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Irregularity Strength of Dense Graphs

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Irregularity strength of dense graphs

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Abstract

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It is proved that if t is a fixed positive integer and n is sufficiently large, then each graph of order n with minimum degree $n-t$ has an assignment of weights 1, 2 or 3 to the edges in such a way that weighted degrees of the vertices become distinct.

1. Introduction

Let G be a simple graph and assign positive integer weights to the edges in such a way that the (weighted) degrees become distinct. Let $s(G)$ denote the minimum of the largest weight over all such irregular assignments of G. The problem of studying $s(G)$, called the irregularity strength of G, was proposed by Chartrand et al. in [2].

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Determining the strength proved to be rather hard, even for very simple graphs. Several open questions and quite a few answered cases could be cited from [2, 6–9]. We know, among others, that $s(K_n)=s(K_{2k,2k})=3$ and $s(K_{2k+1,2k+1})=4$ for every $n \geq 3$ and $k \geq 1$, where K_n and $K_{n,n}$ denotes the complete and the complete bipartite graph of order n and $2n$, respectively, (see [6, 8]).

In [11], a general upper bound is proved: $s(G) \leq n-1$ for every connected graph G of order $n \geq 4$. We do not know whether this bound is valid or not for disconnected graphs. For the irregularity strength of regular graphs $n-1$ is an upper bound, however, not the sharpest one (cf. [7]). In this paper we consider the problem of improving this general upper bound for graphs 'close' to cliques and regular graphs, such as the disjoint union of cliques and graphs of order n with minimum degree constant times n.

If weights are assigned to the edges by some assignment $w: E(G) \to \mathbb{Z}^+$, then the weighted degree of a vertex $x \in V(G)$ called also its degree is denoted by $d_w(x)$ or $d(x)$. A graph G with edge weights is called a *network* and denoted by $G(w)$, or simply G. The *strength* of a network is $s(G(w)) = \max\{w(e): e \in E(G)\}$. Note that in [1] a network of strength s is also called an s-graph.

A network $G(w)$ defined on G is *irregular* iff $d_w(x)=d_w(y)$ implies $x=y$. The *irregularity strength* of a graph G is $s(G)=\min\{s(G(w)): G(w) \text{ is irregular}\}$.

Let G be a graph of order n and α be a real with $0<\alpha<1$. The main question we are considering here is the following.

Question. Does there exist a constant c_α such that $s(G) \leq c_\alpha$ for every graph G with minimum degree $\delta(G) \geq (1-\alpha)n$?

We strongly believe that the answer is affirmative, however, at this time there does not exist much evidence to support this conjecture. This question motivated the results we present here.

In [7], $s(G) \leq n/2+9$ is proved for r-regular graphs of order n. Here we improve this result for $r=(1-\alpha)n$ and $0<\alpha<1$. The bound we are able to prove here is much smaller than $n-1$, however, it still depends on n (Theorem 10):

$$s(G) \leq (1/(\lfloor 1/\alpha \rfloor - 1))n + 5 \textit{ holds for sufficiently large } n.$$

($\lfloor x \rfloor$ is the floor function, the largest integer not larger than x; $\lceil x \rceil$ denotes the ceiling function, the smallest integer not smaller than x).

Note that the lower bound $(n+r-1)/r$ given in [2] is conjectured to be the right value of the strength of r-regular graphs of order n, up to an additive constant.

In Section 4, we investigate graphs of irregularity strength at most 3. It is proved that $s(G) \leq 3$ if $\delta(G) \geq n-2$, i.e., if $G=K_n-F$ where F is a matching and $n \geq 3$ (Theorem 7).

A result pertaining to less special graphs is: If t is fixed and n is sufficiently large, then the irregularity strength of every graph of order n and minimum degree $n-t$ is at most 3 (Corollary 9).

In some sense Corollary 9 gives a sharp result on the minimum degree of a graph of irregularity strength 3 or less. In fact, if G is an r-regular graph of order n, and $r=n-t$ ($t \geq 2$ is fixed), then obviously

$$s(G) \geq (n+r-1)/r = (2n-t-1)/(n-t) > 2.$$

Thus the strength of infinitely many graphs of minimum degree $n-t$ is equal to 3.

If we let t depend on n, e.g., if $t=t(n) < \sqrt{n/18}$, the irregularity strength of a graph with minimum degree $n-t$ is still at most 3 or less (see Theorem 3). In this respect Theorem 8, is not sharp at all, and one can try to expand further the restriction imposed on $t(n)$.

In Section 2, we develop a weighting technique for complete graphs. In Section 3, we show a method using the results of Section 2 which gives an upper bound on the irregularity strength of disjoint unions of complete graphs. In particular, if $G=tK_p$ ($1 \leq t$, $3 \leq p$) and $n=tp$, $r=p-1$, then $s(G) \leq \lceil n/r \rceil + 2$ follows from Proposition 5. Note that this bound is sharp up to an additive constant.

Results in Section 2 also show a remarkable relationship between irregularity and graphical degree sequences. One of them, a special case of Proposition 3, may have some interest in its own right.

