subset of the edges weighted zero results in a new graph where the restriction of the original $\{0,1\}$-VCTW to the subgraph still colors the vertices. This fact also plays an important role in a restriction

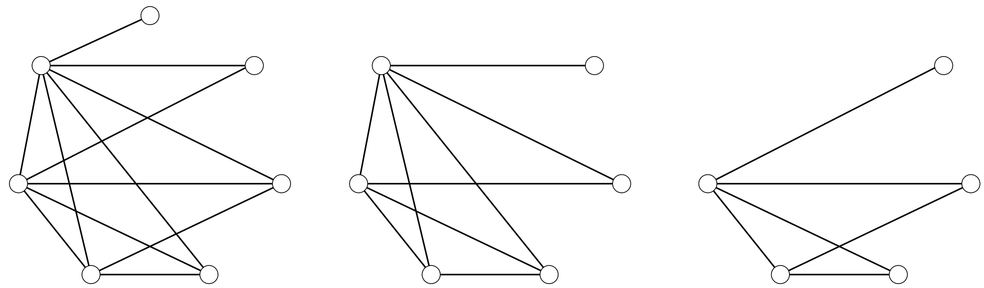

Figure 3.1: H_7 Decomposed Into H_6 and H_5

of VCTW's where every pair of vertices, regardless of adjacency, have distinct weight sums.

Definition 3.16. Let $G = (V, E)$ be a graph and $S \subseteq \mathbb{R}$. We call a total weighting $w : (V \cup E) \to S$ a vertex distinguishing total S-weighting of G, S-VDTW for short, if $W(u) \neq W(v)$, for every $u, v \in V$.

Note that VCTW's and VDTW's are identical for complete graphs. If a $\{0, 1\}$-VCTW of K_n is represented by a symmetric $n \times n$ matrix $A = (a_{ij})$ where $a_{ii} = w(v_i)$ and $a_{ij} = a_{ji} = w(v_i v_j)$, observe that the vertices can be permuted in such a way so that A forms an "anti-triangular" matrix: that is, the anti-diagonal running from the bottom left to top right has entries all the same value say 1, all entries above the anti-diagonal are 0, and all the entries below it are 1. This can be easily seen since each row sum, corresponding to the weight sum of that vertex, must be distinct from all other rows. If all of the edges weighted zero are removed from the complete graph, the following graph is produced:

Definition 3.17. Let H_n be the graph defined as follows: $V(H_n) = \{v_1, v_2, \ldots, v_n\}$, and $E(H_n) = \{v_j v_k : j + k \geq n + 1\}$.

This family of graphs has an interesting recursion property within it.

Proposition 3.18. *For $n \geq 3$, $H_n - v_n \cong H_{n-2} \cup \{v_0\}$.*

Proof. Denote $u_j = v_{j-1}$, $j = 1, \ldots, n-1$, a vertex of $H_{n-2} \cup \{v_0\}$, and use v_i, $i = 1, \ldots, n-1$, to denote the vertices of $H_n - v_n$. We claim that the bijection $v_i \leftrightarrow u_i, 1 \leq i \leq n-1$, is an isomorphism between $H_n - v_n$ and $H_{n-2} \cup \{v_0\}$.

By the definition of H_n, we have $v_i v_j \in E(H_n - v_n)$ if and only if $i + j \geq n + 1$. Similarly, by the definition of H_{n-2}, $u_k u_\ell \in E(H_{n-2} \cup \{v_0\})$ if and only if $v_{k-1} v_{\ell-1} \in E(H_{n-2})$ if and only if $k + \ell \geq n + 1$. Therefore $v_i v_j \in E(H_n - v_n)$ if and only if $u_i u_j \in E(H_{n-2} \cup \{v_0\})$. □

The graphs H_n have several interesting properties that can be obtained easily by the definition. There is exactly one maximum matching in H_n, the set $M_n = \{v_j v_k : j + k = n + 1\}$. Furthermore, $\overline{H_n} \cong H_n \setminus M_n \cong H_{n-1} \cup \{v_0\}$. This last fact implies that H_n contains H_i as a subgraph for every $1 \leq i \leq n$. This can perhaps be seen more clearly by considering the vertices of H_n, call them $v_1, v_2, \ldots, v_n$, and reducing each index by 1: since $(i-1) + (j-1) \geq n$ implies $i + j \geq n + 1$, if $v_{i-1} v_{j-1}$ is an edge in H_{n-1} then $v_i v_j$ is an edge in H_n for $1 \leq i, j \leq n$. Therefore $v_2, v_3, \ldots, v_n$ produce a copy of H_{n-1} in H_n (see Figure 3.1).

