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ABSTRACT

A graph G with at least $2k$ vertices is k-path pairable if for any k pairs of distinct vertices of G there are k edge disjoint paths between the pairs. For $k = 2$ and 3, and for any $\Delta \geq 9$, we will determine the minimal number of edges in a graph G of maximum degree Δ that is k-path pairable.

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1. INTRODUCTION

We shall consider graphs without loops or multiple edges. Any such graph can quite naturally represent a computer or communication network. There are various reasonable ways to measure the capability of the network represented by this graph to transfer information and handle communications. We will consider the capability of the network to allow messages to be passed simultaneously between any fixed number of pairs of nodes of the network. With this in mind, we give the following formal definition.

Definition . Given a fixed positive integer k, a graph G is *k-path pairable* if for any pair of disjoint ordered sets of vertices $X = \{x_1, x_2, \cdots, x_k\}$ and $Y = \{y_1, y_2, \cdots, y_k\}$ of G there are k edge-disjoint paths P_i, where P_i is a path from x_i to y_i, for $1 \leq i \leq k$.

We note that the concept of k-path pairable is related to several other concepts. It is closely related to but is not the same as weakly k-linkable (see [H]). (In both weakly linked and path pairable graphs, k edge disjoint paths are required, but duplication of the pairs is allowed in the weakly linked case and prohibited in the path pairable case.)

Let $p_k(n, \Delta)$ be the minimum number of edges in a graph G of order n and maximum degree at most Δ that is k-path pairable. Our objective is to evaluate the function $p_k(n, \Delta)$. Useful in the determination of this function is the function $p_k(n, \Delta, \delta)$, which is the minimum number of edges in a graph G of order n with maximum degree at most Δ and minimum degree at least δ that is k-path pairable.

Any connected graph is 1-path pairable, so $p_1(n, \Delta) = n - 1$ for any $2 \leq \Delta < n$. We will prove the following theorems concerning 2-path pairable and 3-path pairable graphs (see Theorems 9 and 12) which determine $p_2(n, \Delta)$ and $p_3(n, \Delta)$ assymptotically.

Theorem A. *For $\Delta \geq 3$ a fixed integer and $n \geq 12\Delta - 15$,*

$$p_2(n, \Delta) = (1 + \epsilon)n + \alpha,$$

where $0 \leq \alpha < 13/6$, and ϵ depends upon Δ and approaches 0 as Δ increases.

Theorem B. *For $\Delta \geq 9$ a fixed integer and $n \geq 10\Delta - 21$,*

$$p_3(n, \Delta) = (1 + \epsilon)n + \alpha,$$

where $0 \leq \alpha \leq 3/2$, and ϵ depends upon Δ and approaches 0 as Δ increases.

In case of $\Delta = 3$ and for $n > 20$, sharp results are proved, namely that $p_2(n, 3) = \lceil \frac{11n}{10} \rceil$ (Theorem 7), and $p_3(n, 3) = \lceil \frac{5n}{4} \rceil + \alpha$, where $\alpha = 0$ or 1 (Theorem 11).

In [FGL], $p_k(n,3)$ is investigated for $k > 3$. For these cases exact results are not obtained. However, it is proved that for n sufficiently large there exist ϵ_1 and ϵ_2 (that depend upon k and approach 0 as k increases) such that

$$\left(\frac{3}{2} - \epsilon_1\right) n < p_k(n,3) < \left(\frac{3}{2} - \epsilon_2\right) n.$$

Of course the major problem is to determine the function $p_k(n,\Delta)$ precisely, but this is probably very difficult for $k > 3$. What can be said about the structure of k-path pairable graphs, and is there a characterization of these graphs for small values of k are both interesting questions as well.

We begin the determination of the function $p_k(n,\Delta)$, for$k = 2$ and 3, by describing some classes of k-path pairable graphs that will give upper bounds for this function. To prove that our innocent looking graphs are 2– or 3–path pairable a tedious case analysis is required. We feel appropriate to omit two proofs of this type (Propositions 3 and 5) – they are available in the preprint version of this paper.

We will generally follow the notation of [CL]. For a graph G, the vertex and edge set will be denoted by $V(G)$ and $E(G)$ respectively. The cardinality of $V(G)$ and $E(G)$ will be called the *order* and *size* respectively of the graph G. If X is a collection of vertices and edges of G, then $G - X$ will denote the graph obtained from G by deleting the edges in X and by deleting the vertices in X and the edges incident to a vertex in X. If u and v are vertices in G, then the edge determined by this pair of vertices will be denoted by uv.

2. UPPER BOUNDS AND EXAMPLES

For p even, let F_p denote the graph with p vertices and $3p/2$ edges obtained from a C_p by adding the long chords (each vertex is adjacent to the unique vertex that is a maximum distance away on the cycle). This graph has two kinds of edges, *cycle edges* and *chord edges*, and we will refer to them in that way.

The graph F_p is easily seen to be 2-path pairable. Let a, a' and b, b' be two pairs of vertices of F_p, and consider the cycle in F_p oriented. If the order of these four vertices on the cycle is a, a', b, b', then clearly there are the appropriate paths (in fact vertex disjoint paths) between pairs using only cycle edges. If the order on the cycle is a, b, a', b', then one can assume with no loss of generality that a, b and a' are in the first $(p/2) + 1$ vertices of the cycle. Therefore b is adjacent by a chord to a vertex b^* with the order on the cycle now being a, a', b^*, and b' (or with the order of the last two vertices reversed). Now, using the chord from b and cycle edges the appropriate edge disjoint paths can be constructed. This verifies the claim.

Let $C_1 = (x_1, x_2, \cdots, x_m, x_1)$ and $C_2 = (y_1, y_2, \cdots, y_m, y_1)$ denote two vertex disjoint cycles of length m. Let $p = 2m$ with m odd, and let G_p denote the graph obtained from these cycles by adding the edges $\{(x_i, y_{2i}) : 1 \le i \le m\}$, where the indices are taken modulo m. We will call the edges that were added between the cycles chords. Thus G_p is a graph with $p = 2m$ vertices and $3p/2 = 3m$ edges (p cycle edges and $p/2$ chord edges). In a very similar way as before one can verify easily that G_p is 2-path pairable.

