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# Multipartite Graph-Sparse Graph Ramsey Numbers

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# MULTIPARTITE GRAPH—SPARSE GRAPH RAMSEY NUMBERS

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The Ramsey number $r(F, G)$ is determined in the case where $F$ is an arbitrary fixed graph and $G$ is a sufficiently large sparse connected graph with a restriction on the maximum degree of its vertices. An asymptotically correct upper bound is obtained for $r(F, T)$ where $T$ is a sufficiently large, but otherwise arbitrary, tree.

## 1. Introduction

Let $F$ be a graph with chromatic number $\chi(F)=m$. The number $s=s(F)$ is defined to be the least number of vertices in a color class under any $m$-coloring of the vertices of $F$. Then for every connected graph $G$ with $n \geqq s$ vertices, the *Ramsey number* $r(F, G)$ satisfies the inequality

(1) $$r(F, G) \geqq (m-1)(n-1)+s.$$

The Ramsey number $r(F, G)$ is the least number $N$ such that in every two-coloring $(R, B)=$*(red, blue)* of the edges of $K_N$, either there is a red copy of $F$ or else a blue copy of $G$. The inequality (1) holds in view of the fact that the edges of the complete graph of order $(m-1)(n-1)+s-1$ can be given the coloring in which the blue graph, denoted $\langle B\rangle$, is isomorphic to $(m-1)K_{n-1}\cup K_{s-1}$. Then the red graph, denoted $\langle R\rangle$, does not contain $F$ since $\langle R\rangle$ has chromatic number $m$ but the smallest color class has $s-1$ vertices. Likewise, $\langle B\rangle$ does not contain $G$ since no component of $\langle B\rangle$ has more than $n-1$ vertices. A natural line of inquiry asks for the determination of those cases for which (1) holds with equality. The classical result of this type is the simple theorem of Chvátal [4], namely $r(K_m, T)=(m-1)(n-1)+1$ for every tree $T$ of order $n$. Many other examples of equality in (1) can be found when one assumes that $G$ is sufficiently sparse. Previous results of this type can be found in [2], [3] and [5]. The present paper concerns the case in which $F$ is a multipartite graph. Our notation for an $m$-partite graph with parts of size $p_1, p_2, \dots, p_m$ will be $K(p_1, \dots, p_m)$. In case the parts are all of equal size $p$, we shall write $K_m(p, \dots, p)$. The *floor* of $x$ (greatest integer $\leqq x$) and the *ceiling* of $x$ (least integer $\geqq x$) will be denoted $\lfloor x\rfloor$ and $\lceil x\rceil$, respectively. A path which is a subgraph of $G$ is a *suspended path* of $G$ if each vertex of the path, except for its endvertices, has degree 2 in $G$. Throughout $F$ and $G$ will have no isolates.

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## 2. Sparse graphs with restricted maximum degree

This section is devoted to a consideration of $r(K(p_1, \ldots, p_m), G)$ where $G$ is a connected graph which is sparse, i.e. with relatively few edges, and in which the maximum degree, $\Delta(G)$, is restricted. The starting point is the following general-purpose lemma which gives certain key "sparse graph" techniques.

**Lemma.** (i) *Let* $X=\{x_1, \ldots, x_a\}$ and $Y=\{y_1, y_2, \ldots, y_b\}$ *be disjoint sets of vertices in a two-colored complete graph in which* $(x_1, \ldots, x_a)$ *is a blue monochromatic path. If* $a \geqq b(c-1)+d$ *and there is no longer blue path from* $x_1$ *to* $x_a$, *then either* $\langle R\rangle \supset K_c$ *or else there is a set* $X' \subset X$ *with* $|X'|=d$ *which is completely joined with* $Y$ *in* $\langle R\rangle$. (ii) *Suppose that* $(R, B)$ *is a two-coloring of the edges joining* $X=\{x_1, \ldots, x_a\}$ *with* $Y=\{y_1, \ldots, y_b\}$ *in which each vertex of* $X$ *is adjacent in* $\langle B\rangle$ *to at least* $c$ *vertices of* $Y$. *Then either there is a blue matching of* $X$ *into* $Y$ *or else* $\langle R\rangle \supset K(c+1, b-a+1)$. (iii) *If* $G$ *is a graph with* $n$ *vertices and* $q>n$ *edges in which there is no suspended path with more than* $a$ *vertices, then* $G$ *has at least* $[(n/2a)-(3(q-n)/2)]$ *vertices of degree one.*