If $D=(d, d-1, \ldots, d-n+1)$ is a sequence of n consecutive integers with even sum, then there exists a complete network of strength $\lceil d/(n-1) \rceil$ with degree sequence D, for each $n \geq 5$ and $d > 2(n-1)$. (Note that $\lceil d/(n-1) \rceil$ is obviously the smallest possible strength a network can have with degree sequence D.)

In Section 5, we conclude the paper with some questions inspiring further research on the irregularity strength of dense graphs.

2. Irregular complete networks

There are several irregular assignments of the complete graph K_n, and each obviously is of strength at least 3. One is the *alternating scheme* defined in [2] as follows. For convenience we denote by $w(i, j)$ the weight assigned to edge ij.

Let $V(K_n) = (1, 2, \ldots, n)$, and assign weights $w(1, 2) = 1$, $w(2, 3) = 2$, $w(3, 1) = 3$; furthermore, let

$$w(j, i) = \begin{cases} 1 & \text{if } j \text{ even}, \\ 3 & \text{if } j \text{ odd}, \end{cases}$$

for every $j = 4, 5, \ldots, n$ and $1 \leq i < j$. The result is an irreguiar complete network of strength 3 for every $n \geq 3$. Moreover, the degree sequence of the network, has the following form for large n:

$$D=(a, a+2, \ldots, b-2, b, b+1, b+2, b+3, b+6, b+8, \ldots, c-2, c).$$

(It has four consecutive integers in the middle and all the other neighboring degrees in the sequence differ by at least 2, for $n \geqslant 4$.) This property of the alternating scheme is formulated in the following proposition.

Proposition 1. *The alternating scheme gives an irregular complete network of strength 3 with degree sequence* $d_1=a$, $d_2=a+2, \ldots, d_k=b$, $d_{k+1}=b+1$, $d_{k+2}=b+2$, $d_{k+3}=b+3, \ldots, d_{n-1}=c-2$, $d_n=c$, *where* $k=\lfloor n/2\rfloor-1$, *and* $a=n+1$, $b=2n-4$, $c=3n-3$ *if n is odd*, $a=n-1$, $b=2n-5$, $c=3n-5$ *if n is even.*

In this section we study irregular complete networks with degree sequence consisting of consecutive or almost consecutive integers.

Let $a \geqslant 0$, $a+1 \leqslant b \leqslant a+2$ and $b+n-2 \leqslant c \leqslant b+n-1$. If $D=(a, b, b+1, \ldots, b+n-3, c)$ is the degree sequence of some (not necessarily complete) irregular network G of order n, then obviously, $s(G) \geqslant \lceil c/(n-1)\rceil$. Our aim is to show that there exist complete irregular networks having this minimum strength, for every D, of course with $a \geqslant n-1$, except the case $D=(0, 1, 2, \ldots, n-1)$. The results here will be used in subsequent sections and their proofs are based on the following technical lemma.

Lemma 2. *For $n \geqslant 5$ let $D=(a, b, b+1, \ldots, b+n-3, c)$ be a sequence of integers with $0 \leqslant a \leqslant n-2$, $a+1 \leqslant b \leqslant a+2$, $b+n-2 \leqslant c \leqslant b+n-1$ and with even sum. Then there exists a network with degree sequence D and with strength*

$$s=\begin{cases}\lceil c/(n-1)\rceil & \text{if } c \neq n-1,\\ 2 & \text{if } c=n-1.\end{cases}$$

Note that $s=2$ unless $c=2n-1$, in which case $s=3$.

Proof. We will show that if $D=(n-2, n, \ldots, 2n-3, 2n-1)$, then there exists a network of strength 3 with degree sequence D; otherwise, there exists one of strength 2.

For $n=5$ the lemma is true, see Fig. 1.

Let $n>5$ and assume that the lemma is true for every $n'<n$. If $a=0$ or $c \geqslant 2n-2$, we give constructions by induction.

Case a: $a=0$.

To obtain an irregular network with degree sequence $D=(0, b, \ldots, c)$ (with $1 \leqslant b \leqslant 2$), add a new isolated vertex to the irregular network of strength two with degree sequence $D'=(b, \ldots, c)$.

Case b: $c=2n-2$.

In this case $a \geqslant c-(n-1)-2=n-3 \geqslant 2$, thus every member of D is at least two. Start with a network of order $n-1$ with degrees two less than the elements

