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Hamiltonian Cycles Containing Ordered Linear Forests

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Abstract

Given integers k, s, t with $0 \leq s \leq t$ and $k \geq 0$, a (k, t, s)-linear forest F is a graph that is the vertex disjoint union of t paths with a total of k edges and with s of the paths being single vertices. A graph G of order n is (k, t, s)-hamiltonian if for any (k, t, s)-linear forest F there is a hamiltonian cycle containing the linear forest F. If the paths of the forest F are required to appear on the hamiltonian cycle in a specified order, then the graph is said to be (k, t, s)-ordered hamiltonian. If, in addition, each path in the system is oriented and must be traversed in the order of the orientation, then the graph is said to be strongly (k, t, s)-ordered hamiltonian. Minimum degree conditions and minimum sum of degree conditions for nonadjacent vertices that imply a graph is (strongly) (k, t, s)-ordered hamiltonian are proved. Examples showing that these constraints are best possible are provided.

1 INTRODUCTION

In this paper we will deal only with finite graphs without loops or multiple edges. Notation will be standard, and we will generally follow the notation of Chartrand and Lesniak in [CL96]. For a graph G we will use G to represent the vertex set $V(G)$ and the edge set $E(G)$ when the meaning is clear from the context. Given a subset (or subgraph) H of a graph G and a vertex v, let $d_H(v)$ denote the degree of v relative to H. Given a subset H of vertices of a graph G, the subgraph induced by H will also be denoted by H. Thus, for example, $G - H$ will denote a set of vertices in G not in H as well as a subgraph spanned by these vertices, depending on the context. The connectivity of G with be denoted by $\kappa(G)$.

Various degree conditions have been investigated which imply that a graph has hamiltonian type properties. The most common condition involves the minimum degree of a graph G, denoted by $\delta(G)$. Another condition studied extensively is the sum of degrees of nonadjacent vertices. For a graph G, $\sigma_2(G) \geq p$ means that $d(u) + d(v) \geq p$ for each pair u and v of nonadjacent vertices in G.

The classical results of Dirac [D52] and Ore [O60] gave degree conditions that imply the existence of a spanning cycle, and so the graph is hamiltonian. These classical results were generalized in many ways, but we will center our attention on two generalizations. The first generalization we will consider was introduced by Posa in [P64] by considering degree conditions that imply the existence of hamiltonian cycles that contained specified edges, or more generally specified vertex disjoint paths (linear forests). This result is stated in Section 2. This leads to a series of definitions which will formalize this concept.

Definition 1 : *Let $k \geq 0$, $t \geq 1$, and $0 \leq s \leq t$ be fixed integers. A* (k, t, s)-**linear forest** *F is a vertex disjoint union of t paths with a total of k edges and with s of the paths just a single vertex. When the number of single vertex paths is not critical for F, it will be denoted as simply a (k, t)-linear forest.*

Definition 2 : *Let $k \geq 0$ and $t \geq 1$ be fixed integers. A graph G is* (k, t, s)-**hamiltonian** *if for each (k, t, s)-linear forest F of G, there is a hamiltonian cycle of G containing F.*

The next generalization of the classical results of Dirac and Ore we will consider is a consequence of the introduction of the concept of *k-ordered (hamiltonian)* by Ng and Schultz in [NS97].

Definition 3 : *A graph G is k-ordered (hamiltonian) if given any ordered set S of k vertices, there is a (hamiltonian) cycle that contains S and the vertices of S are encountered on the cycle in the specified order.*

Conditions on $\delta(G)$ and $\sigma_2(G)$ that imply a graph G is k-ordered or k-ordered hamiltonian can be found in [KSS99] and [FGKLSS03], and the results are stated in Section 2. The concept of k-ordered was extended from vertices to linear forests in [CFGJLP04]. The following two definitions form the basis of the extension.

Definition 4 : *Let $k \geq 0$ and $0 \leq s \leq t$ be fixed integers. A (k,t,s)-linear forest F is said to be* **ordered** *if the set of paths is ordered, and the linear forest is* **strongly ordered**, *if in addition, there is an orientation on each of the paths.*

Definition 5 : *Let $k \geq 0$ and $0 \leq s \leq t$ be fixed integers and G be a graph of order n. A graph G is* **(k,t,s)-ordered** *if for each ordered (k,t,s)-linear forest F of G, there is a cycle C of G containing F that encounters the paths in the correct order. If F is also strongly ordered, and the orientation of each path is respected as it appears on C, then G is* **strongly (k,t,s)-ordered**. *If the cycle C is hamiltonian, then G is said to be* **(strongly) (k,t,s)-ordered hamiltonian**.

In [CFGJLP04] sharp conditions on sum of degrees of nonadjacent vertices that imply a graph is strongly (k,t,s)-ordered hamiltonian were proved, and the results are stated in Section 2. The results in [CFGJLP04] will be extended in this paper to include minimum degree conditions that imply strongly (k,t,s)-ordered hamiltonian, and also minimum degree and sum of degrees of nonadjacent vertices conditions that imply (k, t, s)-ordered hamiltonian. More specifically, the following three theorems will be proved in Section 4.

The following result can also be considered as an extension of the main result in [KSS99] which implies a graph G of order n with $\delta(G) \geq \lceil n/2 \rceil + \lfloor t/2 \rfloor - 1$ is strongly $(0, t, t)$-ordered hamiltonian.

Theorem 1 : *Let $k \geq 1$ and $0 \leq s < t$ be integers, and G a graph of sufficiently large order n. The graph G is strongly (k, t, s)-ordered hamiltonian if $\delta(G)$ satisfies any of the following conditions:*

(i)	$\delta(G) \geq (n+k+t-3)/2$	*when* $t \geq 3$,
(ii)	$\delta(G) \geq (n+k)/2$	*when* $t \leq 2$.

Also, all of the conditions on $\delta(G)$ are sharp.

The next two results deal with the weaker condition of (k, t, s)-ordered hamiltonian, and so the strong requirement is dropped. In the following $\epsilon_{s,t} = 0, 1$ or 2 for $s = t$, $3(t-1)/4 \leq s < t$, or $0 \leq s < 3(t-1)/4$ respectively.

Theorem 2 : *Let $k \geq 1$ and $0 \leq s \leq t$ be integers, and G a graph of sufficiently large order n. The graph G is (k, t, s)-ordered hamiltonian if $\sigma_2(G)$ satisfies any of the following conditions:*

(i)	$\sigma_2(G) \geq n+k+t-5$	*when* $s = 0$ *and* $t \geq 5$,
(ii)	$\sigma_2(G) \geq n+k+t+s-6$	*when* $0 < 2s \leq t$, $s \geq 3$, *and* $t \geq 6$,
(iii)	$\sigma_2(G) \geq n+k+(3t-9)/2-\epsilon_{s,t}$	*when* $3 \leq t < 2s$, *and* $(s,t) \neq (2,3)$,
(iv)	$\sigma_2(G) \geq n+k+t+s-5$	*when* $0 < 2s \leq t$, $1 \leq s \leq 2$,
(v)	$\sigma_2(G) \geq n+k$	*when* $t \leq 2$, $t = 3$ *and* $s \leq 2$, *or* $t = 4$ *and* $s \leq 1$.

Also, all of the conditions on $\sigma_2(G)$ are sharp.

This next result can also be considered an extension the main result in [KSS99] dealing with the case when $k = 0$.