**Proof.** (i) Here we make use of a familiar argument which occurs in the Bondy—Erdős paper on cycle Ramsey numbers [1] and elsewhere. If one of the vertices $y \in Y$ is adjacent in $\langle B\rangle$ to $c$ vertices on the path, then there are at least $c-1$ successor vertices (the last vertex may be $x_a$ and not have a successor) which together with $y$ span a $K_c$ in $\langle R\rangle$. Otherwise, the path could be lengthened. It follows that with no $K_c$ in $\langle R\rangle$ there are at most $b(c-1)$ blue edges joining $X$ with $Y$. However, if we assume that there is no set of $d$ vertices in $X$ which is completely joined with $Y$ in $\langle R\rangle$, then there are at least $a-d+1 \geqq b(c-1)+1$ blue edges joining $Y$ with $X$ and a contradiction has been reached. (ii) This is an immediate consequence of Hall's theorem [6]. If there is no blue matching, then for some subset $X' \subset X$ Hall's matching condition fails. By the assumed degree condition, $|X'|>c$. Then $X'$ is completely joined in $\langle R\rangle$ with $Y' \subset Y$ where $|Y'| \geqq b-a+1$. (iii) This is Lemma 2 of [5]. The short proof can be found in this reference. ∎

The following proposition, while not giving an exact Ramsey number, is a stepping stone to a case of equality in (1).

**Proposition.** *Given* $p_1 \leqq p_2 \leqq \ldots \leqq p_m$, $k$ *and* $\Delta$, *there exists a corresponding number* $l$ *such that for every graph* $G$ *with* $n$ *vertices,* $q \leqq n+k$ *edges and maximum degree* $\leqq \Delta$

$$r(K(p_1, \ldots, p_m), G) \leqq (m-1)(n-1)+l.$$

**Proof.** Let $p=p_1+p_2+\ldots+p_m$ denote the order of the multipartite graph. The proof is by double induction on $m$ and $n$. Although it represents a momentary departure from our disavowal of graphs with isolates, the trivial $m=1$ case will serve as an anchor. It is clear that a proper choice of $l$ will ensure that the proposition is true for all $n \leqq p^2[2(p^2-1)\Delta+3k]$. Hence, we take $m>1$, $n>p^2[2(p^2-1)\Delta+3k]$ and assume that the proposition is true for all smaller values of these parameters. Set $N=(m-1)(n-1)+l$ and suppose that $(R, B)$ is a two-coloring of the edges of $K_N$ in which $\langle R\rangle \not\supset K(p_1, \ldots, p_m)$ and $\langle B\rangle \not\supset G$. We shall demonstrate that this assumption leads to a contradiction. The proof is divided into three cases.

(i) *G contains a suspended path with $p^2$ vertices.*
Let $H$ be the graph obtained from $G$ by shortening the length of this path by one. Then $H$ has $n-1$ vertices, no more than $(n-1)+k$ edges and has maximum degree $\leqq \Delta$. Since we may assume that $l$ is an increasing function of $p$, $k$ and $\Delta$, it follows from the induction hypothesis that there is a blue copy of $H$ and, disjointly, a red copy of $K(p_1, \ldots, p_{m-1})$. Now apply part (i) of the lemma by letting $X$ be the vertex set of the suspended path and $Y$ be the vertex set of the red $K(p_1, \ldots, p_{m-1})$. Now we set $a=p^2-1$, $b=p-p_m$, $c=p$ and $d=p_m$. The requisite inequality is clearly satisfied and so from the lemma we find that either $\langle R\rangle \supset K(p_1, \ldots, p_m)$ (or perhaps even $K_p$) or else $\langle B\rangle \supset G$.