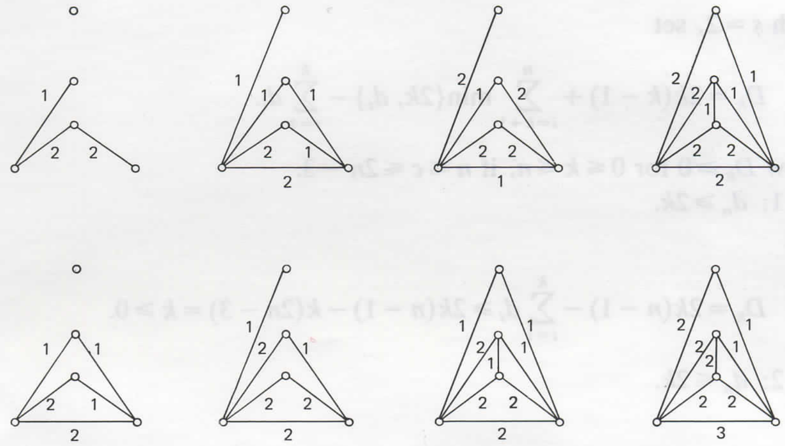

Fig. 1. Irregular networks of order five.

of $D' = D \setminus \{c\}$. (Note that D' satisfies the conditions of the lemma.) Then, add a new vertex joined to each vertex with an edge of weight 2.

Case c: $c = 2n - 1$.

From the condition in the lemma, $a \leqslant n - 2$. On the other hand $a \geqslant c - (n-1) - 2 = n - 2$ with equality if and only if $D = (n-2, n, \ldots, 2n-3, 2n-1)$. Therefore, $a = n - 2$ and $b = n$. Let G_1 be a network of strength two on $n - 2$ vertices with degree sequence $D' = (n-5, n-4, \ldots, 2n-9, 2n-7)$. (Note that D' has even sum and also satisfies further conditions of the lemma.) Add a new vertex x_1 to G_1 with $n - 2$ edges of weights $1, 2, 2, \ldots, 2, 1$ (1 is assigned to the edges going to vertices of G_1 with the smallest and largest degree). Then, x_1 has degree $2n - 6$ and the degree sequence of G_2 that we obtain is $(n-4, n-2, \ldots, 2n-7, 2n-6, 2n-6)$. Now add a vertex x_2 with $n - 1$ edges to G_2 and assign weights $2, 2, \ldots, 2, 3$ (3 is assigned to the edge going to one of the vertices of largest degree). Then x_2 has degree $2n - 1$, the network we obtain has strength 3 and degree sequence $(n-2, n, \ldots, 2n-4, 2n-3, 2n-1)$.

Assume now that $a \geqslant 1$ and $c \leqslant 2n - 3$. To prove the lemma we use a variant of the Erdős–Gallai inequality [4], adapted for multigraphs in [3] as follows (see also in [1]).

The sequence $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n$ with even sum is the degree sequence of a network of strength s if and only if

$$\sum_{i=1}^{k} d_i \leqslant sk(k-1) + \sum_{i=k+1}^{n} \min\{sk, d_i\} \quad \text{for all } k,\ 1 \leqslant k \leqslant n.$$

Our sequence in non-increasing order is $c, b+n-3, b+n-4, \ldots, b, a$, so $d_1 = c$, $d_n = a$. Setting $c' = b + n - 2$ and $a' = b - 1$, if $1 < i < n$, then $d_i = c' - (i-1)$. Set $\delta = (c - c') + (a' - a)$ and note $0 \leqslant \delta \leqslant 2$. Finally, for $0 \leqslant k \leqslant n$, and with $s=2$, set

$$D_k = 2k(k-1) + \sum_{i=k+1}^{n} \min\{2k, d_i\} - \sum_{i=1}^{k} d_i.$$

We show $D_k \geqslant 0$ for $0 \leqslant k \leqslant n$, if $n \leqslant c \leqslant 2n-3$.

Case 1: $d_n \geqslant 2k$.

Then

$$D_k = 2k(n-1) - \sum_{i=1}^{k} d_i \geqslant 2k(n-1) - k(2n-3) = k \geqslant 0.$$

Case 2: $d_n \leqslant 2k$.

Then

$$\begin{aligned} D_k &= 2k(k-1) + \sum_{i=k+1}^{n} d_i - \sum_{i=1}^{k} d_i \\ &= 2k(k-1) + \sum_{i=k+1}^{n} (c' - (i-1)) - \sum_{i=1}^{k} (c' - (i-1)) - \delta \\ &= 6\binom{k}{2} - \binom{n}{2} + (n-2k)c' - \delta. \end{aligned}$$

Set $e' = 3(n-1) - c'$ and $k' = n-1$, we obtain

$$\begin{aligned} D_k &= 6\binom{n-k}{2} - \binom{n}{2} + (2k-n)e' - \delta \\ &= 6\binom{k'}{2} - \binom{n}{2} + (n-2k')e' - \delta. \end{aligned}$$

Since $n \leqslant c' \leqslant 2n-3$, we also have $n \leqslant e' \leqslant 2n-3$. Also, if $k > n/2$, then $k' < n/2$. Therefore, we may assume that $k \leqslant n/2$. Since $c' \geqslant n$,

$$D_k \geqslant 6\binom{k}{2} - \binom{n}{2} + (n-2k)n - \delta = \binom{n-2k+1}{2} + k(k-2) - \delta.$$

If $k > 2$, then $D_k > 0$ follows. If $k \leqslant 2$, then

$$D_k \geqslant \binom{n-3}{2} - 1 - \delta.$$

Having already proved the lemma for $n=5$, we may assume that $n \geqslant 6$, so again $D_k \geqslant 0$ follows.