Theorem 3 : *Let $k \geq 1$ and $0 \leq s < t$ be integers, and G a graph of sufficiently large order n. The graph G is (k, t, s)-ordered hamiltonian if $\delta(G)$ satisfies any of the following conditions:*

(i)	$\delta(G) \geq (n+k+t-5)/2$	*when* $s = 0$ *and* $t \geq 5$,
(ii)	$\delta(G) \geq (n+k+t-4)/2$	*when* $s = 1$ *and* $t \geq 4$ *or* $s = 0$ *and* $t = 4$,
(iii)	$\delta(G) \geq (n+k+t-3)/2$	*when* $1 < s < t$ *and* $t \geq 3$ *or* $s = 0, 1$ *and* $t = 3$,
(iv)	$\delta(G) \geq (n+k)/2$	*when* $t \leq 2$.

Also, all of the conditions on $\delta(G)$ are sharp.

2 KNOWN RESULTS

In this section a summary of the results dealing with linear forests, k-ordered, and hamiltonian that lead to the main results Theorem 1, Theorem 2, and Theorem 3 are listed. The following result of Posa [P64] and extended by Kronk in [K69] implies the existence of a hamiltonian cycle containing a specified linear forest.

Theorem 4 : **Posa [P64]** *Let $0 \leq t \leq k$ be integers and G a graph of order n. If $\sigma_2(G) \geq n + k$ (or $\delta(G) \geq (n+k)/2$), then G is $(k, t, 0)$-hamiltonian. Also, both the σ_2 bound and the δ bound are sharp with respect to general n and general $(k, t, 0)$-linear forests.*

After the definition of k-ordered hamiltonian was introduced by Ng and Schultz [NS97], the following theorem was the first degree result with a sharp bound for k-ordered hamiltonian. The lower bound for n was improved by to $n \geq 5k + 6$ in [KY05].

Theorem 5 : [KSS99] *Let $k \geq 2$ and G a graph of order $n \geq 11k-3$. If $\delta(G) \geq \lceil n/2 \rceil + \lfloor k/2 \rfloor - 1$, then G is k-ordered hamiltonian. The condition on $\delta(G)$ is sharp.*

A corresponding k-ordered hamiltonian result using the degree sum condition, $\sigma_2(G)$, was proved in [FGKLSS03]. It is important to note that the $\delta(G)$ condition is not just one-half the $\sigma_2(G)$ condition.

Theorem 6 : [FGKLSS03] *Let k be an integer with $3 \leq k \leq n/2$. If $\sigma_2(G) \geq n + (3k-9)/2$, then G is k-ordered hamiltonian. The condition on $\sigma_2(G)$ is sharp.*

The extension from ordered sets of vertices on hamiltonian cycles to strongly ordered linear forests was accomplished in [CFGJLP04] for the sum of degree condition σ_2.

Theorem 7 : [CFGJLP04] *Let $k \geq 0$ and $0 \leq s \leq t$ be integers, and G a graph of order $n \geq \max\{179t + k, 8t^2 + t + k\}$. The graph G is strongly (k, t, s)-ordered hamiltonian if $\sigma_2(G)$ satisfies any of the following conditions:*

(i)	$\sigma_2(G) \geq n + k + t - 3$	*when* $s = 0$ *and* $t \geq 3$,
(ii)	$\sigma_2(G) \geq n + k + t + s - 4$	*when* $0 < 2s \leq t$ *and* $t \geq 3$,
(iii)	$\sigma_2(G) \geq n + k + (3t-9)/2$	*when* $3 \leq t < 2s$,
(iv)	$\sigma_2(G) \geq n + k$	*when* $t \leq 2$,

Also, all of the conditions on $\sigma_2(G)$ are sharp.

3 EXAMPLES AND PRELIMINARY RESULTS

This section begins with examples that will verify the sharpness of the main results, and illustrate the structure of the extremal graphs. Since most of these examples are variations of extremal examples of the results stated in Section 2, we also describe the examples used to verify the sharpness of the Theorems in Section 2.

EXAMPLES 1: The minimum degree condition that implies that a graph G is k-ordered is given in Theorem 5. The graph H_0 in Figure 1, which is $K_{2\lfloor k/2\rfloor-1} + (K_{\lceil (n-2\lfloor k/2\rfloor+1)/2\rceil} \cup K_{\lfloor (n-2\lfloor k/2\rfloor+1)/2\rfloor})$, verifies that Theorem 5 is sharp. The graph H_0 is not k-ordered and $\delta(G) \geq \lceil n/2\rceil + \lfloor k/2\rfloor - 2$. If consecutive vertices of the set $S = \{x_1, x_2, \cdots, x_k\}$ are in alternate components of $H_0 - A$, then a cycle encountering S in the appropriate order cannot exist.

The graph H_0' in Figure 2, which was given by Ng and Schultz [NS97], verifies that the degree condition in Theorem 6 cannot be reduced. There is no cycle containing the vertices $S = \{x_1, x_2, \cdots, x_k\}$

of H_0', since there is a set of $k-1$ vertices that separates alternate vertices of S from the remainder of the graph. Also, $\sigma_2(H_0') \geq n + (3k-10)/2$.

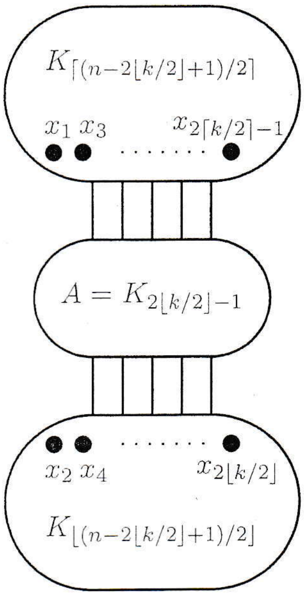

Figure 1: H_0

Theorem 7 extends the results in Theorem 6 from just vertices to linear forests and the linear forests allow for paths with a single vertex. The graphs verifying the sharpness of conditions (i), (ii), and (iii) of Theorem 7 are described as graphs H_1, H_2, and H_3 in Examples 2. The sharpness of condition (iv) is demonstrated in graph H_4.

EXAMPLES 2: Let F be a (k,t,s)-linear forest with x_i and y_i the endvertices of the i^{th} path of F for $1 \leq i \leq t$. Let $H_1 = K + (A \cup B)$, where $A = K_{\lceil \frac{n-k-t+2}{2} \rceil}$, $K = K_{k+t-2}$, and $B = K_{\lfloor \frac{n-k-t+2}{2} \rfloor}$. The degree sum $\sigma_2(H_1) = n+k+t-4$, and H_1 is not strongly (k,t,s)-ordered if $x_1 \in A$, $y_t \in B$, and $F - \{x_1, y_t\} \subseteq K$. When $s = 0$, H_1 is the extremal example for case (i).

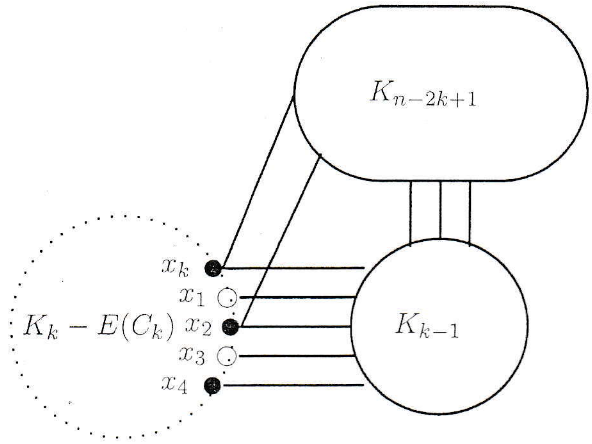

Figure 2: H_0'

For $t \geq 2s \geq 2$ let H_2 be the graph derived from $A+(T+(S \cup B))$, where $S = K_s, T = K_{k+t-s}, A = K_{2s-1}$, and $B = K_{n-k-t-2s+1}$. For each vertex $s_i \in S$, pick two vertices $u_i, v_i \in T$. Delete the edges s_iu_i, s_iv_i for each i to form H_2. We have $\sigma_2(H_2) = n+k+t+s-5$, but if we pick $F \subseteq S \cup T$, such that $x_{2i} = y_{2i} = s_i, x_{2i+1} = u_i, y_{2i-1} = v_i$ for all $i \leq s$, then H_2 is not strongly (k, t, s)-ordered.