(ii) *G contains a set of $p^2$ independent end edges.*
Let $H$ denote the graph obtained from $G$ by deleting a set of $p^2$ independent end edges. In order to be able to use part (ii) of the lemma, we shall use the following device. Let $V$ denote the vertex set of the two-colored $K_N$ and note that either $\langle R\rangle \supset K_p$ or else there is a set $W$ with $|W| \leqq c(p-1)$ such that every vertex in $V-W$ is adjacent in $\langle B\rangle$ to at least $c$ vertices of $W$. Such a set $W$ may be found by an algorithm in which $W$ is initially empty and the process of finding a red clique in $V-W$ and then placing the vertices of that clique in $W$ is repeated $c$ times. As long as $c$ does not exceed $p$ it is certainly true that $N-c(p-1) \geqq (m-1)(n-p^2-1)+l$. Thus, by the induction hypothesis, there is a blue copy of $H$ which is vertex disjoint from $W$. Set $a=p^2$ and let $X=\{x_1, \ldots, x_a\}$ denote the set of vertices in this copy which are incident with the edges which were deleted in going from $G$ to $H$. Let $Y \supset W$ denote the set of all vertices of $V$ which are not vertices of the copy of $H$. Thus $|Y|=b=N-(n-p^2)$ and every vertex of $X$ is adjacent in $\langle B\rangle$ to at least $c$ vertices of $Y$. It follows from part (ii) of the lemma that either $\langle B\rangle \supset G$ or else $\langle R\rangle \supset K(p, N-n+1)$. In view of the induction hypothesis, $N-n+1 \geqq r(K(p_1, \ldots, p_{m-1}), G)$. Thus either $\langle R\rangle \supset K(p_1, \ldots, p_m)$ or else $\langle B\rangle \supset G$.

(iii) *Neither* (i) *nor* (ii) *occur.*
Note that if $G$ had $(p^2-1)\Delta+1$ vertices of degree one, then it would necessarily have $p^2$ independent end edges. Thus, in view of part (iii) of the lemma, we must have $(n/(2p^2))-(3k/2) \leqq (p^2-1)\Delta$ and so $n \leqq p^2[2(p^2-1)\Delta+3k]$. But $l$ was chosen so as to ensure that the proposition is true for all $n \leqq p^2[2(p^2-1)\Delta+3k]$ and so we have reached a contradiction. ∎

Now we are ready to prove a case of equality in (1).

**Theorem 1.** *Given $s=p_1 \leqq \ldots \leqq p_m$, $k$ and $\Delta$, there exists a corresponding number $n_0$ such that for every connected graph $G$ with $n>n_0$ vertices, $q \leqq n+k$ edges and maximum degree $\leqq \Delta$,*

$$r(K(p_1, \ldots, p_m), G) = (m-1)(n-1)+s.$$

**Proof.** The proof is by induction on $m$. Again, it is trivial for $m=1$. Thus we set $N=(m-1)(n-1)+s$ and assume that $(R, B)$ is a two-coloring of the edges of $K_N$ in which $\langle R\rangle \not\supset K(p_1, \ldots, p_m)$ and $\langle B\rangle \not\supset G$. In what follows, $l$ will denote the function whose existence was established in the preceding proposition. Again, we have three cases. As before, $p=p_1+\ldots+p_m$.

(i) *G has a suspended path with $p^2+l$ vertices.*
Let $H$ be the graph obtained from $G$ by shortening this path by $l$. By use of the induction hypothesis together with the result of the preceding proposition we find a blue

copy of $H$ and, disjointly, a red copy of $K(p_1, \ldots, p_{m-1})$. The argument is completed by an application of part (i) of the lemma.

(ii) *$G$ has a set of $p^2+l$ independent end edges.*

The structure of this argument is the same as that in the proof of the proposition. The desired result follows by an application of the induction hypothesis, the preceding proposition, and part (ii) of the lemma.

(iii) *Neither* (i) *nor* (ii) *occur.*

By means of part (iii) of the lemma, we are now able to conclude that $n \leq n_0 \equiv \equiv (p^2+l)[2(p^2+l-1)\Delta+3k]$. In view of (1), it follows that $r(K(p_1, \ldots, p_m), G) = (m-1)(n-1)+s$ holds for all $n > n_0$.