We will always assume that the cycles in each of these graphs is oriented. If x and y are vertices on the cycle, then $[x, y]$ (or (x, y)) will denote the closed (or open) interval of vertices on the path from x to y.

In this section additional 2– and 3–path pairable graphs will be derived from F_p and G_p.

Denote by F_p^* the graph obtained from F_p by double subdividing (placing two vertices on the edge) each of the chord edges and subdividing each of the cycle edges. Thus F_p^* has $3p$ vertices and $7p/2$ edges.

Proposition 1. *For $p \ge 6$ and even, F_p^* is 2-path pairable. Moreover, the graphs obtained from F_p^* adding vertex disjoint stars to its vertices are still 2-path pairable.*

Proof. We verify first that F_p^* is 2-path pairable for $p \ge 6$ and even. Let a, a' and b, b' be two pairs of vertices in F_p^*. First consider the case when a and a' are vertices of degree three in F_p^*. We will construct three paths from a to a', which we will denote by P_1, P_2, and P_3. If a and a' are endvertices of the same chord path, then P_1 will be the chord path, and P_2 and P_3 will be the two vertex disjoint paths (except for endvertices) from a to a' using only cycle edges. If a and a' are not associated with the same chord, then P_1 will be the shortest cycle path between a and a', P_2 will be the path using the chord path from a followed by the shortest cycle path from the endvertex of this chord path to a', and P_3 will be the corresponding path using the chord path from a'.

Observe that the removal of the edges of any of these paths from F_p^* leaves some isolated vertices (those of degree two) and one more connected component. Since the pairwise intersection of these three paths contains no inner vertices of degree two, one of P_1, P_2, and P_3 is such that after removing its edges from F_p^* both b and b' are contained in the same component of the remaining graph. Therefore, F_p^* contains the required edge disjoint paths.

If a is a vertex of degree two, then associate with it a vertex a^* of degree three that is adjacent to a. Otherwise, just let $a^* = a$. In the same way, there is an a'^* associated with a'. We can also construct three paths between a and a'. These paths will be derived from the three paths between a^* and a'^* by either adding or deleting the edges aa^* and $a'a'^*$. These paths will have the same properties as the three paths between the vertices of degree three. This completes the proof of the fact that F_p^* is 2-path pairable.

Let G be a graph obtained from F_p^* adding vertex disjoint stars to its vertices. With no loss of generality we can assume that a and b are vertices of degree one with the same neighbor u. It is easy to check that if the shortest path from u to $\{a', b'\}$ is P, then the removal of the edges of P does not disconnect G. Thus the required paths for a's and b's exist. ■

Denote by $\mathcal{F}_p^*(n)$ the class of graphs with n vertices that is obtained from F_p^* by adding vertex disjoint stars to its vertices until a total of n vertices are obtained. Of course there is not a unique graph in $\mathcal{F}_p^*(n)$, but each of these graphs has $n-3p$ vertices of degree one and $n+p/2$ edges. If the stars are placed to minimize the maximum degree of a graph in $\mathcal{F}_p^*(n)$, then the maximum degree will be $\lceil((2n+p)-(n-3p))/3p\rceil = \lceil(n+4p)/3p\rceil$. The following proposition summarizes these observations.

Proposition 2. *Let $p \geq 6$ be a fixed even integer. Then for each $n \geq 3p$ any graph in $\mathcal{F}_p^*(n)$ is 2-path pairable, and has $n+p/2$ edges. Moreover, there is a graph in $\mathcal{F}_p^*(n)$ of maximum degree $\lceil(n+4p)/3p\rceil$.*

Proposition 3. *For $p \geq 10$, the graphs obtained from F_p by adding vertex disjoint stars to its vertices are 3-path pairable.*

Notice that for $p \geq 9$ and odd, the graph F_{p+1}^- obtained from F_{p+1} by adding vertex disjoint stars to its vertices, then collapsing two consecutive vertices on the cycle still remains 3–path pairable.

Theorem 4. *For $\Delta \geq 3$ a fixed positive integer and $n \geq 10\Delta - 21$,*

$$p_3(n,\Delta) \leq n + \lceil r/2\rceil + \gamma,$$

where $r = \lceil n/(\Delta-2)\rceil$, and $\gamma = 0$ or 1 according to the even or odd parity of r.

Proof. Let $\mathcal{F}_r(n)$ be the class of graphs obtained from F_r and F_{r+1}^- by adding vertex disjoint stars to the vertices until a total of n vertices are obtained.

Observe that $r \geq 9$, hence by Proposition 3 and the note in the previous paragraph, each graph in $\mathcal{F}_r(n)$ is 3-path pairable. Also these graphs have $n-r$ vertices of degree one and $n+r/2$ or $n+\lceil r/2\rceil+1$ edges, depending on the parity of r.

Since by the definition of r, $r(\Delta-3) \geq n-r$, $\mathcal{F}_r(n)$ contains a graph with maximum degree Δ, which proves the theorem. ■

Let G_p^* be the graph obtained from G_p by subdividing each of the chord edges. Therefore, G_p^* has $3p/2$ vertices and $2p$ edges.

Proposition 5. *For $p = 4m+2$ and $m \geq 3$, G_p^* is 3-path pairable.*

One cannot arbitrarily add stars to the vertices of G_p^* and keep the 3-path pairable property. However, a vertex of degree one can be added to the vertices of degree two in G_p^*, as one can verify easily. The following proposition summarizes that observation.

Proposition 6. *Let $p = 4m+2$ a fixed integer and $m \geq 3$. Then for each integer n with $3p/2 \leq n \leq 2p$, there is 3-path pairable graph $G_p^*(n)$ with n vertices, $n+p/2$ edges and maximal degree three.*

3. LOWER BOUNDS

We begin with a general observation about graphs that are k-path pairable. Let G be a k-path pairable graph, and let X be a set of cut edges of G that separates the vertices into two sets A and B. If t vertices in one part are paired with t vertices in the other part, then there must be at least t edges in X. Thus,

$$|X| \geq \min \{|A|, |B|, k\}.$$

This condition is called the *Cut Condition* for a k-pairable graph, and is clearly a necessary condition for a graph to be k-path pairable. The Cut Condition implies that certain induced subgraphs are forbidden in a k-path pairable graph. There are, however, other forbidden subgraphs that are not implied by the Cut Condition.