For $2s > t$ let H_3 be derived from $A + (T + (S \cup B))$, where $S = K_{\lceil \frac{t}{2} \rceil}, T = K_{t+k-\lceil \frac{t}{2} \rceil}, A = K_{t-1}$, and $B = K_{n-k-2t+1}$. For every vertex $s_i \in S$, pick two vertices $u_i, v_i \in T$, with the exception that $v_{i+1} = u_i$ for $1 \leq i \leq s-\lceil \frac{t}{2} \rceil$. For each i delete the edges s_iu_i, s_iv_i between S and T to form H_3. We have $\sigma_2(G) = n+k+\left\lfloor \frac{3t}{2} \right\rfloor - 5$, but if we pick $F \subseteq S \cup T$, such that $x_{2i} = y_{2i} = s_i, x_{2i+1} = u_i, y_{2i-1} = v_i$ for all $i \leq \lceil \frac{t}{2} \rceil$, then H_3 is not strongly (k, t, s)-ordered.

For general t, but in particular for $t \leq 2$ of Theorem 7, the graph $H_4 = \overline{K}_{\lceil (n-k+1)/2 \rceil} + K_{\lfloor (n+k-1)/2 \rfloor}$ with the the linear forest F contained in $K_{\lfloor (n+k-1)/2 \rfloor}$ is not (strongly) (k, t, s)-ordered hamiltonian, and $\sigma_2(H_4) = 2\lfloor (n+k-1)/2 \rfloor$. Note also that $\delta(H_4) = \lfloor (n+k-1)/2 \rfloor$. Thus, for n and k of the same parity, $\sigma_2(H_4) = n + k - 1$ and $\delta(H_4) = (n + k - 1)/2$.

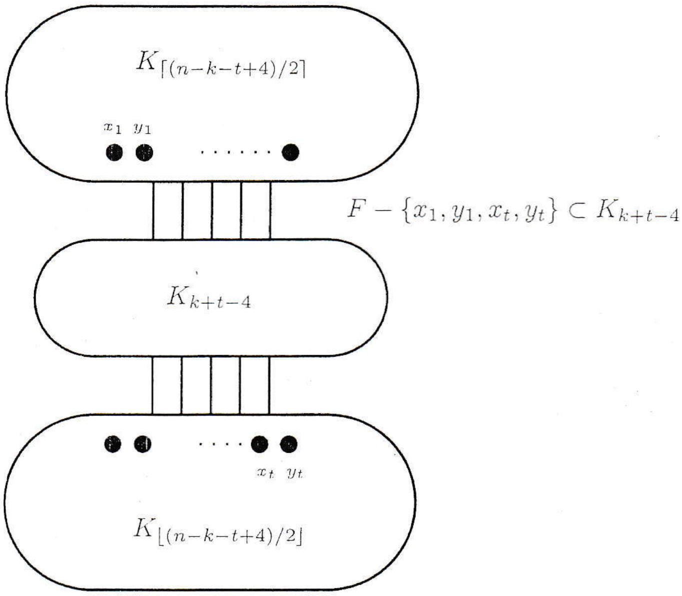

Figure 3: H_1'

The next set of examples gives the sharpness of the results of Theorem 1.

EXAMPLES 3: The graph H_1 in Examples 2 also applies when $s \geq 0$. Also, $\delta(H_1) = \lfloor (n+k+t-4)/2 \rfloor$, and H_1 is not (k,t,s)-ordered.

For $t \leq 2$ the graph H_4 in Examples 2 is not strongly (k,t,s)-ordered hamiltonian, and $\delta(H_4) = \lfloor (n+k-1)/2 \rfloor$.

The following collection of examples $H_1', H_2', H_3', H_4^*, H_4^{**}, H_4$ verify the sharpness of Theorem 2. Each of the examples is a slight variation of the corresponding graphs in Examples 2, and the first three graphs are pictured in Figures 3, 4 and 5.

EXAMPLES 4: Let F be a (k,t,s)-linear forest with x_i and y_i the endvertices of the i^{th} path of F for $1 \leq i \leq t$. When $s = 0$,

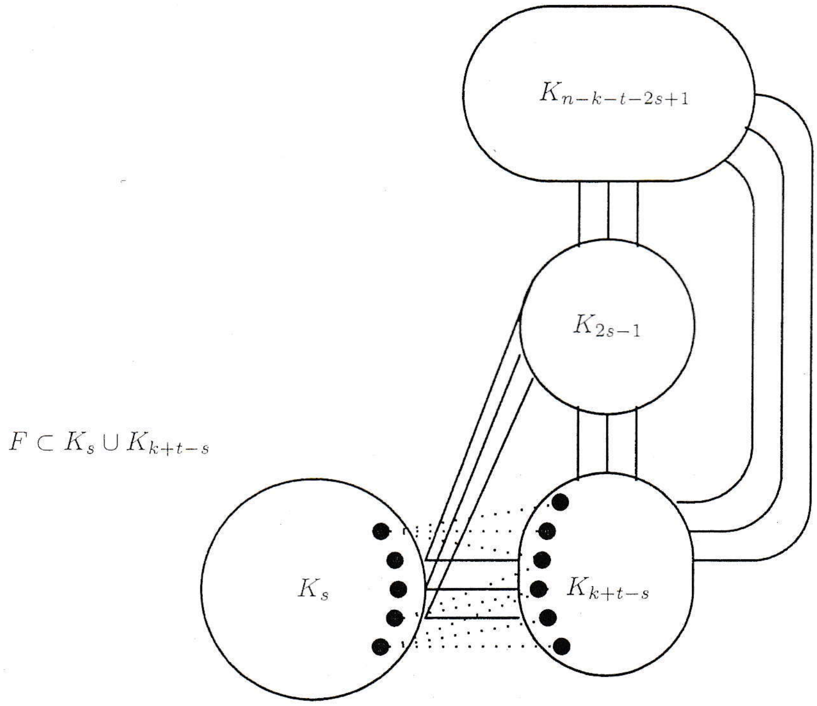

Figure 4: H_2'

let $H_1' = K + (A \cup B)$, where $A = K_{\lceil \frac{n-k-t+4}{2} \rceil}$, $K = K_{k+t-4}$, and $B = K_{\lfloor \frac{n-k-t+4}{2} \rfloor}$. The degree sum $\sigma_2(H_1') = n+k+t-6$, and H_1' is not $(k,t,0)$-ordered if $x_1, y_1 \in A$, $x_t, y_t \in B$, and $F - \{x_1, y_1, x_t, y_t\} \subseteq K$. For $t \geq 5$ this is the extremal example for case (i).

For $0 < 2s \leq t$ let H_2' be the graph derived from $A+(T+(S \cup B))$, where $S = K_s, T = K_{k+t-s}, A = K_{2s-1}$, and $B = K_{n-k-t-2s+1}$. For each vertex $s_i \in S$, pick four vertices $u_i', u_i, v_i, v_i' \in T$. Delete the edges $s_iu_i', s_iu_i, s_iv_is_iv_i'$ for each i to form H_2'. We have $\sigma_2(H_2') = n+k+t+s-7$, but if we pick $F \subseteq S \cup T$, such that $x_{2i} = y_{2i} = s_i, y_{2i+1} = u_i', x_{2i+1} = u_i, x_{2i-1} = v_i', y_{2i-1} = v_i$ for all $i \leq s$, then H_2' is not (k,t,s)-ordered. For $s \geq 3$ this is the extremal example for case (ii).