**Corollary.** *Let $F$ be a fixed graph of order $p$, chromatic number $\chi(F)=m$ and such that in every $m$-coloring of $V(F)$ each color class has at least $s$ vertices. Set $\alpha = 1/(2p-1)$. There exist constants $C_1$ and $C_2$ such that for all sufficiently large $n$ $q(G) \leq n + C_1 n^\alpha$ and $\Delta(G) \leq C_2 n^\alpha$ imply that the connected graph $G$ satisfies*

$$r(F, G) = (m-1)(n-1)+s.$$

**Proof.** To see that this result is true, it suffices to work with crude upper bounds, first for $l$ (Proposition) and then for $n_0$ (Theorem 1). Review of the proof of the proposition shows that it suffices to take

$$l = r(K_p, K_j) \leq \binom{p+j-2}{p-1}, \tag{2}$$

where $j = p^2[2(p^2-1)\Delta+3k]$. In what follows, $C_3$, $C_4$ and $C_5$ will represent numbers which are independent of $n$ and which approach 0 as $C_1$ and $C_2$ approach 0. Using the above hypothesis and the relationship between $n_0$ and $l$, we have $j \leq C_3 n^\alpha$, $l \leq \leq C_4 n^{(p-1)\alpha}$ and finally $n_0 \leq C_5 n^{2(p-1)\alpha} < n$ when $C_1$ and $C_2$ are chosen to be sufficiently small. ∎

## 3. Trees

The extreme case of a sparse connected graph is a tree. In the case of trees, we can obtain some interesting results without assuming a restriction on the maximum degree. In the lemma which follows, the *join* of $G$ and $H$, denoted $G+H$, is defined for $V(G) \cap V(H) = \emptyset$ to be the graph obtained by adding all edges joining $V(G)$ and $V(H)$.

**Lemma.** *Let $X = \{x_1, \ldots, x_a\}$ and $W$ be disjoint sets of vertices and let $W_k$ denote the set of vertices in $W$ to which $x_k$ is adjacent, $k=1, \ldots, a$. Suppose that for $k=1, \ldots, a$, $\langle W_k \rangle$ contains $b_k \geq b$ subgraphs isomorphic to $G$ but that $\langle W \rangle$ contains a total of $c \leq d$ such subgraphs. Then in $\langle X \cup W \rangle$ the number of subgraphs isomorphic to $G+\bar{K}_p$ is at least $(ab-pd)^p d^{1-p}/p!$ when $ab > pd$.*

**Proof.** Let $G_1, G_2, \ldots, G_c$ be a listing of the subgraphs of $\langle W \rangle$ which are isomorphic to $G$. For each $j=1, \ldots, c$ let $M_j$ denote the number of $k$ for which $G_j \subset \langle W_k \rangle$. Then

$$\sum_{j=1}^{c} M_j = \sum_{k=1}^{a} b_k \geq ab. \tag{3}$$

Using the convexity of $f(x)=\binom{x}{p}$, Jensen's inequality yields the fact that the number of subgraphs isomorphic to $G+\bar{K}_p$ is at least (under the assumption $ab>pd$)

$$(4)\qquad \sum_{j=1}^{c}\binom{M_j}{p} \geqq c\binom{ab/c}{p} \geqq d\binom{ab/d}{p} > (ab-pd)^p d^{1-p}/p!. \quad \blacksquare$$

The Ramsey number $r(K_m(p, \dots, p),T)$ is well known to be complicated. Chances for exact results are slim unless restrictions are placed on $T$. Therefore, we are content with asymptotic results.

**Theorem 2.** *Let* $p\geqq 2$ *be fixed and for* $m=1, 2, \dots$ *set*

$$\alpha(m) = \frac{p^m-p}{p^m-1}$$

*and*

$$\beta(m) = \frac{p(p^m-1)}{p-1}.$$

*For each* $m=1, 2, \dots$ *there exists positive constants* $A_m$ *and* $B_m$ *such that if* $N=(m-1)n+k$ *where* $k=O(n)$ *but* $k\geqq A_m n^{\alpha(m)}$, *n sufficiently large will imply that in every two-coloring* $(R, B)$ *of the edges of* $K_N$ *either* $\langle B\rangle$ *contains every tree T of order n or else* $\langle B\rangle$ *contains at least* $B_m(k/n)^{\beta(m)}n^{mp}$ *subgraphs isomorphic to* $K_m(p, \dots, p)$.