Case 3: for some x, $k < x \leqslant n$, $d_x = 2k$ (or $a' = 2k$ and $a = 2k-1$).

Note that $d_x = 2k = c' - (x-1)$ and

$$D_k = 2k(x-1) + \sum_{i=x+1}^{n} d_i + \sum_{i=1}^{k} d_i.$$

Case 3.1: $k+x \leqslant n$. Then

$$\begin{aligned} D_k &\geqslant 2k(x-1) + \sum_{i=x+1}^{x+k} d_i - \sum_{i=1}^{k} d_i \\ &\geqslant 2k(x-1) - kx - \delta = k(x-2) - \delta. \end{aligned}$$

Now $x = c' - 2k + 1 \geqslant n - 2k + 1$. If $k = 1$, then $x \geqslant n - 1 \geqslant 5$, so $D_k > 0$. If $k \geqslant 2$, then $x \geqslant 3$, so $D_k \geqslant 0$.

Case 3.2: $k + x > n$ (note, $k > 0$). For each i, $1 < i < n$, $d_i = c' - (i-1) = 2k + x - i$ (d_i and d_n differs from this by at most 1). Thus

$$\begin{aligned} D_k &= 2k(x-1) + \sum_{i=x+1}^{n} (2k+x-i) - \sum_{i=1}^{k} (2k+x-i) - \delta \\ &= k(2n-2k-x-2) + \binom{k+1}{2} - \binom{n-x+1}{2} - \delta. \end{aligned}$$

Now $c' = 2k + x - 1 \leqslant 2n - 3$, so $2n - 2k - x - 2 \geqslant 0$, and $n - x + 1 \leqslant k$, hence $D_k \geqslant k(2n - 2k - x - 2) + k - \delta$. If $2n - 2k - x - 2 \geqslant 1$, then $D_k \geqslant 0$. If $2n - 2k - x - 2 = 0$, then $c' = 2n - 3 = c$, so $\delta \leqslant 1$ and $D_k \geqslant k - \delta \geqslant 0$. □

Note that Lemma 2 is sharp in the sense that the strength given by the formula for s is the smallest possible for each degree sequence D. The following corollary of Lemma 2, which concerns irregular complete networks with consecutive or almost consecutive degree sequences, is formulated for determining the irregularity strength of tK_p.

Proposition 3. *Let* $D = (d_1, d_2, \ldots, d_p)$ *be a sequence of* $p \geqslant 5$ *positive integers with even sum. If* $d_1 \geqslant p - 1$, $1 \leqslant d_2 - d_1 \leqslant 2$, $1 \leqslant d_p - d_{p-1} \leqslant 2$, *and* $d_{j+1} - d_j = 1$ *for* $j = 2, \ldots, p-2$, *then there exists a complete network of strength at most* $\lceil d_p/(p-1) \rceil + 1$ *with degree sequence* D. *Moreover, if* $p - 1$ *does not divide* d_1, *then* $s = \lceil d_p/(p-1) \rceil$.

Proof. Put $q = \lfloor d_1/(p-1) \rfloor$, and apply Lemma 2 with $n = p$, $a = d_1 - q(p-1)$, $b = d_2 - q(p-1)$ and $c = d_p - q(p-1)$. Consider non-edges of the network obtained as edges of zero weight and increase the weight of all edges by q. Note that $q \geqslant 1$ and each degree increases by $q(p-1)$. Thus we obviously get a complete network with degree sequence D and with strength $\lceil c/(p-1) \rceil + 1 + q = \lceil d_p/(p-1) \rceil + 1$ or less. Furthermore, if $p - 1$ does not divide d_1, then the sequence differs from $(0, 1, \ldots, p-1)$, therefore $c \neq n - 1$ and $s = \lceil d_p/(p-1) \rceil$ follows from Lemma 2. □

3. The irregularity strength of the union of cliques

The consecutive scheme developed in Section 2 allows us to give an irregular assignment for graphs that are the disjoint unions of cliques of order at least 3.

Suppose that $G = K_{p_1} \cup K_{p_2} \cup \cdots \cup K_{p_t}$, where $t \geq 2$, $5 \leq p_1 \leq p_2 \leq \cdots \leq p_t$ and denote $G_k = K_{p_1} \cup \cdots \cup K_{p_k}$ for $1 \leq k \leq t$ ($G = G_t$). (Note that for the case when there are 3- and 4-cliques among the components one can use similar ideas.) Assume that G_k has an irregular weighting with consecutive or almost consecutive degree sequence $(d, d+1, \ldots, e_{k-1}, e_k, e_k')$ with $e_k + 1 \leq e_k' \leq e_k + 2$.