A *suspended path* in a graph is a path in which all of the interior vertices have degree two in the graph. A suspended path with five vertices is forbidden by the Cut Condition for $k \geq 3$, but it does not violate the Cut Condition for $k = 2$. Assume that x_1, x_2 and x_3 are the vertices of degree two (in the order indicated) of a suspended path with five vertices, and that x_1 and x_3 are paired and x_2 is paired with some other vertex. Then any path from x_1 to x_3 will destroy the possibility of an edge disjoint path from x_2 to any other vertex. Thus, this suspended path is forbidden for $k = 2$ as well.

3.1 2-path pairable graphs

Let G be a 2-path pairable graph. We will first describe three forbidden induced subgraphs for such a graph G. The Cut Condition implies that a vertex of degree one cannot be adjacent to a vertex of degree two, so this is the first forbidden structure. We have already observed that G cannot contain a suspended path with five vertices, which is a second forbidden structure. This means that G cannot be derived from any graph by triple subdividing any edge (placing three vertices on an edge).

The third forbidden structure is obtained from double subdividing two of the edges incident to a vertex of degree three in any graph. Observe that if x is the vertex of degree 3, x_1, x_2 are the vertices on the first subdivided edge, and y_1, y_2 are the vertices on the second edge, then it is impossible to find edge disjoint paths between x_1 and y_2, and between x_2 and y_1. Note also that since a vertex of degree one is never the interior vertex of a path, that the addition of any vertex of degree one to any of the forbidden subgraphs yields another forbidden subgraph.

With this information on forbidden subgraphs for 2-path pairable graphs, and with the examples of section 2, we are prepared to prove the following theorem.

Theorem 7. *For $n > 20$, $p_2(n, 3) = \lceil \frac{11n}{10} \rceil$.*

Proof . Let $n = 10r - s$ for $r \geq 3$ and $0 \leq s \leq 9$. Since $\lceil \frac{11(10r-s)}{10} \rceil = 11r - s$, it is sufficient to show $p_2(n, 3) = 11r - s$.

By Proposition 2, there is a graph in $\mathcal{F}^*_{2r}(10r-s)$ that is 2-path pairable, has maximum degree $\lceil(10r-s+8r)/6r\rceil = 3$, and has $11r-s$ edges. Thus it is sufficient to show that any 2-path pairable graph of maximum degree at most three has at least $11r-s$ edges.

Let G be a 2-path pairable graph with $10r-s$ vertices and of maximum degree at most three. Let G' be the graph obtained from G by deleting the vertices of degree one, and let G^* be the graph obtained from G' by replacing each suspended path in G' by an edge. Each of the graphs G' and G^* are 2-path pairable graphs, and G^* is a 3-regular graph, say of order t.

The graph G' can be obtained from G^* by subdividing, double subdividing, etc. the edges of G^*. Since the forbidden structures do not allow an edge to be triple subdivided, we can assume that a edges are double subdivided and b edges are subdivided. Thus the graph G' has $t+2a+b$ vertices and $3t/2+2a+b$ edges. The forbidden subgraphs imply that $a \leq t/2$, since at most one edge incident to any vertex can be double subdivided. Of course, $b \leq 3t/2-a$. The graph G can be obtained from G' by adding vertices of degree one to the vertices of degree two. Hence, if c is the number of vertices of degree one added, then $c \leq 2a+b$. The graph G has $t+2a+b+c$ vertices and $3t/2+2a+b+c$ edges.

To complete the proof it is sufficient to show with the restrictions previously listed on a, b, and c, that if $t+2a+b+c = 10r-s$, then $3t/2+2a+b+c \geq 11r-s$. Assume that this is not true. Then we must have $t/2 < r$, or equivalently $t \leq 2r-2$ since t is even. Thus, using the inequalities satisfied by a, b, c, and t, we have

$$10r-s = t+2a+b+c \leq 5t \leq 10r-10.$$

This is a contradiction that completes the proof of Theorem 7. ■

In Theorem 7 there was the restriction $n > 20$. The same techniques can be used to show that $p_2(n,3) = \lceil\frac{11n}{10}\rceil$ for $4 \leq n \leq 16$, and that $p_2(n,3) = \lceil\frac{11n}{10}\rceil + 1$ for $17 \leq n \leq 20$.

The minimal graphs of Theorem 7 have vertices of degree one. Therefore, if only graphs of minimal degree at least two are considered, then one would expect that the minimal number of edges in a 2-path pairable graph of maximum degree three will be greater. The following Proposition 8 verifies this. The proof of Theorem 8 is identical to that of Theorem 7, except that the step of adding vertices of degree one is not used, and only the step of subdividing the edges is needed. Thus the proof is left to the reader.

Proposition 8. *For $n > 12$, $p_2(n,3,2) = \lceil\frac{7n}{6}\rceil$.*

In Proposition 8 the restriction $n > 12$ is necessary. However, the same techniques can be used to show that $p_2(n,3,2) = \lceil\frac{7n}{6}\rceil$ for $4 \leq n \leq 10$, and $p_2(n,3,2) = \lceil\frac{11n}{10}\rceil + 1$ for $11 \leq n \leq 12$.

With Proposition 8 we can determine a more general result about minimal 2-path pairable graphs.

Theorem 9. *For $\Delta \geq 3$ a fixed positive integer and $n \geq 12\Delta - 15$,*

$$p_2(n,\Delta) = n + \lceil r/6 \rceil,$$

where r is the minimum integer such that

$$r\Delta - 2p_2(r,3,2) \geq n - r. \tag{1}$$

Proof. Let G be a 2-pairable graph of order n and maximum degree at most Δ that has $p_2(n,\Delta)$ edges. Delete every vertex of degree one from G, and assume that the obtained graph G' has r' vertices and q edges. Note that because of the forbidden structures in 2-path pairable graphs, G' has no vertices of degree one. Since each vertex in G has degree at most Δ and there are $n - r'$ vertices of degree one in G, a graph G' can be obtained from G by removing all of the vertices of degree one iff r' satisfies the following inequality

$$r'\Delta - 2q \geq n - r'. \tag{2}$$

If all of the vertices of G' have degree two or three, then clearly $q = p_2(r',3,2)$.