For $t < 2s$ let H_3' be derived from $A + (T + (S \cup B))$, where

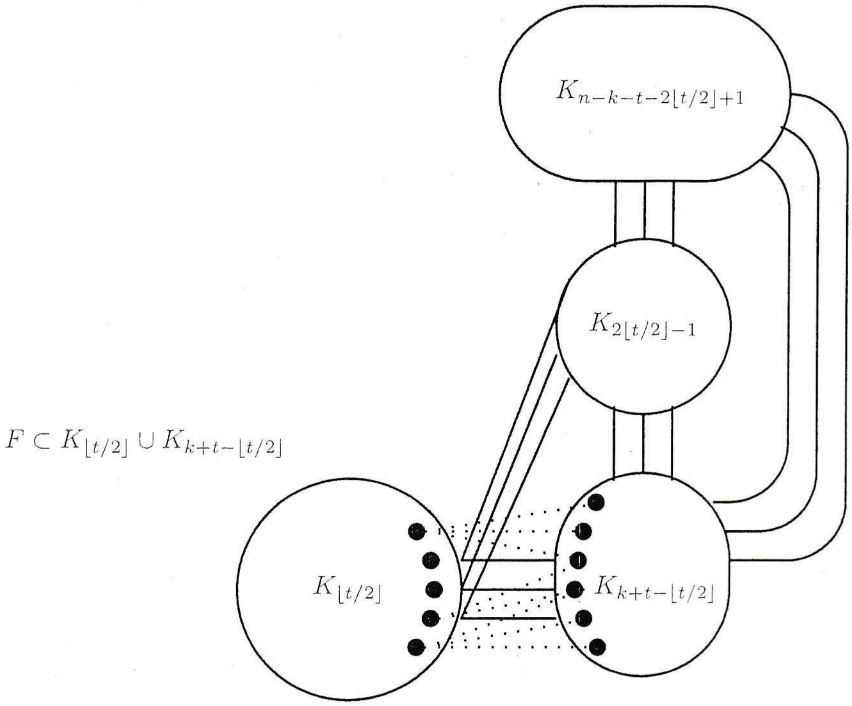

Figure 5: H_3'

$S = K_{\lfloor \frac{t}{2} \rfloor}, T = K_{t+k-\lfloor \frac{t}{2} \rfloor}, A = K_{2\lfloor t/2\rfloor-1}$, and $B = K_{n-k-t-2\lfloor t/2\rfloor+1}$. Pick $F \subseteq S \cup T$, such that $x_{2i} = y_{2i} = s_i \in S$, $i \leq \lfloor \frac{t}{2} \rfloor$, and the remaining vertices of F are in T. For each i delete the edges $s_i x_{2i-1}, s_i y_{2i-1}, s_i x_{2i+1}, s_i y_{2i+1}$ between S and T to form H_3'. The number of edges deleted adjacent to s_i will be either 2, 3, or 4 depending on whether $x_{2i-1} = y_{2i-1}$ and/or $x_{2i+1} = y_{2i+1}$. If $s = t$, then two edges will be deleted from each s_i, if $3(t-1)/4 \leq s < t$, then the vertices of F can be arranged such that no more than three edges will be deleted from any vertex s_i, and in the remaining cases some vertex s_i will have four edges deleted. Therefore with $\epsilon_{s,t} = 0, 1$ or 2 for $s = t$, $3(t-1)/4 \leq s < t$, or $0 \leq s < 3(t-1)/4$ respectively we have

$$\sigma_2(H_3') = n + k + \left\lfloor \frac{3t}{2} \right\rfloor - 5 - \epsilon_{s,t},$$

and H'_3 is not (k,t,s)-ordered. The graph H'_3 is an extremal graph for case (iii).

The graph $H_4 = \overline{K}_{\lceil (n-k+1)/2 \rceil} + K_{\lfloor (n+k-1)/2 \rfloor}$ of Examples 2 with the the linear forest F contained in $K_{\lfloor (n+k-1)/2 \rfloor}$ is not (strongly) (k,t,s)-ordered hamiltonian, with $\sigma_2(H_4) = n+k-1$. Thus, when $s \leq 1$ and $t = 4$, $\sigma_2(H_4) = n+k+t-5$, when $s \leq 2$ and $t = 3$, $\sigma_2(G) = n+k+t-4$, and when $t \leq 2$, $\sigma_2(H_4) = n+k-1$. Hence, the graph H_4 provides an extremal example for case (v).

For $s = 2$ the graph H'_1 can be modified to form $H^*_4 = K_{k+t-2} + (K_{\lceil (n-k-t+2)/2 \rceil} \cup K_{\lfloor (n-k-t+2)/2 \rfloor})$ with all of the forest F in K_{k+t-2} except for two single vertices of F that are in different components of the graph $H^*_4 - K_{k+t-2}$. Then H^*_4 is not (k,t,s)-ordered if the single vertices of F are consecutive in the ordering of F, and $\sigma_2(H^*_4) = n+k+t-4$. The graph H^*_4 is an extremal graph when $s = 2$ and $t \geq 3$. Likewise, the graph H'_1 can be modified to form $H^{**}_4 = K_{k+t-3} + (K_{\lceil (n-k-t+3)/2 \rceil} \cup K_{\lfloor (n-k-t+3)/2 \rfloor})$ with all of the forest F in K_{k+t-3} except for a single vertex and the endvertices of a path of F that are in different components of the graph $H^{**}_4 - K_{k+t-3}$. Then H^{**}_4 is not (k,t,s)-ordered if the single vertex and the two other endvertices of F are in consecutive components in the ordering of F, and $\sigma_2(H^{**}_4) = n+k+t-5$. The graph H^*_4 is an extremal graph when $s = 1$ and $t \geq 3$.

For $t \leq 2$, the graph H_4 of Examples 2 is the extremal example for case (iv), since it is not (k,t,s)-ordered hamiltonian and $\sigma_2(H_4) = n+k-1$.

The examples showing the sharpness of Theorem 3 follow immediately from Examples 2 and Examples 4, or are slight variations and are identified in the following.

EXAMPLES 5: When $s = 0$ in Theorem 3 the graph H'_1 in Examples 4 also applies and H'_1 is not $(k,t,0)$-ordered, and $\delta(H'_1) = \lfloor (n+k+t-6)/2 \rfloor$. In the case when $s = 1$ and $t \geq 4$ the graph H'_1 can be modified to form a graph H''_1, which considers the case when $x_1 = y_1$ and $x_t \neq y_t$. In this case $\delta(H''_1) = \lfloor (n+k+t-5)/2 \rfloor$ and H''_1 is not (k,t,s)-ordered. For case (iii) and $1 < s < t$. H'_1 can be modified to H'''_1 where $x_1 = y_1$ and $x_t = y_t$. In this case $\delta(H'''_1) = \lfloor (n+k+t-4)/2 \rfloor$ and H'''_1 is not (k,t,s)-ordered. The graphs H'_1, H''_2 and H'''_1 are extremal graphs for the cases (i), (ii), and (iii) except for some small values of s and t.

For $t \leq 2$ the graph H_4 in Examples 2 is not (k, t, s)-ordered hamiltonian, and $\delta(H_4) = \lfloor (n + k - 1)/2 \rfloor$. When $s = 0$ and $t = 4$, the graph H_4 is also not (k, t, s)-ordered hamiltonian and $\delta(H_4) = \lfloor (n + k - 1)/2 \rfloor = \lfloor (n + k + t - 5)/2$. Likewise, when $s = 0$ or 1 and $t = 3$, $\delta(H_4) = \lfloor (n + k - t - 4)/2 \rfloor$, and H_4 is not (k, t, s)-ordered hamiltonian. Thus H_4 is the extremal graph for the remaining cases in Theorem 3.

Before stating some results that will be needed in the proofs, one critical definition on t-linked graphs will be introduced.