Consider the clique component $K_{p_{k+1}}$ which is to be added to G_k in order to obtain G_{k+1}. If the minimum degree of $G - G_k$, that is $p_{k+1} - 1$ is larger than the largest weighted degree of G_k, then start with a consecutive weighting for $K_{p_{k+1}}$. Otherwise assign weights to the edges of $K_{p_{k+1}}$ in such a way that the degrees extend the degree sequence of G_k with consecutive integers. Since $p_{k+1} \geq 5$, one can apply Proposition 3 with initial values $d_2 = e_k + 3$ and $d_1 = d_2 - (e_k' - e_k)$; if the last two degrees are e_{k+1} and e_{k+1}', then the largest weight is at most $\lceil e_{k+1}'/(p_{k+1} - 1) \rceil + 1$.

Now we consider some special cases. Denote by tK_p the disjoint union of t cliques of order p. The counting argument given in [2] gives that $s(tK_p) \geq \lceil (pt + p - 2)/(p - 1) \rceil$.

The irregularity strength for $p = 3$ was determined in [6]:

$$s(tK_3) = \begin{cases} \lceil (3t+1)/2 \rceil + 2 & \text{for } t \equiv 3 \pmod 4, \\ \lceil (3t+1)/2 \rceil + 1 & \text{otherwise.} \end{cases}$$

Proposition 4. *$s(tK_4) = \lceil (4t + 2)/3 \rceil$ for $t \geq 2$.*

Proof. We give a consecutive scheme for every t based on the irregular weighting of the tK_4 with $t \leq 4$. Denote by v_k, x_k, y_k, z_k the vertices of the kth 4-clique, $k = 0$, 1, 2 and 3, and assign weights to the edges as is shown in Fig. 2.

If $i = 3j + k$ with $1 \leq k \leq 3$, $j \geq 0$, then define the edge weights of the ith component as follows: $w(v_i, x_i) = 4j + w(v_k, x_k), \ldots, w(y_i, z_i) = 4j + w(y_k, z_k)$. Defining the weights in this way for every i, $0 \leq i \leq t - 1$, the network has the desired strength. Since $d(v_i) = 4j + 3$, $d(x_i) = 4j + 4$, $d(y_i) = 4j + 5$ and $d(z_i) = 4j + 6$, the obtained network on tK_4 is irregular. $\square$

Proposition 5. *If $p \geq 5$ then $s(tK_p) \leq \lceil (pt + p - 1)/(p - 1) \rceil$.*

Proof. We use the consecutive scheme to define the edge weights of the subsequent cliques. The smallest degree is $p - 1$, so the largest one (in the last

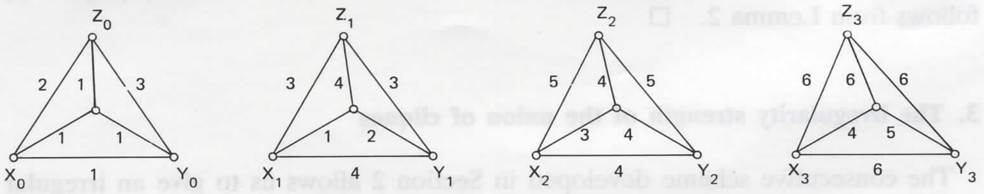

Fig. 2. Consecutive scheme for tK_4, $t \leq 4$.

component) satisfies $d_n \leqslant n + p - 1$. By Proposition 3, $s(G) \leqslant \lceil d_n/(p-1) \rceil + 1$ and the theorem follows. □

Remark. The scheme similar to that of Proposition 4 gives $s(tK_5) = \lceil (5t+3)/4 \rceil$ for $t \geqslant 2$. We conjecture that for all $t \geqslant 2$ and $p > 3$, $s(tK_p) = \lceil (pt + p - 2)/(p-1) \rceil$.

A technical result used in the next section is when each clique is either a K_p or a K_{p+1} with $p \geqslant 3$. You will observe that we slightly modify the consecutive scheme allowing one jump in the middle of the degree sequence.

Proposition 6. *Let G be the disjoint union of cliques of order p and $p+1$, $p \geqslant 3$; and consider the partition $V(G) = P \cup Q$, where $P = \{x: d(x) = p\}$ and $Q = \{y: d(y) = p-1\}$. If G is of order n, then there is an irregular assignment w with weights at most $\lceil n/(p-1) \rceil + 4$ such that $d_w(x) < d_w(y)$ holds for every $x \in P$ and $y \in Q$.*

Proof. Let $|P| = n_1$, $|Q| = n_2$, $(n_1 + n_2 = n)$; furthermore, let $G = G(P) \cup G(Q)$. Define weights on $G(P)$ by using the consecutive scheme to obtain a network with degree sequence $(p, p+1, \ldots, p+n_1-1, p+n_1')$, where $n_1' = n_1$ or $n_1 + 1$. Then define an irregular network on $G(Q)$ with consecutive or almost consecutive degree sequence $(p + n_1' + 1, \ldots, p + n_1' + n_2')$, where $n_2' = n_2$ or $n_2 + 1$. This assignment w satisfies the proposition. Moreover, according to Propositions 3 and 4, the largest weight (coming from the last clique of $G(P)$ or that of $G(Q)$ is at most

$$\max\{\lceil p + n_1 + 2)/p \rceil + 1, \lceil (p + n + 3)/(p-1) \rceil + 1\} \leqslant \lceil n/(p-1) \rceil + 4. \quad \square$$

4. The strength of dense graphs

Now we go back to our main question: does there exist a constant c_α such that $s(G) \leqslant c_\alpha$ for every graph G with minimum degree $\delta(G) \geqslant (1-\alpha) - n$?