Since r satisfies (2) and $n \geq 12\Delta - 15$ implies that $r > 12$, we can apply Proposition 8 to obtain the required upper bound on $p_2(n,\Delta)$. Thus an optimal graph on r vertices has $q = \lceil 7r/6 \rceil$ edges, so for the number of edges in G we have

$$p_2(n,\Delta) = n - r + q \leq n + \lceil r/6 \rceil.$$

To obtain the same lower bound we will study G' further. If $q \geq \lceil 7r'/6 \rceil$, then r' also satisfies (1), and we have $r' \geq r$. Thus $p_2(n,\Delta) = n - r' + q \geq n + \lceil r/6 \rceil$ follows.

Hence we will assume that $q < \lceil 7r'/6 \rceil$. Let G^* be the graph obtained from G' by replacing each of the suspended paths by an edge. Thus, G^* is a graph of minimal degree three. Let s and t be the number of vertices and edges in G^* respectively. If a is the number of subdivisions of edges (some edges are subdivided twice) needed to obtain G' from G^*, then $r' = s + a$ and $q = t + a$. Since $t + a < \lceil \frac{7(s+a)}{6} \rceil$ is assumed, $t + a' < \lceil \frac{7(s+a')}{6} \rceil$ also holds for any $a' \geq a$.

Let b be the number of vertices of G^* of degree three. Because of the forbidden subgraphs, no more than one edge incident to a vertex of degree three can be double subdivided. Thus $a \leq 2t - b$, and so $t + 2t - b \leq 7(s + 2t - b)/6$. It follows that $t \leq (7s - b)/4$. On the other hand, since G^* has minimal degree at least 3, we have $t \geq (3b + 4(s - b))/2 = 2s - b/2$. This implies that $(7s - b)/4 \geq 2s - b/2$. Therefore $b = s$, G^* is a 3-regular graph, and $t = 3s/2$. Again, because of the forbidden subgraphs (no more than one edge incident to a vertex of degree three can be subdivided), $a \leq 2s$, which implies $t + a \geq \lceil \frac{7(s+a)}{6} \rceil$. This gives a contradiction, and concludes the proof of Theorem 9. ∎

Let r be defined as in the previous theorem, and assume that $r = 6m - p$, with $0 \leq p \leq 5$. Then by (1),

$$\left\lceil \frac{r}{6} \right\rceil = \left\lceil \frac{(3n + 7p)}{6(3\Delta - 4)} \right\rceil .$$

Hence Theorem 9 has the corollary that

$$p_2(n,\Delta) = \left(\frac{6\Delta - 7}{6\Delta - 8}\right) n + \alpha,$$

where $0 \leq \alpha < 13/6$ is the remainder term, which proves Theorem A in the introduction.

3.2 3-path pairable graphs

There are many forbidden induced subgraphs for any graph G that is 3-path pairable. We start by describing 11 such forbidden induced subgraphs. The Cut Condition implied that a vertex of degree one cannot be adjacent to a vertex of degree two. Also by the Cut Condition a vertex of degree three or four cannot be adjacent to two vertices of degree one. We have already observed that G cannot contain a suspended path with five vertices. Thus, the first three forbidden structures follow from the Cut Condition.

The remaining forbidden subgraphs do not follow from the Cut Condition. The next four forbidden subgraphs deal with how many edges incident to a vertex in a 3-path pairable graph can be subdivided. The three edges incident to a vertex of degree three cannot be subdivided. This follows from the fact that if the vertex of degree three is paired with a vertex that it is not adjacent to, then any path between this pair of vertices destroys the possibility of the vertex on the subdivided edge used in this path to be paired with another vertex.

Similar reasoning implies that all of the edges incident to a vertex of degree four cannot be subdivided. These are the next two forbidden subgraphs. The subgraph obtained by subdividing one edge and double subdividing a second edge of the edges incident to a vertex of degree three is forbidden. A pairing for which the required paths cannot be found is when the vertex of degree three is paired with the vertex at a distance two on the double subdivided edge, and the other vertices on subdivided edges are paired with other vertices in the graph. For similar reasons the graph obtained by double subdividing two edges and subdividing a third edge of the edges incident to a vertex of degree four is a forbidden subgraph.

The forbidden induced graphs imply the following properties. In a graph G that is 3-path pairable there are no triple subdivided edges. On the edges incident to a vertex of degree three at most two vertices can be on subdivided edges or the 3-path pairable property is lost. Also, at most four vertices can be on subdivided edges incident to a vertex of degree four. There are no restrictions for vertices of degree five or more.

The next four forbidden subgraphs deal with the distance between vertices of degree three which are incident to subdivided edges. Before describing these forbidden subgraphs we need some additional notation.

The *weight* of a vertex v of degree three will refer to the number of subdivisions of edges incident to the vertex v. Since a vertex of degree three in a 3-path pairable graph can have at most two incident edges subdivided or one edge subdivided, the posssible weights are 0, 1 or 2. Those vertices of weight 2 with two edges subdivided with be called *type 1* and those with a single edge double subdivided will be called *type 2.* The first forbidden subgraphs is a result of the fact that a vertex v of degree three that is of weight 2 and type 1 cannot be adjacent any vertex of degree three of weight at least 1. Also, such a vertex v cannot be at distance two from a vertex u of degree three and weight 2. This situation gives three more forbidden graphs.

The first case is when u is a type 2 vertex, and the last two cases are when u is a type 1 vertex and the path from u to v contains either a vertex of degree three or a vertex of degree two. Vertification that these subgraphs are forbidden are similar to the previous cases mentioned.

With this information on forbidden subgraphs for 3-path pairable graphs, and with the examples of section 2, we are prepared to prove the following theorem.

Theorem 10. *For $n \geq 16$,*

$$\left\lceil \frac{9n}{7} \right\rceil \leq p_3(n,3,2) \leq \left\lceil \frac{4n}{3} \right\rceil + 1.$$

Proof. Assume $n = 3m - p$ for $m \geq 7$ and odd $(0 \leq p \leq 5)$. Then from Proposition 5 (replacing p suspended paths of length two in the graph G^*_{2m} with just edges) we have a 3-path pairable graph with $3m - p$ vertices, $4m - p$ edges. This graph has at most $\lceil \frac{4n}{3} \rceil + 1$ edges for every $n \geq 16$.