The graph H_n is almost degree irregular: it has all degrees $1, \ldots, n-1$, and just two vertices are of the same degree $\lfloor \frac{n}{2} \rfloor$. By defining $w(v_i v_j) = 1$, for $1 \leq i < j \leq n$, $w(v_i) = 0$ for $i \leq \lfloor \frac{n}{2} \rfloor$, and $w(v_i) = 1$ for $i > \lfloor \frac{n}{2} \rfloor$ we obtain a $\{0, 1\}$-VDTW of H_n.

Proposition 3.19. *Let G be a graph of order n. If G contains a copy H_{n-1}, then it has a $\{0, 1\}$-VDTW.*

Proof. Weight all elements of the copy of H_{n-1} according to the previously described weighting; weight all other elements of G with 0. Let v_0 be the vertex not included by the copy of H_{n-1}. Clearly, $W(v_0) = 0$ and the vertices in the copy of H_{n-1} will have distinct positive weights, thus we have obtained a $\{0,1\}$-VDTW. $\square$

Proposition 3.20. *Let G be a graph of order n. If G has a $\{0,1\}$-VCTW then it contains a copy of H_{n-1}.*

Proof. We proceed by induction on n. The statement is clearly true for $n = 2$, since $H_2 \cong P_2$ and H_1 is a single vertex. Assume the statement is true for $n = k - 1$. Let G be a graph of order k and let w be a $\{0,1\}$-VDTW of G. Observe that by the pigeonhole principle, G must contain either a vertex v with $W(v) = 0$ or $W(v) = k$.

Suppose G has a vertex v such that $W(v) = 0$. Let $G' = G - v$. Observe that w', the restriction of the weighting w of G to G', is a $\{0,1\}$-VDTW of G'; otherwise w would not be a $\{0,1\}$-VDTW of G since all edges incident to v in G are weighted 0. Furthermore, there cannot exist a vertex in G' with color 0, otherwise G would have two vertices with color 0; so there must exist a vertex v' in G' such that $W'(v') = W(v') = k - 1$. Since $|V(G')| = k - 1$, by hypothesis G' contains a copy of H_{k-1}, and thus so does G.

Suppose G has a vertex v such that $W(v) = k$. Let $G' = G - v$. Observe that w', the restriction of the weighting w of G to G', is a $\{0,1\}$-VDTW of G'; otherwise w would not be a $\{0,1\}$-VDTW of G since v must be adjacent to all other vertices in G and each incident edge is weighted 1. Furthermore, there cannot exist a vertex in G' with color $k-1$, otherwise G would have two vertices with color k; therefore there exists a vertex v' in G' such that $W'(v') = W(v') - 1 = 0$. Since $|V(G')| = k - 1$, by hypothesis G' contains a copy of H_{k-2} that misses some vertex. By Proposition 3.18, since v is adjacent to every vertex in G', G contains a copy of H_k and thus a copy of H_{k-1}. $\square$

By Propositions 3.19 and 3.20, we conclude the following:

Theorem 3.21. *A graph G of order n has a $\{0,1\}$-VDTW if and only if G contains a copy of H_{n-1}.* $\square$

3.5 Conclusion

Here we have confirmed all bipartite graphs, complete multipartite graphs, and graphs with maximum degree 2 have an $\{a,b\}$-VCTW for any pair of distinct real values a and b; we proved the same for three-colorable graphs when $\frac{a}{b} \neq -\frac{1}{2}$. Furthermore, we have shown that every graph with maximum degree at most four has a $\{0,1\}$-VCTW. In light of these, I propose the following:

Conjecture 3.22. *Every graph has an $\{a,b\}$-VCTW for any distinct $a, b \in \mathbb{R}$.*

Conjecture 3.23. *Every graph has a $\{0,1\}$-VCTW.*

Conjecture 3.22 is a generalization of Conjecture 3.2, and thus is likely a bit more difficult to approach. The inclusion of the additive identity as a weight allows for some useful techniques, as demonstrated by the proofs in Sections 3.3 and 3.4; as such, I suspect that proving Conjecture 3.23 will be considerably easier than the original 1,2 conjecture.