Although we believe that the answer is affirmative, we have been able to prove only weaker variants of the original question.

In [5], necessary conditions are established for graphs to have irregularity strength 2. Now we give sufficient conditions on a graph G that imply $s(G) \leqslant 3$. The first result concerns the case $\delta(G) \geqslant n-2$. A somewhat simpler proof is in [9].

Theorem 7. *If G is a graph of order n $(n \geqslant 3)$ with $\delta(G) \geqslant n-2$, then $s(G) \leqslant 3$.*

Proof. Let $G = K_n - F$, where $F = \{y_i z_i: i = 1, \ldots, r\}$, and let $X = \{x_i: i = 1, \ldots, n-2r\}$ be the vertex set of the $(n-2r)$-clique K disjoint from F.

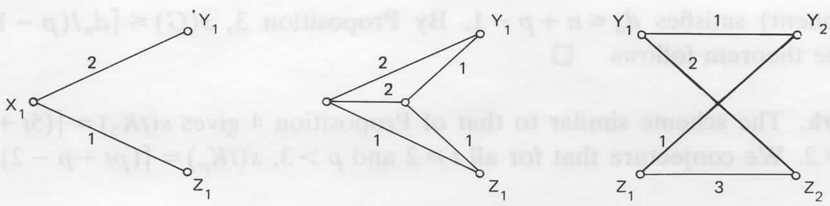

Fig. 3. Theorem 7 is true for $n = 3$ and 4.

Note that if $r = 0$, then $G = K_n$ and $s(G) = 3$ follows from Section 2. Moreover, the theorem is true for $n = 3$ or 4, see Fig. 3. Assume now that $n \geqslant 5$.

In the case of $r = 1$, by Proposition 3, there is an assignment of the complete subgraph $K + \{y_1\}$ induced by $X \cup \{y_1\}$ with weights at most 3 resulting in distinct degrees $n-2, n, \ldots, c$, with $c \leqslant 2n - 2$. Let y_1 be the vertex of weighted degree n, and assign unit-weight to each edge of G incident with z_1. Then we obtain an almost consecutive degree sequence for G: $(n-2, n-1, n, n+2, \ldots, c+1)$.

To prove the theorem for $r \geqslant 2$, we first introduce a relation $u > v$ between $u, v \in V(G)$ as follows: for every $j = 1, \ldots, r$,

$$y_j > x_k \text{ and } z_j > x_k \quad \text{for each } 1 \leqslant k \leqslant n - 2r;$$

$$y_j > y_i,\ y_j > z_i,\ z_j > z_i \text{ and } z_j > y_i \quad \text{iff} \quad 1 \leqslant i < j.$$

Case 1: $n \geqslant 2r + 1$.

Observe that X is not empty. Let G_1 be the network induced by $X \cup \{y_1, z_1\}$ with the almost consecutive assignment described above. If K has just one vertex ($n = 2r + 1$), then we start with the path on vertices x_1, y_1 and z_1 weighted as in Fig. 3. Otherwise, observe that the degrees of G_1 belong to the interval $D_1 = [n - 2r, 2n - 4r + 3]$ for each r and $n > 2r + 1$, and to the interval $D_1 = [n - 2r, 2n - 4r + 1]$ for $n = 2r + 1$. We extend this initial assignment by the alternating scheme.

Let G_j be the network induced by $G_{j-1} \cup \{y_j, z_j\}$ for $j = 2, \ldots, r$. When $u = y_j$ or z_j, $j = 2, \ldots, r$, define the weight $w(u, v)$ of edge uv for each $u > v$ as follows:

$$w(u, v) = \begin{cases} 1 & \text{if } u = y_j, \\ 3 & \text{if } u = z_j, \end{cases}$$

Let $d_j(v)$ be the degree of a vertex v in G_j $(1 \leqslant j \leqslant r)$. Then for $j = 2, \ldots, r$,

$$d_j(y_j) = n - 2r + 2j - 2, \qquad d_j(z_j) = 3(n - 2r + 2j - 2);$$

furthermore, $d_j(v) = d_{j-1}(v) + 4$ for every vertex of G_j different from y_j and z_j. Therefore,

$$d_j(y_j) < n - 2r + 2j \leqslant d_j(v),$$

and we need to show that $d_j(v) < d_j(z_j)$ for $j = 2, \ldots, r$. In the case of $n > 2r + 1$

$$d_2(v) \leqslant 2n - 4r + 7 < 3n - 6r + 6 = d_2(z_2),$$

and for $n = 2r + 1$,

$$d_2(v) \leq 2n - 4r + 5 < 3n - 6r + 6 = d_2(z_2).$$

Hence, for every $j = 3, \ldots, r$,

$$d_j(v) = d_{j-1}(v) + 4 \leq 3(n - 2r + 2j - 2) - 2 < d_j(z_j).$$

Thus, the network $G = G_r$ is irregular.