To complete the proof it is sufficient to show that any graph G of order n that is 3-path pairable and has no vertices of degree one has at least $\lceil \frac{9n}{7} \rceil$ edges. We will first partition the vertices of G into five parts N_0, N_1, N_2, N_3, and N_4, where N_0 and N_1 are the vertices of degree three and weight 0 and 1 respectivly, N_2 are the weight 2 and type 1 vertices, N_3 are the weight 2 and type 2 vertices, and N_4 are the remaining vertices which have degree two. For $0 \leq i \leq 4$, let $n_i = |N_i|$. The number of vertices n_4 of degree two in G is determined by the weights of the vertices of degree three of weight at least 1, and so

$$n_4 = \frac{n_1 + 2n_2 + 2n_3}{2}.$$

Our first objective is to give an upper bound on n_4. We claim $n_4 \leq 3n/7$. The forbidden subgraphs imply that each vertex of degree three, weight 2, and type 1 is adjacent to a vertex of weight 0, and no other vertex of degree three and weight 2 is adjacent to this weight 0 vertex. Hence $n_0 \geq n_2$.

Again partition the vertices of N_3 into N'_3 and N^*_3 where N'_3 are those vertices adjacent to a vertex of weight 0 and N^*_3 the remaining vertices. Let n'_3 and n^*_3 be the number of vertices in these sets respecively. Each vertex v in N'_3 is naturally associated with a vertex in N_0, (possibly 3 vertices in N'_3 could be associated with the same vertex in N_0), so we now have $n_0 \geq n_2 + n'_3/3$. The forbidden structures imply that each vertex in N_3^* is adjacent to 2 vertices in N_1. Since no vertex in N_1 can be adjacent to more than 2 vertices in N_3^*, we have $n_1 \geq n_3^*$. Therefore we have

$$n_4 = \frac{n_1}{2} + n_2 + n_3' + n_3^* = \frac{n_1}{2} + \frac{n_2 + n_3' + n_3^*}{4} + 3\frac{n_2 + n_3' + n_3^*}{4}.$$

Since $\frac{n_3^*}{4} \leq \frac{n_1}{4}$ and $\frac{n_2+n_3'}{4} \leq 3\frac{n_0}{4}$, this gives

$$n_4 \leq \frac{3}{4}(n_0 + n_1 + n_2 + n_3' + n_3^*) = \frac{3}{4}(n - n_4).$$

This implies that $n_4 \leq 3n/7$, which is the bound on n_4 claimed earlier.

The number of edges in G is at least

$$\frac{2\frac{3n}{7} + 3\frac{4n}{7}}{2} = \frac{9n}{7}.$$

This completes the proof of Theorem 10. ■

Note that if in the proof of Theorem 10 there are no double subdivided edges ($n_3 = 0$), then a sharper lower bound can be obtained. In this case, using the notation of Theorem 10, we have

$$n_4 = (n_1 + 2n_2)/2 \leq (n_0 + n_1 + n_2)/2 = (n - n_4)/2.$$

This implies that $n_4 \leq n/3$, and that G has at least $4n/3$ edges. The following theorem will make use of the note.

Theorem 11. *For* $n \geq 21$, $p_3(n,3) = \lceil \frac{5n}{4} \rceil + \alpha$, *where* $\alpha = 0$ *or* 1.

Proof. The existence of a 3-path pairable graph G of order n and maximum degree 3 with $\lceil \frac{5n}{4} \rceil + \alpha$ edges follows from Proposition 6 using the the class of graphs $G_p^*(n)$ for an appropriate p.

Let G be a 3-path pairable graph of order n, size m, and maximum degree 3. Consider the case when G has a suspended path with two vertices of degree two. Suppose (x_1, x_2, x_3, x_4) is the path; thus, x_2 and x_3 have degree two and the other vertices have degree three. It is straightforward to verify that if x_2 and x_3 are replaced by x^* and x', where x^* is adjacent to x_1, x_4, and x', then you get a graph of the same order and size that is 3-path pairable. Therefore, we can assume that G has no suspended paths with two interior vertices of degree two. It also follows from the Cut Condition that there is no suspended paths in the graph G^* that remains after deleting all the vertices of degree one from G.

If s is the number of vertices of degree one in G, then G^* is a 3-path pairable graph of order $n - s$ and size $m - s$ with only vertices of degree two and three. Also, by the observation of the previous paragraph, we can assume that G^* has no double subdivided edges. By the note after Theorem 10, $m - s \geq \lceil \frac{4(n-s)}{3} \rceil$. Hence, G^* has at most $(n-s)/3$ vertices of degree three. Since a vertex of degree one can only be attached to vertices of degree two, we must have $s \leq (n-s)/3$. This implies $s \leq n/4$. Therefore

$$m \geq s + 4(n-s)/3 = 4n/3 - s/3 \geq 4n/3 - n/12 \geq 5n/4.$$

This completes the proof of Theorem 11. ■

The note after Theorem 10 will also be the basis for determining a more general result about minimal 3-path pairable graphs.

Theorem 12. *For $\Delta \geq 4$ a fixed positive integer and $n \geq 10\Delta - 21$,*

$$p_3(n, \Delta) \geq n + \lceil r/2 \rceil,$$

where $r = \lceil n/(\Delta - 2) \rceil$.

Proof . Let G be a 3-pairable graph of order n and maximum degree at most Δ with a minimal number of edges. Let G' be the graph obtained from G by deleting all of the vertices of degree one. We can suppose that G' does not contain any double subdivided edges. Assume G' has s vertices and $s + \lceil s/2 \rceil - C$ edges.