Extending the problem to other commutative semigroups would likely yield several fruitful results. Observe that proving Conjecture 3.22 would also solve the related problem where weights are multiplied: a solution to the additive problem using weight set $\{\ln|a|, \ln|b|\}$ would provide a solution to the multiplicative problem with weight set $\{a,b\}$ for $0 < |a| < |b|$ while an additive solution with weight set $\{-1,1\}$ would provide a multiplicative weighting for the case $a = -b$.

4 Edge List Multi-Coloring of Graphs with Measurable Sets

A variation on traditional graph colorings are list colorings. These were first introduced in the late 1970's by Vizing [37] and independently by Erdős, Rubin, and Taylor [16]. In these problems, rather than having a communal collection of colors to distribute among the elements of the graph, each particular edge or vertex has a specified set from which its color can be chosen. The chosen colors must still satisfy the requirements of a proper coloring. A multi-coloring, as the name suggests, is one in which a set of colors are assigned rather than just a single one. Each element is assigned a demand: a natural number indicating the number of colors to be assigned. For edge multi-colorings, adjacent edges are required to have disjoint color sets.

Edge list multi-colorings of graphs combine both of these elements: edges are colored with subsets of colors from a pre-determined list of colors for each edge. This problem was investigated by Cropper, Gyárfás, and Lehel in [15] where they determined that the class of connected graphs for which a relatively simple necessary condition for such a coloring is also sufficient is exactly the trees. This idea was later extended by Hilton and Johnson in [22] to consider measurable sets of colors, rather than discrete ones.

4.1 Basics and Terminology

The problem of edge list multi-coloring is a generalization of the well-known problem of finding a system of distinct representatives: given a collection of sets, when is it possible to pick an element from each set so that every set has a distinct element? In graph theoretic terms, the problem can be stated as follows: in a bipartite graph with equal partite sets, when is it possible to find a perfect matching; that is, a spanning 1-regular subgraph. However, the problem is perhaps most well-known when stated in a more traditional manner: in a town with an equal number of men and women where each man is acquainted with a subset of the women and vice-versa, when is it possible for everyone to be paired up in marriage? A simple necessary condition is that each subset of men must be acquainted with at least as many women: surely if four men know only three women between them, one man will remain a bachelor. In fact, this obvious necessary condition is also sufficient for a well-married town. This result is known as Hall's marriage theorem [18].

Theorem 4.1. *Let $A_1, A_2, \ldots, A_n$ be a collection of discrete sets. There exist distinct $x_i \in A_i$ for every $1 \leq i \leq n$ if and only if every $I \subseteq [n]$ satisfies $|\bigcup_{i \in I} A_i| \geq |I|$.* $\square$

This classic result has been extended in many ways. The well-known proof of Halmos and Vaughan in [19] actually extends to consider the case where there are infinitely many sets and the case where each set is to have multiple representatives, rather than just one. Bollobás and Varopoulos in [11] examined the problem when infinitely many measurable sets are used.

To restructure this problem in terms of edge list colorings, consider each set to correspond to a list of colors available to some edge of a star $S_n = K_{1,n}$. The case where each set requires multiple representatives is exactly an edge list multi-coloring problem in this setting. The aim of Cropper, Gyárfás, and Lehel in [15] was to generalize this condition described in Theorem 4.1 for all graphs, not just stars, and classify those graphs for which this generalized Hall's condition was sufficient for the existence of a list multi-coloring.