Case 2: $n = 2r$.

We use the alternating scheme starting with the 4-cycle on vertices y_1, y_2, z_1 and z_2 weighted as in Fig. 3. For each $j = 3, \ldots, r$ and $u > v$, let $w(u, v)$ be defined as in Case 1. This assignment is clearly irregular. □

Note that by a result in [5], $s(K_n - F_r) = 3$ if F_r is an r-matching with $r < \lceil (n-1)/4 \rceil$, so Theorem 7 is sharp. Our next result is an extension of Theorem 7 where we will apply the following theorem.

Theorem (Hajnal, Szemerédi [10]). *The vertices of a graph of order n with minimum degree $n - t$ can be partitioned into sets inducing t cliques of order $p = \lfloor n/t \rfloor$ or $p' = \lceil n/t \rceil$.*

Theorem 8. *Let G be a graph of order n and minimum degree $\delta(G) = n - t$. If $1 \leq t \leq \sqrt{n/18}$, then $s(G) \leq 3$.*

Proof. The case $t \leq 2$ is proved in Theorem 7, so suppose that $t \geq 3$.

Consider the t cliques given by the Hajnal–Szemerédi theorem and let v_{ij} denote the ith vertex of the jth clique of order $k_j = p$ or p', $1 \leq i \leq k_j$ and $1 \leq j \leq t$.

Identify the jth clique of G with the jth vertex of a complete graph K_t. Use the alternating scheme (Proposition 1) to assign weights of at most 3 to the edges of this K_t resulting in degrees

$$d_1^* = a,\ d_2^* = a + 2, \ldots,$$
$$d_g^* = b,\ d_{g+1}^* = b + 1,\ d_{g+2}^* = b + 2,\ d_{g+3}^* = b + 3, \ldots,$$
$$d_{t-1}^* = c - 2,\ d_t^* = cm$$

where $g = \lfloor t/2 \rfloor - 1$. For $j = 1, \ldots, t - 1$,

$$d_{j+1}^* - d_j^* = 1 \quad \text{if } j = g + h, \text{ with } h = 0, 1 \text{ or } 2;$$

and

$$d_{j+1}^* - d_j^* \geq 2, \quad \text{otherwise.}$$

Now, we expand the vertices of K_t back to cliques of G, and assign to an edge between a pair of cliques in G the same weight that we assigned to the edge between the corresponding pair of vertices in K_t. Denote by $d^*(v)$ the sum of these weights at vertex v.

First we show that

$$d_j^*p - 3(t-1) \leqslant d^*(v_{ij}) \leqslant d_j^*p + 3(t-1)$$

for $1 \leqslant j \leqslant t$ and $1 \leqslant i \leqslant k_j$.

Observe, if v_{ij} is joined to all points of every clique and each clique is of order p, then $d^*(v_{ij}) = d_j^*p$. Since $\delta(G) \geqslant n-t$, there are at most $t-1$ missing edges of weight 3 or less at v_{ij}, and the lower bound follows. Furthermore, if the size of each clique is $p' = p+1$, then we have to consider at most $t-1$ more edges of weight 3 or less (one for each clique different from that containing v_{ij}) which yields the upper bound.

Now we apply Proposition 3 to obtain an irregular network on the jth clique of G with consecutive or almost consecutive degree sequence as follows:

$$d_j(v_{ij}) = x_j + i - 1 \qquad \text{for } i = 1, \ldots, k_j - 1,$$

$$d_j(v_{ij}) = x_j + k_j - 1 \text{ or } x_j + k_j \text{ for } i = k_j,$$

where

$$x_j = \begin{cases} p+1 & \text{if } 1 \leqslant j \leqslant g, \\ p+1+h(\lfloor p/3 \rfloor - 2) & \text{if } j = g+h,\ h = 1, 2 \text{ or } 3, \\ p+1+3(\lfloor p/3 \rfloor - 2) & \text{if } g+4 \leqslant j \leqslant t. \end{cases}$$

Observe that $k_j \geqslant n/t - 1 \geqslant 5$ follows from $t < \sqrt{n/18}$. Furthermore, the weights we have to use in Proposition 3 are not greater than 3, since

$$k_j - 1 < p + 1 \leqslant x_j \leqslant 2p - 5 < 2k_j - 3$$

for every $1 \leqslant j \leqslant t$, so $k_j -$ does not divide x_j and $(x_j + k_j)/(k_j - 1) \leqslant 3$.

Now we show that the degrees $d(v_{ij}) = d^*(v_{ij}) + d_j(v_{ij})$ become distinct. Obviously,

$$d(v_{ij}) \geqslant x_j + d_j^*p - 3(t-1) = A(j)$$

and

$$d(v_{ij}) \leqslant x_j + p + 1 + d_j^*p + 3(t-1) = B(j).$$

Thus the network becomes irregular if $B(j) < A(j+1)$ holds for every $j = 1, \ldots, t-1$.