Definition 6 : *A graph G of order at least $2t$ is t-linked, if for every vertex set $X = \{x_1, x_2, \cdots, x_t, y_1, y_2, \cdots, y_t\}$ of $2t$ vertices, there are t vertex disjoint $x_i - y_i$ paths.*

It should be noted that the linked property remains the same if we allow repetition in X, and ask for t internally disjoint $x_i - y_i$ paths. Thus, as an easy consequence, every t-linked graph with $t = k + s$ is (k, t, s)-ordered.

Theorem 8 : [TW05] *Every $10t$-connected graph is t-linked.*

This is an improvement of a result by Bollobás and Thomason [BT96] that $22t$-connected is sufficient for t-linked. Another result by Kawarabayshi, Kostochka, and Yu [KKY06] implies that 12-connected is sufficient for t-linked.

Lemma 1 : [CFGJLP04] *If a $2t$-connected graph G has a t-linked subgraph H, then G is t-linked.*

Lemma 2 : [CFGJLP04] *If G is a graph, v is a vertex of G with $d(v) \geq 2t - 1$, and $G - v$ is t-linked, then G is t-linked.*

Theorem 9 : [M72] *Every graph G with $|V(G)| = n \geq 2t - 1$ and $|E(G)| \geq 2tn$ has a t-connected subgraph.*

Theorem 10 : [CFGJLP04] *Let k, s, t be integers with $0 \leq s < t$ and $k \geq 0$. If G is a (strongly) (k, t, s)-ordered graph on $n \geq k$ vertices with*

(i) $\sigma_2(G) \geq n + k$ *if $s = 0$, or*

(ii) $\sigma_2(G) \geq n + k + s - 1$ *if $s > 0$,*

then G is (strongly) (k, t, s)-ordered hamiltonian

Theorem 11 : [CFGJLP04] *For $t \geq 1$, if G is a (strongly) (k, t, s)-ordered graph on $n \geq k$ vertices with $\delta(G) \geq \frac{n+k+s}{2}$, then G is (strongly) (k, t, s)-ordered hamiltonian.*

A similar result was also proved in [CFGJLP04] that will be useful in extending from (strongly) (k, t, s)-ordered to (strongly) (k, t, s)-ordered hamiltonian.

Theorem 12 : [CFGJLP04] *Let k, s, t be integers with $1 < t/2 < s \leq t$. If G is a (strongly) (k, t, s)-ordered graph of sufficiently large order n with $\sigma_2(G) \geq n+k+\frac{t-3}{2}$, then G is (strongly) (k, t, s)-ordered hamiltonian.*

4 PROOFS

We begin with a lemma that allows us to reduce the proofs of Theorems 1, 2 and 3 to the consideration of linear forests in which all of the paths are either edges or vertices.

Lemma 3 : *Let $0 \leq k$, $0 \leq s \leq t$ be integers. If for any graph G' of order n' the condition $\sigma_2(G') \geq q'$ implies that G' is (strongly) $(t-s, t, s)$-ordered hamiltonian, then the graph G of order $n = n' + (k-t+s)$ with $\sigma_2(G) \geq q = q' + 2(k-t+s)$ is (strongly) (k, t, s)-ordered hamiltonian. Likewise, if $\delta(G') \geq q'$ implies that G' is (strongly) $(t-s, t, s)$-ordered hamiltonian, then the graph G of order $n = n' + (k-t+s)$ with $\delta(G) \geq q = q' + (k-t+s)$ is (strongly) (k, t, s)-ordered hamiltonian.*

Proof: Let F be a (strongly) ordered (k, t, s)-linear forest in the graph G or order n with $\sigma_2(G) \geq q$. Let G' be the graph obtained from G by deleting the interior vertices of each of the paths of F of length at least two and replacing each path of F with an edge. Thus, G' has $n' = n - (k - t + s)$ vertices, $\sigma_2(G') \geq q - 2(k - t + s) = q'$, and the forest F' replacing F is a (strongly) ordered $(t - s, t, s)$-linear forest. By assumption, G' is (strongly) $(t - s, t, s)$-ordered hamiltonian and has an appropriate hamiltonian cycle containing F'. Now, by replacing each edge of the forest F' by the appropriate path of F gives the required hamiltonian cycle in G. The proof for the conditions $\delta(G) \geq q$ and $\delta(G') \geq q - (k - t + s)$ is identical. This completes the proof of Lemma 3. $\square$

The next lemma, which insures the existence of a small cycle containing a linear forest, will be useful in several of the remaining proofs.

Lemma 4 *: Let $2 \leq t$, $0 \leq s \leq t$ be integers, and let F be a (strongly) ordered $(t - s, t, s)$-linear forest. If G is a graph of order n with $\sigma_2(G) \geq n$ that contains a nonhamiltonian cycle containing F as a (strongly) ordered linear forest, then G also contains such a cycle of order at most $6t$.*

Proof: Let $F = x_1y_1 \cup x_2y_2 \cup \cdots \cup x_ty_t$ with of course in s of these cases $x_i = y_i$. Let C be the smallest order cycle that contains F as a (strongly) ordered linear forest, and let P_i be the path in C between y_i and x_{i+1} with the indices taken modulo t. Let $H = G - C$.

We will first show that $|C| \leq 3t^2 + t$. Assume not. Note that if $h \in H$, then h can be adjacent to a most 3 vertices of any path P_i of C, for otherwise the cycle C is not minimum in length. Thus, $d_C(h) \leq 3t$. If $h_1, h_2 \in H$ and $h_1h_2 \notin H$, then $d_H(h_1) + d_H(h_2) \geq n - 6t > |H|$, and so each pair of vertices of H have a path of length at most 2 between them. Observe that if some path P_i of C has as many as 6 vertices with adjacencies in H, then the cycle C is not of minimum length since some path of length at least 5 can be replaced by a path of length at most 4. Hence, there is a vertex, say $z \in C$, that has no adjacencies in H, and we can assume that z is in the longest path P_j for some j. Also, the vertex z is not adjacent to any other vertices of P_j except for the two vertices adjacent on the cycle C. Thus, $d_C(z) \leq (t-1)/t)|C| + 1$. This implies that

$$n \leq d(h) + d(z) \leq n - |C| - 1 + 3t + ((t-1)/t)|C| + 1 < n,$$

a contradiction.

We can now assume that $|C| \leq 3t^2 + t$. First consider the case when at least one of the paths, say P_1, has as many as 7 vertices. Thus, $P_1 = (z_1, z_2, \cdots, z_6, z_7, \cdots)$. The set of vertices $Z = \{z_1, z_4, z_7\}$ is independent. Since $\sigma_2(G) \geq n$ and n is sufficiently large, some pair of vertices of Z has a common adjacency outside of C. This contradicts the minimum length of C. Therefore, we can assume that each P_i has at most 6 vertices, and so $|C| \leq 6t$. This completes the proof of Lemma 4. □

The following Lemma determines the structure of a graph that is dense but is not (strongly) $(t-s, t, s)$-ordered.