Definition 4.2. Let $G = (V, E)$ be a graph, $L : E \to 2^{\mathbb{N}}$ be an assignment of color lists, and $w : E \to \mathbb{N}$ be an assignment of demands. For any subgraph $H \subseteq G$ and color $\gamma \in \mathbb{N}$, define the subgraph of H induced by all edges containing γ in its color list given by L to be the support of γ in H, denoted by $H_{L,\gamma}$. The maximum number of edges in a 1-regular subgraph of H is called the matching number and is denoted $\nu(H)$. We say that the ordered triple (G, L, w) satisfies the generalized Hall's condition if for every subgraph $H \subseteq G$ we have

$$\sum_{\gamma \in \mathbb{N}} \nu(H_{L,\gamma}) \geq \sum_{e \in E(H)} w(e).$$

If there exists a coloring $\phi : E \to 2^{\mathbb{N}}$ such that $\phi(e) \subseteq L(e)$, $|\phi(e)| = w(e)$ and $\phi(e) \cap \phi(f) = \emptyset$ for any pair of adjacent edges $e, f \in E$, then we say that ϕ is an edge list multi-coloring of (G, L, w).

As in the original problem, it is clear that satisfying this inequality is necessary for a list multi-coloring of G to exist: the right hand side denotes the total number of colors needed to satisfy the demands of H while the left hand side denotes the maximum number of colors that can be assigned, without regard for satisfying specific color demands, such that no two adjacent edges share a color. As we have just discussed, it was known that this condition is also sufficient for stars. In [15], the authors proved that the class of connected graphs for which satisfying generalized Hall's condition was also sufficient for a list multi-coloring to exist were the trees.

Motivated by this result, Hilton and Johnson examined in [22] the corresponding problem wherein the colors used are measurable sets.

Definition 4.3. Let $(X, \mathcal{A}, \mu)$ be a measure space. We say that $(X, \mathcal{A}, \mu)$ is non-atomic if for every $A \in \mathcal{A}$ with $\mu(A) > 0$, there exists a $B \subset A$ with $0 < \mu(B) < \mu(A)$. Given a non-atomic measure space $(X, \mathcal{A}, \mu)$ and a graph $G = (V, E)$, let $L : E \to \mathcal{A}$ be an assignment of color lists and $w : E \to \mathbb{R}^+$ be an assignment of demands. We say that (G, L, w) satisfies the measurable generalized Hall's condition if for every subgraph $H \subseteq G$ we have

$$\int_X \nu(H_{L,\gamma})\, d\mu(\gamma) \geq \sum_{e \in E(H)} w(e).$$

If there exists a coloring $\phi : E \to \mathcal{A}$ such that $\phi(e) \subseteq L(e)$, $\mu(\phi(e)) = w(e)$, and $\mu(\phi(e) \cap \phi(f)) = 0$ for any pair of adjacent edges $e, f \in E$, then we say that ϕ is an edge list multi-coloring of (G, L, w).

One important aspect of non-atomic measure spaces is that given any set A of positive measure $k > 0$, for every $0 \leq \ell < k$ there exists some subset $B \subset A$ such that $\mu(B) = \ell$. Since we are considering only finite graphs and $L : E \to \mathcal{A}$, it follows that $\nu(H_{L,(\cdot)}) : X \to \mathbb{N}$ is a measurable function. Note also that the condition that $\mu(\phi(e) \cap \phi(f)) = 0$ could be replaced by $\phi(e) \cap \phi(f) = \emptyset$ for finite graphs: the latter clearly implies the former, while to get from the former to the latter we would simply remove these sets of measure zero without affecting Hall's condition. The Hilton and Johnson in [22] actually stated the problem as a vertex coloring one; here I have transposed the terms and results into the more familiar problem of edge colorings. The authors confirmed that the class of connected graphs for which satisfying the measurable generalized Hall's condition was both necessary and sufficient for an edge list multi-coloring are the trees.

4.2 Hall's Theorem for Finitely Many Measurable Sets

The proof of the main result in [22] requires the use of certain methods of functional analysis. A somewhat simplified proof of the same result was presented by Hladký, Kráal, Serenit, and Stiebitz in [24], although it still relied on analytical methods. Since the problem is still mostly combinatorial in nature and especially since only finite graphs are considered, it seemed that a more elementary proof of the result avoiding limit arguments must be possible.

In pursuit of this, we considered proofs of the problem for stars; that is, the finite case of the result of Bollobás and Varopoulos [11]. Both [22] and [24] used this result. Although certainly true, we were somewhat dissatisfied with the existing proofs that we found: it seemed as though the correct proofs required too much machinery for our purposes while the more elementary proofs were either incomplete or incorrect. Interestingly enough, this result has applications in mathematical economics; in the book [21] about a topic in this area, Hildenbrand gave a seemingly incomplete proof of a somewhat weaker result. Here we give a full new proof of Hall's theorem for measurable sets in the language of edge list multi-colorings.