**Proof.** The proof is by induction on $m$. The case $m=1$ is easily checked. Now let $m$ exceed 1 and suppose that the theorem is true in all prior cases. Throughout the proof, $A_m, B_m, C_m, \dots$ will represent positive numbers which may depend on $m$ but which are independent of $n$. By an obvious parameter scaling, it suffices to prove the existence of a positive $A_m$ which ensures the stated conclusion when $N=(m-1)n+2k$ and $k\geqq A_m n^{\alpha(m)}$.

It is a well-known fact that if every vertex of $\langle B\rangle$ has degree at least $n-1$ then $\langle B\rangle$ contains every tree $T$ of order $n$. By assuming that $\langle B\rangle$ does not contain every tree of order $n$, repeated use of this fact leads to a sequence of vertices $x_1, \dots, x_k$ such that each of these vertices is adjacent in $\langle R\rangle$ to at least $(m-2)n+k$ of the remaining $(m-1)n+k$ vertices. Let $W$ denote the set of these remaining vertices and note that $k=O(n)$ implies that in the red subgraph induced by $W$ the number of copies of $K_{m-1}(p, \dots, p)$ is at most $O(n^{(m-1)p})$, $n\to\infty$. Thus, invoking the induction hypothesis, we may apply the lemma with

$$(5)\qquad a = k \geqq A_m n^{\alpha(m)}$$

$$(6)\qquad b \geqq B_{m-1}(k/n)^{\beta(m-1)}n^{(m-1)p}$$

and

$$(7)\qquad d \leqq C_m n^{(m-1)p}.$$

Note that the exponents $\alpha(m)$ and $\beta(m)$ have been defined so that

$$(8)\qquad \alpha(m)\{1+\beta(m-1)\} = \beta(m-1)$$

and

(9) $$p\{1+\beta(m-1)\} = \beta(m).$$

It follows from (5), (6) and (8) that

(10) $$ab \geqq B_{m-1}k(k/n)^{\beta(m-1)}n^{(m-1)p} \geqq D_m n^{(m-1)p}.$$

Consequently, (7) yields

(11) $$ab-pd \geqq E_m k(k/n)^{\beta(m-1)}n^{(m-1)p}.$$

In view of the lemma and relations (8) and (9), we find that $\langle R\rangle$ contains at least

(12) $$\{E_m k(k/n)^{\beta(m-1)}n^{(m-1)p}\}^p\{C_m n^{(m-1)p}\}^{1-p}/p! \geqq B_m(k/n)^{\beta(m)}n^{mp}$$

subgraphs isomorphic to $K_m(p, \dots, p)$. ∎

The crucial fact is that for each fixed $p$ and $m$, $\alpha(m)<1$ so that $n^{\alpha(m)}=o(n)$, $n\to\infty$. The theorem clearly gives the bound $r(K_m(p, \dots, p), T)\leqq(m-1)n+A_m n^{\alpha(m)}$. In view of (1) the consequences of the theorem can be stated in the following uniform asymptotic sense.

**Corollary.** *Let $F$ be a fixed graph with chromatic number $\chi(F)=m$. For every $\varepsilon>0$ there exists $N(\varepsilon)$ such that*

$$\left|\frac{r(F, T)}{n}-(m-1)\right|<\varepsilon$$

*for every tree $T$ of order $n>N(\varepsilon)$.* ∎

In the bound $r(K_m(p, \dots, p), T)\leqq(m-1)n+A_m n^{\alpha(m)}$, the exponent $(p^m-p)/(p^m-1)$ can hardly be expected to be best possible. However, in the case $p=m=2$ we can obtain a sharp result.

**Theorem 3.** *For every tree $T$ of order $n$,*

$$r(K(2,2), T)\leqq n+\lceil\sqrt{n}\rceil .$$

**Proof.** Before giving the proof of this result, let us note that it is best possible sincn $r(K(2,2), K(1, p^2+1))>p^2+p+1$ whenever $p$ is a power of a prime [7]. It is known as well [7] that $r(K(2,2), K(1, n-1))\leqq n+\lceil\sqrt{n}\rceil$ so it is enough for us to prove the result when $T\not\simeq K(1, n-1)$.