Lemma 5 : *Let $2 \leq t$, $0 \leq s \leq t$ be integers. If G is a graph of sufficiently large order n with $\sigma_2(G) \geq n + q$ that is not (strongly) $(t-s, t, s)$-ordered and is edge maximal with respect to this property, then $G \supseteq X + (A \cup B)$ where X is a minimum cut set of G with $q + 2 \leq |X| \leq 2t - 1$, and X can be partitioned into three sets X', X_A, X_B such that the vertices of X' are adjacent to all of the vertices of $A \cup B$, and the graphs $A \cup X_A$ and $B \cup X_B$ are complete graphs. Also, each vertex of X_A (X_B) has at most $6t$ adjacencies in B (A), and if $|B| \geq |A| > 12t^2$, then $|X'| \geq q + 2$ and the vertices of X' have degree $n - 1$. In addition, if $\delta(G) \geq (n+q)/2$, then $|A|, |B| \geq (n+q)/2 - 2t + 2$.*

Proof: Let F be a (strongly) ordered $(t-s, t, s)$-linear forest. Assume G is not (strongly) $(t-s, t, s)$-ordered and edge-maximal with respect to this property. If G has a vertex v of degree strictly less than $n/2$, then all of the vertices nonadjacent to v have degree at least $n/2$, and otherwise all of the vertices have degree at least $n/2$. Thus, in either case G always has at least $n^2/8$ edges. Theorem 9 implies that G has a $10t$-connected subgraph, and by Theorem 8, G has a t-linked subgraph H. If G is $2t$-connected, then G is t-linked by Lemma 1, and thus G is (strongly) (t - s, t, s)-ordered, a contradiction. Hence, we can assume that G is at most $2t-1$ connected. Therefore,

$$q+2 \leq \kappa(G) \leq 2t-1.$$

Let the components of F be $x_1y_1, x_2y_2, \cdots, x_ty_t$ with the understanding that if the i^{th} component is a vertex, then $x_i = y_i$, and this will occur s times.

Select vertices u and v such that $uv \notin G$. Since G is edge-maximal, $G^* = G + uv$ has a cycle C^* containing F such that the edge uv is on the cycle, say between y_1 and x_2. Select vertices u and v such that the cycle C^* is of minimum length. Note that C^* will not be hamiltonian, since for any choice of uv with $u, v \notin F$ the smallest cycle containing uv and the forest F will not be hamiltonian. By Lemma 4 the cycle C^* has length at most $6t$. If there is a vertex w such that $uw, vw \in G$, but $w \notin C^*$, then there is the required cycle in G. Thus, we can assume that all of the common adjacencies of u and v are in C^*, and so $|N(u) \cap N(v)| \leq 6t$. This implies that if x and y are two vertices of G with $|N(x) \cap N(y)| > 6t$, then $xy \in G$.

Let X be a minimal cut set of G, and so $q+2 \leq |X| \leq 2t-1$. Let A and B be the components of $G - X$, and we can assume that $|A| \leq |B|$. We will show that A and B are complete. First, we will show that any vertex $a \in A$ (or $b \in B$) can be nonadjacent to at most $2t-2$ vertices of $A \cup X$ (or $B \cup X$). Let $a \in A$ and assume $d(a) = |A| + |X| - m$ for some $m > 0$. Let $b \in B$. Then, since a_1 and a_2 have at most $6t$ common adjacencies,

$$n \leq n+q \leq d(a)+d(b) \leq [|A|+|X|-m]+[|B|+|X|-1] \leq n+2t-m-2.$$

So, $m \leq 2t-2$. Thus, for every $a \in A$, $d(a) \geq |A| + |X| - 2t + 2$ and for every $b \in B$, $d(b) \geq |B| + |X| - 2t + 2$. Now, assume there exist nonadjacent vertices $a_1, a_2 \in A$. Then,

$$n \leq n+q \leq d(a_1) + d(a_2) \leq (2t-2) + 2(6t) + (2t-2) = 16t - 4$$

which is a contradiction for sufficiently large n. Thus, A is complete. An analogous argument shows B is complete.

Next we will show that for all $x \in X$, $d_A(x) = |A|$ or $d_B(x) = |B|$. First observe that $d_B(x) \leq 6t$ or $d_B(x) = |B|$ since B is complete.

If $d_B(x) \leq 6t < |B|$ and $d_A(x) < |A|$, then there exists $a \in A$ such that $n + q \leq d(a) + d(x) \leq |A| + (2t - 1) - 2 + 6t + d_B(x)$. Thus, $|A| \geq n + q + 3 - 14t$, a contradiction.

Thus, each vertex of X will be adjacent to all of the vertices of A or to all of the vertices of B. Thus, the vertices of X can be partitioned into three sets X', X_A and X_B, such that the vertices of X' are those vertices of X adjacent to all of the vertices of both A and B, the vertices of X_A are adjacent to all of the vertices of A, and the same is true of X_B. Each vertex of X_A has at most $6t$ adjacencies in B, and so nearly all of the vertices of B have no adjacencies in X_A. Also, if A has more than $(6t)(2t)$ vertices, then A will also have a vertex that is not adjacent to any vertex of X_B.

Consider the case when $|A| > 12t^2$. Vertices $x \in A$ and $y \in B$ have at least $q + 2$ common adjacencies, so this implies that $|X'| \geq q+2$, since for appropriate x and y all of their common adjacencies are in X'. Also note that because of large neighborhood intersections the vertices of both X_A and X_B are adjacent to all of the vertices of X'. Hence, each vertex of X' is adjacent to all of the other vertices of G. If $\delta(G) \geq (n+q)/2$, then each vertex of A has at least $(n+q)/2-(2t-1)$ adjacencies in A, and so $|B| \geq |A| \geq (n+q)/2-2t+2$. This completes the proof of Lemma 5. □

We next give the proof of Theorem 1.

Proof: By Lemma 3 we need to consider only linear forests whose components are only edges and vertices. Let F be such a $(t-s, t, s)$-linear forest. The case (ii) when $t \leq 2$ follows directly from Theorem 7 (iv). In Case (i) when $t = 3$ the result follows directly from Theorem 7 (i), (ii), and (iii) depending on whether $s = 0, 1, 2$ or 3. Thus, we can assume that $t \geq 4$. Also, if $s = 0$, then Theorem 7 (i) implies Theorem 1 directly, so we can assume that $s > 0$.

If G is strongly (k, t, s)-ordered graph, then G is strongly (k, t, s)-ordered hamiltonian by Theorem 10 in the case when $s \leq t-2$. When $s = t - 1$ or $s = t$, then $2s > t$ and Theorem 12 implies that G is strongly (k, t, s)-ordered hamiltonian.

Thus, it is sufficient to show that G is a strongly (k, t, s)-ordered graph with $k = t - s$. Assume that G does not have a cycle C containing the strongly ordered (k, t, s)-linear forest F and is edge-maximal with respect to this property. By Lemma 5 we have that

$G \supseteq X+(A \cup B)$ such that X is partitioned into 3 sets $X' \cup X_A \cup X_B$ with $|X'| \geq k+t-1$, each vertex of X' has degree $n-1$, and $A \cup X_A$ and $B \cup X_B$ form complete graphs.

Let the components of F be x_1y_1, x_2, y_2, $\cdots, x_ty_t$ where $x_i = y_i$ in s of the components and indices are modulo t. Observe that from the previous paragraph we know y_ix_{i+1} is an edge in G unless one vertex is in $A \cup X_A$ and the other is in $B \cup X_B$. Let r be the number of pairs of this type. If $|X' - F| \geq r$ then for every nonadjacent pair y_ix_{i+1} there exists a vertex $v \in X' - F$ such that y_ivx_{i+1} is a path in G and a cycle containing F in order can be formed using edges and paths of length two.

Note $|X' - F| = |X'| - |F \cap X'| \geq (k+t-1) - (k+t-2r) \geq r$ provided $r \geq 1$. This completes the proof of Theorem 1. □

The proof of Theorem 2, which is very similar to the proof of Theorem 1, follows.

Proof: By Lemma 3 we need only consider linear forests whose components are edges and vertices. Let F be a $(t-s, t, s)$-linear forest with $k = t-s$. The general structure of the proofs of cases (i)through (iv) are the same, so we will deal with them collectively. We will first show that G is a (k, t, s)-ordered graph, and then later show that this cycle can be extended to a hamiltonian cycle. By Lemma 5 we can assume that $G \supseteq X + (A \cup B)$ such that X is partitioned into 3 sets $X' \cup X_A \cup X_B$ such that each vertex of X' adjacent to all of the vertices of $A \cup B$, and $A \cup X_A$, and $B \cup X_B$ are complete graphs. We also have for the various cases

$$k + t - r \leq \kappa(G) \leq 2t - 1,$$

where $r = 3$ in case (i), $r = 4 - s$ in case (ii), $r = \lceil 4(1-(s/t)) \rceil - (t-5)/2$ in case (iii), $r = 3 - s$ in case (iv), and in case (v) $r = t - 2$ for $2 \leq t \leq 4$.