Let n_i be the number of vertices of G' of degree i for $2 \leq i < s$. Because of the forbidden structures in 3-path pairable graphs, G' has no vertices of degree one. Note also that each vertex of degree two in G' can be adjacent to at most one vertex of degree one in G. Since $\Delta \geq 4$,

$$n - s \leq n_2 + (\Delta - 3) \sum_{i=3}^{s-1} n_i \leq (\Delta - 3)s.$$

Hence $s \geq \lceil n/(\Delta - 2) \rceil = r$. In the case of $C \leq 0$ we obtain from this that the number of edges of G is $n + \lceil s/2 \rceil - C \geq n + \lceil r/2 \rceil$. Thus we can assume that $C > 0$.

Let n_3' be the number of vertices of degree three in G' that are not adjacent to any vertex of degree one in G. We now have the following inequality regarding the number of edges in G.

$$\left(\sum_{i=4}^{s-1} \Delta n_i \right) + \Delta(n_3 - n_3') + 3(n_2 + n_3') \geq n + 2s - 2C.$$

Since $\sum_{i=2}^{s-1} n_i = s$, the previous inequality reduces to the following inequality.

$$(\Delta - 2)s - (\Delta - 3)(n_2 + n_3') \geq n - 2C. \tag{3}$$

Using the fact that $\sum_{i=2}^{s-1} i n_i = 3s - 2C$, and the fact that $3s = \sum_{i=2}^{s-1} 3n_i$, we have that $n_2 = 2C + \sum_{i=4}^{s-1}(i-3)n_i$.

If a vertex of degree two in G' is adjacent to two vertices of degree three in G', then both the vertices of degree three cannot be adjacaent to a vertex of degree one in G, by the Cut condition. Therefore, each vertex of degree two is adjacent to either a vertex of degree at least four in G', or it is adjacent to a vertex of degree three in G' that is not adjacent to any vertex of degree one. Associate with each vertex of degree two one of these special vertices of degree three or a vertex of degree at least four. From the forbidden structures we know that no vertex of degree three can be adjacent to three vertices of degree two and no vertex of degree four can be adjacent to four vertices of degree two.

This implies that $\sum_{i=4}^{s-1}(i-3)n_i + n_3' \geq n_2/3$. The extreme case of the previous inequality is the one in which all of the vertices of degree two are associated with vertices of degree four. A stronger inequality would result in any other case. Hence $n_2 + n_3' \geq 2C + n_2/3$. Since G' has s vertices and at most $3s/2 - C$ edges, we must have $n_2 \geq 2C$. Thus, we have $n_2 + n_3' \geq 8C/3$. It follows from inequality (3) that

$$s \geq \frac{n - 10C + \frac{8C}{3}\Delta}{\Delta - 2} \geq \frac{n}{\Delta - 2} + 2C\frac{4\Delta - 15}{3\Delta - 6} \geq \frac{n}{\Delta - 2} + 2C$$

holds for $\Delta \geq 9$.

Hence $s \geq r + 2C$, and $\lceil s/2 \rceil - C \geq \lceil r/2 \rceil$ implies that G has at least $n + \lceil r/2 \rceil$ edges. This concludes the proof of Theorem 12. ■

Combining Theorems 4 and 12 we obtain

$$\left(\frac{2\Delta - 3}{2\Delta - 4}\right) n \leq p_3(n, \Delta) \leq \left(\frac{2\Delta - 3}{2\Delta - 4}\right) n + 1.5,$$

which proves Theorem B in the introduction. Notice that $p_3(n, \Delta)$ equals the lower bound for infinitely many n.

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4. REFERENCES

[CL] *G. Chartrand and L. Lesniak:* **Graphs and Digraphs**, 2nd edition, Wadsworth and Brooks/Cole, Belmont, 1986.

[FGL] *R.J. Faudree, A. Gyárfás and J. Lehel:* Three — regular path pairable graphs, manuscript.

[H] *A. Huck:* A sufficient condition for graphs to be weakly k-linked, manuscript.

The proof of Propositions 3 and 5

Proposition 3. *For $p \geq 10$, the graphs obtained from F_p by adding vertex disjoint stars to its vertices are 3-path pairable.*

Proof. To verify the 3-path pairable property of a graph assume that we have three pairs of vertices labeled a, a', b, b', c, c', and denote this set of vertices by X.

A. We will verify first that F_p is 3-path pairable for $p \geq 10$ and even. We will start with some observations that will allow us to reduce the order of the F_p considered.

I. If uv is a chord of F_p and both $u, v \notin X$, then the deletion of u and v and the appropriate addition of two edges between the vertices of degree 2 gives an F_{p-2}. If F_{p-2} is 3-path pairable, then these paths can be used to obtain the required paths in F_p. Hence, we can assume that each chord in F_p has an endvertex in X.

II. If aa' is a chord in F_p, then remove aa' from F_p. The graph obtained in this way is 2-path pairable since it is equivalent with a F_{p-2}. The paths in this 2-path pairable graph can be used together with aa' to get the 3 required paths. Thus, we can assume that aa', bb', and cc' are not chords in F_p.

We will call a chord of F_p *full* if both of its endpoints belong to X. We will now consider five cases that depend on the number of full chords in F_p.

Case A.1: There are 3 full chords in F_p.

According to I and II, the only possibility to be considered is $n = 6$, and the order of the vertices is a, b, c, b', c', a'. Then $[b, b']$, $[a, a']$ and $[c', a']$ along with the chord ca' are the required 3 paths.

Case A.2: The only full chord of F_p is ab.

Let a^* and b^* be the endvertices of the chords containing a' and b' respectively, such that $[a, a^*] \cap [b^*, b] = \emptyset$. After removing the chords at a^* and b^* and the edges of $[a, a^*]$ and $[b^*, b]$ the remaining graph is still connected. Thus any path from c to c' in this remaining graph, the path $[a, a^*]$ (with a^*a' if $a^* \neq a'$), and the path $[b, b^*]$ (with b^*b' if $b^* \neq b'$) are the required edge disjoint paths.

Case A.3: There are no full chords in F_p.

According to I, $p = 12$, and we can find two chords, say aa^* and bb^*, such that ab^* and a^*b are edges of the cycle F_{12}. Now, by identifying a with b^* and b with a^* we obtain an F_{10} with just one full chord ab, which was settled in Case A.2. The 3 paths in F_{10} define the required path system in F_{12}.

Case A.4: F_p has two full chords ab and $a'b'$.