The proof is by induction on $n$. For $n\leqq 3$, $T\simeq K(1, n-1)$ so we know the result to be true. Take $n>3$ and let $T$ be a tree of order $n$ other than $K(1, n-1)$. With $N=n+\lceil\sqrt{n}\rceil$ suppose that there exists a two-coloring $(R, B)$ of the edges of $K_N$ in which $\langle R\rangle\not\supseteq K(2,2)$ and $\langle B\rangle\not\supseteq T$. Consider the tree to be rooted at a vertex of degree $\Delta(T)$. Since $T\not\simeq K(1, n-1)$, we may select an end vertex $u$ which is at distance at least two from the root vertex. Let $T'$ denote the tree obtained from $T$ by deleting $u$. Let $v$ denote the immediate predecessor of $u$ in the rooted tree and let $w$ denote the immediate predecessor of $v$. Note that $w$ is possibly the root vertex. By the induction hypothesis, there must be an embedding of $T'$ in $\langle B\rangle$. For simplicity of notation, the vertices in this embedding will have the same designation as in $T'$. Set $a=\lceil\sqrt{n}\rceil$ and suppose that $v$ has $b$ successors in $T$. Consider the set of vertices $P$ in the two-colored $K_N$ which are either immediate successors of $v$ in the blue embedding of $T'$ or else disjoint from $V(T')$ in this embedding. Form the partition of this set $(RR, RB, BR, BB)$ according to the adjacency pattern with respect to $v-w$. Thus $RB$ is the set of vertices in $P$ which are adjacent to $v$ in $\langle R\rangle$ and adjacent to $w$ in $\langle B\rangle$. In terms of the previously introduced parameters, we have $|RR|+|RB|=a+1$ and $|BR|+|BB|=b-1$. Let $|RR|=c$ and $|BB|=d$. In view of the fact that $\langle R\rangle \not\supset K(2, 2)$, $c\leqq 1$. Let $Q=P\cup\{v\}$. Note that no vertex of $Q$ which is adjacent to $w$ in $\langle B\rangle$ can be adjacent in $\langle B\rangle$ to as many as $b$ vertices of $Q$. Otherwise, such a vertex could play the role of $v$ and provide an embedding of $T$ into $\langle B\rangle$. Thus, every such vertex is adjacent in $\langle R\rangle$ to at least $a+1$ vertices of $Q$. In terms of our parameters, the number of these vertices is $a-c+d+2$. At the same time, $w$ is adjacent in $\langle R\rangle$ to $b+c-d-1$ vertices of $Q$. Now, by the usual pigeonhole argument using the assumption that $\langle R\rangle \not\supset K(2, 2)$

(13) $$(a-c+d+2)\binom{a+1}{2}+\binom{b+c-d-1}{2} \leqq \binom{a+b+1}{2}.$$

Using the identity

(14) $$\binom{x+p}{2}-\binom{x-q}{2} = \frac{1}{2}(p+q)[2x+(p-q+1)]$$

with $x=b, p=a+1$ and $q=d-c+1$, we see that (13) is equivalent to

(15) $$2b \geqq a^2+d-c+1.$$

However, $a^2\geqq n, d\geqq 0$ and $c\leqq 1$. Since the root vertex was chosen to have maximum degree and $v$ is necessarily distinct from the root, (15) implies that $T$ has at least $2b\geqq n$ edges. Thus, we have arrived at a contradiction and the proof is complete. ∎

## 4. Questions

The *edge density* of a graph $G$ is defined to be $\max\{q(H)/p(H)\}$ where $p$ and $q$ denote the number of vertices and number of edges respectively and the maximum is taken over all subgraphs $H\subset G$. Does bounded edge density and $\Delta(G)=o(n)$ imply that

$$r(F, G) = (m-1)(n-1)+s$$

for all sufficiently large $n$? Does bounded degree have such an implication?

## 5. Dedication

The three last named authors [R. J. F, C. C. R., R. H. S] gratefully dedicate this paper to Paul Erdős in honor of his 70th birthday.

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