Consider the difference

$$A(j+1) - B(j) = x_{j+1} - x_j + p(d_{j+1}^* - d_j^* - 1) - 6(t-1).$$

If $j = g+h$, with $h = 0$, 1 or 2, then

$$d_{j+1}^* - d_j^* = 1, \quad \text{and} \quad x_{j+1} - x_j = \lfloor p/3 \rfloor - 2.$$

Therefore,

$$A(j+1) - B(j) \geqslant \lfloor p/3 \rfloor + 4 - 6t > n/(3t) - 6t > 0$$

holds when $t \leqslant \sqrt{n/18}$.

If $1 \leq j \leq g-1$ or $g+3 \leq j \leq t-1$, then

$$d^*_{j+1} - d^*_j \geq 2, \quad \text{and} \quad x_{j+1} - x_j = 0.$$

Therefore,

$$A(j+1) - B(j) > p + 4 - 6t > n/t - 6t > 0. \qquad \square$$

Theorem 8 has the following immediate consequence.

Corollary 9. *If t is a fixed positive integer and n is sufficiently large, then $s(G) \leq 3$ for each graph G of order n with minimum degree $n-t$.*

As it was mentioned earlier, we conjecture that Corollary 9 has an extension in the following direction: if $0 < \alpha < 1$ is fixed and n is sufficiently large, then $s(G) \leq c_\alpha$ for each graph G or order n with minimum degree $\delta(G) \geq (1-\alpha)n$, where c_α is a constant.

A weaker result follows by using similar techniques as in the proof of Theorem 8.

Theorem 10. *Let $0 < \alpha < 1$ and G be a $(1-\alpha)n$-regular graph of order n. Then $s(G) \leq (1/(\lfloor 1/\alpha \rfloor - 1))n + 5$ for sufficiently large n.*

Proof. Set $t = \alpha n$, so that $\delta(G) = n - t$. According to the theorem of Hajnal–Szemerédi, there is a spanning subgraph G^* of G which is the union of t cliques of order $p = \lfloor n/t \rfloor$ or $p' = \lceil n/t \rceil$.

Denote by k_j the order of the jth clique and assume that for $j = 1, \ldots, t$, $k_j = p+1$ if $1 \leq j \leq g$, and $k_j = p$ otherwise.

By Proposition 6, G^* has an irregular assignment with weights at most

$$\lceil n/(p-1) \rceil + 4 < (1/(\lfloor 1/\alpha \rfloor - 1))n + 5$$

such that if x and y are vertices of the ith and jth clique, respectively, then $d^*(x) < d^*(y)$ holds iff $i < j$.

Now assign unit-weights to all edges of G not belonging to G^*. The weighted degree of a vertex v belonging to the jth clique, $j = 1, \ldots, t$ becomes

$$d(v) = \begin{cases} d^*(v) + (n-t) - p & \text{if } 1 \leq j \leq g, \\ d^*(v) + (n-t) - p + 1 & \text{otherwise.} \end{cases}$$

The network described can be seen to be irregular. $\square$

5. Concluding remarks

If F_k is a k-matching and $G = K_n - F_k$, then $s(G) \leqslant 3$ for $n \geqslant 3$, by Theorem 7. Furthermore, it is stated in [5], that if $k = \lceil (n-1)/4 \rceil$, then $s(G) = 2$ for $n \equiv 0$, 1, 3 (mod 4). Gyárfás proved in [9] that except for the cases above, $s(K_n - F_k) = 3$. We summarize that in the following.

Proposition 11. *If $s(K_n - F_k) = 2$, then $n = 4m \pm 1$ or $4m$, and $k = m$.*

The following question arises from Theorem 8.

Question 12. What is the largest integer $t = t(n)$ such that $s(G) \leqslant 3$ holds for every graph G of order n with minimum degree $n - t$ and if n is sufficiently large?

If $t = t(n) > 2n/3$, it follows that $s(G) \geqslant 4$ for every $(n-t)$-regular graph G of order n. We propose the following question.

Question 13. What is the smallest integer $t = t(n)$ such that there exist infinitely many graphs of order n with minimum degree $n - t$ and having irregularity strength at least 4?

In [7], $s(G) \leqslant n/2 +$ constant is proved for every r-regular graph G. We expect, however, that $n/r +$ constant is an upper bound. In Theorem 5, $s(G) \leqslant n/r + 4$ is proved for a very special case when all the connected components of G are cliques of order $r + 1$. Thus we restate the following open problem.

Question 14. Does $s(G) \leqslant n/r + c$ hold for some constant c for every r-regular graph G of order n?

Finally we repeat the main question motivating this paper.

Question 15. Let α be a real with $0 < \alpha < 1$. Does there exist a constant c_α such that $s(G) \leqslant c_\alpha$ for every graph G with minimum degree $\delta(G) \geqslant (1 - \alpha)n$?

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