With no loss of generality we can assume that $b \in [a', b']$. After removing $[b, b']$, $[a', b]$ and ba from F_p, the remaining graph is connected and contains the third path from c to c'.

Case A.5: F_p **has two full chords.**

According to I, $n = 8$. We can suppose that Case A.4 does not apply, so the chords are ab and $a'c$. Assign the same label b or c to both endvertices of the chord containing b' or c', respectively. Then we have some cyclic ordering of the labels a, a', b, b, b, c, c, c along the cycle of F_8. Suppose that there exist two pairwise disjoint paths on the cycle between two b's and between two c's, such that both are disjoint from the path $[a, a']$. Then, these paths (possibly with the chords at b' and c') clearly define the required edge disjoint paths in F_8. One can easily check that there is only one ordering, namely

$$a, c, b, a', b, c, b, c$$

for which the above argument does not apply. In this case (a, b, a'), (b, b, c, b) and (c, c, a, c) are paths that contain the required paths. This concludes the proof of this case and the fact that F_p is 3-path pairable.

B. Let G be a graph obtained from F_p by adding vertex disjoint stars to the vertices of degree three. We will verify that G is 3-path pairable for $p \geq 10$.

For a vertex $x \in X$ of degree one assigne the same label x to its neighbor and remove every vertex of degree one from G. When we have the required three edge disjoint paths in F_p we only have to add the necessary pendant edges to get the required paths in G. In particular, if a vertex is labelled with both members of a pair, then the path between them will consist of pendant edge(es) at that vertex. One can assume that this situation doesn't occure, consequently a vertex has at most three labels. If the proof in **A** does not work, then with no loss of generality we can suppose that a and b are labels of the same vertex u. Denote by u^* the other vertex of the chord at u.

Case B.1: $[a', b'] \subset [u, u^*]$.

Let c^* be the other vertex of the chord at vertex c if $c \notin [u^*, u]$, and let $c^* = c$ otherwise. Define c'^* for c' in a similar way. Then the path between c^* and c'^* contained by $[u^*, u]$ together with the chords at c^* and c'^* (if necessary) defines the path for c's. The other two edge disjoint paths are $[u, a']$ and $[b', u^*] \cup u^*u$.

Case B.2: $b' \in (u, u^*)$ and $a' \in (u^*, u)$.

If the other endvertex of the chord at a' has neither c nor c' as a label, then denote it by a'^*, and apply Case 2 with a'^* in the role of a'. By symmetry, the same argument can be used for b'. Thus the only case we have to check is when $a'c$ and $b'c'$ are chords. If the paths (u, b'), (a', u) and (c, c') are pairwise disjoint, then we are done. Otherwise the three required paths are (u, b'), $uu^* \cup (u^*, a')$ and $ca' \cup (a', c')$. ■

Proposition 5. *For $p = 4m + 2$ and $m \geq 3$, G_p^* is 3-path pairable.*

Proof. We will verify that G_p^* is 3-path pairable for $p = 4m + 2$ and $m \geq 3$. The chords that have been subdivided or double subdivided will be referred to in G_p^* as *chord paths.*

Let a, a', b, b' and c, c' be three pairs of vertices in G_p^*. Set $X = \{a, a', b, b', c, c'\}$. If every vertex of a chord path P belongs to X then P is called *full*, and if $P \cap X = \emptyset$ then P is called *empty*.

Case a: every chord path contains at most one vertex from X.

If a chord path contains $x \in X$ then its endvertex in C_i is denoted by x_i $(i = 1, 2)$. We suppose that the cyclic ordering of the a's and b's on C_2 is a_2, b_2, a'_2 and b'_2. Otherwise, the pairwise edge disjoint paths $[a_2, a'_2], [b_2, b'_2]$ and $[c_1, c'_1]$ together with the distinct chord paths at their endvertices define the required three paths.

With no loss of generality we can assume that the length of $[b_1, b'_1]$ is larger than $n/4$. If $a_1, a'_1 \notin [b'_1, b_1]$, in particular if $b_1 b'_1$ is an edge of C_1, then the pairwise edge disjoint paths $[a'_1, a_1], [b'_1, b_1]$ and $[c'_2, c_2]$ together with the attached chord paths define the required three paths.

Thus we can suppose that $a_1 \in [b_1, b'_1]$ and $a'_1 \in [b'_1, b_1]$; in particular, $[b'_2, b_2]$ contains two consecutive inner vertices of C_2, say a'_2 and x_2. Let x_1 be the other endvertex of the chord path P at x_2. Clearly $x_1 \in [b_1, b'_1]$; hence, the paths $[b'_1, b_1]$ and $[a_1, x_1]$ are disjoint. This last path together with P and $a'_2 x_2$ define the path for a's, and the third path will be obtained on C_2.

Case b: no chord path is full.

If Case a doesn't apply then some chord path P_1 contains two vertices of X. For the case $a, a' \in P_1$ the required three paths can be found easily. Suppose from now on that a's are on distinct chords and the same is true for b's and c's. With no loss of generality we can assume that $a \in C_2 \cap P_1$ and $b \in P_1$.

Let $[x, y]$ be the maximal cycle path of C_2 containing a such that every chord path at an inner vertex of $[x, y]$ different from a is empty.

We label the endvertices of every nonempty chord path P as follows. If u is the only vertex of $P \cap X$ then the endvertex in C_i is denoted by u_i $(i = 1, 2)$. If $u, v \in X$ are on $P = (u, v, z)$ with $u \in C_i$ then the endvertices of P are $u_i = u$ and $v_j = z$ $(\{i, j\} = \{1, 2\})$. By symmetry, we have to distinguish between the following essentially different cases.

Case b.1: $x = a'_2$ and $y = b'_2$.

Then $P_1 \cup [a'_2, b'_2]$ together with the chord paths containing a' and b' define edge disjoint paths for the a's and b's. The third path from c to c' can be found in the connected graph we get after removing the edges of the previous two paths from G_p^*.

Case b.2: $x = a'_2$ and $y = c_2$.

Let P_2 and P_3 be the chord paths at a'_2 and c_2.

Case b.2.1: X is not in $P_1 \cup P_2 \cup P_3$.