We will first deal with the situation when $|A| > (6t)(2t)$, in which case $|X'| \geq \sigma_2(G) + 2 - n$, and the vertices of X' have degree $n-1$. Thus in case (i) $|X'| \geq k+t-3$, in case (ii) $|X'| \geq k+t+s-4$, in case (iii) $|X'| \geq k+(3t-5)/2 - \lceil 4(1-s/t) \rceil$, in case (iv) $|X'| \geq k+t+s-3$, and in case (v) $|X'| \geq k+2$.

Let X'' be the vertices of X' that are also vertices of F, let $p = |X''|$, and let k' be the number of edges of F in X''. Recall that we must find t paths connecting consecutive components of F to form a cycle.

If $k' = 0$, then at least p of the paths connecting components of F can be formed from edges incident to X''. Now $|X' - X''| = |X'| - p \geq t - p$ in all cases since $|X'| \geq t$ in all cases. Thus there are $t - p$ vertex disjoint paths of length 2 with the central vertex in $X' - X''$ from any $t - p$ vertices of $A \cup X_A$ to any $t - p$ vertices of $B \cup X_B$. Thus, an appropriate cycle containing F can be formed using such paths and possibly edges in $A \cup X_A$ and $B \cup X_B$.

If $k' > 0$, then at least $p - k' + 1$ paths connecting components of F can be formed from edges incident to vertices in X''. If $|X' - X''| = |X'| - p \geq t - (p - k' + 1)$, we can form the remaining paths in the cycle using paths of length 2 with central vertex in $X' - X''$ as before. Thus, we need $|X'| \geq t + k' - 1$. In case (ii), $|X'| \geq t + k - 1$ and since $k' \leq k$ the desired inequality always holds. In cases (i) and (iv), $|X'| \geq t + k - 1$ unless $|X'| = t + k - 2$ or $|X'| = t + k - 3$. But then $k' \leq k - 1$ or $k' \leq k - 2$ respectively. Thus, in either instance, $|X'| \geq t + k' - 1$. In case (iii), $|X'| \geq t + \frac{t-3}{2} + k - \lceil 4(1 - s/t) \rceil$. Thus, we need $\lceil (k - k') + \frac{t-3}{2} - \lceil 4(1 - s/t) \rceil \rceil \geq 0$ which certainly holds if $s = t$ or $t \geq 6$. This leaves only the two cases where $t = 5, s = 4$ and $t = 4, s = 3$ and by inspection, the inequality holds for both. Finally, in case (v) when $t = 3$ or 4, it is straightforward to check that $|X'| \geq t + k' - 1$ just as in cases (i) and (iv).

Thus in all cases, if $|A| > 12t^2$ there exists a cycle in G containing F in order.

We now consider the case when $|A| \leq 12t^2$. By Lemma 5 we have the vertices of G partitioned into sets A, B and X. Also, X is partitioned into sets X', X_A and X_B such that $A \cup X_A$ and $B \cup X_B \cup X'$ form complete graphs. Each vertex of $A \cup X_A$ has "small degree", for it would be in X' otherwise. Hence, only a small number of vertices of B are adjacent to any vertex of $A \cup X_A$, and so the set of vertices of B nonadjacent to all vertices of $A \cup X_A$, which will be denoted by B^*, is large. Let $A^* = A \cup X_A$, and $X^* = X' \cup X_B \cup (B - B^*)$. For any vertex $u \in A^*$ and $v \in B^*$, $uv \notin G$, and so u and v have at least $\sigma_2(G) - n + 2$ common adjacencies in X^*. The vertices of G are partitioned into three sets A^*, B^* and X^* with $B^* \cup X^*$ and A^* forming complete graphs, and each vertex of A^* has at least $\sigma_2(G) - n + 2$ adjacencies in X^*.

For the linear forest F let $A'' = F \cap A^*, B'' = B^* \cap F$, and $X'' = X^* \cap F$ and let a, b and c be the number of vertices in these respective sets. Since $B^* \cup X^*$ spans a complete graph, to form the t paths for the appropriate cycle containing F, it is sufficient to construct the disjoint paths from the a vertices of A'' to X^*, since there are edges between the remaining vertices of F. Thus, if $a = 0$, then $F \subseteq (B^* \cup X^*)$, so there is a cycle containing F. Thus, we can assume that $a > 0$.

Let s' be the number of singletons in A''. Begin constructing paths from vertices in A'' as follows. Order the vertices of A'' consistent with that on F and such that the last in this ordering is not a singleton (if such an endvertex appears in A''.) Find paths from A'' to $X^* \cup F$ by first choosing an edge to another vertex in A'' if possible, secondly an edge to a vertex in X'' if possible, and then finally a vertex in $X^* - X''$. Note that if u is a vertex in A'' that is a singleton in F, we must find two paths in $X^* \cup F$. Let r be the number of vertices in the components of F just preceeding and just following the last vertex in A''. [That is, if u is a vertex in A'' that, in F, is preceeded by a singleton and followed by an edge, then $r = 3$. So for all vertices, $r = 2, 3$, or 4.] Then, all vertices in A'', including the last, have at least M vertices available in $X^* \cup F$ where $M = \sigma_2(G) - n + 2 - c - 2(a-1) + (r - \min\{r, b\})$ if A'' contains all singletons of F and $M = \sigma_2(G) - n + 2 - c - (a + s' - 1) + (r - \min\{r, b\})$ otherwise. Note in the expressions for M, the term $(r - \min\{r, b\})$ adds back the number of vertices we (may) have mistakenly deleted by ignoring consecutive components in F. If the last vertex in A'' is a singleton, we need $M \geq 2$. If the last vertex in A'' is the endvertex of an edge in F, we need $M \geq 1$.

Case (i):

Since $s = 0$, we know $s' = 0$, $r = 4$. Thus, $M \geq k + t - 3 - c - (a - 1) + (4 - \min\{4, b\}) = (b - \min\{4, b\}) + 2 \geq 1$.

Cases (ii) and (iv):

Note that in both cases, $\sigma_2 \geq n + k + t + s - 6$ so each vertex of A' has at least $k + t + s - 4$ adjacencies in X^*.

If $s' = a$, then

$$M = k + t + s - 4 - c - 2(a-1) + (r - \min\{r, b\})$$
$$= (\text{b-}\min\{r, b\}) + (s - s') + r - 2.$$

If the last vertex in A'' has no vertex of A'' that is consecutive in the order of F, then $s' = s$ forces $r = 4$; $s' = s - 1$ forces $r \geq 3$; $s' \leq s - 2$ forces $r \geq 2$. So $M \geq 2$. If the last vertex in A'' has precisely one vertex of A'' that is consecutive in F, then $s' = s$ forces $r = 3$; $s' = s - 1$ forces $r \geq 2$; $s' \leq s - 2$ forces $r \geq 2$. So $M \geq 1$. In all of these situations the cycle can be completed. If the last vertex in A'' has two vertices of A'' consecutive in the order of F, then the cycle clearly can be formed.

If $s' < a$, then

$$M = k + t + s - 4 - c - (a + s' - 1) + (r - \min\{r, b\})$$

$$= (\text{b-min}\{r, b\}) + (s - s') + r - 3.$$

If $s' \leq s - 2$, then $M \geq 1$. If $s' = s - 1$, either the last vertex u is adjacent to a singleton in A'' or $r \geq 3$. In either case, $M \geq 1$. If $s' = s$, either u is adjacent to a singleton of A'' or $r \geq 4$ and $M \geq 1$.