Lemma 4.4. *Suppose S_ℓ is a star and $(X, \mathcal{A}, \mu)$ is a non-atomic measure space. Let $L : E(S_\ell) \to \mathcal{A}$ be an assignment of color lists and $w : E(S_\ell) \to \mathbb{R}^+$ an assignment of demands. If (S_ℓ, L, w) satisfies the measurable generalized Hall's condition, then it has an edge list multi-coloring.*

Proof. We proceed by induction on ℓ. For $\ell = 1$, the result is trivial. Suppose the result holds for all $\ell < m$. Consider some (S_m, L, w) satisfying Hall's condition.

For each subgraph $H \subseteq S_m$, define $\delta_H := \int_X \nu(H_{L,\gamma})\, d\mu - \sum_{e \in E(H)} w(e)$; since (S_m, L, w) satisfies Hall's condition, $\delta_H \geq 0$ for every $H \subset S_m$. Let $J \subsetneq S_m$ be such that δ_J is minimum among all proper subgraphs. Denote by I the subgraph $S_m \setminus E(J)$ with the isolated vertices removed; thus I and J are both stars with fewer than m edges. Let $R = \left(\bigcup_{e \in E(J)} L(e)\right) \cap \left(\bigcup_{e \in E(I)} L(e)\right)$. Choose $D \subseteq R$ such that $\mu(D) = \min\{\mu(R), \delta_J\}$.

Define $L' : E(S_m) \to \mathcal{A}$ as follows: for $e \in E(J)$, $L'(e) = L(e) \setminus D$; for $e \in E(I)$, $L'(e) = L(e) \setminus \bigcup_{f \in E(J)} L'(f)$. Every edge in J has a color list in L' disjoint from the color list of each edge in I, thus it suffices to show Hall's condition holds for (J, L', w) and (I, L', w).

For every subgraph $H \subseteq J$, observe

$$\begin{aligned} \nu(H_{L',\gamma}) &\geq \nu(H_{L\gamma}) - 1 \text{ for } \gamma \in D \\ \nu(H_{L',\gamma}) &= \nu(H_{L,\gamma}) \text{ for } \gamma \notin D \ . \end{aligned}$$

It follows that,

$$\begin{aligned} \int_X \nu(H_{L',\gamma})\ d\mu &\geq \int_X \nu(H_{L,\gamma})\ d\mu - \mu(D) \\ &\geq \sum_{e\in E(H)} w(e) + \delta_H - \delta_J \\ &\geq \sum_{e\in E(H)} w(e). \end{aligned}$$

Thus (J, L', w) satisfies Hall's condition.

Consider $H \subseteq I$; define $H^* = H \cup J$. If $\mu(D) = \mu(R)$, then $\mu(L(e) \setminus L'(e)) = 0$ for all $e \in E(H)$, and so Hall's condition is still satisfied. Suppose $\mu(D) = \delta_J$, and so $\int_X \nu(J_{L',\gamma})\ d\mu = \sum_{e\in E(J)} w(e)$. Define $D_H = D \setminus \bigcup_{e\in E(H)} L'(e)$; these are colors originally present in $\bigcup_{e\in E(J)} L(e)$ that were removed in the definition of L' and are not present in $\bigcup_{e\in E(H)} L'(e)$. Equivalently, $D_H = \left(\bigcup_{e\in E(H^*)} L(e)\right) \setminus \left(\bigcup_{e\in E(H^*)} L'(e)\right)$. Recall too that $\left(\bigcup_{e\in E(J)} L'(e)\right) \cap \left(\bigcup_{e\in E(H)} L'(e)\right) = \emptyset$. Observe

$$\begin{aligned} \nu(H_{L',\gamma}) + \nu(J_{L',\gamma}) &= \nu(H^*_{L',\gamma}) \text{ for } \gamma \in X \\ \nu(H^*_{L',\gamma}) &= \nu(H^*_{L,\gamma}) - 1 \text{ for } \gamma \in D_H \\ \nu(H^*_{L',\gamma}) &= \nu(H^*_{L,\gamma}) \text{ for } \gamma \notin D_H \ . \end{aligned}$$