If b' is contained by a new chord path P_4 with endvertex b'_2, then $C_2 \cup P_1 \cup P_2 \cup P_4$ define edge disjoint paths for a's and b's. If c'_2 is the endvertex of a chord path P_4, then there are edge disjoint paths for a's and c's on $C_2 \cup P_2 \cup P_3 \cup P_4$. In both cases the third required path can be obtained on C_1.

Case b.2.2: $X \subset P_1 \cup P_2 \cup P_3$.

Clearly, $c' \in P_2$ and $b' \in P_3$. If $a' \in C_2$, then $P_1 \cup C_1 \cup P_3$ and $P_3 \cup [c_2, a'] \cup P_2$ contain edge disjoint paths for b's and c's that are edge disjoint from $[a', a]$.

If $b' \in C_1$, then let P_4 be a new chord path with endvertex $u \in C_1$. With no loss of generality we can assume that $[c', b']$ and $[b', u]$ are edge disjoint. Thus $C_2 \cup [b', u] \cup P_4 \cup P_1 \cup P_2$ contains the paths for a's and b's. The third one can be found on $[c', b'] \cup P_3$.

The remaining case when $P_2 = (a'_2, a', c')$ and $P_3 = (c, b', b'_1)$ can be verified easily by using the fact that $p > 10$.

Case b.3: $x = c'_2$ and $y = c_2$.

Let P_2 and P_3 be the chord paths containing c' and c, respectively. If $X \subseteq P_1 \cup P_2 \cup P_3$ then the argument in Case b.2.2 can be used after permuting the labels in X and/or exchanging the role of C_1 and C_2. Assume now that $P_3 \cap X = \{c\}$.

Case b.3.1: P_2 contains only c' from X.

Then $P_3 \cup [c'_1, c_1] \cup P_2$ contains a path for c's. The other two required paths are obtained on the union of P_1, C_2 and the chord paths containing a' and b', except the case when a' and b' belong to the same chord path P_4. With no loss of generality we can assume that $P_4 \cap C_1 = \{a'\}$.

If c_1 and c'_1 are consecutive on C_1, then there is a path $[a', u]$ of C_1 disjoint from $[c'_1, c_1]$ such that the chord path P_5 at u is empty. Then $C_2 \cup [a', u] \cup P_5 \cup P_4 \cup P_1$ contains the paths for a's and b's. The third path is contained in $P_2 \cup [c'_1, c] \cup P_3$.

If c_1 and c'_1 are not consecutive on C_1, then either $a+1$ or $a-1$ is an interior vertex of $[c'_2, c_2]$, say $u = a+1$ is the endvertex of an empty chord path P_5. Then $P_3 \cup [c_2, c'_2] \cup P_2$ contains a path for c's and $C_1 \cup [a, a+1] \cup P_5 \cup P_4 \cup P_1$ contains the required paths for the a's and b's.

Case b.3.2: P_2 contains two vertices of X.

If $c' \notin C_2$, then essentially the same argument works as in Case b.3.1. If $a' \in C_2$ we can get easily the required three paths. The argument similar to that in Case b.3.1 handles the case when $[a'_2, c_2]$ contains an interior vertex different from c' and a. The remaining case when a'_2, c', a and c_2 are consecutive vertices on C_2 can be verified by using the fact that $p \geq 14$.

Case b.4: $x = b'_2$ and $y = c_2$.

Let P_2 and P_3 be the chord paths containing b' and c, respectively.

Case b.4.1: $X \subset P_1 \cup P_2 \cup P_3$.

If one of the cycles C_2 and C_1 contains three vertices of X, say $b', a, c \in C_2$, then the three required paths are contained by $[a,c] \cup P_3, [b',a] \cup P_1$ and $[c,b'] \cup P_2$. By symmetry, we can assume that $b', a \in C_2$ and $c \notin C_2$. Since $p \geq 14$, one of the vertices $b'-1, b'+1, a-1$ and $a+1$ is such that the chord path P_4 at that vertex is empty. With no loss of generality we can assume that $a+1 \in P_4$. Than the path for a's is contained in $[a, a+1] \cup P_4 \cup C_1$, and the paths for b's and c's are the same as before.

Case b.4.2: X is not in $P_1 \cup P_2 \cup P_3$.

We can assume that $P_2 \cap X = \{b'\}$. Let P_4 be the chord path containing c'. If a' doesn't belong to $P_3 \cup P_4$, then for the a's and b's there are edge disjoint paths in $C_2 \cup P_1 \cup P_2 \cup P_5$, where P_5 is the chord path containing a'. The path for the c's is contained by $P_3 \cup [a'_1, c'_1] \cup P_4$.

Suppose now that $a' \in P_3$ and let P_5 be an empty chord path with endvertex $u \in C_1$. With no loss of generality we can assume that $[u, a'_1]$ and $[a'_1, c'_1]$ are edge disjoint. Then $C_2 \cup P_1 \cup P_2 \cup P_5 \cup [u, a'_1]$ contains paths for a's and b's, and the third required path can be found in $P_3 \cup [c_1, c'_1] \cup P_4$.

Case c: G_p^* has just one full chord path.

We omit the details of the argument which is very similar to that of Case b.

Case d: G_p^* has two full chord paths.

If a pair of X, say a and a', are consecutive vertices on a chord path, then the paths for the b's and c's that avoid aa' can be found easily.

Assume that one of the full chord paths is (a,b,c), with $a \in C_2$. If the second chord path is different from (c', b', a'), with $c' \in C_2$, then it is straightforward to get the required three paths. To handle the missing case, with no loss of generality we can assume that $[a, c']$ contains at least two interior vertices. Then the empty chord paths P_1 and P_2 at $a+1$ and at $c'-1$ are distinct. Let a^* and c^* be the other endvertices of P_1 and P_2. Clearly $[a^*, a']$ and $[c, c^*]$ are edge disjoint on C_1. The three required paths are $[a, a+1] \cup P_1 \cup [a^*, a']$, $[c'-1, c'] \cup P_2 \cup [c, c^*]$ and $[a', c] \cup \{bc, b'a'\}$.

The remaining case when the two full chord paths are (a, b, a') and (c, b', c') can be verified easily.

This proves the fact that G_p^* is 3-path pairable. ∎