Case (iii):

So each vertex of A'' has at least $k + (3t - 5)/2 - \epsilon_{s,t}$ adjacencies in X^*. As before, we will form the cycle containing F by sequentially finding paths from the vertices of A'' to X^*, and the notation of a, b, c and s' of the previous case will be used in this case. The number of adjacencies in X^* available to form the paths for the last vertex of A'' is

$$k + (3t - 5)/2 - \epsilon_{s,t} - (c - (r - \min\{r, b\})) - (a + s' - 1 - \alpha)$$

$$\geq (t - 5)/2 + (b - \min\{r, b\}) - \epsilon_{s,t} + r + 1 + \alpha - s', \quad (1)$$

where $\alpha = 1$ if the last vertex of A'' is a singleton. If the number in the preceding equation is at least $1 + \delta$, then the cycle can be formed. Also, the minimal case in the previous inequality is when $r = 2$, and so it is sufficient to consider the $r = 2$ case. Hence, we can assume that $s' > (t - 1)/2 - \epsilon_{s,t} + (b - \min\{2, b\})$, for the required cycle could be formed otherwise. Also, as observed earlier, we can assume that $c > k + (t - 5)/2 - \epsilon_{s,t}$ and so $a \leq (t + 4)/2 + \epsilon_{s,t} - b$, for otherwise the required cycle could be formed. Thus,

$$t/2 - \epsilon_{s,t} + (b - \min\{2, b\}) \leq s' \leq a \leq (t + 4)/2 + \epsilon_{s,t} - b. \quad (2)$$

Since G is edge maximal with respect to having the required cycle containing F, all edges between pairs of vertices of F that are not in consecutive components of F cannot be used in any cycle containing F. Hence, if there are vertices from three components of F in A'' or two consecutive components of F, then there will be no vertices in B'', since any such vertex would have to be in a component consecutive to each of the components represented in A''. Also, if there are vertices in as many as two components of F in A'', then $b \leq 2$, since at most two vertices can be consecutive components of F of two distinct components.

If $b \geq 5$, then the inequality (2) gives a contradiction. If $b = 4$, then inequality (2) becomes an equality which implies that t is even and $a = s' = t/2$, and $\epsilon_{s,t} = 2$. However, this implies that $t \geq 6$, $s' \geq 3$, and so by the remarks of the previous paragraph $b = 0$. Hence, we can assume that $b \leq 3$. The same argument applies if $b = 3$ and $\epsilon_{s,t} = 1$, and so if $b = 3$, then $\epsilon_{s,t} = 2$ and $t/2 - 1 \leq s' \leq t/2 + 1$. Since $b = 3$ implies that $s' = 1$, and so $t = 4$. Also $\epsilon_{s,t} = 2$ implies $s = 2$, a contradiction, so we can assume that $b \leq 2$.

Observe that if $s' = t/2 + p$ for some integer $p > 0$ with t even, then the term $(a + s' - 2)$ in displayed expression (1) becomes $(a + s' - 2 - 4p)$, since at least $2p$ of the isolated pairs of vertices of F in A'' would be consecutive in F, adjacent in A'', and would need no adjacencies in X^* to form the cycle. Thus, the modified expression for the lower bound of (1) would be $(t - 4)/2 + (b - \min\{2, b\}) - \epsilon_{s,t} + 4 + 4p - (t/2 + p) \geq 3p \geq 2$ and for $p \geq 1$. Thus, for t even, we can assume that $s' \leq t/2$. If $s' = (t - 1)/2 + p$ for t odd, then the modified expression (1) becomes $(t - 5)/2 + (b - \min\{2, b\}) - \epsilon_{s,t} + 4 + 4p - 2 - ((t - 1)/2 + p) \geq 3p - \epsilon_{s,t}$. Thus, if $p \geq 2$ or if $\epsilon_{s,t} = 1$, this implies there are two choices for the selection of adjacencies in X^* for the last vertex of A''. If $p = 1$ and $\epsilon_{s,t} = 2$, then either there are more that one pair of the isolated vertices of $F \cap A''$ that are consecutive or the last vertex used to build the path can be assumed to have four, not just two, potential adjacencies in F, which increases the count in the expression (1) by two. Thus, we can assume for t odd that $s' \leq (t - 1)/2$.

If $s' = t/2$, then either two of the isolated vertices of $F \cap A''$ are consecutive or the last vertex of A'' used to complete the cycle has $2 + \epsilon_{s,t}$ potential vertices of F it can be adjacent to. Based on the modification of expression (1) this implies that the possible choices for the edges needed to complete the cycle coming from this last vertex is at least $(t-4)/2+4-t/2 \geq 2$, so the cycle can be completed. The same arguments apply for the other cases that are left, namely $s' = (t-4)/2, (t-3)/2, (t-2)/2$, and $(t-1)/2$. This completes the proof of this case.

The case (v) when $t \leq 2$ follows directly from Theorem 7 (iv). The subcase $t = 3$ and $s \leq 2$ and the subcase $t = 4$ and $s = 1$ is a consequence of the argument used in the proof of cases (ii) and (iv). Also, the subcase $t = 4$ and $s = 0$ follows directly from the argument for case (i).

Thus, in all of the cases, the graph G is (k, t, s)-ordered, so it remains to shown that the cycle can be extended to a hamiltonian cycle.

In case (*i*), $\sigma_2(G) \geq n + k$ and $s = 0$. So by Theorem 10, the ordered cycle containing F can be extended to a hamiltonian cycle. In cases (ii) and (iv) $\sigma_2(G) \geq n + k + s - 1$ and $s \geq 1$. Thus, by Theorem 10 the ordered cycle containing F can be extended to a hamiltonian cycle. In case (iii), $\sigma_2(G) \geq n + k + (t-3)/2$ for all values of s and t of this case, and so by Theorem 12, the ordered cycle containing F can be extended to a hamiltonian cycle. Case (v) when $t \leq 2$ follows directly from Theorem 7 (iv), and the remaining subcases of (v) follow from Theorems 10 and 12. This completes the proof of Theorem 2. □

Finally for the proof of Theorem 3, observe that the only part of Theorem 3 that does not follow immediately from Theorem 2 is part (*iii*) where $s > t/2$. The proof of this part is exactly the same as the proof of case (*ii*) of Theorem 2 in the instance that $|A| > 12t^2$. □

5 QUESTIONS

There are numerous questions that arise from these results and some natural generalizations. Sharp minimum degree conditions and sum of degrees of nonadjacent vertices conditions have been proved that imply a graph is (strongly) (k, t, s)-ordered hamiltonian. It is natural to consider other conditions that imply hamiltonicity, such as closure conditions, to see if they imply (strongly) (k,t,s)-ordered hamiltonian.

Question 1 *What degree conditions or closure conditions that imply hamiltonian type conditions also imply that a graph is (strongly) (k,t,s)-ordered hamiltonian?*

It is also natural to consider placing linear forests on many smaller cycles, and in particular are there pancyclic type properties analogous to those introduced by Bondy in [B71] that replace the hamiltonian property.

Question 2 *What are the minimum degree conditions and sum of degrees of nonadjacent vertices that imply that a graph contains cycles of all possible lengths containing a (strongly) ordered (k,t,s)-linear forest?*

Linear forests have been placed on hamiltonian cycles, but no requirement on the components of the linear forest being distributed, maybe uniformly, on the hamiltonian cycle.

Question 3 *Do conditions similar to the the minimum degree conditions and sum of degrees of nonadjacent vertices that imply that a graph is (strongly) (k,t,s)-ordered hamiltonian also imply that the components can be distributed uniformly on the cycle.*

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