Note that if $H^* = S_\ell$, then $\mu(D_H) = 0$; otherwise $\delta_{H^*} - \mu(D_H) \geq \delta_{H^*} - \delta_J \geq 0$ by the minimality of δ_J. It follows then that

$$\begin{aligned}
\int_X \nu(H_{L',\gamma})\ d\mu + \int_X \nu(J_{L',\gamma})\ d\mu &= \int_X \nu(H^*_{L',\gamma})\ d\mu \\
&= \int_X \nu(H_{L,\gamma^*})\ d\mu - \mu(D_H) \\
&= \sum_{e \in E(H^*)} w(e) + \delta_{H^*} - \mu(D_H) \\
&\geq \sum_{e \in E(H)} w(e) + \sum_{e \in E(J)} w(e).
\end{aligned}$$

and so $\int_X \nu(H_{L',\gamma})\ d\mu \geq \sum_{e \in E(H)} w(e)$. Thus (I, L', w) satisfies Hall's condition and the lemma holds. $\square$

Bibliography

[1] Martin Aigner, Eberhard Triesch, and Zsolt Tuza. Irregular assignments and vertex-distinguishing edge-colorings of graphs. In *Combinatorics '90*, volume 52 of *Annals of Discrete Mathematics*, pages 1–9. North-Holland Amsterdam, 1992.

[2] Kenneth Appel and Wolfgang Haken. Every planar map is four colorable. Part I: Discharging. *Illinois Journal of Mathematics*, 21(3):429–490, 1977.

[3] Kenneth Appel, Wolfgang Haken, and John Koch. Every planar map is four colorable. Part II: Reducibility. *Illinois Journal of Mathematics*, 21(3):491–567, 1977.

[4] Paul N. Balister. Vertex-distinguishing edge colorings of random graphs. *Random Structures and Algorithms*, 20(1):89–97, 2002.

[5] Paul N. Balister, Béla Bollobás, and Richard H. Schelp. Vertex distinguishing colorings of graphs with $\Delta(G) = 2$. *Discrete Mathematics*, 252(2):17–29, 2002.

[6] Paul N. Balister, Ervin Győri, Jenő Lehel, and Richard H. Schelp. Adjacent vertex distinguishing edge-colorings. *SIAM Journal on Discrete Mathematics*, 21(1):237–250, 2007.

[7] Paul N. Balister, Alexandr Kostochka, Hao Li, and Richard H. Schelp. Balanced edge colorings. *Journal of Combinatorial Theory, Series B*, 90(1):3–20, 2004.

[8] Cristina Bazgan, Amel Harkat-Benhamdine, Hao Li, and Mariusz Woźniak. On the vertex-distinguishing proper edge-coloring of graphs. *Journal of Combinatorial Theory, Series B*, 75(2):288–301, 1999.

[9] Mehdi Behzad. *Graphs and their chromatic numbers.* Ph.D. thesis, Michigan State University, 1965.

[10] Béla Bollobás. *Modern Graph Theory*, volume 184 of *Graduate Texts in Mathematics.* Springer, New York, New York, 1998.

[11] Béla Bollobás and Nicolas Th. Varopoulos. Representation of systems of measurable sets. *Mathematical Proceedings of the Cambridge Philosophical Society*, 78:323–325, 1975.

[12] Roland Leonard Brooks. On colouring the nodes of a network. *Proceedings of the Cambridge Philosophical Society*, 37:194–197, 1941.

[13] Anita Burris and Richard H. Schelp. Vertex-distinguishing proper edge-colorings. *Journal of Graph Theory*, 26(2):73–82, 1997.

[14] Xiang'en Chen. On the adjacent vertex distinguishing total coloring numbers of graphs with $\delta = 3$. *Discrete Mathematics*, 308:4003–4007, 2008.

[15] Matthew Cropper, András Gyárfás, and Jenő Lehel. Edge list multicoloring trees: an extension of Hall's theorem. *Journal of Graph Theory*, 42:246–255, 2003.

[16] Paul Erdős, Arthur L. Rubin, and Herbert Taylor. Choosability in graphs. In *Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing, Arcata*, volume 26 of *Congressus Numerantium*, pages 125–157, 1979.

[17] Odile Favaron, Hao Li, and Richard H. Schelp. Strong edge colorings of graphs. *Discrete Mathematics*, 159(1):103–109, November 1996.

[18] Philip Hall. On representatives of subsets. *Journal of the London Mathematical Society*, 10:26–30, 1935.

[19] Paul R. Halmos and Herbert E. Vaughan. The marriage problem. *American Journal of Mathematics*, 72(1):214–215, 1950.

[20] Hamed Hatami. Delta + 300 is a bound on the adjacent vertex distinguishing edge chromatic number. *Journal of Combinatorial Theory, Series B*, 95(2):246–256, November 2005.

[21] Werner Hildenbrand. *Core and Equilibria of a Large Economy.* Princeton University Press, Princeton, New Jersey, 1974. Lemma 4, page 74.

[22] Anthony J.W. Hilton and Peter D. Johnson, Jr. List multicolorings of graphs with measurable sets. *Journal of Graph Theory*, 54:179–193, 2007.

[23] Hugh Hind, Michael Molloy, and Bruce Reed. Total coloring with Δ + poly($log\Delta$) colors. *SIAM Journal on Computing*, 28(3):816–821, 1999.

[24] Jan Hladký, Daniel Kráal, Jean-Sébastien Serenit, and Michael Stiebitz. List colorings with measurable sets. *Journal of Graph Theory*, 59(3):229–238, 2008.

[25] Jonathan Hulgan. Concise proofs for adjacent vertex-distinguishing total colorings. *Discrete Mathematics*, 309(8):2548–2550, 2009.

[26] Jonathan Hulgan, Jenő Lehel, Kenta Ozeki, and Kiyoshi Yoshimoto. Vertex coloring of graphs by total 2-weightings. Submitted, 2009.

[27] Tommy R. Jensen and Bjarne Toft. *Graph Coloring Problems.* Wiley-Interscience, New York, New York, 1995.

[28] Maciej Kalkowski, Michał Karoński, and Florian Pfender. Vertex-coloring edge-weightings: towards the 1-2-3-conjecture. *Journal of Combinatorial Theory, Series B*, 2009. Article in Press, Corrected Proof.

[29] Michał Karoński, Tomasz Łuczak, and Andrew Thomason. Edge weights and vertex colours. *Journal of Combinatorial Theory, Series B*, 91:151–157, 2004.

[30] Dénes Kőnig. Über graphen und ihre anwendung auf determinantentheorie und mengenlehre. *Mathematische Annalen*, 77:453–465, 1916.

[31] Michael Molloy and Bruce Reed. A bound on the total chromatic number. *Combinatorica*, 18(2):241–280, 1998.

[32] Jakub Przybyło and Mariusz Woźniak. 1,2 conjecture. Preprint MD 024, 2007.

[33] Jakub Przybyło and Mariusz Woźniak. 1,2 conjecture, ii. Preprint MD 026, 2007.

[34] Ján Černý, Mirko Horňák, and Roman Soták. Observability of a graph. *Mathematica Slovaca*, 46(1):21–31, 1996.

[35] Vadim G. Vizing. On an estimate of the chromatic class of a p-graph (in Russian). *Metody Diskretnogo Analiza*, 3:25–30, 1964.

[36] Vadim G. Vizing. The chromatic class of a multigraph. *Cybernetics and Systems Analysis*, 1(3):32–41, 1965.

[37] Vadim G. Vizing. Vertex colorings with given colors (in Russian). *Metody Diskretnogo Analiza*, 29:3–10, 1976.

[38] Haiying Wang. On the adjacent vertex-distinguishing total chromatic numbers of the graphs with $\Delta(G) = 3$. *Journal of Combinatorial Optimization*, 14:87–109, 2007.

[39] Zhongfu Zhang, Xiang'en Chen, Jingwen Li, Bing Yao, Xinzhong Lu, and Jianfang Wang. On adjacent-vertex-distinguishing total coloring of graphs. *Science in China Series A Mathematics*, 48:289–299, 2005.

[40] Zhongfu Zhang, Linzhong Liu, and Jianfang Wang. Adjacent strong edge coloring of graphs. *Applied Mathematics Letters*, 15(5):623–626, 2002.